TOPICS IN TOPOLOGY AND HOMOTOPY THEORY

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PREFACE

This book is addressed to those readers who have been through Rotman[†] (or its equivalent), possess a wellthumbed copy of Spanier[‡], and have a good background in algebra and general topology.

Granted these prerequisites, my intention is to provide at the core a state of the art treatment of the homotopical foundations of algebraic topology. The depth of coverage is substantial and I have made a point to include material which is ordinarily not included, for instance, an account of algebraic K-theory in the sense of Waldhausen. There is also a systematic treatment of ANR theory (but, reluctantly, the connections with modern geometric topology have been omitted). However, truly advanced topics are not considered (e.g., equivariant stable homotopy theory, surgery, infinite dimensional topology, étale K-theory, ...). Still, one should not get the impression that what remains is easy: There are numerous difficult technical results that have to be brought to heel.

Instead of laying out a synopsis of each chapter, here is a sample of some of what is taken up.

(1) Nilpotency and its role in homotopy theory.

(2) Bousfield's theory of the localization of spaces and spectra.

(3) Homotopy limits and colimits and their applications.

(4) The James construction, symmetric products, and the Dold-Thom theorem.

(5) Brown and Adams representability in the setting of triangulated categories.

(6) Operads and the May-Thomason theorem on the uniqueness of infinite loop space machines.

(7) The plus construction and theorems A and B of Quillen.

(8) Hopkins' global picture of stable homotopy theory.

(9) Model categories, cofibration categories, and Waldhausen categories.

(10) The Dugundji extension theorem and its consequences.

A book of this type is not meant to be read linearly. For example, a reader wishing to study stable homotopy theory could start by perusing §12 and §15 and then proceed to §16 and §17 or a reader who wants to learn the theory of dimension could immediately turn to §19 and §20. One could also base a second year course in algebraic topology on §3–§11. Many other combinations are possible.

[†] *An Introduction to Algebraic Topology*, Springer Verlag (1988).

[‡] *Algebraic Topology*, Springer Verlag (1989).

Structurally, each § has its own set of references (both books and articles). No attempt has been made to append remarks of a historical nature but for this, the reader can do no better than turn to Dieudonne[†]. Finally, numerous exercises and problems (in the form of "examples" and "facts") are scattered throughout the text, most with partial or complete solutions.

[†] *A History of Algebraic and Differential Topology* 1900–1960, Birkhäuser (1989); see also, Adams, *Proc. Sympos. Pure Math.* **22** (1971), 1–22 and Whitehead, *Bull. Amer. Math. Soc.* **8** (1983), 1–29.

NOTATION

(1) $\mathbf{N}$, the positive integers; $\mathbf{Z}$, the integers; $\mathbf{Q}$, the rational numbers; $\mathbf{P}$, the irrational numbers; $\mathbf{R}$, the real numbers; $\mathbf{C}$, the complex numbers; $\mathbf{H}$, the quaternions; $\mathbf{\Pi}$, the prime numbers.

(2) $\mathbf{R}^n = \mathbf{R} \times \cdots \times \mathbf{R}$ (n factors); $\mathbf{D}^n = \{x \in \mathbf{R}^n : \|x\| \leq 1\}$; $\mathbf{B}^n = \{x \in \mathbf{R}^n : \|x\| < 1\}$; $\mathbf{S}^{n-1} = \{x \in \mathbf{R}^n : \|x\| = 1\}$; $\mathbf{T}^n = \mathbf{S}^1 \times \cdots \times \mathbf{S}^1$ (n factors).

(3) $\Delta^n = \{x \in \mathbf{R}^{n+1} : \sum_i x_i = 1 \ \& \ \forall\, i, x_i \geq 0\}$; $\mathring{\Delta}^n = \{x \in \mathbf{R}^{n+1} : \sum_i x_i = 1 \ \& \ \forall\, i,\, x_i > 0\}$; $\dot{\Delta}^n = \{x \in \mathbf{R}^{n+1} : \sum_i x_i = 1 \ \& \ \exists\, i,\, x_i = 0\}$.

(4) ω = first infinite ordinal; Ω = first uncountable ordinal.

(5) cl = closure, fr = frontier, wt = weight, int = interior, osc = oscillation.

(6) Given a set S, χ_S is the characteristic function of S and $\#(S)$ is the cardinality of S.

(7) Given a topological space X, $C(X)$ is the set of real valued continuous functions on X and $BC(X)$ is the set of real valued bounded continuous functions on X.

(8) Given a topological space X, X_∞ is the one point compactification of X.

(9) Given a completely regular Hausdorff space X, βX is the Stone-Čech compactification of X.

(10) Given a completely regular Hausdorff space X, υX is the $\mathbf{R}$-compactification of X.

§0. CATEGORIES AND FUNCTORS

In addition to establishing notation and fixing terminology, background material from the theory relevant to the work as a whole is collected below and will be referred to as the need arises.

Given a category $\mathbf{C}$, denote by $\operatorname{Ob}\mathbf{C}$ its class of objects and by $\operatorname{Mor}\mathbf{C}$ its class of morphisms. If X, $Y \in \operatorname{Ob}\mathbf{C}$ is an ordered pair of objects, then $\operatorname{Mor}(X,Y)$ is the set of morphisms (or arrows) from X to Y. An element $f \in \operatorname{Mor}(X,Y)$ is said to have domain X and codomain Y. One writes $f: X \to Y$ or $X \xrightarrow{f} Y$. Functors preserve the arrows, while cofunctors reverse the arrows, i.e., a cofunctor is a functor on $\mathbf{C}^{\mathrm{OP}}$, the category opposite to $\mathbf{C}$.

Here is a list of frequently occurring categories.

(1) **SET**, the category of sets, and $\mathbf{SET}_*$, the category of pointed sets. If $X, Y \in \operatorname{Ob}\mathbf{SET}$, then $\operatorname{Mor}(X,Y) = F(X,Y)$, the functions from X to Y, and if (X,x_0), $(Y,y_0) \in \operatorname{Ob}\mathbf{SET}_*$, then $\operatorname{Mor}((X,x_0),(Y,y_0)) = F(X,x_0;Y,y_0)$, the base point preserving functions from X to Y.

(2) **TOP**, the category of topological spaces, and $\mathbf{TOP}_*$, the category of pointed topological spaces. If $X,Y \in \operatorname{Ob}\mathbf{TOP}$, then $\operatorname{Mor}(X,Y) = C(X,Y)$, the continuous functions from X to Y, and if (X,x_0), $(Y,y_0) \in \operatorname{Ob}\mathbf{TOP}_*$, then $\operatorname{Mor}((X,x_0),(Y,y_0)) = C(X,x_0;Y,y_0)$, the base point preserving continuous functions from X to Y.

(3) $\mathbf{SET}^2$, the category of pairs of sets, and $\mathbf{SET}^2_*$, the category of pointed pairs of sets. If (X,A), $(Y,B) \in \operatorname{Ob}\mathbf{SET}^2$, then $\operatorname{Mor}((X,A),(Y,B)) = F(X,A;Y,B)$, the functions from X to Y that take A to B, and if $(X,A,x_0),(Y,B,y_0) \in \operatorname{Ob}\mathbf{SET}^2_*$, then $\operatorname{Mor}((X,A,x_0),(Y,B,y_0)) = F(X,A,x_0;Y,B,y_0)$, the base point preserving functions from X to Y that take A to B.

(4) $\mathbf{TOP}^2$, the category of pairs of topological spaces, and $\mathbf{TOP}^2_*$, the category of pointed pairs of topological spaces. If (X,A), $(Y,B) \in \operatorname{Ob}\mathbf{TOP}^2$, then $\operatorname{Mor}((X,A),(Y,B)) = C(X,A;Y,B)$, the continuous functions from X to Y that take A to B, and if (X,A,x_0), $(Y,B,y_0) \in \operatorname{Ob}\mathbf{TOP}^2_*$, then $\operatorname{Mor}((X,A,x_0),(Y,B,y_0)) = C(X,A,x_0;Y,B,y_0)$, the base point preserving continuous functions from X to Y that take A to B.

(5) **HTOP**, the homotopy category of topological spaces, and $\mathbf{HTOP}_*$, the homotopy category of pointed topological spaces. If $X,Y \in \operatorname{Ob}\mathbf{HTOP}$, then $\operatorname{Mor}(X,Y) = [X,Y]$, the homotopy classes in $C(X,Y)$, and if $(X,x_0),(Y,y_0) \in \operatorname{Ob}\mathbf{HTOP}_*$, then $\operatorname{Mor}((X,x_0),(Y,y_0)) = [X,x_0;Y,y_0]$, the homotopy classes in $C(X,x_0;Y,y_0)$.

(6) $\mathbf{HTOP}^2$, the homotopy category of pairs of topological spaces, and $\mathbf{HTOP}^2_*$, the homotopy category of pointed pairs of topological spaces. If (X,A), $(Y,B) \in \operatorname{Ob}\mathbf{HTOP}^2$,

then Mor $((X, A), (Y, B)) = [X, A; Y, B]$, the homotopy classes in $C(X, A; Y, B)$, and if $(X, A, x_0), (Y, B, y_0) \in \mathrm{Ob}\, \mathbf{HTOP}^2_*$, then Mor $((X, A, x_0), (Y, B, y_0)) = [X, A, x_0; Y, B, y_0]$, the homotopy classes in $C(X, A, x_0; Y, B, y_0)$.

(7) **HAUS**, the full subcategory of **TOP** whose objects are the Hausdorff spaces and **CPTHAUS**, the full subcategory of **HAUS** whose objects are the compact spaces.

(8) ΠX, the fundamental groupoid of a topological space X.

(9) **GR**, **AB**, **RG** (A-**MOD** or **MOD**-A), the category of groups, abelian groups, rings with unit (left or right A-modules, $A \in \mathrm{Ob}\, \mathbf{RG}$).

(10) **0**, the category with no objects and no arrows. **1**, the category with one object and one arrow. **2**, the category with two objects and one arrow not the identity.

A category is said to be discrete if all its morphisms are identities. Every class is the class of objects of a discrete category.

[Note: A category is small if its class of objects is a set; otherwise it is large. A category is finite (countable) if its class of morphisms is a finite (countable) set.]

In this book, the foundation for category theory is the "one universe" approach taken by Herrlich-Strecker and Osborne (referenced at the end of the §). The key words are "set", "class", and "conglomerate". Thus the issue is not only one of size but also of membership (every set is a class and every class is a conglomerate). Example: $\{\mathrm{Ob}\, \mathbf{SET}\}$ is a conglomerate, not a class (the members of a class are sets).

[Note: A functor $F : \mathbf{C} \to \mathbf{D}$ is a function from Mor $\mathbf{C}$ to Mor $\mathbf{D}$ that preserves identities and respects composition. In particular: F is a class, hence $\{F\}$ is a conglomerate.]

A metacategory is defined in the same way as a category except that the objects and the morphisms are allowed to be conglomerates and the requirement that the conglomerate of morphisms between two objects be a set is dropped. While there are exceptions, most categorical concepts have metacategorical analogs or interpretations. Example: The "category of categories" is a metacategory.

[Note: Every category is a metacategory. On the other hand, it can happen that a metacategory is isomorphic to a category but is not itself a category. Still, the convention is to overlook this technical nicety and treat such a metacategory as a category.]

Given categories $\mathbf{A}, \mathbf{B}, \mathbf{C}$ and functors $\begin{cases} T : \mathbf{A} \to \mathbf{C} \\ S : \mathbf{B} \to \mathbf{C} \end{cases}$, the comma category $|T, S|$ is the category whose objects are the triples $(X, f, Y) : \begin{cases} X \in \mathrm{Ob}\, \mathbf{A} \\ Y \in \mathrm{Ob}\, \mathbf{B} \end{cases}$ & $f \in \mathrm{Mor}\,(TX, SY)$ and whose morphisms $(X, f, Y) \to (X', f', Y')$ are the pairs $(\phi, \psi) : \begin{cases} \phi \in \mathrm{Mor}\,(X, X') \\ \psi \in \mathrm{Mor}\,(Y, Y') \end{cases}$ for which the square $$\begin{tikzcd} TX \arrow[r, "f"] \arrow[d, "T\phi"'] & SY \arrow[d, "S\psi"] \\ TX' \arrow[r, "f'"'] & SY' \end{tikzcd}$$ commutes. Composition is defined componentwise and the identity attached to (X, f, Y) is $(\mathrm{id}_X, \mathrm{id}_Y)$.

$(A\backslash\mathbf{C})$ Let $A \in \mathrm{Ob}\,\mathbf{C}$ and write K_A for the constant functor $\mathbf{1} \rightarrow \mathbf{C}$ with value A—then $A\backslash\mathbf{C} \equiv |K_A, \mathrm{id}_{\mathbf{C}}|$ is the category of objects under A.

$(\mathbf{C}/B)$ Let $B \in \mathrm{Ob}\,\mathbf{C}$ and write K_B for the constant functor $\mathbf{1} \rightarrow \mathbf{C}$ with value B—then $\mathbf{C}/B \equiv |\mathrm{id}_{\mathbf{C}}, K_B|$ is the category of objects over B.

Putting together $A\backslash\mathbf{C}$ & $\mathbf{C}/B$ leads to the category of objects under A and over B: $A\backslash\mathbf{C}/B$. The notation is incomplete since it fails to reflect the choice of the structural morphism $A \rightarrow B$. Examples: (1) $\emptyset\backslash\mathbf{TOP}/* = \mathbf{TOP}$; (2) $*\backslash\mathbf{TOP}/* = \mathbf{TOP}_*$; (3) $A\backslash\mathbf{TOP}/* = A\backslash\mathbf{TOP}$; (4) $\emptyset\backslash\mathbf{TOP}/B = \mathbf{TOP}/B$; (5) $B\backslash\mathbf{TOP}/B = \mathbf{TOP}(B)$, the "exspaces" of James (with structural morphism id_B).

The arrow category $\mathbf{C}(\rightarrow)$ of $\mathbf{C}$ is the comma category $|\mathrm{id}_{\mathbf{C}}, \mathrm{id}_{\mathbf{C}}|$. Examples: (1) $\mathbf{TOP}^2$ is a subcategory of $\mathbf{TOP}(\rightarrow)$; (2) $\mathbf{TOP}_*^2$ is a subcategory of $\mathbf{TOP}_*(\rightarrow)$.

[Note: There are obvious notions of homotopy in $\mathbf{TOP}(\rightarrow)$ or $\mathbf{TOP}_*(\rightarrow)$, from which $\mathbf{HTOP}(\rightarrow)$ or $\mathbf{HTOP}_*(\rightarrow)$.]

The comma category $|K_A, K_B|$ is $\mathrm{Mor}\,(A, B)$ viewed as a discrete category.

A morphism $f : X \rightarrow Y$ in a category $\mathbf{C}$ is said to be an isomorphism if there exists a morphism $g : Y \rightarrow X$ such that $g \circ f = \mathrm{id}_X$ and $f \circ g = \mathrm{id}_Y$. If g exists, then g is unique. It is called the inverse of f and is denoted by f^{-1}. Objects $X, Y \in \mathrm{Ob}\,\mathbf{C}$ are said to be isomorphic, written $X \approx Y$, provided that there is an isomorphism $f : X \rightarrow Y$. The relation "isomorphic to" is an equivalence relation on $\mathrm{Ob}\,\mathbf{C}$.

The isomorphisms in **SET** are the bijective maps, in **TOP** the homeomorphisms, in **HTOP** the homotopy equivalences. The isomorphisms in any full subcategory of **TOP** are the homeomorphisms.

Let $\begin{cases} F : \mathbf{C} \rightarrow \mathbf{D} \\ G : \mathbf{C} \rightarrow \mathbf{D} \end{cases}$ be functors—then a natural transformation Ξ from F to G is a function that assigns to each $X \in \mathrm{Ob}\,\mathbf{C}$ an element $\Xi_X \in \mathrm{Mor}\,(FX, GX)$ such that for every $f \in \mathrm{Mor}\,(X, Y)$ the square $$\begin{tikzcd} FX \arrow[r, "\Xi_X"] \arrow[d, "Ff"'] & GX \arrow[d, "Gf"] \\ FY \arrow[r, "\Xi_Y"'] & GY \end{tikzcd}$$ commutes, Ξ being termed a natural isomorphism if all the Ξ_X are isomorphisms, in which case F and G are said to be naturally isomorphic, written $F \approx G$.

Given categories $\left\{\begin{array}{l}\mathbf{C}\\ \mathbf{D}\end{array}\right.$, the functor category $[\mathbf{C},\mathbf{D}]$ is the metacategory whose objects are the functors $F:\mathbf{C}\to\mathbf{D}$ and whose morphisms are the natural transformations $\mathrm{Nat}(F,G)$ from F to G. In general, $[\mathbf{C},\mathbf{D}]$ need not be isomorphic to a category, although this will be true if $\mathbf{C}$ is small.

[Note: The isomorphisms in $[\mathbf{C},\mathbf{D}]$ are the natural isomorphisms.]

Given categories $\left\{\begin{array}{l}\mathbf{C}\\ \mathbf{D}\end{array}\right.$ and functors $\left\{\begin{array}{l}K:\mathbf{A}\to\mathbf{C}\\ L\ :\mathbf{D}\to\mathbf{B}\end{array}\right.$, there are functors $\left\{\begin{array}{l}[K,\mathbf{D}]:[\mathbf{C},\mathbf{D}]\to[\mathbf{A},\mathbf{D}]\\ [\mathbf{C},L]:[\mathbf{C},\mathbf{D}]\to[\mathbf{C},\mathbf{B}]\end{array}\right.$ defined by $\left\{\begin{array}{l}\text{precomposition}\\ \text{postcomposition}\end{array}\right.$. If $\Xi\in\mathrm{Mor}\,([\mathbf{C},\mathbf{D}])$, then we shall write $\left\{\begin{array}{l}\Xi K\\ L\Xi\end{array}\right.$ in place of $\left\{\begin{array}{l}[K,\mathbf{D}]\Xi\\ [\mathbf{C},L]\Xi\end{array}\right.$, so $L(\Xi K)=(L\Xi)K$.

There is a simple calculus that governs these operations:

$$\left\{\begin{array}{l}\Xi(K\circ K')=(\Xi K)K'\\ (\Xi'\circ\Xi)K=(\Xi'K)\circ(\Xi K)\end{array}\right. \quad\text{and}\quad \left\{\begin{array}{l}(L'\circ L)\Xi=L'(L\Xi)\\ L(\Xi'\circ\Xi)=(L\Xi')\circ(L\Xi)\end{array}\right..$$

A functor $F:\mathbf{C}\to\mathbf{D}$ is said to be faithful (full) if for any ordered pair $X,Y\in\mathrm{Ob}\,\mathbf{C}$, the map $\mathrm{Mor}\,(X,Y)\to\mathrm{Mor}\,(FX,FY)$ is injective (surjective). If F is full and faithful, then F is conservative, i.e., f is an isomorphism iff Ff is an isomorphism.

A category $\mathbf{C}$ is said to be concrete if there exists a faithful functor $U:\mathbf{C}\to\mathbf{SET}$. Example: **TOP** is concrete but **HTOP** is not.

[Note: A category is concrete iff it is isomorphic to a subcategory of **SET**.]

Associated with any object X in a category $\mathbf{C}$ is the functor $\mathrm{Mor}\,(X,—)\in\mathrm{Ob}\,[\mathbf{C},\mathbf{SET}]$ and the cofunctor $\mathrm{Mor}\,(—,X)\in\mathrm{Ob}\,[\mathbf{C}^{\mathbf{OP}},\mathbf{SET}]$. If $F\in\mathrm{Ob}\,[\mathbf{C},\mathbf{SET}]$ is a functor or if $F\in\mathrm{Ob}\,[\mathbf{C}^{\mathbf{OP}},\mathbf{SET}]$ is a cofunctor, then the Yoneda lemma establishes a bijection ι_X between $\mathrm{Nat}(\mathrm{Mor}\,(X,—),F)$ or $\mathrm{Nat}(\mathrm{Mor}\,(—,X),F)$ and FX, viz. $\iota_X(\Xi)=\Xi_X(\mathrm{id}_X)$. Therefore the assignments $\left\{\begin{array}{l}X\to\mathrm{Mor}\,(X,—)\\ X\to\mathrm{Mor}\,(—,X)\end{array}\right.$ lead to functors $\left\{\begin{array}{l}\mathbf{C}^{\mathbf{OP}}\to[\mathbf{C},\mathbf{SET}]\\ \mathbf{C}\to[\mathbf{C}^{\mathbf{OP}},\mathbf{SET}]\end{array}\right.$ that are full, faithful, and injective on objects, the Yoneda embeddings. One says that F is representable (by X) if F is naturally isomorphic to $\mathrm{Mor}\,(X,—)$ or $\mathrm{Mor}\,(—,X)$. Representing objects are isomorphic.

The forgetful functors $\mathbf{TOP}\to\mathbf{SET}$, $\mathbf{GR}\to\mathbf{SET}$, $\mathbf{RG}\to\mathbf{SET}$ are representable. The power set cofunctor $\mathbf{SET}\to\mathbf{SET}$ is representable.

A functor $F : \mathbf{C} \rightarrow \mathbf{D}$ is said to be an isomorphism if there exists a functor $G : \mathbf{D} \rightarrow \mathbf{C}$ such that $G \circ F = \mathrm{id}_{\mathbf{C}}$ and $F \circ G = \mathrm{id}_{\mathbf{D}}$. A functor is an isomorphism iff it is full, faithful, and bijective on objects. Categories $\mathbf{C}$ and $\mathbf{D}$ are said to be isomorphic provided that there is an isomorphism $F : \mathbf{C} \rightarrow \mathbf{D}$.

[Note: An isomorphism between categories is the same as an isomorphism in the "category of categories".]

The full subcategory of TOP whose objects are the A spaces is isomorphic to the category of ordered sets and order preserving maps (reflexive + transitive = order).

[Note: An A space is a topological space X in which the intersection of every collection of open sets is open. Each $x \in X$ is contained in a minimal open set U_x and the relation $x \leq y$ iff $x \in U_y$ is an order on X. On the other hand, if $\leq$ is an order on a set X, then X becomes an A space by taking as a basis the sets $U_x = \{y : y \leq x\}$ $(x \in X)$.]

A functor $F : \mathbf{C} \rightarrow \mathbf{D}$ is said to be an equivalence if there exists a functor $G : \mathbf{D} \rightarrow \mathbf{C}$ such that $G \circ F \approx \mathrm{id}_{\mathbf{C}}$ and $F \circ G \approx \mathrm{id}_{\mathbf{D}}$. A functor is an equivalence iff it is full, faithful, and has a representative image, i.e., for any $Y \in \mathrm{Ob}\,\mathbf{D}$ there exists an $X \in \mathrm{Ob}\,\mathbf{C}$ such that FX is isomorphic to Y. Categories $\mathbf{C}$ and $\mathbf{D}$ are said to be equivalent provided that there is an equivalence $F : \mathbf{C} \rightarrow \mathbf{D}$. The object isomorphism types of equivalent categories are in a one-to-one correspondence.

[Note: If F and G are injective on objects, then $\mathbf{C}$ and $\mathbf{D}$ are isomorphic (categorical "Schroeder-Bernstein").]

The functor from the category of metric spaces and continuous functions to the category of metrizable spaces and continuous functions which assigns to a pair (X, d) the pair (X, τ_d), τ_d the topology on X determined by d, is an equivalence but not an isomorphism.

[Note: The category of metric spaces and continuous functions is not a subcategory of TOP.]

A category is skeletal if isomorphic objects are equal. Given a category $\mathbf{C}$, a skeleton of $\mathbf{C}$ is a full, skeletal subcategory $\overline{\mathbf{C}}$ for which the inclusion $\overline{\mathbf{C}} \rightarrow \mathbf{C}$ has a representative image (hence is an equivalence). Every category has a skeleton and any two skeletons of a category are isomorphic. A category is skeletally small if it has a small skeleton.

The full subcategory of SET whose objects are the cardinal numbers is a skeleton of SET.

A morphism $f : X \rightarrow Y$ in a category $\mathbf{C}$ is said to be a monomorphism if it is left cancellable with respect to composition, i.e., for any pair of morphisms $u, v : Z \rightarrow X$ such that $f \circ u = f \circ v$, there follows $u = v$.

A morphism $f : X \to Y$ in a category $\mathbf{C}$ is said to be an epimorphism if it is right cancellable with respect to composition, i.e., for any pair of morphisms $u, v : Y \to Z$ such that $u \circ f = v \circ f$, there follows $u = v$.

A morphism is said to be a bimorphism if it is both a monomorphism and an epimorphism. Every isomorphism is a bimorphism. A category is said to be balanced if every bimorphism is an isomorphism. The categories **SET**, **GR**, and **AB** are balanced but the category **TOP** is not.

In **SET**, **GR**, and **AB**, a morphism is a monomorphism (epimorphism) iff it is injective (surjective). In any full subcategory of **TOP**, a morphism is a monomorphism iff it is injective. In the full subcategory of $\mathbf{TOP}_*$ whose objects are the connected spaces, there are monomorphisms that are not injective on the underlying sets (covering projections in this category are monomorphisms). In **TOP**, a morphism is an epimorphism iff it is surjective but in **HAUS**, a morphism is an epimorphism iff it has a dense range. The homotopy class of a monomorphism (epimorphism) in **TOP** need not be a monomorphism (epimorphism) in **HTOP**.

Given a category $\mathbf{C}$ and an object X in $\mathbf{C}$, let $M(X)$ be the class of all pairs (Y, f), where $f : Y \to X$ is a monomorphism. Two elements (Y, f) and (Z, g) of $M(X)$ are deemed equivalent if there exists an isomorphism $\phi : Y \to Z$ such that $f = g \circ \phi$. A representative class of monomorphisms in $M(X)$ is a subclass of $M(X)$ that is a system of representatives for this equivalence relation. $\mathbf{C}$ is said to be wellpowered provided that each of its objects has a representative class of monomorphisms which is a set.

Given a category $\mathbf{C}$ and an object X in $\mathbf{C}$, let $E(X)$ be the class of all pairs (Y, f), where $f : X \to Y$ is an epimorphism. Two elements (Y, f) and (Z, g) of $E(X)$ are deemed equivalent if there exists an isomorphism $\phi : Y \to Z$ such that $g = \phi \circ f$. A representative class of epimorphisms in $E(X)$ is a subclass of $E(X)$ that is a system of representatives for this equivalence relation. $\mathbf{C}$ is said to be cowellpowered provided that each of its objects has a representative class of epimorphisms which is a set.

SET, **GR**, **AB**, **TOP** (or **HAUS**) are wellpowered and cowellpowered. The category of ordinal numbers is wellpowered but not cowellpowered.

A monomorphism $f : X \to Y$ in a category $\mathbf{C}$ is said to be extremal provided that in any factorization $f = h \circ g$, if g is an epimorphism, then g is an isomorphism.

An epimorphism $f : X \to Y$ in a category $\mathbf{C}$ is said to be extremal provided that in any factorization $f = h \circ g$, if h is a monomorphism, then h is an isomorphism.

In a balanced category, every monomorphism (epimorphism) is extremal. In any category, a morphism is an isomorphism iff it is both a monomorphism and an extremal epimorphism iff it is both an extremal monomorphism and an epimorphism.

In TOP, a monomorphism is extremal iff it is an embedding but in HAUS, a monomorphism is extremal iff it is a closed embedding. In TOP or HAUS, an epimorphism is extremal iff it is a quotient map.

A source in a category **C** is a collection of morphisms $f_i : X \to X_i$ indexed by a set I and having a common domain. An n-source is a source for which $\#(I) = n$.

A sink in a category **C** is a collection of morphisms $f_i : X_i \to X$ indexed by a set I and having a common codomain. An n-sink is a sink for which $\#(I) = n$.

A diagram in a category **C** is a functor $\Delta : \mathbf{I} \to \mathbf{C}$, where **I** is a small category, the indexing category. To facilitate the introduction of sources and sinks associated with Δ, we shall write Δ_i for the image in $\operatorname{Ob}\mathbf{C}$ of $i \in \operatorname{Ob}\mathbf{I}$.

(lim) Let $\Delta : \mathbf{I} \to \mathbf{C}$ be a diagram—then a source $\{f_i : X \to \Delta_i\}$ is said to be natural if for each $\delta \in \operatorname{Mor}\mathbf{I}$, say $i \xrightarrow{\delta} j$, $\Delta\delta \circ f_i = f_j$. A limit of Δ is a natural source $\{\ell_i : L \to \Delta_i\}$ with the property that if $\{f_i : X \to \Delta_i\}$ is a natural source, then there exists a unique morphism $\phi : X \to L$ such that $f_i = \ell_i \circ \phi$ for all $i \in \operatorname{Ob}\mathbf{I}$. Limits are essentially unique. Notation: $L = \lim_{\mathbf{I}} \Delta$ (or $\lim \Delta$).

(colim) Let $\Delta : \mathbf{I} \to \mathbf{C}$ be a diagram—then a sink $\{f_i : \Delta_i \to X\}$ is said to be natural if for each $\delta \in \operatorname{Mor}\mathbf{I}$, say $i \xrightarrow{\delta} j$, $f_i = f_j \circ \Delta\delta$. A colimit of Δ is a natural sink $\{\ell_i : \Delta_i \to L\}$ with the property that if $\{f_i : \Delta_i \to X\}$ is a natural sink, then there exists a unique morphism $\phi : L \to X$ such that $f_i = \phi \circ \ell_i$ for all $i \in \operatorname{Ob}\mathbf{I}$. Colimits are essentially unique. Notation: $L = \operatorname{colim}_{\mathbf{I}} \Delta$ (or $\operatorname{colim} \Delta$).

There are a number of basic constructions that can be viewed as a limit or colimit of a suitable diagram.

Let I be a set; let **I** be the discrete category with $\operatorname{Ob}\mathbf{I} = I$. Given a collection $\{X_i : i \in I\}$ of objects in **C**, define a diagram $\Delta : \mathbf{I} \to \mathbf{C}$ by $\Delta_i = X_i$ $(i \in I)$.

(Products) A limit $\{\ell_i : L \to \Delta_i\}$ of Δ is said to be a product of the X_i. Notation: $L = \prod\limits_i X_i$ (or X^I if $X_i = X$ for all i), $\ell_i = \mathrm{pr}_i$, the projection from $\prod\limits_i X_i$ to X_i. Briefly put: Products are limits of diagrams with discrete indexing categories. In particular, the limit of a diagram having **0** for its indexing category is a final object in **C**.

[Note: An object X in a category **C** is said to be final if for each object Y there is exactly one morphism from Y to X.]

(Coproducts) A colimit $\{\ell_i : \Delta_i \to L\}$ of Δ is said to be a coproduct of the X_i. Notation: $L = \coprod\limits_i X_i$ (or $I \cdot X$ if $X_i = X$ for all i), $\ell_i = \text{in}_i$, the injection from X_i to $\coprod\limits_i X_i$. Briefly put: Coproducts are colimits of diagrams with discrete indexing categories. In particular, the colimit of a diagram having **0** for its indexing category is an initial object in **C**.

[Note: An object X in a category **C** is said to be initial if for each object Y there is exactly one morphism from X to Y.]

In the full subcategory of **TOP** whose objects are the locally connected spaces, the product is the product in **SET** equipped with the coarsest locally connected topology that is finer than the product topology. In the full subcategory of **TOP** whose objects are the compact Hausdorff spaces, the coproduct is the Stone-Čech compactification of the coproduct in **TOP**.

Let **I** be the category $1 \bullet \overset{a}{\underset{b}{\rightrightarrows}} \bullet 2$. Given a pair of morphisms $u, v : X \to Y$ in **C**, define a diagram $\Delta : \mathbf{I} \to \mathbf{C}$ by $\begin{cases} \Delta_1 = X \\ \Delta_2 = Y \end{cases}$ & $\begin{cases} \Delta a = u \\ \Delta b = v \end{cases}$.

(Equalizers) An equalizer in a category **C** of a pair of morphisms $u, v : X \to Y$ is a morphism $f : Z \to X$ with $u \circ f = v \circ f$ such that for any morphism $f' : Z' \to X$ with $u \circ f' = v \circ f'$ there exists a unique morphism $\phi : Z' \to Z$ such that $f' = f \circ \phi$. The 2-source $X \xleftarrow{f} Z \xrightarrow{u \circ f} Y$ is a limit of Δ iff $Z \xrightarrow{f} X$ is an equalizer of $u, v : X \to Y$. Notation: $Z = \text{eq}(u, v)$.

[Note: Every equalizer is a monomorphism. A monomorphism is regular if it is an equalizer. A regular monomorphism is extremal. In **SET**, **GR**, **AB**, **TOP** (or **HAUS**), an extremal monomorphism is regular.]

(Coequalizers) A coequalizer in a category **C** of a pair of morphisms $u, v : X \to Y$ is a morphism $f : Y \to Z$ with $f \circ u = f \circ v$ such that for any morphism $f' : Y \to Z'$ with $f' \circ u = f' \circ v$ there exists a unique morphism $\phi : Z \to Z'$ such that $f' = \phi \circ f$. The 2-sink $Y \xrightarrow{f} Z \xleftarrow{f \circ u} X$ is a colimit of Δ iff $Y \xrightarrow{f} Z$ is a coequalizer of $u, v : X \to Y$. Notation: $Z = \text{coeq}(u, v)$.

[Note: Every coequalizer is an epimorphism. An epimorphism is regular if it is a coequalizer. A regular epimorphism is extremal. In **SET**, **GR**, **AB**, **TOP** (or **HAUS**), an extremal epimorphism is regular.]

There are two aspects to the notion of equalizer or coequalizer, namely: (1) Existence of f and (2) Uniqueness of ϕ. Given (1), (2) is equivalent to requiring that f be a monomorphism or an epimorphism. If (1) is retained and (2) is abandoned, then the terminology is weak equalizer or weak coequalizer.

For example, **HTOP**$_*$ has neither equalizers nor coequalizers but does have weak equalizers and weak coequalizers.

Let $\mathbf{I}$ be the category $1 \bullet \xrightarrow{a} \underset{3}{\bullet} \xleftarrow{b} \bullet 2$. Given morphisms $\begin{cases} f : X \to Z \\ g : Y \to Z \end{cases}$ in $\mathbf{C}$, define a diagram $\Delta : \mathbf{I} \to \mathbf{C}$ by $\begin{cases} \Delta_1 = X \\ \Delta_2 = Y \\ \Delta_3 = Z \end{cases}$ & $\begin{cases} \Delta a = f \\ \Delta b = g \end{cases}$.

(Pullbacks) Given a 2-sink $X \xrightarrow{f} Z \xleftarrow{g} Y$, a commutative diagram

$$\begin{tikzcd} P \arrow[r, "\eta"] \arrow[d, "\xi"'] & Y \arrow[d, "g"] \\ X \arrow[r, "f"'] & Z \end{tikzcd}$$

is said to be a pullback square if for any 2-source $X \xleftarrow{\xi'} P' \xrightarrow{\eta'} Y$ with $f \circ \xi' = g \circ \eta'$ there exists a unique morphism $\phi : P' \to P$ such that $\xi' = \xi \circ \phi$ and $\eta' = \eta \circ \phi$. The 2-source $X \xleftarrow{\xi} P \xrightarrow{\eta} Y$ is called a pullback of the 2-sink $X \xrightarrow{f} Z \xleftarrow{g} Y$. Notation: $P = X \times_Z Y$. Limits of Δ are pullback squares and conversely.

Let $\mathbf{I}$ be the category $1 \bullet \xleftarrow{a} \underset{3}{\bullet} \xrightarrow{b} \bullet 2$. Given morphisms $\begin{cases} f : Z \to X \\ g : Z \to Y \end{cases}$ in $\mathbf{C}$, define a diagram $\Delta : \mathbf{I} \to \mathbf{C}$ by $\begin{cases} \Delta_1 = X \\ \Delta_2 = Y \\ \Delta_3 = Z \end{cases}$ & $\begin{cases} \Delta a = f \\ \Delta b = g \end{cases}$.

(Pushouts) Given a 2-source $X \xleftarrow{f} Z \xrightarrow{g} Y$, a commutative diagram

$$\begin{tikzcd} Z \arrow[r, "g"] \arrow[d, "f"'] & Y \arrow[d, "\eta"] \\ X \arrow[r, "\xi"'] & P \end{tikzcd}$$

is said to be a pushout square if for any 2-sink $X \xrightarrow{\xi'} P' \xleftarrow{\eta'} Y$ with $\xi' \circ f = \eta' \circ g$ there exists a unique morphism $\phi : P \to P'$ such that $\xi' = \phi \circ \xi$ and $\eta' = \phi \circ \eta$. The 2-sink $X \xrightarrow{\xi} P \xleftarrow{\eta} Y$ is called a pushout of the 2-source $X \xleftarrow{f} Z \xrightarrow{g} Y$. Notation: $P = X \underset{Z}{\sqcup} Y$. Colimits of Δ are pushout squares and conversely.

The result of dropping uniqueness in ϕ is weak pullback or weak pushout. Examples are the commutative squares that define fibration and cofibration in **TOP**.

Let $\mathbf{I}$ be a small category, $\Delta : \mathbf{I}^{\mathrm{OP}} \times \mathbf{I} \to \mathbf{C}$ a diagram.

(Ends) A source $\{f_i : X \to \Delta_{i,i}\}$ is said to be dinatural if for each $\delta \in \operatorname{Mor}\mathbf{I}$, say $i \xrightarrow{\delta} j$, $\Delta(\mathrm{id}, \delta) \circ f_i = \Delta(\delta, \mathrm{id}) \circ f_j$. An end of Δ is a dinatural source $\{e_i : E \to \Delta_{i,i}\}$ with the property that if $\{f_i : X \to \Delta_{i,i}\}$ is a dinatural source, then there exists a unique morphism $\phi : X \to E$ such that $f_i = e_i \circ \phi$ for all $i \in \operatorname{Ob}\mathbf{I}$. Every end is a limit (and every limit is an end). Notation: $E = \int_i \Delta_{i,i}$ (or $\int_{\mathbf{I}} \Delta$.

(Coends) A sink $\{f_i : \Delta_{i,i} \to X\}$ is said to be dinatural if for each $\delta \in \operatorname{Mor}\mathbf{I}$, say $i \xrightarrow{\delta} j$, $f_i \circ \Delta(\delta, \mathrm{id}) = f_j \circ \Delta(\mathrm{id}, \delta)$. A coend of Δ is a dinatural sink $\{e_i : \Delta_{i,i} \to E\}$ with the property that if $\{f_i : \Delta_{i,i} \to X\}$ is a dinatural sink, then there exists a unique morphism $\phi : E \rightarrow X$ such that $f_i = \phi \circ e_i$ for all $i \in \operatorname{Ob} \mathbf{I}$. Every coend is a colimit (and every colimit is a coend). Notation: $E = \int^i \Delta_{i,i}$ (or $\int^{\mathbf{I}} \Delta$).

Let $\begin{cases} F : \mathbf{I} \rightarrow \mathbf{C} \\ G : \mathbf{I} \rightarrow \mathbf{C} \end{cases}$ be functors—then the assignment $(i,j) \rightarrow \operatorname{Mor}(Fi, Gj)$ defines a diagram $\mathbf{I}^{\mathrm{OP}} \times \mathbf{I} \rightarrow$ **SET** and $\operatorname{Nat}(F,G)$ is the end $\int_i \operatorname{Mor}(Fi, Gi)$.

INTEGRAL YONEDA LEMMA Let $\mathbf{I}$ be a small category, $\mathbf{C}$ a complete and cocomplete category—then for every F in $[\mathbf{I}^{\mathrm{OP}}, \mathbf{C}]$, $\int^i \operatorname{Mor}(-, i) \cdot Fi \approx F \approx \int_i Fi^{\operatorname{Mor}(i,-)}$.

Let $\mathbf{I} \neq \mathbf{0}$ be a small category—then $\mathbf{I}$ is said to be filtered if

(F$_1$) Given any pair of objects i, j in $\mathbf{I}$, there exists an object k and morphisms $\begin{cases} i \rightarrow k \\ j \rightarrow k \end{cases}$;

(F$_2$) Given any pair of morphisms $a, b : i \rightarrow j$ in $\mathbf{I}$, there exists an object k and a morphism $c : j \rightarrow k$ such that $c \circ a = c \circ b$.

Every nonempty directed set $(I, \leq)$ can be viewed as a filtered category $\mathbf{I}$, where $\operatorname{Ob} \mathbf{I} = I$ and $\operatorname{Mor}(i,j)$ is a one element set when $i \leq j$ but is empty otherwise.

Example: Let [N] be the filtered category associated with the directed set of nonnegative integers. Given a category $\mathbf{C}$, denote by **FIL(C)** the functor category [[N], **C**]—then an object $(\mathbf{X}, \mathbf{f})$ in **FIL(C)** is a sequence $\{X_n, f_n\}$, where $X_n \in \operatorname{Ob} \mathbf{C}$ & $f_n \in \operatorname{Mor}(X_n, X_{n+1})$, and a morphism $\phi : (\mathbf{X}, \mathbf{f}) \rightarrow (\mathbf{Y}, \mathbf{g})$ in **FIL(C)** is a sequence $\{\phi_n\}$, where $\phi_n \in \operatorname{Mor}(X_n, Y_n)$ & $g_n \circ \phi_n = \phi_{n+1} \circ f_n$.

(Filtered Colimits) A filtered colimit in $\mathbf{C}$ is the colimit of a diagram $\Delta : \mathbf{I} \rightarrow \mathbf{C}$, where $\mathbf{I}$ is filtered.

(Cofiltered Limits) A cofiltered limit in $\mathbf{C}$ is the limit of a diagram $\Delta : \mathbf{I} \rightarrow \mathbf{C}$, where $\mathbf{I}$ is cofiltered.

[Note: A small category $\mathbf{I} \neq \mathbf{0}$ is said to be cofiltered provided that $\mathbf{I}^{\mathrm{OP}}$ is filtered.]

A Hausdorff space is compactly generated iff it is the filtered colimit in **TOP** of its compact subspaces. Every compact Hausdorff space is the cofiltered limit in **TOP** of compact metrizable spaces.

Given a small category $\mathbf{C}$, a path in $\mathbf{C}$ is a diagram σ of the form $X_0 \rightarrow X_1 \leftarrow \cdots \rightarrow X_{2n-1} \leftarrow X_{2n}$ $(n \geq 0)$. One says that σ begins at X_0 and ends at X_{2n}. The quotient of $\operatorname{Ob} \mathbf{C}$ with respect to the equivalence relation obtained by declaring that $X' \sim X''$ iff there exists a path in $\mathbf{C}$ which begins at X' and ends at X'' is the set $\pi_0(\mathbf{C})$ of components of $\mathbf{C}$, $\mathbf{C}$ being called connected when the cardinality of $\pi_0(\mathbf{C})$ is one. The full subcategory of $\mathbf{C}$ determined by a component is connected and is maximal with respect to this property. If $\mathbf{C}$ has an initial object or a final object, then $\mathbf{C}$ is connected.

[Note: The concept of "path" makes sense in any category.]

Let $\mathbf{I} \neq \mathbf{0}$ be a small category—then $\mathbf{I}$ is said to be pseudofiltered if

(PF$_1$) Given any pair of morphisms $\begin{cases} a : i \rightarrow j \\ b : i \rightarrow k \end{cases}$ in $\mathbf{I}$, there exists an object ℓ and morphisms $\begin{cases} c : j \rightarrow \ell \\ d : k \rightarrow \ell \end{cases}$ such that $c \circ a = d \circ b$;

(PF$_2$) Given any pair of morphisms $a, b : i \rightarrow j$ in $\mathbf{I}$, there exists a morphism $c : j \rightarrow k$ such that $c \circ a = c \circ b$.

$\mathbf{I}$ is filtered iff $\mathbf{I}$ is connected and pseudofiltered. $\mathbf{I}$ is pseudofiltered iff its components are filtered.

Given small categories $\begin{cases} \mathbf{I} \\ \mathbf{J} \end{cases}$, a functor $\nabla : \mathbf{J} \rightarrow \mathbf{I}$ is said to be final provided that for every $i \in \operatorname{Ob} \mathbf{I}$, the comma category $|K_i, \nabla|$ is nonempty and connected. If $\mathbf{J}$ is filtered and $\nabla : \mathbf{J} \rightarrow \mathbf{I}$ is final, then $\mathbf{I}$ is filtered.

[Note: A subcategory of a small category is final if the inclusion is a final functor.]

Let $\nabla : \mathbf{J} \rightarrow \mathbf{I}$ be final. Suppose that $\Delta : \mathbf{I} \rightarrow \mathbf{C}$ is a diagram for which $\operatorname{colim} \Delta \circ \nabla$ exists—then $\operatorname{colim} \Delta$ exists and the arrow $\operatorname{colim} \Delta \circ \nabla \rightarrow \operatorname{colim} \Delta$ is an isomorphism. Corollary: If i is a final object in $\mathbf{I}$, then $\operatorname{colim} \Delta \approx \Delta_i$.

[Note: Analogous considerations apply to limits so long as "final" is replaced throughout by "initial".]

Let $\mathbf{I}$ be a filtered category—then there exists a directed set $(J, \leq)$ and a final functor $\nabla : \mathbf{J} \rightarrow \mathbf{I}$.

Limits commute with limits. In other words, if $\Delta : \mathbf{I} \times \mathbf{J} \rightarrow \mathbf{C}$ is a diagram, then under the obvious assumptions

$$\lim_{\mathbf{I}} \lim_{\mathbf{J}} \Delta \approx \lim_{\mathbf{I} \times \mathbf{J}} \Delta \approx \lim_{\mathbf{J} \times \mathbf{I}} \Delta \approx \lim_{\mathbf{J}} \lim_{\mathbf{I}} \Delta.$$

Likewise, colimits commute with colimits. In general, limits do not commute with colimits. However, if $\Delta : \mathbf{I} \times \mathbf{J} \rightarrow \mathbf{SET}$ and if $\mathbf{I}$ is finite and $\mathbf{J}$ is filtered, then the arrow $\operatorname{colim}_{\mathbf{J}} \lim_{\mathbf{I}} \Delta \rightarrow \lim_{\mathbf{I}} \operatorname{colim}_{\mathbf{J}} \Delta$ is a bijection, so that in $\mathbf{SET}$ filtered colimits commute with finite limits.

[Note: In $\mathbf{GR}$, $\mathbf{AB}$ or $\mathbf{RG}$, filtered colimits commute with finite limits. But, e.g., filtered colimits do not commute with finite limits in $\mathbf{SET}^{\mathbf{OP}}$.]

In $\mathbf{AB}$ (or any Grothendieck category), pseudofiltered colimits commute with finite limits.

A category **C** is said to be complete (cocomplete) if for each small category **I**, every $\Delta \in \operatorname{Ob}[\mathbf{I}, \mathbf{C}]$ has a limit (colimit). The following are equivalent.

(1) **C** is complete (cocomplete).

(2) **C** has products and equalizers (coproducts and coequalizers).

(3) **C** has products and pullbacks (coproducts and pushouts).

(4) **C** has a final object and multiple pullbacks (initial object and multiple pushouts).

[Note: A source $\{\xi_i : P \rightarrow X_i\}$ (sink $\{\xi_i : X_i \rightarrow P\}$) is said to be a multiple pullback (multiple pushout) of a sink $\{f_i : X_i \rightarrow X\}$ (source $\{f_i : X \rightarrow X_i\}$) provided that $f_i \circ \xi_i = f_j \circ \xi_j$ $(\xi_i \circ f_i = \xi_j \circ f_j)$ $\forall \left\{ \begin{matrix} i \\ j \end{matrix} \right.$ and if for any source $\{\xi'_i : P' \rightarrow X_i\}$ (sink $\{\xi'_i : X_i \rightarrow P'\}$) with $f_i \circ \xi'_i = f_j \circ \xi'_j$ $(\xi'_i \circ f_i = \xi'_j \circ f_j)$ $\forall \left\{ \begin{matrix} i \\ j \end{matrix} \right.$, there exists a unique morphism $\phi : P' \rightarrow P$ $(\phi : P \rightarrow P')$ such that $\forall\, i$, $\xi'_i = \xi_i \circ \phi$ $(\xi'_i = \phi \circ \xi_i)$. Every multiple pullback (multiple pushout) is a limit (colimit).]

The categories **SET**, **GR**, and **AB** are both complete and cocomplete. The same is true of **TOP** and **TOP$_*$** but not of **HTOP** and **HTOP$_*$**.

[Note: **HAUS** is complete; it is also cocomplete, being epireflective in **TOP**.]

A category **C** is said to be finitely complete (finitely cocomplete) if for each finite category **I**, every $\Delta \in \operatorname{Ob}[\mathbf{I}, \mathbf{C}]$ has a limit (colimit). The following are equivalent.

(1) **C** is finitely complete (finitely cocomplete).

(2) **C** has finite products and equalizers (finite coproducts and coequalizers).

(3) **C** has finite products and pullbacks (finite coproducts and pushouts).

(4) **C** has a final object and pullbacks (initial object and pushouts).

The full subcategory of **TOP** whose objects are the finite topological spaces is finitely complete and finitely cocomplete but neither complete nor cocomplete. A nontrivial group, considered as a category, has multiple pullbacks but fails to have finite products.

If **C** is small and **D** is finitely complete and wellpowered (finitely cocomplete and cowellpowered), then $[\mathbf{C}, \mathbf{D}]$ is wellpowered (cowellpowered).

$\mathbf{SET}(\rightarrow), \mathbf{GR}(\rightarrow), \mathbf{AB}(\rightarrow), \mathbf{TOP}(\rightarrow)$ (or $\mathbf{HAUS}(\rightarrow)$) are wellpowered and cowellpowered.

[Note: The arrow category $\mathbf{C}(\rightarrow)$ of any category **C** is isomorphic to $[\mathbf{2}, \mathbf{C}]$.]

Let $F : \mathbf{C} \rightarrow \mathbf{D}$ be a functor.

(a) F is said to preserve a limit $\{\ell_i : L \to \Delta_i\}$ (colimit $\{\ell_i : \Delta_i \to L\}$) of a diagram $\Delta : \mathbf{I} \to \mathbf{C}$ if $\{F\ell_i : FL \to F\Delta_i\}$ ($\{F\ell_i : F\Delta_i \to FL\}$) is a limit (colimit) of the diagram $F \circ \Delta : \mathbf{I} \to \mathbf{D}$.

(b) F is said to preserve limits (colimits) over an indexing category $\mathbf{I}$ if F preserves all limits (colimits) of diagrams $\Delta : \mathbf{I} \to \mathbf{C}$.

(c) F is said to preserve limits (colimits) if F preserves limits (colimits) over all indexing categories $\mathbf{I}$.

The forgetful functor TOP → SET preserves limits and colimits. The forgetful functor GR → SET preserves limits and filtered colimits but not coproducts. The inclusion HAUS → TOP preserves limits and coproducts but not coequalizers. The inclusion AB → GR preserves limits but not colimits.

There are two rules that determine the behavior of $\left\{\begin{matrix}\operatorname{Mor}(X,—)\\ \operatorname{Mor}(—,X)\end{matrix}\right.$ with respect to limits and colimits.

(1) The functor $\operatorname{Mor}(X,—) : \mathbf{C} \to \mathbf{SET}$ preserves limits. Symbolically, therefore, $\operatorname{Mor}(X, \lim \Delta) \approx \lim(\operatorname{Mor}(X,—) \circ \Delta)$.

(2) The cofunctor $\operatorname{Mor}(—,X) : \mathbf{C} \to \mathbf{SET}$ converts colimits into limits. Symbolically, therefore, $\operatorname{Mor}(\operatorname{colim} \Delta, X) \approx \lim(\operatorname{Mor}(—,X) \circ \Delta)$.

REPRESENTABLE FUNCTOR THEOREM Given a complete category $\mathbf{C}$, a functor $F : \mathbf{C} \to \mathbf{SET}$ is representable iff F preserves limits and satisfies the solution set condition: There exists a set $\{X_i\}$ of objects in $\mathbf{C}$ such that for each $X \in \operatorname{Ob} \mathbf{C}$ and each $y \in FX$, there is an i, a $y_i \in FX_i$, and an $f : X_i \to X$ such that $y = (Ff)y_i$.

Take for C the category opposite to the category of ordinal numbers—then the functor C → SET defined by $\alpha \to *$ has a complete domain and preserves limits but is not representable.

Limits and colimits in functor categories are computed "object by object". So, if $\mathbf{C}$ is a small category, then $\mathbf{D}$ (finitely) complete $\Rightarrow [\mathbf{C}, \mathbf{D}]$ (finitely) complete and $\mathbf{D}$ (finitely) cocomplete $\Rightarrow [\mathbf{C}, \mathbf{D}]$ (finitely) cocomplete.

Given a small category $\mathbf{C}$, put $\widehat{\mathbf{C}} = [\mathbf{C}^{\mathrm{OP}}, \mathbf{SET}]$—then $\widehat{\mathbf{C}}$ is complete and cocomplete. The Yoneda embedding $Y_{\mathbf{C}} : \mathbf{C} \to \widehat{\mathbf{C}}$ preserves limits; it need not, however, preserve finite colimits. The image of $\mathbf{C}$ is "colimit dense" in $\widehat{\mathbf{C}}$, i.e., every cofunctor $\mathbf{C} \to \mathbf{SET}$ is a colimit of representable cofunctors.

An indobject in a small category $\mathbf{C}$ is a diagram $\Delta : \mathbf{I} \to \mathbf{C}$, where $\mathbf{I}$ is filtered. Corresponding to an indobject Δ, is the object L_Δ in $\widehat{\mathbf{C}}$ defined by $L_\Delta = \operatorname{colim}(Y_{\mathbf{C}} \circ \Delta)$.

The indcategory $\mathbf{IND}(\mathbf{C})$ of $\mathbf{C}$ is the category whose objects are the indobjects and whose morphisms are the sets $\operatorname{Mor}(\Delta', \Delta'') = \operatorname{Nat}(L_{\Delta'}, L_{\Delta''})$. The functor $L : \mathbf{IND}(\mathbf{C}) \to \widehat{\mathbf{C}}$ that sends Δ to L_Δ is full and faithful (although in general not injective on objects), hence establishes an equivalence between $\mathbf{IND}(\mathbf{C})$ and the full subcategory of $\widehat{\mathbf{C}}$ whose objects are the cofunctors $\mathbf{C} \to \mathbf{SET}$ which are filtered colimits of representable cofunctors. The category $\mathbf{IND}(\mathbf{C})$ has filtered colimits; they are preserved by L, as are all limits. Moreover, in $\mathbf{IND}(\mathbf{C})$, filtered colimits commute with finite limits. If $\mathbf{C}$ is finitely cocomplete, then $\mathbf{IND}(\mathbf{C})$ is complete and cocomplete. The functor $K : \mathbf{C} \to \mathbf{IND}(\mathbf{C})$ that sends X to K_X, where $K_X : \mathbf{1} \to \mathbf{C}$ is the constant functor with value X, is full, faithful, and injective on objects. In addition, K preserves limits and finite colimits. The composition $\mathbf{C} \xrightarrow{K} \mathbf{IND}(\mathbf{C}) \xrightarrow{L} \widehat{\mathbf{C}}$ is the Yoneda embedding $Y_{\mathbf{C}}$. A cofunctor $F \in \operatorname{Ob}\widehat{\mathbf{C}}$ is said to be indrepresentable if it is naturally isomorphic to a functor of the form L_Δ, $\Delta \in \operatorname{Ob}\mathbf{IND}(\mathbf{C})$. An indrepresentable cofunctor converts finite colimits into finite limits and conversely, provided that $\mathbf{C}$ is finitely cocomplete.

[Note: The procategory $\mathbf{PRO}(\mathbf{C})$ is by definition $\mathbf{IND}(\mathbf{C}^{\mathrm{OP}})^{\mathrm{OP}}$. Its objects are the proobjects in $\mathbf{C}$, i.e., the diagrams defined on cofiltering categories.]

The full subcategory of **SET** whose objects are the finite sets is equivalent to a small category. Its indcategory is equivalent to **SET** and its procategory is equivalent to the full subcategory of **TOP** whose objects are the totally disconnected compact Hausdorff spaces.

[Note: There is no small category $\mathbf{C}$ for which $\mathbf{PRO}(\mathbf{C})$ is equivalent to **SET**. This is because in **SET**, cofiltered limits do not commute with finite colimits.]

Given categories $\begin{cases} \mathbf{C} \\ \mathbf{D} \end{cases}$, functors $\begin{cases} F : \mathbf{C} \to \mathbf{D} \\ G : \mathbf{D} \to \mathbf{C} \end{cases}$ are said to be an adjoint pair if the functors $\begin{cases} \operatorname{Mor} \circ (F^{\mathrm{OP}} \times \mathrm{id}_{\mathbf{D}}) \\ \operatorname{Mor} \circ (\mathrm{id}_{\mathbf{C}^{\mathrm{OP}}} \times G) \end{cases}$ from $\mathbf{C}^{\mathrm{OP}} \times \mathbf{D}$ to **SET** are naturally isomorphic, i.e., if it is possible to assign to each ordered pair $\begin{cases} X \in \operatorname{Ob}\mathbf{C} \\ Y \in \operatorname{Ob}\mathbf{D} \end{cases}$ a bijective map $\Xi_{X,Y} : \operatorname{Mor}(FX, Y) \to \operatorname{Mor}(X, GY)$ which is functorial in X and Y. When this is so, F is a left adjoint for G and G is a right adjoint for F. Any two left (right) adjoints for G (F) are naturally isomorphic. Left adjoints preserve colimits; right adjoints preserve limits. In order that (F, G) be an adjoint pair, it is necessary and sufficient that there exist natural transformations $\begin{cases} \mu \in \operatorname{Nat}(\mathrm{id}_{\mathbf{C}}, G \circ F) \\ \nu \in \operatorname{Nat}(F \circ G, \mathrm{id}_{\mathbf{D}}) \end{cases}$ subject to $\begin{cases} (G\nu) \circ (\mu G) = \mathrm{id}_G \\ (\nu F) \circ (F\mu) = \mathrm{id}_F \end{cases}$. The data (F, G, μ, ν) is referred to as an adjoint situation, the natural transformations $\begin{cases} \mu : \mathrm{id}_{\mathbf{C}} \to G \circ F \\ \nu : F \circ G \to \mathrm{id}_{\mathbf{D}} \end{cases}$ being the arrows of adjunction.

(UN) Suppose that G has a left adjoint F—then for each $X \in \operatorname{Ob}\mathbf{C}$, each

$Y \in \operatorname{Ob}\mathbf{D}$, and each $f : X \to GY$, there exists a unique $g : FX \to Y$ such that $f = Gg \circ \mu_X$.
[Note: When reformulated, this property is characteristic.]

The forgetful functor **TOP** $\to$ **SET** has a left adjoint that sends a set X to the pair (X, τ), where τ is the discrete topology, and a right adjoint that sends a set X to the pair (X, τ), where τ is the indiscrete topology.

Let $\mathbf{I}$ be a small category, $\mathbf{C}$ a complete and cocomplete category. Examples: (1) The constant diagram functor $K : \mathbf{C} \to [\mathbf{I}, \mathbf{C}]$ has a left adjoint, viz. $\operatorname{colim} : [\mathbf{I}, \mathbf{C}] \to \mathbf{C}$, and a right adjoint, viz. $\lim : [\mathbf{I}, \mathbf{C}] \to \mathbf{C}$; (2) The functor $\mathbf{C} \to [\mathbf{I}^{\mathrm{OP}} \times \mathbf{I}, \mathbf{C}]$ that sends X to $(i, j) \to \operatorname{Mor}(i, j) \cdot X$ is a left adjoint for end and the functor that sends X to $(i, j) \to X^{\operatorname{Mor}(j,i)}$ is a right adjoint for coend.

GENERAL ADJOINT FUNCTOR THEOREM Given a complete category $\mathbf{D}$, a functor $G : \mathbf{D} \to \mathbf{C}$ has a left adjoint iff G preserves limits and satisfies the solution set condition: For each $X \in \operatorname{Ob}\mathbf{C}$, there exists a source $\{f_i : X \to GY_i\}$ such that for every $f : X \to GY$, there is an i and a $g : Y_i \to Y$ such that $f = Gg \circ f_i$.

The general adjoint functor theorem implies that a small category is complete iff it is cocomplete.

KAN EXTENSION THEOREM Given small categories $\begin{cases} \mathbf{C} \\ \mathbf{D} \end{cases}$, a complete (cocomplete) category $\mathbf{S}$, and a functor $K : \mathbf{C} \to \mathbf{D}$, the functor $[K, \mathbf{S}] : [\mathbf{D}, \mathbf{S}] \to [\mathbf{C}, \mathbf{S}]$ has a right (left) adjoint ran (lan) and preserves limits and colimits.
[Note: If K is full and faithful, then ran (lan) is full and faithful.]

Suppose that $\mathbf{S}$ is complete. Let $T \in \operatorname{Ob}[\mathbf{C}, \mathbf{S}]$—then $\operatorname{ran} T$ is called the right Kan extension of T along K. In terms of ends, $(\operatorname{ran} T)Y = \int_X TX^{\operatorname{Mor}(Y,KX)}$. There is a "universal" arrow $(\operatorname{ran} T) \circ K \to T$. It is a natural isomorphism if K is full and faithful.

Suppose that $\mathbf{S}$ is cocomplete. Let $T \in \operatorname{Ob}[\mathbf{C}, \mathbf{S}]$—then $\operatorname{lan} T$ is called the left Kan extension of T along K. In terms of coends, $(\operatorname{lan} T)Y = \int^X \operatorname{Mor}(KX, Y) \cdot TX$. There is a "universal" arrow $T \to (\operatorname{lan} T) \circ K$. It is a natural isomorphism if K is full and faithful.

Application: If $\mathbf{C}$ and $\mathbf{D}$ are small categories and if $F : \mathbf{C} \to \mathbf{D}$ is a functor, then the precomposition functor $\widehat{\mathbf{D}} \to \widehat{\mathbf{C}}$ has a left adjoint $\widehat{F} : \widehat{\mathbf{C}} \to \widehat{\mathbf{D}}$ and $\widehat{F} \circ Y_{\mathbf{C}} \approx Y_{\mathbf{D}} \circ F$.
[Note: One can always arrange that $\widehat{F} \circ Y_{\mathbf{C}} = Y_{\mathbf{D}} \circ F$.]

The construction of the right (left) adjoint of $[K, \mathbf{S}]$ does not use the assumption that $\mathbf{D}$ is small, its role being to ensure that $[\mathbf{D}, \mathbf{S}]$ is a category. For example, if $\mathbf{C}$ is small and **S** is cocomplete, then taking $K = Y_{\mathbf{C}}$, the functor $[Y_{\mathbf{C}}, \mathbf{S}] : [\widehat{\mathbf{C}}, \mathbf{S}] \to [\mathbf{C}, \mathbf{S}]$ has a left adjoint that sends $T \in \mathrm{Ob}\,[\mathbf{C}, \mathbf{S}]$ to $\Gamma_T \in \mathrm{Ob}\,[\widehat{\mathbf{C}}, \mathbf{S}]$, where $\Gamma_T \circ Y_{\mathbf{C}} = T$. On an object $F \in \widehat{\mathbf{C}}$, $\Gamma_T F = \int^X \mathrm{Nat}(Y_{\mathbf{C}}X, F) \cdot TX = \int^X FX \cdot TX$. Γ_T is the realization functor; it is a left adjoint for the singular functor S_T, the composite of the Yoneda embedding $\mathbf{S} \to [\mathbf{S}^{\mathrm{OP}}, \mathbf{SET}]$ and the precomposition functor $[\mathbf{S}^{\mathrm{OP}}, \mathbf{SET}] \to [\mathbf{C}^{\mathrm{OP}}, \mathbf{SET}]$, thus $(S_T Y)X = \mathrm{Mor}\,(TX, Y)$.

[Note: The arrow of adjunction $\Gamma_T \circ S_T \to \mathrm{id}_{\mathbf{S}}$ is a natural isomorphism iff S_T is full and faithful.]

CAT is the category whose objects are the small categories and whose morphisms are the functors between them: $\mathbf{C}, \mathbf{D} \in \mathrm{Ob}\,\mathbf{CAT} \Rightarrow \mathrm{Mor}\,(\mathbf{C}, \mathbf{D}) = \mathrm{Ob}\,[\mathbf{C}, \mathbf{D}]$. **CAT** is concrete and complete and cocomplete. **0** is an initial object in **CAT** and **1** is a final object in **CAT**.

Let $\pi_0 : \mathbf{CAT} \to \mathbf{SET}$ be the functor that sends **C** to $\pi_0(\mathbf{C})$, the set of components of **C**; let dis : $\mathbf{SET} \to \mathbf{CAT}$ be the functor that sends X to dis$X$, the discrete category on X; let ob : $\mathbf{CAT} \to \mathbf{SET}$ be the functor that sends **C** to $\mathrm{Ob}\,\mathbf{C}$, the set of objects in **C**; let grd : $\mathbf{SET} \to \mathbf{CAT}$ be the functor that sends X to grd$X$, the category whose objects are the elements of X and whose morphisms are the elements of $X \times X$—then π_0 is a left adjoint for dis, dis is a left adjoint for ob, and ob is a left adjoint for grd.

[Note: π_0 preserves finite products; it need not preserve arbitrary products.]

GRD is the full subcategory of **CAT** whose objects are the groupoids, i.e., the small categories in which every morphism is invertible. Example: The assignment $\Pi : \begin{cases} \mathbf{TOP} \to \mathbf{GRD} \\ X \to \Pi X \end{cases}$ is a functor.

Let iso : $\mathbf{CAT} \to \mathbf{GRD}$ be the functor that sends **C** to iso **C**, the groupoid whose objects are those of **C** and whose morphisms are the invertible morphisms in **C**—then iso is a right adjoint for the inclusion $\mathbf{GRD} \to \mathbf{CAT}$. Let $\pi_1 : \mathbf{CAT} \to \mathbf{GRD}$ be the functor that sends **C** to $\pi_1(\mathbf{C})$, the fundamental groupoid of **C**, i.e., the localization of **C** at $\mathrm{Mor}\,\mathbf{C}$—then π_1 is a left adjoint for the inclusion $\mathbf{GRD} \to \mathbf{CAT}$.

$\mathbf{\Delta}$ is the category whose objects are the ordered sets $[n] \equiv \{0, 1, \ldots, n\}$ $(n \geq 0)$ and whose morphisms are the order preserving maps. In $\mathbf{\Delta}$, every morphism can be written as an epimorphism followed by a monomorphism and a morphism is a monomorphism (epimorphism) iff it is injective (surjective). The face operators are the monomorphisms $\delta_i^n : [n-1] \to [n]$ $(n > 0, 0 \leq i \leq n)$ defined by omitting the value i. The degeneracy operators are the epimorphisms $\sigma_i^n : [n+1] \to [n]$ $(n \geq 0, 0 \leq i \leq n)$ defined by repeating the value i. Suppressing superscripts, if $\alpha \in \mathrm{Mor}\,([m], [n])$ is not the identity, then α has a unique factorization $\alpha = (\delta_{i_1} \circ \cdots \circ \delta_{i_p}) \circ (\sigma_{j_1} \circ \cdots \circ \sigma_{j_q})$, where $n \geq i_1 > \cdots > i_p \geq 0$, $0 \leq j_1 < \cdots < j_q < m$, and $m + p = n + q$. Each $\alpha \in \text{Mor}([m],[n])$ determines a linear transformation $\mathbf{R}^{m+1} \to \mathbf{R}^{n+1}$ which restricts to a map $\Delta^\alpha : \Delta^m \to \Delta^n$. Thus there is a functor $\Delta^? : \mathbf{\Delta} \to \mathbf{TOP}$ that sends $[n]$ to Δ^n and α to Δ^α. Since the objects of $\mathbf{\Delta}$ are themselves small categories, there is also an inclusion $\iota : \mathbf{\Delta} \to \mathbf{CAT}$.

Given a category $\mathbf{C}$, write $\mathbf{SIC}$ for the functor category $[\mathbf{\Delta}^{\text{OP}}, \mathbf{C}]$ and $\mathbf{COSIC}$ for the functor category $[\mathbf{\Delta}, \mathbf{C}]$—then by definition, a simplicial object in $\mathbf{C}$ is an object in $\mathbf{SIC}$ and a cosimplicial object in $\mathbf{C}$ is an object in $\mathbf{COSIC}$. Example: $Y_{\mathbf{\Delta}} \equiv \Delta$ is a cosimplicial object in $\widehat{\mathbf{\Delta}}$.

Specialize to $\mathbf{C} = \mathbf{SET}$—then an object in $\mathbf{SISET}$ is called a simplicial set and a morphism in $\mathbf{SISET}$ is called a simplicial map. Given a simplicial set X, put $X_n = X([n])$, so for $\alpha : [m] \to [n]$, $X\alpha : X_n \to X_m$. If $\begin{cases} d_i = X\delta_i \\ s_i = X\sigma_i \end{cases}$, then d_i and s_i are connected by the simplicial identities:

$$\begin{cases} d_i \circ d_j = d_{j-1} \circ d_i & (i < j) \\ s_i \circ s_j = s_{j+1} \circ s_i & (i \leq j) \end{cases}, \qquad d_i \circ s_j = \begin{cases} s_{j-1} \circ d_i & (i < j) \\ \text{id} & (i = j \text{ or } i = j + 1) \\ s_j \circ d_{i-1} & (i > j + 1) \end{cases}.$$

The simplicial standard n-simplex is the simplicial set $\Delta[n] = \text{Mor}(—, [n])$, i.e., $\Delta[n]$ is the result of applying Δ to $[n]$, so for $\alpha : [m] \to [n]$, $\Delta[\alpha] : \Delta[m] \to \Delta[n]$. Owing to the Yoneda lemma, if X is a simplicial set and if $x \in X_n$, then there exists one and only one simplicial map $\Delta_x : \Delta[n] \to X$ that takes $\text{id}_{[n]}$ to x. $\mathbf{SISET}$ is complete and cocomplete, wellpowered and cowellpowered.

Let X be a simplicial set—then one writes $x \in X$ when one means $x \in \bigcup_n X_n$. With this understanding, an $x \in X$ is said to be degenerate if there exists an epimorphism $\alpha \neq \text{id}$ and a $y \in X$ such that $x = (X\alpha)y$; otherwise, $x \in X$ is said to be nondegenerate. The elements of X_0 (= the vertexes of X) are nondegenerate. Every $x \in X$ admits a unique representation $x = (X\alpha)y$, where α is an epimorphism and y is nondegenerate. The nondegenerate elements in $\Delta[n]$ are the monomorphisms $\alpha : [m] \to [n]$ $(m \leq n)$.

A simplicial subset of a simplicial set X is a simplicial set Y such that Y is a subfunctor of X, i.e., $Y_n \subset X_n$ for all n and the inclusion $Y \to X$ is a simplicial map. Notation: $Y \subset X$. The n-skeleton of a simplicial set X is the simplicial subset $X^{(n)}$ $(n \geq 0)$ of X defined by stipulating that $X_p^{(n)}$ is the set of all $x \in X_p$ for which there exists an epimorphism $\alpha : [p] \to [q]$ $(q \leq n)$ and a $y \in X_q$ such that $x = (X\alpha)y$. Therefore $X_p^{(n)} = X_p$ $(p \leq n)$; furthermore, $X^{(0)} \subset X^{(1)} \subset \cdots$ and $X = \text{colim}\, X^{(n)}$. A proper simplicial subset of $\Delta[n]$ is contained in $\Delta[n]^{(n-1)}$, the frontier $\dot{\Delta}[n]$ of $\Delta[n]$. Of course,

$\dot{\Delta}[0] = \emptyset$. $X^{(0)}$ is isomorphic to $X_0 \cdot \Delta[0]$. In general, let $X_n^{\#}$ be the set of nondegenerate elements of X_n. Fix a collection $\{\Delta[n]_x : x \in X_n^{\#}\}$ of simplicial standard n-simplexes indexed by $X_n^{\#}$—then the simplicial maps $\Delta_x : \Delta[n] \rightarrow X$ $(x \in X_n^{\#})$ determine an arrow $X_n^{\#} \cdot \Delta[n] \rightarrow X^{(n)}$ and the commutative diagram

$$\begin{tikzcd} X_n^{\#} \cdot \dot{\Delta}[n] \arrow[r] \arrow[d] & X^{(n-1)} \arrow[d] \\ X_n^{\#} \cdot \Delta[n] \arrow[r] & X^{(n)} \end{tikzcd}$$

is a pushout square. Note too that $\dot{\Delta}[n]$ is a coequalizer: Consider the diagram

$$\coprod_{0 \leq i < j \leq n} \Delta[n-2]_{i,j} \overset{u}{\underset{v}{\rightrightarrows}} \coprod_{0 \leq i \leq n} \Delta[n-1]_i,$$

where u is defined by the $\Delta[\delta_{j-1}^{n-1}]$ and v is defined by the $\Delta[\delta_i^{n-1}]$—then the $\Delta[\delta_i^n]$ define a simplicial map $f : \coprod_{0 \leq i \leq n} \Delta[n-1]_i \rightarrow \Delta[n]$ that induces an isomorphism $\mathrm{coeq}(u, v) \rightarrow \dot{\Delta}[n]$.

Call $\mathbf{\Delta}_n$ the full subcategory of $\mathbf{\Delta}$ whose objects are the $[m]$ $(m \leq n)$. Given a category $\mathbf{C}$, denote by $\mathbf{SIC}_n$ the functor category $[\mathbf{\Delta}_n^{\mathrm{OP}}, \mathbf{C}]$. The objects of $\mathbf{SIC}_n$ are the "n-truncated simplicial objects" in $\mathbf{C}$. Employing the notation of the Kan extension theorem, take for K the inclusion $\mathbf{\Delta}_n^{\mathrm{OP}} \rightarrow \mathbf{\Delta}^{\mathrm{OP}}$ and write $\mathrm{tr}^{(n)}$ in place of $[K, \mathbf{C}]$, so $\mathrm{tr}^{(n)} : \mathbf{SIC} \rightarrow \mathbf{SIC}_n$. If $\mathbf{C}$ is complete (cocomplete), then $\mathrm{tr}^{(n)}$ has a left (right) adjoint $\mathrm{sk}^{(n)}$ $(\mathrm{cosk}^{(n)})$. Put $\mathit{sk}^{(n)} = \mathrm{sk}^{(n)} \circ \mathrm{tr}^{(n)}$ (the n-skeleton), $\mathit{cosk}^{(n)} = \mathrm{cosk}^{(n)} \circ \mathrm{tr}^{(n)}$ (the n-coskeleton). Example: Let $\mathbf{C} = \mathbf{SET}$—then for any simplicial set X, $\mathit{sk}^{(n)} X \approx X^{(n)}$.

(Geometric Realizations) The realization functor $\Gamma_{\Delta^?}$ is a functor $\mathbf{SISET} \rightarrow \mathbf{TOP}$ such that $\Gamma_{\Delta^?} \circ \Delta = \Delta^?$. It assigns to a simplicial set X a topological space $|X| = \int^{[n]} X_n \cdot \Delta^n$, the geometric realization of X, and to a simplicial map $f : X \rightarrow Y$ a continuous function $|f| : |X| \rightarrow |Y|$, the geometric realization of f. In particular, $|\Delta[n]| = \Delta^n$ and $|\Delta[\alpha]| = \Delta^{\alpha}$. There is an explicit description of $|X|$: Equip X_n with the discrete topology and $X_n \times \Delta^n$ with the product topology—then $|X|$ can be identified with the quotient $\coprod_n X_n \times \Delta^n / \sim$, the equivalence relation being generated by writing $((X\alpha)x, t) \sim (x, \Delta^{\alpha} t)$. These relations are respected by every simplicial map $f : X \rightarrow Y$. Denote by $[x, t]$ the equivalence class corresponding to (x, t). The projection $(x, t) \rightarrow [x, t]$ of $\coprod_n X_n \times \Delta^n$ onto $|X|$ restricts to a map $\coprod_n X_n^{\#} \times \mathring{\Delta}^n \rightarrow |X|$ that is in fact a set theoretic bijection. Consequently, if we attach to each $x \in X_n^{\#}$ the subset e_x of $|X|$ consisting of all $[x, t]$ $(t \in \mathring{\Delta}^n)$, then the collection $\{e_x : x \in X_n^{\#}\ (n \geq 0)\}$ partitions $|X|$. It follows from this that a simplicial map $f : X \rightarrow Y$ is injective (surjective) iff its geometric realization $|f| : |X| \rightarrow |Y|$ is injective (surjective). Being a left adjoint, the functor $|?| : \mathbf{SISET} \rightarrow \mathbf{TOP}$ preserves colimits. So, e.g., by taking the geometric realization of the diagram

$$\coprod_{0\leq i<j\leq n} \Delta[n-2]_{i,j} \overset{u}{\underset{v}{\rightrightarrows}} \coprod_{0\leq i\leq n} \Delta[n-1]_i,$$

and unraveling the definitions, one finds that $|\dot{\Delta}[n]|$ can be identified with $\dot{\Delta}^n$.

[Note: It is also true that the arrow $|\Delta[m] \times \Delta[n]| \to |\Delta[m]| \times |\Delta[n]|$ associated with the geometric realization of the projections $\begin{cases} p_m : \Delta[m] \times \Delta[n] \to \Delta[m] \\ p_n : \Delta[m] \times \Delta[n] \to \Delta[n] \end{cases}$ is a homeomorphism but this is not an a priori property of $|?|$.]

(Singular Sets) The singular functor $S_{\Delta^?}$ is a functor **TOP** → **SISET** that assigns to a topological space X a simplicial set $\sin X$, the singular set of $X : \sin X([n]) = \sin_n X = C(\Delta^n, X)$. $|?|$ is a left adjoint for sin. The arrow of adjunction $X \to \sin|X|$ sends $x \in X_n$ to $|\Delta_x| \in C(\Delta^n, |X|)$, where $|\Delta_x|(t) = [x, t]$; it is a monomorphism. The arrow of adjunction $|\sin X| \to X$ sends $[x, t]$ to $x(t)$; it is an epimorphism.

There is a functor T from **SIAB** to the category of chain complexes of abelian groups: Take an X and let TX be $X_0 \overset{\partial}{\leftarrow} X_1 \overset{\partial}{\leftarrow} X_2 \overset{\partial}{\leftarrow} \cdots$, where $\partial = \sum_0^n (-1)^i d_i$ $(d_i : X_n \to X_{n-1})$. That $\partial \circ \partial = 0$ is implied by the simplicial identities. One can then apply the homology functor H_* and end up in the category of graded abelian groups. On the other hand, the forgetful functor **AB** → **SET** has a left adjoint F_{ab} that sends a set X to the free abelian group $F_{\mathrm{ab}}X$ on X. Extend it to a functor F_{ab} : **SISET** → **SIAB**. In this terminology, the singular homology $H_*(X)$ of a topological space X is $H_*(TF_{\mathrm{ab}}(\sin X))$.

(Categorical Realizations) The realization functor Γ_ι is a functor **SISET** → **CAT** such that $\Gamma_\iota \circ \Delta = \iota$. It assigns to a simplicial set X a small category $cX = \int^{[n]} X_n \cdot [n]$ called the categorical realization of X. In particular, $c\Delta[n] = [n]$. In general, cX can be represented as a quotient category $CX/\sim$. Here, CX is the category whose objects are the elements of X_0 and whose morphisms are the finite sequences $(x_1, \ldots, x_n)$ of elements of X_1 such that $d_0 x_i = d_1 x_{i+1}$. Composition is concatenation and the empty sequences are the identities. The relations are $s_0 x = \mathrm{id}_x$ $(x \in X_0)$ and $(d_0 x) \circ (d_2 x) = d_1 x$ $(x \in X_2)$.

(Nerves) The singular functor S_ι is a functor **CAT** → **SISET** that assigns to a small category $\mathbf{C}$ a simplicial set $\mathrm{ner}\,\mathbf{C}$, the nerve of $\mathbf{C} : \mathrm{ner}\,\mathbf{C}([n]) = \mathrm{ner}_n\mathbf{C}$, the set of all diagrams in $\mathbf{C}$ of the form $X_0 \xrightarrow{f_0} X_1 \to \cdots \to X_{n-1} \xrightarrow{f_{n-1}} X_n$. Therefore, $\mathrm{ner}_0\mathbf{C} = \mathrm{Ob}\,\mathbf{C}$ and $\mathrm{ner}_1\mathbf{C} = \mathrm{Mor}\,\mathbf{C}$. c is a left adjoint for ner. Since ner is full and faithful, the arrow of adjunction $c \circ \mathrm{ner} \to \mathrm{id}_{\mathbf{CAT}}$ is a natural isomorphism. The classifying space of $\mathbf{C}$ is the geometric realization of its nerve: $B\mathbf{C} \equiv |\mathrm{ner}\,\mathbf{C}|$. Example: $B\mathbf{C} \approx B\mathbf{C}^{\mathrm{OP}}$.

The composite $\Pi = \pi_1 \circ c$ is a functor **SISET** → **GRD** that sends a simplicial set X to its fundamental groupoid ΠX. Example: If X is a topological space, then $\Pi X \approx \Pi(\sin X)$.

Let $\mathbf{C}$ be a small category. Given a cofunctor $F : \mathbf{C} \to \mathbf{SET}$, the Grothendieck construction on F is the category $\text{gro}_{\mathbf{C}}F$ whose objects are the pairs (X, x), where X is an object in $\mathbf{C}$ with $x \in FX$, and whose morphisms are the arrows $f : (X, x) \to (Y, y)$, where $f : X \to Y$ is a morphism in $\mathbf{C}$ with $(Ff)y = x$. Denoting by π_F the projection $\text{gro}_{\mathbf{C}}F \to \mathbf{C}$, if $\mathbf{S}$ is cocomplete, then for any $T \in \text{Ob}\,[\mathbf{C}, \mathbf{S}]$, $\Gamma_T F \approx \text{colim}(\text{gro}_{\mathbf{C}}F \xrightarrow{\pi_F} \mathbf{C} \xrightarrow{T} \mathbf{S})$. In particular: $F \approx \text{colim}(\text{gro}_{\mathbf{C}}F \xrightarrow{\pi_F} \mathbf{C} \xrightarrow{Y_{\mathbf{C}}} \widehat{\mathbf{C}})$.

[Note: The Grothendieck construction on a functor $F : \mathbf{C} \to \mathbf{SET}$ is the category $\text{gro}_{\mathbf{C}}F$ whose objects are the pairs (X, x), where X is an object in $\mathbf{C}$ with $x \in FX$, and whose morphisms are the arrows $f : (X, x) \to (Y, y)$, where $f : X \to Y$ is a morphism in $\mathbf{C}$ with $(Ff)x = y$. Example: $\text{gro}_{\mathbf{C}}\text{Mor}\,(X, —) \approx X\backslash\mathbf{C}$.]

Let $\gamma : \mathbf{C} \to \mathbf{CAT}$ be the functor that sends X to $\mathbf{C}/X$—then the realization functor Γ_γ assigns to each F in $\widehat{\mathbf{C}}$ its Grothendieck construction, i.e., $\Gamma_\gamma F \approx \text{gro}_{\mathbf{C}}F$.

A full, isomorphism closed subcategory $\mathbf{D}$ of a category $\mathbf{C}$ is said to be a reflective (coreflective) subcategory of $\mathbf{C}$ if the inclusion $\mathbf{D} \to \mathbf{C}$ has a left (right) adjoint R, a reflector (coreflector) for $\mathbf{D}$.

[Note: A full subcategory $\mathbf{D}$ of a category $\mathbf{C}$ is isomorphism closed provided that every object in $\mathbf{C}$ which is isomorphic to an object in $\mathbf{D}$ is itself an object in $\mathbf{D}$.]

SET has precisely three (two) reflective (coreflective) subcategories. **TOP** has two reflective subcategories whose intersection is not reflective. The full subcategory of **GR** whose objects are the finite groups is not a reflective subcategory of **GR**.

Let $\mathbf{D}$ be a reflective subcategory of $\mathbf{C}$, R a reflector for $\mathbf{D}$—then one may attach to each $X \in \text{Ob}\,\mathbf{C}$ a morphism $r_X : X \to RX$ in $\mathbf{C}$ with the following property: Given any $Y \in \text{Ob}\,\mathbf{D}$ and any morphism $f : X \to Y$ in $\mathbf{C}$, there exists a unique morphism $g : RX \to Y$ in $\mathbf{D}$ such that $f = g \circ r_X$. If the r_X are epimorphisms, then $\mathbf{D}$ is said to be an epireflective subcategory of $\mathbf{C}$.

[Note: If the r_X are monomorphisms, then the r_X are epimorphisms, so "monoreflective" ⇒ "epireflective".]

A reflective subcategory $\mathbf{D}$ of a complete (cocomplete) category $\mathbf{C}$ is complete (cocomplete).

[Note: Let $\Delta : \mathbf{I} \to \mathbf{D}$ be a diagram in $\mathbf{D}$.

(1) To calculate a limit of Δ, postcompose Δ with the inclusion $\mathbf{D} \to \mathbf{C}$ and let $\{\ell_i : L \to \Delta_i\}$ be its limit in $\mathbf{C}$—then $L \in \text{Ob}\,\mathbf{D}$ and $\{\ell_i : L \to \Delta_i\}$ is a limit of Δ.

(2) To calculate a colimit of Δ, postcompose Δ with the inclusion $\mathbf{D} \rightarrow \mathbf{C}$ and let $\{\ell_i : \Delta_i \rightarrow L\}$ be its colimit in **C**—then $\{r_L \circ \ell_i : \Delta_i \rightarrow RL\}$ is a colimit of Δ.]

EPIREFLECTIVE CHARACTERIZATION THEOREM If a category **C** is complete, wellpowered, and cowellpowered, then a full, isomorphism closed subcategory **D** of **C** is an epireflective subcategory of **C** iff **D** is closed under the formation in **C** of products and extremal monomorphisms.

[Note: Under the same assumptions on **C**, the intersection of any conglomerate of epireflective subcategories is epireflective.]

A full, isomorphism closed subcategory of **TOP** (**HAUS**) is an epireflective subcategory iff it is closed under the formation in **TOP** (**HAUS**) of products and embeddings (products and closed embeddings).

(hX) **HAUS** is an epireflective subcategory of **TOP**. The reflector sends X to its maximal Hausdorff quotient hX.

(cr$X$) The full subcategory of **TOP** whose objects are the completely regular Hausdorff spaces is an epireflective subcategory of **TOP**. The reflector sends X to its complete regularization cr$X$.

(βX) The full subcategory of **HAUS** whose objects are the compact spaces is an epireflective subcategory of **HAUS**. Therefore the category of compact Hausdorff spaces is an epireflective subcategory of the category of completely regular Hausdorff spaces and the reflector sends X to βX, the Stone-Čech compactification of X.

[Note: If X is Hausdorff, then $\beta(\mathrm{cr}X)$ is its compact reflection.]

(vX) The full subcategory of **HAUS** whose objects are the **R**-compact spaces is an epireflective subcategory of **HAUS**. Therefore the category of **R**-compact spaces is an epireflective subcategory of the category of completely regular Hausdorff spaces and the reflector sends X to vX, the **R**-compactification of X.

[Note: If X is Hausdorff, then $v(\mathrm{cr}X)$ is its **R**-compact reflection.]

A full, isomorphism closed subcategory of GR or AB is an epireflective subcategory iff it is closed under the formation of products and subgroups. Example: AB is an epireflective subcategory of GR, the reflector sending X to its abelianization $X/[X,X]$.

If **C** is a full subcategory of **TOP** (**HAUS**), then there is a smallest epireflective subcategory of **TOP** (**HAUS**) containing **C**, the epireflective hull of **C**. If X is a topological space (Hausdorff topological space), then X is an object in the epireflective hull of

C in **TOP** (**HAUS**) iff there exists a set $\{X_i\} \subset \text{Ob}\,\mathbf{C}$ and an extremal monomorphism $f : X \rightarrow \prod\limits_i X_i$.

The epireflective hull in **TOP** (**HAUS**) of $[0,1]$ is the category of completely regular Hausdorff spaces (compact Hausdorff spaces). The epireflective hull in **TOP** of $[0,1]/[0,1[$ is the full subcategory of **TOP** whose objects satisfy the T_0 separation axiom. The epireflective hull in **TOP** (**HAUS**) of $\{0,1\}$ (discrete topology) is the full subcategory of **TOP** (**HAUS**) whose objects are the zero dimensional Hausdorff spaces (zero dimensional compact Hausdorff spaces). The epireflective hull in **TOP** of $\{0,1\}$ (indiscrete topology) is the full subcategory of **TOP** whose objects are the indiscrete spaces.

[Note: Let E be a nonempty Hausdorff space—then a Hausdorff space X is said to be E-compact provided that X is in the epireflective hull of E in **HAUS**. Example: A Hausdorff space is **N**-compact iff it is **Q**-compact iff it is **P**-compact. There is no E such that every Hausdorff space is E-compact. In fact, given E, there exists a Hausdorff space X_E with $\#(X_E) > 1$ such that every element of $C(X_E, E)$ is a constant.]

A morphism $f : A \rightarrow B$ and an object X in a category **C** are said to be orthogonal ($f \perp X$) if the precomposition arrow $f^* : \text{Mor}(B, X) \rightarrow \text{Mor}(A, X)$ is bijective. Given a class $S \subset \text{Mor}\,\mathbf{C}$, $S^\perp$ is the class of objects orthogonal to each $f \in S$ and given a class $D \subset \text{Ob}\,\mathbf{C}$, $D^\perp$ is the class of morphisms orthogonal to each $X \in D$. One then says that a pair (S, D) is an orthogonal pair provided that $S = D^\perp$ and $D = S^\perp$. Example: Since $\perp\perp\perp = \perp$, for any S, $(S^{\perp\perp}, S^\perp)$ is an orthogonal pair, and for any D, $(D^\perp, D^{\perp\perp})$ is an orthogonal pair.

[Note: Suppose that (S, D) is an orthogonal pair—then (1) S contains the isomorphisms of **C**; (2) S is closed under composition; (3) S is cancellable, i.e., $g \circ f \in S$ & $f \in S \Rightarrow g \in S$ and $g \circ f \in S$ & $g \in S \Rightarrow f \in S$. In addition, if $$\begin{tikzcd} A \arrow[r] \arrow[d, "f"'] & A' \arrow[d, "f'"] \\ B \arrow[r] & B' \end{tikzcd}$$ is a pushout square, then $f \in S \Rightarrow f' \in S$, and if $\Xi \in \text{Nat}(\Delta, \Delta')$, where $\Delta, \Delta' : \mathbf{I} \rightarrow \mathbf{C}$, then $\Xi_i \in S$ $(\forall\, i) \Rightarrow \text{colim}\,\Xi \in S$ (if $\text{colim}\,\Delta$, $\text{colim}\,\Delta'$ exist).]

Every reflective subcategory **D** of **C** generates an orthogonal pair. Thus, with $R : \mathbf{C} \rightarrow \mathbf{D}$ the reflector, put $T = \iota \circ R$, where $\iota : \mathbf{D} \rightarrow \mathbf{C}$ is the inclusion, and denote by $\epsilon : \text{id}_{\mathbf{C}} \rightarrow T$ the associated natural transformation. Take for $S \subset \text{Mor}\,\mathbf{C}$ the class consisting of those f such that Tf is an isomorphism and take for $D \subset \text{Ob}\,\mathbf{C}$ the object class of **D**, i.e., the class consisting of those X such that ϵ_X is an isomorphism—then (S, D) is an orthogonal pair.

A full, isomorphism closed subcategory **D** of a category **C** is said to be an orthogonal subcategory

of C if Ob D = $S^\perp$ for some class $S \subset$ Mor C. If D is reflective, then D is orthogonal but the converse is false (even in TOP).

[Note: Let (S, D) be an orthogonal pair. Suppose that for each $X \in$ Ob C there exists a morphism $\epsilon_X : X \to TX$ in S, where $TX \in D$—then for every $f : A \to B$ in S and for every $g : A \to X$ there exists a unique $t : B \to TX$ such that $\epsilon_X \circ g = t \circ f$. So, for any arrow $X \to Y$, there is a commutative diagram

$$\begin{tikzcd} X \arrow[r, "\epsilon_X"] \arrow[d] & TX \arrow[d] \\ Y \arrow[r, "\epsilon_Y"'] & TY \end{tikzcd}$$

, thus T defines a functor $\mathbf{C} \to \mathbf{C}$ and $\epsilon : \mathrm{id}_{\mathbf{C}} \to T$ is a natural transformation. Since $\epsilon T = T\epsilon$ is a natural isomorphism, it follows that $S^\perp = D$ is the object class of a reflective subcategory of C.]

(κ-DEF) Fix a regular cardinal κ—then an object X in a cocomplete category **C** is said to be κ-definite provided that $\forall$ regular cardinal $\kappa' \geq \kappa$, Mor$(X, —)$ preserves colimits over $[0, \kappa'[$, so for every diagram $\Delta : [0, \kappa'[\to \mathbf{C}$, the arrow colim Mor$(X, \Delta_\alpha) \to$ Mor$(X, \text{colim}\, \Delta_\alpha)$ is bijective.

Given a group G, there is a κ for which G is κ-definite and all finitely presented groups are ω-definite.

REFLECTIVE SUBCATEGORY THEOREM Let **C** be a cocomplete category. Suppose that $S_0 \subset$ Mor **C** is a set with the property that for some κ, the domain and codomain of each $f \in S_0$ are κ-definite—then $S_0^\perp$ is the object class of a reflective subcategory of **C**.

(P-Localization) Let P be a set of primes. Let $S_P = \{1\} \cup \{n > 1 : p \in P \Rightarrow p \nmid n\}$—then a group G is said to be P-local if the map $\begin{cases} G \to G \\ g \to g^n \end{cases}$ is bijective $\forall\, n \in S_P$. $\mathbf{GR}_P$, the full subcategory of **GR** whose objects are the P-local groups, is a reflective subcategory of **GR**. In fact, Ob $\mathbf{GR}_P = S_P^\perp$, where now S_P stands for the set of homomorphisms $\begin{cases} \mathbf{Z} \to \mathbf{Z} \\ 1 \to n \end{cases}$ $(n \in S_P)$. The reflector $L_P : \begin{cases} \mathbf{GR} \to \mathbf{GR}_P \\ G \to G_P \end{cases}$ is called P-localization.

P-localization need not preserve short exact sequences. For example, $1 \to A_3 \to S_3 \to S_3/A_3 \to 1$, when localized at $P = \{3\}$, gives $1 \to A_3 \to 1 \to 1 \to 1$.

A category **C** with finite products is said to be cartesian closed provided that each of the functors $— \times Y : \mathbf{C} \to \mathbf{C}$ has a right adjoint $Z \to Z^Y$, so Mor$(X \times Y, Z) \approx$ Mor(X, Z^Y). The object Z^Y is called an exponential object. The evaluation morphism $\mathrm{ev}_{Y,Z}$ is the morphism $Z^Y \times Y \to Z$ such that for every $f : X \times Y \to Z$ there is a unique $g : X \to Z^Y$ such that $f = \mathrm{ev}_{Y,Z} \circ (g \times \mathrm{id}_Y)$.

In a cartesian closed category:

$$\text{(1)}\quad X^{Y\times Z}\approx (X^Y)^Z; \qquad \text{(3)}\quad X^{\coprod\limits_i Y_i}\approx \prod_i (X^{Y_i});$$

$$\text{(2)}\quad (\prod_i X_i)^Y\approx \prod_i (X_i^Y); \qquad \text{(4)}\quad X\times(\coprod_i Y_i)\approx \coprod_i (X\times Y_i).$$

SET is cartesian closed but $\mathbf{SET}^{\mathbf{OP}}$ is not cartesian closed. **TOP** is not cartesian closed but does have full, cartesian closed subcategories, e.g., the category of compactly generated Hausdorff spaces.

[Note: If $\mathbf{C}$ is cartesian closed and has a zero object, then $\mathbf{C}$ is equivalent to $\mathbf{1}$. Therefore neither $\mathbf{SET}_*$ nor $\mathbf{TOP}_*$ is cartesian closed.]

CAT is cartesian closed: $\mathrm{Mor}\,(\mathbf{C}\times\mathbf{D},\mathbf{E})\approx\mathrm{Mor}\,(\mathbf{C},\mathbf{E}^{\mathbf{D}})$, where $\mathbf{E}^{\mathbf{D}}=[\mathbf{D},\mathbf{E}]$. **SISET** is cartesian closed: $\mathrm{Nat}(X\times Y,Z)\approx\mathrm{Nat}(X,Z^Y)$, where $Z^Y([n])=\mathrm{Nat}(Y\times\Delta[n],Z)$.

[Note: The functor ner : **CAT** → **SISET** preserves exponential objects.]

A monoidal category is a category $\mathbf{C}$ equipped with a functor $\otimes:\mathbf{C}\times\mathbf{C}\to\mathbf{C}$ (the multiplication) and an object $e\in\mathrm{Ob}\,\mathbf{C}$ (the unit), together with natural isomorphisms R, L, and A, where $\begin{cases} R_X: X\otimes e\to X\\ L_X: e\otimes X\to X\end{cases}$ and $A_{X,Y,Z}:X\otimes(Y\otimes Z)\to(X\otimes Y)\otimes Z$, subject to the following assumptions.

(MC_1) The diagram

$$\begin{tikzcd}
X\otimes(Y\otimes(Z\otimes W)) \arrow[r, "A"] \arrow[d, "\mathrm{id}\otimes A"'] & (X\otimes Y)\otimes(Z\otimes W) \arrow[r, "A"] & ((X\otimes Y)\otimes Z)\otimes W \\
X\otimes((Y\otimes Z)\otimes W) \arrow[rr, "A"'] & & (X\otimes(Y\otimes Z))\otimes W \arrow[u, "A\otimes\mathrm{id}"']
\end{tikzcd}$$

commutes.

(MC_2) The diagram

$$\begin{tikzcd}
X\otimes(e\otimes Y) \arrow[r, "A"] \arrow[d, "\mathrm{id}\otimes L"'] & (X\otimes e)\otimes Y \arrow[d, "R\otimes\mathrm{id}"] \\
X\otimes Y \arrow[r, equal] & X\otimes Y
\end{tikzcd}$$

commutes.

[Note: The "coherency" principle then asserts that "all" diagrams built up from instances of R, L, A (or their inverses), and id by repeated application of $\otimes$ necessarily commute. In particular, the diagrams

$$\begin{tikzcd}
e\otimes(X\otimes Y) \arrow[r, "A"] \arrow[d, "L"'] & (e\otimes X)\otimes Y \arrow[d, "L\otimes\mathrm{id}"] \\
X\otimes Y \arrow[r, equal] & X\otimes Y
\end{tikzcd}
\qquad
\begin{tikzcd}
X\otimes(Y\otimes e) \arrow[r, "A"] \arrow[d, "\mathrm{id}\otimes R"'] & (X\otimes Y)\otimes e \arrow[d, "R"] \\
X\otimes Y \arrow[r, equal] & X\otimes Y
\end{tikzcd}$$

commute and $L_e = R_e : e \otimes e \to e$.]

Any category with finite products (coproducts) is monoidal: Take $X \otimes Y$ to be $X \amalg Y$ ($X \amalg Y$) and let e be a final (initial) object. The category **AB** is monoidal: Take $X \otimes Y$ to be the tensor product and let e be $\mathbf{Z}$. The category $\mathbf{SET}_*$ is monoidal: Take $X \otimes Y$ to be the smash product $X \# Y$ and let e be the two point set.

A *symmetry* for a monoidal category **C** is a natural isomorphism $\top$, where $\top_{X,Y} : X \otimes Y \to Y \otimes X$, such that $\top_{Y,X} \circ \top_{X,Y} : X \otimes Y \to X \otimes Y$ is the identity, $R_X = L_X \circ \top_{X,e}$, and the diagram

$$\begin{tikzcd}
X \otimes (Y \otimes Z) \arrow[r, "A"] \arrow[d, "\mathrm{id} \otimes \top"'] & (X \otimes Y) \otimes Z \arrow[r, "\top"] & Z \otimes (X \otimes Y) \arrow[d, "A"] \\
X \otimes (Z \otimes Y) \arrow[r, "A"'] & (X \otimes Z) \otimes Y \arrow[r, "\top \otimes \mathrm{id}"'] & (Z \otimes X) \otimes Y
\end{tikzcd}$$

commutes. A *symmetric monoidal category* is a monoidal category **C** endowed with a symmetry $\top$. A monoidal category can have more than one symmetry (or none at all).

[Note: The "coherency" principle then asserts that "all" diagrams built up from instances of R, L, A, $\top$ (or their inverses), and id by repeated application of $\otimes$ necessarily commute.]

Let **C** be the category of chain complexes of abelian groups; let **D** be the full subcategory of **C** whose objects are the graded abelian groups. **C** and **D** are both monoidal: Take $X \otimes Y$ to be the tensor product and let $e = \{e_n\}$ be the chain complex defined by $e_0 = \mathbf{Z}$ and $e_n = 0$ $(n \neq 0)$. If $\begin{cases} X = \{X_p\} \\ Y = \{Y_q\} \end{cases}$ and if $\begin{cases} x \in X_p \\ y \in Y_q \end{cases}$, then the assignment $\begin{cases} X \otimes Y \to Y \otimes X \\ x \otimes y \to (-1)^{pq}(y \otimes x) \end{cases}$ is a symmetry for **C** and there are no others. By contrast, **D** admits a second symmetry, namely the assignment $\begin{cases} X \otimes Y \to Y \otimes X \\ x \otimes y \to y \otimes x \end{cases}$.

A *closed category* is a symmetric monoidal category **C** with the property that each of the functors $- \otimes Y : \mathbf{C} \to \mathbf{C}$ has a right adjoint $Z \to \hom(Y, Z)$, so $\mathrm{Mor}(X \otimes Y, Z) \approx \mathrm{Mor}(X, \hom(Y, Z))$. The functor $\hom : \mathbf{C}^{\mathrm{OP}} \times \mathbf{C} \to \mathbf{C}$ is called an *internal hom functor*. The *evaluation morphism* $\mathrm{ev}_{Y,Z}$ is the morphism $\hom(Y, Z) \otimes Y \to Z$ such that for every $f : X \otimes Y \to Z$ there is a unique $g : X \to \hom(Y, Z)$ such that $f = \mathrm{ev}_{Y,Z} \circ (g \otimes \mathrm{id}_Y)$. Agreeing to write U_e for the functor $\mathrm{Mor}(e, -)$ (which need not be faithful), one has $U_e \circ \hom \approx \mathrm{Mor}$. Consequently, $X \approx \hom(e, X)$ and $\hom(X \otimes Y, Z) \approx \hom(X, \hom(Y, Z))$.

A cartesian closed category is a closed category. **AB** is a closed category but is not cartesian closed.

TOP admits, to within isomorphism, exactly one structure of a closed category. For let X and Y be topological spaces—then their product $X \otimes Y$ is the cartesian product $X \times Y$ supplied with the final topology determined by the inclusions $\begin{cases} \{x\} \times Y \to X \times Y \\ X \times \{y\} \to X \times Y \end{cases}$ $(x \in X, y \in Y)$, the unit being the one point space. The associated internal hom functor $\hom(X, Y)$ sends (X, Y) to $C(X, Y)$, where $C(X, Y)$ carries the topology of pointwise convergence.

Given a monoidal category **C**, a monoid in **C** is an object $X \in \operatorname{Ob} \mathbf{C}$ together with morphisms $m : X \otimes X \to X$ and $\epsilon : e \to X$ subject to the following assumptions.

(MO$_1$) The diagram

$$\begin{tikzcd}
X \otimes (X \otimes X) \arrow[r, "A"] \arrow[d, "\mathrm{id} \otimes m"'] & (X \otimes X) \otimes X \arrow[r, "m \otimes \mathrm{id}"] & X \otimes X \arrow[d, "m"] \\
X \otimes X \arrow[rr, "m"'] & & X
\end{tikzcd}$$

commutes.

(MO$_2$) The diagrams

$$\begin{tikzcd}
e \otimes X \arrow[r, "\epsilon \otimes \mathrm{id}"] \arrow[d, "L"'] & X \otimes X \arrow[d, "m"] \\
X \arrow[r, equal] & X
\end{tikzcd}
\qquad
\begin{tikzcd}
X \otimes X \arrow[d, "m"'] & X \otimes e \arrow[l, "\mathrm{id} \otimes \epsilon"'] \arrow[d, "R"] \\
X \arrow[r, equal] & X
\end{tikzcd}$$

commute.

$\mathbf{MON_C}$ is the category whose objects are the monoids in **C** and whose morphisms $(X, m, \epsilon) \to (X', m', \epsilon')$ are the arrows $f : X \to X'$ such that $f \circ m = m' \circ (f \otimes f)$ and $f \circ \epsilon = \epsilon'$.

$\mathbf{MON_{SET}}$ is the category of semigroups with unit. $\mathbf{MON_{AB}}$ is the category of rings with unit.

Given a monoidal category **C**, a left action of a monoid X in **C** on an object $Y \in \operatorname{Ob} \mathbf{C}$ is a morphism $l : X \otimes Y \to Y$ such that the diagram

$$\begin{tikzcd}
X \otimes (X \otimes Y) \arrow[r, "A"] \arrow[d, "\mathrm{id} \otimes l"'] & (X \otimes X) \otimes Y \arrow[r, "m \otimes \mathrm{id}"] & X \otimes Y \arrow[d, "l"] & e \otimes Y \arrow[l, "\epsilon \otimes \mathrm{id}"'] \arrow[d, "L"] \\
X \otimes Y \arrow[rr, "l"'] & & Y \arrow[r, equal] & Y
\end{tikzcd}$$

commutes.

[Note: The definition of a right action is analogous.]

$\mathbf{LACT}_X$ is the category whose objects are the left actions of X and whose morphisms $(Y,l) \to (Y',l')$ are the arrows $f : Y \to Y'$ such that $f \circ l = l' \circ (\mathrm{id} \otimes f)$.

If X is a monoid in $\mathbf{SET}$, then $\mathbf{LACT}_X$ is isomorphic to the functor category $[\mathbf{X}, \mathbf{SET}]$, $\mathbf{X}$ the category having a single object $*$ with $\mathrm{Mor}(*,*) = X$.

A triple $\mathbf{T} = (T, m, \epsilon)$ in a category $\mathbf{C}$ consists of a functor $T : \mathbf{C} \to \mathbf{C}$ and natural transformations $\begin{cases} m \in \mathrm{Nat}(T \circ T, T) \\ \epsilon \in \mathrm{Nat}(\mathrm{id}_{\mathbf{C}}, T) \end{cases}$ subject to the following assumptions.

(T$_1$) The diagram

$$\begin{tikzcd}
T \circ T \circ T \arrow[r, "mT"] \arrow[d, "Tm"'] & T \circ T \arrow[d, "m"] \\
T \circ T \arrow[r, "m"'] & T
\end{tikzcd}$$

commutes.

(T$_2$) The diagrams

$$\begin{tikzcd}
T \arrow[r, "\epsilon T"] \arrow[d, "\mathrm{id}"'] & T \circ T \arrow[d, "m"] \\
T \arrow[r, equal] & T
\end{tikzcd}
\qquad
\begin{tikzcd}
T \circ T \arrow[d, "m"'] & T \arrow[l, "T\epsilon"'] \arrow[d, "\mathrm{id}"] \\
T \arrow[r, equal] & T
\end{tikzcd}$$

commute.

[Note: Formally, the functor category $[\mathbf{C}, \mathbf{C}]$ is a monoidal category: Take $F \otimes G$ to be $F \circ G$ and let e be $\mathrm{id}_{\mathbf{C}}$. Therefore a triple in $\mathbf{C}$ is a monoid in $[\mathbf{C}, \mathbf{C}]$ (and a cotriple in $\mathbf{C}$ is a monoid in $[\mathbf{C}, \mathbf{C}]^{\mathrm{OP}}$), a morphism of triples being a morphism in the metacategory $\mathbf{MON}_{[\mathbf{C},\mathbf{C}]}$.]

Given a triple $\mathbf{T} = (T, m, \epsilon)$ in $\mathbf{C}$, a $\mathbf{T}$-algebra is an object X in $\mathbf{C}$ and a morphism $\xi : TX \to X$ subject to the following assumptions.

(TA$_1$) The diagram

$$\begin{tikzcd}
T(TX) \arrow[r, "T\xi"] \arrow[d, "m_X"'] & TX \arrow[d, "\xi"] \\
TX \arrow[r, "\xi"'] & X
\end{tikzcd}$$

commutes.

(TA$_2$) The diagram

$$\begin{tikzcd} X \arrow[r, "\epsilon_X"] \arrow[d, "\text{id}"'] & TX \arrow[d, "\xi"] \\ X \arrow[r, equal] & X \end{tikzcd}$$

commutes.

T-ALG is the category whose objects are the **T**-algebras and whose morphisms $(X,\xi) \rightarrow (Y,\eta)$ are the arrows $f : X \rightarrow Y$ such that $f \circ \xi = \eta \circ Tf$.

[Note: If $\mathbf{T} = (T, m, \epsilon)$ is a cotriple in **C**, then the relevant notion is **T**-coalgebra and the relevant category is **T-COALG**.]

Take $\mathbf{C} = \mathbf{AB}$. Let $A \in \text{Ob}\,\mathbf{RG}$. Define $T : \mathbf{AB} \rightarrow \mathbf{AB}$ by $TX = A \otimes X$, $m \in \text{Nat}(T \circ T, T)$ by $m_X : \begin{cases} A \otimes (A \otimes X) \rightarrow A \otimes X \\ a \otimes (b \otimes x) \rightarrow ab \otimes x \end{cases}$, $\epsilon \in \text{Nat}(\text{id}_{\mathbf{AB}}, T)$ by $\epsilon_X : \begin{cases} X \rightarrow A \otimes X \\ x \rightarrow 1 \otimes x \end{cases}$—then **T-ALG** is isomorphic to A-**MOD**.

Every adjoint situation (F, G, μ, ν) determines a triple in **C**, viz. $(G \circ F, G\nu F, \mu)$ (and a cotriple in **D**, viz. $(F \circ G, F\mu G, \nu)$). Different adjoint situations can determine the same triple. Conversely, every triple is determined by at least one adjoint situation, in general by many. One realization is the construction of Eilenberg-Moore: Given a triple $\mathbf{T} = (T, m, \epsilon)$ in **C**, call $F_{\mathbf{T}}$ the functor $\mathbf{C} \rightarrow \mathbf{T\text{-}ALG}$ that sends $X \xrightarrow{f} Y$ to $(TX, m_X) \xrightarrow{Tf} (TY, m_Y)$, call $G_{\mathbf{T}}$ the functor $\mathbf{T\text{-}ALG} \rightarrow \mathbf{C}$ that sends $(X,\xi) \xrightarrow{f} (Y,\eta)$ to $X \xrightarrow{f} Y$, put $\mu_X = \epsilon_X$, and $\nu_{(X,\xi)} = \xi$—then $F_{\mathbf{T}}$ is a left adjoint for $G_{\mathbf{T}}$ and this adjoint situation determines **T**.

Suppose that $\mathbf{C} = \mathbf{SET}$, $\mathbf{D} = \mathbf{MON}_{\mathbf{SET}}$. Let $F : \mathbf{C} \rightarrow \mathbf{D}$ be the functor that sends X to the free semigroup with unit on X—then F is a left adjoint for the forgetful functor $G : \mathbf{D} \rightarrow \mathbf{C}$. The triple determined by this adjoint situation is $\mathbf{T} = (T, m, \epsilon)$, where $T : \mathbf{SET} \rightarrow \mathbf{SET}$ assigns to each X the set $TX = \bigcup_0^\infty X^n$, $m_X : T(TX) \rightarrow TX$ is defined by concatenation and $\epsilon_X : X \rightarrow TX$ by inclusion. The corresponding category of **T**-algebras is isomorphic to $\mathbf{MON}_{\mathbf{SET}}$.

Let (F, G, μ, ν) be an adjoint situation. If $\mathbf{T} = (G \circ F, G\nu F, \mu)$ is the associated triple in **C**, then the comparison functor Φ is the functor $\mathbf{D} \rightarrow \mathbf{T\text{-}ALG}$ that sends Y to $(GY, G\nu_Y)$ and g to Gg. It is the only functor $\mathbf{D} \rightarrow \mathbf{T\text{-}ALG}$ for which $\Phi \circ F = F_{\mathbf{T}}$ and $G_{\mathbf{T}} \circ \Phi = G$.

Consider the adjoint situation produced by the forgetful functor $\mathbf{TOP} \rightarrow \mathbf{SET}$—then $\mathbf{T\text{-}ALG} = \mathbf{SET}$ and the comparison functor $\mathbf{TOP} \rightarrow \mathbf{SET}$ is the forgetful functor.

Given categories $\begin{cases}\mathbf{C}\\ \mathbf{D}\end{cases}$, a functor $G : \mathbf{D} \to \mathbf{C}$ is said to be monadic (strictly monadic) provided that G has a left adjoint $F : \mathbf{C} \to \mathbf{D}$ and the comparison functor $\Phi : \mathbf{D} \to \mathbf{T}\text{-}\mathbf{ALG}$ is an equivalence (isomorphism) of categories.

In order that G be monadic, it is necessary that G be conservative. So, e.g., the forgetful functor $\mathbf{TOP} \to \mathbf{SET}$ is not monadic. If $\mathbf{D}$ is the category of Banach spaces and linear contractions and if $G : \mathbf{D} \to \mathbf{SET}$ is the "unit ball" functor, then G has a left adjoint and is conservative, but not monadic. Theorems due to Beck, Duskin and others lay down conditions that are necessary and sufficient for a functor to be monadic or strictly monadic. In particular, these results imply that if $\mathbf{D}$ is a "finitary category of algebraic structures", then the forgetful functor $\mathbf{D} \to \mathbf{SET}$ is strictly monadic. Therefore the forgetful functor from $\mathbf{GR}$, $\mathbf{RG}$, $\ldots$, to $\mathbf{SET}$ is strictly monadic.

[Note: No functor from $\mathbf{CAT}$ to $\mathbf{SET}$ can be monadic.]

Among the possibilities of determining a triple $\mathbf{T} = (T, m, \epsilon)$ in $\mathbf{C}$ by an adjoint situation, the construction of Eilenberg-Moore is "maximal". The "minimal" construction is that of Kleisli: $\mathbf{KL(T)}$ is the category whose objects are those of $\mathbf{C}$, the morphisms from X to Y being $\operatorname{Mor}(X, TY)$ with $\epsilon_X \in \operatorname{Mor}(X, TX)$ serving as the identity. Here, the composition of $\begin{cases} X \xrightarrow{f} TY\\ Y \underset{g}{\rightarrow} TZ\end{cases}$ in $\mathbf{KL(T)}$ is $m_Z \circ Tg \circ f$ (calculated in $\mathbf{C}$). If $K_{\mathbf{T}} : \mathbf{C} \to \mathbf{KL(T)}$ is the functor that sends $X \xrightarrow{f} Y$ to $X \xrightarrow{\epsilon_Y \circ f} TY$ and if $L_{\mathbf{T}} : \mathbf{KL(T)} \to \mathbf{C}$ is the functor that sends $X \xrightarrow{f} TY$ to $TX \xrightarrow{m_Y \circ Tf} TY$, then $K_{\mathbf{T}}$ is a left adjoint for $L_{\mathbf{T}}$ with arrows of adjunction $\epsilon_X, \mathrm{id}_{TX}$ and this adjoint situation determines $\mathbf{T}$.

[Note: Let $G : \mathbf{D} \to \mathbf{C}$ be a functor—then the shape of G is the metacategory $\mathbf{S}_G$ whose objects are those of $\mathbf{C}$, the morphisms from X to Y being the conglomerate $\operatorname{Nat}(\operatorname{Mor}(Y, G—), \operatorname{Mor}(X, G—))$. While ad hoc arguments can sometimes be used to show that $\mathbf{S}_G$ is isomorphic to a category, the situation is optimal when G has a left adjoint $F : \mathbf{C} \to \mathbf{D}$ since in this case $\mathbf{S}_G$ is isomorphic to $\mathbf{KL(T)}$, $\mathbf{T}$ the triple in $\mathbf{C}$ determined by F and G.]

Consider the adjoint situation produced by the forgetful functor $\mathbf{GR} \to \mathbf{SET}$—then $\mathbf{KL(T)}$ is isomorphic to the full subcategory of $\mathbf{GR}$ whose objects are the free groups.

A triple $\mathbf{T} = (T, m, \epsilon)$ in $\mathbf{C}$ is said to be idempotent provided that m is a natural isomorphism (hence $\epsilon T = m^{-1} = T\epsilon$). If $\mathbf{T}$ is idempotent, then the comparison functor $\mathbf{KL(T)} \to \mathbf{T}\text{-}\mathbf{ALG}$ is an equivalence of categories. Moreover, $G_{\mathbf{T}} : \mathbf{T}\text{-}\mathbf{ALG} \to \mathbf{C}$ is full, faithful, and injective on objects. Its image is a reflective subcategory of $\mathbf{C}$, the objects being those X such that $\epsilon_X : X \rightarrow TX$ is an isomorphism. On the other hand, every reflective subcategory of $\mathbf{C}$ generates an idempotent triple. Agreeing that two idempotent triples T and T' are equivalent if there exists a natural isomorphism $\tau : T \rightarrow T'$ such that $\epsilon' = \tau \circ \epsilon$ (thus also $\tau \circ m = m' \circ \tau T' \circ T\tau$), the conclusion is that the conglomerate of reflective subcategories of $\mathbf{C}$ is in a one-to-one correspondence with the conglomerate of idempotent triples in $\mathbf{C}$ modulo equivalence.

[Note: An idempotent triple $\mathbf{T} = (T, m, \epsilon)$ determines an orthogonal pair (S, D). Let $f : X \rightarrow Y$ be a morphism—then f is said to be T-localizing if there is an isomorphism $\phi : TX \rightarrow Y$ such that $f = \phi \circ \epsilon_X$. For this to be the case, it is necessary and sufficient that $f \in S$ and $Y \in D$. If $\mathbf{C}'$ is a full subcategory of $\mathbf{C}$ and if $\mathbf{T}' = (T', m', \epsilon')$ is an idempotent triple in $\mathbf{C}'$, then $\mathbf{T}$ (or T) is said to extend $\mathbf{T}'$ (or T') provided that $S' \subset S$ and $D' \subset D$ (in general, $(S')^{\perp} \supset D \supset (D')^{\perp\perp}$, where orthogonality is meant in $\mathbf{C}$).]

Let (F, G, μ, ν) be an adjoint situation—then the following conditions are equivalent: (1) $(G \circ F, G\nu F, \mu)$ is an idempotent triple; (2) μG is a natural isomorphism; (3) $(F \circ G, F\mu G, \nu)$ is an idempotent cotriple; (4) νF is a natural isomorphism. And: (1),...,(4) imply that the full subcategory $\mathbf{C}_\mu$ of $\mathbf{C}$ whose objects are the X such that μ_X is an isomorphism is a reflective subcategory of $\mathbf{C}$ and the full subcategory $\mathbf{D}_\nu$ of $\mathbf{D}$ whose objects are the Y such that ν_Y is an isomorphism is a coreflective subcategory of $\mathbf{D}$.

[Note: $\mathbf{C}_\mu$ and $\mathbf{D}_\nu$ are equivalent categories.]

Given a category $\mathbf{C}$ and a class $S \subset \operatorname{Mor}\mathbf{C}$, a localization of $\mathbf{C}$ at S is a pair $(S^{-1}\mathbf{C}, L_S)$, where $S^{-1}\mathbf{C}$ is a metacategory and $L_S : \mathbf{C} \rightarrow S^{-1}\mathbf{C}$ is a functor such that $\forall\, s \in S$, $L_S s$ is an isomorphism, $(S^{-1}\mathbf{C}, L_S)$ being initial among all pairs having this property, i.e., for any metacategory $\mathbf{D}$ and for any functor $F : \mathbf{C} \rightarrow \mathbf{D}$ such that $\forall\, s \in S$, Fs is an isomorphism, there exists a unique functor $F' : S^{-1}\mathbf{C} \rightarrow \mathbf{D}$ such that $F = F' \circ L_S$. $S^{-1}\mathbf{C}$ exists, is unique up to isomorphism, and there is a representative that has the same objects as $\mathbf{C}$ itself. Example: Take $\mathbf{C} = \mathbf{TOP}$ and let $S \subset \operatorname{Mor}\mathbf{C}$ be the class of homotopy equivalences—then $S^{-1}\mathbf{C} = \mathbf{HTOP}$.

[Note: If $\overline{S}$ is the class of all morphisms rendered invertible by L_S (the saturation of S), then the arrow $S^{-1}\mathbf{C} \rightarrow \overline{S}^{-1}C$ is an isomorphism.]

Fix a class I which is not a set. Let $\mathbf{C}$ be the category whose objects are X, Y, and $\{Z_i : i \in I\}$ and whose morphisms, apart from identities, are $f_i : X \rightarrow Z_i$ and $g_i : Y \rightarrow Z_i$. Take $S = \{g_i : i \in I\}$—then $S^{-1}\mathbf{C}$ is a metacategory that is not isomorphic to a category.

[Note: The localization of a small category at a set of morphisms is again small.]

Let $\mathbf{C}$ be a category and let $S \subset \operatorname{Mor}\mathbf{C}$ be a class containing the identities of $\mathbf{C}$ and closed with respect to composition—then S is said to admit a calculus of left fractions if

(LF$_1$) Given a 2-source $X' \xleftarrow{s} X \xrightarrow{f} Y$ ($s \in S$), there exists a commutative square

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "s"'] & Y \arrow[d, "t"] \\ X' \arrow[r, "f'"'] & Y' \end{tikzcd}$$

, where $t \in S$;

(LF$_2$) Given $f, g : X \rightarrow Y$ and $s : X' \rightarrow X$ ($s \in S$) such that $f \circ s = g \circ s$, there exists $t : Y \rightarrow Y'$ ($t \in S$) such that $t \circ f = t \circ g$.

[Note: Reverse the arrows to define "calculus of right fractions".]

Let $S \subset \operatorname{Mor}\mathbf{C}$ be a class containing the identities of $\mathbf{C}$ and closed with respect to composition such that $\forall\ (s,t) : t \circ s \in S\ \&\ s \in S \Rightarrow t \in S$—then S admits a calculus of left fractions if every 2-source $X' \xleftarrow{s} X \xrightarrow{f} Y$ ($s \in S$) can be completed to a weak pushout square

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "s"'] & Y \arrow[d, "t"] \\ X' \arrow[r, "f'"'] & Y' \end{tikzcd}$$

, where $t \in S$. For an illustration, take $\mathbf{C} = \mathbf{HTOP}$ and consider the class of homotopy classes of homology equivalences.

Let $\mathbf{C}$ be a category and let $S \subset \operatorname{Mor}\mathbf{C}$ be a class admitting a calculus of left fractions. Given $X, Y \in \operatorname{Ob} S^{-1}\mathbf{C}$, $\operatorname{Mor}(X,Y)$ is the conglomerate of equivalence classes of pairs $(s,f) : X \xrightarrow{f} Y' \xleftarrow{s} Y$, two pairs $\left\{\begin{matrix}(s,f)\\(t,g)\end{matrix}\right.$ being equivalent iff there exist $u, v \in \operatorname{Mor}\mathbf{C}$: $\left\{\begin{matrix}u \circ s\\v \circ t\end{matrix}\right. \in S$, with $u \circ s = v \circ t$ and $u \circ f = v \circ g$. Every morphism in $S^{-1}\mathbf{C}$ can be represented in the form $(L_S s)^{-1} L_S f$ and if $L_S f = L_S g$, then there is an $s \in S$ such that $s \circ f = s \circ g$.

[Note: $S^{-1}\mathbf{C}$ is a metacategory. To guarantee that $S^{-1}\mathbf{C}$ is isomorphic to a category, it suffices to impose a solution set condition: For each $X \in \operatorname{Ob}\mathbf{C}$, there exists a source $\{s_i : X \rightarrow X_i'\}$ ($s_i \in S$) such that for every $s : X \rightarrow X'$ ($s \in S$), there is an i and a $u : X' \rightarrow X_i'$ such that $u \circ s = s_i$. This, of course, is automatic provided that $X \backslash S$, the full subcategory of $X \backslash \mathbf{C}$ whose objects are the $s : X \rightarrow X'$ ($s \in S$), has a final object.]

If $\mathbf{C}$ is the full subcategory of $\mathbf{HTOP}_*$ whose objects are the pointed connected CW complexes and if S is the class of pointed homotopy classes of pointed n-equivalences, then S admits a calculus of left fractions and satisfies the solution set condition.

Let (F, G, μ, ν) be an adjoint situation. Assume: G is full and faithful or, equivalently, that ν is a natural isomorphism. Take for $S \subset \operatorname{Mor}\mathbf{C}$ the class consisting of those s such that Fs is an isomorphism (so $F = F' \circ L_S$)—then $\{\mu_X\} \subset S$ and S admits a calculus of left fractions. Moreover, S is saturated and satisfies the solution set condition (in fact, $\forall\ X \in \operatorname{Ob}\mathbf{C}$, $X\backslash S$ has a final object, viz. $\mu_X : X \to GFX$). Therefore $S^{-1}\mathbf{C}$ is isomorphic to a category and $L_S : \mathbf{C} \to S^{-1}\mathbf{C}$ has a right adjoint that is full and faithful, while $F' : S^{-1}\mathbf{C} \to \mathbf{D}$ is an equivalence.

[Note: Suppose that $\mathbf{T} = (T, m, \epsilon)$ is an idempotent triple in $\mathbf{C}$. Let $\mathbf{D}$ be the corresponding reflective subcategory of $\mathbf{C}$ with reflector $R : \mathbf{C} \to \mathbf{D}$, so $T = \iota \circ R$, where $\iota : \mathbf{D} \to \mathbf{C}$ is the inclusion. Take for $S \subset \operatorname{Mor}\mathbf{C}$ the class consisting of those f such that Tf is an isomorphism—then S is the class consisting of those f such that Rf is an isomorphism, hence S admits a calculus of left fractions, is saturated, and satisfies the solution set condition. The Kleisli category of $\mathbf{T}$ is isomorphic to $S^{-1}\mathbf{C}$ and T factors as $\mathbf{C} \to S^{-1}\mathbf{C} \to \mathbf{D} \to \mathbf{C}$, the arrow $S^{-1}\mathbf{C} \to \mathbf{D}$ being an equivalence.]

Let (F, G, μ, ν) be an adjoint situation. Put $\begin{cases} S = \{\mu_X\} \subset \operatorname{Mor}\mathbf{C} \\ T = \{\nu_Y\} \subset \operatorname{Mor}\mathbf{D} \end{cases}$—then $\begin{cases} S^{-1}\mathbf{C} \\ T^{-1}\mathbf{D} \end{cases}$ are isomorphic to categories and $\begin{cases} F \\ G \end{cases}$ induce functors $\begin{cases} F' : S^{-1}\mathbf{C} \to T^{-1}\mathbf{D} \\ G' : T^{-1}\mathbf{D} \to S^{-1}\mathbf{C} \end{cases}$ such that $\begin{cases} G' \circ F' \approx \mathrm{id}_{S^{-1}\mathbf{C}} \\ F' \circ G' \approx \mathrm{id}_{T^{-1}\mathbf{D}} \end{cases}$, thus $\begin{cases} S^{-1}\mathbf{C} \\ T^{-1}\mathbf{D} \end{cases}$ are equivalent. In particular, when G is full and faithful, $S^{-1}\mathbf{C}$ is equivalent to $\mathbf{D}$ (the saturation of S being the class consisting of those s such that Fs is an isomorphism, i.e., $\overline{S}$ is the "S" considered above).

Given a category $\mathbf{C}$, a set $\mathcal{U}$ of objects in $\mathbf{C}$ is said to be a separating set if for every pair $X \underset{g}{\overset{f}{\rightrightarrows}} Y$ of distinct morphisms, there exists a $U \in \mathcal{U}$ and a morphism $\sigma : U \to X$ such that $f \circ \sigma \neq g \circ \sigma$. An object U in $\mathbf{C}$ is said to be a separator if $\{U\}$ is a separating set, i.e., if the functor $\operatorname{Mor}(U, —) : \mathbf{C} \to \mathbf{SET}$ is faithful. If $\mathbf{C}$ is balanced, finitely complete, and has a separating set, then $\mathbf{C}$ is wellpowered. Every cocomplete cowellpowered category with a separator is wellpowered and complete. If $\mathbf{C}$ has coproducts, then a $U \in \operatorname{Ob}\mathbf{C}$ is a separator iff each $X \in \operatorname{Ob}\mathbf{C}$ admits an epimorphism $\coprod U \to X$.

[Note: Suppose that $\mathbf{C}$ is small—then the representable functors are a separating set for $[\mathbf{C}, \mathbf{SET}]$.]

Every nonempty set is a separator for $\mathbf{SET}$. $\mathbf{SET} \times \mathbf{SET}$ has no separators but the set $\{(\emptyset, \{0\}), (\{0\}, \emptyset)\}$ is a separating set. Every nonempty discrete topological space is a separator for $\mathbf{TOP}$ (or $\mathbf{HAUS}$). $\mathbf{Z}$ is a separator for $\mathbf{GR}$ and $\mathbf{AB}$, while $\mathbf{Z}[t]$ is a separator for $\mathbf{RG}$. In A-$\mathbf{MOD}$, A (as a left A-module) is a separator and in $\mathbf{MOD}$-A, A (as a right A-module) is a separator.

Given a category $\mathbf{C}$, a set $\mathcal{U}$ of objects in $\mathbf{C}$ is said to be a coseparating set if for every pair $X \overset{f}{\underset{g}{\rightrightarrows}} Y$ of distinct morphisms, there exists a $U \in \mathcal{U}$ and a morphism $\sigma : Y \to U$ such that $\sigma \circ f \neq \sigma \circ g$. An object U in **C** is said to be a coseparator if $\{U\}$ is a coseparating set, i.e., if the cofunctor Mor$(—, U)$: **C** → **SET** is faithful. If **C** is balanced, finitely cocomplete, and has a coseparating set, then **C** is cowellpowered. Every complete wellpowered category with a coseparator is cowellpowered and cocomplete. If **C** has products, then a $U \in$ Ob **C** is a coseparator iff each $X \in$ Ob **C** admits a monomorphism $X \to \prod U$.

Every set with at least two elements is a coseparator for **SET**. Every indiscrete topological space with at least two elements is a coseparator for **TOP**. **Q/Z** is a coseparator for **AB**. None of the categories **GR**, **RG**, **HAUS** has a coseparating set.

SPECIAL ADJOINT FUNCTOR THEOREM Given a complete wellpowered category **D** which has a coseparating set, a functor $G : \mathbf{D} \to \mathbf{C}$ has a left adjoint iff G preserves limits.

A functor from **SET**, **AB** or **TOP** to a category **C** has a left adjoint iff it preserves limits and a right adjoint iff it preserves colimits.

Given a category **C**, an object P in **C** is said to be projective if the functor Mor$(P, —)$: **C** → **SET** preserves epimorphisms. In other words: P is projective iff for each epimorphism $f : X \to Y$ and each morphism $\phi : P \to Y$, there exists a morphism $g : P \to X$ such that $f \circ g = \phi$. A coproduct of projective objects is projective.

A category **C** is said to have enough projectives provided that for any $X \in$ Ob **C** there is an epimorphism $P \to X$, with P projective. If a category has enough projectives and a separator, then it has a projective separator. If a category has coproducts and a projective separator, then it has enough projectives.

The projective objects in the category of compact Hausdorff spaces are the extremally disconnected spaces. The projective objects in **AB** or **GR** are the free groups. The full subcategory of **AB** whose objects are the torsion groups has no projective objects other than the initial objects. In A-**MOD** or **MOD**-A, an object is projective iff it is a direct summand of a free module (and every free module is a projective separator).

Given a category **C**, an object Q in **C** is said to be injective if the cofunctor Mor$(—, Q)$: **C** → **SET** converts monomorphisms into epimorphisms. In other words: Q is injective iff for each monomorphism $f : X \to Y$ and each morphism $\phi : X \to Q$, there exists a morphism $g : Y \to Q$ such that $g \circ f = \phi$. A product of injective objects is injective.

A category $\mathbf{C}$ is said to have enough injectives provided that for any $X \in \operatorname{Ob}\mathbf{C}$, there is a monomorphism $X \to Q$, with Q injective. If a category has enough injectives and a coseparator, then it has an injective coseparator. If a category has products and an injective coseparator, then it has enough injectives.

The injective objects in the category of compact Hausdorff spaces are the retracts of products $\Pi[0,1]$. The injective objects in the category of Banach spaces and linear contractions are, up to isomorphism, the $C(X)$, where X is an extremally disconnected compact Hausdorff space. In $\mathbf{AB}$, the injective objects are the divisible abelian groups (and $\mathbf{Q}/\mathbf{Z}$ is an injective coseparator) but the only injective objects in $\mathbf{GR}$ or $\mathbf{RG}$ are the final objects. The module $\operatorname{Hom}_{\mathbf{Z}}(A, \mathbf{Q}/\mathbf{Z})$ is an injective coseparator in A-$\mathbf{MOD}$ or $\mathbf{MOD}$-A.

A zero object in a category $\mathbf{C}$ is an object which is both initial and final. The categories $\mathbf{TOP}_*$, $\mathbf{GR}$, and $\mathbf{AB}$ have zero objects. If $\mathbf{C}$ has a zero object $0_{\mathbf{C}}$ (or 0), then for any ordered pair $X, Y \in \operatorname{Ob}\mathbf{C}$ there exists a unique morphism $X \to 0_{\mathbf{C}} \to Y$, the zero morphism 0_{XY} (or 0) in $\operatorname{Mor}(X,Y)$. It does not depend on the choice of a zero object in $\mathbf{C}$. An equalizer (coequalizer) of an $f \in \operatorname{Mor}(X,Y)$ and 0_{XY} is said to be a kernel (cokernel) of f. Notation: $\ker f$ $(\operatorname{coker} f)$.

[Note: Suppose that $\mathbf{C}$ has a zero object. Let $\{X_i : i \in I\}$ be a collection of objects in $\mathbf{C}$ for which $\prod\limits_i X_i$ and $\coprod\limits_i X_i$ exist. The morphisms $\delta_{ij} : X_i \to X_j$ defined by $\begin{cases} \mathrm{id}_{X_i} (i = j) \\ 0_{X_i X_j} (i \neq j) \end{cases}$ then determine a morphism $t : \coprod\limits_i X_i \to \prod\limits_i X_i$ such that $\mathrm{pr}_j \circ t \circ \mathrm{in}_i = \delta_{ij}$. Example: Take $\#(I) = 2$—then this morphism can be a monomorphism (in $\mathbf{TOP}_*$), an epimorphism (in $\mathbf{GR}$), or an isomorphism (in $\mathbf{AB}$).]

A pointed category is a category with a zero object.

Let $\mathbf{C}$ be a category with a zero object. Assume that $\mathbf{C}$ has kernels and cokernels. Given a morphism $f : X \to Y$, an image (coimage) of f is a kernel of a cokernel (cokernel of a kernel) for f. Notation: $\operatorname{im} f$ $(\operatorname{coim} f)$. There is a commutative diagram

$$\begin{tikzcd}
\ker f \arrow[r] & X \arrow[r, "f"] \arrow[d] & Y \arrow[r] & \operatorname{coker} f \\
 & \operatorname{coim} f \arrow[r, "\bar{f}"'] & \operatorname{im} f, \arrow[u] &
\end{tikzcd}$$

where $\bar{f}$ is the morphism parallel to f. If parallel morphisms are isomorphisms, then $\mathbf{C}$ is said to be an exact category.

[Note: In general, $\bar{f}$ need be neither a monomorphism nor an epimorphism and $\bar{f}$ can be a bimorphism without being an isomorphism.]

A category **C** that has a zero object is exact iff every monomorphism is the kernel of a morphism, every epimorphism is the cokernel of a morphism, and every morphism admits a factorization: $f = g \circ h$ (g a monomorphism, h an epimorphism). Such a factorization is essentially unique. An exact category is balanced; it is wellpowered iff it is cowellpowered. Every exact category with a separator or a coseparator is wellpowered and cowellpowered. If an exact category has finite products (finite coproducts), then it has equalizers (coequalizers), hence is finitely complete (finitely cocomplete).

AB is an exact category but the full subcategory of AB whose objects are the torsion free abelian groups is not exact. Neither GR nor TOP$_*$ is exact.

Let **C** be an exact category.

(EX) A sequence $\cdots \rightarrow X_{n-1} \xrightarrow{d_{n-1}} X_n \xrightarrow{d_n} X_{n+1} \rightarrow \cdots$ is said to be exact provided that $\operatorname{im} d_{n-1} \approx \ker d_n$ for all n.

[Note: A short exact sequence is an exact sequence of the form $0 \rightarrow X' \rightarrow X \rightarrow X'' \rightarrow 0$.]

(Ker-Coker Lemma) Suppose that the diagram

$$\begin{tikzcd}
& X_1 \arrow[r] \arrow[d, "f_1"] & X_2 \arrow[r] \arrow[d, "f_2"] & X_3 \arrow[r] \arrow[d, "f_3"] & 0 \\
0 \arrow[r] & Y_1 \arrow[r] & Y_2 \arrow[r] & Y_3 &
\end{tikzcd}$$

is commutative and has exact rows—then there is a morphism $\delta : \ker f_3 \rightarrow \operatorname{coker} f_1$, the connecting morphism, such that the sequence

$$\ker f_1 \rightarrow \ker f_2 \rightarrow \ker f_3 \xrightarrow{\delta} \operatorname{coker} f_1 \rightarrow \operatorname{coker} f_2 \rightarrow \operatorname{coker} f_3$$

is exact. Moreover, if $X_1 \rightarrow X_2$ ($Y_2 \rightarrow Y_3$) is a monomorphism (epimorphism), then $\ker f_1 \rightarrow \ker f_2$ ($\operatorname{coker} f_2 \rightarrow \operatorname{coker} f_3$) is a monomorphism (epimorphism).

(Five Lemma) Suppose that the diagram

$$\begin{tikzcd}
X_1 \arrow[r] \arrow[d, "f_1"] & X_2 \arrow[r] \arrow[d, "f_2"] & X_3 \arrow[r] \arrow[d, "f_3"] & X_4 \arrow[r] \arrow[d, "f_4"] & X_5 \arrow[d, "f_5"] \\
Y_1 \arrow[r] & Y_2 \arrow[r] & Y_3 \arrow[r] & Y_4 \arrow[r] & Y_5
\end{tikzcd}$$

is commutative and has exact rows.

(1) If f_2 and f_4 are epimorphisms and f_5 is a monomorphism, then f_3 is an epimorphism.

(2) If f_2 and f_4 are monomorphisms and f_1 is an epimorphism, then f_3 is a monomorphism.

(Nine Lemma) Suppose that the diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & X' \arrow[r] \arrow[d] & X \arrow[r] \arrow[d] & X'' \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & Y' \arrow[r] \arrow[d] & Y \arrow[r] \arrow[d] & Y'' \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & Z' \arrow[r] \arrow[d] & Z \arrow[r] \arrow[d] & Z'' \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

is commutative, has exact columns, and an exact middle row—then the bottom row is exact iff the top row is exact.

In an exact category **C**, there are two short exact sequences associated with each morphism $f : X \to Y$, viz. $\begin{cases} 0 \to \ker f \to X \to \operatorname{coim} f \to 0 \\ 0 \to \operatorname{im} f \to Y \to \operatorname{coker} f \to 0 \end{cases}$.

An additive category is a category **C** that has a zero object and which is equipped with a function $+$ that assigns to each ordered pair $f, g \in \operatorname{Mor} \mathbf{C}$ having common domain and codomain, a morphism $f + g$ with the same domain and codomain satisfying the following conditions.

(ADD_1) On each morphism set $\operatorname{Mor}(X, Y)$, $+$ induces the structure of an abelian group.

(ADD_2) Composition is distributive over $+$: $\begin{cases} f \circ (g + h) = (f \circ g) + (f \circ h) \\ (g + h) \circ k = (g \circ k) + (h \circ k) \end{cases}$.

(ADD_3) The zero morphisms are identities with respect to $+$: $0 + f = f + 0 = f$.

An additive category has finite products iff it has finite coproducts and when this is so, finite coproducts are finite products.

[Note: If **C** is small and **D** is additive, then $[\mathbf{C}, \mathbf{D}]$ is additive.]

AB is an additive category but GR is not. Any ring with unit can be viewed as an additive category having exactly one object (and conversely). The category of Banach spaces and continuous linear transformations is additive but not exact.

An abelian category is an exact category **C** that has finite products and finite coproducts. Every abelian category is additive, finitely complete, and finitely cocomplete. A category **C** that has a zero object is abelian iff it has pullbacks, pushouts, and every monomorphism (epimorphism) is the kernel (cokernel) of a morphism. In an abelian category, $t : \coprod_{i=1}^{n} X_i \to \prod_{i=1}^{n} X_i$ is an isomorphism.

[Note: If **C** is small and **D** is abelian, then [**C**, **D**] is abelian.]

AB is an abelian category, as is its full subcategory whose objects are the finite abelian groups but there are full subcategories of AB which are exact and additive, yet not abelian.

A Grothendieck category is a cocomplete abelian category **C** in which filtered colimits commute with finite limits or, equivalently, in which filtered colimits of exact sequences are exact. Every Grothendieck category with a separator is complete and has an injective coseparator, hence has enough injectives (however there exist wellpowered Grothendieck categories that do not have enough injectives). In a Grothendieck category, every filtered colimit of monomorphisms is a monomorphism, coproducts of monomorphisms are monomorphisms, and $t : \coprod_i X_i \to \prod_i X_i$ is a monomorphism.

[Note: If **C** is small and **D** is Grothendieck, then [**C**, **D**] is Grothendieck.]

AB is a Grothendieck category but its full subcategory whose objects are the finitely generated abelian groups, while abelian, is not Grothendieck. If *A* is a ring with unit, then *A*-MOD and MOD-*A* are Grothendieck categories.

Given exact categories $\begin{cases} \mathbf{C} \\ \mathbf{D} \end{cases}$, a functor $F : \mathbf{C} \to \mathbf{D}$ is said to be left exact (right exact) if it preserves kernels (cokernels) and exact if it is both right and left exact. F is left exact (right exact) iff for every short exact sequence $0 \to X' \to X \to X'' \to 0$ in **C**, the sequence $0 \to FX' \to FX \to FX''$ ($FX' \to FX \to FX'' \to 0$) is exact in **D**. Therefore F is exact iff F preserves short exact sequences or still, iff F preserves arbitrary exact sequences.

[Note: F is said to be half exact if for every short exact sequence $0 \to X' \to X \to X'' \to 0$ in **C**, the sequence $FX' \to FX \to FX''$ is exact in **D**.]

The projective (injective) objects in an abelian category are those for which Mor $(X, —)$(Mor $(—, X)$) is exact. In AB, $X \otimes —$ is exact iff X is flat or here, torsion free. If I is small and filtered and if C is Grothendieck, then colim : [I, C] → C is exact.

Given additive categories $\begin{cases}\mathbf{C}\\ \mathbf{D}\end{cases}$, a functor $F:\mathbf{C}\to\mathbf{D}$ is said to be additive if for all $X,Y\in\operatorname{Ob}\mathbf{C}$, the map $\operatorname{Mor}(X,Y)\to\operatorname{Mor}(FX,FY)$ is a homomorphism of abelian groups. Every half exact functor between abelian categories is additive. An additive functor between abelian categories is left exact (right exact) iff it preserves finite limits (finite colimits). The additive functor category $[\mathbf{C},\mathbf{D}]^+$ is the full submetacategory of $[\mathbf{C},\mathbf{D}]$ whose objects are the additive functors. There are Yoneda embeddings $\begin{cases}\mathbf{C}^{\mathrm{OP}}\to[\mathbf{C},\mathbf{AB}]^+\\ \mathbf{C}\to[\mathbf{C}^{\mathrm{OP}},\mathbf{AB}]^+\end{cases}$. If $\mathbf{C}$ and $\mathbf{D}$ are abelian categories with $\mathbf{C}$ small, if $K:\mathbf{C}\to\mathbf{D}$ is additive, and if $\mathbf{S}$ is a complete (cocomplete) abelian category, then there is an additive version of Kan extension applicable to $\begin{cases}[\mathbf{C},\mathbf{S}]^+\\ [\mathbf{D},\mathbf{S}]^+\end{cases}$. The functors produced need not agree with those obtained by forgetting the additive structure.

Let A be a ring with unit viewed as an additive category having exactly one object—then A-**MOD** is isomorphic to $[A,\mathbf{AB}]^+$ and **MOD**-A is isomorphic to $[A^{\mathrm{OP}},\mathbf{AB}]^+$.

[Note: A right A-module X and a left A-module Y define a diagram $A^{\mathrm{OP}}\times A\to\mathbf{AB}$ (tensor product over $\mathbf{Z}$) and the coend $\int^A X\otimes Y$ is $X\otimes_A Y$, the tensor product over A.]

If $\mathbf{C}$ is small and additive and if $\mathbf{D}$ is additive, then

(1) $\mathbf{D}$ finitely complete and wellpowered (finitely cocomplete and cowellpowered) $\Rightarrow[\mathbf{C},\mathbf{D}]^+$ wellpowered (cowellpowered);

(2) $\mathbf{D}$ (finitely) complete $\Rightarrow[\mathbf{C},\mathbf{D}]^+$ (finitely) complete and $\mathbf{D}$ (finitely) cocomplete $\Rightarrow[\mathbf{C},\mathbf{D}]^+$ (finitely) cocomplete;

(3) $\mathbf{D}$ abelian (Grothendieck) $\Rightarrow[\mathbf{C},\mathbf{D}]^+$ abelian (Grothendieck).

[Note: Suppose that $\mathbf{C}$ is small. If $\mathbf{C}$ is additive, then $[\mathbf{C},\mathbf{AB}]^+$ is a complete Grothendieck category and if $\mathbf{C}$ is exact and additive, then $[\mathbf{C},\mathbf{AB}]^+$ has a separator which as a functor $\mathbf{C}\to\mathbf{AB}$ is left exact.]

Given a small abelian category $\mathbf{C}$ and an abelian category $\mathbf{D}$, write $\mathbf{LEX}(\mathbf{C},\mathbf{D})$ for the full, isomorphism closed subcategory of $[\mathbf{C},\mathbf{D}]^+$ whose objects are the left exact functors.

DERIVED FUNCTOR THEOREM If $\mathbf{C}$ is a small abelian category and if $\mathbf{D}$ is a wellpowered Grothendieck category, then $\mathbf{LEX}(\mathbf{C},\mathbf{D})$ is a reflective subcategory of $[\mathbf{C},\mathbf{D}]^+$. As such, it is Grothendieck. Moreover, the reflector is an exact functor.

[Note: The reflector sends F to its zeroth right derived functor R^0F.]

If $\mathbf{C}$ is a small abelian category, then $\mathbf{LEX}(\mathbf{C},\mathbf{AB})$ is a Grothendieck category with a separator. Therefore $\mathbf{LEX}(\mathbf{C},\mathbf{AB})$ has enough injectives. Every injective object in

$\mathbf{LEX(C,AB)}$ is an exact functor. The Yoneda embedding $\mathbf{C}^{\mathrm{OP}} \rightarrow [\mathbf{C},\mathbf{AB}]^+$ is left exact. It factors through $\mathbf{LEX(C,AB)}$ and is then exact.

[Note: Since $\mathbf{C}$ is abelian, every object in $[\mathbf{C},\mathbf{AB}]^+$ is a colimit of representable functors and every object in $\mathbf{LEX(C,AB)}$ is a filtered colimit of representable functors. Thus $\mathbf{LEX(C,AB)}$ is equivalent to $\mathbf{IND}(\mathbf{C}^{\mathrm{OP}})$ and so $\mathbf{LEX(C,AB)}^{\mathrm{OP}}$ is equivalent to $\mathbf{PRO(C)}$.]

The full subcategory of AB whose objects are the finite abelian groups is equivalent to a small category. Its procategory is equivalent to the opposite of the full subcategory of AB whose objects are the torsion abelian groups.

Given an abelian category $\mathbf{C}$, a nonempty class $\mathcal{C} \subset \operatorname{Ob}\mathbf{C}$ is said to be a Serre class provided that for any short exact sequence $0 \rightarrow X' \rightarrow X \rightarrow X'' \rightarrow 0$ in $\mathbf{C}$, $X \in \mathcal{C}$ iff $\left\{\begin{matrix} X' \\ X'' \end{matrix}\right. \in \mathcal{C}$ or, equivalently, for any exact sequence $X' \rightarrow X \rightarrow X''$ in $\mathbf{C}$, $\left\{\begin{matrix} X' \\ X'' \end{matrix}\right. \in \mathcal{C} \Rightarrow X \in \mathcal{C}$.

[Note: Since $\mathcal{C}$ is nonempty, $\mathcal{C}$ contains the zero objects of $\mathbf{C}$.]

Given an abelian category $\mathbf{C}$ with a separator and a Serre class $\mathcal{C}$, let $S_{\mathcal{C}} \subset \operatorname{Mor}\mathbf{C}$ be the class consisting of those s such that $\ker s \in \mathcal{C}$ and $\operatorname{coker} s \in \mathcal{C}$—then $S_{\mathcal{C}}$ admits a calculus of left and right fractions and $S_{\mathcal{C}} = \overline{S}_{\mathcal{C}}$, i.e., $S_{\mathcal{C}}$ is saturated. The metacategory $S_{\mathcal{C}}^{-1}\mathbf{C}$ is isomorphic to a category. As such, it is abelian and $L_{S_{\mathcal{C}}} : \mathbf{C} \rightarrow S_{\mathcal{C}}^{-1}\mathbf{C}$ is exact and additive. An object X in $\mathbf{C}$ belongs to $\mathcal{C}$ iff $L_{S_{\mathcal{C}}}X$ is a zero object. Moreover, if $\mathbf{D}$ is an abelian category and $F : \mathbf{C} \rightarrow \mathbf{D}$ is an exact functor, then F can be factored through $L_{S_{\mathcal{C}}}$ iff all the objects of $\mathcal{C}$ are sent to zero objects by F.

[Note: Suppose that $\mathbf{C}$ is a Grothendieck category with a separator U—then for any Serre class $\mathcal{C}$, $L_{S_{\mathcal{C}}} : \mathbf{C} \rightarrow S_{\mathcal{C}}^{-1}\mathbf{C}$ has a right adjoint iff $\mathcal{C}$ is closed under coproducts, in which case $S_{\mathcal{C}}^{-1}\mathbf{C}$ is again Grothendieck and has $L_{S_{\mathcal{C}}}U$ as a separator.]

Take $\mathbf{C} = \mathbf{AB}$ and let $\mathcal{C}$ be the class of torsion abelian groups—then $\mathcal{C}$ is a Serre class and $S_{\mathcal{C}}^{-1}\mathbf{C}$ is equivalent to the category of torsion free divisible abelian groups or still, to the category of vector spaces over $\mathbf{Q}$.

Given a Grothendieck category $\mathbf{C}$ with a separator, a reflective subcategory $\mathbf{D}$ of $\mathbf{C}$ is said to be a Giraud subcategory provided that the reflector $R : \mathbf{C} \rightarrow \mathbf{D}$ is exact. Every Giraud subcategory of $\mathbf{C}$ is Grothendieck and has a separator. There is a one-to-one correspondence between the Serre classes in $\mathbf{C}$ which are closed under coproducts and the Giraud subcategories of $\mathbf{C}$.

[Note: The Gabriel-Popescu theorem says that every Grothendieck category with a separator is equivalent to a Giraud subcategory of A-**MOD** for some A.]

Attached to a topological space X is the category $\mathbf{OP}(X)$ whose objects are the open subsets of X and whose morphisms are the inclusions. The functor category $[\mathbf{OP}(X)^{\mathrm{OP}}, \mathbf{AB}]$ is the category of abelian presheaves on X. It is Grothendieck and has a separator. The full subcategory of $[\mathbf{OP}(X)^{\mathrm{OP}}, \mathbf{AB}]$ whose objects are the abelian sheaves on X is a Giraud subcategory.

Fix a symmetric monoidal category **V**—then a **V**-category **M** consists of a class O (the objects) and a function that assigns to each ordered pair $X, Y \in O$ an object $\mathrm{HOM}(X,Y)$ in **V** plus morphisms $C_{X,Y,Z} : \mathrm{HOM}(X,Y) \otimes \mathrm{HOM}(Y,Z) \to \mathrm{HOM}(X,Z)$, $I_X : e \to \mathrm{HOM}(X,X)$ satisfying the following conditions.

(**V**-cat$_1$) The diagram

$$\begin{tikzcd}
\mathrm{HOM}(X,Y) \otimes (\mathrm{HOM}(Y,Z) \otimes \mathrm{HOM}(Z,W)) \arrow[r, "\mathrm{id}\otimes C"] \arrow[d, "A"'] & \mathrm{HOM}(X,Y) \otimes \mathrm{HOM}(Y,W) \arrow[dd, "C"] \\
(\mathrm{HOM}(X,Y) \otimes \mathrm{HOM}(Y,Z)) \otimes \mathrm{HOM}(Z,W) \arrow[d, "C\otimes\mathrm{id}"'] & \\
\mathrm{HOM}(X,Z) \otimes \mathrm{HOM}(Z,W) \arrow[r, "C"'] & \mathrm{HOM}(X,W)
\end{tikzcd}$$

commutes.

(**V**-cat$_2$) The diagram

$$\begin{tikzcd}
e \otimes \mathrm{HOM}(X,Y) \arrow[r, "L"] \arrow[d, "I\otimes\mathrm{id}"'] & \mathrm{HOM}(X,Y) \arrow[d, equal] & \mathrm{HOM}(X,Y) \otimes e \arrow[l, "R"'] \arrow[d, "\mathrm{id}\otimes I"] \\
\mathrm{HOM}(X,X) \otimes \mathrm{HOM}(X,Y) \arrow[r, "C"'] & \mathrm{HOM}(X,Y) & \mathrm{HOM}(X,Y) \otimes \mathrm{HOM}(Y,Y) \arrow[l, "C"]
\end{tikzcd}$$

commutes.

[Note: The opposite of a **V**-category is a **V**-category and the product of two **V**-categories is a **V**-category.]

The underlying category **UM** of a **V**-category **M** has for its class of objects the class O, $\mathrm{Mor}(X,Y)$ being the set $\mathrm{Mor}(e, \mathrm{HOM}(X,Y))$. Composition $\mathrm{Mor}(X,Y) \times \mathrm{Mor}(Y,Z) \to \mathrm{Mor}(X,Z)$ is calculated from $e \approx e \otimes e \xrightarrow{f\otimes g} \mathrm{HOM}(X,Y) \otimes \mathrm{HOM}(Y,Z) \to \mathrm{HOM}(X,Z)$, while I_X serves as the identity in $\mathrm{Mor}(X,X)$.

[Note: A closed category **V** can be regarded as a **V**-category (take $\mathrm{HOM}(X,Y) = \mathrm{hom}(X,Y)$) and **UV** is isomorphic to **V**.]

Every category is a **SET**-category and every additive category is an **AB**-category.

A morphism $F : \mathbf{V} \to \mathbf{W}$ of symmetric monoidal categories is a functor $F : \mathbf{V} \to \mathbf{W}$, a morphism $\epsilon : e \to Fe$, and morphisms $T_{X,Y} : FX \otimes FY \to F(X \otimes Y)$ natural in X, Y such that the diagrams

$$\begin{tikzcd}
Fe \otimes FX \arrow[r, "T"] & F(e \otimes X) \arrow[d, "FL"] & FX \otimes Fe \arrow[r, "T"] & F(X \otimes e) \arrow[d, "FR"] \\
e \otimes FX \arrow[u, "\epsilon \otimes \mathrm{id}"] \arrow[r, "L"'] & FX & FX \otimes e \arrow[u, "\mathrm{id} \otimes \epsilon"] \arrow[r, "R"'] & FX
\end{tikzcd}$$

$$\begin{tikzcd}
FX \otimes (FY \otimes FZ) \arrow[r, "A"] \arrow[d, "\mathrm{id} \otimes T"'] & (FX \otimes FY) \otimes FZ \arrow[d, "T \otimes \mathrm{id}"] \\
FX \otimes F(Y \otimes Z) \arrow[d, "T"'] & F(X \otimes Y) \otimes FZ \arrow[d, "T"] \\
F(X \otimes (Y \otimes Z)) \arrow[r, "FA"'] & F((X \otimes Y) \otimes Z)
\end{tikzcd}$$

commute with $F\top_{X,Y} \circ T_{X,Y} = T_{Y,X} \circ \top_{FX,FY}$.

Example: Given a symmetric monoidal category $\mathbf{V}$, the representable functor $\mathrm{Mor}\,(e, —)$ determines a morphism $\mathbf{V} \to \mathbf{SET}$ of symmetric monoidal categories.

Let $F : \mathbf{V} \to \mathbf{W}$ be a morphism of symmetric monoidal categories. Suppose that $\mathbf{M}$ is a $\mathbf{V}$-category. Definition: $F_*\mathbf{M}$ is the $\mathbf{W}$-category whose object class is O, the rest of the data being $F\mathrm{HOM}(X,Y)$, $F\mathrm{HOM}(X,Y) \otimes F\mathrm{HOM}(Y,Z) \xrightarrow{T} F(\mathrm{HOM}(X,Y) \otimes \mathrm{HOM}(Y,Z)) \xrightarrow{FC} F\mathrm{HOM}(X,Z)$, $e \xrightarrow{\epsilon} Fe \xrightarrow{FI} F\mathrm{HOM}(X,X)$.

[Note: Take $\mathbf{W} = \mathbf{SET}$ and $F = \mathrm{Mor}\,(e, —)$ to recover $\mathbf{UM}$.]

Fix a symmetric monoidal category $\mathbf{V}$. Suppose given $\mathbf{V}$-categories $\mathbf{M}, \mathbf{N}$—then a $\underline{\mathbf{V}\text{-functor}}$ $F : \mathbf{M} \to \mathbf{N}$ is the specification of a rule that assigns to each object X in $\mathbf{M}$ an object FX in $\mathbf{N}$ and the specification of a rule that assigns to each ordered pair $X, Y \in O$ a morphism $F_{X,Y} : \mathrm{HOM}(X,Y) \to \mathrm{HOM}(FX,FY)$ in $\mathbf{V}$ such that the diagram

$$\begin{tikzcd}
\mathrm{HOM}(X,Y) \otimes \mathrm{HOM}(Y,Z) \arrow[r, "C"] \arrow[d, "F_{X,Y} \otimes F_{Y,Z}"'] & \mathrm{HOM}(X,Z) \arrow[d, "F_{X,Z}"] \\
\mathrm{HOM}(FX,FY) \otimes \mathrm{HOM}(FY,FZ) \arrow[r, "C"'] & \mathrm{HOM}(FX,FZ)
\end{tikzcd}$$

commutes with $F_{X,X} \circ I_X = I_{FX}$.

[Note: The underlined underlying functor $UF : \mathbf{UM} \to \mathbf{UN}$ sends X to FX and $f : e \to \mathrm{HOM}(X,Y)$ to $F_{X,Y} \circ f$.]

Example: $\mathrm{HOM} : \mathbf{M}^{\mathrm{OP}} \times \mathbf{M} \to \mathbf{V}$ is a $\mathbf{V}$-functor if $\mathbf{V}$ is closed.

A **V**-category is small if its class of objects is a set; otherwise it is large. **V-CAT**, the category of small **V**-categories and **V**-functors, is a symmetric monoidal category.

Take **V** = **AB**—then an additive functor between additive categories "is" a **V**-functor.

Fix a symmetric monoidal category **V**. Suppose given **V**-categories **M**, **N** and **V**-functors $F, G : \mathbf{M} \rightarrow \mathbf{N}$—then a V-natural transformation Ξ from F to G is a class of morphisms $\Xi_X : e \rightarrow \mathrm{HOM}(FX, GX)$ for which the diagram

$$\begin{tikzcd}
e \otimes \mathrm{HOM}(X,Y) \arrow[r, "\Xi_X \otimes G_{X,Y}"] & \mathrm{HOM}(FX,GX) \otimes \mathrm{HOM}(GX,GY) \arrow[d, "C"] \\
\mathrm{HOM}(X,Y) \arrow[u, "L^{-1}"] \arrow[d, "R^{-1}"'] & \mathrm{HOM}(FX,GY) \\
\mathrm{HOM}(X,Y) \otimes e \arrow[r, "F_{X,Y} \otimes \Xi_Y"'] & \mathrm{HOM}(FX,FY) \otimes \mathrm{HOM}(FY,GY) \arrow[u, "C"']
\end{tikzcd}$$

commutes.

Assume that **V** is complete and closed. Let **M**, **N** be **V**-categories with **M** small—then the category **V**[**C**, **D**] whose objects are the **V**-functors $\mathbf{M} \rightarrow \mathbf{N}$ and whose morphisms are the **V**-natural transformations is a **V**-category if $\mathrm{HOM}(F,G) = \int_X \mathrm{HOM}(FX,GX)$, the equalizer of $\prod_{X \in O} \mathrm{HOM}(FX,GX) \rightrightarrows \prod_{X',X'' \in O} \mathrm{hom}(\mathrm{HOM}(X',X''), \mathrm{HOM}(FX',GX''))$.

Let **C** be a category with pullbacks—then an internal category (or a category object) in **C** consists of an object M, an object O, and morphisms $s : M \rightarrow O$, $t : M \rightarrow O$, $e : O \rightarrow M$, $c : M \times_O M \rightarrow M$ satisfying the usual category theoretic relations (here,

$$\begin{tikzcd}
M \times_O M \arrow[r] \arrow[d] & M \arrow[d, "t"] \\
M \arrow[r, "s"'] & O
\end{tikzcd}$$

). Notation: $\mathbf{M} = (M, O, s, t, e, c)$.

[Note: There are obvious notions of internal functor and internal natural transformation.]

An internal category in **SET** is a small category. An internal category in **SISET** is a simplicial object in **CAT**.

An internal category in **CAT** is a (small) double category.

[Note: Spelled out, such an entity consists of objects $X, Y, \ldots$, horizontal morphisms $f, g, \ldots$, vertical morphisms $\phi, \psi, \ldots$, and bimorphisms (represented diagramatically by squares). The objects and the horizontal morphisms form a category with identities $X \xrightarrow{h_X} X$. The objects and the vertical morphisms form a category with identities $$\begin{tikzcd} X \arrow[d, "v_X"'] \\ X \end{tikzcd}$$. The bimorphisms have horizontal and vertical laws of composition $$\begin{tikzcd} \bullet \arrow[r] \arrow[d] & \bullet \arrow[r] \arrow[d] & \bullet \arrow[d] \\ \bullet \arrow[r] & \bullet \arrow[r] & \bullet \end{tikzcd}, \quad \begin{tikzcd} \bullet \arrow[r] \arrow[d] & \bullet \arrow[d] \\ \bullet \arrow[r] \arrow[d] & \bullet \arrow[d] \\ \bullet \arrow[r] & \bullet \end{tikzcd}$$ under which they form a category with identities $$\begin{tikzcd} X \arrow[r, "h_X"] \arrow[d, "\phi"'] \arrow[dr, phantom, "\mathrm{id}_\phi"] & X \arrow[d, "\phi"] \\ Y \arrow[r, "h_Y"'] & Y \end{tikzcd}, \quad \begin{tikzcd} X \arrow[r, "f"] \arrow[d, "v_X"'] \arrow[dr, phantom, "\mathrm{id}_f"] & Y \arrow[d, "v_Y"] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$. In the situation $$\begin{tikzcd} \bullet \arrow[r] \arrow[d] & \bullet \arrow[r] \arrow[d] & \bullet \arrow[d] \\ \bullet \arrow[r] \arrow[d] & \bullet \arrow[r] \arrow[d] & \bullet \arrow[d] \\ \bullet \arrow[r] & \bullet \arrow[r] & \bullet \end{tikzcd}$$, the result of composing horizontally and then vertically is the same as the result of composing vertically and then horizontally. Furthermore, horizontal composition of vertical identities gives a vertical identity and vertical composition of horizontal identities gives a horizontal identity. Finally, the horizontal and vertical identities $$\begin{tikzcd} X \arrow[r, "h_X"] \arrow[d, "v_X"'] \arrow[dr, phantom, "\mathrm{id}_{v_X}"] & X \arrow[d, "v_X"] \\ X \arrow[r, "h_X"'] & X \end{tikzcd}, \quad \begin{tikzcd} X \arrow[r, "h_X"] \arrow[d, "v_X"'] \arrow[dr, phantom, "\mathrm{id}_{h_X}"] & X \arrow[d, "v_X"] \\ X \arrow[r, "h_X"'] & X \end{tikzcd}$$ coincide.]

Example: Let **C** be a small category—then db **C** is the double category whose objects are those of **C**, whose horizontal and vertical morphisms are those of **C**, and whose bimorphisms are the commutative squares in **C**. All sources, targets, identities, and compositions come from **C**.

Let **C** be a category with pullbacks. Given an object O in **C**, an O-graph is an object A and a pair of morphisms $s, t : A \to O$. O-**GR** is the category whose objects are the O-graphs and whose morphisms $(A, s, t) \to (A', s', t')$ are the arrows $f : A \to A'$ such that $s = s' \circ f$, $t = t' \circ f$. If $A \times_O A'$ is defined by the pullback square $$\begin{tikzcd} A \times_O A' \arrow[r] \arrow[d] & A' \arrow[d, "t'"] \\ A \arrow[r, "s"'] & O \end{tikzcd}$$ and if the structural morphisms are $A \times_O A' \to A' \xrightarrow{s'} O$, $A \times_O A' \to A \xrightarrow{t} O$, then $A \times_O A'$ is an O-graph. Therefore O-**GR** is a monoidal category: Take $A \otimes A'$ to be $A \times_O A'$ and let e be $(O, \mathrm{id}_O, \mathrm{id}_O)$. A monoid **M** in O-**GR** is an internal category in **C** with object element O.

Let **C** be a category with pullbacks. Given an internal category **M** in **C**, the nerve

$\operatorname{ner} \mathbf{M}$ of $\mathbf{M}$ is the simplicial object in $\mathbf{C}$ defined by $\operatorname{ner}_0 \mathbf{M} = O$, $\operatorname{ner}_1 \mathbf{M} = M$, $\operatorname{ner}_n \mathbf{M} = M \times_O \cdots \times_O M$ (n factors). At the bottom, $\begin{cases} d_0 \\ d_1 \end{cases} : \operatorname{ner}_1 \mathbf{M} \to \operatorname{ner}_0 \mathbf{M}$ is $\begin{cases} t \\ s \end{cases}$, while higher up, in terms of the underlying projections, $d_0 = (\pi_1, \ldots, \pi_{n-1})$, $d_n = (\pi_2, \ldots, \pi_n)$, $d_i = (\pi_1, \ldots, c \circ \pi_{n-i,n-i+1}, \ldots, \pi_n)$ $(0 < i < n)$, and at the bottom, $s_0 : \operatorname{ner}_0 \mathbf{M} \to \operatorname{ner}_1 \mathbf{M}$ is e, while higher up, $s_i = e_i \circ \sigma_i$, where σ_i inserts O at the $n - i + 1$ spot and e_i is $\mathrm{id} \times_O \cdots \times_O e \times_O \cdots \times_O \mathrm{id}$ placed accordingly $(0 \leq i \leq n)$.

[Note: An internal functor $\mathbf{M} \to \mathbf{M}'$ induces a morphism $\operatorname{ner} \mathbf{M} \to \operatorname{ner} \mathbf{M}'$ of simplicial objects.]

Suppose that $\mathbf{C}$ is a small category. Consider $\operatorname{ner} \mathbf{C}$—then an element f of $\operatorname{ner}_n \mathbf{C}$ is a diagram of the form $X_0 \xrightarrow{f_0} X_1 \to \cdots \to X_{n-1} \xrightarrow{f_{n-1}} X_n$ and

$$d_i f = \begin{cases} X_1 \to \cdots \to X_n & (i = 0) \\ X_0 \to \cdots \to X_{i-1} \xrightarrow{f_i \circ f_{i-1}} X_{i+1} \to \cdots \to X_n & (0 < i < n), \\ X_0 \to \cdots \to X_{n-1} & (i = n) \end{cases}$$

$s_i f = X_0 \to \cdots \to X_i \xrightarrow{\mathrm{id}_{X_i}} X_i \to \cdots \to X_n$. The abstract definition thus reduces to these formulas since f corresponds to the n-tuple $(f_{n-1}, \ldots, f_0)$.

Let $\mathbf{C}$ be a category with pullbacks. Given an internal category $\mathbf{M}$ in $\mathbf{C}$, a left $\mathbf{M}$-object is an object $T : Y \to O$ in $\mathbf{C}/O$ and a morphism $\lambda : M \times_O Y \to Y$ such that

$$\begin{tikzcd}
M \times_O M \times_O Y \arrow[r, "c \times_O \mathrm{id}"] \arrow[d, "\mathrm{id} \times_O \lambda"'] & M \times_O Y \arrow[d, "\lambda"] & O \times_O Y \arrow[l, "e \times_O \mathrm{id}"'] \arrow[d, "L"] \\
M \times_O Y \arrow[r, "\lambda"'] & Y \arrow[r, equal] & Y
\end{tikzcd}$$

and $\begin{tikzcd} M \times_O Y \arrow[r, "\lambda"] \arrow[d] & Y \arrow[d, "T"] \\ M \arrow[r, "t"'] & O \end{tikzcd}$ commute, where $M \times_O Y$ is defined by the pullback square $\begin{tikzcd} M \times_O Y \arrow[r] \arrow[d] & Y \arrow[d, "T"] \\ M \arrow[r, "s"'] & O \end{tikzcd}$. Example: Take $\mathbf{C} = \mathbf{SET}$—then $\mathbf{M}$ is a small category and the category of left $\mathbf{M}$-objects is equivalent to the functor category $[\mathbf{M}, \mathbf{SET}]$.

[Note: A right $\mathbf{M}$-object is an object $S : X \to O$ in $\mathbf{C}/O$ and a morphism $\rho : X \times_O M \to X$ such that the analogous diagrams commute, where $X \times_O M$ is defined by the pullback square

$$\begin{tikzcd} X \times_O M \arrow[r] \arrow[d] & M \arrow[d, "t"] \\ X \arrow[r, "s"'] & O \end{tikzcd}$$

. Example: Take $\mathbf{C} = \mathbf{SET}$—then $\mathbf{M}$ is a small category and the category of right $\mathbf{M}$-objects is equivalent to the functor category $[\mathbf{M}^{\text{OP}}, \mathbf{SET}]$.]

Let $\mathbf{C}$ be a category with pullbacks. Given an internal category $\mathbf{M}$ in $\mathbf{C}$ and a left $\mathbf{M}$-object Y, the translation category tran$Y$ of Y is the category object $\mathbf{M}_Y = (M_Y, O_Y, s_Y, t_Y, e_Y, c_Y)$ in $\mathbf{C}$, where $M_Y = M \times_O Y, O_Y = Y, s_Y$ is the projection $M \times_O Y \to Y$, t_Y is the action $\lambda : M \times_O Y \to Y$, and e_Y, c_Y are derived from $e : O \to M$, $c : M \times_O M \to M$. Example: Take $\mathbf{C} = \mathbf{SET}$, let $\mathbf{M}$ be a small category, and suppose that $G : \mathbf{M} \to \mathbf{SET}$ is a functor—then G determines a left $\mathbf{M}$-object Y_G and the translation category of Y_G can be identified with the Grothendieck construction on G.

Let G be a semigroup with unit, $\mathbf{G}$ the category having a single object $*$ with $\text{Mor}(*,*) = G$. Suppose that Y is a left G-set, i.e., an object in $\mathbf{LACT}_G$ or still, a left $\mathbf{G}$-object. The translation category of Y is $(G \times Y, Y, s_Y, t_Y, e_Y, c_Y)$, where $s_Y(g, y) = y$, $t_Y(g, y) = g \cdot y$, $e_Y(y) = (e, y)$, $c_Y((g_2, y_2), (g_1, y_1)) = (g_2 g_1, y_1)$. Specialize and let $Y = G$—then the objects of the translation category of G are the elements of G and $\text{Mor}(g_1, g_2) \approx \{g : g g_1 = g_2\}$.

Let $\mathbf{C}$ be a category with pullbacks. Given an internal category $\mathbf{M}$ in $\mathbf{C}$, and a right $\mathbf{M}$-object X and a left $\mathbf{M}$-object Y, the bar construction $\text{bar}(X; \mathbf{M}; Y)$ on (X, Y) is the simplicial object in $\mathbf{C}$ defined by $\text{bar}_n(X; \mathbf{M}; Y) = X \times_O \text{ner}_n \mathbf{M} \times_O Y$. Note that ρ appears only in d_n and λ appears only in d_0. The translation category $\text{tran}(X, Y)$ of (X, Y) is the category object $\mathbf{M}_{X,Y} = (M_{X,Y}, O_{X,Y}, s_{X,Y}, t_{X,Y}, e_{X,Y}, c_{X,Y})$ in $\mathbf{C}$, where $M_{X,Y} = X \times_O M \times_O Y, O_{X,Y} = X \times_O Y, s_{X,Y} = \rho \times_O \text{id}_Y, t_{X,Y} = \text{id}_X \times_O \lambda, e_{X,Y}$ & $c_{X,Y}$ being definable in terms of e & c. Therefore $\text{bar}(X; \mathbf{M}; Y) \approx \text{ner}\, \mathbf{M}_{X,Y}$. Example: O can be viewed as a right $\mathbf{M}$-object via $O \times_O M \xrightarrow{L} M \xrightarrow{s} O$ and as a left $\mathbf{M}$-object via $M \times_O O \xrightarrow{R} M \xrightarrow{t} O$, and M can be viewed as a right $\mathbf{M}$-object via $M \times_O M \xrightarrow{c} M \xrightarrow{s} O$ and as a left $\mathbf{M}$-object via $M \times_O M \xrightarrow{c} M \xrightarrow{t} O$, so $\text{bar}(O; \mathbf{M}; O)$, $\text{bar}(O; \mathbf{M}; M)$, $\text{bar}(M; \mathbf{M}; O)$, $\text{bar}(M; \mathbf{M}; M)$ are meaningful.

Let G be a group, $\mathbf{G}$ the groupoid having a single object $*$ with $\text{Mor}(*,*) = G$. View G as a left G-set—then $\text{bar}(*; \mathbf{G}; G)$ is isomorphic to the nerve of grd$G$. In fact, the objects of grd$G$ are the elements of G and the morphisms of grd$G$ are the elements of $G \times G$ ($s(g, h) = g$, $t(g, h) = h$, $\text{id}_g = (g, g)$, $(h, k) \circ (g, h) = (g, k)$), thus $\text{ner}_n\, \text{grd}G = G \times \cdots \times G$ ($n+1$ factors) and $d_i(g_0, \ldots, g_n) = (g_0, \ldots, \widehat{g_i}, \ldots, g_n)$, $s_i(g_0, \ldots, g_n) = (g_0, \ldots, g_i, g_i, \ldots, g_n)$. On the other hand, $\text{bar}(*; \mathbf{G}; G)$ is the nerve of the translation category of G. The functor tran$G \to$ grd$G$ which is the identity on objects and sends a morphism (g, h)

in tran$G$ to the morphism $(h, g \cdot h)$ in grd$G$ induces an isomorphism ner tran$G \to$ ner grd$G$ of simplicial sets. For $(g_0, \ldots, g_n) \to (g_n, g_{n-1}g_n, \ldots, g_0 \cdots g_n)$ is the arrow ner_n tran$G \to \text{ner}_n$ grd$G$, its inverse being $(g_0, \ldots, g_n) \to (g_n g_{n-1}^{-1}, g_{n-1} g_{n-2}^{-1}, \ldots, g_0)$. Both ner tran$G$ and ner grd$G$ are simplicial right G-sets, viz. $(g_0, \ldots, g_n) \cdot g = (g_0, \ldots, g_n g)$ and $(g_0, \ldots, g_n) \cdot g = (g_0 g, \ldots, g_n g)$, and the isomorphism ner tran$G \to$ ner grd$G$ is equivariant.

Let $\mathbf{T} = (T, m, \epsilon)$ be a triple in a category $\mathbf{C}$—then a right $\mathbf{T}$-functor in a category $\mathbf{V}$ is a functor $F : \mathbf{C} \to \mathbf{V}$ plus a natural transformation $\rho : F \circ T \to F$ such that the diagrams

$$\begin{tikzcd} F \circ T \circ T \arrow[r, "\rho T"] \arrow[d, "Fm"'] & F \circ T \arrow[d, "\rho"] \\ F \circ T \arrow[r, "\rho"'] & F \end{tikzcd} , \quad \begin{tikzcd} F \arrow[r, "F\epsilon"] \arrow[dr, equal] & F \circ T \arrow[d, "\rho"] \\ & F \end{tikzcd}$$

commute and a left $\mathbf{T}$-functor in a category $\mathbf{U}$ is a functor $G : \mathbf{U} \to \mathbf{C}$ plus a natural transformation $\lambda : T \circ G \to G$ such that the diagrams

$$\begin{tikzcd} T \circ T \circ G \arrow[r, "T\lambda"] \arrow[d, "mG"'] & T \circ G \arrow[d, "\lambda"] \\ T \circ G \arrow[r, "\lambda"'] & G \end{tikzcd} , \quad \begin{tikzcd} G \arrow[r, "\epsilon G"] \arrow[dr, equal] & T \circ G \arrow[d, "\lambda"] \\ & G \end{tikzcd}$$

commute. The bar construction bar$(F; \mathbf{T}; G)$ on (F, G) is the simplicial object in $[\mathbf{U}, \mathbf{V}]$ defined by $\text{bar}_n(F; \mathbf{T}; G) = F \circ T^n \circ G$, where $d_0 = \rho T^{n-1} G$, $d_i = F T^{i-1} m T^{n-i-1} G$ $(0 < i < n)$, $d_n = F T^{n-1} \lambda$, and $s_i = F T^i \epsilon T^{n-i} G$. In particular: $\text{bar}_1(F; \mathbf{T}; G) = F \circ T \circ G$, $\text{bar}_0(F; \mathbf{T}; G) = F \circ G$, and $d_0, d_1 : F \circ T \circ G \to F \circ G$ are $\rho G, F\lambda$, while $s_0 : F \circ G \to F \circ T \circ G$ is $F \epsilon G$.

Example: If X is a $\mathbf{T}$-algebra in $\mathbf{C}$ with structural morphism $\xi : TX \to X$, then X determines a left $\mathbf{T}$-functor $G : \mathbf{1} \to \mathbf{C}$ and one writes bar$(F; \mathbf{T}; X)$ for the associated bar construction.

Take $\mathbf{V} = \mathbf{C}$, $F = T$, $\rho = m$, and put $\tau = \epsilon T G$ (thus $\tau : T \circ G \to T \circ T \circ G$). There is a commutative diagram

$$\begin{tikzcd} T \circ G \arrow[rrr, "\lambda"] \arrow[dd, equal] \arrow[dr, "\tau"] & & & G \arrow[dd, equal] \arrow[dl, "\epsilon G"'] \\ & T \circ T \circ G \arrow[r, "T\lambda"] \arrow[dl, "mG"] & T \circ G \arrow[dr, "\lambda"'] & \\ T \circ G \arrow[rrr, "\lambda"'] & & & G \end{tikzcd}$$

from which it follows that $\lambda : T \circ G \to G$ is a coequalizer of $(d_0, d_1) = (mG, T\lambda)$. Consider the string of arrows $T \circ T^n \circ G \xrightarrow{d_0} T \circ T^{n-1} \circ G \to \cdots \to T \circ T \circ G \xrightarrow{d_0} T \circ G \xrightarrow{\lambda} G \xrightarrow{\epsilon G} T \circ G \xrightarrow{s_0} T \circ T \circ G \to \cdots \to T \circ T^{n-1} \circ G \xrightarrow{s_0} T \circ T^n \circ G$. Viewing G as a constant simplicial object in $[\mathbf{\Delta}^{\text{OP}}, [\mathbf{C}, \mathbf{V}]]$, there are simplicial morphisms $G \to$ bar$(T; \mathbf{T}; G)$, bar$(T; \mathbf{T}; G) \to G$, viz. $s_0^n \circ \epsilon G : G \to T \circ T^n \circ G$, $\lambda \circ d_0^n : T \circ T^n \circ G \to G$, and the composition $G \to$ bar$(T; \mathbf{T}; G) \to G$ is the identity. On the other hand, if $h_i : T \circ T^n \circ G \to T \circ T^{n+1} \circ G$

is defined by $h_i = s_0^i(\epsilon T^{n-i+1} G) d_0^i$ $(0 \leq i \leq n)$, then $d_0 \circ h_0 = \text{id}$, $d_{n+1} \circ h_n = s_0^n \circ \epsilon G \circ \lambda \circ d_0^n$, and

$$d_i \circ h_j = \begin{cases} h_{j-1} \circ d_i & (i < j) \\ d_i \circ h_{i-1} & (i = j > 0) \\ h_j \circ d_{i-1} & (i > j+1) \end{cases}, s_i \circ h_j = \begin{cases} h_{j+1} \circ s_i & (i \leq j) \\ h_j \circ s_{i-1} & (i > j) \end{cases}.$$

[Note: Take instead $\mathbf{U} = \mathbf{C}$, $G = T$, $\lambda = m$—then with $\tau = FT\epsilon$, $\rho : F \circ T \to F$ is a coequalizer of $(d_1, d_0) = (Fm, \rho T)$ and the preceding observations dualize.]

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§1. COMPLETELY REGULAR HAUSDORFF SPACES

The reader is assumed to be familiar with the elements of general topology. Even so, I think it best to provide a summary of what will be needed in the sequel. Not all terms will be defined; most proofs will be omitted.

Let X be a locally compact Hausdorff space (LCH space).

PROPOSITION 1 A subspace of X is locally compact iff it is locally closed, i.e., has the form $A \cap U$, where A is closed and U is open in X.

The class of nonempty LCH spaces is closed under the formation in **TOP** of finite products and arbitrary coproducts.

[Note: An arbitrary product of nonempty LCH spaces is a LCH space iff all but finitely many of the factors are compact.]

In practice, various additional conditions are often imposed on a LCH space X. The connections among the most common of these can be summarized as follows:

$$\begin{tikzcd}
 & \text{metrizable} \arrow[rr] & & \text{paracompact} \arrow[r] & \text{normal} \\
\text{compact metrizable} \arrow[ur] \arrow[dr] & & & & \\
 & \text{compact} \arrow[rr] \arrow[dr] & & \sigma\text{-compact} \arrow[uu] & \\
 & & \text{Lindelöf} \arrow[ur, leftrightarrow] & &
\end{tikzcd}$$

EXAMPLE Let Ω be the first uncountable ordinal and consider $[0, \Omega]$ (in the order topology)—then $[0, \Omega]$ is Hausdorff. And: (i) $[0, \Omega]$ is compact but not metrizable; (ii) $[0, \Omega[$ is locally compact and normal but not paracompact; (iii) $[0, \Omega] \times [0, \Omega[$ is locally compact but not normal.

Here are some important points to keep in mind.

(LCH_1) X is completely regular, i.e., X has enough real valued continuous functions to separate points and closed sets in the sense that for every point $x \in X$ and for every closed subset $A \subset X$ not containing x, there exists a continuous function $\phi : X \to [0, 1]$ such that $\phi(x) = 1$, $\phi | A = 0$.

(LCH_2) X is σ-compact iff X possesses a sequence of exhaustion, i.e., an increasing sequence $\{U_n\}$ of relatively compact open sets $U_n \subset X$ such that $\overline{U}_n \subset U_{n+1}$ and $X = \bigcup_n U_n$.

(LCH$_3$) X is paracompact iff X admits a representation $X = \coprod_i X_i$, where the X_i are pairwise disjoint nonempty open σ-compact subspaces of X.

(LCH$_4$) X is second countable iff X is σ-compact and metrizable.

(a) If X is metrizable, then X is completely metrizable.

(b) If X is metrizable and connected, then X is second countable.

Let X be a topological space—then a collection $\mathcal{S} = \{S\}$ of subsets of X is said to be:

point finite if each $x \in X$ belongs to at most finitely many $S \in \mathcal{S}$;

neighborhood finite if each $x \in X$ has a neighborhood meeting at most finitely many $S \in \mathcal{S}$;

discrete if each $x \in X$ has a neighborhood meeting at most one $S \in \mathcal{S}$.

A collection which is the union of a countable number of $\left\{\begin{array}{l}\text{point finite}\\ \text{neighborhood finite}\\ \text{discrete}\end{array}\right.$ subcollections is said to be

$$\left\{\begin{array}{l}\sigma\text{-point finite}\\ \sigma\text{-neighborhood finite}\\ \sigma\text{-discrete.}\end{array}\right.$$

A collection $\mathcal{S} = \{S\}$ of subsets of X is said to be closure preserving if for every subcollection $\mathcal{S}_0 \subset \mathcal{S}$, $\bigcup \overline{\mathcal{S}}_0 = \overline{\bigcup \mathcal{S}_0}$, $\overline{\mathcal{S}}_0$ the collection $\{\overline{S} : S \in \mathcal{S}_0\}$.

A collection which is the union of a countable number of closure preserving subcollections is said to be σ-closure preserving.

Every neighborhood finite collection of subsets of X is closure preserving but the converse is certainly false since any collection of subsets of a discrete space is closure preserving. A point finite closure preserving closed collection is neighborhood finite. However, this is not necessarily true if "closed" is replaced by "open" as can be seen by taking $X = [0, 1]$, $\mathcal{S} = \{]0, 1/n[: n \in \mathbf{N}\}$.

Let $\mathcal{S} = \{S\}$ be a collection of subsets of X. The order of a point $x \in X$ with respect to $\mathcal{S}$, written $\operatorname{ord}(x, \mathcal{S})$, is the cardinality of $\{S \in \mathcal{S} : x \in S\}$. $\mathcal{S}$ is of finite order if $\operatorname{ord}(\mathcal{S}) = \sup\limits_{x \in X} \operatorname{ord}(x, \mathcal{S}) < \omega$. The star of a subset $Y \subset X$ with respect to $\mathcal{S}$, written $\operatorname{st}(Y, \mathcal{S})$, is the set $\bigcup\{S \in \mathcal{S} : S \cap Y \neq \emptyset\}$. $\mathcal{S}$ is star finite if $\forall\, S_0 \in \mathcal{S} : \#\{S \in \mathcal{S} : S \cap S_0 \neq \emptyset\} < \omega$.

Suppose that $\mathcal{U} = \{U_i : i \in I\}$ is a covering of X—then a covering $\mathcal{V} = \{V_j : j \in J\}$ of X is a refinement (star refinement) of $\mathcal{U}$ if each V_j ($\operatorname{st}(V_j, \mathcal{V})$) is contained in some U_i and is a precise refinement of $\mathcal{U}$ if $I = J$ and $V_i \subset U_i$ for every i. If $\mathcal{U}$ admits a point finite (open) or a neighborhood finite (open, closed) refinement, then $\mathcal{U}$ admits a precise point finite (open) or neighborhood finite (open, closed) refinement.

To illustrate the terminology, recall that if X is metrizable, then every open covering of X has an open refinement that is both neighborhood finite and σ-discrete.

Let X be a completely regular Hausdorff space (CRH space).

(C) X is compact iff every open covering of X has a finite (neighborhood finite, point finite) subcovering.

(P) X is paracompact iff every open covering of X has a neighborhood finite open (closed) refinement.

(M) X is metacompact iff every open covering of X has a point finite open refinement.

The following conditions are equivalent to paracompactness.

(P_1) Every open covering of X has a closure preserving open refinement.

(P_2) Every open covering of X has a σ-closure preserving open refinement.

(P_3) Every open covering of X has a closure preserving closed refinement.

(P_4) Every open covering of X has a closure preserving refinement.

PROPOSITION 2 A LCH space X is paracompact iff every open covering of X has a star finite open refinement.

[Suppose that X is paracompact. Given an open covering $\mathcal{U} = \{U_i\}$ of X, choose a relatively compact open refinement $\mathcal{V} = \{V_j\}$ of $\mathcal{U}$ such that each $\overline{V}_j$ is contained in some U_i—then every neighborhood finite open refinement of $\mathcal{V}$ is necessarily star finite.]

A collection $\mathcal{S} = \{S\}$ of subsets of a CRH space X is said to be directed if for all $S_1, S_2 \in \mathcal{S}$, there exists $S_3 \in \mathcal{S}$ such that $S_1 \cup S_2 \subset S_3$.

The following condition is equivalent to metacompactness.

$(M)_D$ Every directed open covering of X has a closure preserving closed refinement.

Given an open covering $\mathcal{U}$ of X, denote by $\mathcal{U}_F$ the collection whose elements are the unions of the finite subcollections of $\mathcal{U}$—then $\mathcal{U}_F$ is directed and refines $\mathcal{U}$ if $\mathcal{U}$ itself is directed. So the above characterization of metacompactness can be recast:

$(M)_F$ For every open covering $\mathcal{U}$ of X, $\mathcal{U}_F$ has a closure preserving closed refinement.

It is therefore clear that a LCH space X is metacompact iff X admits a representation $X = \bigcup_i K_i$, where $\{K_i\}$ is a closure preserving collection of compact subsets of X.

A CRH space X is said to be subparacompact if every open covering of X has a σ-discrete closed refinement.

[Note: This definition is partially suggested by the fact that X is paracompact iff every open covering of X has a σ-discrete open refinement.]

Suppose that X is subparacompact. Let $\mathcal{U} = \{U\}$ be an open covering of X—then $\mathcal{U}$ has a closed refinement $\mathcal{A} = \bigcup\limits_n \mathcal{A}_n$, where each $\mathcal{A}_n$ is discrete. Every $A \in \mathcal{A}_n$ is contained in some $U_A \in \mathcal{U}$. The collection

$$\mathcal{V}_n = \{U_A - (\cup\mathcal{A}_n - A) : A \in \mathcal{A}_n\} \cup \{U - \cup\mathcal{A}_n : U \in \mathcal{U}\}$$

is an open refinement of $\mathcal{U}$ and $\forall\ x \in X\ \exists\ n_x : \operatorname{ord}(x, \mathcal{V}_{n_x}) = 1$.

FACT X is subparacompact iff every open covering of X has a σ-closure preserving closed refinement.

A CRH space X is said to be submetacompact if for every open covering $\mathcal{U}$ of X there exists a sequence $\{\mathcal{V}_n\}$ of open refinements of $\mathcal{U}$ such that $\forall\ x \in X\ \exists\, n_x : \operatorname{ord}(x, \mathcal{V}_{n_x}) < \omega$.

FACT X is submetacompact iff every directed open covering of X has a σ-closure preserving closed refinement.

These properties are connected by the implications:

$$\begin{tikzcd}
 & & \text{metacompact} \arrow[r] & \text{submetacompact} \\
\text{compact} \arrow[r] & \text{paracompact} \arrow[ur] \arrow[dr] & & \\
 & & \text{subparacompact} \arrow[uur] &
\end{tikzcd}$$

Each is hereditary with respect to closed subspaces and, apart from compactness, each is hereditary with respect to F_σ-subspaces (and all subspaces if this is so of open subspaces).

EXAMPLE (The Thomas Plank) Let $L_0 = \{(x,0) : 0 < x < 1\}$ and for $n \geq 1$, let $L_n = \{(x, 1/n) : 0 \leq x < 1\}$. Put $X = \bigcup\limits_0^\infty L_n$. Topologize X as follows: For $n \geq 1$, each point of L_n except for $(0, 1/n)$ is isolated, basic neighborhoods of $(0, 1/n)$ being subsets of L_n containing $(0, 1/n)$ and having finite complements, while for $n = 0$, basic neighborhoods of $(x, 0)$ are sets of the form $\{(x,0)\} \cup \{(x, 1/m) : m \geq n\}$ $(n = 1, 2, \ldots)$. X is a LCH space. Moreover, X is metacompact: Every open covering of X has an open refinement consisting of one basic neighborhood for each $x \in X$ and any such refinement is point finite since the order of each $x \in X$ with respect to it is at most three. But X is not paracompact. In fact, X is not even normal: $A = \{(0, 1/n) : n = 1, 2, \ldots\}$ and $B = L_0$ are disjoint closed subsets of X and every neighborhood of A contains all but countably many points of $\bigcup\limits_1^\infty L_n$, while every neighborhood of B contains uncountably many points of $\bigcup\limits_1^\infty L_n$. Finally, X is subparacompact. This is because X is a countable union of closed paracompact subspaces.

EXAMPLE (The Burke Plank) Take $X = [0,\Omega^+[\times[0,\Omega^+[-\{(0,0)\}$, Ω^+ the cardinal successor of Ω. For $0 < \alpha < \Omega^+$, put

$$\begin{cases} H_\alpha = [0,\Omega^+[\times\{\alpha\} \\ V_\alpha = \{\alpha\}\times[0,\Omega^+[. \end{cases}$$

Topologize X as follows: Isolate all points except those on the vertical or horizontal axis, the basic neighborhoods of $\begin{cases}(0,\alpha)\\(\alpha,0)\end{cases}$ being the subsets of $\begin{cases}H_\alpha\\V_\alpha\end{cases}$ containing $\begin{cases}(0,\alpha)\\(\alpha,0)\end{cases}$ and having finite complements. X is a metacompact LCH space. But X is not subparacompact. To see this, first observe that if S and T are subsets of X such $S\cap H_\alpha$ and $T\cap V_\alpha$ are countable for every $\alpha < \Omega^+$, then $X \neq S\cup T$. Let $\mathcal{U} = \{H_\alpha : 0 < \alpha < \Omega^+\}\cup\{V_\alpha : 0<\alpha<\Omega^+\}$. $\mathcal{U}$ is an open covering of X and the claim is: $\mathcal{U}$ does not have a σ-discrete closed refinement $\mathcal{V} = \bigcup\limits_n \mathcal{V}_n$. To get a contradiction suppose that such a $\mathcal{V}$ does exist. Let $\mathcal{S}_n$ and $\mathcal{T}_n$ be the elements of $\mathcal{V}_n$ which are contained in $\{H_\alpha : 0<\alpha<\Omega^+\}$ and $\{V_\alpha : 0<\alpha<\Omega^+\}$, respectively—then $\mathcal{V}_n = \mathcal{S}_n\cup\mathcal{T}_n$. Write $\begin{cases} S = \bigcup\limits_n S_n \\ T = \bigcup\limits_n T_n \end{cases}$, where $\begin{cases} S_n = \cup\mathcal{S}_n \\ T_n = \cup\mathcal{T}_n \end{cases}$. Since the $\mathcal{V}_n$ are discrete, $S\cap H_\alpha$ and $T\cap V_\alpha$ are countable for every $\alpha<\Omega^+$, thus $X\neq S\cup T = \cup\mathcal{V}$ and so $\mathcal{V}$ does not cover X.

[Note: Why does one work with Ω^+ rather than Ω? Reason: In general, if the weight of X is $\leq \Omega$, then X is subparacompact iff X is submetacompact.]

EXAMPLE (Isbell-Mrówka Space) Let D be an infinite set. Choose a maximal infinite collection $\mathcal{S}$ of almost disjoint countably infinite subsets of D, almost disjoint meaning that $\forall\, S_1 \neq S_2 \in \mathcal{S}$, $\#(S_1\cap S_2) < \omega$. Observe that $\mathcal{S}$ is uncountable. Put $\Psi(D) = \mathcal{S}\cup D$. Topologize $\Psi(D)$ as follows: Isolate the points of D and take for the basic neighborhoods of a point $S\in\mathcal{S}$ all sets of the form $\{S\}\cup(S-F)$, F a finite subset of S. $\Psi(D)$ is a LCH space. In addition: $\mathcal{S}$ is closed and discrete, while D is open and dense. Specialize and let $D = \mathbf{N}$—then $X = \Psi(\mathbf{N})$ is subparacompact, being a Moore space (cf. p. 1–17), but is not metacompact. In fact, since $\mathcal{S}$ is uncountable, the open covering $\{\mathbf{N}\}\cup\{\{S\}\cup S : S\in\mathcal{S}\}$ cannot have a point finite open refinement.

[Note: The Isbell-Mrówka space $\Psi(\mathbf{N})$ depends on $\mathcal{S}$. Question: Up to homeomorphism how many distinct $\Psi(\mathbf{N})$ are there? Answer: 2^{2^ω}.]

The coproduct of the Burke plank and the Isbell-Mrówka space provides an example of a submetacompact X that is neither metacompact nor subparacompact.

EXAMPLE (The van Douwen Line) The object is to equip $X = \mathbf{R}$ with a first countable, separable topology that is finer than the usual topology (hence Hausdorff) and under which $X = \mathbf{R}$ is locally compact but not submetacompact. Given $x\in\mathbf{R}$, choose a sequence $\{q_n(x)\}\subset\mathbf{Q}$ such that $|x - q_n(x)| < 1/n$. Next, let $\{C_\alpha : \alpha < 2^\omega\}$ be an enumeration of the countable subsets C_α of $\mathbf{R}$ with $\#(\overline{C}_\alpha) = 2^\omega$. For $\alpha < 2^\omega$, $N = 0,1,2,\ldots$, pick inductively a point

$$x_{\alpha N}\in\overline{C}_\alpha - (\mathbf{Q}\cup\{x_{\beta M} : \beta<\alpha \text{ or } \beta=\alpha \text{ and } M<N\}).$$

Put

$$\begin{cases} S_0 = \{x_{\alpha 0} : \alpha < 2^\omega\} \\ S_N = \{x_{\alpha N} : \alpha < 2^\omega \text{ and } C_\alpha \subset S_0\} \ (N = 1, 2, \ldots) \end{cases}$$

and write S in place of $\mathbf{R} - \bigcup_1^\infty S_N$. Observe that $\mathbf{Q} \cup S_0 \subset S$ and that the S_N are pairwise disjoint. Given $x = x_{\alpha N} \in \mathbf{R} - S$, choose a sequence $\{c_m(x)\} \subset C_\alpha (\subset S_0 \subset S)$ such that $|x - c_m(x)| < 1/m$. Topologize $X = \mathbf{R}$ as follows: Isolate the points of $\mathbf{Q}$ and take for the basic neighborhoods of $\begin{cases} x \in S - \mathbf{Q} \\ x \in \mathbf{R} - S \end{cases}$ the sets

$$\begin{cases} K_k(x) = \{x\} \cup \{q_n(x) : n \geq k\} \\ K_k(x) = \{x\} \cup \{c_m(x) : m \geq k\} \cup \{q_n(c_m(x)) : m \geq k, n \geq m\} \end{cases} \quad (k = 1, 2, \ldots).$$

This prescription defines a first countable, separable topology on the line that is finer than the usual topology. And, since the K_k are compact, it is a locally compact topology. However, it is not a submetacompact topology. Thus let $U_N = S \cup S_N$—then U_N is open and $\mathcal{U} = \{U_N\}$ is an open covering of X. Consider any sequence $\{\mathcal{V}_M\}$ of open refinements of $\mathcal{U}$. For $M = 1, 2, \ldots$, and $N = 1, 2, \ldots$, let $W_{MN} = \bigcup \{V \in \mathcal{V}_M : V \cap S_N \neq \emptyset\}$ and form $W_0 = S_0 \cap \bigcap_{M,N} W_{MN} = S_0 - \bigcup_{M,N} (S_0 - W_{MN})$. Since $\#(S_0) = 2^\omega$ and since the $S_0 - W_{MN}$ are countable, W_0 is nonempty. But any x_0 in W_0 necessarily belongs to infinitely many distinct elements of $\mathcal{V}_M$ $(M = 1, 2, \ldots)$. Consequently, the topology is not submetacompact.

JONES' LEMMA If a Hausdorff space X contains a dense set D and a closed discrete subspace S with $\#(S) \geq 2^{\#(D)}$, then X is not normal.

Application: The van Douwen line is not normal.
[In fact, each S_N is closed and discrete with $\#(S_N) = 2^\omega$.]

Let X be a LCH space. Under what conditions is it true that X metacompact $\Rightarrow$ X paracompact? For example, is it true that if X is normal and metacompact, then X is paracompact? This is an open question. There are no known counterexamples in ZFC or under any additional set theoretic assumptions. Two positive results have been obtained.

(1) (Daniels[†]) A normal LCH space X is paracompact provided that it is boundedly metacompact, i.e., every open covering of X has an open refinement of finite order.

(2) (Gruenhage[‡]) A normal LCH space X is paracompact provided that it is locally connected and submetacompact.

[†] *Canad. J. Math.* **35** (1983), 807–823; see also *Topology Appl.* **28** (1988), 113–125.

[‡] *Topology Proc.* **4** (1979), 393–405.

Suppose that X is normal and metacompact—then on general grounds all that one can say is this. Consider any open covering $\mathcal{U}$ of X: By metacompactness, $\mathcal{U}$ has a point finite open refinement $\mathcal{V}$ which, by normality, has a precise open refinement $\mathcal{W}$ with the property that $\overline{\mathcal{W}}$ is a precise closed refinement of $\mathcal{V}$.

FACT Let X be a CRH space. Suppose that X is submetacompact—then X is normal iff every open covering of X has a precise closed refinement.

A Hausdorff space X is said to be perfect if every closed subset of X is a G_δ. The Isbell-Mrówka space $\Psi(\mathrm{N})$ is perfect; however, it is not normal (cf. p. 1–12).

A Hausdorff space X is said to be perfectly normal if it is perfect and normal. The ordinal space $[0,\Omega]$, while normal, is not perfectly normal since the point $\{\Omega\}$ is not a G_δ. On the other hand, X metrizable $\Rightarrow$ X perfectly normal. Every perfectly normal LCH space X is first countable.

[Note: The assumption of perfect normality can be used to upgrade the strength of a covering property.

(1) (Arhangel'skii[†]) Let X be a LCH space. If X is perfectly normal and metacompact, then X is paracompact.

(2) (Bennett-Lutzer[‡]) Let X be a LCH space. If X is perfectly normal and submetacompact, then X is subparacompact.]

A CRH space X is said to be countably paracompact if every countable open covering of X has a neighborhood finite open refinement. The ordinal space $[0,\Omega[$ is countably paracompact (being countably compact) and normal, whereas the ordinal space $[0,\Omega]\times[0,\Omega[$ is countably paracompact (being compact × countably compact ≡ countably compact) but not normal. On the other hand, X perfectly normal $\Rightarrow$ X countably paracompact.

To recapitulate:

$$\begin{tikzcd}
 & \text{paracompact} \arrow[d] \arrow[dr] & \\
\text{metrizable} \arrow[ur] \arrow[dr] & \text{normal} & \text{countably paracompact} \\
 & \text{perfectly normal} \arrow[u] \arrow[ur] &
\end{tikzcd}$$

FACT Suppose that X is normal—then X is countably paracompact iff every countable open covering of X has a σ-discrete closed refinement.

[†] *Soviet Math. Dokl.* **13** (1972), 517–520.

[‡] *General Topology Appl.* **2** (1972), 49–54.

So: In the presence of normality, X subparacompact $\Rightarrow$ X countably paracompact. This implication is strict since the ordinal space $[0,\Omega[$ is normal and countably paracompact; however, it is not even submetacompact (cf. p. 1–12). On the other hand: (i) The ordinal space $[0,\Omega] \times [0,\Omega[$ is nonnormal and countably paracompact but not subparacompact; (ii) The Isbell-Mrówka space $\Psi(\mathbf{N})$ is nonnormal and subparacompact but not countably paracompact (cf. p. 1–12).

[Note: To verify that $X = [0,\Omega] \times [0,\Omega[$ is not subparacompact, let $A = \{(\Omega,\alpha) : \alpha < \Omega\}$ and $B = \{(\alpha,\alpha) : \alpha < \Omega\}$—then A and B are disjoint closed subsets of X. Therefore $X = U \cup V$, where $U = X - A$ and $V = X - B$. Since the open covering $\{U, V\}$ has no σ-discrete closed refinement, X is not subparacompact.]

Is every normal LCH space countably paracompact? This question is a reinforcement of the "Dowker problem". Dropping the supposition of local compactness, a Dowker space is by definition a normal Hausdorff space which fails to be countably paracompact or, equivalently, whose product with $[0,1]$ is not normal. Do such spaces exist? The answer is "yes", the first such example within ZFC being a construction due to M.E. Rudin[†]. Her example is not locally compact and only by imposing assumptions beyond ZFC has it been possible to produce locally compact examples.

The ordinal space $[0,\Omega] \times [0,\Omega[$ is neither first countable nor separable. Can one construct an example of a nonnormal countably paracompact LCH space with both of these properties? The answer is "yes".

Let S and T be subsets of $\mathbf{N}$. Write $S \leq T$ if $\#(S-T) < \omega$; write $S < T$ if $S \leq T$ and $\#(T-S) = \omega$.

LEMMA (Hausdorff) There exist collections $\begin{cases} \mathcal{S}^+ = \{S_\alpha^+ : \alpha < \Omega\} \\ \mathcal{S}^- = \{S_\alpha^- : \alpha < \Omega\} \end{cases}$ of subsets of $\mathbf{N}$ with the following properties:

(1) $\forall\, \alpha : \#(\mathbf{N} - (S_\alpha^+ \cup S_\alpha^-)) = \omega$.

(2) $\forall\, \alpha, \forall\, \beta : \beta < \alpha \Rightarrow S_\beta^+ < S_\alpha^+$ and $S_\beta^- < S_\alpha^-$.

(3) $\forall\, \alpha : \#(S_\alpha^+ \cap S_\alpha^-) < \omega$.

(4) $\forall\, \alpha, \forall\, n \in \mathbf{N}$: $\#\{\beta : \beta < \alpha \;\&\; S_\alpha^+ \cap S_\beta^- \subset F_n\} < \omega$ $(F_n = \{1,\ldots,n\})$.

There is then no $H \subset \mathbf{N}$ such that $\forall\, \alpha : S_\alpha^+ \leq H$ and $S_\alpha^- \leq \mathbf{N} - H$.

[We shall establish the existence of $\mathcal{S}^+$ and $\mathcal{S}^-$ by constructing their elements via induction on α. Start by setting $S_0^+ = \emptyset$ and $S_0^- = \emptyset$. Given S_α^+ and S_α^-, decompose $\mathbf{N} - (S_\alpha^+ \cup S_\alpha^-)$ into three infinite pairwise disjoint sets N_α^+, N_α^-, and N_α. Put

$$\begin{cases} S_{\alpha+1}^+ = S_\alpha^+ \cup N_\alpha^+ \\ S_{\alpha+1}^- = S_\alpha^- \cup N_\alpha^- \end{cases} \qquad (\Rightarrow \mathbf{N} - (S_{\alpha+1}^+ \cup S_{\alpha+1}^-) \supset N_\alpha).$$

[†] *Fund. Math.* **73** (1971), 179–186; see also Balogh, *Proc. Amer. Math. Soc.* **124** (1996), 2555–2560.

Then this definition handles the successor ordinals $< \Omega$. Suppose now that $0 < \Lambda < \Omega$ is a limit ordinal. Choose a strictly increasing sequence $\{\alpha_i\} \subset [0,\Omega[$: $\alpha_1 = 0$, $\sup \alpha_i = \Lambda$. Fix $n_i \in \mathrm{N}$ such that $S^+_{\alpha_i} \cap \bigcup\limits_{j \le i} S^-_{\alpha_j} \subset F_{n_i}$ and write T^+_Λ for $\bigcup\limits_i (S^+_{\alpha_i} - F_{n_i})$. Note that $\forall\, \alpha < \Lambda : S^+_\alpha < T^+_\Lambda$ and $\forall\, i : \#(T^+_\Lambda \cap S^-_{\alpha_i}) < \omega$. If $I_i = \{\alpha : \alpha_i \le \alpha < \alpha_{i+1} \ \&\ T^+_\Lambda \bigcap S^-_\alpha \subset F_i\}$ and if $I = \bigcup\limits_i I_i$, then each I_i is finite and so $I \cap [0,\alpha[$ is finite for every $\alpha < \Lambda$. Assign to each nonzero $\alpha \in I_i$ the infinite set $S^-_\alpha - \bigcup\{S^-_{\alpha_j} : \alpha_j < \alpha\}$ and denote by $n(\alpha)$ its minimum element in $\mathrm{N} - F_i$. Relative to this data, define $S^+_\Lambda = T^+_\Lambda \cup \{n(\alpha) : \alpha \in I \ (\alpha \neq 0)\}$. Then it is not difficult to verify that

$$\begin{cases} \forall\, \alpha < \Lambda : S^+_\alpha < S^+_\Lambda \text{ and } \forall\, i : \#(S^+_\Lambda \cap S^-_{\alpha_i}) < \omega \\ \forall\, n \in \mathrm{N} : \#\{\alpha : \alpha < \Lambda \ \&\ S^+_\Lambda \cap S^-_\alpha \subset F_n\} < \omega. \end{cases}$$

As for S^-_Λ, observe that $(\mathrm{N} - S^+_\Lambda) - \bigcup\limits_{j \le i} S^-_{\alpha_j}$ is infinite, thus there exists an infinite set $L_\Lambda \subset (\mathrm{N} - S^+_\Lambda)$ such that $L_\Lambda \cap S^-_{\alpha_i}$ is finite for every i. Defining $S^-_\Lambda = \mathrm{N} - (S^+_\Lambda \cup L_\Lambda)$, we have

$$\begin{cases} \forall\, \alpha < \Lambda : S^-_\alpha < S^-_\Lambda \\ S^+_\Lambda \cap S^-_\Lambda = \emptyset, \#(\mathrm{N} - (S^+_\Lambda \cup S^-_\Lambda)) = \omega, \end{cases}$$

which completes the induction. There remains the assertion of nonseparation. To deal with it, assume that there exists an $H \subset \mathrm{N}$ such that $S^+_\alpha - H$ and $S^-_\alpha \cap H$ are both finite for every $\alpha < \Omega$. Choose an $n \in \mathrm{N} : W = \{\alpha : S^-_\alpha \cap H \subset F_n\}$ is uncountable. Fix an $\alpha \in W$ with the property that $W \cap [0,\alpha[$ is infinite. If $S^+_\alpha - H \subset F_m$, then $\{\beta : \beta < \alpha \ \&\ S^+_\alpha \cap S^-_\beta \subset F_{\max(m,n)}\}$ contains $W \cap [0,\alpha[$. Contradiction.]

EXAMPLE (van Douwen Space) Let

$$\begin{cases} X^+ = \{+1\} \times]0,\Omega[\\ X^- = \{-1\} \times]0,\Omega[\end{cases}$$

and put $X = X^+ \cup X^- \cup \mathrm{N}$. Topologize X as follows: Isolate the points of N and take for the basic neighborhoods of a point $\begin{cases} (+1,\alpha) \in X^+ \\ (-1,\alpha) \in X^- \end{cases}$ all sets of the form

$$\begin{cases} K(+1,\alpha : \beta, F) = \{(+1,\gamma) : \beta < \gamma \le \alpha\} \cup ((S^+_\alpha - S^+_\beta) - F) \\ K(-1,\alpha : \beta, F) = \{(-1,\gamma) : \beta < \gamma \le \alpha\} \cup ((S^-_\alpha - S^-_\beta) - F), \end{cases}$$

where $\beta < \alpha$ and $F \subset \mathrm{N}$ is finite. Since the $K(\pm 1, \alpha : \beta, F)$ are compact, X is a LCH space. Obviously, X is first countable and separable; in addition, X is countably paracompact, $X^\pm$ being a copy of $]0,\Omega[$. Still, X is not normal.

[Suppose that the disjoint closed sets X^+ and X^- can be separated by disjoint open sets U^+ and U^-. Given $\alpha \in]0,\Omega[$, select an ordinal $f(\alpha) < \alpha$ and a finite subset $F(\alpha) \subset \mathrm{N}$ such that $K(\pm 1, \alpha : f(\alpha), F(\alpha)) \subset U^\pm$. Choose a $\kappa < \Omega$ and a cofinal $\mathcal{K} \subset [0,\Omega[$ such that $f | \mathcal{K} = \kappa$ (by "pressing down", i.e., Fodor's lemma). Put

$$\begin{cases} H^+ = (S^+_\kappa \cup (\mathrm{N} \cap U^+)) - S^-_\kappa \\ H^- = (S^-_\kappa \cup (\mathrm{N} \cap U^-)) - S^+_\kappa. \end{cases}$$

Then $H^+ \cap H^- = \emptyset$. Let $\alpha < \Omega$ be arbitrary. Using the cofinality of $\mathcal{K}$ and the relation $f|\mathcal{K} = \kappa$, one finds that $S_\alpha^\pm \leq H^\pm$. Contradiction.]

A CRH space X is said to be countably compact if every countable open covering of X has a finite subcovering or, equivalently, if every neighborhood finite collection of nonempty subsets of X is finite. The ordinal space $[0, \Omega[$ is countably compact but not compact. The van Douwen space is not countably compact but is countably paracompact.

Associated with this ostensibly simple concept are some difficult unsolved problems. Sample: Within ZFC, does there exist a first countable, separable, countably compact LCH space X that is not compact? This is an open question. But under CH, e.g., such an X does exist (cf. p. 1–17). Consider the assertion: Every perfectly normal, countably compact LCH space X is compact. While innocent enough, this statement is undecidable in ZFC (Ostaszewski[†], Weiss[‡]).

PROPOSITION 3 X is countably compact iff every point finite open covering of X has a finite subcovering.

[Suppose that X is countably compact. Let $\mathcal{U}$ be a point finite open covering of X—then, on general grounds, $\mathcal{U}$ admits an irreducible subcovering $\mathcal{V}$. This minimal covering must be finite: For otherwise there would exist an infinite subset $S \subset X$ such that each $x \in X$ has a neighborhood containing exactly one point of S, an impossibility.

Suppose that X is not countably compact—then there exists a countably infinite discrete closed subset $D \subset X$, say $D = \{x_n\}$. Choose a sequence $\{U_n\}$ of nonempty open sets whose closures are pairwise disjoint such that $\forall\, n : x_n \in U_n$. The collection $\{X - D, U_1, U_2, \ldots\}$ is a point finite open covering of X which has no finite subcovering.]

A CRH space X is said to be pseudocompact if every countable open covering of X has a finite subcollection whose closures cover X or, equivalently, if every neighborhood finite collection of nonempty open subsets of X is finite. The Isbell-Mrówka space $\Psi(\mathbf{N})$ is pseudocompact but not countably compact (cf. p. 1–12).

PROPOSITION 4 X is pseudocompact iff every real valued continuous function on X is bounded.

[Suppose that X is not pseudocompact—then there exists a countably infinite neighborhood finite collection $\{U_n\}$ of nonempty open subsets of X. Choose a point $x_n \in U_n$.

[†] *J. London Math. Soc.* **14** (1976), 505–516.

[‡] *Canad. J. Math.* **30** (1978), 243–249.

Since X is completely regular, there exists a continuous function $f_n : X \rightarrow [0, n]$ such that $f_n(x_n) = n$, $f_n|X - U_n = 0$. Put $f = \sum\limits_n f_n$: f is continuous and unbounded.]

A CRH space X is said to be countably metacompact if every countable open covering of X has a point finite open refinement. The ordinal space $[0, \Omega[$ is countably metacompact but not metacompact (cf. p. 1–12). Every perfect X is countably metacompact.

The relative position of these conditions is shown by:

$$\begin{tikzcd}
\text{compact} \arrow[r] \arrow[d] & \text{paracompact} \arrow[r] \arrow[d] & \text{metacompact} \arrow[d] \\
\text{countably compact} \arrow[r] \arrow[d] & \text{countably paracompact} \arrow[r] & \text{countably metacompact} \\
\text{pseudocompact} & &
\end{tikzcd}$$

FACT X is countably metacompact iff for every countable open covering $\mathcal{U}$ of X there exists a sequence $\{\mathcal{V}_n\}$ of open refinements of $\mathcal{U}$ such that $\forall\, x \in X\ \exists\, n_x : \text{ord}(x, \mathcal{V}_{n_x}) < \omega$.

[The point here is to show that the stated condition forces X to be countably metacompact. Enumerate the elements of $\mathcal{U}$: U_n $(n = 1, 2, \ldots)$. Write W_n for the set of all $x \in U_n$ such that $\forall\, m \leq n\ \exists\, V \in \mathcal{V}_m : x \in V$ and $V \not\subset \bigcup\limits_{i<n} U_i$. Then $\mathcal{W} = \{W_n\}$ is a point finite open refinement of $\mathcal{U} = \{U_n\}$.]

So: X submetacompact $\Rightarrow$ X countably metacompact. The van Douwen line is not countably metacompact (inspect the argument used to establish nonsubmetacompactness). The Tychonoff plank is countably metacompact but is neither submetacompact nor countably paracompact (cf. p. 1–12).

PROPOSITION 5 If X is pseudocompact and either normal or countably paracompact, then X is countably compact.

[Suppose that X is normal. If X is not countably compact, then there exists a countably infinite discrete closed subset $D \subset X$, say $D = \{x_n\}$. By the Tietze extension theorem, there exists a continuous function $f : X \rightarrow \mathbf{R}$ such that $f(x_n) = n$ $(n = 1, 2, \ldots)$. Contradiction.

Suppose that X is countably paracompact. If X is not countably compact, then there exists a countable open covering $\{U_n\}$ of X that cannot be reduced to a finite covering. Let $\{V_n\}$ be a precise neighborhood finite open refinement of $\{U_n\}$—then there exists a finite subset $F \subset \mathbf{N}$ such that $V_n \neq \emptyset$ iff $n \in F$. But $\bigcup\limits_n V_n = X$. Contradiction.]

EXAMPLE The Isbell-Mrówka space $\Psi(\mathbf{N})$ is not countably compact. However, $\Psi(\mathbf{N})$ is pseudocompact so, by the above, it is neither normal nor countably paracompact.

[Put $X = \Psi(\mathbf{N})$ and suppose that $f : X \to \mathbf{R}$ is continuous but unbounded. Since $\forall\ S \in \mathcal{S}$, $\{S\} \cup S$ is compact, $f|S$ is bounded. This means that there exists a sequence $\{x_n\}$ of distinct points in X such that (i) $|f(x_n)| \geq n$ and (ii) $\forall\ S \in \mathcal{S}, \#(\{x_n\} \cap S) < \omega$. The maximality of $\mathcal{S}$ then implies that $\{x_n\} \in \mathcal{S}$. Contradiction.]

EXAMPLE (The Tychonoff Plank) Let $X = [0, \Omega] \times [0, \omega] - \{(\Omega, \omega)\}$. X is not countably compact (consider $\{(\Omega, n) : 0 \leq n < \omega\}$). However, X is pseudocompact so, by the above, it is neither normal nor countably paracompact.

[Suppose that $f : X \to \mathbf{R}$ is continuous—then it suffices to show that f extends continuously to $\{(\Omega, \omega)\}$. Because every real valued continuous function on $[0, \Omega[$ is constant on some tail $[\alpha, \Omega[$, $\forall n \leq \omega$, there exists an $\alpha_n < \Omega$ and a constant r_n such that $f(\alpha, n) = r_n\ \forall\ \alpha \geq \alpha_n$. Put $\alpha_0 = \sup \alpha_n$—then $\alpha_0 < \Omega$. One can therefore let $f(\Omega, \omega) = r_\omega$.]

PROPOSITION 6 If X is countably compact and submetacompact, then X is compact.

[Let $\mathcal{U}$ be an open covering of X. Let $\{\mathcal{V}_n\}$ be a sequence of open refinements of $\mathcal{U}$ such that $\forall\ x \in X\ \exists\ n_x$: $\operatorname{ord}(x, \mathcal{V}_{n_x}) < \omega$. Write A_{mn} for $\{x : \operatorname{ord}(x, \mathcal{V}_n) \leq m\}$—then A_{mn} is a closed subspace of X, hence is countably compact, and $\mathcal{V}_n$ is point finite on A_{mn}. Proposition 3 therefore implies that A_{mn} can be covered by finitely many elements of $\mathcal{V}_n$. Every $x \in X$ is in some A_{mn}, so there is a countable open covering of X made up of elements from the sequence $\{\mathcal{V}_n\}$. This covering has a finite subcovering, thus so does $\mathcal{U}$.]

Consequently, the ordinal space $[0, \Omega[$ is not submetacompact. It then follows from this that the Tychonoff plank is not submetacompact (since $[0, \Omega[$ sits inside it as a closed subspace).

Let X be a CRH space. A π-basis for X is a collection $\mathcal{P}$ of nonempty open subsets of X such that if O is a nonempty open subset of X, then for some $P \in \mathcal{P}$, $P \subset O$.

LEMMA Suppose that X is Baire. Let $\mathcal{U}$ be a point finite open covering of X—then there exists a π-basis $\mathcal{P}$ for X such that $\forall\, P \in \mathcal{P}$ and $\forall\, U \in \mathcal{U}$, either $P \subset U$ or $P \cap U = \emptyset$.

[For $n = 1, 2, \ldots$, denote by X_n the subset of X consisting of those points that are in at most n elements of $\mathcal{U}$. Each X_n is closed and $X = \bigcup\limits_n X_n$. Let O be a nonempty open subset of X. Since $O = \bigcup\limits_n O \cap X_n$, there will be an n such that $O \cap X_n$ has a nonempty interior. Let $n(O)$ be the smallest such n. Let $U_O \subset O \cap X_{n(O)}$ be a nonempty open subset of X. Choose an $x_O \in U_O$ that belongs to exactly $n(O)$ elements of $\mathcal{U}$ and write P for their intersection with U_O—then $\mathcal{P} = \{P\}$ is a π-basis for X with the stated properties.]

Suppose that X is pseudocompact—then X is Baire. To see this, let $\{O_n\}$ be a decreasing sequence of dense open subsets of X. Let U be a nonempty open subset of X. Inductively choose nonempty open sets $V_n : V_1 = U$ & $\overline{V}_{n+1} \subset U \cap O_n \cap V_n$. By pseudocompactness, $\bigcap\limits_n \overline{V}_n \neq \emptyset$, hence $U \cap (\bigcap\limits_n O_n) \neq \emptyset$.

PROPOSITION 7 If X is pseudocompact and metacompact, then X is compact.

[Let $\mathcal{O}$ be an open covering of X. Let $\mathcal{U} = \{U\}$ be a point finite open refinement of $\mathcal{O}$ with the property that $\overline{\mathcal{U}} = \{\overline{U}\}$ refines $\mathcal{O}$. Use the lemma to determine a π-basis $\mathcal{P}$ for X per $\mathcal{U}$. Fix $P_1 \in \mathcal{P}$. Consider $\{U \in \mathcal{U} : U \cap P_1 \neq \emptyset\}$. Since $U \cap P_1 \neq \emptyset \Rightarrow P_1 \subset U$ and since $\mathcal{U}$ is point finite, it is clear that this is a finite set. If $X = \overline{\text{st}(P_1, \mathcal{U})}$, then finitely many elements of $\mathcal{O}$ cover X and we are done. Otherwise, proceed inductively and, using the fact that $\mathcal{P}$ is a π-basis for X, given $n \in \mathbf{N}$ choose a $P_{n+1} \in \mathcal{P}$ such that

$$P_{n+1} \subset X - \bigcup_{m \leq n} \overline{\text{st}(P_m, \mathcal{U})}.$$

We claim that the process terminates, from which the result. Suppose the opposite—then, due to the pseudocompactness of X, $\{P_n\}$ cannot be neighborhood finite. Therefore there exists $x \in U_x \in \mathcal{U}$ with $U_x \cap P_n \neq \emptyset$ for infinitely many n, contrary to construction.]

One cannot replace "metacompact" by "submetacompact" in the preceding result: The Isbell-Mrówka space $\Psi(\mathbf{N})$ is pseudocompact and submetacompact but not compact. However, the argument does go through under the weaker condition: Every open covering of X has a σ-point finite open refinement.

PROPOSITION 8 If X is normal and countably metacompact, then X is countably paracompact.

One can check:

(CP) X is countably paracompact iff for every decreasing sequence $\{A_n\}$ of closed sets such that $\bigcap\limits_n A_n = \emptyset$, there exists a decreasing sequence $\{U_n\}$ of open sets with $A_n \subset U_n$ for every n and such that $\bigcap\limits_n \overline{U}_n = \emptyset$.

(CM) X is countably metacompact iff for every decreasing sequence $\{A_n\}$ of closed sets such that $\bigcap\limits_n A_n = \emptyset$, there exists a decreasing sequence $\{U_n\}$ of open sets with $A_n \subset U_n$ for every n and such that $\bigcap\limits_n U_n = \emptyset$.

It remains only to note that for normal X, CP $\Leftrightarrow$ CM.

If X is the Tychonoff plank, then $X = Y \cup Z$, where $Y = \bigcup\limits_{n<\omega} [0,\Omega] \times \{n\}$ and $Z = [0,\Omega[\times\{\omega\}$. Since Y is an open paracompact subspace of X and Z is a closed countably compact subspace of X, it is clear that X is countably metacompact. Because X is not countably paracompact, Proposition 8 allows one to infer once again that X is not normal (cf. Proposition 5).

A Hausdorff space X is said to be collectionwise normal if for every discrete collection $\{A_i : i \in I\}$ of closed subsets of X there exists a pairwise disjoint collection $\{U_i : i \in I\}$ of open subsets of X such that $\forall\, i \in I : A_i \subset U_i$.

Of course, X collectionwise normal $\Rightarrow$ X normal. On the other hand, X normal and countably compact $\Rightarrow$ X collectionwise normal. So, the ordinal space $[0,\Omega[$ is collectionwise normal. However, it is not perfectly normal since the set of all limit ordinals $\alpha < \Omega$, while closed, is not a G_δ. Rudin's Dowker space is collectionwise normal.

LEMMA Suppose that X is collectionwise normal. Let $\{A_i : i \in I\}$ be a discrete collection of closed subsets of X—then there exists a discrete collection $\{O_i : i \in I\}$ of open subsets of X such that $\forall\, i \in I : A_i \subset O_i$.

[Let $\{U_i : i \in I\}$ be a pairwise disjoint collection of open subsets of X such that $\forall\, i \in I : A_i \subset U_i$. Choose an open set U subject to $\bigcup\limits_i A_i \subset U \subset \overline{U} \subset \bigcup\limits_i U_i$ and then put $O_i = U_i \cap U$.]

Suppose that X is normal. Let $\{A_n\}$ be a countable discrete collection of closed subsets of X—then there exists a countable pairwise disjoint collection $\{U_n\}$ of open subsets of X such that $\forall\, n : A_n \subset U_n$. In fact, given $n \in \mathrm{N}$, choose a pair (O_n, P_n) of disjoint open subsets of X such that $O_n \supset A_n$, $P_n \supset \bigcup\limits_{m \neq n} A_m$ and then put $U_n = O_n \cap \bigcap\limits_{m<n} P_m$.

PROPOSITION 9 If X is paracompact, then X is collectionwise normal.

[Let $\{A_i : i \in I\}$ be a discrete collection of closed subsets of X. Put $O_i = X - \bigcup\limits_{j \neq i} A_j$—then the collection $\{O_i : i \in I\}$ is an open covering of X, hence, in view of the paracompactness of X, has a precise neighborhood finite closed refinement $\{C_i : i \in I\}$. If $U_i = X - \bigcup\limits_{j \neq i} C_j$, then $\{U_i : i \in I\}$ is a pairwise disjoint collection of open subsets of X such that $\forall\, i \in I : A_i \subset U_i$. Therefore X is collectionwise normal.]

PROPOSITION 10 If X is collectionwise normal and metacompact, then X is paracompact.

[It is enough to prove that a given point finite open covering $\mathcal{O} = \{O\}$ of X has a σ-discrete open refinement $\mathcal{U} = \bigcup\limits_n \mathcal{U}_n$. Put $A_n = \{x : \text{ord}(x, \mathcal{O}) \leq n\}$—then A_n is a closed subspace of X and $X = \bigcup\limits_n A_n$. Assign to each $x \in X$ the open set $O_x = \bigcap\{O \in \mathcal{O} : x \in O\}$. Using the O_x, we shall construct the $\mathcal{U}_n$ by induction. To start off, observe that $\{O_x \cap A_1 : x \in A_1\}$ is a discrete collection of closed subsets of X covering A_1. So, by collectionwise normality, there exists a discrete collection $\mathcal{U}_1$ of open subsets of X covering A_1 such that each element of $\mathcal{U}_1$ is contained in some element of $\mathcal{O}$. Proceeding, suppose that $\bigcup\limits_{m=1}^{n} \mathcal{U}_m$ is a covering of A_n by open subsets of X, each of which is contained in some element of $\mathcal{O}$, with $\mathcal{U}_m$ discrete. Let $U_n = \bigcup\{U : U \in \mathcal{U}_m, 1 \leq m \leq n\}$—then $U_n \supset A_n$ and $\{O_x \cap (A_{n+1} - U_n) : x \in A_{n+1} - U_n\}$ is a discrete collection of closed subsets of X covering $A_{n+1} - U_n$. Once again, by collectionwise normality, there exists a discrete collection $\mathcal{U}_{n+1}$ of open subsets of X covering $A_{n+1} - U_n$ such that each element of $\mathcal{U}_{n+1}$ is contained in some element of $\mathcal{O}$. And $A_{n+1} \subset \bigcup\limits_{m=1}^{n+1} \mathcal{U}_m$.]

Trifling modifications in the preceding argument allow one to replace "metacompact" by "submetacompact" and still arrive at the same conclusion.

Kemoto[†] has shown by very different methods that if a normal LCH space X is submetacompact, then X is subparacompact. Example: The Burke plank is not normal.

Let X be a LCH space. Does the chart

$$\begin{tikzcd}
\text{paracompact} \arrow[r] & \text{collectionwise normal} \arrow[r] & \text{normal} \\
& \text{perfectly normal} \arrow[ur] &
\end{tikzcd}$$

admit any additional arrows? We do know that there exists a paracompact X that is not perfectly normal and a collectionwise normal X that is not paracompact.

(Q$_a$) Is every normal LCH space X collectionwise normal?

[There are counterexamples under MA $+\neg$ CH (cf. p. 1–18). Consistency has been established modulo the consistency of the existence of a supercompact cardinal.]

(Q$_b$) Is every perfectly normal LCH space X collectionwise normal?

[This is undecidable in ZFC.]

(Q$_c$) Is every perfectly normal LCH space X paracompact?

[†] *Fund. Math.* **132** (1989), 163–169.

[The Kunen line under CH and the rational sequence topology over a CUE-set under MA +¬ CH are counterexamples. However, under ZFC alone, the issue has not been resolved.]

These questions (and many others) are discussed by Watson[†].

The construction of topologies by transfinite recursion is an important technique that can be used to produce a variety of illuminating examples.

EXAMPLE [Assume CH] (The Kunen Line) The object is to equip $X = \mathbf{R}$ with a first countable, separable topology that is finer than the usual topology (hence Hausdorff) and under which $X = \mathbf{R}$ is locally compact and perfectly normal but not Lindelöf, hence not paracompact (since paracompact + separable ⇒ Lindelöf). It will then turn out that the resulting topology is even hereditarily separable and collectionwise normal.

Let $\{x_\alpha : \alpha < \Omega\}$ be an enumeration of $\mathbf{R}$ and put $X_\alpha = \{x_\beta : \beta < \alpha\}$, so $X_\Omega = \mathbf{R}$. Let $\{C_\alpha : \alpha < \Omega\}$ be an enumeration of the countable subsets of $\mathbf{R}$ such that $\forall\ \alpha : C_\alpha \subset X_\alpha$. We shall now construct by induction on $\alpha \leq \Omega$ a collection $\{\tau_\alpha : \alpha \leq \Omega\}$, where τ_α is a topology on X_α (with closure operator cl_α) subject to:

(a) $\forall\ \alpha : \tau_\alpha$ is a first countable, zero dimensional, locally compact topology on X_α that is finer than the usual topology on X_α (as a subspace of $\mathbf{R}$) and, if $\alpha < \Omega$, is metrizable.

(b) $\forall\ \beta < \alpha : (X_\beta, \tau_\beta)$ is an open subspace of (X_α, τ_α).

(c) $\forall\ \gamma \leq \beta < \alpha$: If $x_\beta \in \text{cl}_{\mathbf{R}}(C_\gamma)$, then $x_\beta \in \text{cl}_\alpha(C_\gamma)$.

First, take τ_α discrete if $\alpha \leq \omega$. Assume next that $\omega < \alpha \leq \Omega$. If α is a limit ordinal, take for τ_α the topology on X_α generated by $\bigcup\limits_{\beta<\alpha} \tau_\beta$. If α is a successor ordinal, say $\alpha = \beta + 1$, then the problem is to define τ_α on $X_\alpha = X_\beta \cup \{x_\beta\}$ and for that we distinguish two cases.

(*) If there is no $\gamma \leq \beta$ such that $x_\beta \in \text{cl}_{\mathbf{R}}(C_\gamma)$, isolate x_β and take for τ_α the topology generated by τ_β and $\{x_\beta\}$.

¬(*) Let $\{\gamma_n\}$ enumerate $\{\gamma \leq \beta : x_\beta \in \text{cl}_{\mathbf{R}}(C_\gamma)\}$, each γ being listed ω times. Put $I_n =]x_\beta - 1/n, x_\beta + 1/n[$ and pick a sequence $\{y_n\}$ of distinct points $y_n \in C_{\gamma_n} \cap I_n$. Choose a discrete collection $\{K_{n,\beta}\}$ of τ_β-clopen compact sets $K_{n,\beta} : y_n \in K_{n,\beta} \subset I_n$. To complete the induction, take for τ_α the topology generated by τ_β and the sets $\{x_\beta\} \cup \bigcup\limits_{m \geq n} K_{m,\beta}$ $(n = 1, 2, \ldots)$.

It follows that $\mathbf{R}$ or still, $X_\Omega = \bigcup\limits_{\alpha<\Omega} X_\alpha$ is a first countable, LCH space under τ_Ω. Because each X_α is τ_Ω-open, X_Ω is not Lindelöf. Every $x \in X_\Omega$ has a countable clopen neighborhood.

Claim: Let $S \subset \mathbf{R}$—then $\#(\text{cl}_{\mathbf{R}}(S) - \text{cl}_\Omega(S)) \leq \omega$.

[†] In: *Open Problems in Topology*, J. van Mill and G. Reed (ed.), North Holland (1990), 37–76.

[Fix a countable subset $C \subset S$ such that $\mathrm{cl}_{\mathbf{R}}(C) = \mathrm{cl}_{\mathbf{R}}(S)$. Write $C = C_{\alpha_0}$ (some $\alpha_0 < \Omega$). If $\alpha > \alpha_0$ and if $x_\alpha \in \mathrm{cl}_{\mathbf{R}}(C)$, then $x_\alpha \in \mathrm{cl}_\Omega(C)$. Therefore $\mathrm{cl}_{\mathbf{R}}(S) - \mathrm{cl}_\Omega(S) \subset \{x_\alpha : \alpha \leq \alpha_0\}$.]

The fact that X_Ω is hereditarily separable is thus immediate. To establish perfect normality, suppose that $A \subset X_\Omega$ is closed—then it is a question of finding a sequence $\{U_n\} \subset \tau_\Omega$ such that $A = \bigcap\limits_n U_n = \bigcap\limits_n \mathrm{cl}_\Omega(U_n)$. Since $\mathbf{R}$ is perfectly normal, there exists a sequence $\{O_n\}$ of $\mathbf{R}$-open sets such that $\mathrm{cl}_{\mathbf{R}}(A) = \bigcap\limits_n O_n = \bigcap\limits_n \mathrm{cl}_{\mathbf{R}}(O_n)$. From the claim, $\mathrm{cl}_{\mathbf{R}}(A) - A$ can be enumerated: $\{a_n\}$. Each $a_n \in X_\Omega - A$, so $\exists\ K_n \in \tau_\Omega$: $a_n \in K_n \subset X_\Omega - A$, K_n clopen. Bearing in mind that τ_Ω is finer than the usual topology on $\mathbf{R}$, we then have

$$A = \bigcap_n O_n \ \cap\ \bigcap_n (X_\Omega - K_n) = \bigcap_n \mathrm{cl}_\Omega(O_n) \ \cap\ \bigcap_n (X_\Omega - K_n).$$

The final point is collectionwise normality. But as CH is in force, Jones' lemma implies that X_Ω, being separable and normal, has no uncountable closed discrete subspaces.

[Note: X_Ω is not metacompact (cf. Proposition 10). However, X_Ω is countably paracompact (being perfectly normal).]

Retaining the assumption CH and working with

$$\begin{cases} X_\Omega = \mathbf{N} \cup (\{0\} \times [0, \Omega[) \\ X_\alpha = \mathbf{N} \cup \{(0, \beta) : \beta < \alpha\}, \end{cases}$$

one can employ the foregoing methods and construct an example of a first countable, separable, countably compact, noncompact LCH space (cf. p. 1–10). Recursive techniques can also be used in conjunction with set theoretic hypotheses other than CH to manufacture the same type of example.

A CRH space X is said to be a Moore space if it admits a development.

[Note: A development for X is a sequence $\{\mathcal{U}_n\}$ of open coverings of X such that $\forall\ x \in X : \{\mathrm{st}(x, \mathcal{U}_n)\}$ is a neighborhood basis at x.]

Every Moore space is first countable and perfect. Any first countable X that is expressible as a countable union of closed discrete subspaces X_n is Moore, so, e.g., the Isbell-Mrówka space $\Psi(\mathbf{N})$ is Moore.

FACT Suppose that X is a Moore space—then X is subparacompact.

[Let $\mathcal{O} = \{O_i : i \in I\}$ be an open covering of X—then the claim is that $\mathcal{O}$ has a σ-discrete closed refinement. Fix a development $\{\mathcal{U}_n\}$ for X. Equip I with a well ordering $<$ and put

$$A_{i,n} = X - \left(\mathrm{st}(X - O_i, \mathcal{U}_n) \ \cup\ \bigcup_{j<i} O_j\right) \subset O_i.$$

Each $A_{i,n}$ is closed and their totality $\mathcal{A}$ covers X. Denote by $\mathcal{A}_n$ the collection $\{A_{i,n} : i \in I\}$—then $\mathcal{A}_n$ is discrete, so $\mathcal{A} = \bigcup\limits_n \mathcal{A}_n$ is a σ-discrete closed refinement of $\mathcal{O}$.]

The metrization theorem of Bing says: X is metrizable iff X is a collectionwise normal Moore space. Equivalently: X is metrizable iff X is a paracompact Moore space (cf. Proposition 9).

The Kunen line is not a Moore space. For if it were, then, being collectionwise normal, it would be metrizable, hence paracompact, which it is not. Variant: The Kunen line is not submetacompact, therefore is not subparacompact (cf. the remark following the proof of Proposition 10), proving once again that it is not a Moore space.

Let X be a LCH space. If X is locally connected, normal, and Moore, then X is metrizable (Reed-Zenor). Proof: (1) X Moore $\Rightarrow$ X subparacompact; (2) X locally connected, normal, and subparacompact (hence submetacompact) $\Rightarrow$ X paracompact (via the result of Gruenhage mentioned on p. 1–6). Now cite Bing.

Question: Is every locally compact normal Moore space metrizable? It turns out that this question is undecidable in ZFC.

(1) Under V = L, every locally compact normal Moore space is metrizable.

[Watson[†] proved that under V = L, every normal submetacompact LCH space X is paracompact. This leads at once to the result.]

(2) Under MA $+\neg$ CH, there exist locally compact normal Moore spaces that are not metrizable.

[Many examples are known that illustrate this phenomenon. A particularly simple case in point is that of the rational sequence topology over a CUE-set. By definition, a CUE-set S is an uncountable subset of $\mathbf{R}$ with the property that $\forall\, T \subset S$, there exists a sequence $\{U_n\}$ of open subsets of $\mathbf{R}$ such that $T = S \cap (\bigcap_n U_n)$, i.e., T is a relative G_δ. Assuming MA $+\neg$ CH, it can be shown that every uncountable subset of $\mathbf{R}$ having cardinality $< 2^\omega$ is a CUE-set. This said, let S be any uncountable subset of the irrationals of cardinality $< 2^\omega$. Put $X = (\mathbf{Q} \times \mathbf{Q}) \cup (S \times \{0\})$. Topologize X as follows: Isolate the points of $\mathbf{Q} \times \mathbf{Q}$ and take for the basic neighborhoods of $(s,0) (s \in S)$ the sets $\{(s,0)\} \cup \{(s_m, 1/m) : m \geq n\}$ $(n = 1, 2, \ldots)$, where $\{s_n\}$ is a fixed sequence of rationals converging to s in the usual sense. X is a separable LCH space. It is clear that X is Moore but not metrizable, hence (i) X is perfect but not collectionwise normal and (ii) X is subparacompact but not metacompact (since separable + metacompact $\Rightarrow$ Lindelöf $\Rightarrow$ paracompact). Nevertheless, X is normal. Indeed, given $T \subset S$, it suffices to produce disjoint open sets $U, V \subset X$: $U \supset T$ and $V \supset S - T$. Using the fact that S is a CUE-set, write $T = S \cap (\bigcap_n U_n)$ and $S - T = S \cap (\bigcap_n V_n)$, where $\{U_n\}$ and $\{V_n\}$ are sequences of open subsets of $\mathbf{R}$: $\forall\, n$, $U_n \supset U_{n+1}$ & $V_n \supset V_{n+1}$. Choose open sets

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[†] *Canad. J. Math.* **34** (1982), 1091–1096.

$O_n, P_n \subset X$:

$$\begin{cases} T - V_n \subset O_n \\ (S-T) \cap \overline{O}_n = \emptyset, \end{cases} \qquad \begin{cases} (S-T) - U_n \subset P_n \\ T \cap \overline{P}_n = \emptyset. \end{cases}$$

Then put

$$\begin{cases} U = \bigcup\limits_n (O_n - \bigcup\limits_{m \leq n} \overline{P}_m) \\ V = \bigcup\limits_n (P_n - \bigcup\limits_{m \leq n} \overline{O}_m).] \end{cases}$$

A topological space X is said to be locally metrizable if every point in X has a metrizable neighborhood. If X is paracompact and locally metrizable, then X is metrizable. Proof: Fix a neighborhood finite open covering $\mathcal{U} = \{U_i : i \in I\}$ of X consisting of metrizable U_i and choose a development $\{\mathcal{U}_i(n)\}$ for U_i such that $\forall\, n : \mathcal{U}_i(n+1)$ refines $\mathcal{U}_i(n)$—then the sequence $\{\bigcup\limits_i \mathcal{U}_i(1), \bigcup\limits_i \mathcal{U}_i(2), \ldots\}$ is a development for X.

FACT Suppose that X is submetacompact and locally metrizable—then X is a Moore space.

[Under the stated conditions, every open covering of X has a closed refinement that is neighborhood countable (obvious definition). Construct a σ-closure preserving closed refinement for the latter and thus conclude that X is subparacompact (by the characterization mentioned on p. 1–4). Suppose, then, that X is subparacompact and locally metrizable or, more generally, locally developable in the sense that every $x \in X$ has a neighborhood U_x with a development $\{\mathcal{U}_n(x)\}$. Let $\mathcal{V} = \bigcup\limits_n \mathcal{V}_n$ be a σ-discrete closed refinement of $\{U_x : x \in X\}$. Assign to each $V \in \mathcal{V}_n$ an element $x_V \in X$ for which $V \subset U_{x_V}$, put $U_V = X - (\cup \mathcal{V}_n - V)$, and let $\mathcal{U}_{m,n}(V) = U_V \cap \mathcal{U}_m(x_V)$. The collection $\mathcal{U}_{m,n} = \{U : U \in \mathcal{U}_{m,n}(V)(V \in \mathcal{V}_n)\} \cup \{X - \cup \mathcal{V}_n\}$ is an open covering of X and the sequence $\{\mathcal{U}_{m,n}\}$ is a development for X.]

A topological manifold (or an n-manifold) is a Hausdorff space X for which there exists a nonnegative integer n such that each point of X has a neighborhood that is homeomorphic to an open subset of $\mathbf{R}^n$.

[Note: We shall refer to n as the euclidean dimension of X. Homeomorphic topological manifolds have the same euclidean dimension (cf. p. 19–25).]

Let X be a topological manifold—then X is a LCH space. As such, X is locally connected. The components of X are therefore clopen. Note too that X is locally metrizable.

FACT Let X be a second countable topological manifold of euclidean dimension n. Assume: X is connected—then there exists a surjective local homeomorphism $\mathbf{R}^n \to X$.

PROPOSITION 11 Let X be a topological manifold—then X is metrizable iff X is paracompact.

[Note: Taking into account the results mentioned on p. 1–2, it is also clear that X is metrizable iff each component of X is σ-compact or, equivalently, iff each component of X is second countable.]

A topological manifold is a Moore space iff it is submetacompact.

EXAMPLE (The Long Line) Put $X = [0,\Omega[\times[0,1[$ and order X by stipulating that $(\alpha,x) < (\beta,y)$ if $\alpha < \beta$ or $\alpha = \beta$ and $x < y$. Give X the associated order topology—then the long ray L^+ is $X - \{(0,0)\}$ and the long line L is $X \amalg X/\sim$, $\sim$ meaning that the two origins are identified. Both L and L^+ are normal connected 1-manifolds. Neither L nor L^+ is σ-compact, so neither L nor L^+ is metrizable. Therefore neither L nor L^+ is Moore: Otherwise, Reed-Zenor would imply that they are metrizable. Variant: Moore $\Rightarrow$ perfect, which they are not. So, neither L nor L^+ is submetacompact. Finally, observe that L is not homeomorphic to L^+. Reason: L is countably compact but L^+ is not.

EXAMPLE (The Prüfer Manifold) Assign to each $r \in \mathbf{R}$ a copy of the plane: $\mathbf{R}_r^2 = \mathbf{R}^2 \times \{r\} = \{(a,b,r) \equiv (a,b)_r\}$. Denote by $\overline{L}_r$ the closed lower half plane in $\mathbf{R}_r^2$, L_r the open lower half plane in $\mathbf{R}_r^2$, and $\partial\overline{L}_r$ the horizontal axis in $\mathbf{R}_r^2$. Let H stand for the open upper half plane in $\mathbf{R}^2$. Put $X = H \cup \bigcup\limits_r \overline{L}_r$. Topologize X as follows: Equip H and each L_r with the usual topology and take for the basic neighborhoods of a typical point $(a,0)_r \in \partial\overline{L}_r$ the sets $N(a:r:\epsilon)$, a given such being the union of the open rectangle in $\overline{L}_r$ with corners at $(a \pm \epsilon, 0)_r$ and $(a \pm \epsilon, -\epsilon)_r$ and the open wedge consisting of all points within ϵ of $(r,0)$ in the open sector of H bounded by the lines of slope $1/(a-\epsilon)$ and $1/(a+\epsilon)$ emanating from $(r,0)$. So, e.g., the sequence $(r+1/n,\ 1/n(a+\epsilon))$ converges to $(a+\epsilon,0)_r$ in the topology of X (although it converges to $(r,0)$ in the usual topology). The subspace $H \cup \{(0,0)_r : r \in \mathbf{R}\}$ (which is not locally compact) is homeomorphic to the Niemytzki plane: $\begin{cases} (x,y) \rightarrow (x,y^2) \\ (0,0)_r \rightarrow (r,0) \end{cases}$. X is a connected 2-manifold. Reason: A closed wedge with its apex removed is homeomorphic to a closed rectangle with one side removed. It is clear that X is not separable. Moreover, X is not second countable, hence is not metrizable (and therefore is not paracompact). But X is a Moore space: Let $\mathcal{U}_n$ be the collection comprised of all open disks of radius $1/n$ in H and the L_r together with all the $N(a:r:1/n)$—then $\{\mathcal{U}_n\}$ is a development for X. This remark allows one to infer that X is not normal: Otherwise, Reed-Zenor would imply that X is metrizable. Explicitly, if $\begin{cases} A = \{(0,0)_r : r \text{ rational}\} \\ B = \{(0,0)_r : r \text{ irrational}\} \end{cases}$, then A and B are disjoint closed subsets of X that fail to have disjoint neighborhoods. Since A is countable, this means that X cannot be countably paracompact. However, X is Moore, thus is subparacompact. Still, X is not metacompact. For X is locally separable (being locally euclidean) and locally separable + metacompact $\Rightarrow$ paracompact. Apart from all this, X is contractible and so is simply connected.

[Note: There are two other nonmetrizable, nonnormal, connected 2-manifolds associated with this construction.

(1) Take two disjoint copies of $H \cup \bigcup_r \partial \overline{L}_r$ and identify the corresponding points on the various $\partial \overline{L}_r$. The result is Moore and separable but has an uncountable fundamental group.

(2) Take $H \cup \bigcup_r \partial \overline{L}_r$ and $\forall\ r$ identify $(a, 0)_r$ and $(-a, 0)_r$. The result is Moore and separable but has a trivial fundamental group.]

According to Reed-Zenor, every normal topological Moore manifold is metrizable. What happens if we drop "Moore" but retain perfection? In other words: Is every perfectly normal topological manifold metrizable? It turns out that this question is undecidable in ZFC.

(1) Under MA $+\neg$ CH, every perfectly normal topological manifold is metrizable.

[Lane[†] proved that under MA $+\neg$ CH, every perfectly normal, locally connected LCH space X is paracompact. This leads at once to the result.]

(2) Under CH, there exist perfectly normal topological manifolds that are not metrizable.

[Let $D = \{(x, y) \in \mathbf{R}^2 : -1 < x < 1 \ \&\ 0 < y < 1\}$—then the idea here is to coherently paste Ω copies of $[0, 1[$ to D via a modification of the Kunen technique (cf. p. 1-16). So let $\{I_\alpha : \alpha < \Omega\}$ be a collection of copies of $[0, 1[$ that are unrelated to $\overline{D}$ or to each other. Let $\{x_\alpha : \alpha < \Omega\}$ be an enumeration of $\overline{D} - D$. Put $X_\alpha = D \cup (\bigcup\limits_{\beta < \alpha} I_\beta)$ and $X = \bigcup\limits_{\alpha < \Omega} X_\alpha$. Let $\{C_\alpha : \alpha < \Omega\}$ be an enumeration of the countable subsets of X such that $\forall\ \alpha : C_\alpha \subset X_\alpha$. Define a function $\phi : X \rightarrow \overline{D} : \phi | D = \mathrm{id}_D \ \&\ \phi | I_\alpha = x_\alpha$. We shall now construct by induction on $\alpha < \Omega$ a topology τ_α on X_α subject to:

(a) $\forall\ \alpha : (X_\alpha, \tau_\alpha)$ is homeomorphic to D and $\phi_\alpha = \phi | X_\alpha$ is continuous.

(b) $\forall\ \beta < \alpha : (X_\beta, \tau_\beta)$ is an open dense subspace of (X_α, τ_α).

(c) $\forall\ \gamma \leq \beta < \alpha$: If x_β is a limit point of $\phi(C_\gamma)$ in $\overline{D}$, then every element of I_β is a limit point of C_γ in (X_α, τ_α).

Assign to $D = X_0$ the usual topology. If α is a limit ordinal, take for τ_α the topology on X_α generated by $\bigcup\limits_{\beta < \alpha} \tau_\beta$. Only condition (a) of the induction hypothesis requires verification. This can be dealt with by appealing to a generality: Any topological space expressible as the union of an increasing sequence of open subsets, each of which is homeomorphic to $\mathbf{R}^n$, is itself homeomorphic to $\mathbf{R}^n$ (Brown[‡]). If α is a successor ordinal, say $\alpha = \beta + 1$, then $X_\alpha = X_\beta \cup I_\beta$ and the problem is to define τ_α knowing τ_β.

Write $\mathbf{N} = \coprod\limits_1^\infty N_k : \forall\ k, \#(N_k) = \omega$ and fix a bijection $\iota_k : N_k \rightarrow \mathbf{Q} \cap]-1, 1[$.

Claim: Let $\{U_n\}$ be a sequence of connected open subsets of D and let $\{p_n\}$ be a sequence of distinct points of $D : \forall\ n$,

$$U_n \supset U_{n+1} \ \&\ D \cap \bigcap_n \overline{U}_n = \emptyset, \quad p_n \in U_n.$$

[†] *Proc. Amer. Math. Soc.* **80** (1980), 693–696; see also Balogh-Bennett, *Houston J. Math.* **15** (1989), 153–162.

[‡] *Proc. Amer. Math. Soc.* **12** (1961), 812–814.

Then there exists an embedding $\mu : D \to D$ such that $D - \mu(D)$ is homeomorphic to $[0,1[$ and

(i) $\forall\ k$: Each point of $D - \mu(D)$ is a limit point of $\{\mu(p_n) : n \in N_k\}$;

(ii) $\forall\ n : D - \mu(D)$ is contained in the interior of the closure of $\mu(U_n)$.

[To begin with, there exists a homeomorphism $h : D \to D$ such that $\forall\ n : h(U_n) \supset D_n$ & $h(p_n) \in D_n - D_{n+1}$, where $D_n = \{(x,y) \in D : 0 < y < 1/2n\}$. Choose next a homeomorphism $g : D \to D$ for which the second coordinate of $g(x,y)$ is again y but for which the first coordinate of $g(h(p_n))$ is $\iota_k(n)$ $(n \in N_k, k = 1,2,\ldots)$. Each point of $\{(x,0) : -1 < x < 1\}$ is therefore a limit point of $\{g(h(p_n)) : n \in N_k\}$. Finally, if F is the map with domain $D \cup \{(x,0) : -1 < x < 1\}$ defined by $\begin{cases} F|D = \mathrm{id}_D \\ F(x,0) = (|x|,0) \end{cases}$, then the image $D \cup \{(x,0) : 0 \leq x < 1\}$, when given the quotient topology, is homeomorphic to D via f, say. The embedding $\mu = f \circ g \circ h$ satisfies all the assertions of the claim.]

To apply the claim, we must specify the U_n and the p_n in terms of X_β. Start by letting $U_n = \phi_\beta^{-1}(O_n(x_\beta))$, where $O_n(x_\beta)$ is the intersection of $\overline{D}$ with the open disk of radius $1/n$ centered at x_β. Fix a bijection $\iota : [0,\beta] \to \mathrm{N}$ and choose the $p_n \in U_n$ so that if $\gamma \leq \beta$ and if x_β is a limit point of $\phi(C_\gamma)$ in $\overline{D}$, then $p_n \in C_\gamma \cap U_n$ for all $n \in N_{\iota(\gamma)}$. By assumption, there is a homeomorphism $\eta_\beta : X_\beta \to D$. Use this to transfer the data from X_β to D and determine an embedding $\mu : D \to D$. Put $\mu_\beta = \mu \circ \eta_\beta$, write D as $\mu_\beta(X_\beta) \cup (D - \mu_\beta(X_\beta))$ and let $\nu_\beta : I_\beta \to D - \mu_\beta(X_\beta)$ be a homeomorphism. The pair (μ_β, ν_β) defines a bijection $X_\alpha = X_\beta \cup I_\beta \to D$. Take then for τ_α the topology on X_α that renders this bijection a homeomorphism and thereby complete the induction.

Give $X = \bigcup\limits_{\alpha<\Omega} X_\alpha$ the topology generated by $\bigcup\limits_{\alpha<\Omega} \tau_\alpha$—then X is a connected 2-manifold. It is clear that X is not Lindelöf. Because X is separable (in fact is hereditarily separable), it follows that X is not paracompact, thus is not metrizable. There remains the verification of perfect normality. Let A be a closed subset of X. Fix an $\alpha < \Omega : \overline{C}_\alpha = A$. Choose a sequence $\{O_n\}$ of open subsets of $\overline{D}$ such that $\overline{\phi(C_\alpha)} = \bigcap\limits_n O_n = \bigcap\limits_n \overline{O}_n$. Obviously, $A \subset \phi^{-1}(\overline{\phi(C_\alpha)}) = \bigcap\limits_n \phi^{-1}(O_n) = \bigcap\limits_n \overline{\phi^{-1}(O_n)}$. But thanks to condition (c) of the induction hypothesis, $\phi^{-1}(\overline{\phi(C_\alpha)}) - A$ is contained in X_α. So write $X_\alpha - A = \bigcup\limits_n K_n, K_n$ compact, and let P_n be a relatively compact open subset of $X : K_n \subset P_n \subset \overline{P}_n \subset X - A$. To finish, simply note that $A = \bigcap\limits_n \cdots_n = \bigcap\limits_n \overline{\cdots_n}$, $\cdots_n$ being $\phi^{-1}(O_n) - \overline{P}_n$. Corollary: X is not submetacompact.]

The preceding construction is due to Rudin-Zenor[†]. Rudin[‡] employed similar methods to produce within ZFC an example of a topological manifold that is both normal and separable, yet is not metrizable.

Is every normal topological manifold collectionwise normal? Recall that this question was asked of an arbitrary LCH space X on p. 1–15. Using the combinatorial principle $\diamondsuit^+$, Rudin (ibid.) established the existence of a normal topological manifold that is not collectionwise normal. On the other hand, since the cardinality of a connected topological manifold is 2^{ω}, there are axioms that imply a positive answer but I shall not discuss them here.

[†] *Houston J. Math.* **2** (1976), 129–134.

[‡] *Topology Appl.* **35** (1990), 137–152.

Let X be a topological space. A collection $\{\kappa_i : i \in I\}$ of continuous functions $\kappa_i : X \to [0,1]$ is said to be a partition of unity on X if the supports of the κ_i form a neighborhood finite closed covering of X and for every $x \in X, \sum_i \kappa_i(x) = 1$. If $\mathcal{U} = \{U_i : i \in I\}$ is a covering of X, then a partition of unity $\{\kappa_i : i \in I\}$ on X is said to be subordinate to $\mathcal{U}$ if $\forall\ i : \operatorname{spt} \kappa_i \subset U_i$.

[Note: Given a map $f : X \to \mathbf{R}$, the support of f, written $\operatorname{spt} f$, is the closure of $\{x : f(x) \neq 0\}$.]

A numerable covering of X is a covering that has a subordinated partition of unity. Examples: Suppose that X is Hausdorff—then (1) Every neighborhood finite open covering of a normal X is numerable; (2) Every σ-neighborhood finite open covering of a countably paracompact normal X is numerable; (3) Every point finite open covering of a collectionwise normal X is numerable; (4) Every open covering of a paracompact X is numerable.

[Note: Numerable coverings and their associated partitions of unity allow one to pass from the "local" to the "global" without the necessity of imposing a paracompactness assumption, a point of some importance in, e.g., fibration theory.]

The requirement on the functions determining a numeration can be substantially weakened.

(NU) Suppose given a collection $\{\sigma_i : i \in I\}$ of continuous functions $\sigma_i : X \to [0,1]$ such that $\sum_i \sigma_i(x) = 1$ ($\forall\ x \in X$)—then there exists a collection $\{\rho_i : i \in I\}$ of continuous functions $\rho_i : X \to [0,1]$ such that $\forall\ i \in I : \operatorname{cl}(\rho_i^{-1}(]0,1])) \subset \sigma_i^{-1}(]0,1])$ and (a) $\{\rho_i^{-1}(]0,1]) : i \in I\}$ is neighborhood finite and (b) $\sum_i \rho_i(x) = 1$ ($\forall\ x \in X$).

[Of course, at any particular $x \in X$, the cardinality of the set of $i \in I$ such that $\sigma_i(x) \neq 0$ is $\leq \omega$. Put $\mu = \sup_i \sigma_i$—then μ is strictly positive. Claim: μ is continuous. In fact, $\forall\ \epsilon > 0$, every $x \in X$ has a neighborhood $U : \sigma_i|U < \epsilon$ for all but a finite number of i, thus μ agrees locally with the maximum of finitely many of the σ_i and so μ is continuous. Let $\sigma = \sum_i \max\{0, \sigma_i - \mu/2\}$ and take for ρ_i the normalization $\max\{0, \sigma_i - \mu/2\}/\sigma$.]

Suppose that H is a Hilbert space with orthonormal basis $\{e_i : i \in I\}$. Let X be the unit sphere in H and set $\sigma_i(x) = |\langle x, e_i\rangle|^2 (x \in X)$—then the σ_i satisfy the above assumptions.

PROPOSITION 12 Every numerable open covering $\mathcal{U} = \{U_i : i \in I\}$ of X has a numerable open refinement that is both neighborhood finite and σ-discrete.

[Let $\{\kappa_i : i \in I\}$ be a partition of unity on X subordinate to $\mathcal{U}$. Denote by $\mathcal{F}$ the collection of all nonempty finite subsets of I. Assign to each $F \in \mathcal{F}$ the functions $\begin{cases} m_F = \min \kappa_i & (i \in F) \\ M_F = \max \kappa_i & (i \notin F) \end{cases}$ and put $\mu = \max\limits_F (m_F - M_F)$, which is strictly positive. Write μ_F in place of $m_F - M_F - \mu/2$, σ_F in place of $\max\{0, \mu_F\}$ and set $V_F = \{x : \sigma_F(x) > 0\}$—then $\overline{V}_F \subset \{x : m_F(x) > M_F(x)\} \subset \bigcap\limits_{i \in F} U_i$. The collection $\mathcal{V} = \{V_F : F \in \mathcal{F}\}$ is a neighborhood finite open refinement of $\mathcal{U}$ which is in fact σ-discrete as may be seen by defining $\mathcal{V}_n = \{V_F : \#(F) = n\}$. In this connection, note that $F' \neq F''$ & $\#(F') = \#(F'') \Rightarrow \{x : m_{F'}(x) > M_{F'}(x)\} \cap \{x : m_{F''}(x) > M_{F''}(x)\} = \emptyset$. The numerability of $\mathcal{V}$ follows upon considering the σ_F / σ $(\sigma = \sum\limits_F \sigma_F)$.]

Implicit in the proof of Proposition 12 is the fact that if $\mathcal{U}$ is a numerable open covering of X, then there exists a countable numerable open covering $\mathcal{O} = \{O_n\}$ of X such that $\forall\, n, O_n$ is the disjoint union of open sets each of which is contained in some member of $\mathcal{U}$.

FACT (Domino Principle) Let $\mathcal{U}$ be a numerable open covering of X. Assume:

(D_1) Every open subset of a member of $\mathcal{U}$ is a member of $\mathcal{U}$.

(D_2) The union of each disjoint collection of members of $\mathcal{U}$ is a member of $\mathcal{U}$.

(D_3) The union of each finite collection of members of $\mathcal{U}$ is a member of $\mathcal{U}$.

Conclusion: X is a member of $\mathcal{U}$.

[Work with the O_n introduced above, noting that there is no loss of generality in assuming that $O_n \subset O_{n+1}$. Choose a precise open refinement $\mathcal{P} = \{P_n\}$ of $\mathcal{O}$: $\forall\, n$, $\overline{P}_n \subset P_{n+1}$. Put $Q_n = \begin{cases} P_n & (n = 1, 2) \\ P_n - \overline{P}_{n-2} & (n \geq 3) \end{cases}$ and write $X = \bigcup\limits_1^\infty Q_n = (\bigcup\limits_1^\infty Q_{2n-1}) \cup (\bigcup\limits_1^\infty Q_{2n}) = X_1 \cup X_2$.]

Let X be a topological space—then by $\begin{cases} C(X) \\ C(X, [0,1]) \end{cases}$ we shall understand the set of all continuous functions $\begin{cases} f : X \rightarrow \mathbf{R} \\ f : X \rightarrow [0,1] \end{cases}$. Bear in mind that $C(X)$ can consist of constants alone, even if X is regular Hausdorff.

A zero set in X is a set of the form $Z(f) = \{x : f(x) = 0\}$, where $f \in C(X)$. The complement of a zero set is a cozero set. Since $Z(f) = Z(\min\{1, |f|\})$, $C(X)$ and $C(X, [0,1])$ determine the same collection of zero sets. All sets of the form $\begin{cases} \{x : f(x) \geq 0\} \\ \{x : f(x) \leq 0\} \end{cases}$ $(f \in C(X))$ are zero sets and all sets of the form $\begin{cases} \{x : f(x) > 0\} \\ \{x : f(x) < 0\} \end{cases}$ $(f \in C(X))$ are cozero sets. The collection of zero sets in X is closed under the formation of finite unions and countable intersections and the collection of cozero sets in X is closed under the formation of countable unions and finite intersections. The union of a neighborhood finite collection of cozero sets is a cozero set. On the other hand, the union of a neighborhood finite collection of zero sets need not be a zero set. But this will be the case if each zero set in the collection is contained in a cozero set, the totality of which is neighborhood finite.

[Note: Suppose that X is Hausdorff—then X is completely regular iff the collection of cozero sets in X is a basis for X. Every compact G_δ in a CRH space is a zero set. If X is normal, then $\left\{\begin{array}{l}\text{closed } G_\delta = \text{zero set}\\ \text{open } F_\sigma = \text{cozero set}\end{array}\right.$, so if X is perfectly normal, then $\left\{\begin{array}{l}\text{closed set} = \text{zero set}\\ \text{open set} = \text{cozero set}\end{array}\right.$.]

A $\left\{\begin{array}{l}\text{zero set}\\ \text{cozero set}\end{array}\right.$ covering of X is a covering consisting of $\left\{\begin{array}{l}\text{zero sets}\\ \text{cozero sets}\end{array}\right.$. The numerable coverings of X are those coverings that have a neighborhood finite cozero set refinement. Example: Every countable cozero set covering $\mathcal{U} = \{U_n\}$ of X is numerable. Proof: Choose $f_n \in C(X,[0,1])$: $U_n = f_n^{-1}(]0,1])$, put $\phi_n = 1/2^n \bullet f_n/1 + f_n$ & $\phi = \sum_n \phi_n$, let $\sigma_n = \phi_n/\phi$, and apply NU.

[Note: Every countable cozero set covering $\mathcal{U} = \{U_n\}$ of X has a countable star finite cozero set refinement. Proof: Choose $f_n \in C(X,[0,1]) : U_n = f_n^{-1}(]0,1])$, put $f = \sum_n 2^{-n} f_n$ and define

$$V_{m,n} = f_n^{-1}(]0,1]) \cap (f^{-1}(]\frac{1}{m+1},1]) - f^{-1}([\frac{1}{m-1},1])) \quad (1 \leq n \leq m),$$

with the obvious understanding if $m = 1$—then the collection $\{V_{m,n}\}$ has the properties in question.]

LEMMA Let $\mathcal{U} = \{U_i : i \in I\}$ be a neighborhood finite cozero set covering of X—then there exists a zero set covering $\mathcal{Z} = \{Z_i : i \in I\}$ and a cozero set covering $\mathcal{V} = \{V_i : i \in I\}$ such that $\forall\, i : Z_i \subset V_i \subset \overline{V}_i \subset U_i$.

[Choose a partition of unity $\{\kappa_i : i \in I\}$ on X subordinate to $\mathcal{U}$. Put $V_i = \kappa_i^{-1}(]0,1])$ and take for Z_i the zero set of the function $\max_i \kappa_i - \kappa_i$.]

Let $\mathcal{U} = \{U_i : i \in I\}$ be a neighborhood finite cozero set covering of X; let $\mathcal{Z} = \{Z_i : i \in I\}$ and $\mathcal{V} = \{V_i : i \in I\}$ be as in the lemma. Denote by $\mathcal{F}$ the collection of all nonempty finite subsets of I. Assign to each $F \in \mathcal{F} : W_F = \bigcap_{i \in F} V_i \cap (X - \bigcup_{i \notin F} Z_i)$. The collection $\mathcal{W} = \{W_F : F \in \mathcal{F}\}$ is a neighborhood finite cozero set covering of X such that $\forall\, i : \mathrm{st}(Z_i, \mathcal{W}) \subset V_i$. Therefore $\{\mathrm{st}(x, \mathcal{W}) : x \in X\}$ refines $\mathcal{V}$, hence $\mathcal{U}$. Now repeat the entire procedure with $\mathcal{W}$ playing the role of $\mathcal{U}$. The upshot is the following conclusion.

PROPOSITION 13 Every numerable open covering of X has a numerable open star refinement that is neighborhood finite.

FACT Let $\mathcal{U} = \{U_i : i \in I\}$ be an open covering of X—then $\mathcal{U}$ is numerable iff there exists a metric space Y, an open covering $\mathcal{V}$ of Y, and a continuous function $f : X \rightarrow Y$ such that $f^{-1}(\mathcal{V})$ refines $\mathcal{U}$.

[The condition is clearly sufficient. As for the necessity, let $\{\kappa_i : i \in I\}$ be a partition of unity on X subordinate to $\mathcal{U}$. Let Y be the subset of $[0,1]^I$ comprised of those $y = \{y_i : i \in I\} : \sum\limits_i y_i = 1$. The prescription $d(y', y'') = \sum\limits_i |y_i' - y_i''|$ is a metric on Y. Define a continuous function $f : X \rightarrow Y$ by sending x to $\{\kappa_i(x) : i \in I\}$. Consider the collection $\mathcal{V} = \{V_i : i \in I\}$, where $V_i = \{y : y_i > 0\}$.]

Application: Let $\mathcal{U} = \{U_i : i \in I\}$ be an open covering of X—then $\mathcal{U}$ is numerable iff there exists a numerable open covering $\mathcal{O} = \{O_i : i \in I\}$ of cr$X$ such that $\forall\ i : \text{cr}^{-1}(O_i) \subset U_i$.

EXAMPLE Let G be a topological group; let U be a neighborhood of the identity in G—then the open covering $\{xU : x \in G\}$ is numerable.

Suppose given a set X and a collection $\{X_i : i \in I\}$ of topological spaces X_i.

(FT) Let $\{f_i : i \in I\}$ be a collection of functions $f_i : X_i \rightarrow X$—then the final topology on X determined by the f_i is the largest topology for which each f_i is continuous. The final topology is characterized by the property that if Y is a topological space and if $f : X \rightarrow Y$ is a function, then f is continuous iff $\forall\ i$ the composition $f \circ f_i : X_i \rightarrow Y$ is continuous.

(IT) Let $\{f_i : i \in I\}$ be a collection of functions $f_i : X \rightarrow X_i$—then the initial topology on X determined by the f_i is the smallest topology for which each f_i is continuous. The initial topology is characterized by the property that if Y is a topological space and if $f : Y \rightarrow X$ is a function, then f is continuous iff $\forall\ i$ the composition $f_i \circ f : Y \rightarrow X_i$ is continuous.

For example, in the category of topological spaces, coproducts carry the final topology and products carry the initial topology. The discrete topology on a set X is the final topology determined by the function $\emptyset \rightarrow X$ and the indiscrete topology on a set X is the initial topology determined by the function $X \rightarrow *$. If X is a topological space and if $f : X \rightarrow Y$ is a surjection, then the final topology on Y determined by f is the quotient topology, while if Y is a topological space and if $f : X \rightarrow Y$ is an injection, then the initial topology on X determined by f is the induced topology.

EXAMPLE Let E be a vector space over $\mathbf{R}$—then the finite topology on E is the final topology determined by the inclusions $F \rightarrow E$, where F is a finite dimensional linear subspace of E endowed with its natural euclidean topology. E, in the finite topology, is a perfectly normal paracompact Hausdorff space. Scalar multiplication $\mathbf{R} \times E \rightarrow E$ is jointly continuous; vector addition $E \times E \rightarrow E$ is separately continuous but jointly continuous iff $\dim E \leq \omega$. For a concrete illustration, put $\mathbf{R}^\infty = \bigcup\limits_0^\infty \mathbf{R}^n$, where $\{0\} = \mathbf{R}^0 \subset \mathbf{R}^1 \subset \cdots$. The elements of $\mathbf{R}^\infty$ are therefore the real valued sequences having a finite number of nonzero values. Besides the finite topology, one can also give $\mathbf{R}^\infty$ the inherited product topology τ_P or any of the topologies $\tau_p (1 \leq p \leq \infty)$ derived from the usual ℓ^p norm. It is clear that $\tau_P \subset \tau_{p'} \subset \tau_{p''}$ $(1 \leq p'' < p' \leq \infty)$, each inclusion being proper. Moreover, τ_1 is strictly smaller than the finite topology. To see this, let $U = \{x \in \mathbf{R}^\infty : \forall\ i, |x_i| < 2^{-i}\}$—then U is a neighborhood of the origin in the finite topology but U is not open in τ_1. These considerations exhibit uncountably many distinct topologies on $\mathbf{R}^\infty$. Nevertheless, under each of them, $\mathbf{R}^\infty$ is contractible, so they all lead to the same homotopy type.

[Note: The finite topology on $\mathbf{R}^\infty$ is not first countable, thus is not metrizable.]

PROPOSITION 14 Suppose that X is Hausdorff—then X is completely regular iff X has the initial topology determined by the elements of $C(X)$ (or, equivalently, $C(X, [0,1])$.

[Note: Therefore, if τ' and τ'' are two completely regular topologies on X, then $\tau' = \tau''$ iff, in obvious notation, $C'(X) = C''(X)$.]

When constructing the initial topology, it is not necessary to work with functions whose domain is all of X.

Suppose given a set X, a collection $\{U_i : i \in I\}$ of subsets $U_i \subset X$, and a collection $\{X_i : i \in I\}$ of topological spaces X_i. Let $\{f_i : i \in I\}$ be a collection of functions $f_i : U_i \rightarrow X_i$—then the initial topology on X determined by the f_i is the smallest topology for which each U_i is open and each f_i is continuous. The initial topology is characterized by the property that if Y is a topological space and if $f : Y \rightarrow X$ is a function, then f is continuous iff $\forall\ i$ the composition $f^{-1}(U_i) \xrightarrow{f} U_i \xrightarrow{f_i} X_i$ is continuous.

EXAMPLE Let X and Y be nonempty topological spaces—then the join $X * Y$ is the quotient of $X \times Y \times [0,1]$ with respect to the relations $\begin{cases} (x, y', 0) \sim (x, y'', 0) \\ (x', y, 1) \sim (x'', y, 1) \end{cases}$. Conventionally $\begin{cases} X * \emptyset = X \\ \emptyset * Y = Y \end{cases}$, so $*$ is a functor $\mathbf{TOP} \times \mathbf{TOP} \rightarrow \mathbf{TOP}$. The projection $p : \begin{cases} X \times Y \times [0,1] \rightarrow X * Y \\ (x, y, t) \rightarrow [x, y, t] \end{cases}$ sends $X \times Y \times \{0\}$ (or $X \times Y \times \{1\}$) onto a closed subspace homeomorphic to X (or Y). Consider now $X * Y$ as merely a set. Let $t : X * Y \rightarrow [0,1]$ be the function $[x, y, t] \rightarrow t$; let $\begin{cases} x : t^{-1}([0,1[) \rightarrow X \\ y : t^{-1}(]0,1]) \rightarrow Y \end{cases}$ be the functions $\begin{cases} [x, y, t] \rightarrow x \\ [x, y, t] \rightarrow y \end{cases}$—then the coarse join $X *_c Y$ is $X * Y$ equipped with the initial topology determined by t, x, and y. The identity map $X * Y \rightarrow X *_c Y$ is continuous; it is a homeomorphism if X and Y are compact Hausdorff but not in general. The coarse join $X *_c Y$ of Hausdorff X and Y is Hausdorff, thus so is $X * Y$. The join $X * Y$ of path connected X and Y is path connected, thus so is $X *_c Y$. Examples: (1)

The cone ΓX of X is the join of X and a single point; (2) The suspension ΣX of X is the join of X and a pair of points. There are also coarse versions of both the cone and the suspension, say $\begin{cases} \Gamma_c X \\ \Sigma_c X \end{cases}$. Complete the picture by setting $\begin{cases} X *_c \emptyset = X \\ \emptyset *_c Y = Y \end{cases}$.

[Note: Analogous definitions can be made in the pointed category **TOP**$_*$.]

FACT Let X and Y be topological spaces—then the identity map $X * Y \to X *_c Y$ is a homotopy equivalence.

[A homotopy inverse $X *_c Y \to X * Y$ is given by $[x, y, t] \to \begin{cases} [x, y, 0] \ (0 \leq t \leq 1/3) \\ [x, y, 3t - 1] \ (1/3 \leq t \leq 2/3) \\ [x, y, 1] \ (2/3 \leq t \leq 1) \end{cases}$. Since the homotopy type of $X * Y$ depends only on the homotopy types of X and Y and since the coarse join is associative, it follows that the join is associative up to homotopy equivalence.]

EXAMPLE (Star Construction) The cone ΓX of a topological space X is contractible and there is an embedding $X \to \Gamma X$. However, one drawback to the functor Γ : **TOP** $\to$ **TOP** is that it does not preserve embeddings or finite products. Another drawback is that while Γ does preserve **HAUS**, within **HAUS** it need not preserve complete regularity (consider ΓX, where X is the Tychonoff plank). The star construction eliminates these difficulties. Thus put $\emptyset^* = \emptyset$ and for $X \neq \emptyset$, denote by X^* the set of all right continuous step functions $f : [0, 1[\to X$. So, $f \in X^*$ iff there is a partition $a_0 = 0 < a_1 < \cdots < a_n < 1 = a_{n+1}$ of $[0, 1[$ such that f is constant on $[a_i, a_{i+1}[$ $(i = 0, 1, \ldots, n)$. There is an injection $i : X \to X^*$ that sends $x \in X$ to $i(x) \in X^*$, the constant step function with value x. Given $a, b : 0 \leq a < b < 1$, U an open subset of X, and $\epsilon > 0$, let $O(a, b, U, \epsilon)$ be the set of $f \in X^*$ such that f is constant on $[a, b[$, U is a neighborhood of $f(a)$, and the Lebesgue measure of $\{t \in [a, b[: f(t) \notin U\}$ is $< \epsilon$. Topologize X^* by taking the $O(a, b, U, \epsilon)$ as a subbasis—then $i : X \to X^*$ is an embedding, which is closed if X is Hausdorff. The assignment $X \to X^*$ defines a functor **TOP** $\to$ **TOP** that preserves embeddings and finite products. It restricts to a functor **HAUS** $\to$ **HAUS** that respects complete regularity.

Claim: Suppose that X is not empty—then X^* is contractible and has a basis of contractible open sets.

[Fix $f_0 \in X^*$ and define $H : X^* \times [0, 1] \to X^*$ by $H(f, T)(t) = \begin{cases} f_0(t) & (0 \leq t < T) \\ f(t) & (T \leq t < 1) \end{cases}$.]

An expanding sequence of topological spaces is a system consisting of a sequence of topological spaces X^n linked by embeddings $f^{n,n+1}$: $X^n \to X^{n+1}$. Denote by X^∞ the colimit in **TOP** associated with this data—then for every n there is an arrow $f^{n,\infty}$: $X^n \to X^\infty$ and the topology on X^∞ is the final topology determined by the $f^{n,\infty}$. Each $f^{n,\infty}$ is an embedding and $X^\infty = \bigcup\limits_n f^{n,\infty}(X^n)$. One can therefore identify X^n with $f^{n,\infty}(X^n)$ and regard the $f^{n,n+1}$ as inclusions.

[Note: If all the $f^{n,n+1}$ are open (closed) embeddings, then the same holds for all the $f^{n,\infty}$.]

If all the X^n are T_1, then X^∞ is T_1. If all the X^n are Hausdorff, then X^∞ need not be Hausdorff but there are conditions that lead to this conclusion.

(A) If all the X^n are LCH spaces, then X^∞ is a Hausdorff space.

[Let $x, y \in X^\infty : x \neq y$. Fix an index n_0 such that $x, y \in X^{n_0}$. Choose open relatively compact subsets $U_{n_0}, V_{n_0} \subset X^{n_0} : x \in U_{n_0}$ & $y \in V_{n_0}$, with $\overline{U}_{n_0} \cap \overline{V}_{n_0} = \emptyset$. Since $\overline{U}_{n_0}$ and $\overline{V}_{n_0}$ are compact disjoint subsets of X^{n_0+1}, there exist open relatively compact subsets $U_{n_0+1}, V_{n_0+1} \subset X^{n_0+1} : U_{n_0} \subset U_{n_0+1}$ & $V_{n_0} \subset V_{n_0+1}$, with $\overline{U}_{n_0+1} \cap \overline{V}_{n_0+1} = \emptyset$. Iterate the procedure to build disjoint neighborhoods $U = \bigcup\limits_{n \geq n_0} U_n$ and $V = \bigcup\limits_{n \geq n_0} V_n$ of x and y in X^∞.]

(B) Suppose that all the X^n are Hausdorff. Assume: $\forall\, n, X^n$ is a neighborhood retract of X^{n+1}—then X^∞ is Hausdorff.

(C) If all the X^n are normal (normal and countably paracompact, perfectly normal, collectionwise normal, paracompact) Hausdorff spaces and if $\forall\, n$, X^n is a closed subspace of X^{n+1}, then X^∞ is a normal (normal and countably paracompact, perfectly normal, collectionwise normal, paracompact) Hausdorff space.

[The closure preserving closed covering $\{X^n\}$ is absolute, so the generalities on p. 5–4 can be applied.]

LEMMA Given an expanding sequence of T_1 spaces, let $\phi : K \rightarrow X^\infty$ be a continuous function such that $\phi(K)$ is a compact subset of X^∞—then there exists an index n and a continuous function $\phi^n : K \rightarrow X^n$ such that $\phi = f^{n,\infty} \circ \phi^n$.

EXAMPLE Working in the plane, fix a countable dense subset $S = \{s_n\}$ of $\{(x,y) : x = 0\}$. Put $X^n = \{(x,y) : x > 0\} \cup \{s_0, \ldots, s_n\}$ and let $f^{n,n+1} : X^n \rightarrow X^{n+1}$ be the inclusion—then X^∞ is Hausdorff but not regular.

EXAMPLE (Marciszewski Space) Topologize the set $[0,2]$ by isolating the points in $]0,2[$, basic neighborhoods of 0 or 2 being the usual ones. Call the resulting space X_0. Given $n > 0$, topologize the set $]0,2[\times[0,1]$ by isolating the points of $]0,2[\times]0,1]$ along with the point $(1,0)$, basic neighborhoods of $(t,0)$ $(0 < t < 1$ or $1 < t < 2)$ being the subsets of L_n that contain $(t,0)$ and have a finite complement, where L_n is the line segment joining $(t,0)$ and $(t+1-1/n,1)$ $(0 < t < 1)$ or $(t,0)$ and $(t-1+1/n,1)$ $(1 < t < 2)$. Call the resulting space X_n. Form $X_0 \amalg X_1 \amalg \cdots \amalg X_n$ and let X^n be the quotient obtained by identifying points in $]0,2[$. Each X^n is Hausdorff and there is an embedding $f^{n,n+1} : X^n \rightarrow X^{n+1}$. But X^∞ is not Hausdorff.

FACT Suppose that $\begin{cases} X^0 \subset X^1 \subset \cdots \\ Y^0 \subset Y^1 \subset \cdots \end{cases}$ are expanding sequences of LCH spaces—then $X^\infty \times Y^\infty = \operatorname{colim}(X^n \times Y^n)$.

Let X be a topological space—then a filtration on X is a sequence $X^0, X^1, \ldots$ of subspaces of X such that $\forall\, n : X^n \subset X^{n+1}$. Here, one does not require that $\bigcup\limits_n X^n = X$. A filtered space $\mathbf{X}$ is a topological space X equipped with a filtration $\{X^n\}$. A filtered map $\mathbf{f} : \mathbf{X} \to \mathbf{Y}$ of filtered spaces is a continuous function $f : X \to Y$ such that $\forall\, n : f(X^n) \subset Y^n$. Notation: $\mathbf{f} \in \mathbf{C}(\mathbf{X}, \mathbf{Y})$. **FILSP** is the category whose objects are the filtered spaces and whose morphisms are the filtered maps. **FILSP** is a symmetric monoidal category: Take $\mathbf{X} \otimes \mathbf{Y}$ to be $X \times Y$ supplied with the filtration $n \to \bigcup\limits_{p+q=n} X^p \times Y^q$, let e be the one point space filtered by specifying that the initial term is $\neq \emptyset$, and make the obvious choice for T. There is a notion of homotopy in **FILSP**. Write $\mathbf{I}$ for $I = [0,1]$ endowed with its skeletal filtration, i.e., $I^0 = \{0,1\}$, $I^n = [0,1]$ $(n \geq 1)$—then filtered maps $\mathbf{f}, \mathbf{g} : \mathbf{X} \to \mathbf{Y}$ are said to be filter homotopic if there exists a filtered map $\mathbf{H} : \mathbf{X} \otimes \mathbf{I} \to \mathbf{Y}$ such that $\begin{cases} H(x,0) = f(x) \\ H(x,1) = g(x) \end{cases} (x \in X)$.

Geometric realization may be viewed as a functor $|?| :$ **SISET** $\to$ **FILSP** via consideration of skeletons. To go the other way, equip Δ^n with its skeletal filtration and let $\mathbf{\Delta}^n$ be the associated filtered space. Given a filtered space $\mathbf{X}$, write $\mathbf{sin}\,\mathbf{X}$ for the simplicial set defined by $\mathbf{sin}\,\mathbf{X}([n]) = \mathbf{sin}_n \mathbf{X} = \mathbf{C}(\mathbf{\Delta}^n, \mathbf{X})$—then the assignment $\mathbf{X} \to \mathbf{sin}\,\mathbf{X}$ is a functor **FILSP** $\to$ **SISET** and $(|?|, \mathbf{sin})$ is an adjoint pair.

If $\mathbf{C}$ is a full subcategory of **TOP** (**HAUS**) and if X is a topological space (Hausdorff topological space), then X is an object in the monocoreflective hull of $\mathbf{C}$ in **TOP** (**HAUS**) iff there exists a set $\{X_i\} \subset \operatorname{Ob} \mathbf{C}$ and an extremal epimorphism $f : \coprod\limits_i X_i \to X$ (cf. p. 0–21 ff.). Example: The monocoreflective hull in **TOP** of the full subcategory of **TOP** whose objects are the locally connected, connected spaces is the category of locally connected spaces.

[Note: The categorical opposite of "epireflective" is "monocoreflective".]

EXAMPLE (A Spaces) The monocoreflective hull in **TOP** of $[0,1]/[0,1[$ is the category of A spaces.

EXAMPLE (Sequential Spaces) A topological space X is said to be sequential provided that a subset U of X is open iff every sequence converging to a point of U is eventually in U. Every first countable space is sequential. On the other hand, a compact Hausdorff space need not be sequential (consider $[0,\Omega]$). Example: The one point compactification of the Isbell-Mrówka space $\Psi(\mathbf{N})$ is sequential but there is no sequence in $\mathbf{N}$ converging to $\infty \in \overline{\mathbf{N}}$. If **SEQ** is the full, isomorphism closed subcategory of **TOP** whose objects are the sequential spaces, then **SEQ** is closed under the formation in **TOP** of coproducts and quotients. Therefore **SEQ** is a monocoreflective subcategory of **TOP** (cf. p. 0-21), hence is complete and cocomplete. The coreflector sends X to its sequential modification sX. Topologically, sX is X equipped with the final topology determined by the $\phi \in C(\mathbf{N}_\infty, X)$, where $\mathbf{N}_\infty$ is the one point compactification of $\mathbf{N}$ (discrete topology). The monocoreflective hull in **TOP** of $\mathbf{N}_\infty$ is **SEQ**, so a topological space is sequential iff it is a quotient of a first countable space. **SEQ** is cartesian closed: $C(s(X \times Y), Z) \approx C(X, Z^Y)$. Here, $s(X \times Y)$ is the product in **SEQ** (calculate the product in **TOP** and apply s). As for the exponential object Z^Y, given any open subset $P \subset Z$ and any continuous function $\phi : \mathbf{N}_\infty \to Y$, put $O(\phi, P) = \{g \in C(Y,Z) : g(\phi(\mathbf{N}_\infty)) \subset P\}$ and call $C_s(Y,Z)$ the result of topologizing $C(Y,Z)$ by letting the $O(\phi,P)$ be a subbasis—then $Z^Y = sC_s(Y,Z)$.

[Note: Every CW complex is sequential.]

A Hausdorff space X is said to be compactly generated provided that a subset U of X is open iff $U \cap K$ is open in K for every compact subset K of X. Examples: (1) Every LCH space is compactly generated; (2) Every first countable Hausdorff space is compactly generated; (3) The product $\mathbf{R}^\kappa$, $\kappa > \omega$, is not compactly generated. A Hausdorff space is compactly generated iff it can be represented as the quotient of a LCH space. Open subspaces and closed subspaces of a compactly generated Hausdorff space are compactly generated, although this is not the case for arbitrary subspaces (consider $\mathbf{N} \cup \{p\} \subset \beta\mathbf{N}$, where $p \in \beta\mathbf{N} - \mathbf{N}$). However, Arhangel'skii[†] has shown that if X is a Hausdorff space, then X and all its subspaces are compactly generated iff for every $A \subset X$ and each $x \in \overline{A}$ there exists a sequence $\{x_n\} \subset A : \lim x_n = x$. The product $X \times Y$ of two compactly generated Hausdorff spaces may fail to be compactly generated (consider $X = \mathbf{R} - \{1/2, 1/3, \ldots\}$ and $Y = \mathbf{R}/\mathbf{N}$) but this will be true if one of the factors is a LCH space or if both factors are first countable.

EXAMPLE (Sequential Spaces) A Hausdorff sequential space is compactly generated. In fact, a Hausdorff space is sequential provided that a subset U of X is open iff $U \cap K$ is open in K for every second countable compact subset K of X.

EXAMPLE Equip $\mathbf{R}^\infty$ with the finite topology and let $H(\mathbf{R}^\infty)$ be its homeomorphism group.

[†] *Czech. Math. J.* **18** (1968), 392–395.

Give $H(\mathbf{R}^\infty)$ the compact open topology—then $H(\mathbf{R}^\infty)$ is a perfectly normal paracompact Hausdorff space. But $H(\mathbf{R}^\infty)$ is not compactly generated.

[The set of all linear homeomorphisms $\mathbf{R}^\infty \to \mathbf{R}^\infty$ is a closed subspace of $H(\mathbf{R}^\infty)$. Show that it is not compactly generated. Incidentally, $H(\mathbf{R}^\infty)$ is contractible.]

For certain purposes of algebraic topology, it is desirable to single out a full, isomorphism closed subcategory of **TOP**, small enough to be "convenient" but large enough to be stable for the "standard" constructions. A popular candidate is the category **CGH** of compactly generated Hausdorff spaces (Steenrod[†]). Since **CGH** is closed under the formation in **HAUS** of coproducts and quotients, **CGH** is a monocoreflective subcategory of **HAUS** (cf. p. 0–21). As such, it is complete and cocomplete. The coreflector sends X to its compactly generated modification kX. Topologically, kX is X equipped with the final topology determined by the inclusions $K \to X$, K running through the compact subsets of X. The identity map $kX \to X$ is continuous and induces isomorphisms of homotopy and singular homology and cohomology groups. If X and Y are compactly generated, then their product in **CGH** is $X \times_k Y \equiv k(X \times Y)$. Each of the functors $- \times_k Y : \mathbf{CGH} \to \mathbf{CGH}$ has a right adjoint $Z \to Z^Y$, the exponential object Z^Y being $kC(Y, Z)$, where $C(Y, Z)$ carries the compact open topology. So one of the advantages of **CGH** is that it is cartesian closed. Another advantage is that if $\begin{cases} X, X' \\ Y, Y' \end{cases}$ are in **CGH** and if $\begin{cases} f : X \to X' \\ g : Y \to Y' \end{cases}$ are quotient, then $f \times_k g : X \times_k Y \to X' \times_k Y'$ is quotient. But there are shortcomings as well. Item: The forgetful functor $\mathbf{CGH} \to \mathbf{TOP}$ does not preserve colimits. For let A be a compactly generated subspace of X and consider the pushout square

$$\begin{tikzcd} A \arrow[r] \arrow[d] & * \arrow[d] \\ X \arrow[r] & P \end{tikzcd}$$

in **CGH**—then $P = h(X/A)$, the maximal Hausdorff quotient of the ordinary quotient computed in **TOP**. To appreciate the point, let $X = [0, 1]$, $A = [0, 1[$—then $[0, 1]/[0, 1[$ is not Hausdorff and $h([0, 1]/[0, 1[)$ is a singleton. Finally, it is clear that **CGH** is the monocoreflective hull in **HAUS** of the category of compact Hausdorff spaces.

$\mathbf{CGH}_*$, the category of pointed compactly generated Hausdorff spaces, is a closed category: Take $X \otimes Y$ to be the smash product $X \#_k Y$ (cf. p. 3–28) and let e be $\mathbf{S}^0$. Here, the internal hom functor sends (X, Y) to the closed subspace of $kC(X, Y)$ consisting of the base point preserving continuous functions.

FACT Let X be a CRH space. Suppose that there exists a sequence $\{\mathcal{U}_n\}$ of open coverings of X such that $\forall\, x \in X : K_x \equiv \bigcap_n \operatorname{st}(x, \mathcal{U}_n)$ is compact and $\{\operatorname{st}(x, \mathcal{U}_n)\}$ is a neighborhood basis at K_x (i.e., any open U containing K_x contains some $\text{st}(x, \mathcal{U}_n)$)—then X is compactly generated. Example: Every Moore space is compactly generated.

[†] *Michigan Math. J.* **14** (1967), 133–152.

[Note: Jiang[†] has shown that any CRH space X realizing this assumption is necessarily submetacompact.]

In practice, it can be troublesome to prove that a given space is Hausdorff and while this is something which is nice to know, there are situations when it is irrelevant. We shall therefore enlarge **CGH** to its counterpart in **TOP**, the category **CG** of compactly generated spaces (Vogt[‡]), by passing to the monocoreflective hull in **TOP** of the category of compact Hausdorff spaces. It is thus immediate that a topological space is compactly generated iff it can be represented as the quotient of a LCH space. Consequently, if X is a topological space, then X is compactly generated provided that a subset U of X is open iff $\phi^{-1}(U)$ is open in K for every $\phi \in C(K, X)$, K any compact Hausdorff space. What has been said above in the Hausdorff case is now applicable in general, the main difference being that the forgetful functor $\mathbf{CG} \rightarrow \mathbf{TOP}$ preserves colimits. Also, like **CGH**, **CG** is cartesian closed: $C(X \times_k Y, Z) \approx C(X, Z^Y)$. Of course, $X \times_k Y \equiv k(X \times Y)$ and the exponential object Z^Y is defined as follows. Given any open subset $P \subset Z$ and any continuous function $\phi : K \rightarrow Y$, where K is a compact Hausdorff space, put $O(\phi, P) = \{g \in C(Y, Z) : g(\phi(K)) \subset P\}$ and call $C_k(Y, Z)$ the result of topologizing $C(Y, Z)$ by letting the $O(\phi, P)$ be a subbasis—then $Z^Y = kC_k(Y, Z)$. Example: A sequential space is compactly generated.

[Note: If X and Y are compactly generated and if $f : X \rightarrow Y$ is a continuous injection, then f is an extremal monomorphism iff the arrow $X \rightarrow kf(X)$ is a homeomorphism, where $f(X)$ has the induced topology. Therefore an extremal monomorphism in **CG** need not be an embedding (= extremal monomorphism in **TOP**). Extremal monomorphisms in **CG** are regular. Call them CG embeddings.]

EXAMPLE Partition $[-1,1]$ by writing $[-1,1] = \{-1\} \cup \bigcup\limits_{0 \leq x < 1} \{x, -x\} \cup \{1\}$. Let X be the associated quotient space—then X is compactly generated (in fact, first countable). Moreover, X is compact and T_1 but not Hausdorff; X is also path connected.

FACT Let X and Y be compactly generated—then the projections $\begin{cases} X \times_k Y \rightarrow X \\ X \times_k Y \rightarrow Y \end{cases}$ are open maps.

[†] *Topology Proc.* **11** (1986), 309–316.

[‡] *Arch. Math.* **22** (1971), 545–555; see also Wyler, *General Topology Appl.* **3** (1973), 225–242.

Given any class $\mathcal{K}$ of compact spaces containing at least one nonempty space, denote by $\mathbf{M}$ the monocoreflective hull of $\mathcal{K}$ in $\mathbf{TOP}$ and let $R : \mathbf{TOP} \rightarrow \mathbf{M}$ be the associated coreflector. If X is a topological space, then a subset U of RX is open provided that $\phi^{-1}(U)$ is open in K for every $\phi \in C(K,X)$, K any element of $\mathcal{K}$. Write $\mathbf{\Delta}$-$\mathbf{K}$ for the full, isomorphism closed subcategory of $\mathbf{TOP}$ whose objects are those X which are Δ-separated by $\mathcal{K}$, i.e., such that $\Delta_X \equiv \{(x,x) : x \in X\}$ is closed in $R(X \times X)$—then $\mathbf{\Delta}$-$\mathbf{K}$ is closed under the formation in $\mathbf{TOP}$ of products and embeddings. Therefore $\mathbf{\Delta}$-$\mathbf{K}$ is an epireflective subcategory of $\mathbf{TOP}$ (cf. p. 0–21). Examples: (1) Take for $\mathcal{K}$ the class of all finite indiscrete spaces—then an X in $\mathbf{TOP}$ is Δ-separated by $\mathcal{K}$ iff it is T_0; (2) Take for $\mathcal{K}$ the class of all finite spaces—then an X in $\mathbf{TOP}$ is Δ-separated by $\mathcal{K}$ iff it is T_1.

[Note: Recall that a topological space X is Hausdorff iff its diagonal is closed in $X \times X$ (product topology).]

EXAMPLE (Sequential Spaces) Let X be a topological space—then every sequence in X has at most one limit iff Δ_X is sequentially closed in $X \times X$, i.e., iff X is Δ-separated by $\mathcal{K} = \{\mathbf{N}_\infty\}$. When this is so, X must be T_1 and if X is first countable, then X must be Hausdorff.

[Note: Recall that a topological space X is Hausdorff iff every net in X has at most one limit.]

If K is a compact space, then for any $\phi \in C(K,X)$, $\phi(K)$ is a compact subset of X. In general, $\phi(K)$ is neither closed nor Hausdorff.

($\mathcal{K}_1$) A topological space X is said to be $\mathcal{K}_1$ provided that $\forall\ \phi \in C(K,X)$ $(K \in \mathcal{K})$, $\phi(K)$ is a closed subspace of X.

($\mathcal{K}_2$) A topological space X is said to be $\mathcal{K}_2$ provided that $\forall\ \phi \in C(K,X)$ $(K \in \mathcal{K})$, $\phi(K)$ is a Hausdorff subspace of X.

A topological space X which is simultaneously $\mathcal{K}_1$ and $\mathcal{K}_2$ is necessarily Δ-separated by $\mathcal{K}$.

Specialize the setup and take for $\mathcal{K}$ the class of compact Hausdorff spaces (McCord[†]), so $\mathbf{M} = \mathbf{CG}$. Suppose that X is $\mathcal{K}_1$ (hence T_1)—then X is $\mathcal{K}_2$. Proof: Let $\left\{\begin{array}{l} x \\ y \end{array}\right. \in \phi(K)$ $(\phi \in C(K,X)) : x \neq y$, choose disjoint open sets $\left\{\begin{array}{l} U \\ V \end{array}\right. \subset K : \left\{\begin{array}{l} \phi^{-1}(x) \subset U \\ \phi^{-1}(y) \subset V \end{array}\right.$ and consider $\left\{\begin{array}{l} \phi(K) - \phi(K-U) \\ \phi(K) - \phi(K-V) \end{array}\right.$. Denote by $\mathbf{\Delta}$-$\mathbf{CG}$ the full subcategory of $\mathbf{CG}$ whose objects are Δ-separated by $\mathcal{K}$. There are strict inclusions $\mathbf{CGH} \subset \mathbf{\Delta}$-$\mathbf{CG} \subset \mathbf{CG}$. Example: Every first countable X in $\mathbf{\Delta}$-$\mathbf{CG}$ is Hausdorff.

[†] *Trans. Amer. Math. Soc.* **146** (1969), 273–298; see also Hoffmann, *Arch. Math.* **32** (1979), 487–504.

LEMMA Let X be a Δ-separated compactly generated space—then X is $\mathcal{K}_1$.

[Let K, $L \in \mathcal{K}$; let $\phi \in C(K,X)$, $\psi \in C(L,X)$. Since $\phi \times \psi : K \times L \to X \times_k X$ is continuous, $(\phi\times\psi)^{-1}(\Delta_X)$ is closed in $K\times L$. Therefore $\psi^{-1}(\phi(K)) = \text{pr}_L\,((\phi\times\psi)^{-1}(\Delta_X))$ is closed in L.]

It follows from the lemma that every Δ-separated compactly generated space X is T_1. More is true: Every compact subspace A of X is closed in X. Proof: For any $\phi \in C(K,X)$ $(K \in \mathcal{K})$, $A \cap \phi(K)$ is a closed subspace of A, thus is compact, so $A \cap \phi(K)$ is a closed subspace of $\phi(K)$, implying that $\phi^{-1}(A) = \phi^{-1}(A \cap \phi(K))$ is closed in K. Corollary: The intersection of two compact subsets of X is compact.

Equalizers in **CGH** and **Δ-CG** are closed (e.g., retracts) but **Δ-CG** is better behaved than **CGH** when it comes to quotients. Indeed, if X is in **Δ-CG** and if E is an equivalence relation on X, then X/E is in **Δ-CG** iff $E \subset X \times_k X$ is closed. To see this, let $p : X \to X/E$ be the projection. Because $p \times_k p : X \times_k X \to X/E \times_k X/E$ is quotient, $\Delta_{X/E}$ is closed in $X/E \times_k X/E$ iff $(p \times_k p)^{-1}(\Delta_{X/E}) = E$ is closed in $X \times_k X$. Consequently, if $A \subset X$ is closed, then X/A is in **Δ-CG**.

[Note: Recall that if X is a topological space, then for any equivalence relation E on X, X/E Hausdorff $\Rightarrow E \subset X \times X$ closed and $E \subset X \times X$ closed plus $p : X \to X/E$ open $\Rightarrow X/E$ Hausdorff.]

Δ-CG, like **CG** and **CGH**, is cartesian closed. For **Δ-CG** has finite products and if X is in **CG** and if Y is in **Δ-CG**, then $kC_k(X,Y)$ is in **Δ-CG**.

[Note: Suppose that B is Δ-separated—then **CG**/B is cartesian closed (Booth-Brown[†]).]

CG$_*$ and **Δ-CG**$_*$ are the pointed versions of **CG** and **Δ-CG**. Both are closed categories.

[Note: The pointed exponential object Z^Y is $\hom(Y, Z)$.]

EXAMPLE Let X be a nonnormal LCH space. Fix nonempty disjoint closed subsets A and B of X that do not have disjoint neighborhoods—then X/A and X/B are compactly generated Hausdorff spaces but neither X/A nor X/B is regular. Put $E = A \times A \cup B \times B \cup \Delta_X$. The quotient X/E is a Δ-separated compactly generated space which is not Hausdorff. Moreover, X/E is not the continuous image of any compact Hausdorff space.

[Note: Take for X the Tychonoff plank. Let $A = \{(\Omega, n) : 0 \leq n < \omega\}$ and $B = \{(\alpha, \omega) : 0 \leq \alpha < \Omega\}$—then X/E is compact and all its compact subspaces are closed. By comparison, the product $X/E \times X/E$, while compact, has compact subspaces that are not closed.]

[†] *General Topology Appl.* **8** (1978), 181–195.

EXAMPLE (k-Spaces) The monocoreflective hull in **TOP** of the category of compact spaces is the category of k-spaces. In other words, a topological space X is a k-space provided that a subset U of X is open iff $U \cap K$ is open in K for every compact subset K of X. Every compactly generated space is a k-space. The converse is false: Let X be the subspace of $[0, \Omega]$ obtained by deleting all limit ordinals except Ω—then X is not discrete. Still, the only compact subsets of X are the finite sets, thus kX is discrete. The one point compactification X_∞ of X is compact and contains X as an open subspace. Therefore X_∞ is not compactly generated but is a k-space (being compact). The category of k-spaces is similar in many respects to the category of compactly generated spaces. However, there is one major difference: It is not cartesian closed (Činčura[†]).

[Note: If $\mathcal{K}$ is the class of compact spaces, then **HAUS** $\subset$ **Δ-K** and the inclusion is strict. Reason: A topological space X is in **Δ-K** iff every compact subspace of X is Hausdorff.]

FACT Let $X^0 \subset X^1 \subset \cdots$ be an expanding sequence of topological spaces. Assume: $\forall\, n, X^n$ is in **Δ-CG** and is a closed subspace of X^{n+1}—then X^∞ is in **Δ-CG**.

[That X^∞ is in **CG** is automatic. Let K be a compact Hausdorff space; let $\phi \in C(K, X^\infty)$—then, from the lemma on p. 1–29, $\phi(K) \subset X^n\ (\exists\, n) \Rightarrow \phi(K)$ is closed in $X^n \Rightarrow \phi(K)$ is closed in X^∞.]

EXAMPLE (Weak Products) Let (X_0, x_0), $(X_1, x_1), \ldots$ be a sequence of pointed spaces in **Δ-CG**$_*$. Put $X^n = X_0 \times_k \cdots \times_k X_n$—then X^n is in **Δ-CG**$_*$ with base point $(x_0, \ldots, x_n)$. The pointed map $X^n \rightarrow X^{n+1}$ is a closed embedding. One writes $(w) \prod\limits_1^\infty X_n$ in place of X^∞ and calls it the weak product of the X_n. By the above, $(w) \prod\limits_1^\infty X_n$ is in **Δ-CG**$_*$ (the base point is the infinite string made up of the x_n).

[Note: The same construction can be carried out in **TOP**, the only difference being that X^n is the ordinary product of $X_0, \ldots, X_n$.]

Every Hausdorff topological group is completely regular. In particular, every Hausdorff topological vector space is completely regular. Every Hausdorff locally compact topological group is paracompact.

[Note: Every topological group which satisfies the T_0 separation axiom is necessarily a CRH space.]

EXAMPLE Take $G = \mathbf{R}^\kappa (\kappa > \omega)$—then G is a Hausdorff topological group but G is not compactly generated. Consider kG: Inversion $kG \rightarrow kG$ is continuous, as is multiplication $kG \times_k kG \rightarrow kG$. But kG is not a topological group, i.e., multiplication $kG \times kG \rightarrow kG$ is not continuous. In fact, kG, while Hausdorff, is not regular.

[†] *Topology Appl.* **41** (1991), 205–212.

Let E be a normed linear space; let E^* be its dual, i.e., the space of continuous linear functionals on E—then E^* is also a normed linear space. The elements of E can be regarded as scalar valued functions on E^*. The initial topology on E^* determined by them is called the weak* topology. It is the topology of pointwise convergence. In the weak* topology, E^* is a Hausdorff topological vector space, thus is completely regular. If $\dim E \geq \omega$, then every nonempty weak* open set in E^* is unbounded in norm. By contrast, Alaoglu's theorem says that the closed unit ball in E^* is compact in the weak* topology (and second countable if E is separable). However, the weak* topology is metrizable iff $\dim E \leq \omega$.

[Note: Let E be a vector space over $\mathbf{R}$—then Kruse[†] has shown that E admits a complete norm (so that E is a Banach space) iff $\dim E < \omega$ or $(\dim E)^\omega = \dim E$. Therefore, the weak* topology on the dual of an infinite dimensional Banach space is not metrizable.]

The forgetful functor from the category of topological groups to the category of topological spaces (pointed topological spaces) has a left adjoint $X \rightarrow F_{\text{gr}}X((X, x_0) \rightarrow F_{\text{gr}}(X, x_0))$, where $F_{\text{gr}}X$ $(F_{\text{gr}}(X, x_0))$ is the free topological group on $X((X, x_0))$. Algebraically, $F_{\text{gr}}X$ $(F_{\text{gr}}(X, x_0))$ is the free group on X $(X - \{x_0\})$. Topologically, $F_{\text{gr}}X$ $(F_{\text{gr}}(X, x_0))$ carries the finest topology compatible with the group structure for which the canonical injection $X \rightarrow F_{\text{gr}}X$ $((X, x_0) \rightarrow F_{\text{gr}}(X, x_0))$ is continuous. There is a commutative triangle

$$\begin{tikzcd}
X \arrow[r] \arrow[dr] & F_{\text{gr}}X \arrow[d] \\
& F_{\text{gr}}(X, x_0)
\end{tikzcd}$$

and $F_{\text{gr}}(X, x_0) \approx F_{\text{gr}}X/\langle x_0 \rangle$ $(\langle x_0 \rangle$ the normal subgroup generated by the word $x_0)$. On the other hand, $F_{\text{gr}}X \approx F_{\text{gr}}(X, x_0) \amalg \mathbf{Z}$ ($\amalg$ the coproduct in the category of topological groups) and, of course, $F_{\text{gr}}X \approx F_{\text{gr}}(X \amalg *, *)$.

[Note: The arrow of adjunction $X \rightarrow F_{\text{gr}}X$ $((X, x_0) \rightarrow F_{\text{gr}}(X, x_0))$ is an embedding iff X is completely regular and is a closed embedding iff X is completely regular + Hausdorff (Thomas[‡]).]

LEMMA If X is a compact Hausdorff space, then $F_{\text{gr}}(X)$ $(F_{\text{gr}}(X, x_0))$ is a Hausdorff topological group.

Application: If X is a CRH space, then $F_{\text{gr}}(X)$ $(F_{\text{gr}}(X, x_0))$ is a Hausdorff topological group.

[Consider $X \rightarrow F_{\text{gr}}(\beta X)$ $((X, x_0) \rightarrow F_{\text{gr}}(\beta X, \beta x_0))$.]

[†] *Math. Zeit.* **83** (1964), 314–320.

[‡] *General Topology Appl.* **4** (1974), 51–72; see also *Quaestiones Math.* **2** (1977), 355–377.

EXAMPLE It is easy to construct nonnormal Hausdorff topological groups. Thus, given a topological space X, let $F_{\text{gr}}X$ be the free topological group on X—then, for X a CRH space, the arrow $X \rightarrow F_{\text{gr}}X$ is a closed embedding and $F_{\text{gr}}X$ is a Hausdorff topological group, so X not normal $\Rightarrow F_{\text{gr}}X$ not normal.

FACT Given a topological space X, $F_{\text{gr}}(X, x_0') \approx F_{\text{gr}}(X, x_0'') \ \forall \ x_0', x_0'' \in X$.

[Let $\mu' : (X, x_0') \rightarrow F_{\text{gr}}(X, x_0'), \mu'' : (X, x_0'') \rightarrow F_{\text{gr}}(X, x_0'')$ be the arrows of adjunction and consider the pointed continuous functions $f' : (X, x_0') \rightarrow F_{\text{gr}}(X, x_0'')$, $f'' : (X, x_0'') \rightarrow F_{\text{gr}}(X, x_0')$ defined by $f'(x) = \mu''(x)\mu''(x_0')^{-1}$, $f''(x) = \mu'(x)\mu'(x_0'')^{-1}$.]

The forgetful functor from the category of abelian topological groups to the category of topological spaces (pointed topological spaces) has a left adjoint $X \rightarrow F_{\text{ab}}X((X, x_0) \rightarrow F_{\text{ab}}(X, x_0))$ and when given the quotient topology, $F_{\text{gr}}X/[F_{\text{gr}}X, F_{\text{gr}}X] \approx F_{\text{ab}}X$ $(F_{\text{gr}}(X, x_0)/[F_{\text{gr}}(X, x_0), F_{\text{gr}}(X, x_0)] \approx F_{\text{ab}}(X, x_0))$.

§ 1

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§2. CONTINUOUS FUNCTIONS

Apart from an important preliminary, namely a characterization of the exponential objects in **TOP**, the emphasis in this § is on the properties possessed by $C(X)$, where X is a CRH space.

A topological space Y is said to be cartesian if the functor $— \times Y : \mathbf{TOP} \rightarrow \mathbf{TOP}$ has a right adjoint $Z \rightarrow Z^Y$. Example: A LCH space is cartesian.

PROPOSITION 1 A topological space Y is cartesian iff $— \times Y$ preserves colimits (cf. p. 0–33) or, equivalently, iff $— \times Y$ preserves coproducts and coequalizers.

[Note: The preservation of coproducts is automatic and the preservation of coequalizers reduces to whether $— \times Y$ takes quotient maps to quotient maps.]

Notation: Given topological spaces $X, Y, Z, \Lambda : F(X \times Y, Z) \rightarrow F(X, F(Y, Z))$ is the bijection defined by the rule $\Lambda(f)(x)(y) = f(x, y)$.

Let τ be a topology on $C(Y, Z)$—then τ is said to be splitting if $\forall\ X,\ f \in C(X \times Y, Z) \Rightarrow \Lambda(f) \in C(X, C(Y, Z))$ and τ is said to be cosplitting if $\forall\ X,\ g \in C(X, C(Y, Z)) \Rightarrow \Lambda^{-1}(g) \in C(X \times Y, Z)$.

LEMMA If τ' is a splitting topology on $C(Y, Z)$ and τ'' is a cosplitting topology on $C(Y, Z)$, then $\tau' \subset \tau''$.

Application: $C(Y, Z)$ admits at most one topology which is simultaneously splitting and cosplitting, the exponential topology.

EXAMPLE $\forall\, Y\ \&\ \forall\, Z$, the compact open topology on $C(Y, Z)$ is splitting.

EXAMPLE If Y is locally compact, then $\forall\, Z$ the exponential topology on $C(Y, Z)$ exists and is the compact open topology.

[Note: A topological space Y is said to be locally compact if $\forall$ open set P and $\forall\, y \in P$, there exists a compact set $K \subset P$ with $y \in \operatorname{int} K$. Example: The one point compactification $\mathbf{Q}_\infty$ of $\mathbf{Q}$ is compact but not locally compact.]

FACT Let Y be a locally compact space—then for all X and Z, the operation of composition $C(X, Y) \times C(Y, Z) \rightarrow C(X, Z)$ is continuous if the function spaces carry the compact open topology.

PROPOSITION 2 A topological space Y is cartesian iff the exponential topology on $C(Y, Z)$ exists for all Z.

EXAMPLE A locally compact space is cartesian.

FACT Suppose that Y is cartesian. Assume: $\forall\ Z$, the exponential topology on $C(Y, Z)$ is the compact open topology—then Y is locally compact.

Let Y be a topological space, τ_Y its topology—then the open sets in the continuous topology on τ_Y are those collections $\mathcal{V} \subset \tau_Y$ such that (1) $V \in \mathcal{V}$, $V' \in \tau_Y \Rightarrow V' \in \mathcal{V}$ if $V \subset V'$ and (2) $V_i \in \tau_Y$ $(i \in I)$, $\bigcup\limits_i V_i \in \mathcal{V} \Rightarrow \exists\, i_1, \ldots, i_n : V_{i_1} \cup \cdots \cup V_{i_n} \in \mathcal{V}$.

LEMMA Let $f \in F(X, \tau_Y)$, where X is a topological space and τ_Y has the continuous topology—then f is continuous if $\{(x, y) : y \in f(x)\}$ is open in $X \times Y$.

Let $\mathrm{T} = \{(P, y) : y \in P\} \subset \tau_Y \times Y$—then a topology on τ_Y is said to have property T if T is open in $\tau_Y \times Y$. Example: The discrete topology on τ_Y has property T.

FACT The continuous topology on τ_Y is the largest topology in the collection of all topologies on τ_Y that are smaller than every topology on τ_Y which has property T.

[If $\tau_Y(\mathrm{T})$ is τ_Y in a topology having property T, then by the lemma, the identity function $\tau_Y(\mathrm{T}) \rightarrow \tau_Y$ is continuous if τ_Y has the continuous topology.]

Let Y be a topological space—then Y is said to be core compact if $\forall$ open set P and $\forall\, y \in P$, there exists an open set $V \subset P$ with $y \in V$ such that every open covering of P contains a finite covering of V. Example: A locally compact space is core compact.

There exists a core compact space with the property that every compact subset has an empty interior (Hofman-Lawson[†]).

FACT Equip τ_Y with the continuous topology—then Y is core compact iff $\forall$ open set P and $\forall\, y \in P$, there exists an open $\mathcal{V} \subset \tau_Y$ such that $P \in \mathcal{V}$ and $y \in \operatorname{int} \cap \mathcal{V}$.

EXAMPLE A topological space Y is core compact iff the continuous topology on τ_Y has property T.

Let Y, Z be topological spaces—then the Isbell topology on $C(Y, Z)$ is the initial topology on $C(Y, Z)$ determined by the $e_Q : \begin{cases} C(Y, Z) \rightarrow \tau_Y \\ f \rightarrow f^{-1}(Q) \end{cases}$ $(Q \in \tau_Z)$, where τ_Y has the continuous topology. Notation: is$C(Y,Z)$. Examples: (1) is$C(Y,[0,1]/[0,1[) \approx \tau_Y$; (2) is$C(*,Z) \approx Z$.

[†] *Trans. Amer. Math. Soc.* **246** (1978), 285–310 (cf. 304–306).

LEMMA The compact open topology on $C(Y,Z)$ is smaller than the Isbell topology.

EXAMPLE $\forall\, Y$ & $\forall\, Z$, the Isbell topology on $C(Y,Z)$ is splitting.

[Fix an $f \in C(X \times Y, Z)$ and let $g = \Lambda(f)$—then the claim is that $g \in C(X, \text{is}C(Y,Z))$. From the definitions, this amounts to showing that $\forall\, Q \in \tau_Z$, $e_Q \circ g$ is continuous. Write $f^{-1}(Q)$ as a union of rectangles $R_i = U_i \times V_i \subset X \times Y$. Take an $x \in X$ and consider any $\mathcal{V} : e_Q(g(x)) \in \mathcal{V}$. Since $e_Q(g(x)) = \bigcup\limits_i \{y : (x,y) \in R_i\}$, $\exists\, i_k$ $(k = 1, \ldots, n) : \bigcup\limits_{k=1}^{n} \{y : (x,y) \in R_{i_k}\} \in \mathcal{V}$, so $\forall\, u \in \bigcap\limits_{k=1}^{n} U_{i_k}$, $e_Q(g(u)) \in \mathcal{V}$.]

FACT Let Y be a core compact space—then for all X and Z, the operation of composition $C(X,Y) \times C(Y,Z) \rightarrow C(X,Z)$ is continuous if the function spaces carry the Isbell topology.

PROPOSITION 3 Let Y be a topological space—then Y is cartesian iff Y is core compact.

[Necessity: Let τ_i run through the topologies on τ_Y which have property T and put $X_i = (\tau_Y, \tau_i)$. Form the coproduct $X = \coprod\limits_i X_i$ and let $f : X \rightarrow \tau_Y$ be the function whose restriction to each X_i is the identity, where τ_Y carries the continuous topology—then f is a quotient map (cf. p. 2–2). Since Y is cartesian, it follows from Proposition 1 that $f \times \text{id}_Y : X \times Y \rightarrow \tau_Y \times Y$ is also quotient. But $X \times Y \approx \coprod\limits_i X_i \times Y$ and, by hypothesis, T is open in $X_i \times Y$ $\forall\, i$. Therefore T must be open in $\tau_Y \times Y$ as well, i.e., the continuous topology on τ_Y has property T, thus Y is core compact (cf. p. 2–2).

Sufficiency: As has been noted above, the Isbell topology on $C(Y,Z)$ is splitting, so to prove that Y is cartesian it suffices to prove that the Isbell topology on $C(Y,Z)$ is cosplitting when Y is core compact (cf. Proposition 2). Fix $g \in C(X, \text{is}C(Y,Z))$ and put $f = \Lambda^{-1}(g)$. Given a point $(x,y) \in X \times Y$, let Q be an open subset of Z such that $f(x,y) \in Q$. Choose an open $P \subset Y : y \in P$ & $f(\{x\} \times P) \subset Q$. Because Y is core compact, there exists an open $\mathcal{V} \subset \tau_Y : P \in \mathcal{V}$ and $y \in \text{int} \cap \mathcal{V}$. But $e_Q(g(x)) \supset P \Rightarrow e_Q(g(x)) \in \mathcal{V}$ and, from the continuity of $e_Q \circ g$, $\exists$ a neighborhood O of $x : e_P(g(O)) \subset \mathcal{V}$, hence $f(O \times \text{int} \cap \mathcal{V}) \subset Q$.]

Remark: Suppose that Y is core compact—then $\forall\, Z$, "the" exponential object Z^Y is is$C(Y,Z)$, the exponential topology on $C(Y,Z)$ being the Isbell topology.

[Note: The Isbell topology and the compact open topology on $C(Y,Z)$ are one and the same if Y is locally compact.]

FACT Let $f, g \in C(Y, Z)$. Assume: f, g are homotopic—then f, g belong to the same path component of is$C(Y, Z)$.

FACT Let $f, g \in C(Y, Z)$. Assume: f, g belong to the same path component of is$C(Y, Z)$—then f, g are homotopic if Y is core compact.

What follows is a review of the elementary properties possessed by $C(X, Y)$ when equipped with the compact open topology (omitted proofs can be found in Engelking[†]).

Notation: Given Hausdorff spaces X and Y, let co$C(X, Y)$ stand for $C(X, Y)$ in the compact open topology.

[Note: The point open topology on $C(X, Y)$ is smaller than the compact open topology. Therefore co$C(X, Y)$ is necessarily Hausdorff. Of course, if X is discrete, then "point open" = "compact open".]

PROPOSITION 4 Suppose that Y is regular—then co$C(X, Y)$ is regular.

PROPOSITION 5 Suppose that Y is completely regular—then co$C(X, Y)$ is completely regular.

EXAMPLE It is false that Y normal $\Rightarrow$ co$C(X, Y)$ normal. Thus take $X = \{0, 1\}$ (discrete topology)—then co$C(\{0, 1\}, Y) \approx Y \times Y$ and there exists a normal Hausdorff space Y whose square is not normal (e.g., the Sorgenfrey line (cf. p. 5–11)).

O'Meara[‡] has shown that if X is a second countable metrizable space and Y is a metrizable space, then co$C(X, Y)$ is perfectly normal and hereditarily paracompact.

EXAMPLE The loop space ΩY of a pointed metrizable space (Y, y_0) is paracompact.

A Hausdorff space X is said to be countable at infinity if there is a sequence $\{K_n\}$ of compact subsets of X such that if K is any compact subset of X, then $K \subset K_n$ for some n. Example: A LCH space is countable at infinity iff it is σ-compact.

[Note: X countable at infinity $\Rightarrow$ X σ-compact. Example: $\mathbf{P}$ is not σ-compact, hence is not countable at infinity.]

FACT Suppose that X is countable at infinity. Assume: X is first countable—then X is locally compact.

[†] *General Topology*, Heldermann Verlag (1989).

[‡] *Proc. Amer. Math. Soc.* **29** (1971), 183–189.

EXAMPLE **Q** is σ-compact but **Q** is not countable at infinity.

EXAMPLE Fix a point $x \in \beta\mathbf{N} - \mathbf{N}$—then $X = \mathbf{N} \cup \{x\}$, viewed as a subspace of $\beta\mathbf{N}$, is countable at infinity but it is not first countable.

[Note: The compact subsets of X are finite. However X is not compactly generated.]

EXAMPLE Let E be an infinite dimensional Banach space—then E^* in the weak* topology is countable at infinity.

PROPOSITION 6 Suppose that X is countable at infinity—then for every metrizable Y, $\mathrm{co}C(X,Y)$ is metrizable.

PROPOSITION 7 Suppose that X is countable at infinity and compactly generated—then for every completely metrizable Y, $\mathrm{co}C(X,Y)$ is completely metrizable.

Notation: Given a topological space X, write $H(X)$ for its set of homeomorphisms—then $H(X)$ is a group under composition.

Let us assume that X is a LCH space. Endow $H(X)$ with the compact open topology. Question: Is $H(X)$ thus topologized a topological group? In general, the answer is "no" (cf. infra) but there are situations in which the answer is "yes".

[Note: The composition $\begin{cases} H(X) \times H(X) \rightarrow H(X) \\ (f,g) \rightarrow g \circ f \end{cases}$ is continuous, so the problem is whether the inversion $f \rightarrow f^{-1}$ is continuous.]

Remark: The evaluation $\begin{cases} H(X) \times X \\ (f,x) \rightarrow f(x) \end{cases}$ is continuous.

Given subsets A and B of X, put $\langle A,B \rangle = \{f \in H(X) : f(A) \subset B\}$—then by definition, the collection $\{\langle K,U \rangle\}$ (K compact and U open) is a subbasis for the compact open topology on $H(X)$.

PROPOSITION 8 If X is a compact Hausdorff space, then $H(X)$ is a topological group in the compact open topology.

[For $f \in \langle K,U \rangle \Leftrightarrow f^{-1} \in \langle X - U, X - K \rangle$.]

FACT If X is a compact metric space, then $H(X)$ is completely metrizable.

LEMMA Let X be a locally connected LCH space—then the collection $\{\langle L,V \rangle\}$, where L is compact & connected with $\mathrm{int}\, L \neq \emptyset$ and V is open, constitute a subbasis for the compact open topology on $H(X)$.

PROPOSITION 9 If X is a locally connected LCH space, then $H(X)$ is a topological group in the compact open topology.

[Fix an $f \in H(X)$ and choose $\langle L, V\rangle$ per the lemma: $f^{-1} \in \langle L, V\rangle$. Determine relatively compact open O & P : $f^{-1}(L) \subset O \subset \overline{O} \subset P \subset \overline{P} \subset V$ ($\Rightarrow f((X - O) \cap \overline{P}) \subset (X - L) \cap f(V)$). Let x be any point such that $f(x) \in \operatorname{int} L$—then $\langle \{x\}, \operatorname{int} L\rangle \cap \langle (X - O) \cap \overline{P}, (X - L) \cap f(V)\rangle$ is a neighborhood of f in $H(X)$, call it H_f. Claim: $g \in H_f \Rightarrow g^{-1} \in \langle L, V\rangle$. To check this, note that $g((X - O) \cap \overline{P}) \subset (X - L) \cap f(V) \Rightarrow L \cup (X - f(V)) \subset g(O) \cup g(X - \overline{P})$. But $g(O)$, $g(X - \overline{P})$ are nonempty disjoint open sets, so L is contained in either $g(O)$ or $g(X - \overline{P})$ (L being connected). Since the containment $L \subset g(X - \overline{P})$ is impossible ($g(x) \in \operatorname{int} L$ and $x \notin X - \overline{P}$), it follows that $L \subset g(O)$ or still, $g^{-1}(L) \subset O \subset V$, i.e., $g^{-1} \in \langle L, V\rangle$. Therefore inversion is a continuous function.]

Application: The homeomorphism group of a topological manifold is a topological group in the compact open topology.

EXAMPLE Let $X = \{0, 2^n (n \in \mathbf{Z})\}$—then in the induced topology from $\mathbf{R}$, X is a LCH space but $H(X)$ in the compact open topology is not a topological group.

Suppose that X is a LCH space, X_∞ its one point compactification—then $H(X)$ can be identified with the subgroup of $H(X_\infty)$ consisting of those homeomorphisms $X_\infty \to X_\infty$ which leave ∞ fixed. In the compact open topology, $H(X_\infty)$ is a topological group (cf. Proposition 8). Therefore $H(X)$ is a topological group in the induced topology. As such, $H(X)$ is a closed subgroup of $H(X_\infty)$.

[Note: This topology on $H(X)$ is the complemented compact open topology. It has for a subbasis all sets of the form $\langle K, U\rangle$, where K is compact and U is open, as well as all sets of the form $\langle X - V, X - L\rangle$, where V is open and L is compact.]

An isotopy of a topological space X is a collection $\{h_t : 0 \leq t \leq 1\}$ of homeomorphisms of X such that $\begin{cases} h : X \times [0,1] \to X \\ h(x,t) = h_t(x) \end{cases}$ is continuous.
[Note: When X is a LCH space, isotopies correspond to paths in $H(X)$ (compact open topology).]

EXAMPLE A homeomorphism $h : \mathbf{R}^n \to \mathbf{R}^n$ is said to be stable if $\exists$ homeomorphisms $h_1, \ldots, h_k : \mathbf{R}^n \to \mathbf{R}^n$ such that $h = h_1 \circ \cdots \circ h_k$, where each h_i has the property that for some nonempty open $U_i \subset \mathbf{R}^n$, $h_i | U_i = \mathrm{id}_{U_i}$. Every stable homeomorphism of $\mathbf{R}^n$ is isotopic to the identity.

[Take $k = 1$ and consider a homeomorphism $h : \mathbf{R}^n \to \mathbf{R}^n$ for which $h|U = \mathrm{id}_U$. Define an isotopy $\{h_t : 0 \leq t \leq 1\}$ of $\mathbf{R}^n$ as follows. Fix $u \in U$ and put $h_t(x) = \begin{cases} h(x + 2tu) - 2tu & (0 \leq t \leq 1/2) \\ \dfrac{1}{2 - 2t} h_{1/2}((2 - 2t)x) & (1/2 \leq t < 1) \end{cases}$ & $h_1(x) = x$.]

FACT Equip $H(\mathbf{R}^n)$ with the compact open topology and write $H_{\mathrm{ST}}(\mathbf{R}^n)$ for the subspace of $H(\mathbf{R}^n)$ consisting of the stable homeomorphisms—then $H_{\mathrm{ST}}(\mathbf{R}^n)$ is an open subgroup of $H(\mathbf{R}^n)$.

[Note: Therefore $H_{\mathrm{ST}}(\mathbf{R}^n)$ is also a closed subgroup of $H(\mathbf{R}^n)$ (since $H(\mathbf{R}^n)$ is a topological group in the compact open topology).]

Application: The path component of $\mathrm{id}_{\mathbf{R}^n}$ in $H(\mathbf{R}^n)$ is $H_{\mathrm{ST}}(\mathbf{R}^n)$.

[In view of the example, there is a path from every element of $H_{\mathrm{ST}}(\mathbf{R}^n)$ to $\mathrm{id}_{\mathbf{R}^n}$. On the other hand, if $\tau : [0,1] \to H(\mathbf{R}^n)$ is a path with $\tau(1) = \mathrm{id}_{\mathbf{R}^n}$ but $\tau(0) \notin H_{\mathrm{ST}}(\mathbf{R}^n)$, then $\tau^{-1}(H_{\mathrm{ST}}(\mathbf{R}^n))$ would be a nontrivial clopen subset of $[0,1]$.]

[Note: It can be shown that $H(\mathbf{R}^n)$ is locally path connected (indeed, locally contractible (cf. p. 6–17)).]

An isotopy $\{h_t : 0 \leq t \leq 1\}$ is said to be invertible if the collection $\{h_t^{-1} : 0 \leq t \leq 1\}$ is an isotopy.

LEMMA An isotopy $\{h_t : 0 \leq t \leq 1\}$ is invertible iff the function $H : X \times [0,1] \to X \times [0,1]$ defined by the rule $(x,t) \to (h_t(x), t)$ is a homeomorphism.

[Note: H is necessarily one-to-one, onto, and continuous.]

FACT Let X be a LCH space—then every isotopy $\{h_t : 0 \leq t \leq 1\}$ of X is invertible.

[Show first that $\forall\, x \in X$, $h_t^{-1}(x)$ is a continuous function of t.]

FACT Let X be a LCH space—then every isotopy $\{h_t : 0 \leq t \leq 1\}$ of X extends to an isotopy of X_∞.

[Define $\overline{h}_t : X_\infty \to X_\infty$ by $\overline{h}_t | X = h_t$ & $\overline{h}_t(\infty) = \infty$. To verify that $\overline{h}$ is continuous, extend H to $X_\infty \times [0,1]$ via the prescription $\overline{H}(\infty, t) = (\overline{h}_t(\infty), t)$, so $\overline{h} = \pi_\infty \circ \overline{H}$, where π_∞ is the projection of $X_\infty \times [0,1]$ onto X_∞. Establish the continuity of $\overline{H}$ by utilizing the continuity of H^{-1} (the substance of the previous result).]

EXAMPLE Every isotopy $\{h_t : 0 \leq t \leq 1\}$ of $\mathbf{R}^n$ extends to an isotopy of $\mathbf{S}^n$.

Let X be a CRH space, (Y, d) a metric space. Given $f \in C(X,Y)$ and $\phi \in C(X, \mathbf{R}_{>0})$, put $N_\phi(f) = \{g : d(f(x), g(x)) < \phi(x)\ \forall\, x\}$.

Observations: (1) If $\phi_1, \phi_2 \in C(X, \mathbf{R}_{>0})$, then $N_\phi(f) \subset N_{\phi_1}(f) \cap N_{\phi_2}(f)$, where $\phi(x) = \min\{\phi_1(x), \phi_2(x)\}$; (2) If $g \in N_\phi(f)$, then $N_\psi(g) \subset N_\phi(f)$, where $\psi(x) = \phi(x) - d(f(x), g(x))$.

Therefore the collection $\{N_\phi(f)\}$ is a basic system of neighborhoods at f. Accordingly, varying f leads to a topology on $C(X,Y)$, the majorant topology.

[Note: Each $\phi \in C(X, \mathbf{R}_{>0})$ determines a metric d_ϕ on $C(X,Y)$, viz. $d_\phi(f,g) = \min\{1, \sup\limits_{x \in X} \dfrac{d(f(x),g(x))}{\phi(x)}\}$, and their totality defines the majorant topology on $C(X,Y)$, which is thus completely regular. However, in general, the majorant topology on $C(X,Y)$ need not be normal (Wegenkittl[†]).]

Here is a proof that $C(X,Y)$ (majorant topology) is completely regular. Fix a closed subset $A \subset C(X,Y)$ and an $f \in C(X,Y) - A$. Choose $\phi \in C(X, \mathbf{R}_{>0}) : N_\phi(f) \subset C(X,Y) - A$. Define a function $\Phi : C(X,Y) \rightarrow [0,1]$ by $\Phi(g) = \sup\limits_{x \in X} \dfrac{d(f(x),g(x))}{\phi(x)}$ if $g \in N_\phi(f)$ and let it be 1 otherwise—then Φ is continuous and $\Phi(f) = 0$, $\Phi | A = 1$.]

[Note: The verification of the continuity of Φ hinges on the observation that $g \in \overline{N_\phi(f)} \Rightarrow d(f(x), g(x)) \leq \phi(x) \ \forall\, x$, hence $\forall\, g \in \overline{N_\phi(f)} - N_\phi(f)$, $\sup\limits_{x \in X} \dfrac{d(f(x),g(x))}{\phi(x)} = 1$.]

Example: Suppose that the sequence $\{f_k\}$ converges to f in $C(\mathbf{R}^n, \mathbf{R}^n)$ (majorant topology)—then $\exists$ a compact $K \subset \mathbf{R}^n$ and an index k_0 such that $f_k(x) = f(x) \ \forall\, k > k_0$ & $\forall\, x \in \mathbf{R}^n - K$.

EXAMPLE Suppose that $f : \mathbf{R}^n \rightarrow \mathbf{R}^n$ is a homeomorphism—then f has a neighborhood of surjective maps in $C(\mathbf{R}^n, \mathbf{R}^n)$ (majorant topology).

EXAMPLE Equip $H(\mathbf{R}^n)$ with the majorant topology—then the path component of $\mathrm{id}_{\mathbf{R}^n}$ in $H(\mathbf{R}^n)$ consists of those homeomorphisms that are the identity outside some compact set.

FACT The majorant topology on $C(\mathbf{R}^n, \mathbf{R}^n)$ is not first countable.

LEMMA The compact open topology on $C(X,Y)$ is smaller than the majorant topology.

[Fix a compact $K \subset X$, an open $V \subset Y$, and a continuous $f : X \rightarrow Y$ such that $f(K) \subset V$. Choose $\epsilon > 0$ such that $\forall\, y \in f(K), d(y,y') < \epsilon \Rightarrow y' \in V$. Let $\phi \in C(X, \mathbf{R}_{>0})$ be the constant function $x \rightarrow \epsilon$—then $\forall\, g \in N_\phi(f)$, $g(K) \subset V$.]

Remark: The uniform topology on $C(X,Y)$ is the topology induced by the metric $d(f,g) = \min\{1, \sup\limits_{x \in X} d(f(x),g(x))\}$. The proof of the lemma shows that the compact open topology on $C(X,Y)$ is smaller than the uniform topology (which in turn is smaller than the majorant topology).

[†] *Ann. Global Anal. Geom.* **7** (1989), 171–178; see also van Douwen, *Topology Appl.* **39** (1991), 3–32.

FACT The compact open topology on $C(X,Y)$ equals the uniform topology if X is compact.

FACT The uniform topology on $C(X,Y)$ equals the majorant topology if X is pseudocompact.

Let $M(Y)$ be the set of all metrics on Y which are compatible with the topology of Y—then the limitation topology on $C(X,Y)$ has for a neighborhood basis at f the $N_m(f)$ $(m \in M(Y))$, where $N_m(f) = \{g : \sup\limits_{x \in X} m(f(x), g(x)) < 1\}$.

[Note: If $m_1, m_2 \in M(Y)$, then $N_{m_1+m_2}(f) \subset N_{m_1}(f) \cap N_{m_2}(f)$ and if $g \in N_m(f)$, then $N_{(\frac{2}{\epsilon})m}(g) \subset N_m(f)$, where $m(f(x), g(x)) \leq 1 - \epsilon \; \forall \; x$.]

The limitation topology is defined by the metrics $(f,g) \to \min\{1, \sup\limits_{x \in X} m(f(x), g(x))\}$ $(m \in M(Y))$, thus the uniform topology on $C(X,Y)$ is smaller than the limitation topology.

LEMMA Suppose that X is paracompact—then the limitation topology on $C(X,Y)$ is smaller than the majorant topology.

[Fix $m \in M(Y)$ and let $f \in C(X,Y)$. By compatibility, $\forall \; x \in X$, $\exists \; \epsilon(x) > 0$: $d(f(x), y) < \epsilon(x) \Rightarrow m(f(x), y) < \frac{1}{4}$. Put $O_x = \{x' : d(f(x), f(x')) < \frac{\epsilon(x)}{2}\}$—then $\{O_x\}$ is an open covering of X. Let $\{U_x\}$ be a precise neighborhood finite open refinement and choose a subordinated partition of unity $\{\kappa_x\}$. Definition: $\phi = \sum\limits_x \frac{\epsilon(x)}{2} \kappa_x$. Consider now any $x_0 \in X$ and assume that $d(f(x_0), y) < \phi(x_0)$. Let $\kappa_{x_1}, \ldots, \kappa_{x_n}$ be an enumeration of those κ_x whose support contains x_0 and fix i between 1 and n : $\frac{\epsilon(x_j)}{2} \leq \frac{\epsilon(x_i)}{2}$ $(j = 1, \ldots, n)$ to get $\phi(x_0) \leq \frac{\epsilon(x_i)}{2}$. But $x_0 \in U_{x_i} \subset O_{x_i}$. Therefore $d(f(x_i), f(x_0)) < \frac{\epsilon(x_i)}{2}$ $(\Rightarrow m(f(x_i), f(x_0)) < \frac{1}{4}) \Rightarrow d(f(x_i), y) < \epsilon(x_i) \Rightarrow m(f(x_i), y) < \frac{1}{4} \Rightarrow m(f(x_0), y) < \frac{1}{2}$. And this shows that $N_\phi(f) \subset N_m(f)$.]

[Note: In general, the limitation topology is strictly smaller than the majorant topology. To see this, observe that $C(\mathbf{R}, \mathbf{R})$ is a topological group under addition in the majorant topology. On the other hand, there is a countable basis at a given $f \in C(\mathbf{R}, \mathbf{R})$ (limitation topology) iff f is bounded, thus $C(\mathbf{R}, \mathbf{R})$ is not a topological group under addition in the limitation topology.]

FACT Take $X = Y$—then in the limitation topology, $H(X)$ is a topological group.

REFINEMENT PRINCIPLE Let (Y, d) be a metric space—then for any open covering $\mathcal{V} = \{V\}$ of Y, $\exists \; m \in M(Y)$ such that the collection $\{V_y\}$ is a refinement of $\mathcal{V}$, where $V_y = \{y' : m(y, y') < 1\}$.

[A proof can be found in Dugundji[†].]

LEMMA Let (Y,d) be a metric space—then for any $\delta \in C(Y, \mathbf{R}_{>0})$, $\exists\, m \in M(Y)$: $d(y,y') < \delta(y)$ whenever $m(y,y') < 1$.

[Choose an open covering $\mathcal{V} = \{V\}$ of Y such that the diameter of a given V is $\leq \frac{1}{2} \inf \delta(V)$. Using the refinement principle, fix an $m \in M(Y)$ such that the collection $\{V_y\}$ refines $\mathcal{V}$. If (y,y') is a pair with $m(y,y') < 1$, then $V_y \subset V$ for some V, hence $y, y' \in V \Rightarrow d(y,y') \leq \frac{1}{2}\delta(y) < \delta(y)$.]

PROPOSITION 10 Take $X = Y$—then the limitation topology on $H(X)$ is equal to the majorant topology.

[Fix $f \in H(X)$ and $\phi \in C(X, \mathbf{R}_{>0})$. Thanks to the lemma, $\exists\, m \in M(X) : d(x,x') < \phi \circ f^{-1}(x)$ whenever $m(x,x') < 1$. If $g \in H(X)$ and $\sup\limits_{x \in X} m(f(x), g(x)) < 1$, then $d(f(x), g(x)) < \phi \circ f^{-1}(f(x)) = \phi(x)\ \forall\ x$, i.e., $N_\phi(f) \cap H(X)$ is open in $H(X)$ (limitation topology).]

Application: The homeomorphism group of a metric space is a topological group in the majorant topology.

EXAMPLE Let X be a second countable topological manifold of euclidean dimension n—then in the majorant topology, $H(X)$ is a topological group. Moreover, Černavskiĭ[‡] has shown that $H(X)$ is locally contractible.

[Note: X is metrizable (cf. §1, Proposition 11), so $\exists\, d : (X,d)$ is a metric space.]

Notation: $\forall\, f \in C(X,Y)$, $\mathrm{gr}_f \subset X \times Y$ is its graph.

Given an open subset $O \subset X \times Y$, let $\Gamma_O = \{f : \mathrm{gr}_f \subset O\}$—then the collection $\{\Gamma_O\}$ is a basis for a topology on $C(X,Y)$, the graph topology.

[Note: In this connection, observe that $\Gamma_O \cap \Gamma_P = \Gamma_{O \cap P}$.]

LEMMA The majorant topology on $C(X,Y)$ is smaller than the graph topology.

[The function $(x,y) \to \phi(x) - d(f(x),y)$ from $X \times Y$ to $\mathbf{R}$ is continuous, thus $O = \{(x,y) : d(f(x),y) < \phi(x)\}$ is an open subset of $X \times Y$. But $\Gamma_O = N_\phi(f)$.]

[†] *Topology*, Allyn and Bacon (1966), 196; see also Bessaga-Pełczyński, *Selected Topics in Infinite Dimensional Topology*, PWN (1975), 63.

[‡] *Math. Sbornik* **8** (1969), 287–333.

Rappel: A function $f : X \to \mathbf{R}$ is lower semicontinuous (upper semicontinuous) if for each real number c, $\{x : f(x) > c\}$ ($\{x : f(x) < c\}$) is open. Example: The characteristic function of a subset S of X is lower semicontinuous (upper semicontinuous) iff S is open (closed).

HAHN'S EINSCHIEBUNGSATZ Suppose that X is paracompact. Let $g : X \to \mathbf{R}$ be lower semicontinuous and $G : X \to \mathbf{R}$ upper semicontinuous. Assume: $G(x) < g(x)\ \forall\ x \in X$—then $\exists$ a continuous function $f : X \to \mathbf{R}$ such that $G(x) < f(x) < g(x)\ \forall\ x \in X$.

[Put $U_r = \{x : G(x) < r\} \cap \{x : g(x) > r\}$ (r rational). Each U_r is open and $X = \bigcup\limits_r U_r$. Let $\{\kappa_r\}$ be a partition of unity subordinate to $\{U_r\}$ and take $f = \sum\limits_r r\kappa_r$.]

The following result characterizes the class of X satisfying the conditions of Hahn's einschiebungsatz.

FACT Let X be a CRH space—then X is normal and countably paracompact iff for every lower semicontinuous $g : X \to \mathbf{R}$ and upper semicontinuous $G : X \to \mathbf{R}$ such that $G(x) < g(x)\ \forall\ x \in X$, $\exists\ f \in C(X, \mathbf{R}) : G(x) < f(x) < g(x)\ \forall\ x \in X$.

[Necessity: With r running through the rationals, there exists a neighborhood finite open covering $\{O_r\}$ of $X : O_r \subset \{x : G(x) < r < g(x)\}\ \forall\ r$ and a neighborhood finite open covering $\{P_r\}$ of $X : \overline{P}_r \subset O_r\ \forall\ r$. Fix a continuous function $f_r : X \to [-\infty, r]$ such that $f_r(x) = \begin{cases} -\infty & (x \notin O_r) \\ r & (x \in \overline{P}_r) \end{cases}$. Put $f(x) = \sup\limits_r f_r(x)$—then f has the required properties.

Sufficiency: There are two parts.

X is normal. Thus let A, B be disjoint closed subsets of X. With G the characteristic function of A, let g be defined by $\begin{cases} g(x) = 1 & (x \in B) \\ g(x) = 2 & (x \notin B) \end{cases}$: g is lower semicontinuous, G is upper semicontinuous, and $G(x) < g(x)\ \forall\ x \in X$. Choose $f \in C(X, \mathbf{R})$ per the assumption and let $U = \{x : f(x) > 1\}$, $V = \{x : f(x) < 1\}$—then $\begin{cases} U \\ V \end{cases}$ are disjoint open subsets of X and $\begin{cases} A \subset U \\ B \subset V \end{cases}$, hence X is normal.

X is countably paracompact. Thus consider any decreasing sequence $\{A_n\}$ of closed sets such that $\bigcap\limits_n A_n = \emptyset$. Put $g(x) = \dfrac{1}{n+1}(x \in A_n - A_{n+1}, n = 0, 1, \ldots)$ $(A_0 = X)$: g is lower semicontinuous. Take $f \in C(X, \mathbf{R}) : 0 < f(x) < g(x)$ and let $U_n = \{x : f(x) < \dfrac{1}{n+1}\}$—then $\{U_n\}$ is a decreasing sequence of open sets with $A_n \subset U_n$ for every n and $\bigcap\limits_n U_n = \emptyset$. Since X is normal, this guarantees that X is also countably paracompact (via CP (cf. p. 1–13)).]

LEMMA Assume that X is paracompact and suppose given a neighborhood finite closed covering $\{A_j : j \in J\}$ of X and $\forall\ j$, a positive real number a_j—then $\exists$ a continuous function $\phi : X \to \mathbf{R}_{>0}$ such that $\phi(x) < a_j$ if $x \in A_j$.

[The function from X to $\mathbf{R}$ defined by the rule $x \to \min\{a_j : x \in A_j\}$ is lower semicontinuous and strictly positive.]

PROPOSITION 11 The majorant topology on $C(X,Y)$ is independent of the choice of d provided that X is paracompact.

[It suffices to show that the graph topology on $C(X,Y)$ is smaller than the majorant topology (cf. p. 2–10). So fix an $f \in \Gamma_O$ and consider any $x_0 \in X$. Choose a neighborhood U_0 of x_0 and a positive real number a_0 such that $x \in U_0$ & $d(f(x_0),y) < 2a_0 \Rightarrow (x,y) \in O$. Choose further a neighborhood V_0 of x_0 such that $V_0 \subset U_0$ & $d(f(x_0),f(x)) < a_0 \ \forall\, x \in V_0$—then $\{(x,y) : x \in V_0 \ \&\ d(f(x),y) < a_0\} \subset O$. From this, it follows that one can find a neighborhood finite closed covering $\{A_j : j \in J\}$ of X and a set $\{a_j : j \in J\}$ of positive real numbers for which $\{(x,y) : x \in A_j \ \&\ d(f(x),y) < a_j\} \subset O$. In view of the lemma, $\exists$ a continuous function $\phi : X \to \mathbf{R}_{>0}$ with $\phi(x) < a_j$ whenever $x \in A_j$, hence $N_\phi(f) \subset \Gamma_O$, i.e., every point of Γ_O is an interior point in the majorant topology.]

To reiterate: If X is paracompact, then the majorant topology on $C(X,Y)$ equals the graph topology.

[Note: The assumption of paracompactness can be relaxed (see below).]

Let X be a CRH space, (Y,d) a metric space. Given $f \in C(X,Y)$ and a lower semicontinuous $\sigma : X \to \mathbf{R}_{>0}$, put $N_\sigma(f) = \{g : d(f(x),g(x)) < \sigma(x) \ \forall\, x\}$.

Observations: (1) If $\sigma_1, \sigma_2 : X \to \mathbf{R}_{>0}$ are lower semicontinuous, then $N_\sigma(f) \subset N_{\sigma_1}(f) \cap N_{\sigma_2}(f)$, where $\sigma(x) = \min\{\sigma_1(x), \sigma_2(x)\}$; (2) If $g \in N_\sigma(f)$, then $N_\tau(g) \subset N_\sigma(f)$, where $\tau(x) = \sigma(x) - d(f(x),g(x))$.

[Note: The minimum of two lower semicontinuous functions is lower semicontinuous, so σ is lower semicontinuous. On the other hand, the sum of two lower semicontinuous functions is lower semicontinuous. But $x \to d(f(x),g(x))$ is continuous, thus $x \to -d(f(x),g(x))$ is lower semicontinuous, so τ is lower semicontinuous.]

Therefore the collection $\{N_\sigma(f)\}$ is a basic system of neighborhoods at f. Accordingly, varying f leads to a topology on $C(X,Y)$, the semimajorant topology.

LEMMA The semimajorant topology on $C(X,Y)$ is smaller than the graph topology.

[Let $O = \{(x,y) : d(f(x),y) < \sigma(x)\}$—then Γ_O is open in $C(X,Y)$. Proof: Fix $(x_0,y_0) \in O$, put $\epsilon = \dfrac{1}{3}(\sigma(x_0) - d(f(x_0),y_0))$, and note that the subset of O consisting of those (x,y) such that $\sigma(x) > \sigma(x_0) - \epsilon$, $d(f(x),f(x_0)) < \epsilon$, and $d(y,y_0) < \epsilon$ is open. And: $N_\sigma(f) = \Gamma_O$.]

LEMMA The graph topology on $C(X,Y)$ is smaller than the semimajorant topology.

[Fix an $f \in \Gamma_O$. Define a strictly positive function $\sigma : X \to \mathbf{R}$ by letting $\sigma(x_0)$ be the supremum of those $a_0 \in]0,1]$ for which x_0 has a neighborhood U_0 such that $x \in U_0$ & $d(f(x_0),y) < a_0 \Rightarrow (x,y) \in O$. Since $N_\sigma(f) \subset \Gamma_O$, the point is to prove that σ is lower semicontinuous, i.e., that $\forall\, c \in \mathbf{R}$, $\{x : c < \sigma(x)\}$ is open. This is trivial if $c \leq 0$ or $c \geq 1$, so take $c \in]0,1[$ and fix $x_0 : c < \sigma(x_0)$. Put $\epsilon = (\sigma(x_0) - c)/3$—then

$c + 2\epsilon < \sigma(x_0)$, thus $\exists$ a neighborhood U_0 of x_0 such that $x \in U_0$ & $d(f(x_0), y) < c + 2\epsilon \Rightarrow (x, y) \in O$. Supposing further that $x \in U_0 \Rightarrow d(f(x_0), f(x)) < \epsilon$, one has $x \in U_0$ & $d(f(x), y) < c + \epsilon \Rightarrow (x, y) \in O \Rightarrow c < c + \epsilon \leq \sigma(x)$.]

FACT The semimajorant topology on $C(X, Y)$ equals the graph topology.

A CRH space X is said to be a CB space if for every strictly positive lower semicontinuous $\sigma : X \to \mathbf{R}$ there exists a strictly positive continuous $\phi : X \to \mathbf{R}$ such that $0 < \phi(x) \leq \sigma(x)\ \forall\, x \in X$.

Example: If X is normal and countably paracompact, then X is a CB space (cf. p. 2–11).

Examples (Mack[†]): (1) Every countably compact space is a CB space; (2) Every CB space is countably paracompact.

EXAMPLE The Isbell-Mrówka space $\Psi(\mathbf{N})$ is a pseudocompact LCH space which is not countably paracompact (cf. p. 1–12), hence is not a CB space.

FACT The majorant topology on $C(X, Y)$ equals the graph topology $\forall$ pair (Y, d) iff X is a CB space.

[Necessity: Fix a strictly positive lower semicontinuous $\sigma : X \to \mathbf{R}$. Specialized to the case $Y = \mathbf{R}$, the assumption is that the majorant topology on $C(X)$ equals the semimajorant topology, so working with $N_\sigma(0)$, $\exists\, \phi : N_\phi(0) \subset N_\sigma(0) \Rightarrow (1 - \epsilon)\phi \in N_\phi(0) \subset N_\sigma(0)$ $(0 < \epsilon < 1) \Rightarrow 0 < \phi(x) \leq \sigma(x)\ \forall\, x \in X$, thus X is a CB space.

Sufficiency: Since $N_\phi(f) \subset N_\sigma(f)$, the semimajorant topology on $C(X, Y)$ is smaller than the majorant topology.]

If (Y, d) is a complete metric space, then $\mathrm{co}C(X, Y)$ need not be Baire. Examples: (1) $\mathrm{co}C([0, \Omega[, \mathbf{R})$ is not Baire; (2) $\mathrm{co}C(\mathbf{Q}, \mathbf{R})$ is not Baire.

[Note: Recall, however, that if X is countable at infinity and compactly generated, then $\mathrm{co}C(X, Y)$ is completely metrizable (cf. Proposition 7), hence is Baire.]

PROPOSITION 12 Assume: (Y, d) is a complete metric space—then $C(X, Y)$ (majorant topology) is Baire.

[Let $\{O_n\}$ be a sequence of dense open subsets of $C(X, Y)$. Let U be a nonempty open subset of $C(X, Y)$. Since $U \cap O_1$ is nonempty and open and since $C(X, Y)$ is completely regular (cf. p. 2–8), $\exists\, f_1 \in U \cap O_1$ & $\phi_1 \in C(X, \mathbf{R}_{>0}) : \{g : d(f_1(x), g(x)) \leq \phi_1(x)\ \forall\, x\} \subset U \cap O_1$, where $\phi_1 < 1$. Next, $\exists\, f_2 \in N_{\phi_1}(f_1) \cap O_2$ & $\phi_2 \in C(X, \mathbf{R}_{>0}) : \{g : d(f_2(x), g(x)) \leq$

[†] *Proc. Amer. Math. Soc.* **16** (1965), 467–472.

$\phi_2(x)\ \forall\ x\} \subset N_{\phi_1}(f_1) \cap O_2$, where $\phi_2 < \phi_1/2$. Proceeding, $\exists\ f_{n+1} \in N_{\phi_n}(f_n) \cap O_{n+1}$ & $\phi_{n+1} \in C(X, \mathbf{R}_{>0}) : \{g : d(f_{n+1}(x), g(x)) \leq \phi_{n+1}(x)\ \forall\ x\} \subset N_{\phi_n}(f_n) \cap O_{n+1}$, where $\phi_{n+1} < \phi_n/2$. So, $\forall\ x$, $d(f_{n+1}(x), f_n(x)) \leq \frac{1}{2^{n-1}}$, thus $\{f_n(x)\}$ is a Cauchy sequence in Y. Definition: $f(x) = \lim f_n(x)$. Because the convergence is uniform, $f \in C(X,Y)$. Moreover, $d(f_n(x), f(x)) \leq \phi_n(x)\ \forall\ n$ & $\forall\ x$, which implies that $f \in U \cap (\bigcap_n O_n)$.]

FACT Assume: (Y, d) is a complete metric space—then $C(X,Y)$ (limitation topology) is Baire.

Convention: Maintaining the assumption that X is a CRH space, $C(X)$ henceforth carries the compact open topology.

Let K be a compact subset of X. Put $p_K(f) = \sup_K |f| (f \in C(X))$—then $p_K : C(X) \to \mathbf{R}$ is a seminorm on $C(X)$, i.e., $p_K(f) \geq 0$, $p_K(f+g) \leq p_K(f) + p_K(g)$, $p_K(cf) = |c| p_K(f)$.

[Note: More is true, viz. p_K is multiplicative in the sense that $p_K(fg) \leq p_K(f) p_K(g)$.]

Remark: The initial topology on $C(X)$ determined by the p_K as K runs through the compact subsets of X is the compact open topology.

[Note: In the compact open topology, $C(X)$ is a Hausdorff locally convex topological vector space.]

Observation: If $K \subset X$ is compact and if $f \in C(K)$, then $\exists\ F \in BC(X) : F|K = f$. Proof: Apply the Tietze extension theorem to K regarded as a compact subset of βX.

A CRH space X is said to be a $k_{\mathbf{R}}$-space provided that a real valued function $f : X \to \mathbf{R}$ is continuous whenever its restriction to each compact subset of X is continuous. Example: A compactly generated X is a $k_{\mathbf{R}}$-space (but not conversely (cf. infra)).

EXAMPLE Let X be a $k_{\mathbf{R}}$-space. Assume: X is countable at infinity—then X is compactly generated.

[Fix a "defining" sequence $\{K_n\}$ of compact subsets of X with $K_n \subset K_{n+1}\ \forall\ n$. Claim: A subset A of X is closed if $A \cap K_n$ is closed in K_n for each n. For if not, then A has an accumulation point $a_0 : a_0 \notin A$, which can be taken in K_1 (adjust the notation). Choose a continuous function $f_1 : K_1 \to \mathbf{R}$ such that $f_1(A \cap K_1) = \{0\}$ and $f_1(a_0) = 1$. Extend f_1 to a continuous function $f_2 : K_2 \to \mathbf{R}$ such that $f_2(A \cap K_2) = \{0\}$. Repeat the process to get a function $f : X \to \mathbf{R}$ such that $f(x) = f_n(x)$ $(x \in K_n)$. Since X is a $k_{\mathbf{R}}$-space, f is continuous. This, however, is a contradiction: $f(A) = \{0\}$, $f(a_0) = 1$.]

FACT A $k_{\mathbf{R}}$-space X is compactly generated iff kX is completely regular.

[If X is a $k_{\mathbf{R}}$-space, then $C(X) = C(kX)$. So, the supposition that kX is completely regular forces $X = kX$ (cf. §1, Proposition 14).]

[Note: Recall that in general, X completely regular $\not\Rightarrow kX$ completely regular (cf. p. 1–36).]

PROPOSITION 13 $C(X)$ is complete as a topological vector space iff X is a $k_{\mathbf{R}}$-space.

[Necessity: Suppose that $f : X \to \mathbf{R}$ is a real valued function such that $f|K$ is continuous $\forall$ compact $K \subset X$. Let $f_K \in C(X)$ be an extension of $f|K$—then $\{f_K\}$ is a Cauchy net in $C(X)$, thus is convergent, say $\lim f_K = F$. But $f = F$.

Sufficiency: Let $\{f_i\}$ be a Cauchy net in $C(X)$—then $\forall$ compact $K \subset X$, the net $\{f_i|K\}$ is Cauchy in $C(K)$, hence has a limit, call it f_K. If $K_1 \subset K_2$, then $f_{K_2}|K_1 = f_{K_1}$, so the prescription $f(x) = f_K(x)$ $(x \in K)$ defines a function $f : X \to \mathbf{R}$. Since X is a $k_{\mathbf{R}}$-space, f is continuous. And: $\lim f_i = f$.]

EXAMPLE Let κ be a cardinal $> \omega$—then $\mathbf{N}^\kappa$ is a $k_{\mathbf{R}}$-space but $\mathbf{N}^\kappa$ is not compactly generated.

[Note: $\mathbf{N}^\omega$ is homeomorphic to $\mathbf{P}$, thus is compactly generated.]

FACT Suppose that the closed bounded subsets of $C(X)$ are complete—then X is a $k_{\mathbf{R}}$-space.

PROPOSITION 14 $C(X)$ is metrizable iff X is countable at infinity (cf. Proposition 6).

[Let d be a compatible metric on $C(X)$. Put $U_n = \{f : d(f,0) < 1/n\}$. Choose a compact $K_n \subset X$ and a positive $\epsilon_n : f(K_n) \subset]-\epsilon_n, \epsilon_n[\Rightarrow f \in U_n$—then for any compact subset K of X, $\exists\, n : K \subset K_n$. Therefore X is countable at infinity.]

PROPOSITION 15 $C(X)$ is completely metrizable iff X is countable at infinity and compactly generated (cf. Proposition 7).

[If $C(X)$ is completely metrizable, then $C(X)$ is complete as a topological vector space, so X is a $k_{\mathbf{R}}$-space (cf. Proposition 13), thus X, being countable at infinity, is compactly generated (cf. p. 2–14).]

A CRH space X is said to be topologically complete if X is a G_δ in βX or still, if X is a G_δ in any Hausdorff space containing it as a dense subspace. Example: $\mathbf{P}$ is topologically complete but $\mathbf{Q}$ is not.

Examples: (1) Every completely metrizable space is topologically complete and every topologically complete metrizable space is completely metrizable; (2) Every LCH space is topologically complete.

[Note: A topologically complete space is necessarily compactly generated and Baire (Engelking[†]).]

[†] *General Topology*, Heldermann Verlag (1989), 197–198.

Remark: It can be shown that Proposition 15 goes through if the hypothesis "completely metrizable" is weakened to "topologically complete" (McCoy-Ntantu[†]).

EXAMPLE Let X be a LCH space. Assume: X is paracompact—then $C(X)$ is Baire.

[Using LCH$_3$ (cf. p. 1–2), write $X = \coprod_i X_i$, where the X_i are pairwise disjoint nonempty open σ-compact subspaces of X. Each X_i is countable at infinity and there is a homeomorphism $C(X) \approx \prod_i C(X_i)$. But the $C(X_i)$ are completely metrizable (cf. Proposition 15), hence are topologically complete, and it is a fact that a product of topologically complete spaces is Baire (Oxtoby[‡]).]

[Note: The paracompactness assumption on X cannot be dropped. Example: Take $X = [0, \Omega[$—then $C(X)$ is not Baire. Proof: Since X is pseudocompact, $O_n = \bigcup_x \{f : n < f(x) < n+1\}$ is a dense open subset of $C(X)$ and $\bigcap_n O_n = \emptyset$.]

FACT Suppose that X is first countable and $C(X)$ is Baire—then X is locally compact.

STONE-WEIERSTRASS THEOREM Let X be a compact Hausdorff space. Suppose that $\mathcal{A}$ is a subalgebra of $C(X)$ which contains the constants and separates the points of X—then $\mathcal{A}$ is uniformly dense in $C(X)$.

EXAMPLE Let $0 < a < b < 1$—then every $f \in C([a,b])$ can be uniformly approximated by polynomials $\sum_1^d n_k x^k$, n_k integral.

[It is enough to show that $f = \frac{1}{2}$ can be so approximated. Given an odd prime p, put $\phi_p(x) = \frac{1}{p}(1 - x^p - (1-x)^p)$: ϕ_p is a polynomial with integral coefficients, no constant term, and $p\phi_p \to 1$ uniformly on $[a,b]$ as $p \to \infty$. Now write $p = 2q+1$, note that $\left|\frac{1}{2} - \frac{q}{p}\right| < \frac{1}{p}$, and consider $q\phi_p$.]

PROPOSITION 16 Suppose that X is a compact Hausdorff space—then $C(X)$ is separable iff X is metrizable.

[Necessity: If $\{f_n\}$ is a uniformly dense sequence in $C(X)$, then the $\{x : |f_n(x)| > \frac{1}{2}\}$ constitute a basis for the topology on X, therefore X is second countable, hence metrizable.

Sufficiency: Let d be a compatible metric on X. Choose a countable basis $\{U_n\}$ for its topology and put $f_n(x) = d(x, X - U_n)$ $(x \in X)$—then the f_n separate the points of X, thus the subalgebra of $C(X)$ generated by 1 and the f_n is uniformly dense in $C(X)$, so the same is true of the rational subalgebra of $C(X)$ generated by 1 and the f_n. But the latter is a countable set.]

[†] *SLN* **1315** (1988), 75.

[‡] *Fund. Math.* **49** (1961), 157–166.

EXAMPLE Assume that X is not compact and consider $BC(X)$, viewed as a Banach space in the supremum norm: $||f|| = \sup_X |f|$—then $BC(X)$ can be identified with $C(\beta X)$ $(f \rightarrow \beta f : ||f|| = ||\beta f||)$. Since βX is not metrizable, it follows that $BC(X)$ is not separable.

[Note: To see that βX is not metrizable, fix a point $x_0 \in \beta X - X$ and, arguing by contradiction, choose a sequence $\{x_n\} \subset X$ of distinct x_n having x_0 for their limit. Put $A = \{x_{2n}\}$, $B = \{x_{2n+1}\}$—then A and B are disjoint closed subsets of X, so, by Urysohn, $\exists\ \phi \in BC(X)$ such that $0 \leq \phi \leq 1$ with $\phi = 1$ on A and $\phi = 0$ on B. Therefore $1 = \phi(x_{2n}) \rightarrow \beta\phi(x_0)$ & $0 = \phi(x_{2n+1}) \rightarrow \beta\phi(x_0)$, an absurdity.]

PROPOSITION 17 $C(X)$ is separable iff X admits a smaller separable metrizable topology.

[Necessity: Fix a countable dense set $\{f_n\}$ in $C(X)$—then $\{f_n\}$ separates the points of X and the initial topology on X determined by the f_n is a separable metrizable topology. Reason: The arrow $X \rightarrow \mathbf{R}^\omega$ defined by the rule $x \rightarrow \{f_n(x)\}$ is an embedding.

Sufficiency: Let X_0 stand for X equipped with a smaller separable metrizable topology. Embed X_0 in $[0,1]^\omega$. Fix a countable dense set $\{\phi_n\}$ in $C([0,1]^\omega)$ (cf. Proposition 16) and put $f_n = \phi_n|X_0$—then the sequence $\{f_n\}$ is dense in $C(X_0)$, thus $C(X_0)$ is separable. Indeed, given a compact subset K_0 of X_0 and $f_0 \in C(X_0)$, $\exists\ \phi_0 \in C([0,1]^\omega) : \phi_0|K_0 = f_0|K_0$ & $\forall\ \epsilon > 0$, $\exists\ \phi_n : p_{K_0}(\phi_n - \phi_0) < \epsilon \Rightarrow p_{K_0}(f_n - f_0) < \epsilon$. Finally, the separability of $C(X_0)$ forces the separability of $C(X)$. This is because a compact subset K of X is a compact subset of X_0 and the two topologies induce the same topology on K.]

Example: Take $X = \mathbf{R}$ (discrete topology)—then $C(X)$ is separable.

EXAMPLE If $X = \bigcup_n K_n$, where each K_n is compact and metrizable, then $C(X)$ is separable.

[There is no loss of generality in supposing that $K_n \subset K_{n+1}\ \forall\ n$. Choose a countable dense subset $\{f_{n,m}\}$ in $C(K_n)$ (cf. Proposition 16) and let $F_{n,m}$ be a continuous extension of $f_{n,m}$ to X—then the initial topology on X determined by the $F_{n,m}$ is a separable metrizable topology which is smaller than the given topology on X, so $C(X)$ is separable (cf. Proposition 17).]

FACT Let X be a LCH space—then $C(X)$ is separable and metrizable iff X is separable and metrizable.

FACT Let X be a LCH space—then $C(X)$ is separable and completely metrizable iff X is separable and completely metrizable.

PROPOSITION 18 $C(X)$ is first countable iff X is countable at infinity.

PROPOSITION 19 $C(X)$ is second countable iff X is countable at infinity and all the compact subsets of X are metrizable.

[Necessity: $C(X)$ second countable $\Rightarrow$ $C(X)$ first countable $\Rightarrow$ X countable at infinity (cf. Proposition 18). In addition, $C(X)$ second countable $\Rightarrow$ $C(X)$ separable. So, by Proposition 17, X admits a smaller separable metrizable topology which, however, induces the same topology on each compact subset of X.

Sufficiency: The hypotheses on X guarantee that $C(X)$ is separable (via the example above) and metrizable (cf. Proposition 14).]

EXAMPLE Let E be an infinite dimensional locally convex topological vector space. Assume: E is second countable and completely metrizable—then the Anderson-Kadec theorem says that E is homeomorphic to $\mathbf{R}^\omega$ (for a proof, see Bessaga-Pełczyński[†]). Consequently, if X is countable at infinity and compactly generated and if all the compact subsets of X are metrizable, then $C(X)$ is homeomorphic to $\mathbf{R}^\omega$.

FACT Suppose that X is second countable—then $C(X)$ is Lindelöf.

Up until this point, the playoff between X and $C(X)$ has been primarily "topological", little use having been made of the fact that $C(X)$ is also a locally convex topological vector space. It is thus only natural to ask: Can one characterize those X for which $C(X)$ has a certain additional property (e.g., barrelled or bornological)? While this theme has generated an extensive literature, I shall present just two results, namely Propositions 20 and 21, these being due independently to Nachbin[‡] and Shirota[||].

FACT $C(X)$ is reflexive iff X is discrete.

[Assuming that $C(X)$ is reflexive, its bounded weakly closed subsets are weakly compact. Therefore the compact subsets of X are finite which means that $C(X)$ is a dense subspace of $\mathbf{R}^X$ (product topology). But the reflexiveness of $C(X)$ also implies that its closed bounded subsets are complete, hence X is a $k_{\mathbf{R}}$-space (cf. p. 2–15). Thus $C(X)$ is complete (cf. Proposition 13), so $C(X) = \mathbf{R}^X$ and X is discrete.]

A subset A of X is said to be bounding if every $f \in C(X)$ is bounded on A. Example: X is pseudocompact iff X is bounding.

[†] *Selected Topics in Infinite Dimensional Topology*, PWN (1975), 189.

[‡] *Proc. Nat. Acad. Sci. U.S.A.* **40** (1954), 471–474.

[||] *Proc. Japan Acad. Sci.* **30** (1954), 294–298.

Given a subset W of $C(X)$, let $K(W)$ be the subset of X consisting of those x with the property that for every neighborhood O_x of x there exists an $f \in C(X)$: $f(X - O_x) = \{0\}$ & $f \notin W$.

BOUNDING LEMMA If W is a barrel in $C(X)$, then $K(W)$ is bounding.

[Suppose that $K(W)$ is not bounding and fix an infinite discrete collection $\mathcal{O} = \{O\}$ of open subsets of X such that $O \cap K(W) \neq \emptyset\ \forall\, O \in \mathcal{O}$. Choose an element $O_1 \in \mathcal{O}$. Since $O_1 \cap K(W) \neq \emptyset$, $\exists\, f_1 \in C(X) : f_1(X - O_1) = \{0\}$ & $f_1 \notin W$. On the other hand, W, being a barrel, is closed, so $\exists$ a compact $K_1 \subset X$ and a positive $\epsilon_1 : \{g : p_{K_1}(f_1 - g) < \epsilon_1\} \cap W = \emptyset$. Choose next an element $O_2 \in \mathcal{O} : O_2 \cap K_1 = \emptyset$ and continue. The upshot is that there exist sequences $\{O_n\}$, $\{f_n\}$, $\{K_n\}$, $\{\epsilon_n\}$ with the following properties: (1) $O_{n+1} \cap (\bigcup_{i=1}^{n} K_i) = \emptyset$; (2) $f_n(X - O_n) = \{0\}$ & $f_n \notin W$; (3) $\{g : p_{K_n}(f_n - g) < \epsilon_n\} \cap W = \emptyset$. Take $c_1 = 1$ and determine $c_{n+1} : 0 < c_{n+1} < \dfrac{1}{n+1}$, subject to the requirement that $c_{n+1} p_{K_{n+1}}(\sum_{i=1}^{n} \frac{1}{c_i} f_i) < \epsilon_{n+1}\ \forall\, n$. Put $f = \sum_{i=1}^{\infty} \frac{1}{c_i} f_i$—then by (2) and the discreteness of $\{O_n\}$, f is continuous, and (1)–(3) combine to imply that $c_{n+1} f \notin W\ \forall\, n$, thus W does not absorb the function f, a contradiction.]

LEMMA OF DETERMINATION If W is a barrel in $C(X)$ and if f is an element of $C(X)$ such that $f(x) = 0\ \forall\, x \in U$, where U is an open set containing $K(W)$, then $f \in W$.

[Suppose false. Choose a compact $K \subset X$ and a positive $\epsilon : \{g : p_K(f - g) < \epsilon\} \cap W = \emptyset$, and for each $x \in K - U$, choose a neighborhood O_x of $x : g(X - O_x) = \{0\} \Rightarrow g \in W$. Fix $f_x \in C(X, [0,1]) : f_x(x) = 1$ & $f_x | X - O_x = 0$, and let $U_x = \{y : f_x(y) > 1/2\}$. The U_x comprise an open covering of $K - U$, thus one can extract a finite subcovering $U_{x_1}, \ldots, U_{x_n}$. Put $\kappa_{x_i} = \dfrac{f_{x_i}}{\max\{1/2, f_{x_1} + \cdots + f_{x_n}\}}$ $(i = 1, \ldots, n)$— then $\sum_{i=1}^{n} \kappa_{x_i} | K - U = 1$. Since $\kappa_{x_i}(X - O_{x_i}) = \{0\}$, $c\kappa_{x_i} f \in W (c \in \mathbf{R})$, therefore $F = \kappa_{x_1} f + \cdots + \kappa_{x_n} f = \dfrac{1}{n}(n\kappa_{x_1} f + \cdots + n\kappa_{x_n} f) \in W$. But by its very construction, $F | K = f | K \Rightarrow F \notin W$.]

PROPOSITION 20 $C(X)$ is barrelled iff every bounding subset of X is relatively compact.

[Necessity: Rephrased, the assertion is that for any closed noncompact subset S of X, $\exists\, f \in C(X) : f$ is unbounded on S. Thus let $B_S = \{f : \sup_S |f| \leq 1\}$—then B_S is balanced and convex. Since B_S is also closed and since the requirement that there be some $f \in C(X)$ which is unbounded on S amounts to the failure of B_S to be absorbing, it need only be shown that B_S does not contain a neighborhood of 0. Assuming the opposite, choose a compact K and a positive $\epsilon : \{f : p_K(f) < \epsilon\} \subset B_S$. Claim: $S \subset K$. Proof:

If $x \in S - K$, $\exists\, f \in C(X) : f(K) = \{0\}$ & $f(x) = 2$, an impossibility. Therefore S is compact (being closed), contrary to hypothesis.

Sufficiency: Fix a barrel W in $C(X)$—then the contention is that W contains a neighborhood of 0. Owing to the bounding lemma, $K(W)$ is compact (inspect the definitions to see that $K(W)$ is closed). Accordingly, it suffices to produce a positive $\epsilon : \{f : p_{K(W)}(f) < \epsilon\} \subset W$. To this end, consider $BC(X)$ viewed as a Banach space in the supremum norm. Because $BC(X)$ is barrelled and $W \cap BC(X)$ is a barrel in $BC(X)$, $\exists\, \epsilon > 0 : \|\phi\| \leq 2\epsilon \Rightarrow \phi \in W$ $(\phi \in BC(X))$. Assuming that $p_{K(W)}(f) < \epsilon$, fix an open set U containing $K(W)$ such that $|f(x)| < \epsilon \; \forall\, x \in U$. Let $F(x) = \max\{\epsilon, f(x)\} + \min\{-\epsilon, f(x)\}$—then $2F(x) = 0$ $(x \in U)$, thus the lemma of determination implies that $2F \in W$. But $\forall\, x \in X$, $|2(f(x) - F(x))| < 2\epsilon \Rightarrow \|2(f - F)\| \leq 2\epsilon \Rightarrow 2(f - F) \in W$, so $\dfrac{1}{2}(2F) + \dfrac{1}{2}(2(f - F)) \in W$, i.e., $f \in W$.]

Example: $C([0, \Omega[)$ is not barrelled.

EXAMPLE If X is a paracompact LCH space, then $C(X)$ is Baire (cf. p. 2-16). Since Baire $\Rightarrow$ barrelled, it follows from Proposition 20 that the bounding subsets of X are relatively compact.

Notation: Every $f \in C(X)$ can be regarded as an element of $C(X, \mathbf{R}_\infty)$, hence admits a unique continuous extension $f_\infty : \beta X \to \mathbf{R}_\infty$.

[Note: Put $v_f X = \{x \in \beta X : f_\infty(x) \in \mathbf{R}\}$—then the intersection $\bigcap\limits_{f \in C(X)} v_f X$ is vX.]

FACT The elements of $\beta X - vX$ are those x with the property that there exists a G_δ in βX containing x which does not meet X.

Let W be a balanced, convex subset of $C(X)$—then W is said to contain a ball if $\exists\, r > 0 : \{f : \sup\limits_X |f| \leq r\} \subset W$.

Example: Every balanced, convex bornivore W in $C(X)$ contains a ball.

[Given $f, g \in C(X)$ with $f \leq g$, let $[f, g] = \{\phi : f \leq \phi \leq g\}$. Since $\forall$ compact $K \subset X$, $p_K(\phi) \leq \max\{p_K(f), p_K(g)\}$, $[f, g]$ is bounded, thus is absorbable by W. In particular: $\exists\, r > 0$ such that $[-r1, r1] \subset W$.]

FACT Suppose that W contains a ball. Let K be a compact subset of X. Assume: $f(K) = \{0\} \Rightarrow f \in W$—then $\exists\, \epsilon > 0 : \{f : p_K(f) < \epsilon\} \subset W$.

Let W be a balanced, convex subset of $C(X)$—then a compact subset K of βX is said to be a hold of W if $f \in W$ whenever $f_\infty(K) = \{0\}$. Example: βX is a hold of W.

LEMMA Suppose that W contains a ball—then a compact subset K of βX is a hold of W provided that $f \in W$ whenever f_∞ vanishes on some open subset O of βX containing K.

Application: Under the assumption that W contains a ball, if K and L are holds of W, then so is $K \cap L$.

[Consider any $f : f_\infty(O) = \{0\}$, where O is some open subset of βX containing $K \cap L$. Choose disjoint open subsets U, V of $\beta X : K \subset U$, $L - O \subset V$ and let U', V' be open subsets of $\beta X : K \subset U' \subset \overline{U}' \subset U$, $L - O \subset V' \subset \overline{V}' \subset V$. Fix $\phi \in C(X, [0,1]) : \beta\phi(\overline{U}') = \{1\}$, $\beta\phi(\overline{V}') = \{0\}$. Note that $2f\phi$ vanishes on $(O \cup V') \cap X$. But $O \cup V' \subset \overline{(O \cup V') \cap X} \Rightarrow (2f\phi)_\infty(O \cup V') = \{0\}$. On the other hand, $L \subset O \cup V'$, thus by the lemma, $2f\phi \in W$. Similarly, $2f(1 - \phi) \in W$. Therefore $f = \frac{1}{2}(2f\phi) + \frac{1}{2}(2f(1 - \phi)) \in W$.]

Let W be a balanced, convex subset of $C(X)$—then the support of W, written spt W, is the intersection of all the holds of W.

LEMMA Suppose that W contains a ball—then spt W is a hold of W.

[Since βX is a compact Hausdorff space, for any open $O \subset \beta X$ containing spt W, $\exists$ holds $K_1, \ldots, K_n$ of W such that $\bigcap\limits_{i=1}^{n} K_i \subset O$.]

PROPOSITION 21 $C(X)$ is bornological iff X is $\mathbf{R}$-compact.

[Necessity: Assuming that X is not $\mathbf{R}$-compact, fix a point $x_0 \in \upsilon X - X$—then the assignment $f \rightarrow f_\infty(x_0)$ defines a nontrivial homomorphism $\widehat{x}_0 : C(X) \rightarrow \mathbf{R}$, which is necessarily discontinuous (cf. p. 2–24). So, to conclude that $C(X)$ is not bornological, it suffices to show that $\widehat{x}_0$ takes bounded sets to bounded sets. If this were untrue, then there would be a bounded subset $B \subset C(X)$ and a sequence $\{f_n\} \subset B$ such that $\widehat{x}_0(f_n) \rightarrow \infty$. The intersection $\bigcap\limits_n \{x \in \beta X : (f_n)_\infty(x) > (f_n)_\infty(x_0) - 1\}$ is a G_δ in βX containing x_0, thus it must meet X (cf. p. 2–20), say at x_{00} hence $f_n(x_{00}) \rightarrow \infty$. But then, as B is bounded, $\dfrac{f_n}{f_n(x_{00})} \rightarrow 0$ in $C(X)$, which is nonsense.

Sufficiency: It is a question of proving that every balanced, convex bornivore W in $C(X)$ contains a neighborhood of 0. Because W contains a ball, the lemma implies that spt W is a hold of W, thus the key is to establish the containment spt $W \subset X$ since this will allow one to say that $\exists\ \epsilon > 0 : \{f : p_{\text{spt}\,W}(f) < \epsilon\} \subset W$ (cf. p. 2–20). So take a point $x_0 \in \beta X - X$ and choose closed subsets $A_1 \supset A_2 \supset \cdots$ of $\beta X : \forall\ n$, $x_0 \in \text{int}\, A_n$ & $(\bigcap\limits_n A_n) \cap X = \emptyset$ (possible, X being $\mathbf{R}$-compact (cf. p. 2–20)). Claim: At least one of the $\beta X - \text{int}\, A_n$ is a hold of W ($\Rightarrow x_0 \notin \text{spt}\, W \Rightarrow \text{spt}\, W \subset X$). If not, then $\forall\ n$,

$\exists\ f_n : (f_n)_\infty(\beta X - \operatorname{int} A_n) = 0$ & $f_n \notin W$. The sequence $\{X - A_n\}$ is an increasing sequence of open subsets of X whose union is X. Therefore $f = \sup\limits_n n|f_n|$ is in $C(X)$. Fix $d > 0$: $[-f, f] \subset dW$—then $nf_n \in dW\ \forall\, n \Rightarrow f_n \in W\ \forall\, n \geq d$, a contradiction.]

LEMMA A subset A of X is bounding iff its closure in βX is contained in υX.

FACT If $C(X)$ is bornological, then $C(X)$ is barrelled.

[Note: Recall that in general, bornological $\not\Rightarrow$ barrelled and barrelled $\not\Rightarrow$ bornological.]

Remark: There are completely regular Hausdorff spaces X whose bounding subsets are relatively compact but that are not $\mathbf{R}$-compact (Gillman-Henriksen[†]). For such X, $C(X)$ is therefore barrelled but not bornological.

Given a closed subset A of X, let $I_A = \{f : f|A = 0\}$—then I_A is a closed ideal in $C(X)$. Examples: (1) $I_\emptyset = C(X)$; (2) $I_X = \{0\}$.

SUBLEMMA Suppose that X is compact. Let $I \subset C(X)$ be an ideal. Assume: $\forall\, x \in X, \exists\, f_x \in I : f_x(x) \neq 0$—then $I = C(X)$.

[$\forall\, x \in X$, $\exists$ a neighborhood U_x of x : $f_x|U_x \neq 0$. Choose points $x_1, \ldots, x_n : X = \bigcup\limits_{i=1}^{n} U_{x_i}$ and let $f = \sum\limits_{i=1}^{n} f_{x_i}^2 : f \in I \Rightarrow 1 = f \cdot \dfrac{1}{f} \in I \Rightarrow I = C(X)$.]

LEMMA Suppose that X is compact. Let $I \subset C(X)$ be an ideal and put $A = \bigcap\limits_{f \in I} Z(f)$. Assume: $A \subset U \subset Z(\phi)$, where U is open and $\phi \in C(X)$—then $\phi \in I$.

[The restriction $I|X - U$ is an ideal in $C(X - U)$ (Tietze), hence by the sublemma, equals $C(X - U)$. Choose an $f \in I : f|X - U = 1$ to get $\phi = f\phi \in I$.]

PROPOSITION 22 Suppose that X is compact. Let $I \subset C(X)$ be an ideal—then $\overline{I} = I_A$, where $A = \bigcap\limits_{f \in I} Z(f)$.

[Since $I \subset I_A$, it need only be shown that $I_A \subset \overline{I}$. So let f be a nonzero element of I_A and fix $\epsilon > 0$. Choose $\phi \in C(X, [0,1]) : \{x : |f(x)| \leq \epsilon/2\} \subset Z(\phi)$ & $\{x : |f(x)| \geq 3\epsilon/4\} \subset Z(1 - \phi)$. Because $A \subset \{x : |f(x)| < \epsilon/4\} \subset Z(f\phi)$, the lemma gives $f\phi \in I$. And: $\|f - f\phi\| = \sup\limits_X |f - f\phi| < \epsilon \Rightarrow f \in \overline{I}$.]

PROPOSITION 23 The closed subsets of X are in a one-to-one correspondence with the closed ideals of $C(X)$ via $A \rightarrow I_A$.

[†] *Trans. Amer. Math. Soc.* **77** (1954), 340–362 (cf. 360–362).

[Due to the complete regularity of X, the map $A \to I_A$ is injective. To see that it is surjective, it suffices to prove that for any closed ideal I in $C(X) : I = I_A$, where $A = \bigcap_{f \in I} Z(f)$. Obviously, $I \subset I_A$. On the other hand, $\forall$ compact $K \subset X$, the restriction $I|K$ is an ideal in $C(K)$ (cf. p. 2–14), thus $\overline{I|K} = I_{A \cap K}$ (cf. Proposition 22), and from this it follows that $I_A \subset \overline{I} = I$.]

Application: The points of X are in a one-to-one correspondence with the closed maximal ideals of $C(X)$ via $x \to I_{\{x\}}$.

By comparison, recall that the points of βX are in a one-to-one correspondence with the maximal ideals of $C(X)$.

[Note: Assign to each $x \in \beta X$ the subset m_x of $C(X)$ consisting of those f such that $x \in \mathrm{cl}_{\beta X}(Z(f))$—then m_x is a maximal ideal and all such have this form. For the details, see Walker[†].]

A character of $C(X)$ is a nonzero multiplicative linear functional on $C(X)$, i.e., a homomorphism $C(X) \to \mathbf{R}$ of algebras.

LEMMA If $\chi : \mathbf{R} \to \mathbf{R}$ is a nonzero ring homomorphism, then $\chi = \mathrm{id}_{\mathbf{R}}$.
[In fact, χ is order preserving and the identity on $\mathbf{Q}$.]

Application: Every ring homomorphism $C(X) \to \mathbf{R}$ is $\mathbf{R}$-linear, thus is a character.

LEMMA If $\chi : C(X) \to \mathbf{R}$ is a character of $C(X)$, then $\forall\, f$, $|\chi(f)| = \chi(|f|)$.
[For $|\chi(f)|^2 = \chi(f)^2 = \chi(f^2) = \chi(|f|^2) = \chi(|f|)^2$ and $\chi(|f|)$ is ≥ 0.]

By way of a corollary, if $\chi : C(X) \to \mathbf{R}$ is a character of $C(X)$ and if $\chi(f) = 0$, then $\chi(\min\{1, |f|\}) = 0$. Proof: $2\chi(\min\{1, |f|\}) = \chi(1) + \chi(f) - \chi(|1 - f|) = 1 - |\chi(1 - f)| = 1 - 1 = 0$.

FACT Write υf for the unique extension of $f \in C(X)$ to $C(\upsilon X)$—then $C(X)$ "is" $C(\upsilon X)$ and the characters of $C(X)$ are parameterized by the points of $\upsilon X : f \to \upsilon f(x)$ $(x \in \upsilon X)$.

[If X is $\mathbf{R}$-compact and if $\chi : C(X) \to \mathbf{R}$ is a character, then in the terminology of p. 19–6 & p. 19–7, $\mathcal{F}_\chi = \{Z(f) : \chi(f) = 0\}$ is a zero set ultrafilter on X. Claim: $\mathcal{F}_\chi$ has the countable intersection property. Thus let $\{Z(f_n)\} \subset \mathcal{F}_\chi$ be a sequence and put $f = \sum_1^\infty \frac{\min\{1, |f_n|\}}{2^n}$ —then $\bigcap_1^\infty Z(f_n) = Z(f)$. To prove that $\chi(f) = 0$, write $f = \sum_{i=1}^{n} \frac{\min\{1, |f_i|\}}{2^i} + g_n$, where $0 \leq g_n \leq 2^{-n}$, apply χ to get $\chi(f) =$

[†] *The Stone-Čech Compactification*, Springer Verlag (1974), 18.

$\chi(g_n) \leq 2^{-n}$, and let $n \to \infty$. It therefore follows that $\cap\mathcal{F}_\chi$ is nonempty, say $x \in \cap\mathcal{F}_\chi$ (cf. p. 19–7). And: $\chi(f - \chi(f)) = 0 \Rightarrow x \in Z(f - \chi(f)) \Rightarrow \chi(f) = f(x)$.]

Notation: $\widehat{C(X)}$ is the set of continuous characters of $C(X)$.

From the above, there is a one-to-one correspondence $X \to \widehat{C(X)}$, viz. $x \to \chi_x$, where $\chi_x(f) = f(x)$.

If X is not $\mathbf{R}$-compact, then the elements of $\upsilon X - X$ correspond to the discontinuous characters of $C(X)$.

Topologize $\widehat{C(X)}$ by giving it the initial topology determined by the functions $\chi \to \chi(f)$ $(f \in C(X))$—then the correspondence $X \to \widehat{C(X)}$ is a homeomorphism (cf. §1, Proposition 14).

PROPOSITION 24 Let $\begin{cases} X \\ Y \end{cases}$ be CRH spaces. Assume: $\begin{cases} C(X) \\ C(Y) \end{cases}$ are isomorphic as topological algebras—then $\begin{cases} X \\ Y \end{cases}$ are homeomorphic.

[Schematically,
$$\begin{tikzcd} X \arrow[d, equal] & Y \arrow[d, equal] \\ \widehat{C(X)} \arrow[r, leftrightarrow] & \widehat{C(Y)} \end{tikzcd}$$
and $\longleftrightarrow$ is a homeomorphism.]

FACT Let $\begin{cases} X \\ Y \end{cases}$ be CRH spaces. Assume: $\begin{cases} C(X) \\ C(Y) \end{cases}$ are isomorphic as algebras—then $\begin{cases} \upsilon X \\ \upsilon Y \end{cases}$ are homeomorphic.

§ 2

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§3. COFIBRATIONS

The machinery assembled here is the indispensable technical prerequisite for the study of homotopy theory in **TOP** or **TOP**$_*$.

Let X and Y be topological spaces. Let $A \rightarrow X$ be a closed embedding and let $f : A \rightarrow Y$ be a continuous function—then the adjunction space $X \sqcup_f Y$ corresponding to the 2-source $X \leftarrow A \xrightarrow{f} Y$ is defined by the pushout square

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d] & Y \arrow[d] \\ X \arrow[r] & X \sqcup_f Y \end{tikzcd}$$

, f being the attaching map. Agreeing to identify A with its image in X, the restriction of the projection $p : X \amalg Y \rightarrow X \sqcup_f Y$ to $\begin{cases} X - A \\ Y \end{cases}$ is a homeomorphism of $\begin{cases} X - A \\ Y \end{cases}$ onto an $\begin{cases} \text{open} \\ \text{closed} \end{cases}$ subset of $X \sqcup_f Y$ and the images $\begin{cases} p(X - A) \\ p(Y) \end{cases}$ partition $X \sqcup_f Y$.

[Note: The adjunction space $X \sqcup_f Y$ is unique only up to isomorphism. For example, if $\phi : X \rightarrow X$ is a homeomorphism such that $\phi | A = \mathrm{id}_A$, then there arises another pushout square equivalent to the original one.]

(AD_1) If A is not empty and if X and Y are connected (path connected), then $X \sqcup_f Y$ is connected (path connected).

(AD_2) If X and Y are T_1, then $X \sqcup_f Y$ is T_1 but if X and Y are Hausdorff, then $X \sqcup_f Y$ need not be Hausdorff.

(AD_3) If X and Y are Hausdorff and if A is compact, then $X \sqcup_f Y$ is Hausdorff.

(AD_4) If X and Y are Hausdorff and if A is a neighborhood retract of X such that each $x \in X - A$ has a neighborhood U with $A \cap \overline{U} = \emptyset$, then $X \sqcup_f Y$ is Hausdorff.

(AD_5) If X and Y are normal (normal and countably paracompact, perfectly normal, collectionwise normal, paracompact) Hausdorff spaces, then $X \sqcup_f Y$ is a normal (normal and countably paracompact, perfectly normal, collectionwise normal, paracompact) Hausdorff space.

(AD_6) If X and Y are in **CG** (**Δ-CG**), then $X \sqcup_f Y$ is in **CG** (**Δ-CG**).

EXAMPLE Working with the Isbell-Mrówka space $\Psi(\mathbf{N}) = S \cup \mathbf{N}$, consider the pushout square

$$\begin{tikzcd} S \arrow[r, "f"] \arrow[d] & \beta S \arrow[d] \\ \Psi(\mathbf{N}) \arrow[r] & \Psi(\mathbf{N}) \sqcup_f \beta S \end{tikzcd}$$

. Due to the maximality of S, every open covering of $\Psi(\mathbf{N}) \sqcup_f \beta S$ has a finite subcovering. Still, $\Psi(\mathbf{N}) \sqcup_f \beta S$ is not Hausdorff.

The cylinder functor I is the functor $I : \begin{cases} \mathbf{TOP} \to \mathbf{TOP} \\ X \to X \times [0,1] \end{cases}$, where $X \times [0,1]$ carries the product topology. There are embeddings $i_t : \begin{cases} X \to IX \\ x \to (x,t) \end{cases}$ $(0 \leq t \leq 1)$ and a projection $p : \begin{cases} IX \to X \\ (x,t) \to x \end{cases}$. The path space functor P is the functor $P : \begin{cases} \mathbf{TOP} \to \mathbf{TOP} \\ X \to C([0,1],X) \end{cases}$, where $C([0,1],X)$ carries the compact open topology. There is an embedding $j : \begin{cases} X \to PX \\ x \to j(x) \end{cases}$, with $j(x)(t) = x$, and projections $p_t : \begin{cases} PX \to X \\ \sigma \to p_t(\sigma) \end{cases}$ $(0 \leq t \leq 1)$, with $p_t(\sigma) = \sigma(t)$. (I,P) is an adjoint pair: $C(IX,Y) \approx C(X,PY)$. Accordingly, two continuous functions $\begin{cases} f : X \to Y \\ g : X \to Y \end{cases}$ determine the same morphism in **HTOP**, i.e. are homotopic ($f \simeq g$), iff $\exists\ H \in C(IX,Y)$ such that $\begin{cases} H \circ i_0 = f \\ H \circ i_1 = g \end{cases}$ or, equivalently, iff $\exists\ G \in C(X,PY)$ such that $\begin{cases} p_0 \circ G = f \\ p_1 \circ G = g \end{cases}$.

Let A and X be topological spaces—then a continuous function $i : A \to X$ is said to be a cofibration if it has the following property: Given any topological space Y and any pair (F,h) of continuous functions $\begin{cases} F : X \to Y \\ h : IA \to Y \end{cases}$ such that $F \circ i = h \circ i_0$, there is a continuous function $H : IX \to Y$ such that $F = H \circ i_0$ and $H \circ Ii = h$. Thus H is a filler for the diagram

$$\begin{tikzcd}
A \arrow[rr, "i_0"] \arrow[dd, "i"'] & & IA \arrow[dl] \arrow[dd, "Ii"] \\
& Y & \\
X \arrow[ur] \arrow[rr, "i_0"'] & & IX \arrow[ul, dashed]
\end{tikzcd} .$$

[Note: One can also formulate the definition in terms of the path space functor, viz.

$$\begin{tikzcd}
A \arrow[r] \arrow[d, "i"'] & PY \arrow[d, "p_0"] \\
X \arrow[r] \arrow[ur, dashed] & Y
\end{tikzcd} .]$$

A continuous function $i : A \to X$ is a cofibration iff the commutative diagram

$$\begin{tikzcd} A \arrow[r, "i"] \arrow[d, "i_0"'] & X \arrow[d, "i_0"] \\ IA \arrow[r, "Ii"'] & IX \end{tikzcd}$$

is a weak pushout square. Homeomorphisms are cofibrations. Maps with an empty domain are cofibrations. The composite of two cofibrations is a cofibration.

EXAMPLE Let $p : X \to B$ be a surjective continuous function. Consider $C_p = IX \amalg B/\sim$, where $(x', 0) \sim (x'', 0)$ & $(x, 1) \sim p(x)$ (no topology). Let $t : C_p \to [0,1]$ be the function $[x,t] \to t$; let $x : t^{-1}(]0,1[) \to X$ be the function $[x,t] \to x$; let $p : t^{-1}(]0,1]) \to B$ be the function $[x,t] \to p(x)$. Definition: The coordinate topology on C_p is the initial topology determined by t, x, p. There is a closed embedding $j : B \to C_p$ which is a cofibration. For suppose that $\begin{cases} F : C_p \to Y \\ h : IB \to Y \end{cases}$ are continuous functions such that $F \circ j = h \circ i_0$—then the formulas $H(j(b), T) = h(b, T)$,

$$H([x,t],T) = \begin{cases} F[x, t + \frac{T}{2}] & (t \geq 1/2, T \leq 2 - 2t) \\ h(p(x), 2t + T - 2) & (t \geq 1/2, T \geq 2 - 2t) \\ F[x, t + tT] & (t \leq 1/2) \end{cases}$$

specify a continuous function $H : IC_p \to Y$ such that $F = H \circ i_0$ and $H \circ Ij = h$.

[Note: C_p also carries another (finer) topology (cf. p. 3–22). When $X = B$ & $p = \mathrm{id}_X$, C_p is $\Gamma_c X$, and when $B = *$ & $p(X) = *$, C_p is $\Sigma_c X$, i.e., the coordinate topology is the coarse topology (cf. p. 1–27 ff.).]

LEMMA Suppose that $i : A \to X$ is a cofibration—then i is an embedding.

[Form the pushout square

$$\begin{tikzcd} A \arrow[r, "i"] \arrow[d, "i_0"'] & X \arrow[d, "F"] \\ IA \arrow[r, "h"'] & Y \end{tikzcd}$$

corresponding to the 2-source $IA \xleftarrow{i_0} A \xrightarrow{i} X$. The definitions imply that there is a continuous function $G : Y \to IX$ such that $\begin{cases} G \circ F = i_0 \\ G \circ h = Ii \end{cases}$ and a continuous function $H : IX \to Y$ such that $\begin{cases} H \circ i_0 = F \\ H \circ Ii = h \end{cases}$. Because $H \circ G = \mathrm{id}_Y$, G is an embedding. On the other hand, $h \circ i_1 : A \to Y$ is an embedding, hence $G \circ h \circ i_1 : A \to i(A) \times \{1\}$ is a homeomorphism.]

For a subspace A of X, the cofibration condition is local in the sense that if there exists a numerable covering $\mathcal{U} = \{U\}$ of X such that $\forall\, U \in \mathcal{U}$, the inclusion $A \cap U \to U$ is a cofibration, then the inclusion $A \to X$ is a cofibration (cf. p. 4–5).

When A is a subspace of X and the inclusion $A \to X$ is a cofibration, the commutative diagram

$$\begin{tikzcd} i_0 A \arrow[r] \arrow[d] & IA \arrow[d] \\ i_0 X \arrow[r] & i_0 X \cup IA \end{tikzcd}$$

is a pushout square and there is a retraction $r : IX \to$

$i_0X \cup IA$. If $\rho : i_0X \cup IA \to IX$ is the inclusion and if $\begin{cases} u : X \to IX \\ v : X \to IX \end{cases}$ are defined by $\begin{cases} u = i_1 \\ v = \rho \circ r \circ i_1 \end{cases}$, then A is the equalizer of (u, v). Therefore the inclusion $A \to X$ is a closed cofibration provided that X is Hausdorff or in **Δ-CG**.

PROPOSITION 1 Let A be a subspace of X—then the inclusion $A \to X$ is a cofibration iff $i_0X \cup IA$ is a retract of IX.

Why should the inclusion $A \to X$ be a cofibration if $i_0X \cup IA$ is a retract of IX? Here is the problem. Suppose that $\phi : i_0X \cup IA \to Y$ is a function such that $\phi|i_0X$ & $\phi|IA$ are continuous. Is ϕ continuous? That the answer is "yes" is a consequence of a generality (which is obvious if A is closed).

LEMMA If $i_0X \cup IA$ is a retract of IX, then a subset O of $i_0X \cup IA$ is open in $i_0X \cup IA$ iff its intersection with $\begin{cases} i_0X \\ IA \end{cases}$ is open in $\begin{cases} i_0X \\ IA \end{cases}$.

[Let r be the retraction in question and assume that O has the stated property. Put $X_O = \{x : (x,0) \in O\}$. Write U_n for the union of all open $U \subset X : A \cap U \times [0, 1/n[\subset O$. Note that $A \cap X_O = A \cap \bigcup\limits_1^\infty U_n$ and $X - \bigcup\limits_1^\infty U_n \subset \overline{A}$. Claim: $X_O \subset \bigcup\limits_1^\infty U_n$. Turn it around and take an $x \in X - \bigcup\limits_1^\infty U_n$—then for any $t \in\,]0,1]$, $r(\overline{A} \times \{t\}) = A \times \{t\}$, so $r(x,t) \in (A - \bigcup\limits_1^\infty U_n) \times [0,1] = (A - X_O) \times [0,1] \subset (X - X_O) \times [0,1] \Rightarrow (x,0) = r(x,0) \in (X - X_O) \times [0,1] \Rightarrow x \in X - X_O$, from which the claim. Thus $O = O' \cup O''$, where $O' = O \cap (A \times]0,1])$ and $O'' = (i_0X \cup IA) \cap \bigcup\limits_1^\infty (X_O \cap U_n \times [0, 1/n[)$ are open in $i_0X \cup IA$.]

EXAMPLE Not every closed embedding is a cofibration: Take $X = \{0\} \cup \{1/n : n \geq 1\}$ and let $A = \{0\}$. Not every cofibration is a closed embedding: Take $X = [0,1]/[0,1[= \{[0],[1]\}$ and let $A = \{[0]\}$.

EXAMPLE Given nonempty topological spaces $\begin{cases} X \\ Y \end{cases}$, form their coarse join $X *_c Y$—then the closed embeddings $\begin{cases} X \\ Y \end{cases} \to X *_c Y$ are cofibrations.

[It suffices to exhibit a retraction $r : I(X *_c Y) \to i_0(X *_c Y) \cup IY$. To this end, consider $r([x,y,1],T) = ([x,y,1],T)$,

$$r([x,y,t],T) = \begin{cases} ([x, y, \frac{2t}{2-T}], 0) & (0 \leq t \leq \frac{2-T}{2}) \\ ([x,y,1], \frac{T+2t-2}{t}) & (\frac{2-T}{2} \leq t \leq 1) \end{cases} .]$$

FACT Let $X^0 \subset X^1 \subset \cdots$ be an expanding sequence of topological spaces. Assume: $\forall\, n$, the inclusion $X^n \to X^{n+1}$ is a cofibration—then $\forall\, n$, the inclusion $X^n \to X^\infty$ is a cofibration.

[Fix retractions $r_k : IX^{k+1} \to i_0X^{k+1} \cup IX^k$. Noting that $IX^\infty = \operatorname{colim} IX^n$, work with the r_k to exhibit $i_0X^\infty \cup IX^n$ as a retract of IX^∞.]

Application: Let X and Y be topological spaces; let $A \subset X$ and $B \subset Y$ be subspaces. Suppose that the inclusions $\begin{cases} A \to X \\ B \to Y \end{cases}$ are cofibrations—then the inclusion $A \times B \to X \times Y$ is a cofibration.

[Consider the inclusions figuring in the factorization $A \times B \to X \times B \to X \times Y$.]

Given $t : 0 \leq t \leq 1$, the inclusion $\{t\} \to [0,1]$ is a closed cofibration and therefore, for any topological space X, the embedding $i_t : X \to IX$ is a closed cofibration. Analogously, the inclusion $\{0,1\} \to [0,1]$ is a closed cofibration and it too can be multiplied.

PROPOSITION 2 Let

$$\begin{tikzcd} Z \arrow[r, "g"] \arrow[d, "f"'] & Y \arrow[d, "\eta"] \\ X \arrow[r, "\xi"'] & P \end{tikzcd}$$

be a pushout square and assume that f is a cofibration—then η is a cofibration.

[The cylinder functor preserves pushouts.]

Application: Let $A \to X$ be a closed cofibration and let $f : A \to Y$ be a continuous function—then the embedding $Y \to X \sqcup_f Y$ is a closed cofibration.

The inclusion $\mathbf{S}^{n-1} \to \mathbf{D}^n$ is a closed cofibration. Proof: Define a retraction $r : I\mathbf{D}^n \to i_0\mathbf{D}^n \cup I\mathbf{S}^{n-1}$ by letting $r(x,t)$ be the point where the line joining $(0,2) \in \mathbf{R}^n \times \mathbf{R}$ and (x,t) meets $i_0\mathbf{D}^n \cup I\mathbf{S}^{n-1}$. Consequently, if $f : \mathbf{S}^{n-1} \to A$ is a continuous function, then the embedding $A \to \mathbf{D}^n \sqcup_f A$ is a closed cofibration. Examples: (1) The embedding $\mathbf{D}^n \to \mathbf{S}^n$ of $\mathbf{D}^n$ as the northern or southern hemisphere of $\mathbf{S}^n$ is a closed cofibration; (2) The embedding $\mathbf{S}^{n-1} \to \mathbf{S}^n$ of $\mathbf{S}^{n-1}$ as the equator of $\mathbf{S}^n$ is a closed cofibration, so $\forall\, m \leq n$, the embedding $\mathbf{S}^m \to \mathbf{S}^n$ is a closed cofibration.

FACT Let $f : \mathbf{S}^{n-1} \to A$ be a continuous function. Suppose that A is path connected—then $\mathbf{D}^n \sqcup_f A$ is path connected and the homomorphism $\pi_q(A) \to \pi_q(\mathbf{D}^n \sqcup_f A)$ is an isomorphism if $q < n-1$ and an epimorphism if $q = n-1$.

VAN KAMPEN THEOREM Suppose that the inclusion $A \to X$ is a closed cofibration. Let

$f : A \to Y$ be a continuous function—then the commutative diagram

$$\begin{tikzcd} \Pi A \arrow[r, "\Pi f"] \arrow[d] & \Pi Y \arrow[d] \\ \Pi X \arrow[r] & \Pi(X \sqcup_f Y) \end{tikzcd}$$

is a pushout square in **GRD**.

[Note: If in addition, A, X and Y are path connected, then for every $x \in A$, the commutative diagram

$$\begin{tikzcd} \pi_1(A, x) \arrow[r, "f_*"] \arrow[d] & \pi_1(Y, f(x)) \arrow[d] \\ \pi_1(X, x) \arrow[r] & \pi_1(X \sqcup_f Y, f(x)) \end{tikzcd}$$

is a pushout square in **GR**.]

Let A be a subspace of X, $i : A \to X$ the inclusion.

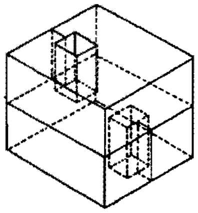
(DR) A is said to be a deformation retract of X if there is a continuous function $r : X \to A$ such that $r \circ i = \mathrm{id}_A$ and $i \circ r \simeq \mathrm{id}_X$.

(SDR) A is said to be a strong deformation retract of X if there is a continuous function $r : X \to A$ such that $r \circ i = \mathrm{id}_A$ and $i \circ r \simeq \mathrm{id}_X \operatorname{rel} A$.

If $i_0X \cup IA$ is a retract of IX, then $i_0X \cup IA$ is a strong deformation retract of IX. Proof: Fix a retraction $r : IX \to i_0X \cup IA$, say $r(x,t) = (p(x,t), q(x,t))$, and consider the homotopy $H : I^2X \to IX$ defined by $H((x,t),T) = (p(x,tT), (1-T)t + Tq(x,t))$.

PROPOSITION 3 Let A be a closed subspace of X and let $f : A \to Y$ be a continuous function. Suppose that A is a strong deformation retract of X—then the image of Y in $X \sqcup_f Y$ is a strong deformation retract of $X \sqcup_f Y$.

EXAMPLE The house with two rooms



is a strong deformation retract of $[0,1]^3$.

LEMMA Suppose that the inclusion $A \to X$ is a cofibration—then the inclusion $i_0X \cup IA \cup i_1X \to IX$ is a cofibration.

[Fix a homeomorphism $\Phi : I[0,1] \to I[0,1]$ that sends $I\{0\} \cup i_0[0,1] \cup I\{1\}$ to $i_0[0,1]$—then the homeomorphism $\mathrm{id}_X \times \Phi : I^2X \to I^2X$ sends $i_0IX \cup I(i_0X \cup IA \cup i_1X)$ to $i_0IX \cup I^2A$. Since the inclusion $IA \to IX$ is a cofibration, $i_0IX \cup I^2A$ is a retract of I^2X and Proposition 1 is applicable.]

[Note: A similar but simpler argument proves that the inclusion $i_0X \cup IA \to IX$ is a cofibration.]

PROPOSITION 4 If A is a deformation retract of X and if $i : A \to X$ is a cofibration, then A is a strong deformation retract of X.

[Choose a homotopy $H : IX \to X$ such that $H \circ i_0 = \mathrm{id}_X$ and $H \circ i_1 = i \circ r$, where $r : X \to A$ is a retraction. Define a function $h : I(i_0X \cup IA \cup i_1X) \to X$ by

$$\begin{cases} h((x,0),T) = x\,(x \in X) \\ h((a,t),T) = H(a,(1-T)t) \quad (a \in A) \\ h((x,1),T) = H(r(x),1-T) \quad (x \in X) \end{cases}.$$

Observing that $i_0X \cup IA \cup i_1X$ can be written as the union of $i_0X \cup A \times [0,1/2]$ and $A \times [1/2,1] \cup i_1X$, the lemma used in the proof of Proposition 1 implies that h is continuous. But the restriction of H to $i_0X \cup IA \cup i_1X$ is $h \circ i_0$, so there exists a continuous function $G : IX \to X$ which extends $h \circ i_1$. Obviously, $G \circ i_0 = \mathrm{id}_X$, $G \circ i_1 = i \circ r$, and $\forall\ a \in A$, $\forall\ t \in [0,1] : G(a,t) = a$. Therefore A is a strong deformation retract of X.]

PROPOSITION 5 If $i : A \to X$ is both a homotopy equivalence and a cofibration, then A is a strong deformation retract of X.

[To say that $i : A \to X$ is a homotopy equivalence means that there exists a continuous function $r : X \to A$ such that $r \circ i \simeq \mathrm{id}_A$ and $i \circ r \simeq \mathrm{id}_X$. However, due to the cofibration assumption, the homotopy class of r contains an honest retraction, thus A is a deformation retract of X or still, a strong deformation retract of X (cf. Proposition 4).]

EXAMPLE (The Comb) Consider the subspace X of $\mathbf{R}^2$ consisting of the union $([0,1] \times \{0\}) \cup (\{0\} \times [0,1])$ and the line segments joining $(1/n, 0)$ and $(1/n, 1)$ $(n = 1, 2, \ldots)$—then X is contractible. Moreover, $\{0\} \times [0,1]$ is a deformation retract of X. But it is not a strong deformation retract. Therefore the inclusion $\{0\} \times [0,1] \to X$, while a homotopy equivalence, is not a cofibration.

Let A be a subspace of X—then a Strøm structure on (X,A) consists of a continuous function $\phi : X \to [0,1]$ such that $A \subset \phi^{-1}(0)$ and a homotopy $\Phi : IX \to X$ of $\mathrm{id}_X \operatorname{rel} A$ such that $\Phi(x,t) \in A$ whenever $t > \phi(x)$.

[Note: If the pair (X,A) admits a Strøm structure (ϕ,Φ) and if A is closed in X, then $A = \phi^{-1}(0)$. Proof: $\phi(x) = 0 \Rightarrow x = \Phi(x,0) = \lim \Phi(x,1/n) \in A$.]

If the pair (X,A) admits a Strøm structure (ϕ_0,Φ_0) for which $\phi_0 < 1$ throughout X, then A is a strong deformation retract of X. Conversely, if A is a strong deformation retract of X and if the pair (X,A) admits a Strøm structure (ϕ,Φ), then the pair (X,A) admits a Strøm structure (ϕ_0,Φ_0) for which $\phi_0 < 1$ throughout X. Proof: Choose a homotopy $H : IX \to X$ of $\mathrm{id}_X \operatorname{rel} A$ such that $H \circ i_1(X) \subset A$ and put $\phi_0(x) = \min\{\phi(x), 1/2\}$, $\Phi_0(x,t) = H(\Phi(x,t), \min\{2t,1\})$.

COFIBRATION CHARACTERIZATION THEOREM The inclusion $A \to X$ is a cofibration iff the pair (X, A) admits a Strøm structure (ϕ, Φ).

[Necessity: Fix a retraction $r : IX \to i_0X \cup IA$ and let $X \xleftarrow{p} IX \xrightarrow{q} [0,1]$ be the projections. Consider $\phi(x) = \sup\limits_{0 \leq t \leq 1} |t - qr(x,t)|$, $\Phi(x,t) = pr(x,t)$.

Sufficiency: Given a Strøm structure (ϕ, Φ) on (X, A), define a retraction $r : IX \to i_0X \cup IA$ by

$$r(x,t) = \begin{cases} (\Phi(x,t), 0) & (t \leq \phi(x)) \\ (\Phi(x,t), t - \phi(x)) & (t \geq \phi(x)) \end{cases}.]$$

One application of this criterion is the fact that if the inclusion $A \to X$ is a cofibration, then the inclusion $\overline{A} \to X$ is a closed cofibration. For let (ϕ, Φ) be a Strøm structure on (X, A)—then $(\phi, \overline{\Phi})$, where $\overline{\Phi}(x,t) = \Phi(x, \min\{t, \phi(x)\})$, is a Strøm structure on $(X, \overline{A})$. Another application is that if the inclusion $A \to X$ is a closed cofibration, then the inclusion $kA \to kX$ is a closed cofibration. Indeed, a Strøm structure on (X, A) is also a Strøm structure on (kX, kA).

EXAMPLE Let $A \subset [0,1]^n$ be a compact neighborhood retract of $\mathbf{R}^n$—then the inclusion $A \to [0,1]^n$ is a cofibration.

EXAMPLE Take $X = [0,1]^\kappa (\kappa > \omega)$ and let $A = \{0_\kappa\}$, 0_κ the "origin" in X—then A is a strong deformation retract of X but the inclusion $A \to X$ is not a cofibration (A is not a zero set in X).

FACT Let A be a nonempty closed subspace of X. Suppose that the inclusion $A \to X$ is a cofibration—then $\forall\, q$, the projection $(X, A) \to (X/A, *_A)$ induces an isomorphism $H_q(X, A) \to H_q(X/A, *_A)$, $*_A$ the image of A in X/A.

[With U running over the neighborhoods of A in X, show that $H_q(X, A) \approx \lim H_q(X, U)$ and then use excision.]

LEMMA Let X and Y be Hausdorff topological spaces. Let A be a closed subspace of X and let $f : A \to Y$ be a continuous function. Assume: The inclusion $A \to X$ is a cofibration—then $X \sqcup_f Y$ is Hausdorff.

As we shall now see, the deeper results in cofibration theory are best approached by implementation of the cofibration characterization theorem.

PROPOSITION 6 Let K be a compact Hausdorff space. Suppose that the inclusion $A \to X$ is a cofibration—then the inclusion $C(K, A) \to C(K, X)$ is a cofibration (compact open topology).

[Let (ϕ, Φ) be a Strøm structure on (X, A). Define $\phi_K : C(K,X) \rightarrow [0,1]$ by $\phi_K(f) = \sup\limits_K \phi \circ f$ and $\Phi_K : IC(K,X) \rightarrow C(K,X)$ by $\Phi_K(f,t)(k) = \Phi(f(k),t)$—then (ϕ_K, Φ_K) is a Strøm structure on $(C(K,X), C(K,A))$.]

EXAMPLE If A is a subspace of X, then the inclusion $PA \rightarrow PX$ is a cofibration provided that the inclusion $A \rightarrow X$ is a cofibration.

EXAMPLE Take $A = \{0,1\}$, $X = [0,1]$—then the inclusion $A \rightarrow X$ is a cofibration but the inclusion $C(\mathbf{N}, A) \rightarrow C(\mathbf{N}, X)$ is not a cofibration (compact open topology).

[The Hilbert cube is an AR but the Cantor set is not an ANR.]

PROPOSITION 7 Let $\begin{cases} A \subset X \\ B \subset Y \end{cases}$, with A closed, and assume that the corresponding inclusions are cofibrations—then the inclusion $A \times Y \cup X \times B \rightarrow X \times Y$ is a cofibration.

[Let (ϕ, Φ) and (ψ, Ψ) be Strøm structures on (X, A) and (Y, B). Define $\omega : X \times Y \rightarrow [0,1]$ by $\omega(x,y) = \min\{\phi(x), \psi(y)\}$ and define $\Omega : I(X \times Y) \rightarrow X \times Y$ by

$$\Omega((x,y),t) = (\Phi(x, \min\{t, \psi(y)\}), \Psi(y, \min\{t, \phi(x)\})).$$

Since A is closed in X, $\phi(x) < 1 \Rightarrow \Phi(x, \phi(x)) \in A$, so (ω, Ω) is a Strøm structure on $(X \times Y, A \times Y \cup X \times B)$.]

[Note: If in addition, $A\,(B)$ is a strong deformation retract of $X\,(Y)$, then $A \times Y \cup X \times B$ is a strong deformation retract of $X \times Y$. Reason: $\phi < 1$ $(\psi < 1)$ throughout $X\,(Y) \Rightarrow \omega < 1$ throughout $X \times Y$.]

EXAMPLE If the inclusion $A \rightarrow X$ is a cofibration, then the inclusion $A \times X \cup X \times A \rightarrow X \times X$ need not be a cofibration. To see this, let $X = [0,1]/[0,1[= \{[0],[1]\}$, $A = \{[0]\}$ and, to get a contradiction, assume that the pair $(X \times X, A \times X \cup X \times A)$ admits a Strøm structure (ϕ, Φ). Obviously, $\phi^{-1}([0,1[) \supset \overline{A \times X} \cup \overline{X \times A} = X \times X$ (since $\overline{A} = X$), so there exists a retraction $r : X \times X \rightarrow A \times X \cup X \times A$. But $([1],[1]) \in \overline{\{([0],[1])\}} \Rightarrow r([1],[1]) \in \overline{\{r([0],[1])\}} = \overline{\{([0],[1])\}} = \overline{\{[0]\}} \times \{[1]\} \Rightarrow r([1],[1]) = ([0],[1])$ and $([1],[1]) \in \overline{\{([1],[0])\}} \Rightarrow \cdots \Rightarrow r([1],[1]) = ([1],[0])$.

LEMMA Let A be a subspace of X and assume that the inclusion $A \rightarrow X$ is a cofibration. Suppose that $K, L : IX \rightarrow Y$ are continuous functions that agree on $i_0X \cup IA$—then $K \simeq L \operatorname{rel} i_0X \cup IA$.

[The inclusion $i_0X \cup IA \cup i_1X \rightarrow IX$ is a cofibration (cf. the lemma preceding the proof of Proposition 4). With this in mind, define a continuous function $F : IX \rightarrow Y$ by $F(x,t) = K(x,0)$ and a continuous function $h : I(i_0X \cup IA \cup i_1X) \rightarrow Y$ by $\begin{cases} h((x,0),T) = K(x,T) \\ h((x,1),T) = L(x,T) \end{cases}$

& $h((a,t),T) = K(a,T) = L(a,T)$. Since the restriction of F to $i_0X \cup IA \cup i_1X$ is equal to $h \circ i_0$, there exists a continuous function $H : I^2X \rightarrow Y$ such that $F = H \circ i_0$ and $H|I(i_0X \cup IA \cup i_1X) = h$. Let $\iota : [0,1] \times [0,1] \rightarrow [0,1] \times [0,1]$ be the involution $(t,T) \rightarrow (T,t)$—then $H \circ (\mathrm{id}_X \times \iota) : I^2X \rightarrow Y$ is a homotopy between K and $L \operatorname{rel} i_0X \cup IA$.]

PROPOSITION 8 Let A and B be closed subspaces of X. Suppose that the inclusions $\begin{cases} A \rightarrow X \\ B \rightarrow X \end{cases}$, $A \cap B \rightarrow X$ are cofibrations—then the inclusion $A \cup B \rightarrow X$ is a cofibration.

[In IX, write $(x,t) \sim (x,0)$ $(x \in A \cap B)$, call $\widetilde{X}$ the quotient $IX/\sim$, and let $p : IX \rightarrow \widetilde{X}$ be the projection. Choose continuous functions $\phi, \psi : X \rightarrow [0,1]$ such that $A = \phi^{-1}(0)$, $B = \psi^{-1}(0)$. Define $\lambda : X \rightarrow \widetilde{X}$ by $\lambda(x) = \left[x, \dfrac{\phi(x)}{\phi(x) + \psi(x)}\right]$ if $x \notin A \cap B$, $\lambda(x) = [x,0]$ if $x \in A \cap B$—then λ is continuous and $\begin{cases} \lambda(x) = [x,0] \text{ on } A \\ \lambda(x) = [x,1] \text{ on } B \end{cases}$. Consider now a pair (F,h) of continuous functions $\begin{cases} F : X \rightarrow Y \\ h : I(A \cup B) \rightarrow Y \end{cases}$ for which $F|(A \cup B) = h \circ i_0$. Fix homotopies $\begin{cases} H_A : IX \rightarrow Y \\ H_B : IX \rightarrow Y \end{cases}$ such that $\begin{cases} H_A|IA = h|IA \\ H_B|IB = h|IB \end{cases}$ & $F = H_A \circ i_0 = H_B \circ i_0$ and, using the lemma, fix a homotopy $H : I^2X \rightarrow Y$ between H_A and $H_B \operatorname{rel} i_0X \cup I(A \cap B)$. With ι as in the proof above, the composite $H \circ (\mathrm{id}_X \times \iota)$ factors through $I^2X \xrightarrow{p \times \mathrm{id}} I\widetilde{X}$, thus there is a continuous function $\widetilde{H} : I\widetilde{X} \rightarrow Y$ that renders the diagram

$$\begin{tikzcd}
I^2X \arrow[r, "\mathrm{id}_X \times \iota"] \arrow[d, "p \times \mathrm{id}"'] & I^2X \arrow[d, "H"] \\
I\widetilde{X} \arrow[r, "\widetilde{H}"'] & Y
\end{tikzcd}$$

commutative. An extension of (F,h) is then given by the composite $\widetilde{H} \circ (\lambda \times \mathrm{id}) : IX \rightarrow I\widetilde{X} \rightarrow Y$.]

FACT Let A and B be closed subspaces of a metrizable space X. Suppose that the inclusions $A \cap B \rightarrow A$, $A \cap B \rightarrow B$, $B \rightarrow X$, $A - B \rightarrow X - B$ are cofibrations—then the inclusion $A \rightarrow X$ is a cofibration.

Let A be a subspace of X. Suppose given a continuous function $\psi : X \rightarrow [0,\infty]$ such that $A \subset \psi^{-1}(0)$ and a homotopy $\Psi : I\psi^{-1}([0,1]) \rightarrow X$ of the inclusion $\psi^{-1}([0,1]) \rightarrow X \operatorname{rel} A$ such that $\Psi(x,t) \in A$ whenever $t > \psi(x)$—then the inclusion $A \rightarrow X$ is a cofibration. Proof: Define a Strøm structure (ϕ, Φ) on (X,A) by $\phi(x) = \min\{2\psi(x), 1\}$,

$$\Phi(x,t) = \begin{cases} \Psi(x,t) \quad (2\psi(x) \leq 1) \\ \Psi(x, t(2 - 2\psi(x))) \quad (1 \leq 2\psi(x) \leq 2) \\ x \quad (\psi(x) \geq 1) \end{cases}.$$

LEMMA Let A be a subspace of X and assume that the inclusion $A \to X$ is a cofibration. Suppose that U is a subspace of X with the property that there exists a continuous function $\pi : X \to [0,1]$ for which $\overline{A} \cap U \subset \pi^{-1}(]0,1]) \subset U$—then the inclusion $A \cap U \to U$ is a cofibration.

[Fix a Strøm structure (ϕ, Φ) on (X, A). Set $\pi_0(x) = \inf\limits_{0 \leq t \leq 1} \pi(\Phi(x,t))$ $(x \in X)$. Define a continuous function $\psi : U \to [0, \infty]$ by $\psi(x) = \phi(x)/\pi_0(x)$. This makes sense since $\phi(x) = 0 \Rightarrow \pi_0(x) > 0$ $(x \in U)$. Next, $\psi(x) \leq 1 \Rightarrow \pi_0(x) > 0 \Rightarrow \pi(\Phi(x,t)) > 0 \Rightarrow \Phi(x,t) \in U$ $(\forall\ t)$. One can therefore let $\Psi : I\psi^{-1}([0,1]) \to U$ be the restriction of Φ and apply the foregoing remark to the pair $(U, A \cap U)$.]

Let A, U be subspaces of a topological space X—then U is said to be a halo of A in X if there exists a continuous function $\pi : X \to [0,1]$ (the haloing function) such that $A \subset \pi^{-1}(1)$ and $\pi^{-1}(]0,1]) \subset U$. For example, if X is normal (but not necessarily Hausdorff), then every neighborhood of a closed subspace A of X is a halo of A in X but in a nonnormal X, a closed subspace A of X may have neighborhoods that are not halos.

(HA_1) If U is a halo of A in X, then U is a halo of $\overline{A}$ in X.

(HA_2) If U is a halo of A in X, then there exists a closed subspace B of $X : A \subset B \subset X$, such that B is a halo of A in X and U is a halo of B in X.

[A haloing function for $\pi^{-1}([1/2, 1])$ is $\max\{2\pi(x) - 1, 0\}$.]

Observation: If the inclusion $A \to X$ is a cofibration and if U is a halo of A in X, then the inclusion $A \to U$ is a cofibration.

[This is a special case of the lemma.]

PROPOSITION 9 If $j : B \to A$ and $i : A \to X$ are continuous functions such that i and $i \circ j$ are cofibrations, then j is a cofibration.

[Take i and j to be inclusions. Using the cofibration characterization theorem, fix a halo U of A in X and a retraction $r : U \to A$. Since U is also a halo of B in X, the inclusion $B \to U$ is a cofibration. Consider a commutative diagram

$$\begin{tikzcd} B \arrow[r, "g"] \arrow[d, "j"'] & PY \arrow[d, "p_0"] \\ A \arrow[r, "F"'] & Y \end{tikzcd}.$$

To construct a filler for this, pass to its counterpart

$$\begin{tikzcd} B \arrow[r, "g"] \arrow[d] & PY \arrow[d, "p_0"] \\ U \arrow[r, "F \circ r"'] & Y \end{tikzcd}$$

over U, which thus admits a filler $G : U \to PY$. The restriction $G|A : A \to PY$ will then do the trick.]

EXAMPLE (Telescope Construction) Let $X^0 \subset X^1 \subset \cdots$ be an expanding sequence of topological spaces. Assume: $\forall\, n$, the inclusion $X^n \to X^{n+1}$ is a closed cofibration—then $\forall\, n$, the inclusion $X^n \to X^\infty$ is a closed cofibration (cf. p. 3–5). Write $\mathrm{tel}\, X^\infty$ for the quotient $\coprod\limits_0^\infty X^n \times [n, n+1]/\sim$. Here, $\sim$ means that the pair $(x, n+1) \in X^n \times \{n+1\}$ is identified with the pair $(x, n+1) \in X^{n+1} \times \{n+1\}$. One calls $\mathrm{tel}\, X^\infty$ the telescope of X^∞. It can be viewed as a closed subspace of $X^\infty \times [0, \infty[$. The inclusion $\mathrm{tel}_n X^\infty \equiv \bigcup\limits_{k=0}^{n} X^k \times [k, k+1] \to X^\infty \times [0, \infty[$ is a closed cofibration (cf. Proposition 8), so the same is true of the inclusion $\mathrm{tel}_n X^\infty \to \mathrm{tel}_{n+1} X^\infty$ (cf. Proposition 9) and $\mathrm{tel}\, X^\infty = \mathrm{colim}\, \mathrm{tel}_n X^\infty$. Denote by p^∞ the composite $\mathrm{tel}\, X^\infty \to X^\infty \times [0, \infty[\to X^\infty$.

Claim: p^∞ is a homotopy equivalence.

[It suffices to establish that $\mathrm{tel}\, X^\infty$ is a strong deformation retract of $X^\infty \times [0, \infty[$. One approach is to piece together strong deformation retractions $X^{n+1} \times [0, n+1] \to X^{n+1} \times \{n+1\} \cup X^n \times [0, n+1]$.]

Let $\begin{cases} X^0 \subset X^1 \subset \cdots \\ Y^0 \subset Y^1 \subset \cdots \end{cases}$ be expanding sequences of topological spaces. Assume: $\forall\, n$, the inclusions $\begin{cases} X^n \to X^{n+1} \\ Y^n \to Y^{n+1} \end{cases}$ are closed cofibrations. Suppose given a sequence of continuous functions $\phi^n : X^n \to Y^n$ such that $\forall\, n$, the diagram

$$\begin{tikzcd} X^n \arrow[r] \arrow[d, "\phi^n"'] & X^{n+1} \arrow[d, "\phi^{n+1}"] \\ Y^n \arrow[r] & Y^{n+1} \end{tikzcd}$$

commutes. Associated with the ϕ^n is a continuous function $\phi^\infty : X^\infty \to Y^\infty$ and a continuous function $\mathrm{tel}\, \phi : \mathrm{tel}\, X^\infty \to \mathrm{tel}\, Y^\infty$, the latter being defined by

$$\mathrm{tel}\, \phi(x, n+t) = \begin{cases} (\phi^n(x), n+2t) \in Y^n \times [n, n+1] & (0 \leq t \leq 1/2) \\ (\phi^n(x), n+1) \in Y^{n+1} \times \{n+1\} & (1/2 \leq t \leq 1) \end{cases}.$$

There is then a commutative diagram

$$\begin{tikzcd} \mathrm{tel}\, X^\infty \arrow[r] \arrow[d, "\mathrm{tel}\, \phi"'] & X^\infty \arrow[d, "\phi^\infty"] \\ \mathrm{tel}\, Y^\infty \arrow[r] & Y^\infty \end{tikzcd}.$$

The horizontal arrows are homotopy equivalences. Moreover, $\mathrm{tel}\, \phi$ is a homotopy equivalence if this is the case of the ϕ^n, thus, under these circumstances, $\phi^\infty : X^\infty \to Y^\infty$ itself is a homotopy equivalence.

[Note: One can also make the deduction from first principles (cf. Proposition 15).]

PROPOSITION 10 Let A be a closed subspace of a topological space X. Suppose that A admits a halo U with $A = \pi^{-1}(1)$ for which there exists a homotopy $\Pi : IU \to X$ of the inclusion $U \to X \,\mathrm{rel}\, A$ such that $\Pi \circ i_1(U) \subset A$—then the inclusion $A \to X$ is a closed cofibration.

[Define a retraction $r : IX \to i_0 X \cup IA$ as follows: (i) $r(x, t) = (x, 0)$ $(\pi(x) = 0)$; (ii) $r(x, t) = (\Pi(x, 2\pi(x)t), 0)$ $(0 < \pi(x) \leq 1/2)$; (iii) $r(x, t) = (\Pi(x, t/2(1 - \pi(x))), 0)$ $(1/2 \leq \pi(x) < 1$ & $0 \leq t \leq 2(1 - \pi(x)))$ and $r(x, t) = (\Pi(x, 1), t - 2(1 - \pi(x)))$ $(1/2 \leq \pi(x) < 1$ & $2(1 - \pi(x)) \leq t \leq 1)$; (iv) $r(x, t) = (x, t)$ $(\pi(x) = 1)$.]

EXAMPLE If A is a subcomplex of a CW complex X, then the inclusion $A \to X$ is a closed cofibration.

A topological space X is said to be locally contractible provided that for any $x \in X$ and any neighborhood U of x there exists a neighborhood $V \subset U$ of x such that the inclusion $V \to U$ is inessential. If X is locally contractible, then X is locally path connected. Example: $\forall\, X$, X^* is locally contractible (cf. p. 1–28).

[Note: The empty set is locally contractible but not contractible.]

A topological space X is said to be numerably contractible if it has a numerable covering $\{U\}$ for which each inclusion $U \to X$ is inessential. Example: Every locally contractible paracompact Hausdorff space is numerably contractible.

[Note: The product of two numerably contractible spaces is numerably contractible.]

FACT Numerable contractibility is a homotopy type invariant. Proof: If X is dominated in homotopy by Y and if Y is numerably contractible, then X is numerably contractible.

Examples: (1) Every topological space having the homotopy type of a CW complex is numerably contractible; (2) If the X^n of the telescope construction are numerably contractible, then X^∞ is numerably contractible (consider $\text{tel}\, X^\infty$).

A topological space X is said to be uniformly locally contractible provided that there exists a neighborhood U of the diagonal $\Delta_X \subset X \times X$ and a homotopy $H : IU \to X$ between $p_1|U$ and $p_2|U \operatorname{rel} \Delta_X$, where p_1 and p_2 are the projections onto the first and second factors. Examples: (1) $\mathbf{R}^n$, $\mathbf{D}^n$, and $\mathbf{S}^{n-1}$ are uniformly locally contractible; (2) The long ray L^+ is not uniformly locally contractible.

EXAMPLE (Stratifiable Spaces) Suppose that X is stratifiable and in NES(stratifiable)—then X is uniformly locally contractible. Thus put $A = X \times i_0 X \cup (I\Delta_X) \cup X \times i_1 X$, a closed subspace of the stratifiable space $I(X \times X)$. Define a continuous function $\phi : A \to X$ by $\begin{cases} (x, y, 0) \to x \\ (x, y, 1) \to y \end{cases}$ & $(x, x, t) \to x$—then ϕ extends to a continuous function $\Phi : O \to X$, where O is a neighborhood of A in $I(X \times X)$. Fix a neighborhood U of Δ_X in $X \times X : IU \subset O$ and consider $H = \Phi|IU$.

[Note: Every CW complex is stratifiable (cf. p. 6–30) and in NES(stratifiable) (cf. p. 6–43). Every metrizable topological manifold is stratifiable (cf. p. 6–29 ff.: metrizable $\Rightarrow$ stratifiable) and, being an ANR (cf. p. 6–28), is in NES(stratifiable) (cf. p. 6–44: stratifiable $\Rightarrow$ perfectly normal + paracompact).]

FACT Let K be a compact Hausdorff space. Suppose that X is uniformly locally contractible—then $C(K, X)$ is uniformly locally contractible (compact open topology).

LEMMA A uniformly locally contractible topological space X is locally contractible.

[Take a point $x_0 \in X$ and let U_0 be a neighborhood of x_0—then $I\{(x_0, x_0)\} \subset H^{-1}(U_0)$. Since $H^{-1}(U_0)$ is open in IU, hence open in $I(X \times X)$, there exists a neighborhood $V_0 \subset U_0$ of $x_0 : I(V_0 \times V_0) \subset H^{-1}(U_0)$. To see that the inclusion $V_0 \rightarrow U_0$ is inessential, define $H_0 : IV_0 \rightarrow U_0$ by $H_0(x,t) = H((x, x_0), t)$.]

[Note: The homotopy H_0 keeps x_0 fixed throughout the entire deformation. In addition, the argument shows that an open subspace of a uniformly locally contractible space is uniformly locally contractible.]

EXAMPLE (A Spaces) Every A space is locally contractible. In fact, if X is a nonempty A space, then $\forall\ x \in X$, U_x is contractible, thus X has a basis of contractible open sets, so X is locally contractible. But an A space need not be uniformly locally contractible. Consider, e.g., $X = \{a, b, c, d\}$, where $\begin{cases} c \leq a \\ d \leq a \end{cases}, \begin{cases} c \leq b \\ d \leq b \end{cases}$.

FACT Let X be a perfectly normal paracompact Hausdorff space. Suppose that X admits a covering by open sets U, each of which is uniformly locally contractible—then X is uniformly locally contractible.

[Use the domino principle.]

When is X uniformly locally contractible? A sufficient condition is that the inclusion $\Delta_X \rightarrow X \times X$ be a cofibration. Proof: Fix a Strøm structure (ϕ, Φ) on the pair $(X \times X, \Delta_X)$, put $U = \phi^{-1}([0, 1[)$ and define $H : IU \rightarrow X$ by

$$H((x,y),t) = \begin{cases} p_1(\Phi((x,y),2t)) & (0 \leq t \leq 1/2) \\ p_2(\Phi((x,y),2-2t)) & (1/2 \leq t \leq 1) \end{cases}.$$

FACT Suppose that X is a perfectly normal Hausdorff space with a perfectly normal square—then X is uniformly locally contractible iff the diagonal embedding $X \rightarrow X \times X$ is a cofibration.

[Use Proposition 10, noting that Δ_X is a zero set.]

Application: If X is a CW complex or a metrizable topological manifold, then the diagonal embedding $X \rightarrow X \times X$ is a cofibration.

FACT Let A be a closed subspace of a metrizable space X such that the inclusion $A \rightarrow X$ is a cofibration. Suppose that A and $X - A$ are uniformly locally contractible—then X is uniformly locally contractible.

[Show that the inclusion $\Delta_X \rightarrow X \times X$ is a cofibration by applying the result on p. 3–10 to the triple $(X \times X, \Delta_X, A \times A)$.]

PROPOSITION 11 Suppose that $A \subset X$ admits a halo U such that the inclusion $\Delta_U \rightarrow U \times U$ is a cofibration. Assume that the inclusion $A \rightarrow X$ is a cofibration—then the inclusion $\Delta_A \rightarrow A \times A$ is a cofibration.

[Consider the commutative diagram

$$\begin{tikzcd} A \arrow[r, "\Delta_A"] \arrow[d] & A \times A \arrow[d] \\ U \arrow[r, "\Delta_U"'] & U \times U \end{tikzcd}$$

. The vertical arrows are cofibrations, as is Δ_U. That Δ_A is a cofibration is therefore implied by Proposition 9.]

PROPOSITION 12 Let X be a Hausdorff space and suppose that the inclusion $\Delta_X \rightarrow X \times X$ is a cofibration. Let $f : X \rightarrow [0,1]$ be a continuous function such that $A = f^{-1}(0)$ is a retract of $f^{-1}([0,1[)$—then the inclusion $A \rightarrow X$ is a closed cofibration.

[Write r for the retraction $f^{-1}([0,1[) \rightarrow A$, fix a Strøm structure (ϕ, Φ) on the pair $(X \times X, \Delta_X)$, and let $H : IU \rightarrow X$ be as above. Define $\phi_f : X \rightarrow [0,1]$ by $\phi_f(x) = \max\{f(x), \phi(x, r(x))\}$ $(f(x) < 1)$ & $\phi_f(x) = 1$ $(f(x) = 1)$—then $\phi_f^{-1}(0) = A$. Put $H_f(x,t) = H((x, r(x)), t)$ to obtain a homotopy $H_f : I\phi_f^{-1}([0,1[) \rightarrow X$ of the inclusion $\phi_f^{-1}([0,1[) \rightarrow X$ rel A such that $H_f \circ i_1(\phi_f^{-1}([0,1[)) \subset A$. Finish by citing Proposition 10.]

Application: Let X be a Hausdorff space and suppose that the inclusion $\Delta_X \rightarrow X \times X$ is a cofibration. Let $e \in C(X,X)$ be idempotent: $e \circ e = e$—then the inclusion $e(X) \rightarrow X$ is a closed cofibration.

[Define $f : X \rightarrow [0,1]$ by $f(x) = \phi(x, e(x))$.]

So, if X is a Hausdorff space and if the inclusion $\Delta_X \rightarrow X \times X$ is a cofibration, then for any retract A of X, the inclusion $A \rightarrow X$ is a closed cofibration. In particular: $\forall\, x_0 \in X$, the inclusion $\{x_0\} \rightarrow X$ is a closed cofibration, which, as seen above, is a condition realized by every CW complex or metrizable topological manifold.

[Note: Let X be the Cantor set—then $\forall\, x_0 \in X$, the inclusion $\{x_0\} \rightarrow X$ is closed but not a cofibration.]

FACT Let X be in **Δ-CG** and suppose that the inclusion $\Delta_X \rightarrow X \times_k X$ is a cofibration—then for any retract A of X, the inclusion $A \rightarrow X$ is a closed cofibration.

[Rework Proposition 12, noting that for any continuous function $f : X \rightarrow X$, the function $X \rightarrow X \times_k X$ defined by $x \rightarrow (x, f(x))$ is continuous.]

LEMMA Suppose that the inclusions $\begin{cases} A \rightarrow X \\ A' \rightarrow X' \end{cases}$ are closed cofibrations and that X is a closed subspace of X' with $A = X \cap A'$. Let $\begin{cases} f : A \rightarrow Y \\ f' : A' \rightarrow Y' \end{cases}$ be continuous functions. Assume that the diagram

$$\begin{tikzcd} X & A \arrow[l] \arrow[r, "f"] \arrow[d] & Y \arrow[d] \\ X' \arrow[from=u] & A' \arrow[l] \arrow[r, "f'"'] & Y' \end{tikzcd}$$

commutes and that the vertical arrows are cofibrations—then the induced map $X \sqcup_f Y \rightarrow X' \sqcup_{f'} Y'$ is a cofibration and $(X \sqcup_f Y) \cap Y' = Y$.

[Consider a commutative diagram

$$\begin{tikzcd} X \sqcup_f Y \arrow[r, "g"] \arrow[d] & PZ \arrow[d, "p_0"] \\ X' \sqcup_{f'} Y' \arrow[r, "F'"'] & Z \end{tikzcd}$$

. To construct a filler H' for this, work first with

$$\begin{tikzcd} Y \arrow[r] \arrow[d] & X \sqcup_f Y \arrow[r, "g"] \arrow[d] & PZ \arrow[d, "p_0"] \\ Y' \arrow[r] & X' \sqcup_{f'} Y' \arrow[r, "F'"'] & Z \end{tikzcd}$$

to get an arrow $G : Y' \rightarrow PZ$. Next, look at

$$\begin{cases} A' \xrightarrow{f'} Y' \xrightarrow{G} PZ \\ X \longrightarrow X \sqcup_f Y \xrightarrow{g} PZ \end{cases}$$

. Since equality obtains on $A = X \cap A'$, $\exists\, G' \in C(X \cup A', PZ) : G'|A' = G \circ f'$. But the inclusion $X \cup A' \rightarrow X'$ is a cofibration (cf. Proposition 8), so the commutative diagram

$$\begin{tikzcd} X \cup A' \arrow[rr, "G'"] \arrow[d] & & PZ \arrow[d, "p_0"] \\ X' \arrow[r] & X' \sqcup_{f'} Y' \arrow[r, "F'"'] & Z \end{tikzcd}$$

admits a filler $H : X' \rightarrow PZ$ which agrees with $G \circ f'$ on A' and therefore determines $H' : X' \sqcup_{f'} Y' \rightarrow PZ$.]

FACT Let $A \rightarrow X$ be a closed cofibration and let $f : A \rightarrow Y$ be a continuous function. Suppose that $\begin{cases} X \\ Y \end{cases}$ are in **Δ-CG** and that the inclusions $\begin{cases} \Delta_X \rightarrow X \times_k X \\ \Delta_Y \rightarrow Y \times_k Y \end{cases}$ are cofibrations—then the inclusion $\Delta_Z \rightarrow Z \times_k Z$ is a cofibration, Z the adjunction space $X \sqcup_f Y$.

[There are closed cofibrations $\begin{cases} A \times_k A \rightarrow X \times_k A \cup A \times_k X \\ Y \times_k Y \rightarrow Z \times_k Y \cup Y \times_k Z \end{cases}$. Precompose these arrows with the diagonal embeddings, form the commutative diagram

$$\begin{tikzcd} X \arrow[d] & A \arrow[l] \arrow[r, "f"] \arrow[d] & Y \arrow[d] \\ X \times_k X & X \times_k A \cup A \times_k X \arrow[l] \arrow[r] & Z \times_k Y \cup Y \times_k Z \end{tikzcd} ,$$

and apply the lemma.]

[Note: Proposition 7 remains in force if the product in **TOP** is replaced by the product in **Δ-CG**. Take $U = X$ in Proposition 11 to see that the inclusion $\Delta_A \rightarrow A \times_k A$ is a cofibration.]

Application: Let X and Y be CW complexes. Let A be a subcomplex of X and let $f : A \rightarrow Y$ be a continuous function—then the inclusion $\Delta_Z \rightarrow Z \times_k Z$ is a cofibration, Z the adjunction space $X \sqcup_f Y$.

[The inclusions $\begin{cases}\Delta_X \to X \times X \\ \Delta_Y \to Y \times Y\end{cases}$ are cofibrations (cf. p. 3–14), thus the same is true of the inclusions $\begin{cases}\Delta_X \to X \times_k X \\ \Delta_Y \to Y \times_k Y\end{cases}$ (cf. p. 3–8). Z itself need not be a CW complex but, in view of the skeletal approximation theorem, Z at least has the homotopy type of a CW complex.]

FACT Let $A \to X$ be a closed cofibration and let $f : A \to Y$ be a continuous function. Suppose that $\begin{cases} X \\ Y \end{cases}$ are uniformly locally contractible perfectly normal Hausdorff spaces with perfectly normal squares—then $X \sqcup_f Y$ is uniformly locally contractible provided that its square is perfectly normal.

[Note: A priori, $X \sqcup_f Y$ is a perfectly normal Hausdorff space (cf. AD_5).]

A pointed space (X, x_0) is said to be wellpointed if the inclusion $\{x_0\} \to X$ is a cofibration. $\overline{\Pi}X$ is the full subgroupoid of ΠX whose objects are the $x_0 \in X$ such that (X, x_0) is wellpointed. Example: Let X be a CW complex or a metrizable topological manifold—then $\forall\ x_0 \in X$, (X, x_0) is wellpointed (cf. p. 3–15).

[Note: Take $X = [0, \Omega]$, $x_0 = \Omega$—then (X, x_0) is not wellpointed.]

The full subcategory of $\mathbf{HTOP}_*$ whose objects are the wellpointed spaces is not isomorphism closed, i.e., if $(X, x_0) \approx (Y, y_0)$ in $\mathbf{HTOP}_*$, then it can happen that the inclusion $\{x_0\} \to X$ is a cofibration but the inclusion $\{y_0\} \to Y$ is not a cofibration (cf. p. 3–8).

EXAMPLE Let X be a topological manifold—then $\forall\ x_0 \in X$, (X, x_0) is wellpointed.

FACT Let K be a compact Hausdorff space. Suppose that (X, x_0) is wellpointed—then $\forall\ k_0 \in K$, $C(K, k_0; X, x_0)$ is wellpointed (compact open topology).

[Note: The base point in $C(K, k_0; X, x_0)$ is the constant map $K \to x_0$.]

Given topological spaces $\begin{cases} X \\ Y \end{cases}$, the base point functor $\overline{\Pi}X \times \Pi Y \to \mathbf{SET}$ sends an object (x_0, y_0) to the set $[X, x_0; Y, y_0]$. To describe its behavior on morphisms, let $\begin{cases} x_0, x_1 \in X \\ y_0, y_1 \in Y \end{cases}$ and suppose that both (X, x_0) and (X, x_1) are wellpointed. Let $\sigma \in PX : \begin{cases} \sigma(0) = x_0 \\ \sigma(1) = x_1 \end{cases}$ & let $\tau \in PY : \begin{cases} \tau(0) = y_0 \\ \tau(1) = y_1 \end{cases}$—then the pair (σ, τ) determines a bijection $[\sigma, \tau]_\# : [X, x_0; Y, y_0] \to [X, x_1; Y, y_1]$ that depends only on the path classes of $\begin{cases} \sigma \\ \tau \end{cases}$ in $\begin{cases} \Pi X \\ \Pi Y \end{cases}$. Here is the procedure. Fix a homotopy $H : IX \to X$ such that $H \circ i_0 = \text{id}_X$, $H(x_1, t) = \sigma(1 - t)$, and put $e = H \circ i_1$. Take an $f \in C(X, x_0; Y, y_0)$ and define a continuous function $F : i_0 X \cup I\{x_1\} \to X \times Y$ by $\begin{cases} (x, 0) \to (e(x), f(e(x))) \\ (x_1, t) \to (\sigma(t), \tau(t)) \end{cases}$—then the diagram

$$\begin{tikzcd} i_0X \cup I\{x_1\} \arrow[r, "F"] \arrow[d] & X \times Y \arrow[d, "p"] \\ IX \arrow[r, "G"'] & X \end{tikzcd}$$

commutes, where $G(x,t) = H(x,1-t)$. To construct a filler $H_f : IX \to X \times Y$, let $q : X \times Y \to Y$ be the projection, choose a retraction $r : IX \to i_0X \cup I\{x_1\}$ and set $H_f(x,t) = (G(x,t), qF(r(x,t)))$. Write $f_\# = q \circ H_f \circ i_1 \in C(X,x_1;Y,y_1)$. Definition: $[\sigma,\tau]_\#[f] = [f_\#]$. The fundamental group $\pi_1(Y,y_0)$ thus operates to the left on $[X,x_0;Y,y_0] : ([\tau],[f]) \to [\sigma_0,\tau]_\#[f]$, σ_0 the constant path in X at x_0. If f, $g \in C(X,x_0;Y,y_0)$, then $f \simeq g$ in **TOP** iff $\exists\ [\tau] \in \pi_1(Y,y_0) : [\sigma_0,\tau]_\#[f] = [g]$. Therefore the forgetful function $[X,x_0;Y,y_0] \to [X,Y]$ passes to the quotient to define an injection $\pi_1(Y,y_0)\backslash[X,x_0;Y,y_0] \to [X,Y]$ which, when Y is path connected, is a bijection. The forgetful function $[X,x_0;Y,y_0] \to [X,Y]$ is one-to-one iff the action of $\pi_1(Y,y_0)$ on $[X,x_0;Y,y_0]$ is trivial. Changing Y to Z by a homotopy equivalence in **TOP** : $\left\{\begin{array}{l} Y \to Z \\ y_0 \to z_0 \end{array}\right.$ leads to an arrow $[X,x_0;Y,y_0] \to [X,x_0;Z,z_0]$. It is a bijection.

FACT Suppose that X and Y are path connected. Let $f \in C(X,Y)$ and assume that $\forall\ x \in X$, $f_* : \pi_1(X,x) \to \pi_1(Y,f(x))$ is surjective—then $\forall\ x \in X$, $f_* : \pi_n(X,x) \to \pi_n(Y,f(x))$ is injective (surjective) iff $f_* : [\mathbf{S}^n,X] \to [\mathbf{S}^n,Y]$ is injective (surjective).

LEMMA Suppose that the inclusion $i : A \to X$ is a cofibration. Let $f \in C(X,X) : f \circ i = i$ & $f \simeq \mathrm{id}_X$—then $\exists\ g \in C(X,X) : g \circ i = i$ & $g \circ f \simeq \mathrm{id}_X \,\mathrm{rel}\, A$.

[Let $H : IX \to X$ be a homotopy with $H \circ i_0 = f$ and $H \circ i_1 = \mathrm{id}_X$; let $G : IX \to X$ be a homotopy with $G \circ i_0 = \mathrm{id}_X$ and $G \circ Ii = H \circ Ii$. Define $F : IX \to X$ by $F(x,t) = \left\{\begin{array}{ll} G(f(x),1-2t) & (0 \leq t \leq 1/2) \\ H(x,2t-1) & (1/2 \leq t \leq 1) \end{array}\right.$ and put

$$k((a,t),T) = \left\{\begin{array}{ll} G(a,1-2t(1-T)) & (0 \leq t \leq 1/2) \\ H(a,1-2(1-t)(1-T)) & (1/2 \leq t \leq 1) \end{array}\right.$$

to get a homotopy $k : I^2A \to X$ with $F \circ Ii = k \circ i_0$. Choose a homotopy $K : I^2X \to X$ such that $F = K \circ i_0$ and $K \circ I^2 i = k$. Write $K_{(t,T)} : X \to X$ for the function $x \to K((x,t),T)$. Obviously, $K_{(0,0)} \simeq K_{(0,1)} \simeq K_{(1,1)} \simeq K_{(1,0)}$, all homotopies being $\mathrm{rel}\, A$. Set $g = G \circ i_1$—then $g \circ f = F \circ i_0 = K_{(0,0)}$ is homotopic $\mathrm{rel}\, A$ to $K_{(1,0)} = F \circ i_1 = \mathrm{id}_X$.]

PROPOSITION 13 Suppose that $\left\{\begin{array}{l} i : A \to X \\ j : A \to Y \end{array}\right.$ are cofibrations. Let $\phi \in C(X,Y) : \phi \circ i = j$. Assume that ϕ is a homotopy equivalence—then ϕ is a homotopy equivalence in $A\backslash$**TOP**.

[Since j is a cofibration, there exists a homotopy inverse $\psi : Y \to X$ for ϕ with $\psi \circ j = i$, thus, from the lemma, $\exists\ \psi' \in C(X,X) : \psi' \circ i = i\ \&\ \psi' \circ \psi \circ \phi \simeq \mathrm{id}_X \operatorname{rel} i(A)$. This says that $\phi' = \psi' \circ \psi$ is a homotopy left inverse for ϕ under A. Repeat the argument with ϕ replaced by ϕ' to conclude that ϕ' has a homotopy left inverse ϕ'' under A, hence that ϕ' is a homotopy equivalence in $A\backslash$**TOP** or still, that ϕ is a homotopy equivalence in $A\backslash$**TOP**.]

Application: Suppose that $\left\{\begin{matrix} (X, x_0) \\ (Y, y_0) \end{matrix}\right.$ are wellpointed. Let $f \in C(X, x_0; Y, y_0)$—then f is a homotopy equivalence in **TOP** iff f is a homotopy equivalence in $\mathbf{TOP}_*$.

FACT Suppose that (X, x_0) is wellpointed. Let $f \in C(X,Y)$ be inessential—then f is homotopic in $\mathbf{TOP}_*$ to the function $x \to f(x_0)$.

LEMMA Suppose given a commutative diagram

$$\begin{tikzcd} A \arrow[r, "i"] \arrow[d, "\phi"'] & X \arrow[d, "\psi"] \\ B \arrow[r, "j"'] & Y \end{tikzcd}$$

in which $\left\{\begin{matrix} i \\ j \end{matrix}\right.$ are cofibrations and $\left\{\begin{matrix} \phi \\ \psi \end{matrix}\right.$ are homotopy equivalences. Fix a homotopy inverse ϕ' for ϕ and a homotopy $h_A : IA \to A$ between $\phi' \circ \phi$ and id_A—then there exists a homotopy inverse ψ' for ψ with $i \circ \phi' = \psi' \circ j$ and a homotopy $H_X : IX \to X$ between $\psi' \circ \psi$ and id_X such that $H_X(i(a), t) = \begin{cases} i(h_A(a, 2t)) & (0 \le t \le 1/2) \\ i(a) & (1/2 \le t \le 1) \end{cases}$.

[Fix some ψ' with $i \circ \phi' = \psi' \circ j$ (possible, j being a cofibration). Put $h = i \circ h_A$: $h \circ i_0 = i \circ h_A \circ i_0 = i \circ \phi' \circ \phi = \psi' \circ j \circ \phi = \psi' \circ \psi \circ i \Rightarrow \exists\ H : IX \to X$ such that $\psi' \circ \psi = H \circ i_0$ and $H \circ Ii = h$. Put $f = H \circ i_1 : f \circ i = i \circ h_A \circ i_1 = i\ \&$ $f \simeq H \circ i_0 = \psi' \circ \psi \simeq \mathrm{id}_X \Rightarrow \exists\ g \in C(X,X) : g \circ i = i\ \&\ g \circ f \simeq \mathrm{id}_X \operatorname{rel} i(A)$. Let $G : IX \to X$ be a homotopy between $g \circ f$ and $\mathrm{id}_X \operatorname{rel} i(A)$. Define $H_X : IX \to X$ by $H_X(x,t) = \begin{cases} g(H(x, 2t)) & (0 \le t \le 1/2) \\ G(x, 2t-1) & (1/2 \le t \le 1) \end{cases}$: H_X is a homotopy between $g \circ \psi' \circ \psi$ and id_X and $H_X \circ Ii = i \circ h'_A$, where $h'_A(a,t) = h_A(a, \min\{2t, 1\})$ is a homotopy between $\phi' \circ \phi$ and id_A. Make the substitution $\psi' \to g \circ \psi'$ to complete the proof.]

PROPOSITION 14 Suppose given a commutative diagram

$$\begin{tikzcd} A \arrow[r, "i"] \arrow[d, "\phi"'] & X \arrow[d, "\psi"] \\ B \arrow[r, "j"'] & Y \end{tikzcd}$$

in which $\left\{\begin{matrix} i \\ j \end{matrix}\right.$ are cofibrations and $\left\{\begin{matrix} \phi \\ \psi \end{matrix}\right.$ are homotopy equivalences—then (ϕ, ψ) is a homotopy equivalence in $\mathbf{TOP}(\to)$.

[The lemma implies that (ϕ', ψ') is a homotopy left inverse for (ϕ, ψ) in **TOP**$(\rightarrow)$.]

EXAMPLE Let $\begin{cases} f : X \rightarrow Y \\ f' : X' \rightarrow Y' \end{cases}$ be objects in **TOP**$(\rightarrow)$. Write $[f, f']$ for the set of homotopy classes of maps in **TOP**$(\rightarrow)$ from f to f'. Question: Is it true that if $\begin{cases} f \simeq g \\ f' \simeq g' \end{cases}$ (in **TOP**), then $[f, f'] = [g, g']$? The answer is "no". Let $f = g$ be the constant map $\mathbf{S}^1 \rightarrow (1,0)$; let $f' : \mathbf{S}^1 \rightarrow \mathbf{D}^2$ be the inclusion and let $g' : \mathbf{S}^1 \rightarrow \mathbf{D}^2$ be the constant map at $(1,0)$—then $[f, f'] \neq [g, g']$.

PROPOSITION 15 Let
$$\begin{tikzcd} X^0 \arrow[r] \arrow[d] & X^1 \arrow[r] \arrow[d] & \cdots \\ Y^0 \arrow[r] & Y^1 \arrow[r] & \cdots \end{tikzcd}$$
be a commutative ladder connecting two expanding sequences of topological spaces. Assume: $\forall\ n$, the inclusions $\begin{cases} X^n \rightarrow X^{n+1} \\ Y^n \rightarrow Y^{n+1} \end{cases}$ are cofibrations and the vertical arrows $\phi^n : X^n \rightarrow Y^n$ are homotopy equivalences—then the induced map $\phi^\infty : X^\infty \rightarrow Y^\infty$ is a homotopy equivalence.

[Using the lemma, inductively construct a homotopy left inverse for ϕ^∞.]

FACT Let $X^0 \subset X^1 \subset \cdots$ be an expanding sequence of topological spaces. Assume: $\forall\ n$, the inclusion $X^n \rightarrow X^{n+1}$ is a cofibration and that X^n is a strong deformation retract of X^{n+1}—then X^0 is a strong deformation retract of X^∞.

[Bearing in mind Proposition 5, recall first that the inclusion $X^0 \rightarrow X^\infty$ is a cofibration (cf. p. 3–5). Consider the commutative ladder
$$\begin{tikzcd} X^0 \arrow[r] \arrow[d] & X^0 \arrow[r] \arrow[d] & \cdots \\ X^0 \arrow[r] & X^1 \arrow[r] & \cdots \end{tikzcd}$$
to see that the inclusion $X^0 \rightarrow X^\infty$ is also a homotopy equivalence.]

FACT Let $X^0 \subset X^1 \subset \cdots$ be an expanding sequence of topological spaces. Assume: $\forall\ n$, the inclusion $X^n \rightarrow X^{n+1}$ is a cofibration and inessential—then X^∞ is contractible.

EXAMPLE Take $X^n = \mathbf{S}^n$—then $X^\infty = \mathbf{S}^\infty$ is contractible.

Let $f : X \rightarrow Y$ be a continuous function—then the mapping cylinder M_f of f is defined by the pushout square
$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "i_0"'] & Y \arrow[d] \\ IX \arrow[r] & M_f \end{tikzcd}$$
. Special case: The mapping cylinder of $X \rightarrow *$ is ΓX, the cone of X (in particular, $\Gamma \mathbf{S}^{n-1} = \mathbf{D}^n$, so $\Gamma \emptyset = *$). There is a closed embedding $j : Y \rightarrow M_f$, a homotopy $H : IX \rightarrow M_f$, and a unique continuous function $r : M_f \rightarrow Y$ such that $r \circ j = \mathrm{id}_Y$ and $r \circ H = f \circ p$ $(p : IX \rightarrow X)$. One has

$j \circ r \simeq \mathrm{id}_{M_f} \operatorname{rel} j(Y)$. The composition $H \circ i_1$ is a closed embedding $i : X \to M_f$ and $f = r \circ i$.

Suppose that X is a subspace of Y and that $f : X \to Y$ is the inclusion—then there is a continuous bijection $M_f \to i_0 Y \cup IX$. In general, this bijection is not a homeomorphism (consider $X =]0,1]$, $Y = [0,1]$) but will be if X is closed or f is a cofibration.

LEMMA j is a closed cofibration and $j(Y)$ is a strong deformation retract of M_f.

LEMMA i is a closed cofibration.

[Define $F : X \amalg X \to Y \amalg X$ by $F = f \amalg \mathrm{id}_X$ and form the pushout square

$$\begin{tikzcd}
X \amalg X \arrow[r, "F"] \arrow[d, "i_0"', "i_1"] & Y \amalg X \arrow[d] \\
IX \arrow[r] & IX \sqcup_F (Y \amalg X)
\end{tikzcd}$$

—then $IX \sqcup_F (Y \amalg X)$ can be identified with M_f, i becoming the composite of the closed cofibrations $X \to Y \amalg X \to IX \sqcup_F (Y \amalg X)$.]

It is a corollary that the embedding i of X into its cone ΓX is a closed cofibration.

EXAMPLE The mapping telescope is the functor tel : **FIL(TOP)** $\to$ **FILSP** defined on an object $(\mathbf{X},\mathbf{f})$ by $\mathrm{tel}(\mathbf{X},\mathbf{f}) = \coprod_n IX_n/\sim$, where $(x_n, 1) \sim (f_n(x_n), 0)$, and on a morphism $\phi : (\mathbf{X},\mathbf{f}) \to (\mathbf{Y},\mathbf{g})$ by $\mathrm{tel}\,\phi([x_n, t]) = [\phi_n(x_n), t]$. Let $\mathrm{tel}_n(\mathbf{X},\mathbf{f})$ be the image of $\left(\coprod_{k \leq n-1} IX_k\right) \coprod i_0 X_n$, so $\mathrm{tel}_n(\mathbf{X},\mathbf{f})$ is obtained from X_n via iterated application of the mapping cylinder construction. The embedding $\mathrm{tel}_n(\mathbf{X},\mathbf{f}) \to \mathrm{tel}_{n+1}(\mathbf{X},\mathbf{f})$ is a closed cofibration and $\mathrm{tel}(\mathbf{X},\mathbf{f}) = \operatorname{colim} \mathrm{tel}_n(\mathbf{X},\mathbf{f})$. There is a homotopy equivalence $\mathrm{tel}_n(\mathbf{X},\mathbf{f}) \to X_n$, viz. the assignment $[x_k, t] \to (f_{n-1} \circ \cdots \circ f_k)(x_k)$ $(0 \leq k \leq n-1)$, $[x_n, 0] \to x_n$ and the diagram

$$\begin{tikzcd}
\mathrm{tel}_n(\mathbf{X},\mathbf{f}) \arrow[r] \arrow[d] & \mathrm{tel}_{n+1}(\mathbf{X},\mathbf{f}) \arrow[d] \\
X_n \arrow[r] & X_{n+1}
\end{tikzcd}$$

commutes. Consequently, if all the f_n are cofibrations, then it follows from Proposition 15 that the induced map $\mathrm{tel}(\mathbf{X},\mathbf{f}) \to \operatorname{colim} X_n$ is a homotopy equivalence.

[Note: Up to homeomorphism, the telescope construction is an instance of the above procedure.]

PROPOSITION 16 Every morphism in **TOP** can be written as the composite of a closed cofibration and a homotopy equivalence.

PROPOSITION 17 Let $f : X \to Y$ be a continuous function—then f is a homotopy equivalence iff $i(X)$ is a strong deformation retract of M_f.

[Note that f is a homotopy equivalence iff i is a homotopy equivalence and quote Proposition 5.]

Let $f : X \to Y$ be a continuous function—then the mapping cone C_f of f is defined by the pushout square

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "i"'] & Y \arrow[d] \\ \Gamma X \arrow[r] & C_f \end{tikzcd}$$

. Special case: The mapping cone of $X \to *$ is ΣX, the suspension of X (in particular, $\Sigma \mathbf{S}^{n-1} = \mathbf{S}^n$, so $\Sigma\emptyset = \mathbf{S}^0$). There is a closed cofibration $j : Y \to C_f$ and an arrow $C_f \to \Sigma X$. By construction, $j \circ f$ is inessential and for any $g : Y \to Z$ with $g \circ f$ inessential, there exists a $\phi : C_f \to Z$ such that $g = \phi \circ j$.

[Note: The mapping cone sequence associated with f is given by $X \xrightarrow{f} Y \to C_f \to \Sigma X \to \Sigma Y \to \Sigma C_f \to \Sigma^2 X \to \cdots$. Taking into account the suspension isomorphism $\widetilde{H}_q(X) \approx \widetilde{H}_{q+1}(\Sigma X)$, there is an exact sequence

$$\cdots \to \widetilde{H}_q(X) \to \widetilde{H}_q(Y) \to \widetilde{H}_q(C_f) \to \widetilde{H}_{q-1}(X) \to \widetilde{H}_{q-1}(Y) \to \cdots .]$$

The mapping cylinder and the mapping cone can be viewed as functors $\mathbf{TOP}(\to) \to \mathbf{TOP}$. With this interpretation, i, j, and r are natural transformations.

[Note: Owing to AD_4, these functors restrict to functors $\mathbf{HAUS}(\to) \to \mathbf{HAUS}$. Consequently, if X and Y are in $\mathbf{CGH}$, then for any continuous function $f : X \to Y$, both M_f and C_f remain in $\mathbf{CGH}$. On the other hand, stability relative to $\mathbf{CG}$ or $\mathbf{\Delta}$-$\mathbf{CG}$ is automatic.]

FACT Suppose that $\begin{cases} f : X \to Y \\ g : X \to Y \end{cases}$ are homotopic—then in $\mathbf{HTOP}^2$, $(M_f, i(X)) \approx (M_g, i(X))$, and in $\mathbf{HTOP}$, $C_f \approx C_g$.

FACT Let $f \in C(X, Y)$. Suppose that $\phi : X' \to X$ ($\psi : Y \to Y'$) is a homotopy equivalence—then the arrow $(M_{f\circ\phi}, i(X')) \to (M_f, i(X))$ ($(M_f, i(X)) \to (M_{\psi\circ f}, i(X))$) is a homotopy equivalence (in $\mathbf{TOP}^2$) and the arrow $C_{f\circ\phi} \to C_f$ ($C_f \to C_{\psi\circ f}$) is a homotopy equivalence (in $\mathbf{TOP}$).

EXAMPLE The suspension ΣX of X is the union of two closed subspaces $\Gamma^- X$ and $\Gamma^+ X$, each homeomorphic to the cone ΓX of X, with $\Gamma^- X \cap \Gamma^+ X = X$ (identify the section $i_{1/2} X$ with X). Therefore ΣX is numerably contractible. The commutative diagram

$$\begin{tikzcd} X \arrow[r] \arrow[d] & \Gamma^+ X \arrow[d] \\ \Gamma^- X \arrow[r] & \Sigma X \end{tikzcd}$$

is a pushout square and the inclusions $\begin{cases} \Gamma^- X \to \Sigma X \\ \Gamma^+ X \to \Sigma X \end{cases}$ are closed cofibrations.

FACT Let $f : X \to Y$ be a continuous function. Suppose that Y is numerably contractible—then C_f is numerably contractible.

[The image of $X \times [0,1[$ in C_f is contractible. On the other hand, the image of $X \times]0,1] \amalg Y$ in C_f has the same homotopy type as Y, hence is numerably contractible (cf. p. 3–13).]

[Note: Y and M_f have the same homotopy type, so Y numerably contractible $\Rightarrow$ M_f numerably contractible (cf. p. 3–13).]

Let $X \xleftarrow{f} Z \xrightarrow{g} Y$ be a 2-source—then the double mapping cylinder $M_{f,g}$ of f, g is defined by the pushout square

$$\begin{tikzcd} Z \amalg Z \arrow[r, "f \amalg g"] \arrow[d, "i_0"', "i_1"] & X \amalg Y \arrow[d] \\ IZ \arrow[r] & M_{f,g} \end{tikzcd}$$

. The homotopy type of $M_{f,g}$ depends only on the homotopy classes of f and g and $M_{f,g}$ is homeomorphic to $M_{g,f}$. There are closed cofibrations $\begin{cases} i : X \to M_{f,g} \\ j : Y \to M_{f,g} \end{cases}$ and an arrow $M_{f,g} \to \Sigma Z$. The diagram

$$\begin{tikzcd} Z \arrow[r, "g"] \arrow[d, "f"'] & Y \arrow[d, "j"] \\ X \arrow[r, "i"'] & M_{f,g} \end{tikzcd}$$

is homotopy commutative and if the diagram

$$\begin{tikzcd} Z \arrow[r, "g"] \arrow[d, "f"'] & Y \arrow[d, "\eta"] \\ X \arrow[r, "\xi"'] & W \end{tikzcd}$$

is homotopy commutative, then there exists a $\phi : M_{f,g} \to W$ such that $\begin{cases} \xi = \phi \circ i \\ \eta = \phi \circ j \end{cases}$. Example: The double mapping cylinder of $X \leftarrow X \times Y \rightarrow Y$ is $X * Y$, the join of X and Y.

[Note: The mapping cylinder and the mapping cone are instances of the double mapping cylinder (homeomorphic models arise from the parameter reversal $t \to 1 - t$). Consideration of $\begin{cases} Z \times [0, 1/2] \\ Z \times [1/2, 1] \end{cases}$ leads to a pushout square

$$\begin{tikzcd} Z \arrow[r] \arrow[d] & M_g \arrow[d] \\ M_f \arrow[r] & M_{f,g} \end{tikzcd}$$

.]

EXAMPLE (The Mapping Telescope) $\mathrm{tel}(\mathbf{X}, \mathbf{f})$ can be identified with the double mapping cylinder of the 2-source $\coprod_{n \geq 0} X_{2n} \leftarrow \coprod_{n \geq 0} X_n \rightarrow \coprod_{n \geq 0} X_{2n+1}$. Here, the left hand arrow is defined by $x_{2n} \to x_{2n}$ & $x_{2n+1} \to f_{2n+1}(x_{2n+1})$ and the right hand arrow is defined by $x_{2n+1} \to x_{2n+1}$ & $x_{2n} \to f_{2n}(x_{2n})$.

Every 2-source $X \xleftarrow{f} Z \xrightarrow{g} Y$ determines a pushout square

$$\begin{tikzcd} Z \arrow[r, "g"] \arrow[d, "f"'] & Y \arrow[d, "\eta"] \\ X \arrow[r, "\xi"'] & P \end{tikzcd}$$

and there is an arrow $\phi : M_{f,g} \to P$ characterized by the conditions $\begin{cases} \xi = \phi \circ i \\ \eta = \phi \circ j \end{cases}$ & $IZ \to M_{f,g} \xrightarrow{\phi} P = \begin{cases} \xi \circ f \circ p \\ \quad \| \\ \eta \circ g \circ p \end{cases}$.

PROPOSITION 18 If f is a cofibration, then $\phi : M_{f,g} \to P$ is a homotopy equivalence in $Y\backslash\mathbf{TOP}$.

[The arrow $M_f \to IX$ admits a left inverse $IX \to M_f$.]

Application: Suppose that $f : X \to Y$ is a cofibration—then the projection $C_f \to Y/f(X)$ is a homotopy equivalence.

[Note: If in addition X is contractible, then the embedding $Y \to C_f$ is a homotopy equivalence. Therefore in this case the projection $Y \to Y/f(X)$ is a homotopy equivalence.]

EXAMPLE Let A be a nonempty finite subset of $\mathbf{S}^n$ $(n \geq 1)$—then $\mathbf{S}^n/A$ has the homotopy type of the wedge of $\mathbf{S}^n$ with $(\#(A) - 1)$ circles.

[The inclusion $A \to \mathbf{S}^n$ is a cofibration (cf. Proposition 8).]

Consider the 2-sources $\begin{cases} X \leftarrow A \xrightarrow{f} Y \\ X \leftarrow A \xrightarrow[g]{} Y \end{cases}$, where the arrow $A \to X$ is a closed cofibration. Assume that $f \simeq g$—then Proposition 18 implies that $X \sqcup_f Y$ and $X \sqcup_g Y$ have the same homotopy type rel Y. Corollary: If $f' : A \to Y'$ is a continuous function and if $\phi : Y \to Y'$ is a homotopy equivalence such that $\phi \circ f \simeq f'$, then there is a homotopy equivalence $\Phi : X \sqcup_f Y \to X \sqcup_{f'} Y'$ with $\Phi \mid Y = \phi$.

FACT Suppose that $A \to X$ is a closed cofibration. Let $f : A \to Y$ be a homotopy equivalence—then the arrow $X \to X \sqcup_f Y$ is a homotopy equivalence.

Denote by $|\Delta, \mathrm{id}|_{\mathbf{TOP}}$ the comma category corresponding to the diagonal functor $\Delta : \mathbf{TOP} \to \mathbf{TOP} \times \mathbf{TOP}$ and the identity functor id on $\mathbf{TOP} \times \mathbf{TOP}$. So, an object in $|\Delta, \mathrm{id}|_{\mathbf{TOP}}$ is a 2-source $X \xleftarrow{f} Z \xrightarrow{g} Y$ and a morphism of 2-sources is a commutative diagram

$$\begin{tikzcd} X \arrow[d] & Z \arrow[l, "f"'] \arrow[r, "g"] \arrow[d] & Y \arrow[d] \\ X' & Z' \arrow[l, "f'"] \arrow[r, "g'"'] & Y' \end{tikzcd}$$

. The double mapping cylinder is a functor $|\Delta, \mathrm{id}|_{\mathbf{TOP}} \to \mathbf{TOP}$. It has a right adjoint $\mathbf{TOP} \to |\Delta, \mathrm{id}|_{\mathbf{TOP}}$, viz. the functor that sends X to the 2-source $X \xleftarrow{p_0} PX \xrightarrow{p_1} X$.

FACT Let

$$\begin{tikzcd} X \arrow[d] & Z \arrow[l, "f"'] \arrow[r, "g"] \arrow[d] & Y \arrow[d] \\ X' & Z' \arrow[l, "f'"] \arrow[r, "g'"'] & Y' \end{tikzcd}$$

be a commutative diagram in which the vertical arrows are homotopy equivalences—then the arrow $M_{f,g} \to M_{f',g'}$ is a homotopy equivalence.

Application: Suppose that $\begin{cases} A \to X \\ A' \to X' \end{cases}$ are closed cofibrations. Let $\begin{cases} f : A \to Y \\ f' : A' \to Y' \end{cases}$ be continuous functions. Assume that the diagram

$$\begin{tikzcd} X \arrow[d] & A \arrow[l] \arrow[r, "f"] \arrow[d] & Y \arrow[d] \\ X' & A' \arrow[l] \arrow[r, "f'"'] & Y' \end{tikzcd}$$

commutes and that the vertical arrows are homotopy equivalences—then the induced map $X \sqcup_f Y \rightarrow X' \sqcup_{f'} Y'$ is a homotopy equivalence.

EXAMPLE Suppose that $X = A \cup B$, where $\left\{\begin{array}{l} A \\ B \end{array}\right.$ are closed and the inclusions $\left\{\begin{array}{l} A \cap B \rightarrow A \\ A \cap B \rightarrow B \end{array}\right.$ are cofibrations. Assume: A and B are contractible—then the arrow $\Sigma(A \cap B) \rightarrow X$ is a homotopy equivalence.

SEGAL-STASHEFF CONSTRUCTION Let X be a topological space. Fix a covering $\mathcal{U} = \{U_i : i \in I\}$ of X. Equip I with a well ordering $<$ and put $I[n] = \{[i] \equiv (i_0, \ldots, i_n) : i_0 < \cdots < i_n\}$. Every strictly increasing $\alpha \in \operatorname{Mor}([m],[n])$ defines a map $I[n] \rightarrow I[m]$. Set $U_{[i]} = U_{i_0} \cap \cdots \cap U_{i_n}$ and form $\mathcal{U}([n]) = \coprod_{I[n]} U_{[i]}$, a coproduct in **TOP**. Give $\mathcal{U}([n]) \times \Delta^n$ the product topology and call $B\mathcal{U}$ the quotient $\coprod_n \mathcal{U}([n]) \times \Delta^n/\sim$, the equivalence relation being generated by writing $((x,[i]), \Delta^\alpha t) \sim ((x, \alpha[i]), t)$. Let $B\mathcal{U}^{(n)}$ be the image of $\coprod_{m \leq n} \mathcal{U}([m]) \times \Delta^m$ in $B\mathcal{U}$, so $B\mathcal{U} = \operatorname{colim} B\mathcal{U}^{(n)}$. The commutative diagram

$$\begin{tikzcd} \coprod_{I[n]} U_{[i]} \times \dot{\Delta}^n \arrow[r] \arrow[d] & B\mathcal{U}^{(n-1)} \arrow[d] \\ \coprod_{I[n]} U_{[i]} \times \Delta^n \arrow[r] & B\mathcal{U}^{(n)} \end{tikzcd}$$

is a pushout square in **TOP** and the vertical arrows are closed cofibrations. There is a projection $p_{\mathcal{U}} : B\mathcal{U} \rightarrow X$ induced by the arrows $U_{[i]} \times \Delta^n \rightarrow U_{[i]}$, i.e., $((x,[i]),t) \rightarrow x$. Moreover, $p_{\mathcal{U}}$ is a homotopy equivalence provided that $\mathcal{U}$ is numerable. Indeed, any partition of unity $\{\kappa_i : i \in I\}$ on X subordinate to $\mathcal{U}$ determines a continuous function $s_{\mathcal{U}} : X \rightarrow B\mathcal{U}$ (since $\forall\, x, \#\{i \in I : x \in \operatorname{spt} \kappa_i\} < \omega$). Obviously, $p_{\mathcal{U}} \circ s_{\mathcal{U}} = \mathrm{id}_X$ and $s_{\mathcal{U}} \circ p_{\mathcal{U}}$ can be connected to the identity on $B\mathcal{U}$ via a linear homotopy.

FACT Let $\left\{\begin{array}{l} X \\ Y \end{array}\right.$ be topological spaces and let $f : X \rightarrow Y$ be a continuous function. Suppose that $\left\{\begin{array}{l} \mathcal{U} = \{U_i : i \in I\} \\ \mathcal{V} = \{V_i : i \in I\} \end{array}\right.$ are numerable coverings of $\left\{\begin{array}{l} X \\ Y \end{array}\right.$ such that $\forall\, i : f(U_i) \subset V_i$. Assume: $\forall\, [i]$, the induced map $f_{[i]} : U_{[i]} \rightarrow V_{[i]}$ is a homotopy equivalence—then f is a homotopy equivalence.

[There is an arrow $F : B\mathcal{U} \rightarrow B\mathcal{V}$ and a commutative diagram

$$\begin{tikzcd} B\mathcal{U} \arrow[r, "F"] \arrow[d, "p_{\mathcal{U}}"'] & B\mathcal{V} \arrow[d, "p_{\mathcal{V}}"] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

. Due to the numerability of $\mathcal{U}$ and $\mathcal{V}$, $p_{\mathcal{U}}$ and $p_{\mathcal{V}}$ are homotopy equivalences. Claim: $\forall\, n$, the restriction $F^{(n)} : B\mathcal{U}^{(n)} \rightarrow B\mathcal{V}^{(n)}$ is a homotopy equivalence. This is clear if $n = 0$. For $n > 0$, consider the commutative diagram

$$\begin{tikzcd}
\coprod\limits_{I[n]} U_{[i]} \times \Delta^n \arrow[d] & \coprod\limits_{I[n]} U_{[i]} \times \dot{\Delta}^n \arrow[l] \arrow[r] \arrow[d] & BU^{(n-1)} \arrow[d] \\
\coprod\limits_{I[n]} V_{[i]} \times \Delta^n & \coprod\limits_{I[n]} V_{[i]} \times \dot{\Delta}^n \arrow[l] \arrow[r] & BV^{(n-1)}
\end{tikzcd}$$

By induction, $F^{(n-1)}$ is a homotopy equivalence, thus $F^{(n)}$ is too. Proposition 15 then implies that $F : BU \to BV$ is a homotopy equivalence, so the same is true of f.]

Let $u, v : X \to Y$ be a pair of continuous functions—then the mapping torus $T_{u,v}$ of u, v is defined by the pushout square

$$\begin{tikzcd}
X \amalg X \arrow[r, shift left, "u"] \arrow[r, shift right, "v"'] \arrow[d, "i_0"', "i_1"] & Y \arrow[d] \\
IX \arrow[r] & T_{u,v}
\end{tikzcd}$$

. There is a closed cofibration $j : Y \to T_{u,v}$. From the definitions, $j \circ u \simeq j \circ v$ and for any $g : Y \to Z$ with $g \circ u \simeq g \circ v$, there exists a $\phi : T_{u,v} \to Z$ such that $g = \phi \circ j$.

[Note: If $u = v = \mathrm{id}_X$, then $T_{u,v}$ is the product $X \times \mathbf{S}^1$.]

EXAMPLE (The Scorpion) Let $\pi : \mathbf{S}^n \to \mathbf{D}^n$ be the restriction of the canonical map $\mathbf{R}^{n+1} \to \mathbf{R}^n$; let $p : \mathbf{D}^n \to \mathbf{D}^n/\mathbf{S}^{n-1} = \mathbf{S}^n$ be the projection. Put $f = p \circ \pi$—then $f : \mathbf{S}^n \to \mathbf{S}^n$ is inessential. The scorpion $\mathcal{S}^{n+1}$ is the quotient of $I\mathbf{S}^n$ with respect to the relations $(x, 0) \sim (f(x), 1)$, i.e., $\mathcal{S}^{n+1}$ is the mapping torus of $x \to f(x)$ & $x \to x$ $(x \in \mathbf{S}^n)$. One may also describe $\mathcal{S}^{n+1}$ as the quotient $\mathbf{D}^{n+1}/\sim$, where $x \sim p(2x)$ $(x \in (1/2)\mathbf{D}^n)$. Fix a point $x_0 \in (1/2)\mathbf{S}^{n-1}$, let L_0 be the line segment from x_0 to $p(2x_0)$, and let C_0 be the circle $L_0/\sim$—then the inclusion $C_0 \to \mathcal{S}^{n+1}$ is a homotopy equivalence, thus $\mathcal{S}^{n+1}$ is a homotopy circle. The dunce hat $\mathcal{D}^{n+1}$ is the quotient $\mathcal{S}^{n+1}/C_0$. It is contractible.

The formalities in $\mathbf{TOP}_*$ run parallel to those in $\mathbf{TOP}$, thus a detailed account of the pointed theory is unnecessary. Of course, there is an important difference between $\mathbf{TOP}$ and $\mathbf{TOP}_*$: $\mathbf{TOP}_*$ has a zero object but $\mathbf{TOP}$ does not. Consequently, if $\begin{cases} (X, x_0) \\ (Y, y_0) \end{cases}$ are in $\mathbf{TOP}_*$, then $[X, x_0; Y, y_0]$ is a pointed set with distinguished element $[0]$, the pointed homotopy class of the zero morphism, i.e., of the constant map $X \to y_0$. Functions $f \in [0]$ are said to be nullhomotopic: $f \simeq 0$.

[Note: The forgetful functor $\mathbf{TOP}_* \to \mathbf{TOP}$ has a left adjoint $\mathbf{TOP} \to \mathbf{TOP}_*$ that sends the space X to the pointed space $X_+ = X \amalg *$.]

The computation of pushouts in $\mathbf{TOP}_*$ is expedited by noting that a pushout in $\mathbf{TOP}$ of a 2-source in $\mathbf{TOP}_*$ is a pushout in $\mathbf{TOP}_*$. Examples: (1) The pushout square

$$\begin{tikzcd}
* \arrow[r] \arrow[d] & (Y, y_0) \arrow[d] \\
(X, x_0) \arrow[r] & X \vee Y
\end{tikzcd}$$

defines the wedge $X \vee Y$; (2) The pushout square

$$\begin{tikzcd} X \vee Y \arrow[r] \arrow[d] & * \arrow[d] \\ X \times Y \arrow[r] & X \# Y \end{tikzcd}$$ defines the smash product $X \# Y$.

[Note: Base points are suppressed if there is no need to display them.]

The wedge is the coproduct in **TOP**$_*$. If both of the inclusions $\begin{cases} \{x_0\} \rightarrow X \\ \{y_0\} \rightarrow Y \end{cases}$ are cofibrations and if at least one is closed, then the embedding $X \vee Y \rightarrow X \times Y$ is a cofibration (cf. Proposition 7) and $X \vee Y$ is wellpointed (cf. Proposition 9).

FACT Suppose that $\begin{cases} (X, x_0) \\ (Y, y_0) \end{cases}$ are in **TOP**$_*$—then $\forall\, n > 1$, there is a split short exact sequence

$$0 \rightarrow \pi_{n+1}(X \times Y, X \vee Y) \rightarrow \pi_n(X \vee Y) \rightarrow \pi_n(X \times Y) \rightarrow 0.$$

Griffiths[†] proved that if (X, x_0) is a path connected pointed Hausdorff space which is both first countable and locally simply connected at x_0, then for any path connected pointed Hausdorff space (Y, y_0), the arrow $\pi_1(X, x_0) * \pi_1(Y, y_0) \rightarrow \pi_1((X, x_0) \vee (Y, y_0))$ is an isomorphism.

[Note: X is locally simply connected at x_0 provided that for any neighborhood U of x_0 there exists a neighborhood $V \subset U$ of x_0 such that the induced homomorphism $\pi_1(V, x_0) \rightarrow \pi_1(U, x_0)$ is trivial.]

Eda[‡] has constructed an example of a path connected CRH space X which is locally simply connected at x_0 with the property that $\pi_1(X, x_0) = 1$ but $\pi_1((X, x_0) \vee (X, x_0)) \neq 1$. Moral: The hypothesis of first countability cannot be dropped.

EXAMPLE (The Hawaiian Earring) Let X be the subspace of $\mathbf{R}^2$ consisting of the union of the circles X_n, where X_n has center $(1/n, 0)$ and radius $1/n$ $(n \geq 1)$. Take $x_0 = (0,0)$—then X is first countable at x_0, X is not locally simply connected at x_0, the inclusion $\{x_0\} \rightarrow X$ is not a cofibration, and the arrow $\pi_1(X, x_0) * \pi_1(X, x_0) \rightarrow \pi_1((X, x_0) \vee (X, x_0))$ is injective but not surjective. Denote now by X_0 the result of assigning to X the final topology determined by the inclusions $X_n \rightarrow X$. X_0 is a CW complex. Take $x_0 = (0,0)$—then X_0 is not first countable at x_0, X_0 is locally simply connected at x_0, the inclusion $\{x_0\} \rightarrow X_0$ is a cofibration, and the arrow $\pi_1(X_0, x_0) * \pi_1(X_0, x_0) \rightarrow \pi_1((X_0, x_0) \vee (X_0, x_0))$ is an isomorphism (Van Kampen).

FACT Given a wellpointed space (X, x_0), suppose that $X = A \cup B$, where $x_0 \in A \cap B$ and $A \cap B$ is contractible. Assume: The inclusions $\begin{cases} A \cap B \rightarrow A \\ A \cap B \rightarrow B \end{cases}$ & $\begin{cases} A \rightarrow X \\ B \rightarrow X \end{cases}$ are cofibrations. Take $\begin{cases} a_0 = x_0 \\ b_0 = x_0 \end{cases}$—then the arrow $A \vee B \rightarrow X$ is a pointed homotopy equivalence.

[†] *Quart. J. Math.* 5 (1954), 175–190.

[‡] *Proc. Amer. Math. Soc.* 109 (1990), 237–241; see also Morgan-Morrison, *Proc. London Math. Soc.* 53 (1986), 562–576.

The smash product $\#$ is a functor $\mathbf{TOP}_* \times \mathbf{TOP}_* \to \mathbf{TOP}_*$. It respects homotopies, thus the pointed homotopy type of $X\#Y$ depends only on the pointed homotopy types of X and Y. If both of the inclusions $\begin{cases} \{x_0\} \to X \\ \{y_0\} \to Y \end{cases}$ are cofibrations and if at least one is closed, then $X\#Y$ is wellpointed.

[Note: Suppose that Y is a pointed LCH space—then it is clear that the functor $—\#Y : \mathbf{TOP}_* \to \mathbf{TOP}_*$ has a right adjoint $Z \to Z^Y$ which passes to $\mathbf{HTOP}_* : [X\#Y, Z] \approx [X, Z^Y]$, Z^Y the set of pointed continuous functions from Y to Z equipped with the compact open topology. One can say more: In fact, Cagliari[†] has shown that for any pointed Y, the functor $—\#Y$ has a right adjoint in $\mathbf{TOP}_*$ iff the functor $— \times Y$ has a right adjoint in $\mathbf{TOP}$, i.e., iff Y is core compact (cf. p. 2–2).]

($\#_1$) $X\#Y$ is homeomorphic to $Y\#X$.

($\#_2$) $(X\#Y)\#Z$ is homeomorphic to $X\#(Y\#Z)$ if both X and Z are LCH spaces or if two of X, Y, Z are compact Hausdorff.

[Note: The smash product need not be associative (consider $(\mathbf{Q}\#\mathbf{Q})\#Z$ and $\mathbf{Q}\#(\mathbf{Q}\#Z)$).]

($\#_3$) $(X \vee Y)\#Z$ is homeomorphic to $(X\#Z) \vee (Y\#Z)$.

($\#_4$) $\Sigma(X * Y)$ is homeomorphic to $\Sigma X\#\Sigma Y$ if X and Y are compact Hausdorff.

[Note: The suspension can be viewed as a functor $\mathbf{TOP} \to \mathbf{TOP}_*$. This is because the suspension is the result of collapsing to a point the embedded image of a space in its cone. Example: $\mathbf{S}^{m-1} * \mathbf{S}^{n-1} = \mathbf{S}^{m+n-1} \Rightarrow \mathbf{S}^m\#\mathbf{S}^n = \mathbf{S}^{m+n}$.]

All the homeomorphisms figuring in the foregoing are natural and preserve the base points.

LEMMA The smash product of two pointed Hausdorff spaces is Hausdorff.

The pushout square

$$\begin{tikzcd} X \vee Y \arrow[r] \arrow[d] & * \arrow[d] \\ X \times_k Y \arrow[r] & X\#_k Y \end{tikzcd}$$

defines the smash product $X\#_k Y$ in $\mathbf{CG}$, $\mathbf{\Delta}$-$\mathbf{CG}$, or $\mathbf{CGH}$. It is associative and distributes over the wedge.

[Note: With $\#_k$ as the multiplication and $\mathbf{S}^0$ as the unit, $\mathbf{CG}_*$, $\mathbf{\Delta}$-$\mathbf{CG}_*$, and $\mathbf{CGH}_*$ are closed categories.]

The pointed cylinder functor $I : \mathbf{TOP}_* \to \mathbf{TOP}_*$ is the functor that sends (X, x_0) to the quotient $X \times [0,1]/\{x_0\} \times [0,1]$, i.e., $I(X, x_0) = IX/I\{x_0\}$. Variant: Let $I_+ = [0,1] \amalg *$—then $I(X, x_0)$ is the smash product $X\#I_+$. The pointed path space functor $P : \mathbf{TOP}_* \to \mathbf{TOP}_*$ is the functor that sends (X, x_0) to $C([0,1], X)$ (compact open topology), the base point for the latter being the constant path $[0,1] \to x_0$. As in the unpointed situation, (I, P) is an adjoint pair.

[†] *Proc. Amer. Math. Soc.* **124** (1996), 1265–1269.

Using I and P, one can define the notion of pointed cofibration. Since all maps and homotopies must respect the base points, an arrow $A \to X$ in $\mathbf{TOP}_*$ may be a pointed cofibration without being a cofibration. For example, $\forall\, x_0 \in X$, the arrow $(\{x_0\}, x_0) \to (X, x_0)$ is a pointed cofibration but in general the inclusion $\{x_0\} \to X$ is not a cofibration. On the other hand, an arrow $A \to X$ in $\mathbf{TOP}_*$ which is a cofibration, when considered as an arrow in $\mathbf{TOP}$, is necessarily a pointed cofibration. Pointed cofibrations are embeddings. If $x_0 \in A \subset X$ and if $\{x_0\}$ is closed in X, then the inclusion $A \to X$ is a pointed cofibration iff $i_0X \cup IA/I\{x_0\}$ is a retract of $I(X, x_0)$. Observe that for this it is not necessary that A itself be closed.

Let (X, A, x_0) be a pointed pair—then a Strøm structure on (X, A, x_0) consists of a continuous function $\phi : X \to [0,1]$ such that $A \subset \phi^{-1}(0)$, a continuous function $\psi : X \to [0,1]$ such that $\{x_0\} = \psi^{-1}(0)$, and a homotopy $\Phi : IX \to X$ of $\mathrm{id}_X \operatorname{rel} A$ such that $\Phi(x,t) \in A$ whenever $\min\{t, \psi(x)\} > \phi(x)$.

[Note: Φ is therefore a pointed homotopy.]

POINTED COFIBRATION CHARACTERIZATION THEOREM Let $x_0 \in A \subset X$ and suppose that $\{x_0\}$ is a zero set in X—then the inclusion $A \to X$ is a pointed cofibration iff the pointed pair (X, A, x_0) admits a Strøm structure.

[Necessity: Fix $\psi \in C(X, [0,1]) : \{x_0\} = \psi^{-1}(0)$ and let $X \xleftarrow{p} IX \xrightarrow{q} [0,1]$ be the projections. Put $Y = \{(x,t) \in i_0X \cup IA : t \leq \psi(x)\}$. Define a continuous function $f : i_0X \cup IA \to Y$ by $f(x,t) = (x, \min\{t, \psi(x)\})$ and let $F : IX \to Y$ be some continuous extension of f. Consider $\phi(x) = \sup\limits_{0 \leq t \leq 1} |\min\{t, \psi(x)\} - qF(x,t)|$, $\Phi(x,t) = pF(x,t)$.

Sufficiency: Given a Strøm structure (ϕ, ψ, Φ) on (X, A, x_0), define a retraction $r : I(X, x_0) \to i_0X \cup IA/I\{x_0\}$ by

$$r(x,t) = \begin{cases} (\Phi(x,t), 0) & (t\psi(x) \leq \phi(x)) \\ (\Phi(x,t), t - \phi(x)/\psi(x)) & (t\psi(x) > \phi(x)) \end{cases} .]$$

LEMMA Let (X, A, x_0) be a pointed pair. Suppose that the inclusions $\begin{cases} \{x_0\} \to A \\ \{x_0\} \to X \end{cases}$ are closed cofibrations and that the inclusion $A \to X$ is a pointed cofibration—then the pair (X, x_0) has a Strøm structure (f, F) for which $F(IA) \subset A$.

[Fix a Strøm structure (f_X, F_X) on (X, x_0). Choose a Strøm structure (ϕ, ψ, Φ) on (X, A, x_0) such that $\phi \leq \psi = f_X$. Fix a Strøm structure (f_A, F_A) on (A, x_0). Extend the pointed homotopy $i \circ F_A : IA \to A \xrightarrow{i} X$ to a pointed homotopy $\overline{F} : IX \to X$ with $\overline{F} \circ i_0 = \mathrm{id}_X$. Put

$$\overline{f}(x) = \begin{cases} (1 - \phi(x)/\psi(x)) f_A(\Phi(x,1)) + \phi(x) & (\phi(x) < \psi(x)) \\ \psi(x) & (\phi(x) = \psi(x)) \end{cases} .$$

Then $\overline{f} \in C(X,[0,1]), \overline{f}|A = f_A$, and $\overline{f}^{-1}(0) = \{x_0\}$. Consider $f(x) = \min\{1, \overline{f}(x) + f_X(\overline{F}(x,1))\}$,

$$F(x,t) = \begin{cases} \overline{F}(x, t/\overline{f}(x)) & (t < \overline{f}(x)) \\ F_X(\overline{F}(x,1), t - \overline{f}(x)) & (t \geq \overline{f}(x)) \end{cases} .]$$

PROPOSITION 19 Let (X, A, x_0) be a pointed pair. Suppose that the inclusions $\begin{cases} \{x_0\} \to A \\ \{x_0\} \to X \end{cases}$ are closed cofibrations—then the inclusion $A \to X$ is a cofibration iff it is a pointed cofibration.

[To establish the nontrivial assertion, take (f, F) as in the lemma and choose a Strøm structure $(\overline{\phi}, \overline{\psi}, \overline{\Phi})$ on (X, A, x_0) with $\overline{\phi} \leq \overline{\psi} = f$. Define a Strøm structure (ϕ, Φ) on (X, A) by $\phi(x) = \overline{\phi}(x) - \overline{\psi}(x) + \sup\limits_{0 \leq t \leq 1} \overline{\psi}(\overline{\Phi}(x,t))$,

$$\Phi(x,t) = F(\overline{\Phi}(x,t), \min\{t, \overline{\phi}(x)/\overline{\psi}(x)\}) \quad (x \neq x_0)$$

and $\Phi(x_0, t) = x_0$.]

So, under conditions commonly occurring in practice, the pointed and unpointed notions of cofibration are equivalent.

Let $X \xleftarrow{f} Z \xrightarrow{g} Y$ be a pointed 2-source—then there is an embedding $M_{*,*} \to M_{f,g}$ and the quotient $M_{f,g}/M_{*,*}$ is the pointed double mapping cylinder of f, g. Here, $M_{*,*}$ is the double mapping cylinder of the 2-source $* \leftarrow * \rightarrow *$, which, being $* \times [0,1]$, is contractible. Thus if X, Y, and Z are wellpointed, then $M_{f,g}/M_{*,*}$ is wellpointed and the projection $M_{f,g} \to M_{f,g}/M_{*,*}$ is a homotopy equivalence (cf. p. 3–24).

[Note: The pointed mapping torus of a pair $u, v : X \to Y$ of pointed continuous functions is $T_{u,v}/T_{*,*}$, where $T_{*,*}$ is $* \times \mathbf{S}^1$, which is not contractible.]

The commutative diagram

$$\begin{tikzcd} Iz_0 \arrow[d] & z_0 \amalg z_0 \arrow[l] \arrow[r] \arrow[d] & x_0 \amalg y_0 \arrow[d] \\ IZ & Z \amalg Z \arrow[l, "i_0"', "i_1"] \arrow[r, "f \amalg g"'] & X \amalg Y \end{tikzcd}$$

leads to an induced map of pushouts $Iz_0 \to M_{f,g}$ which we claim is a cofibration. Thus, since $\begin{cases} X \\ Y \end{cases}$ are wellpointed, the arrow $x_0 \amalg y_0 \to X \amalg Y$ is a cofibration. On the other hand, the pushout of the 2-source $Iz_0 \leftarrow z_0 \amalg z_0 \to Z \amalg Z$ can be identified with $i_0 Z \cup Iz_0 \cup i_1 Z$ (even though z_0 is not assumed to be closed) and the inclusion $i_0 Z \cup Iz_0 \cup i_1 Z \to IZ$ is a cofibration (cf. p. 3–6). The claim is then seen to be a consequence of the proof of Proposition 4 in §12 (which depends only on the fact that cofibrations are pushout stable (cf. Proposition 2)). Consideration of the pushout square

$$\begin{tikzcd} Iz_0 \arrow[r] \arrow[d] & * \arrow[d] \\ M_{f,g} \arrow[r] & M_{f,g}/M_{*,*} \end{tikzcd}$$

now implies that $M_{f,g}/M_{*,*}$ is wellpointed. Finally, one can view $M_{f,g}$ itself as a wellpointed space (take $[z_0, 1/2]$ as the base point). The projection $M_{f,g} \rightarrow M_{f,g}/M_{*,*}$ is therefore a homotopy equivalence between wellpointed spaces, hence is actually a pointed homotopy equivalence (cf. p. 3–19).

In particular: There are pointed versions ΓX and ΣX of the cone and suspension of a pointed space X. Each is a quotient of its unpointed counterpart (and has the same homotopy type if X is wellpointed). ΣX is a cogroup object in $\mathbf{HTOP}_*$. In terms of the smash product, $\Gamma X = X\#[0,1]$ (0 the base point of $[0,1]$) and $\Sigma X = X\#\mathbf{S}^1$ ($(1,0)$ the base point of $\mathbf{S}^1$). Example: $\Gamma(X \vee Y) = \Gamma X \vee \Gamma Y$ and $\Sigma(X \vee Y) = \Sigma X \vee \Sigma Y$. The mapping space functor $\Theta : \mathbf{TOP}_* \rightarrow \mathbf{TOP}_*$ is the functor that sends (X, x_0) to the subspace of $C([0,1], X)$ consisting of those σ such that $\sigma(0) = x_0$ and the loop space functor $\Omega : \mathbf{TOP}_* \rightarrow \mathbf{TOP}_*$ is the functor that sends (X, x_0) to the subspace of $C([0,1], X)$ consisting of those σ such that $\sigma(0) = x_0 = \sigma(1)$, the base point in either case being the constant path $[0,1] \rightarrow x_0$. ΩX is a group object in $\mathbf{HTOP}_*$. (Γ, Θ) and (Σ, Ω) are adjoint pairs. Both drop to $\mathbf{HTOP}_*$: $[\Gamma X, Y] \approx [X, \Theta Y]$ and $[\Sigma X, Y] \approx [X, \Omega Y]$.

[Note: If X is wellpointed, then so are ΘX and ΩX.]

The mapping space ΘX is contractible and there is a pullback square

$$\begin{tikzcd} \Omega X \arrow[r] \arrow[d] & \Theta X \arrow[d, "p_1"] \\ \{x_0\} \arrow[r] & X \end{tikzcd}$$

in **TOP**, hence in $\mathbf{TOP}_*$.

EXAMPLE (The Moore Loop Space) Given a pointed space X, let $\Omega_M X$ be the set of all pairs $(\sigma, r_\sigma) : \sigma \in C([0, r_\sigma], X)$ $(0 \leq r_\sigma < \infty)$ and $\sigma(0) = x_0 = \sigma(r_\sigma)$. Attach to each $(\sigma, r_\sigma) \in \Omega_M X$ the function $\overline{\sigma}(t) = \sigma(\min\{t, r_\sigma\})$ on $\mathbf{R}_{\geq 0}$—then the assignment $(\sigma, r_\sigma) \rightarrow (\overline{\sigma}, r_\sigma)$ injects $\Omega_M X$ into $C(\mathbf{R}_{\geq 0}, X) \times \mathbf{R}_{\geq 0}$. Equip $\Omega_M X$ with the induced topology from the product (compact open topology on $C(\mathbf{R}_{\geq 0}, X)$). Define an associative multiplication on $\Omega_M X$ by writing $(\tau + \sigma)(t) = \begin{cases} \sigma(t) & (0 \leq t \leq r_\sigma) \\ \tau(t - r_\sigma) & (r_\sigma \leq t \leq r_{\tau+\sigma}) \end{cases}$, where $r_{\tau+\sigma} = r_\tau + r_\sigma$, the unit thus being $(0,0)$ $(0 \rightarrow x_0)$. Since "+" is continuous, $\Omega_M X$ is a monoid in **TOP**, the Moore loop space of X, and Ω_M is a functor $\mathbf{TOP}_* \rightarrow \mathbf{MON}_{\mathbf{TOP}}$. The inclusion $\Omega X \rightarrow \Omega_M X$ is an embedding (but it is not a pointed map).

Claim: ΩX is a deformation retract of $\Omega_M X$.

[Consider the homotopy $H : I\Omega_M X \rightarrow \Omega_M X$ defined as follows. The domain of $H((\sigma, r_\sigma), t)$ is the interval $[0, (1-t)r_\sigma + t]$ and there

$$H((\sigma, r_\sigma), t)(T) = \sigma\left(\frac{Tr_\sigma}{(1-t)r_\sigma + t}\right)$$

if $r_\sigma > 0$, otherwise $H((0,0),t)(T) = x_0$.]

One can also introduce $\Theta_M X$, the Moore mapping space of X. Like ΘX, $\Theta_M X$ is contractible and evaluation at the free end defines a Hurewicz fibration $\Theta_M X \to X$ whose fiber over the base point is $\Omega_M X$.

Let $f : X \to Y$ be a pointed continuous function, C_f its pointed mapping cone.

LEMMA If f is a pointed cofibration, then the projection $C_f \to Y/f(X)$ is a pointed homotopy equivalence.

In general, there is a pointed cofibration $j : Y \to C_f$ and an arrow $C_f \to \Sigma X$. Iterate to get a pointed cofibration $j' : C_f \to C_j$—then the triangle

$$\begin{tikzcd} C_f \arrow[r] \arrow[dr] & C_j \arrow[d] \\ & \Sigma X \end{tikzcd}$$

commutes and by the lemma, the vertical arrow is a pointed homotopy equivalence. Iterate again to get a pointed cofibration $j'' : C_j \to C_{j'}$—then the triangle

$$\begin{tikzcd} C_j \arrow[r] \arrow[dr] & C_{j'} \arrow[d] \\ & \Sigma Y \end{tikzcd}$$

commutes and by the lemma, the vertical arrow is a pointed homotopy equivalence. Example: Given pointed spaces $\begin{cases} X \\ Y \end{cases}$, let $X \overline{\#} Y$ be the pointed mapping cone of the inclusion $f : X \vee Y \to X \times Y$—then in **HTOP$_*$**, $C_j \approx \Sigma(X \vee Y)$ and $C_{j'} \approx \Sigma(X \times Y)$.

Let $f : X \to Y$ be a pointed continuous function—then the pointed mapping cone sequence associated with f is given by $X \xrightarrow{f} Y \to C_f \to \Sigma X \to \Sigma Y \to \Sigma C_f \to \Sigma^2 X \to \cdots$. Example: When $f = 0$, this sequence becomes $X \xrightarrow{0} Y \to Y \vee \Sigma X \to \Sigma X \to \Sigma Y \to \Sigma Y \vee \Sigma^2 X \to \Sigma^2 X \to \cdots$.

[Note: If the diagram

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d] & Y \arrow[d] \\ X' \arrow[r, "f'"'] & Y' \end{tikzcd}$$

commutes in **HTOP$_*$** and if the vertical arrows are pointed homotopy equivalences, then the pointed mapping cone sequences of f and f' are connected by a commutative ladder in **HTOP$_*$**, all of whose vertical arrows are pointed homotopy equivalences.]

REPLICATION THEOREM Let $f : X \rightarrow Y$ be a pointed continuous function—then for any pointed space Z, there is an exact sequence

$$\cdots \rightarrow [\Sigma Y, Z] \rightarrow [\Sigma X, Z] \rightarrow [C_f, Z] \rightarrow [Y, Z] \rightarrow [X, Z]$$

in **SET**$_*$.

[Note: A sequence of pointed sets and pointed functions $(X, x_0) \xrightarrow{\phi} (Y, y_0) \xrightarrow{\psi} (Z, z_0)$ is said to be exact in **SET**$_*$ if the range of ϕ is equal to the kernel of ψ.]

EXAMPLE Let $f : X \rightarrow Y$ be a pointed continuous function, Z a pointed space. Given pointed continuous functions $\alpha : \Sigma X \rightarrow Z$, $\phi : C_f \rightarrow Z$, write $(\alpha \cdot \phi)[x, t] = \begin{cases} \alpha(x, 2t) & (0 \leq t \leq 1/2) \\ \phi(x, 2t-1) & (1/2 \leq t \leq 1) \end{cases}$ $(x \in X)$ & $(\alpha \cdot \phi)(y) = \phi(y)$ $(y \in Y)$—then this prescription defines a left action of $[\Sigma X, Z]$ on $[C_f, Z]$ and the orbits are the fibers of the arrow $[C_f, Z] \rightarrow [Y, Z]$.

FACT Given a pointed continuous function $f : X \rightarrow Y$ and a pointed space Z, put $f_Z = f \# \mathrm{id}_Z$—then there is a commutative ladder

$$\begin{tikzcd}
X \# Z \arrow[r] \arrow[d, "\mathrm{id}"'] & Y \# Z \arrow[r] \arrow[d, "\mathrm{id}"'] & C_{f_Z} \arrow[r] \arrow[d] & \Sigma(X \# Z) \arrow[r] \arrow[d] & \Sigma(Y \# Z) \arrow[r] \arrow[d] & \cdots \\
X \# Z \arrow[r] & Y \# Z \arrow[r] & C_f \# Z \arrow[r] & \Sigma X \# Z \arrow[r] & \Sigma Y \# Z \arrow[r] & \cdots
\end{tikzcd}$$

in **HTOP**$_*$, all of whose vertical arrows are pointed homotopy equivalences.

[Show that there are mutually inverse pointed homotopy equivalences $\begin{cases} \phi : C_f \# Z \rightarrow C_{f_Z} \\ \psi : C_{f_Z} \rightarrow C_f \# Z \end{cases}$ for which the triangles

$$\begin{tikzcd}
 & C_f \# Z \arrow[dd, "\phi"] \\
Y \# Z \arrow[ur] \arrow[dr] & \\
 & C_{f_Z}
\end{tikzcd}
\qquad
\begin{tikzcd}
 & C_f \# Z \\
Y \# Z \arrow[ur] \arrow[dr] & \\
 & C_{f_Z} \arrow[uu, "\psi"']
\end{tikzcd}$$

commute.]

Given a pointed space (X, x_0), let $\check{X}$ be the mapping cylinder of the inclusion $\{x_0\} \rightarrow X$ and denote by $\check{x}_0$ the image of x_0 under the embedding $i : \{x_0\} \rightarrow \check{X}$—then $(\check{X}, \check{x}_0)$ is wellpointed and $\{\check{x}_0\}$ is closed in $\check{X}$ (cf. p. 3–21). The embedding $j : X \rightarrow \check{X}$ is a closed cofibration (cf. p. 3–21). It is not a pointed map but the retraction $r : \check{X} \rightarrow X$ is both a pointed map and a homotopy equivalence. We shall term (X, x_0) nondegenerate if $r : \check{X} \rightarrow X$ is a pointed homotopy equivalence.

[Note: Consider $X \vee [0, 1]$, where $x_0 = 0$—then $\check{X}$ is homeomorphic to $X \vee [0, 1]$ with $\check{x}_0 \leftrightarrow 1$.]

FACT Suppose that $\left\{\begin{array}{l}(X,x_0)\\(Y,y_0)\end{array}\right.$ are nondegenerate. Assume: $\left\{\begin{array}{l}X\\Y\end{array}\right.$ are numerably contractible—then $X \vee Y$ and $X \# Y$ are numerably contractible.

[To discuss $X \# Y$, take $\left\{\begin{array}{l}(X,x_0)\\(Y,y_0)\end{array}\right.$ wellpointed with $\left\{\begin{array}{l}\{x_0\} \subset X\\\{y_0\} \subset Y\end{array}\right.$ closed. The mapping cone of the inclusion $X \vee Y \to X \times Y$ is numerably contractible (cf. p. 3–22) and has the homotopy type of $X \times Y / X \vee Y = X \# Y$, which is therefore numerably contractible.]

FACT Suppose that $\left\{\begin{array}{l}(X,x_0)\\(Y,y_0)\end{array}\right.$ are nondegenerate. Let $f \in C(X,x_0;Y,y_0)$—then the pointed mapping cone C_f is numerably contractible provided that Y is numerably contractible.

[Consider the commutative diagram

$$\begin{tikzcd}
X \vee [0,1] \arrow[r, "f \vee \mathrm{id}"] \arrow[d] & Y \vee [0,1] \arrow[d] \\
X \arrow[r, "f"'] & Y
\end{tikzcd}$$

. By hypothesis, the vertical arrows are pointed homotopy equivalences, so $C_{f \vee \mathrm{id}}$ and C_f have the same pointed homotopy type. Look at the unpointed mapping cone of $f \vee \mathrm{id}$.]

Application: The pointed suspension of any nondegenerate space is numerably contractible.

A pointed space (X,x_0) is said to satisfy Puppe's condition provided that there exists a halo U of $\{x_0\}$ in X and a homotopy $\Phi : IU \to X$ of the inclusion $U \to X \operatorname{rel} \{x_0\}$ such that $\Phi \circ i_1(U) = \{x_0\}$. Every wellpointed space satisfies Puppe's condition.

LEMMA Let (X,A,x_0) be a pointed pair. Suppose that there exists a pointed homotopy $H : IX \to X$ of id_X such that $H \circ i_1(A) = \{x_0\}$ and $H \circ i_t(A) \subset A$ $(0 \leq t \leq 1)$—then the projection $X \to X/A$ is a pointed homotopy equivalence.

PROPOSITION 20 Let (X,x_0) be a pointed space—then (X,x_0) is nondegenerate iff it satisfies Puppe's condition.

[Necessity: Let $\rho : X \to \check{X}$ be a pointed homotopy inverse for r. Fix a homotopy $H : IX \to X$ of $\mathrm{id}_X \operatorname{rel} \{x_0\}$ such that $H \circ i_1 = r \circ \rho$. Put $U = \rho^{-1}(\{x_0\} \times]0,1])$—then U is a halo of $\{x_0\}$ in X with haloing function π the composite $X \xrightarrow{\rho} \check{X} \to \check{X}/X = [0,1]$. Consider $\Phi = H | IU$.

Sufficiency: One can assume that U is closed (cf. p. 3–11). Set

$$\Phi'(x,t) = \begin{cases} \Phi(x,2t) \quad (\in X \subset \check{X}) & (0 \leq t \leq 1/2) \\ 2t-1 \quad (\in [0,1] \subset \check{X}) & (1/2 \leq t \leq 1) \end{cases} \quad (x \in U).$$

Define a pointed homotopy $H : I\check{X} \to \check{X}$ by

$$(H \circ i_t | X)(x) = \begin{cases} x & (x \notin U) \\ \Phi'(x, t\pi(x)) & (x \in U) \end{cases}$$

and

$$(H \circ i_t | [0,1])(T) = \begin{cases} T & (0 \leq t \leq 1/2) \\ 1 - (1-T)(2-2t) & (1/2 \leq t \leq 1) \end{cases} .$$

The lemma implies that $r : \check{X} \rightarrow \check{X}/[0,1] = X$ is a pointed homotopy equivalence.]

EXAMPLE Take $X = [0,1]^\kappa (\kappa > \omega)$ and let $x_0 = 0_\kappa$, the "origin" in X—then (X, x_0) is not wellpointed (cf. p. 3–8) but is nondegenerate.

FACT A pointed space (X, x_0) is nondegenerate iff it has the same pointed homotopy type as $(\check{X}, \check{x}_0)$.

Application: Nondegeneracy is a pointed homotopy type invariant.

[Note: Compare this with the remark on p. 3–17.]

FACT Suppose that $\begin{cases} (X, x_0) \\ (Y, y_0) \end{cases}$ are nondegenerate. Let $f \in C(X, x_0; Y, y_0)$—then f is a homotopy equivalence in **TOP** iff f is a homotopy equivalence in **TOP**$_*$.

EXAMPLE (The Moore Loop Space) Suppose that the pointed space X is nondegenerate—then ΩX and $\Omega_M X$ are nondegenerate. Since the retraction of $\Omega_M X$ onto ΩX is not only a homotopy equivalence in **TOP** but a pointed map as well, it follows that ΩX and $\Omega_M X$ have the same pointed homotopy type.

PROPOSITION 21 Let (X, x_0) be a pointed space—then (X, x_0) is wellpointed and $\{x_0\}$ is closed in X iff (X, x_0) is nondegenerate and $\{x_0\}$ is a zero set in X.

[This is a consequence of Propositions 10 and 20.]

As noted above, nondegeneracy is a pointed homotopy type invariant. It is also a relatively stable property: X nondegenerate $\Rightarrow \Gamma X, \Sigma X, \Theta X, \Omega X$ nondegenerate and X, Y nondegenerate $\Rightarrow X \times Y, X \vee Y, X \# Y$ nondegenerate.

To illustrate, consider $X \# Y$. In **HTOP**$_*$, $X \# Y \approx \check{X} \# \check{Y}$, and since $\begin{cases} \{\check{x}_0\} \rightarrow \check{X} \\ \{\check{y}_0\} \rightarrow \check{Y} \end{cases}$ are closed cofibrations, $\check{X} \# \check{Y}$ is wellpointed (cf. p. 3–28), hence a fortiori, nondegenerate. Thus the same is true of $X \# Y$.

Given pointed spaces $(X_1, x_1), \ldots, (X_n, x_n)$, write $X_1 \Delta \cdots \Delta X_n$ for the subspace

$$(\{x_1\} \times X_2 \times \cdots \times X_n) \cup \cdots \cup (X_1 \times \cdots \times X_{n-1} \times \{x_n\})$$

of $X_1 \times \cdots \times X_n$ and let $X_1 \# \cdots \# X_n$ be the quotient $X_1 \times \cdots \times X_n / X_1 \Delta \cdots \Delta X_n$.

PROPOSITION 22 Let X, Y, Z be nondegenerate—then $(X\#Y)\#Z$ and $X\#(Y\#Z)$ have the same pointed homotopy type.

[There is a pointed 2-source $(X\#Y)\#Z \leftarrow X\#Y\#Z \rightarrow X\#(Y\#Z)$ arising from the identity. Both arrows are continuous bijections and it will be enough to show that they are pointed homotopy equivalences. For this purpose, consider instead the pointed 2-source $(\check{X}\#\check{Y})\#\check{Z} \leftarrow \check{X}\#\check{Y}\#\check{Z} \rightarrow \check{X}\#(\check{Y}\#\check{Z})$ and, to be specific, work on the left, calling the arrow ϕ. Define pointed continuous functions $\begin{cases} u : \check{X} \rightarrow \check{X} \\ v : \check{Y} \rightarrow \check{Y} \end{cases}$ by $\begin{cases} (u|X)(x) = x \\ (v|Y)(y) = y \end{cases}$ & $\begin{array}{l} (u|[0,1])(t) = \max\{0, 2t-1\} \\ (v|[0,1])(t) = \max\{0, 2t-1\} \end{array}$—then $u \times v \times \text{id}_Z$ induces a pointed function $\psi : (\check{X}\#\check{Y})\#\check{Z} \rightarrow \check{X}\#\check{Y}\#\check{Z}$. To check that ψ is continuous, introduce closed subspaces $\begin{cases} A \\ B \end{cases}$ of $\check{X}\#\check{Y}$: Points of A are represented by pairs (x, y), where $x \geq 1/2$ $(y \in \check{Y})$ or $y \geq 1/2$ $(x \in \check{X})$, and points of B are represented by pairs (x, y), where $\begin{cases} x \in X \\ y \in Y \end{cases}$ or $\begin{cases} x \leq 1/2 \ (y \in Y) \\ y \leq 1/2 \ (x \in X) \end{cases}$ or $x \leq 1/2$ & $y \leq 1/2$. Since the projection $(\check{X}\#\check{Y}) \times \check{Z} \rightarrow (\check{X}\#\check{Y})\#\check{Z}$ is closed, the images $\begin{cases} A_Z \\ B_Z \end{cases}$ of $\begin{cases} A \times \check{Z} \\ B \times \check{Z} \end{cases}$ in $(\check{X}\#\check{Y})\#\check{Z}$ are closed and their union fills out $(\check{X}\#\check{Y})\#\check{Z}$. The continuity of ψ is a consequence of the continuity of $\psi|A_Z$ and $\psi|B_Z$ (B_Z is homeomorphic to $B \times \check{Z}/B \times \{\check{z}_0\}$ and $B \times \check{Z}$ is closed in both $(\check{X}\#\check{Y}) \times \check{Z}$ and $\check{X} \times \check{Y} \times \check{Z}$). To see that $\begin{cases} \phi \\ \psi \end{cases}$ are mutually inverse pointed homotopy equivalences, define pointed homotopies $\begin{cases} H : I\check{X} \rightarrow \check{X} \\ G : I\check{Y} \rightarrow \check{Y} \end{cases}$ by $\begin{cases} (H \circ i_t|X)(x) = x \\ (G \circ i_t|Y)(y) = y \end{cases}$ & $\begin{cases} (H \circ i_t|[0,1])(T) \\ (G \circ i_t|[0,1])(T) \end{cases} = \max\left\{0, \dfrac{2T-t}{2-t}\right\}$. H and G combine with id_Z to define a pointed homotopy on $\check{X} \times \check{Y} \times \check{Z}$ which (i) induces a pointed homotopy on $\check{X}\#\check{Y}\#\check{Z}$ between the identity and $\psi \circ \phi$ and (ii) induces a pointed homotopy on $(\check{X}\#\check{Y})\#\check{Z}$ between the identity and $\phi \circ \psi$.]

Application: If X and Y are nondegenerate, then in **HTOP$_*$**, $\Sigma(X\#Y) \approx \Sigma X\#Y \approx X\#\Sigma Y$.

[Note: Nondegeneracy is not actually necessary for the truth of this conclusion (cf. p. 3–33).]

Within the class of nondegenerate spaces, associativity of the smash product is natural, i.e., if f :

$X \to X'$, $g : Y \to Y'$, $h : Z \to Z'$ are pointed continuous functions, then the diagram

$$\begin{tikzcd}
(X\#Y)\#Z \arrow[r] \arrow[d, "(f\#g)\#h"'] & X\#(Y\#Z) \arrow[d, "f\#(g\#h)"] \\
(X'\#Y')\#Z' \arrow[r] & X'\#(Y'\#Z')
\end{tikzcd}$$

commutes in **HTOP**$_*$.

[Note: The horizontal arrows are the pointed homotopy equivalences figuring in the proof of Proposition 22.]

PROPOSITION 23 Suppose that X and Y are nondegenerate—then the projection $X\overline{\#}Y \to X\#Y$ is a pointed homotopy equivalence.

[Consider the commutative diagram

$$\begin{tikzcd}
\check{X}\overline{\#}\check{Y} \arrow[r] \arrow[d] & \check{X}\#\check{Y} \arrow[d] \\
X\overline{\#}Y \arrow[r] & X\#Y
\end{tikzcd}$$

. The upper horizontal arrow and the two vertical arrows are pointed homotopy equivalences, thus so is the lower horizontal arrow.]

Given pointed spaces $\begin{cases} X \\ Y \end{cases}$, the pointed mapping cone sequence associated with the inclusion $f : X \vee Y \to X \times Y$ reads: $X \vee Y \xrightarrow{f} X \times Y \to X\overline{\#}Y \to \Sigma(X \vee Y) \to \Sigma(X \times Y) \to \cdots$.

LEMMA The arrow $F : X\overline{\#}Y \to \Sigma(X \vee Y)$ is nullhomotopic.

[There is a pointed injection $X\overline{\#}Y \to \Gamma(X \times Y)$. It is continuous (but not necessarily an embedding). Write $\Sigma(X \vee Y) = \Sigma X \vee \Sigma Y$ to realize $F : \begin{cases} F[x, y_0, t] = [x, t] \in \Sigma X \\ F[x_0, y, t] = [y, t] \in \Sigma Y \end{cases}$ & $F[x, y, 1] = *$, the base point. Put $\begin{cases} \overline{\Sigma}X = \Sigma X/\{[x,t] : x \in X, t \leq 1/2\} \\ \underline{\Sigma}Y = \Sigma Y/\{[y,t] : y \in Y,\ t \geq 1/2\} \end{cases}$—then the arrows $\begin{cases} \Sigma X \to \overline{\Sigma}X \\ \Sigma Y \to \underline{\Sigma}Y \end{cases}$ are pointed homotopy equivalences, hence the same holds for their wedge: $\Sigma X \vee \Sigma Y \to \overline{\Sigma}X \vee \underline{\Sigma}Y$. The assignment $[x, y, t] \to \begin{cases} [x, t]\ (t \geq 1/2) \\ [y, t]\ (t \leq 1/2) \end{cases}$ defines a pointed continuous function $\Gamma(X \times Y) \to \overline{\Sigma}X \vee \underline{\Sigma}Y$. The composite $X\overline{\#}Y \to \Gamma(X \times Y) \to \overline{\Sigma}X \vee \underline{\Sigma}Y$ is equal to the composite $X\overline{\#}Y \xrightarrow{F} \Sigma X \vee \Sigma Y \to \overline{\Sigma}X \vee \underline{\Sigma}Y$. But the first composite is nullhomotopic. Therefore the second composite is nullhomotopic and this implies that $F \simeq 0$.]

PUPPE FORMULA Suppose that X and Y are nondegenerate—then in **HTOP**$_*$, $\Sigma(X \times Y) \approx \Sigma X \vee \Sigma Y \vee \Sigma(X\#Y)$.

[The third term of the pointed mapping cone sequence of $0 : X\overline{\#}Y \to \Sigma(X \vee Y)$ is $\Sigma(X \vee Y) \vee \Sigma(X\overline{\#}Y)$, so from the lemma, $C_F \approx \Sigma(X \vee Y) \vee \Sigma(X\overline{\#}Y)$. Using now the notation of p. 3–32, there is a commutative triangle

$$\begin{tikzcd} X\overline{\#}Y \arrow[r, "j'"] \arrow[dr, "F"'] & C_j \arrow[d] \\ & \Sigma(X \vee Y) \end{tikzcd}$$

in which the vertical arrow is a pointed homotopy equivalence, thus $C_{j'} \approx C_F$ or still, $\Sigma(X \times Y) \approx \Sigma(X \vee Y) \vee \Sigma(X\overline{\#}Y) \approx \Sigma X \vee \Sigma Y \vee \Sigma(X\#Y)$ (cf. Proposition 23).]

Thanks to Proposition 22, this result can be iterated. Let $X_1, \ldots, X_n$ be nondegenerate—then $\Sigma(X_1 \times \cdots \times X_n)$ has the same pointed homotopy type as $\bigvee_N \Sigma(\#_{i \in N} X_i)$, where N runs over the nonempty subsets of $\{1, \ldots, n\}$. Example: $\Sigma(\mathbf{S}^{k_1} \times \cdots \times \mathbf{S}^{k_n}) \approx \bigvee_N \mathbf{S}^N, \mathbf{S}^N$ a sphere of dimension $1 + \sum_{i \in N} k_i$.

EXAMPLE (Whitehead Products) Let $\begin{cases} X \\ Y \end{cases}$ be nondegenerate—then for any pointed space E, there is a short exact sequence of groups

$$0 \to [\Sigma(X\#Y), E] \to [\Sigma(X \times Y), E] \to [\Sigma(X \vee Y), E] \to 0.$$

Here, composition is written additively even though the groups involved may not be abelian. This data generates a pairing $[\Sigma X, E] \times [\Sigma Y, E] \to [\Sigma(X\#Y), E]$. Take $\begin{cases} \alpha \in [\Sigma X, E] \\ \beta \in [\Sigma Y, E] \end{cases}$ and use the embeddings $\begin{cases} [\Sigma X, E] \\ [\Sigma Y, E] \end{cases} \to [\Sigma(X \times Y), E]$ to form the commutator $\alpha + \beta - \alpha - \beta$ in $[\Sigma(X \times Y), E]$. Because it lies in the kernel of the homomorphism $[\Sigma(X \times Y), E] \to [\Sigma(X \vee Y), E]$, by exactness there exists a unique element $[\alpha, \beta] \in [\Sigma(X\#Y), E]$ with image $\alpha + \beta - \alpha - \beta$. $[\alpha, \beta]$ is called the Whitehead product of α, β. $[\alpha, \beta]$ and $[\beta, \alpha]$ are connected by the relation $[\alpha, \beta] + [\beta, \alpha] \circ \Sigma \mathrm{T} = 0$, where $\mathrm{T} : X\#Y \to Y\#X$ is the interchange. Of course, $[\alpha, 0] = [0, \beta] = 0$. In general, $[\alpha, \beta] = 0$ if E is an H space (since then $[\Sigma(X \times Y), E]$ is abelian), hence, always $\Sigma[\alpha, \beta] = 0$ (look at the arrow $E \to \Omega\Sigma E$). There are left actions

$$\begin{cases} [\Sigma X, E] \times [\Sigma(X\#Y), E] \to [\Sigma(X\#Y), E] \\ [\Sigma Y, E] \times [\Sigma(X\#Y), E] \to [\Sigma(X\#Y), E] \end{cases} : \begin{cases} (\alpha, \xi) \to \alpha \cdot \xi = \alpha + \xi - \alpha \\ (\beta, \xi) \to \beta \cdot \xi = \beta + \xi - \beta \end{cases} \text{(abuse of notation).}$$

One has $\begin{cases} [\alpha + \alpha', \beta] = \alpha \cdot [\alpha', \beta] + [\alpha, \beta] \\ [\alpha, \beta + \beta'] = [\alpha, \beta] + \beta \cdot [\alpha, \beta'] \end{cases}$. These relations simplify if the cogroup objects $\begin{cases} \Sigma X \\ \Sigma Y \end{cases}$ are commutative (as would be the case, e.g., when $\begin{cases} X = \Sigma X' \\ Y = \Sigma Y' \end{cases}$ for nondegenerate $\begin{cases} X' \\ Y' \end{cases}$). Indeed, under this assumption, $[\Sigma(X\#Y), E]$ is abelian. Therefore the $\begin{cases} \alpha \cdot [\alpha', \beta] - [\alpha', \beta] \\ \beta \cdot [\alpha, \beta'] - [\alpha, \beta'] \end{cases}$ must vanish ("being commutators"), implying that $\begin{cases} [\alpha + \alpha', \beta] = [\alpha, \beta] + [\alpha', \beta] \\ [\alpha, \beta + \beta'] = [\alpha, \beta] + [\alpha, \beta'] \end{cases}$. The Whitehead product also satisfies a form of the

Jacobi identity. Precisely: Suppose given nondegenerate X, Y, Z whose associated cogroup objects ΣX, ΣY, ΣZ are commutative—then

$$[[\alpha,\beta],\gamma] + [[\beta,\gamma],\alpha] \circ \Sigma\sigma + [[\gamma,\alpha],\beta] \circ \Sigma\tau = 0$$

in the group $[\Sigma(X\#Y\#Z), E]$, where $\begin{cases} \sigma : X\#Y\#Z \to Y\#Z\#X \\ \tau : X\#Y\#Z \to Z\#X\#Y \end{cases}$ (cf. Proposition 22). The verification is a matter of manipulating commutator identities.]

A graded Lie algebra over a commutative ring R with unit is a graded R-module $L = \bigoplus_{n\geq 0} L_n$ together with bilinear pairings $[\ ,\ ] : L_n \times L_m \to L_{n+m}$ such that $[x, y] = (-1)^{|x||y|+1}[y, x]$ and

$$(-1)^{|x||z|}[[x,y],z] + (-1)^{|y||x|}[[y,z],x] + (-1)^{|z||y|}[[z,x],y] = 0.$$

L is said to be connected if $L_0 = 0$. Example: Let $A = \bigoplus_{n\geq 0} A_n$ be a graded R-algebra. For $x \in A_n$, $y \in A_m$, put $[x, y] = xy - (-1)^{|x||y|}yx$—then with this definition of the bracket, A is a graded Lie algebra over R.

[Note: As usual, an absolute value sign stands for the degree of a homogenous element in a graded R-module.]

EXAMPLE Let X be a path connected topological space. Given $\begin{cases} \alpha \in \pi_n(X) \\ \beta \in \pi_m(X) \end{cases}$, the Whitehead product $[\alpha, \beta] \in \pi_{n+m-1}(X)$. One has $[\alpha, \beta] = (-1)^{nm+n+m}[\beta, \alpha]$. Moreover, if $\gamma \in \pi_r(X)$, then

$$(-1)^{nr+m}[[\alpha,\beta],\gamma] + (-1)^{mn+r}[[\beta,\gamma],\alpha] + (-1)^{rm+n}[[\gamma,\alpha],\beta] = 0.$$

Assume now that X is simply connected. Consider the graded $\mathbf{Z}$-module $\pi_*(\Omega X) = \bigoplus_{n\geq 0} \pi_n(\Omega X)$. Since $\pi_{n+1}(X) = \pi_n(\Omega X)$, the Whitehead product determines a bilinear pairing $[\ ,\ ] : \pi_n(\Omega X) \times \pi_m(\Omega X) \to \pi_{n+m}(\Omega X)$ with respect to which $\pi_*(\Omega X)$ acquires the structure of a connected graded Lie algebra over $\mathbf{Z}$.

FACT Suppose that X is simply connected—then the Hurewicz homomorphism $\pi_*(\Omega X) \to H_*(\Omega X)$ is a morphism of graded Lie algebras, i.e., preserves the brackets.

[Note: Recall that $H_*(\Omega X)$ is a graded $\mathbf{Z}$-algebra (Pontryagin product), hence can be regarded as a graded Lie algebra over $\mathbf{Z}$.]

A pair (X, A) is said to be n-connected ($n \geq 1$) if each path component of X meets A and $\pi_q(X, A, x_0) = 0$ $(1 \leq q \leq n)$ for all $x_0 \in A$ or, equivalently, if every map $(\mathbf{D}^q, \mathbf{S}^{q-1}) \to (X, A)$ is homotopic $\text{rel}\,\mathbf{S}^{q-1}$ to a map $\mathbf{D}^q \to A$ $(0 \leq q \leq n)$. If A is path connected, then $\forall\ x_0', x_0'' \in A$, $\pi_n(X, A, x_0') \approx \pi_n(X, A, x_0'')$ $(n \geq 1)$. Examples: (1) $(\mathbf{D}^{n+1}, \mathbf{S}^n)$ is n-connected; (2) $(\mathbf{B}^{n+1}, \mathbf{B}^{n+1} - \{0\})$ is n-connected.

[Note: Take $A = \{x_0\}$—then $\pi_q(X, \{x_0\}, x_0) = \pi_q(X, x_0)$, so X is n-connected $(n \geq 1)$ provided that X is path connected and $\pi_q(X) = 0$ $(1 \leq q \leq n)$. Example: $\mathbf{S}^{n+1}$ is n-connected.]

EXAMPLE If X is n-connected and Y is m-connected, then $X * Y$ is $((n+1)+(m+1))$-connected.
[Note: If X is path connected and Y is nonempty but arbitrary, then $X * Y$ is 1-connected.]

EXAMPLE Suppose that $\left\{\begin{matrix} X \\ Y \end{matrix}\right.$ are nondegenerate and X is n-connected and Y is m-connected—then $X \# Y$ is $(n + m + 1)$-connected.

FACT Let $f : \mathbf{S}^n \rightarrow A$ be a continuous function. Put $X = \mathbf{D}^{n+1} \sqcup_f A$—then (X, A) is n-connected.

EXAMPLE The pair $(\mathbf{S}^n \times \mathbf{S}^m, \mathbf{S}^n \vee \mathbf{S}^m)$ is $n + m - 1$ connected.

HOMOTOPY EXCISION THEOREM Suppose that $\left\{\begin{matrix} X_1 \\ X_2 \end{matrix}\right.$ are subspaces of X with $X = \operatorname{int} X_1 \cup \operatorname{int} X_2$. Assume: $\left\{\begin{matrix} (X_1, X_1 \cap X_2) \\ (X_2, X_2 \cap X_1) \end{matrix}\right.$ is $\left\{\begin{matrix} n\text{-connected} \\ m\text{-connected} \end{matrix}\right.$ —then the arrow $\pi_q(X_1, X_1 \cap X_2) \rightarrow \pi_q(X_1 \cup X_2, X_2)$ induced by the inclusion $(X_1, X_1 \cap X_2) \rightarrow (X_1 \cup X_2, X_2)$ is bijective for $1 \leq q < n + m$ and surjective for $q = n + m$.

[This is dealt with at the end of the §.]

LEMMA Let X be a strong deformation retract of Y and let $A \subset X$ be a strong deformation retract of $B \subset Y$—then $\forall\ n \geq 1$, $\pi_n(X, A) \approx \pi_n(Y, B)$.

[Use the exact sequence for a pair and the five lemma.]

PROPOSITION 24 Let $\left\{\begin{matrix} A \\ B \end{matrix}\right.$ be closed subspaces of X with $X = A \cup B$. Put $C = A \cap B$. Assume: The inclusions $\left\{\begin{matrix} C \rightarrow A \\ C \rightarrow B \end{matrix}\right.$ are cofibrations and $\left\{\begin{matrix} (A, C) \\ (B, C) \end{matrix}\right.$ is $\left\{\begin{matrix} n\text{-connected} \\ m\text{-connected} \end{matrix}\right.$ —then the arrow $\pi_q(A, C) \rightarrow \pi_q(X, B)$ is bijective for $1 \leq q < n + m$ and surjective for $q = n + m$.

[Set $\overline{X} = i_0 A \cup IC \cup i_1 B$, $\left\{\begin{matrix} \overline{X}_1 = i_0 A \cup IC \\ \overline{X}_2 = IC \cup i_1 B \end{matrix}\right.$: $\overline{X}_1 \cap \overline{X}_2 = IC$ and $\left\{\begin{matrix} \operatorname{int} \overline{X}_1 \supset \overline{X} - i_1 B \\ \operatorname{int} \overline{X}_2 \supset \overline{X} - i_0 A \end{matrix}\right. \Rightarrow$ $\overline{X} = \operatorname{int} \overline{X}_1 \cup \operatorname{int} \overline{X}_2$. From the lemma, $\left\{\begin{matrix} \pi_q(A, C) \approx \pi_q(\overline{X}_1, IC) \\ \pi_q(B, C) \approx \pi_q(\overline{X}_2, IC) \end{matrix}\right. \Rightarrow \left\{\begin{matrix} (\overline{X}_1, IC) \\ (\overline{X}_2, IC) \end{matrix}\right.$ is $\left\{\begin{matrix} n\text{-connected} \\ m\text{-connected} \end{matrix}\right.$, thus the homotopy excision theorem is applicable to the triple $(\overline{X}, \overline{X}_1, \overline{X}_2)$. Because the inclusions $\left\{\begin{matrix} C \rightarrow A \\ C \rightarrow B \end{matrix}\right.$ are cofibrations, $i_0 A \cup IC$ is a strong deformation retract of IA and $IC \cup i_1 B$ is a strong deformation retract of IB (cf. p. 3–6). Therefore $\overline{X}$ is a strong deformation retract of $IA \cup IB = IX$, so $\pi_q(\overline{X}, \overline{X}_2) \approx \pi_q(IX, IB) \approx \pi_q(X, B)$.]

LEMMA Let $f : (X, A) \rightarrow (Y, B)$ be a homotopy equivalence in $\mathbf{TOP}^2$—then $\forall\, x_0 \in A$ and any $q \geq 1$, the induced map $f_* : \pi_q(X, A, x_0) \rightarrow \pi_q(Y, B, f(x_0))$ is bijective.

PROPOSITION 25 Let A be a nonempty closed subspace of X. Assume: The inclusion $A \rightarrow X$ is a cofibration and A is n-connected, (X, A) is m-connected—then the arrow $\pi_q(X, A) \rightarrow \pi_q(X/A, *)$ is bijective for $1 \leq q \leq n + m$ and surjective for $q = n + m + 1$.

[Denote by C_i the unpointed mapping cone of the inclusion $i : A \rightarrow X$. There are closed cofibrations $\begin{cases} \Gamma A \rightarrow C_i \\ X \rightarrow C_i \end{cases}$ and $C_i = \Gamma A \cup X$, with $\Gamma A \cap X = A$. Since the pair $(\Gamma A, A)$ is $(n+1)$-connected, it follows from Proposition 24 that the arrow $\pi_q(X, A) \rightarrow \pi_q(C_i, \Gamma A)$ is bijective for $1 \leq q \leq n+m$ and surjective for $q = n+m+1$. But ΓA is contractible, hence the projection $(C_i, \Gamma A) \rightarrow (C_i/\Gamma A, *)$ is a homotopy equivalence in $\mathbf{TOP}^2$ (cf. Proposition 14). Taking into account the lemma, it remains only to observe that X/A can be identified with $C_i/\Gamma A$.]

FREUDENTHAL SUSPENSION THEOREM Suppose that X is nondegenerate and n-connnected—then the suspension homomorphism $\pi_q(X) \rightarrow \pi_{q+1}(\Sigma X)$ is bijective for $0 \leq q \leq 2n$ and surjective for $q = 2n + 1$.

[Take X wellpointed with a closed base point and, for the moment, work with its unpointed suspension ΣX. Using the notation of p. 3–22, write $\Sigma X = \Gamma^- X \cup \Gamma^+ X$—then $\forall\, q, \pi_q(X) \approx \pi_q(\Gamma^- X \cap \Gamma^+ X) \approx \pi_{q+1}(\Gamma^- X, \Gamma^- X \cap \Gamma^+ X)$. On the other hand, Proposition 25 implies that the arrow $\pi_{q+1}(\Gamma^- X, \Gamma^- X \cap \Gamma^+ X) \rightarrow \pi_{q+1}(\Sigma X)$ is a bijection for $1 \leq q + 1 \leq 2n + 1$ and a surjection for $q + 1 = 2n + 2$. Moreover, X is wellpointed, therefore its pointed and unpointed suspensions have the same homotopy type.]

[Note: This result is true if X is merely path connected, i.e., $n = 0$ is admissible (inspect the proof of Proposition 25).]

Application: Suppose that $n \geq 1$—then (i) $\pi_q(\mathbf{S}^n) = 0$ $(0 \leq q < n)$; (ii) $\pi_q(\mathbf{S}^n) \approx \pi_{q+1}(\mathbf{S}^{n+1})$ $(0 \leq q \leq 2n - 2)$; (iii) $\pi_n(\mathbf{S}^n) \approx \mathbf{Z}$.

[As regards the last point, note that in the sequence $\pi_1(\mathbf{S}^1) \rightarrow \pi_2(\mathbf{S}^2) \rightarrow \pi_3(\mathbf{S}^3) \rightarrow \cdots$, the first homomorphism is an epimorphism, the others are isomorphisms, and $\pi_1(\mathbf{S}^1) \approx \mathbf{Z}$, $\pi_2(\mathbf{S}^2) \approx \mathbf{Z}$ (a piece of the exact sequence associated with the Hopf map $\mathbf{S}^3 \rightarrow \mathbf{S}^2$ is $\pi_2(\mathbf{S}^3) \rightarrow \pi_2(\mathbf{S}^2) \rightarrow \pi_1(\mathbf{S}^1) \rightarrow \pi_1(\mathbf{S}^3)$).]

The infinite cyclic group $\pi_n(\mathrm{S}^n)$ is generated by $[\iota_n]$, ι_n the identity $\mathrm{S}^n \to \mathrm{S}^n$. Form the Whitehead product $[\iota_n, \iota_n] \in \pi_{2n-1}(\mathrm{S}^n)$—then the kernel of the suspension homomorphism $\pi_{2n-1}(\mathrm{S}^n) \to \pi_{2n}(\mathrm{S}^{n+1})$ is generated by $[\iota_n, \iota_n]$ (Whitehead[†]).

The proof of the homotopy excision theorem is elementary but complicated. This is the downside. The upside is that the highpowered approaches are cluttered with unnecessary assumptions, hence do not go as far.

OPEN HOMOTOPY EXCISION THEOREM Suppose that $\begin{cases} X_1 \\ X_2 \end{cases}$ are open subspaces of X with $X = X_1 \cup X_2$. Assume: $\begin{cases} (X_1, X_1 \cap X_2) \\ (X_2, X_2 \cap X_1) \end{cases}$ is $\begin{cases} n\text{-connected} \\ m\text{-connected} \end{cases}$—then the arrow $\pi_q(X_1, X_1 \cap X_2) \to \pi_q(X_1 \cup X_2, X_2)$ induced by the inclusion $(X_1, X_1 \cap X_2) \to (X_1 \cup X_2, X_2)$ is bijective for $1 \leq q < n+m$ and surjective for $q = n+m$.

[Note: Goodwillie[‡] has extended the open homotopy excision theorem to "$(N+1)$-ads".]

Admit the open homotopy excision theorem.

CW HOMOTOPY EXCISION THEOREM Suppose that $\begin{cases} K_1 \\ K_2 \end{cases}$ are subcomplexes of a CW complex K with $K = K_1 \cup K_2$. Assume: $\begin{cases} (K_1, K_1 \cap K_2) \\ (K_2, K_2 \cap K_1) \end{cases}$ is $\begin{cases} n\text{-connected} \\ m\text{-connected} \end{cases}$—then the arrow $\pi_q(K_1, K_1 \cap K_2) \to \pi_q(K_1 \cup K_2, K_2)$ induced by the inclusion $(K_1, K_1 \cap K_2) \to (K_1 \cup K_2, K_2)$ is bijective for $1 \leq q < n+m$ and surjective for $q = n+m$.

[Fix a neighborhood $\begin{cases} U \\ V \end{cases}$ of $K_1 \cap K_2$ in $\begin{cases} K_1 \\ K_2 \end{cases}$ such that $K_1 \cap K_2$ is a strong deformation retract of $\begin{cases} U \\ V \end{cases}$ and put $\begin{cases} K_1' = K_1 \cup V \\ K_2' = K_2 \cup U \end{cases}$. Write $\begin{cases} U = O \cap K_1 \\ V = P \cap K_2 \end{cases}$, where $\begin{cases} O \\ P \end{cases}$ are open in K—then $\begin{cases} K_1' = P \cup (K - K_2) \\ K_2' = O \cup (K - K_1) \end{cases}$, hence $\begin{cases} K_1' \\ K_2' \end{cases}$ are open in K and $K = K_1' \cup K_2'$. Since $\begin{cases} K_1 \;\&\; V \\ K_2 \;\&\; U \end{cases}$ are closed in $\begin{cases} K_1' \\ K_2' \end{cases}$, the homotopy deforming $\begin{cases} V \\ U \end{cases}$ into $K_1 \cap K_2$ can be extended to all of $\begin{cases} K_1' \\ K_2' \end{cases}$ in the obvious way, so $\begin{cases} K_1 \\ K_2 \end{cases}$ is a strong deformation retract of $\begin{cases} K_1' \\ K_2' \end{cases}$. On the other hand, $K_1' \cap K_2' = U \cup V$ and $\begin{cases} U \\ V \end{cases}$ is closed in $U \cup V$, thus the union of the deforming homotopies is continuous and $K_1 \cap K_2$ is a strong deformation retract of $K_1' \cap K_2'$. Therefore $\begin{cases} (K_1', K_1' \cap K_2') \\ (K_2', K_2' \cap K_1') \end{cases}$ is $\begin{cases} n\text{-connected} \\ m\text{-connected} \end{cases}$ and the open homotopy excision theorem is applicable to the triple (K, K_1', K_2'). Consider the commutative triangle

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[†] *Elements of Homotopy Theory*, Springer Verlag (1978), 549.

[‡] *Memoirs Amer. Math. Soc.* **431** (1990), 1–317.

$$\begin{tikzcd}
\pi_q(K_1, K_1 \cap K_2) \arrow[rr] \arrow[dr] & & \pi_q(K_1', K_1' \cap K_2') \arrow[dl] \\
& \pi_q(K_1 \cup K_2, K_2) &
\end{tikzcd}$$

.]

The CW homotopy excision theorem implies the homotopy excision theorem. For choose a CW resolution $L \to X_1 \cap X_2$. There exist: (1) A CW complex $K_1 \supset L$ and a CW resolution $f_1 : K_1 \to X_1$ such that the square

$$\begin{tikzcd}
K_1 \arrow[r] & X_1 \\
L \arrow[r] \arrow[u] & X_1 \cap X_2 \arrow[u]
\end{tikzcd}$$

commutes; (2) A CW complex $K_2 \supset L$ and a CW resolution $f_2 : K_2 \to X_2$ such that the square

$$\begin{tikzcd}
K_2 \arrow[r] & X_2 \\
L \arrow[r] \arrow[u] & X_2 \cap X_1 \arrow[u]
\end{tikzcd}$$

commutes. Note that $\begin{cases} (K_1, L) \\ (K_2, L) \end{cases}$ is $\begin{cases} n\text{-connected} \\ m\text{-connected} \end{cases}$. Define a CW complex K by the pushout square

$$\begin{tikzcd}
L \arrow[r] \arrow[d] & K_2 \arrow[d] \\
K_1 \arrow[r] & K
\end{tikzcd}$$

: $K = K_1 \cup K_2$ & $L = K_1 \cap K_2$—then there is an arrow $f : K \to X$ determined by $\begin{cases} f_1 \\ f_2 \end{cases}$, viz. $\begin{cases} f|K_1 = f_1 \\ f|K_2 = f_2 \end{cases}$.

LEMMA f is a weak homotopy equivalence.

[Set $\overline{K} = i_0 K_1 \cup IL \cup i_1 K_2$: $\begin{cases} U_1 = \overline{K} - i_1 K_2 \\ U_2 = \overline{K} - i_0 K_1 \end{cases}$—then $\begin{cases} U_1 \\ U_2 \end{cases}$ are open in $\overline{K}$ and $\overline{K} = U_1 \cup U_2$. Let $\overline{p} : \overline{K} \to K$ be the restriction of the projection $p : IK \to K$ and denote by $\overline{f}$ the composite $f \circ \overline{p}$: $\begin{cases} \overline{f}(U_1) \subset X_1 \\ \overline{f}(U_2) \subset X_2 \end{cases}$ and $\begin{cases} \overline{f}|U_1 \\ \overline{f}|U_2 \end{cases}$ & $\overline{f}|U_1 \cap U_2$ are weak homotopy equivalences. But by assumption $X = \operatorname{int} X_1 \cup \operatorname{int} X_2$. Therefore $\overline{f}$ is a weak homotopy equivalence (cf. p. 4–52). The inclusions $\begin{cases} K_1 \to K \\ K_2 \to K \end{cases}$ are closed cofibrations (cf. p. 3–13), hence $\overline{K}$ is a strong deformation retract of IK. Consequently, $\overline{p}$ is a homotopy equivalence, so f is a weak homotopy equivalence.]

The CW homotopy excision theorem is applicable to the triple (K, K_1, K_2). Examination of the commutative square

$$\begin{tikzcd}
\pi_q(K_1, K_1 \cap K_2) \arrow[r] \arrow[d] & \pi_q(K_1 \cup K_2, K_2) \arrow[d] \\
\pi_q(X_1, X_1 \cap X_2) \arrow[r] & \pi_q(X_1 \cup X_2, X_2)
\end{tikzcd}$$

thus justifies the claim. Accordingly, it is the open homotopy excision theorem which is the heart of the matter.

Given a p-dimensional cube C in $\mathbf{R}^q$ ($q \geq 1, 0 \leq p \leq q$), denote by $\mathrm{sk}_d\, C$ its d-dimensional skeleton, i.e., the set of its d-dimensional faces. Put $\dot{C} = \cup\, \mathrm{sk}_{p-1}\, C$—then the inclusion $\dot{C} \rightarrow C$ is a closed cofibration. Analytically, C is specified by a point $(c_1, \ldots, c_q) \in \mathbf{R}^q$, a positive number δ, and a subset P of $\{1, \ldots, q\}$ of cardinality p : C is the set of $x \in \mathbf{R}^q$ such that $c_i \leq x_i \leq c_i + \delta$ ($i \in P$) & $x_i = c_i$ ($i \notin P$). Here, if $P = \emptyset$, then $C = \{(c_1, \ldots, c_q)\}$. For $1 \leq d \leq q$, let
$$\begin{cases} K_d(C) = \{x \in C : x_i < c_i + \dfrac{\delta}{2} \text{ for at least } d \text{ indices } i \in P\} \\ L_d(C) = \{x \in C : x_i > c_i + \dfrac{\delta}{2} \text{ for at least } d \text{ indices } i \in P\} \end{cases}.$$
When $d > p$, it is understood that $\begin{cases} K_d(C) = \emptyset \\ L_d(C) = \emptyset \end{cases}$.

COMPRESSION LEMMA Fix a p-dimensional cube C in $\mathbf{R}^q$ ($q \geq 1, 1 \leq p \leq q$), a positive integer $d \leq p$, and a pair (X, A). Suppose that $f : C \rightarrow X$ is a continuous function such that $\forall\, D \in \mathrm{sk}_{p-1}\, C$, $f^{-1}(A) \cap D \subset K_d(D)$ ($L_d(D)$)—then there exists a continuous function $g : C \rightarrow X$ with $f \simeq g \operatorname{rel} \dot{C}$ and $g^{-1}(A) \subset K_d(C)$ ($L_d(C)$).

[Take $p = q$, $C = [0,1]^q$, and put $x_0 = (1/4, \ldots, 1/4)$. Given an $x \in [0,1]^q$, let $\ell(x_0, x)$ be the ray that starts at x_0 and passes through x. Denote by $P(x)$ the intersection of $\ell(x_0, x)$ with the frontier of $[0, 1/2]^q$, $Q(x)$ the intersection of $\ell(x_0, x)$ with the frontier of $[0,1]^q$. Let $\phi : [0,1]^q \rightarrow [0,1]^q$ be the continuous function that sends the line segment joining $P(x)$ and $Q(x)$ to the point $Q(x)$ and maps the line segment joining x_0 and $P(x)$ linearly onto the line segment joining x_0 and $Q(x)$. Note that $\phi \simeq \mathrm{id}_{[0,1]^q} \operatorname{rel} \mathrm{fr}\,[0,1]^q$. Now set $g = f \circ \phi$. Assume: $x \in g^{-1}(A)$. Case 1: $x_i < 1/2$ ($\forall\, i$) $\Rightarrow x \in K_q([0,1]^q) \subset K_d([0,1]^q)$. Case 2: $x_i \geq 1/2$ ($\exists\, i$) $\Rightarrow \phi(x) \in \mathrm{fr}\,[0,1]^q \Rightarrow \phi(x) \in D$ ($\exists\, D \in \mathrm{sk}_{q-1}\,[0,1]^q$) $\Rightarrow \phi(x) \in K_d(D) \Rightarrow 1/2 > \phi(x)_i = 1/4 + t(x_i - 1/4)$ for at least d indices $i \Rightarrow 1/2 > \phi(x)_i \geq x_i$ ($t \geq 1$) for at least d indices $i \Rightarrow x \in K_d([0,1]^q)$.]

[Note: The parenthetical assertion is analogous.]

Notation: Put $I^q = [0,1]^q$, $\dot{I}^q = \mathrm{fr}\,[0,1]^q$, $I_0^{q-1} = I^{q-1} \times \{0\}$ ($q > 1$) & $I_0^0 = \{0\}$ ($q = 1$), $J^{q-1} = \dot{I}^{q-1} \times I \cup I^{q-1} \times \{1\}$ ($q > 1$) & $J^0 = \{1\}$ ($q = 1$), so $\dot{I}^q = I_0^{q-1} \cup J^{q-1}$ and $\dot{I}_0^{q-1} = I_0^{q-1} \cap J^{q-1}$—then for any pointed pair (X, A, x_0), $\pi_q(X, A, x_0) = [I^q, \dot{I}^q, J^{q-1}; X, A, x_0]$.

[Note: A continuous function $f : (I^q, \dot{I}^q, J^{q-1}) \rightarrow (X, A, x_0)$ represents 0 in $\pi_q(X, A, x_0)$ iff there exists a continuous function $g : I^q \rightarrow A$ such that $f \simeq g \operatorname{rel} \dot{I}^q$.]

There are two steps in the proof of the open homotopy excision theorem: (1) Surjectivity in the range $1 \leq q \leq n + m$; (2) Injectivity in the range $1 \leq q < n + m$. The argument in either situation is founded on the same iterative principle.

Starting with surjectivity, let $\alpha \in \pi_q(X_1 \cup X_2, X_2, x_0)$, $x_0 \in X_1 \cap X_2$ the ambient base point. Represent α by an $f : (I^q, \dot{I}^q, J^{q-1}) \rightarrow (X_1 \cup X_2, X_2, x_0)$. It will be shown below that $\exists\, F \in \alpha$: $\mathrm{pr}(F^{-1}(X - X_1)) \cap \mathrm{pr}(F^{-1}(X - X_2)) = \emptyset$, $\mathrm{pr} : I^q \rightarrow I^{q-1}$ the projection. Granted this, choose a continuous function $\phi : I^{q-1} \rightarrow [0,1]$ which is 1 on $\mathrm{pr}(F^{-1}(X - X_1))$ and 0 on $\dot{I}^{q-1} \cup \mathrm{pr}(F^{-1}(X - X_2))$. Define

$\Phi : I^q \to I^q$ by $\Phi(x_1,\ldots,x_q) = (x_1,\ldots,x_{q-1},t+(1-t)x_q)$, where $t = \phi(x_1,\ldots,x_{q-1})$, and put $g = F\circ\Phi$— then $g : (I^q,\dot{I}^q,J^{q-1}) \to (X_1,X_1\cap X_2,x_0)$ is a continuous function whose class $\beta \in \pi_q(X_1,X_1\cap X_2,x_0)$ is sent to α under the inclusion.

There remains the task of producing F. Since $\{f^{-1}(X_1),f^{-1}(X_2)\}$ is an open covering of I^q, one can subdivide I^q into a collection $\mathcal{C}$ of q-dimensional cubes C such that either $f(C)\subset X_1$ or $f(C)\subset X_2$. Enumerate the elements in $\operatorname{sk}_d C\,(C\in\mathcal{C}, d = 0,1,\ldots,q) : \mathcal{D} = \{D\}$. In $\mathcal{D}$, distinguish two subcollections $\begin{cases}\{D_k : k = 1,\ldots,r\} : f(D_k)\subset X_2 \\ \{D_l : l = 1,\ldots,s\} : f(D_l)\subset X_1\end{cases}$ but $\begin{matrix} f(D_k)\not\subset X_1 \\ f(D_l)\not\subset X_2\end{matrix}$, arranging the indexing so that $\dim D_j \leq \dim D_{j+1}$.

(μ) There exist continuous functions $\mu_0 = f$, $\mu_k : I^q \to X$ $(k = 1,\ldots,r)$ such that $\forall\, k : \mu_k \simeq \mu_0$ (as a map of triples), $\mu_k^{-1}(X_2 - X_1\cap X_2)\cap D_j \subset K_{n+1}(D_j)$ $(j\leq k)$, and $\forall\, D\in\mathcal{D} : \begin{cases}\mu_0(D)\subset X_1\\ \mu_0(D)\subset X_2\end{cases} \Rightarrow \begin{cases}\mu_k(D)\subset X_1\\ \mu_k(D)\subset X_2\end{cases}$ or $\mu_0(D)\subset X_1\cap X_2 \Rightarrow \mu_k(D)\subset X_1\cap X_2$. This is seen via induction on k, $\mu_0 = f$ being the initial step. Assume that μ_{k-1} has been constructed.

Claim: $\exists$ a homotopy $h_k : ID_k \to X_2 \operatorname{rel} \dot{D}_k$ such that $h_k\circ i_0 = \mu_{k-1}|D_k$ and $(h_k\circ i_1)^{-1}(X_2 - X_1\cap X_2)\subset K_{n+1}(D_k)$.

[Case 1: $\dim D_k = 0$. Here, $K_{n+1}(D_k) = \emptyset$ and the point $\mu_{k-1}(D_k)\in X_2$ can be joined by a path in X_2 to some point of $X_1\cap X_2$. Case 2: $0 < \dim D_k < n+1$. Here, $K_{n+1}(D_k) = \emptyset$ and the induction hypothesis forces the containment $\mu_{k-1}(\dot{D}_k)\subset X_1\cap X_2$, hence $\mu_{k-1}|D_k$ represents an element of $\pi_{d_k}(X_2,X_1\cap X_2) = 0$ $(d_k = \dim D_k)$. Case 3: $\dim D_k \geq n+1$. Apply the compression lemma.]

Extend h_k to a homotopy $H_k : I^q\times I\to X$ of μ_{k-1} rel $\cup\{D : f(D)\subset X_1\}\cup \bigcup\limits_{j=1}^{k-1} D_j$ such that $\bigcup\limits_{j=k+1}^{r} H_k(ID_j)\subset X_2$. Complete the induction by taking $\mu_k = H_k\circ i_1$.

(ν) There exist continuous functions $\nu_0 = \mu_r$, $\nu_l : I^q\to X$ $(l = 1,\ldots,s)$ such that $\forall\, l : \nu_l \simeq \nu_0 \operatorname{rel} \cup\{D : f(D)\subset X_2\}$, $\nu_l^{-1}(X_1 - X_1\cap X_2)\cap D_j\subset L_{m+1}(D_j)$ $(j\leq l)$, and $\forall\, D\in\mathcal{D} : \begin{cases}\nu_0(D)\subset X_1\\ \nu_0(D)\subset X_2\end{cases} \Rightarrow \begin{cases}\nu_l(D)\subset X_1\\ \nu_l(D)\subset X_2\end{cases}$ or $\nu_0(D)\subset X_1\cap X_2 \Rightarrow \nu_l(D)\subset X_1\cap X_2$. As above, this is seen via induction on l, $\nu_0 = \mu_r$ being the initial step. Observe that $\cup\{D : f(D)\subset X_2\}\supset \dot{I}^q \supset J^{q-1}$.

Definition: $F = \nu_s$ $(\Rightarrow F\in\alpha)$. If $\operatorname{pr}(F^{-1}(X - X_1))\cap\operatorname{pr}(F^{-1}(X - X_2))$ were nonempty, then there would exist an $x\in I^{q-1}$ and a cube $D\subset I^{q-1} : \begin{cases} x\in K_n(D)\\ x\in L_m(D)\end{cases}$, an impossibility since $q - 1 < n + m$.

Turning to injectivity, let $f, g : (I^q,\dot{I}^q,J^{q-1})\to(X_1,X_1\cap X_2,x_0)$ be continuous functions such that $u\circ f\simeq u\circ g$ as maps of triples, $u : (X_1,X_1\cap X_2,x_0)\to(X_1\cup X_2,X_2,x_0)$ the inclusion. Fix a homotopy $h : (I^q,\dot{I}^q,J^{q-1})\times I\to(X_1\cup X_2,X_2,x_0) : \begin{cases} h\circ i_0 = u\circ f\\ h\circ i_1 = u\circ g\end{cases}$. Using the techniques employed in the proof of surjectivity, one can replace h by another homotopy H such that $\operatorname{pr}\times\operatorname{id}_I(H^{-1}(X - X_1))\cap\operatorname{pr}\times\operatorname{id}_I(H^{-1}(X - X_2)) = \emptyset$. It is this extra dimension that accounts for the restriction $q < n + m$. Choose a continuous function $\phi : I^{q-1}\times I\to[0,1]$ which is 1 on $\operatorname{pr}\times\operatorname{id}_I(H^{-1}(X - X_1))$ and 0 on $(\dot{I}^{q-1}\times I)\cup(I^{q-1}\times\dot{I})\cup\operatorname{pr}\times\operatorname{id}_I(H^{-1}(X - X_2))$. Define $\Phi : I^q\times I\to I^q\times I$ by $\Phi(x_1,\ldots,x_q,x_{q+1}) =$

$(x_1, \ldots, x_{q-1}, t+(1-t)x_q, x_{q+1})$, where $t = \phi(x_1, \ldots, x_{q-1}, x_{q+1})$—then the composite $H \circ \Phi$ is a homotopy between f and $g : H \circ \Phi(\dot{I}^q \times I) \subset X_1 \cap X_2$ & $H \circ \Phi(J^{q-1} \times I) = \{x_0\}$.

Given a pair (X, A), let $\pi_0(X, A)$ be the quotient $\pi_0(X)/\sim$, where $\sim$ means that the path components of X which meet A are identified. With this agreement, $\pi_0(X, A)$ is a pointed set. If $f : (X, A) \rightarrow (Y, B)$ is a map of pairs, then $f_* : \pi_0(X, A) \rightarrow \pi_0(Y, B)$ is a morphism of pointed sets and the sequence $* \rightarrow \pi_0(X, A) \rightarrow \pi_0(Y, B)$ is exact in $\mathbf{SET}_*$ iff $(f_*)^{-1}\operatorname{im}(\pi_0(B) \rightarrow \pi_0(Y)) = \operatorname{im}(\pi_0(A) \rightarrow \pi_0(X))$.

LEMMA Let $f : (X, A) \rightarrow (Y, B)$ be a continuous function. Fix $q \geq 0$—then $\forall\, x_0 \in A$, $f_* : \pi_q(X, A, x_0) \rightarrow \pi_q(Y, B, f(x_0))$ is injective and $f_* : \pi_{q+1}(X, A, x_0) \rightarrow \pi_{q+1}(Y, B, f(x_0))$ is surjective iff in any diagram

$$\begin{tikzcd}
(X, A) \arrow[r, "f"] & (Y, B) \\
(J^q, \dot{I}^q_0) \arrow[u, "\phi"] \arrow[r] & (I^{q+1}, I^q_0) \arrow[u, "\psi"']
\end{tikzcd}$$

, where $f \circ \phi \simeq \psi$ on J^q by $h : (J^q, \dot{I}^q_0) \times I \rightarrow (Y, B)$, there exists a $\Phi : (I^{q+1}, I^q_0) \rightarrow (X, A)$ such that $\Phi|(J^q, \dot{I}^q_0) = \phi$ and an $H : (I^{q+1}, I^q_0) \times I \rightarrow (Y, B)$ such that $H|(J^q, \dot{I}^q_0) \times I = h$ and $f \circ \Phi \simeq \psi$ on I^{q+1} by H.

[Note: When $q = 0$, replace injectivity by the statement "$* \rightarrow \pi_0(X, A) \rightarrow \pi_0(Y, B)$" is exact. Observe that $f \circ \phi = \psi$ on J^q is permissible (h = constant homotopy) and implies by specialization the direct assertion. In addition, if Φ & H exist in this case, then Φ & H exist in general. Thus the point is to show that the direct assertion entails the existence of Φ & H under the assumption that $f \circ \phi = \psi$ on J^q.]

FACT Suppose that $\begin{cases} X_1 \\ X_2 \end{cases}$ & $\begin{cases} Y_1 \\ Y_2 \end{cases}$ are open subspaces of $\begin{cases} X \\ Y \end{cases}$ with $\begin{cases} X = X_1 \cup X_2 \\ Y = Y_1 \cup Y_2 \end{cases}$. Let $f : X \rightarrow Y$ be a continuous function such that $\begin{cases} X_1 = f^{-1}(Y_1) \\ X_2 = f^{-1}(Y_2) \end{cases}$. Fix $n \geq 1$. Assume: The sequence $* \rightarrow \pi_0(X_i, X_1 \cap X_2) \rightarrow \pi_0(Y_i, Y_1 \cap Y_2)$ is exact $(i = 1, 2)$ and that $f_* : \pi_q(X_i, X_1 \cap X_2) \rightarrow \pi_q(Y_i, Y_1 \cap Y_2)$ is bijective for $1 \leq q < n$ and surjective for $q = n$ $(i = 1, 2)$—then the sequence $* \rightarrow \pi_0(X, X_i) \rightarrow \pi_0(Y, Y_i)$ is exact $(i = 1, 2)$ and $f_* : \pi_q(X, X_i) \rightarrow \pi_q(Y, Y_i)$ is bijective for $1 \leq q < n$ and surjective for $q = n$ $(i = 1, 2)$.

[Fix $i_0 \in \{1, 2\}$, $0 \leq q < n$, and maps $\phi : (J^q, \dot{I}^q_0) \rightarrow (X, X_{i_0})$, $\psi : (I^{q+1}, I^q_0) \rightarrow (Y, Y_{i_0})$ satisfying $f \circ \phi = \psi$ on J^q. In view of the lemma, it suffices to exhibit an extension $\Phi : (I^{q+1}, I^q_0) \rightarrow (X, X_{i_0})$ of ϕ and a homotopy $H : (I^{q+1}, I^q_0) \times I \rightarrow (Y, Y_{i_0})$ such that $H|(J^q, \dot{I}^q_0) \times I$ is the constant homotopy at $f \circ \phi$ and $f \circ \Phi \simeq \psi$ on I^{q+1} by H. Subdivide I^{q+1} into a collection $\mathcal{C}$ of $(q+1)$-dimensional cubes $C : \forall\, C \in \mathcal{C}, \exists\, i_C \in \{1, 2\} : \phi(C \cap J^q) \subset X_{i_C}$ and $\psi(C) \subset Y_{i_C}$ (possible, $\begin{cases} \phi^{-1}(X - X_1) \cup \psi^{-1}(Y - Y_1) \\ \phi^{-1}(X - X_2) \cup \psi^{-1}(Y - Y_2) \end{cases}$ being disjoint and closed). Regard I^{q+1} as $I^q \times I$—then $\mathcal{C}$ restricts to a subdivision of I^q and induces a partition of I into subintervals $I_k = [a_{k-1}, a_k] : 0 = a_0 < a_1 < \cdots < a_r = 1$. Break the subdivision of I^q into its skeletal constituents D. Construct Φ on $D \times I_k$ & H on $I(D \times I_k)$ via downward induction on k and for fixed k, via upward induction on $\dim D$. Arrange matters so that: (1) $\psi(D \times I_k) \subset Y_i \Rightarrow$

$\Phi(D \times I_k) \subset X_i$ & $H(I(D \times I_k)) \subset Y_i$; (2) $\psi(D \times \{a_{k-1}\}) \subset Y_1 \cap Y_2 \Rightarrow \Phi(D \times \{a_{k-1}\}) \subset X_1 \cap X_2$ & $H(I(D \times \{a_{k-1}\})) \subset Y_1 \cap Y_2$. The first condition plus the second when $k = 1$ yield $\Phi(I_0^q) \subset X_{i_0}$ & $H(I_0^q \times I) \subset Y_{i_0}$. At each stage, the induction hypothesis secures Φ on $\dot{D} \times I_k \cup D \times \{a_k\}$ & H on $I(\dot{D} \times I_k \cup D \times \{a_k\})$. Case 1: If either $\psi(D \times \{a_{k-1}\})$ is not contained in $Y_1 \cap Y_2$ or $\psi(D \times I_k)$ is contained in $Y_1 \cap Y_2$, use the fact that $\dot{D} \times I_k \cup D \times \{a_k\}$ is a strong deformation retract of $D \times I_k$ to specify Φ on $D \times I_k$ & H on $I(D \times I_k)$. Case 2: If $\psi(D \times \{a_{k-1}\})$ is contained in $Y_1 \cap Y_2$ and $\psi(D \times I_k)$ is contained in just one of the Y_i, realize $\Phi : (\dot{D} \times I_k \cup D \times \{a_k\}, \dot{D} \times \{a_{k-1}\}) \rightarrow (X_i, X_1 \cap X_2)$ & $H : (\dot{D} \times I_k \cup D \times \{a_k\}, \dot{D} \times \{a_{k-1}\}) \times I \rightarrow (Y_i, Y_1 \cap Y_2)$. Apply the lemma to produce the required extension of Φ to $D \times I_k$ & H to $I(D \times I_k)$. Here, of course, the assumption on f comes in.]

§ 3

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§4. FIBRATIONS

The technology developed below, like that in the preceding §, underlies the foundations of homotopy theory in **TOP** or $\mathbf{TOP}_*$.

Let B be a topological space. An object in $\mathbf{TOP}/B$ is a topological space X together with a continuous function $p : X \to B$ called the projection. For $O \subset B$, put $X_O = p^{-1}(O)$, which is therefore an object in $\mathbf{TOP}/O$ (with projection $p_O = p|X_O$). The notation $X|O$ is also used. In particular: $X_b = p^{-1}(b)$ is the fiber over $b \in B$. A morphism in $\mathbf{TOP}/B$ is a continuous function $f : X \to Y$ over B, i.e., an $f \in C(X, Y)$ such that the triangle

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "p"'] & & Y \arrow[dl, "q"] \\ & B & \end{tikzcd}$$

commutes. Notation: $f \in C_B(X, Y), f_O = f|X_O$ $(O \subset B)$. The base space B is an object in $\mathbf{TOP}/B$, where $p = \mathrm{id}_B$. An element $s \in C_B(B, X)$ is called a section of X, written $s \in \mathrm{sec}_B(X)$.

[Note: The product of $\begin{cases} p : X \to B \\ q : Y \to B \end{cases}$ in $\mathbf{TOP}/B$ is the fiber product: $X \times_B Y$. If B' is a topological space and if $\Phi' \in C(B', B)$, then Φ' determines a functor $\mathbf{TOP}/B \to \mathbf{TOP}/B'$ that sends X to $X' = B' \times_B X$. Obviously, $(X \times_B Y)' = X' \times_{B'} Y'$.]

EXAMPLE Let X be in $\mathbf{TOP}/B$—then the assignment $O \to \mathrm{sec}_O(X_O)$, O open in B, defines a sheaf of sets on B, the sheaf of sections Γ_X of X.

[Note: Recall that for any sheaf of sets $\mathcal{F}$ on B, there exists an X in $\mathbf{TOP}/B$ with $p : X \to B$ a local homeomorphism such that $\mathcal{F}$ is isomorphic to Γ_X. In fact, the category of sheaves of sets on B is equivalent to the full subcategory of $\mathbf{TOP}/B$ whose objects are those X for which $p : X \to B$ is a local homeomorphism.]

FACT Let X be in $\mathbf{TOP}/B$—then the projection $p : X \to B$ is a local homeomorphism iff both it and the diagonal embedding $X \to X \times_B X$ are open maps.

FACT Let X be in $\mathbf{TOP}/B$. Assume: X & B are path connected Hausdorff spaces and the projection $p : X \to B$ is a local homeomorphism—then p is a homeomorphism iff p is proper and $p_* : \pi_1(X) \to \pi_1(B)$ is surjective.

There is a functor $\mathbf{TOP} \to \mathbf{TOP}/B$ that sends a topological space T to $B \times T$ (product topology) with projection $B \times T \to B$. An X in $\mathbf{TOP}/B$ is said to be trivial if there exists a T in **TOP** such that X is homeomorphic over B to $B \times T$, locally trivial if there exists an open covering $\{O\}$ of B such that $\forall$ O, X_O is trivial over O.

[Note: Spelled out, local triviality means that $\forall\ O$ there exists a topological space T_O and a homeomorphism $X_O \rightarrow O \times T_O$ over O. If the T_O can be chosen independent of O, so $\forall\ O$, $T_O = T$, then X is said to be locally trivial with fiber T. When B is connected, this can always be arranged.]

FACT Let X be in $\mathbf{TOP}/IB$. Suppose that $X|(B \times [0,1/2])$ and $X|(B \times [1/2,1])$ are trivial—then X is trivial.

EXAMPLE Let X be in $\mathbf{TOP}/[0,1]^n$ $(n \geq 1)$. Suppose that X is locally trivial—then X is trivial.

A fiber homotopy is a homotopy over $B : f \underset{B}{\simeq} g$ $(f, g \in C_B(X,Y))$. Isomorphisms in the associated homotopy category are the fiber homotopy equivalences and any two $\begin{cases} X \\ Y \end{cases}$ in $\mathbf{TOP}/B$ for which there exists a fiber homotopy equivalence $X \rightarrow Y$ have the same fiber homotopy type. The fiber homotopy type of $X \times_B Y$ depends only on the fiber homotopy types of X and Y. The objects in $\mathbf{TOP}/B$ that have the fiber homotopy type of B itself are said to be fiberwise contractible. Example: The path space PB with projection p_0 is in $\mathbf{TOP}/B$ and is fiberwise contractible (consider the fiber homotopy $H : IPB \rightarrow PB$ defined by $H(\sigma,t)(T) = \sigma(tT)$).

[Note: A fiber homotopy with domain IB is called a vertical homotopy.]

LEMMA Let X be in $\mathbf{TOP}/B$. Assume: X is fiberwise contractible—then for any $\Phi' \in C(B', B)$, X' is fiberwise contractible.

Let $f : X \rightarrow Y$ be a continuous function. View its mapping cylinder M_f as an object in $\mathbf{TOP}/Y$ with projection $r : M_f \rightarrow Y$—then $j \in \sec_Y(M_f)$ and M_f is fiberwise contractible.

Let X, Y be in $\mathbf{TOP}/B$—then a fiber preserving function $f : X \rightarrow Y$ is said to be fiberwise constant if $f = t \circ p$ for some section $t : B \rightarrow Y$. Elements of $C_B(X,Y)$ that are fiber homotopic to a fiberwise constant function are fiberwise inessential.

Suppose that B is not in $\mathbf{CG}$—then the identity map $kB \rightarrow B$ is continuous and constant on fibers but not fiberwise constant.

LEMMA Let X be in $\mathbf{TOP}/B$—then X is fiberwise contractible iff id_X is fiberwise inessential.

EXAMPLE Take $X = ([0,1] \times \{0,1\}) \cup (\{0\} \times [0,1])$, $B = [0,1]$, and let p be the vertical projection—then X is contractible but not fiberwise contractible.

EXAMPLE Let X be a subspace of $B \times \mathbf{R}^n$ and suppose that there exists an $s \in \sec_B(X)$, say $b \to (b, s(b))$, such that $\forall\, b \in B$, $\forall\, x \in X_b$, $\{(b, (1-t)s(b) + tx) : 0 \leq t \leq 1\} \subset X_b$—then X is fiberwise contractible.

FACT Let X be in $\mathbf{TOP}/B$; let $f, g \in C_B(X,X)$. Suppose that $\{O, P\}$ is a numerable covering of B for which $\begin{cases} f_O \\ g_P \end{cases}$ are fiberwise inessential—then $g \circ f$ is fiberwise inessential.

[Fix fiber homotopies $\begin{cases} K : IX_O \to X_O \\ L : IX_P \to X_P \end{cases}$ between $\begin{cases} f_O \ \& \ k \circ p_O \\ g_P \ \& \ l \circ p_P \end{cases}$, where $\begin{cases} k \in \sec_O(X_O) \\ l \in \sec_P(X_P) \end{cases}$. Through reparametrization, it can be assumed that $\begin{cases} K \circ i_t \\ L \circ i_t \end{cases}$ are independent of t when $0 \leq t \leq 1/4$, $3/4 \leq t \leq 1$. Choose $\begin{cases} \mu \\ \nu \end{cases} \in C(B, [0,1]) : \begin{cases} \operatorname{spt} \mu \subset O \\ \operatorname{spt} \nu \subset P \end{cases}$ & $\mu + \nu = 1$. Let Δ be the triangle in $\mathbf{R}^2$ with vertexes $(0,0)$, $(1,0)$, $(0,1)$. Note that the transformation $(\xi, \eta) \to (\xi, (1-\xi)\eta)$ takes $I[0,1] - I\{1\}$ homeomorphically onto $\Delta - \{(1,0)\}$. The continuous fiber preserving function $\Phi : I^2 X_{O \cap P} \to X_{O \cap P}$ defined by $\Phi(x, (\xi, \eta)) = L(K(x, \eta), \xi)$ is independent of η when $\xi = 1$, thus it induces a continuous fiber preserving function $\Phi_\Delta : X_{O \cap P} \times \Delta \to X_{O \cap P}$. On $X_{O \cap P} \times \operatorname{fr} \Delta$, one has $\Phi_\Delta(x, (t, 1-t)) = L(k(p(x)), t)$, $\Phi_\Delta(x, (0,t)) = g(K(x,t))$, $\Phi_\Delta(x, (t, 0)) = L(f(x), t)$. Write $s(b) = \begin{cases} L(k(b), \nu(b)) & (b \in O \cap P) \\ g(k(b)) & (b \in O - P) \\ l(b) & (b \in P - O) \end{cases}$—then $s \in \sec_B(X)$ and $g \circ f$ is fiber homotopic to $s \circ p$ via

$$H(x,t) = \begin{cases} \Phi_\Delta(x, t(\nu(b), \mu(b))) & (b \in O \cap P) \\ g(K(x,t)) & (b \in O - P) \quad (x \in X_b).] \\ L(f(x), t) & (b \in P - O) \end{cases}$$

Consequently, if $f_1, \ldots, f_n \in C_B(X,X)$ and if $O_1, \ldots, O_n$ is a numerable covering of B such that $\forall\, i$, f_{O_i} is fiberwise inessential, then $f_1 \circ \cdots \circ f_n$ is fiberwise inessential. Example: X_{O_i} fiberwise contractible $(i = 1, \ldots, n) \Rightarrow X$ fiberwise contractible (cf. p. 4–26).

Let X be in $\mathbf{TOP}/B$—then X is said to have the section extension property (SEP) provided that for each $A \subset B$, every section s_A of X_A which admits an extension s_O to a halo O of A in B can be extended to a section s of $X : s|A = s_A$.

[Note: If X has the SEP, then $\sec_B(X)$ is nonempty (take $A = \emptyset = O$).]

Let X be in $\mathbf{TOP}/B$ and suppose that X has the SEP. Let s be a section of $X|\phi^{-1}(]0,1])$, where $\phi \in C(B, [0,1])$—then $\forall\, \epsilon$, $0 < \epsilon < 1$, $s|\phi^{-1}([\epsilon, 1])$ can be extended to a section s_ϵ of X but it is false in general that s can be so extended.

EXAMPLE Suppose that B is a CW complex of combinatorial dimension $\leq n+1$ and T is n-connected—then $B \times T$ has the SEP.

PROPOSITION 1 Let X, Y be in $\mathbf{TOP}/B$ and suppose that Y has the SEP. Assume: $\exists \begin{cases} f \in C_B(X,Y) \\ g \in C_B(Y,X) \end{cases} : g \circ f \underset{B}{\simeq} \mathrm{id}_X$—then X has the SEP.

[Fix a fiber homotopy $H : IX \to X$ between id_X and $g \circ f$. Given $A \subset B$, let s_A be a section of X_A which admits an extension s_O to a halo O of A in B. Choose a closed halo P of A in $B : A \subset P \subset O$ and O a halo of P in B (cf. HA$_2$, p. 3–11). Since Y has the SEP, there exists a section t of $Y : t|P = f \circ s_O|P$. With π a haloing function of P, define $s : B \to X$ by $s(b) = \begin{cases} g \circ t(b) & (b \in \pi^{-1}(0)) \\ H(s_O(b), 1 - \pi(b)) & (b \in P) \end{cases}$ to get a section s of $X : s|A = s_A$.]

Application: Fiberwise contractible spaces have the SEP.

LEMMA Let X be in $\mathbf{TOP}/B$ and suppose that X has the SEP. Let O be a cozero set in B—then X_O has the SEP.

[There is no loss of generality in assuming that $A = f^{-1}(]0,1])$, where $f \in C(O,[0,1])$. Accordingly, given a section s_A of X_A, it will be enough to construct a section s of X_O which agrees with s_A on $f^{-1}(1)$. Fix $\phi \in C(B,[0,1]) : O = \phi^{-1}(]0,1])$. Claim: There exist sections $s_2, s_3, \ldots$ of X such that $s_{n+1}(b) = s_n(b)$ $(\phi(b) > \frac{1}{n})$ and $s_n(b) = s_A(b)$ $(f(b) > 1 - \frac{1}{n}$ & $\phi(b) > \frac{1}{n+1})$. Granted the claim, we are done. Put $F(b) = \begin{cases} f(b)\phi(b) & (b \in O) \\ 0 & (b \in B - O) \end{cases} : F \in C(B,[0,1])$. Since X has the SEP and s_A is defined on $F^{-1}(]0,1])$, a halo of $F^{-1}([1/6,1])$ in B, there exists a section of X that agrees with s_A on $f^{-1}(]1/2,1]) \cap \phi^{-1}(]1/3,1])$. Call it s_2, thus setting the stage for induction. Choose continuous functions μ_n, $\nu_n : [0,1] \to [0,1]$ subject to $\frac{1}{n+3} < \nu_n(x) < \mu_n(x) \leq \frac{1}{n}$ with $\mu_n(x) \leq \frac{1}{n+2}$ $(x \geq 1 - \frac{1}{n+1})$ and $\nu_n(x) \geq \frac{1}{n+1}$ $(x \leq 1 - \frac{1}{n})$ $(n = 2, 3, \ldots)$. Let $A_n = \{b \in O : \phi(b) > \mu_n(f(b))\}$, $O_n = \{b \in O : \phi(b) > \nu_n(f(b))\}$—then O_n is a halo of A_n in B, a haloing function being 1 on $\{b \in O : \mu_n(f(b)) \leq \phi(b)\}$,

$$\frac{\phi(b) - \nu_n(f(b))}{\mu_n(f(b)) - \nu_n(f(b))} \text{ on } \{b \in O : \nu_n(f(b)) \leq \phi(b) \leq \mu_n(f(b))\},$$

and 0 on $\{b \in O : \phi(b) \leq \nu_n(f(b))\} \cup B - O$. To pass from n to $n+1$, note that the prescription $b \to \begin{cases} s_n(b) & (\phi(b) > \frac{1}{n+1}) \\ s_A(b) & (f(b) > 1 - \frac{1}{n}) \end{cases}$ defines a section of X_{O_n}. Its restriction to A_n can therefore be extended to a section s_{n+1} of X with the required properties.]

SECTION EXTENSION THEOREM Let X be in $\mathbf{TOP}/B$. Suppose that $\mathcal{O} = \{O_i : i \in I\}$ is a numerable covering of B such that $\forall\, i$, X_{O_i} has the SEP—then X has the SEP.

[Given $A \subset B$, let s_A be a section of X_A which admits an extension s_O to a halo O of A in B. Fix a haloing function π for O and let $\{\pi_i : i \in I\}$ be a partition of unity on B subordinate to $\mathcal{O}$. Put $\Pi_S = \sum_{i \in S}(1-\pi)\pi_i + \pi$ $(S \subset I)$. Consider the set $\mathcal{S}$ of all pairs (S, s) : s is a section of $X|\Pi_S^{-1}(]0,1])$ & $s|A = s_A$: $\mathcal{S}$ is nonempty (take $S = \emptyset$, $s = s_O|\pi^{-1}(]0,1])$). Order $\mathcal{S}$ by stipulating that $(S', s') \leq (S'', s'')$ iff $S' \subset S''$ and $s'(b) = s''(b)$ when $\Pi_{S'}(b) = \Pi_{S''}(b) > 0$. One can check that every chain in $\mathcal{S}$ has an upper bound, so by Zorn, $\mathcal{S}$ has a maximal element (S_0, s_0). Since $\Pi_I = 1$, to finish it need only be shown that $S_0 = I$. Suppose not. Select an $i_0 \in I - S_0$, set $\Pi_0 = \Pi_{S_0}$ & $\pi_0 = (1-\pi)\pi_{i_0}$, and define a continuous function $\phi_0 : \pi_0^{-1}(]0,1]) \rightarrow [0,1]$ by $\phi_0(b) = \min\{1, \Pi_0(b)/\pi_0(b)\}$. Owing to the lemma, $X|\pi_0^{-1}(]0,1])$ has the SEP ($\pi_0^{-1}(]0,1]) \subset O_{i_0}$). On the other hand, $\phi_0^{-1}(]0,1])$ is a halo of $\phi_0^{-1}(1)$ in $\pi_0^{-1}(]0,1])$ and $s_0|\phi_0^{-1}(1)$ admits an extension to $\phi_0^{-1}(]0,1])$, viz. $s_0|\phi_0^{-1}(]0,1])$. Therefore $s_0|\phi_0^{-1}(1)$ can be extended to a section s_{i_0} of $X|\pi_0^{-1}(]0,1])$. Let $T = S_0 \cup \{i_0\}$ and write $t(b) = \begin{cases} s_0(b) & (\pi_0(b) \leq \Pi_0(b)) \\ s_{i_0}(b) & (\pi_0(b) \geq \Pi_0(b)) \end{cases}$ $(\Pi_T(b) > 0)$—then $(T, t) \in \mathcal{S}$ and $(S_0, s_0) < (T, t)$, contradicting the maximality of (S_0, s_0).]

FACT Let A be a subspace of X. Suppose that there exists a numerable covering $\mathcal{U} = \{U_i : i \in I\}$ of X such that $\forall$ i, the inclusion $A \cap U_i \rightarrow U_i$ is a cofibration—then the inclusion $A \rightarrow X$ is a cofibration.

[Let $\{\kappa_i : i \in I\}$ be a partition of unity on X subordinate to $\mathcal{U}$. The lemma on p. 3–11 implies that $\forall$ i, the inclusion $A \cap \kappa_i^{-1}(]0,1]) \rightarrow \kappa_i^{-1}(]0,1])$ is a cofibration. Therefore one can assume that $\mathcal{U}$ is numerable and open. Fix a topological space Y and a pair (F, h) of continuous functions $\begin{cases} F : X \rightarrow Y \\ h : IA \rightarrow Y \end{cases}$ such that $F|A = h \circ i_0$. Define a sheaf of sets $\mathcal{F}$ on X by assigning to each open set U the set of all continuous functions $H : IU \rightarrow Y$ such that $F|U = H \circ i_0$ and $H|I(A \cap U) = h|I(A \cap U)$. Choose a topological space E and a local homeomorphism $p : E \rightarrow X$ for which $\mathcal{F}(U) = \text{sec}_U(E_U)$ at each U. Show that $\forall$ i, E_{U_i} has the SEP. The section extension theorem then says that $\exists$ $H \in \mathcal{F}(X)$.]

Let X be in **TOP**/B. Let E be in **TOP**; let $\phi \in C(E, B)$—then a continuous function $\Phi : E \rightarrow X$ is a lifting of ϕ provided that $p \circ \Phi = \phi$. Example: Every $s \in \text{sec}_B(X)$ is a lifting of id_B.

FACT Suppose that X is fiberwise contractible. Let $\phi \in C(E, B)$—then for any halo U of any A in E and all $\psi \in C(U, X) : p \circ \psi = \phi|U$, there exists a lifting Φ of $\phi : \Phi|A = \psi|A$.

[Note: The condition is also characteristic. First take $E = B$, $A = \emptyset = U$, and $\phi = \text{id}_B$ to see that $\exists$ $s \in \text{sec}_B(X)$. Next let $E = IX$, $A = i_0X \cup i_1X$, $U = X \times [0, 1/2[\cup X \times]1/2, 1]$, and define $\phi : IX \rightarrow B$ by $\phi(x, t) = p(x)$, $\psi : U \rightarrow X$ by $\psi(x, t) = \begin{cases} x & (t < 1/2) \\ s \circ p(x) & (t > 1/2) \end{cases}$. Since U is a halo of A in IX, every lifting Φ of ϕ with $\Phi|A = \psi|A$ is a fiber homotopy between id_X and $s \circ p$, i.e., X is fiberwise contractible.]

(HLP) Let Y be a topological space—then the projection $p : X \rightarrow B$ is said to have the homotopy lifting property with respect to Y (HLP w.r.t. Y) if given continuous functions $\begin{cases} F : Y \rightarrow X \\ h : IY \rightarrow B \end{cases}$ such that $p \circ F = h \circ i_0$, there is a continuous function $H : IY \rightarrow X$ such that $F = H \circ i_0$ and $p \circ H = h$.

If $p : X \rightarrow B$ has the HLP w.r.t. Y and if $\begin{cases} f \in C(Y, B) \\ g \in C(Y, B) \end{cases}$ are homotopic, then f has a lifting $F \in C(Y, X)$ iff g has a lifting $G \in C(Y, X)$.

EXAMPLE **Take $X = [0,1] \amalg *$, $B = [0,1]$ and define $p : X \rightarrow B$ by $p(t) = t$, $p(*) = 0$. Fix a nonempty Y and let f be the constant map $Y \rightarrow 0$—then the constant map $Y \rightarrow *$ is a lifting $F \in C(Y, X)$ of f. Put $h(y, t) = t$, so $h : IY \rightarrow B$. Obviously, $p \circ F = h \circ i_0$ but there does not exist $H \in C(IY, X) : F = H \circ i_0$ and $p \circ H = h$.**

Let X be in **TOP/B**. Given a topological space Y and continuous functions $\begin{cases} F : Y \rightarrow X \\ h : IY \rightarrow B \end{cases}$ such that $p \circ F = h \circ i_0$, let W be the subspace of $Y \times PX$ consisting of the pairs $(y, \sigma) : F(y) = \sigma(0)$ & $h(y, t) = p(\sigma(t))$ $(0 \leq t \leq 1)$. View W as an object in **TOP/Y** with projection $(y, \sigma) \rightarrow y$.

LEMMA The commutative diagram

$$\begin{tikzcd}
Y \arrow[r, "F"] \arrow[d, "i_0"'] & X \arrow[d, "p"] \\
IY \arrow[r, "h"'] & B
\end{tikzcd}$$

admits a filler $H : IY \rightarrow X$ iff $\sec_Y(W) \neq \emptyset$.

PROPOSITION 2 Suppose that $p : X \rightarrow B$ has the HLP w.r.t. Y—then $\forall$ pair (F, h), W has the SEP.

[Fix $A \subset Y$ and let V be a halo of A in Y for which there exists a homotopy $H_V : IV \rightarrow X$ such that $F|V = H_V \circ i_0$ and $p \circ H_V = h|IV$. To construct a homotopy $H : IY \rightarrow X$ such that $F = H \circ i_0$ and $p \circ H = h$, with $H|IA = H_V|IA$, take V closed (cf. HA$_2$, p. 3–11) and using a haloing function π, put $\overline{h}(y, t) = h(y, \min\{1, \pi(y) + t\})$, so $\overline{h} : IY \rightarrow B$. Define $\overline{H}_V : i_0 Y \cup IV \rightarrow X$ by $\begin{cases} \overline{H}_V(y, 0) = F(y) \\ \overline{H}_V(y, t) = H_V(y, t) \end{cases}$ and define $\overline{F} : Y \rightarrow X$ by $\overline{F}(y) = \overline{H}_V(y, \pi(y))$. Since $p \circ \overline{F} = \overline{h} \circ i_0$, there is a continuous function $\overline{H} : IY \rightarrow X$ such that $\overline{F} = \overline{H} \circ i_0$ and $p \circ \overline{H} = \overline{h}$. The rule

$$H(y, t) = \begin{cases} \overline{H}_V(y, t) & (0 \leq t \leq \pi(y)) \\ \overline{H}(y, t - \pi(y)) & (\pi(y) \leq t \leq 1) \end{cases}$$

then specifies a homotopy $H : IY \rightarrow X$ having the properties in question.]

Let $\mathcal{Y}$ be a class of topological spaces—then $p : X \rightarrow B$ is said to be a $\mathcal{Y}$ fibration if $\forall\, Y \in \mathcal{Y}$, $p : X \rightarrow B$ has the HLP w.r.t. Y.

(H) Take for $\mathcal{Y}$ the class of topological spaces—then a $\mathcal{Y}$ fibration $p : X \rightarrow B$ is called a Hurewicz fibration.

(S) Take for $\mathcal{Y}$ the class of CW complexes—then a $\mathcal{Y}$ fibration $p : X \rightarrow B$ is called a Serre fibration.

Every Hurewicz fibration is a Serre fibration. The converse is false (cf. p. 4–8).

Observation: Let $Y \in \mathcal{Y}$ and suppose that $p : X \rightarrow B$ is a $\mathcal{Y}$ fibration—then any inessential $f \in C(Y, B)$ admits a lifting $F \in C(Y, X)$.

[Note: It is thus a corollary that if $B \in \mathcal{Y}$ is contractible, then $\sec_B(X)$ is nonempty.]

Other possibilities suggest themselves. For example, one could consider $p : X \rightarrow B$, where both X and B are in **CG**, and work with the class $\mathcal{Y}$ of compactly generated spaces. This leads to the notion of **CG** fibration. Any **CG** fibration is a Serre fibration. In general, if $p : X \rightarrow B$ is a Hurewicz fibration, then $kp : kX \rightarrow kB$ is a **CG** fibration. Another variant would be to consider pointed spaces and pointed homotopies. Via the artifice of adding a disjoint base point (cf. p. 3–26), one sees that every pointed Hurewicz fibration is a Hurewicz fibration. In the opposite direction, an $f \in C_B(X, Y)$ is said to be a fiberwise Hurewicz fibration if it has the fiber homotopy lifting property with respect to all E in **TOP**/B. Of course, if f is a Hurewicz fibration, then f is a fiberwise Hurewicz fibration. On the other hand, for any X in **TOP**/B, the projection $p : X \rightarrow B$ is always a fiberwise Hurewicz fibration.

FACT Suppose that $p : X \rightarrow B$ is a Hurewicz fibration. Let E be a topological space with the homotopy type of a compactly generated space—then a $\phi \in C(E, B)$ has a lifting $E \rightarrow X$ iff $k\phi \in C(kE, kB)$ has a lifting $kE \rightarrow kX$.

[The identity map $kE \rightarrow E$ is a homotopy equivalence.]

EXAMPLE For any topological space T, the projection $B \times T \rightarrow B$ is a Hurewicz fibration. Take, e.g., $T = \mathbf{D}^n$, let $X_0 \subset B \times \mathbf{S}^{n-1}$, and put $X = B \times \mathbf{D}^n - X_0$—then the restriction to X of the projection $B \times \mathbf{D}^n \rightarrow B$ is a Hurewicz fibration.

EXAMPLE (Covering Spaces) A continuous function $p : X \rightarrow B$ is said to be a covering projection if each $b \in B$ has a neighborhood O such that X_O is trivial with discrete fiber. Every covering projection is a Hurewicz fibration.

[Note: A sheaf of sets $\mathcal{F}$ on B is locally constant provided that each $b \in B$ has a basis $\mathcal{B}$ of neighborhoods such that whenever $U, V \in \mathcal{B}$ with $U \subset V$, the restriction map $\mathcal{F}(V) \rightarrow \mathcal{F}(U)$ is a bijection. If $p : X \rightarrow B$ is a covering projection, then its sheaf of sections Γ_X is locally constant. Moreover, every locally constant sheaf of sets $\mathcal{F}$ on B can be so realized.]

EXAMPLE Let X be the triangle in $\mathbf{R}^2$ with vertexes $(0,0)$, $(1,0)$, $(0,1)$—then the vertical projection $p : X \rightarrow [0,1]$ is a Hurewicz fibration but X is not locally trivial.

[Note: Ferry[†] has constructed an example of a Hurewicz fibration $p : X \rightarrow [0,1]$ whose fibers are connected n-manifolds but such that X is not locally trivial.]

LEMMA Let X be in $\mathbf{TOP}/B$—then $p : X \rightarrow B$ is a Serre fibration iff it has the HLP w.r.t. the $[0,1]^n$ $(n \geq 0)$.

EXAMPLE Take $X = \{(x,-x) : 0 \leq x \leq 1\} \cup \bigcup\limits_1^\infty([0,1] \times \{1/n\})$, $B = [0,1]$, and let p be the vertical projection—then p is a Serre fibration but not a Hurewicz fibration.

[Note: $p^{-1}(0)$ and $p^{-1}(1)$ do not have the same homotopy type.]

EXAMPLE Let B be a topological space which is not compactly generated—then ΓB is not compactly generated and the identity map $k\Gamma B \rightarrow \Gamma B$ is a Serre fibration but not a Hurewicz fibration.

[For any compact Hausdorff space K, the arrow $C(K, k\Gamma B) \rightarrow C(K, \Gamma B)$ is a bijection.]

EXAMPLE Let $B = [0,1]^\omega$, the Hilbert cube. Put $X = B \times B - \Delta_B$ and let p be the vertical projection, q the horizontal projection—then $p : X \rightarrow B$ is a Serre fibration. Moreover, B is an AR as are the X_b (each being homeomorphic to $B \times [0,1[$) but $p : X \rightarrow B$ is not a Hurewicz fibration.

[If so, then there would exist an $s \in \sec_B(X)$. Consider $q \circ s$: It is a continuous function $B \rightarrow B$ without a fixed point, contradicting Brouwer.]

Ungar[‡] has shown that if X and B are compact ANRs of finite topological dimension, then a Serre fibration $p : X \rightarrow B$ is necessarily a Hurewicz fibration.

The projection $p : X \rightarrow B$ is a Hurewicz fibration iff the commutative diagram

$$\begin{tikzcd} PX \arrow[r, "p_0"] \arrow[d, "Pp"'] & X \arrow[d, "p"] \\ PB \arrow[r, "p_0"'] & B \end{tikzcd}$$

is a weak pullback square. Homeomorphisms are Hurewicz fibrations. Maps with an empty domain are Hurewicz fibrations. The composite of two Hurewicz fibrations is a Hurewicz fibration.

PROPOSITION 3 Let $\begin{cases} p_1 : X_1 \rightarrow B_1 \\ p_2 : X_2 \rightarrow B_2 \end{cases}$ be Hurewicz fibrations—then $p_1 \times p_2 : X_1 \times X_2 \rightarrow B_1 \times B_2$ is a Hurewicz fibration.

[†] *Trans. Amer. Math. Soc.* **327** (1991), 201–219; see also Husch, *Proc. Amer. Math. Soc.* **61** (1976), 155–156.

[‡] *Pacific J. Math.* **30** (1969), 549–553.

PROPOSITION 4 Let
$$\begin{tikzcd} X' \arrow[r] \arrow[d, "p'"'] & X \arrow[d, "p"] \\ B' \arrow[r] & B \end{tikzcd}$$
be a pullback square. Suppose that p is a Hurewicz fibration—then p' is a Hurewicz fibration.

Application: Let $p : X \rightarrow B$ be a Hurewicz fibration—then $\forall\ O \subset B$, $p_O : X_O \rightarrow O$ is a Hurewicz fibration.

PROPOSITION 5 Let $p : X \rightarrow B$ be a Hurewicz fibration—then for any LCH space Y, the postcomposition arrow $p_* : C(Y, X) \rightarrow C(Y, B)$ is a Hurewicz fibration (compact open topology).

[Convert

$$\begin{tikzcd} E \arrow[r] \arrow[d] & C(Y,X) \arrow[d] \\ IE \arrow[r] \arrow[ur, dashed] & C(Y,B) \end{tikzcd} \quad \text{to} \quad \begin{tikzcd} E \times Y \arrow[r] \arrow[d] & X \arrow[d] \\ I(E \times Y) \arrow[r] \arrow[ur, dashed] & B \end{tikzcd}.]$$

Application: Let $p : X \rightarrow B$ be a Hurewicz fibration—then $Pp : PX \rightarrow PB$ is a Hurewicz fibration.

PROPOSITION 6 Let $i : A \rightarrow X$ be a closed cofibration, where X is a LCH space—then for any topological space Y, the precomposition arrow $i^* : C(X, Y) \rightarrow C(A, Y)$ is a Hurewicz fibration (compact open topology).

[Convert

$$\begin{tikzcd} E \arrow[r] \arrow[d] & C(X,Y) \arrow[d] \\ IE \arrow[r] \arrow[ur, dashed] & C(A,Y) \end{tikzcd} \quad \text{to} \quad \begin{tikzcd} E \times X \arrow[rr] \arrow[d] & & Y \\ I(E \times X) \arrow[urr, dashed] & & I(E \times A) \arrow[ll] \arrow[u] \end{tikzcd} .]$$

Application: Let X be a topological space—then $p_t : PX \rightarrow X$ $(0 \leq t \leq 1)$ is a Hurewicz fibration.

EXAMPLE Let $i : A \rightarrow X$ be a closed cofibration, where X is a LCH space. Fix $a_0 \in A$ and put $x_0 = i(a_0)$—then for any pointed topological space (Y, y_0), the precomposition arrow $i^* : C(X, x_0; Y, y_0) \rightarrow C(A, a_0; Y, y_0)$ is a Hurewicz fibration (compact open topology).

[The commutative diagram
$$\begin{tikzcd} C(X, x_0; Y, y_0) \arrow[r] \arrow[d] & C(X,Y) \arrow[d] \\ C(A, a_0; Y, y_0) \arrow[r] & C(A,Y) \end{tikzcd}$$
is a pullback square.]

FACT Let X be a topological space—then $\Pi : \begin{cases} PX \rightarrow X \times X \\ \sigma \rightarrow (\sigma(0), \sigma(1)) \end{cases}$ is a Hurewicz fibration. Moreover, X is locally path connected iff Π is open.

[Note: Fix $x_0 \in X$—then the fiber of Π over (x_0, x_0) is ΩX, the loop space of (X, x_0).]

STACKING LEMMA Given a topological space Y, let $\{P_i : i \in I\}$ be a numerable covering of IY—then there exists a numerable covering $\{Y_j : j \in J\}$ of Y and positive real numbers ϵ_j $(j \in J)$ such that $\forall\ t', t'' \in [0,1]$ with $t' \leq t''$ & $t'' - t' < \epsilon_j$, $\exists\ i \in I$: $Y_j \times [t', t''] \subset P_i$.

[Let $\{\rho_i : i \in I\}$ be a partition of unity on IY subordinate to $\{P_i : i \in I\}$. Put $J = \bigcup\limits_1^\infty I^r$. Take $j \in J$, say $j = (i_1, \ldots, i_r) \in I^r$, define $\pi_j \in C(Y, [0,1])$ by

$$\pi_j(y) = \prod_{k=1}^{r} \min\left\{\rho_{i_k}(y,t) : t \in \left[\frac{k-1}{r+1}, \frac{k+1}{r+1}\right]\right\}$$

and set $Y_j = \pi_j^{-1}(]0,1])$, $\epsilon_j = 1/2r$. Since $Y_j \subset \bigcap\limits_{k=1}^{r} \left\{y : \{y\} \times \left[\frac{k-1}{r+1}, \frac{k+1}{r+1}\right] \subset P_{i_k}\right\}$, the ϵ_j will work. Moreover, due to the compactness of $[0,1]$, for each $y \in Y$ there is: (1) An index $j \in I^r$ such that $\{y\} \times \left[\frac{k-1}{r+1}, \frac{k+1}{r+1}\right] \subset \rho_{i_k}^{-1}(]0,1])$ $(k = 1, \ldots, r)$ and (2) A neighborhood V of y such that IV meets but a finite number of the $\rho_i^{-1}(]0,1])$. Therefore $\{Y_j : j \in J\} = \bigcup\limits_1^\infty \{Y_j : j \in I^r\}$ is a σ-neighborhood finite cozero set covering of Y, hence is numerable.]

LOCAL-GLOBAL PRINCIPLE Let X be in $\mathbf{TOP}/B$. Suppose that $\mathcal{O} = \{O_i : i \in I\}$ is a numerable covering of B such that $\forall\ i$, $p_{O_i} : X_{O_i} \rightarrow O_i$ is a Hurewicz fibration—then $p : X \rightarrow B$ is a Hurewicz fibration.

[Fix a topological space Y and a pair (F, h) of continuous functions $\begin{cases} F : Y \rightarrow X \\ h : IY \rightarrow B \end{cases}$ such that $p \circ F = h \circ i_0$. To establish the existence of an $H : IY \rightarrow X$ such that $F = H \circ i_0$ and $p \circ H = h$ is equivalent to proving that $\text{sec}_Y(W) \neq \emptyset$ (cf. p. 4–6). For this, we shall use the section extension theorem and show that W has the SEP, which suffices. Set $P_i = h^{-1}(O_i) : \{P_i : i \in I\}$ is a numerable covering of IY and the stacking lemma is applicable. Given j, put $W_j = W|Y_j$, choose $t_k : 0 = t_0 < t_1 < \cdots < t_n = 1$, $t_k - t_{k-1} < \epsilon_j$, and select i accordingly: $h(Y_j \times [t_{k-1}, t_k]) \subset O_i$. The claim is that W_j has the SEP. So let $A \subset Y_j$, let V be a halo of A in Y_j, and let $H_V : IV \rightarrow X$ be a homotopy such that $F|V = H_V \circ i_0$ and $p \circ H_V = h|IV$. With π a haloing function of V, put $A_k = \pi^{-1}([t_k, 1])$ $(k = 1, \ldots, n)$: A_k is a halo of A_{k+1} in Y_j and V is a halo of A_1 in Y_j. Owing to Proposition 2, there exist homotopies $H_k : Y_j \times [t_{k-1}, t_k] \rightarrow X$ having the following properties: $p \circ H_k = h|Y_j \times [t_{k-1}, t_k]$, $H_k(y, t_{k-1}) = H_{k-1}(y, t_{k-1})$ $(k > 1)$, $H_1(y, 0) = F(y)$, $H_k|A_k \times [t_{k-1}, t_k] = H_V|A_k \times [t_{k-1}, t_k]$. The H_k thus combine to determine a homotopy $H : IY_j \rightarrow X$ such that $F|Y_j = H \circ i_0$, $p \circ H = h|IY_j$, and $H|IA = H_V|IA$.]

Application: Suppose that B is a paracompact Hausdorff space. Let X be in $\mathbf{TOP}/B$. Assume: X is locally trivial—then $p : X \rightarrow B$ is a Hurewicz fibration.

EXAMPLE Let $B = L^+$, the long ray. Put $X = \{(x, y) \in L^+ \times L^+ : x < y\}$ and let p be the vertical projection—then X is locally trivial but $p : X \rightarrow B$ is not a Hurewicz fibration.

FACT Let X be in $\mathbf{TOP}/B$. Suppose that $\mathcal{O} = \{O_i : i \in I\}$ is an open covering of B such that $\forall$ i, $p_{O_i} : X_{O_i} \rightarrow O_i$ is a Hurewicz fibration—then the projection $p : X \rightarrow B$ is a $\mathcal{Y}$ fibration, where $\mathcal{Y}$ is the class of paracompact Hausdorff spaces.

[Given $Y \in \mathcal{Y}$ and continuous functions $\begin{cases} F : Y \rightarrow X \\ h : IY \rightarrow B \end{cases}$ such that $p \circ F = h \circ i_0$, consider the pullback square

$$\begin{tikzcd} IY \times_B X \arrow[r] \arrow[d] & X \arrow[d, "p"] \\ IY \arrow[r, "h"'] & B \end{tikzcd}$$

, observing that $IY \in \mathcal{Y}$.]

[Note: It follows that $p : X \rightarrow B$ is a Serre fibration.]

Let $f : X \rightarrow Y$ be a continuous function—then the mapping track W_f of f is defined by the pullback square

$$\begin{tikzcd} W_f \arrow[r] \arrow[d] & PY \arrow[d, "p_0"] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

. Special case: $\forall$ $y_0 \in Y$, the mapping track of the inclusion $\{y_0\} \rightarrow Y$ is the mapping space ΘY of (Y, y_0). There is a projection $p : W_f \rightarrow X$, a homotopy $G : W_f \rightarrow PY$, and a unique continuous function $s : X \rightarrow W_f$ such that $p \circ s = \mathrm{id}_X$ and $G \circ s = j \circ f$ $(j : Y \rightarrow PY)$. One has $s \circ p \underset{X}{\simeq} \mathrm{id}_{W_f}$. The composition $p_1 \circ G$ is a projection $q : W_f \rightarrow Y$ and $f = q \circ s$.

[Note: The mapping track is a functor $\mathbf{TOP}(\rightarrow) \rightarrow \mathbf{TOP}$.]

LEMMA p is a Hurewicz fibration and W_f is fiberwise contractible over X.

LEMMA q is a Hurewicz fibration.

[To construct a filler for
$$\begin{tikzcd} E \arrow[r, "\Phi"] \arrow[d, "i_0"'] & W_f \arrow[d, "q"] \\ IE \arrow[r, "h"'] & Y \end{tikzcd}$$
, write $\Phi(e) = (x_e, \tau_e) : \begin{cases} x_e \in X \\ \tau_e \in PY \end{cases}$ & $f(x_e) = \tau_e(0)$, and define $H : IE \to W_f$ by $H(e,t) = (x_e, \overline{h}(e,t))$, where

$$\overline{h}(e,t)(T) = \begin{cases} \tau_e(2T(2-t)^{-1}) & (T \leq 1 - t/2) \\ h(e, 2T + t - 2) & (T \geq 1 - t/2) \end{cases} .]$$

PROPOSITION 7 Every morphism in **TOP** can be written as the composite of a homotopy equivalence and a Hurewicz fibration.

FACT Let $f : X \to Y$ be a continuous function—then f can be factored as $f = \begin{cases} \Phi \circ k \\ \Psi \circ l \end{cases}$, where $\begin{cases} \Phi \\ \Psi \end{cases}$ is a Hurewicz fibration, $\begin{cases} k \\ l \end{cases}$ is a closed cofibration, and $\begin{cases} k \\ \Psi \end{cases}$ is a homotopy equivalence.

[Per Proposition 7, write $f = q \circ s$, form $S = Is(X) \cup W_f \times]0,1] \subset IW_f$, and let $\omega : IW_f \to [0,1]$ be the projection. The restriction to S of the Hurewicz fibration $IW_f \to W_f$ is a Hurewicz fibration, call it p. Proof: Given continuous functions $\begin{cases} F : Y \to S \\ h : IY \to W_f \end{cases}$ such that $p \circ F = h \circ i_0$, consider $H : IY \to S$, where $H(y,t) = (h(y,t), t + (1-t)\omega(F(y)))$. Next, if $k : X \to S$ is defined by $k(x) = (s(x), 0)$, then $k(X)$ is both a strong deformation retract of S and a zero set in S (being $(\omega|S)^{-1}(0)$). Therefore k is a closed cofibration (cf. §3, Proposition 10). And: $f = q \circ p \circ k$. To derive the other factorization, write $f = r \circ i$ (cf. §3, Proposition 16) and decompose r as above.]

Let X be in $\mathbf{TOP}/B$. Define $\lambda : PX \to W_p$ by $\sigma \to (\sigma(0), p \circ \sigma)$.

PROPOSITION 8 The projection $p : X \to B$ is a Hurewicz fibration iff λ has a right inverse Λ.

[Note: Λ is called a lifting function.]

FACT Let $p : X \to B$ be a Hurewicz fibration. Suppose that A is a subspace of X for which there exists a fiber preserving retraction $r : X \to A$—then the restriction of p to A is a Hurewicz fibration $A \to B$.

EXAMPLE Let X be a nonempty compact subspace of $\mathbf{R}^n$. Realize ΓX in $\mathbf{R}^{n+1}$ by writing $\Gamma X = \bigcup\limits_x \{(t, tx) : 0 \leq t \leq 1\}$, so $\Gamma^2 X$ is $\bigcup\limits_x \{(s, st, stx) : 0 \leq s \leq 1 \ \& \ 0 \leq t \leq 1\}$, a subspace of $\mathbf{R}^{n+2}$. Claim: The projection $p : \begin{cases} \Gamma^2 X \to [0,1] \\ (s, st, stx) \to s \end{cases}$ is a Hurewicz fibration. To see this, consider $[0,1] \times \Gamma X = \bigcup\limits_x \{(s, t, tx) : 0 \leq s \leq 1 \ \& \ 0 \leq t \leq 1\}$ with projection $(s, t, tx) \to s$ and define a fiber preserving retraction

$r : [0,1] \times \Gamma X \to \Gamma^2 X$ by $r(s,t,tx) = \begin{cases} (s,s,sx) & (t \geq s) \\ (s,t,tx) & (t \leq s) \end{cases}$. The fibers of p over the points in $]0,1]$ can be identified with ΓX, while $p^{-1}(0) = *$.

[Note: If X is the Cantor set, then ΓX is not an ANR.]

Let X be in **TOP**$/B$—then there is a morphism

$$\begin{tikzcd} X \arrow[rr, "\gamma"] \arrow[dr, "p"'] & & W_p \arrow[dl, "q"] \\ & B & \end{tikzcd}$$

. Here, in a change of notation, γ sends x to $(x, j(p(x)))$, $j : B \to PB$ the embedding.

PROPOSITION 9 Suppose that $p : X \to B$ is a Hurewicz fibration—then $\gamma : X \to W_p$ is a fiber homotopy equivalence.

[Choose a lifting function $\Lambda : W_p \to PX$. Define a fiber homotopy $H : IX \to X$ by $H(x,t) = \Lambda(\gamma(x))(t)$ and a fiber homotopy $G : IW_p \to W_p$ by $G((x,\tau),t) = (\Lambda(x,\tau)(t), \tau_t)$ $(\tau_t(T) = \tau(t + T - tT))$—then it is clear that the assignment $(x,\tau) \to \Lambda(x,\tau)(1)$ is a fiber homotopy inverse for γ.]

Application: The fibers of a Hurewicz fibration over a path connected base have the same homotopy type.

[Note: This need not be true if "Hurewicz" is replaced by "Serre" (cf. p. 4–8). It can also fail if "path connected" is weakened to "connected". Indeed, for a connected B whose path components are singletons, every $p : X \to B$ is a Hurewicz fibration.]

A Hurewicz fibration $p : X \to B$ is said to be regular if the morphism

$$\begin{tikzcd} X \arrow[rr, "\gamma"] \arrow[dr, "p"'] & & W_p \arrow[dl, "q"] \\ & B & \end{tikzcd}$$

has a left inverse Γ in **TOP**$/B$.

FACT The Hurewicz fibration $p : X \to B$ is regular iff there exists a lifting function $\Lambda_0 : W_p \to PX$ with the property that $\Lambda_0(x,\tau) \in j(X)$ whenever $\tau \in j(B)$.

[Given a left inverse Γ for γ, consider the lifting function $\Lambda_0 : W_p \to PX$ defined by $\Lambda_0(x,\tau)(t) = \Gamma(x,\tau_t)$, where $\tau_t(T) = \tau(tT)$.]

FACT The Hurewicz fibration $p : X \to B$ is regular iff every commutative diagram

$$\begin{tikzcd} Y \arrow[r, "F"] \arrow[d, "i_0"'] & X \arrow[d, "p"] \\ IY \arrow[r, "h"'] & B \end{tikzcd}$$

admits a filler $H : IY \to X$ such that H is stationary with h, i.e., $h|I\{y_0\}$ constant $\Rightarrow$ $H|I\{y_0\}$ constant.

[Note: The local-global principle is valid in the regular situation (work with a suitable subspace of W to factor in the stationary condition).]

A sufficient condition for the regularity of the Hurewicz fibration $p : X \to B$ is that $j(B)$ be a zero set in PB. Thus let $\phi \in C(PB,[0,1]) : j(B) = \phi^{-1}(0)$. Define $\Phi \in C(PB,PB)$ by $\Phi(\tau)(t) = \begin{cases} \tau(t/\phi(\tau)) & t < \phi(\tau) \\ \tau(1) & (\phi(\tau) \leq t \leq 1) \end{cases}$. Take any lifting function Λ and put $\Lambda_0(x,\tau)(t) = \Lambda(x,\Phi(\tau))(\phi(\tau)t)$ to get a lifting function $\Lambda_0 : W_p \to PX$ with the property that $\Lambda_0(x,\tau) \in j(X)$ whenever $\tau \in j(B)$. Example: $j(B)$ is a zero set in PB if Δ_B is a zero set in $B \times B$, e.g., if the inclusion $\Delta_B \to B \times B$ is a closed cofibration, a condition satisfied by a CW complex or a metrizable topological manifold (cf. p. 3–14).

EXAMPLE Let $B = [0,1]/[0,1[$—then the Hurewicz fibration $p_0 : PB \to B$ is not regular.

FACT Suppose that $p : X \to B$ is a regular Hurewicz fibration—then $\forall\, x_0 \in X$, $p : (X,x_0) \to (B,b_0)$ is a pointed Hurewicz fibration ($b_0 = p(x_0)$).

Let X be in **TOP**/B—then the projection $p : X \to B$ is said to have the slicing structure property if there exists an open covering $\mathcal{O} = \{O_i : i \in I\}$ of B and continuous functions $s_i : O_i \times X_{O_i} \to X_{O_i}$ ($i \in I$) such that $s_i(p(x),x) = x$ and $p \circ s_i(b,x) = b$. Note that p is necessarily open. Example: X locally trivial $\Rightarrow p : X \to B$ has the slicing structure property (but not conversely).

Observation: Suppose that $p : X \to B$ has the slicing structure property—then $\forall\, i, p_{O_i} : X_{O_i} \to O_i$ is a regular Hurewicz fibration.

[Consider the lifting function Λ_i defined by $\Lambda_i(x,\tau)(t) = s_i(\tau(t),x)$.]

So, if $p : X \to B$ has the slicing structure property, then $p : X \to B$ must be a Serre fibration and is even a regular Hurewicz fibration provided that B is a paracompact Hausdorff space.

FACT Let X be in **TOP**/B, where B is uniformly locally contractible. Assume: The projection $p : X \to B$ is a regular Hurewicz fibration—then p has the slicing structure property.

Application: Suppose that B is a uniformly locally contractible paracompact Hausdorff space. Let X be in **TOP**/B—then the projection $p : X \to B$ is a regular Hurewicz fibration iff p has the slicing structure property.

[Note: It therefore follows that if B is a CW complex or a metrizable topological manifold, then the Hurewicz fibrations with base B are precisely the $p : X \to B$ which have the slicing structure property.]

FACT Let $p : X \to B$ be a Serre fibration, where X and B are CW complexes—then p is a CG fibration.

[An open subset of a CW complex is homeomorphic to a retract of a CW complex (cf. p. 5–12).]

[Note: If $X \times B$ is compactly generated, then p is a Hurewicz fibration.]

Cofibrations are embeddings (cf. p. 3–3). By analogy, one might expect that surjective Hurewicz fibrations are quotient maps. However, this is not true in general. Example:

Take $X = \mathbf{Q}$ (discrete topology), $B = \mathbf{Q}$ (usual topology), $p = \mathrm{id}_{\mathbf{Q}}$—then $p : X \rightarrow B$ is a surjective Hurewicz fibration but not a quotient map.

PROPOSITION 10 Let $p : X \rightarrow B$ be a Hurewicz fibration. Assume: p is surjective and B is locally path connected—then p is a quotient map.

[Consider the commutative diagram

$$\begin{tikzcd} PX \arrow[r, "\lambda"] \arrow[d, "p_1"'] & W_p \arrow[d, "q"] \\ X \arrow[r, "p"'] & B \end{tikzcd}$$

. Since λ and p_1 have right inverses, they are quotient, so p is quotient iff q is quotient. Take a nonempty subset $O \subset B : W_O$ is open in W_p. Fix $b \in O$, $x \in X_b$, and choose a neighborhood O_b of $b : (\{x\} \times PO_b) \cap W_p \subset W_O$. The path component O_0 of O_b containing b is open. Given $b_0 \in O_0$, $\exists\ \tau \in PO_b$ connecting b and b_0. But $(x, \tau) \in W_O \Rightarrow b_0 = q(x, \tau) \in O \Rightarrow O_0 \subset O$. Therefore O is open in B, hence q is quotient.]

Application: Every connected locally path connected nonempty space B is the quotient of a contractible space.

[Fix $b_0 \in B$ and consider the mapping space ΘB of (B, b_0) with projection $\tau \rightarrow \tau(1)$.]

Let $p : X \rightarrow B$ be a Hurewicz fibration—then for any path component A of X, $p(A)$ is a path component of B and $A \rightarrow p(A)$ is a Hurewicz fibration. Therefore $p(X)$ is a union of path components of B. So, if B is path connected and X is nonempty, then p is surjective.

FACT Let $p : X \rightarrow B$ be a Hurewicz fibration. Assume: B is path connected and X_b is path connected for some $b \in B$—then X is path connected.

[Note: The fibers of a Hurewicz fibration $p : X \rightarrow B$ need not be path connected but if X is path connected, then any two path components of a given fiber have the same homotopy type.]

FACT Suppose that B is path connected—then B is locally path connected iff every Hurewicz fibration $p : X \rightarrow B$ is open.

PROPOSITION 11 Let $p : X \rightarrow B$ be a Hurewicz fibration. Suppose that the inclusion $O \rightarrow B$ is a closed cofibration—then the inclusion $X_O \rightarrow X$ is a closed cofibration.

[Fix a Strøm structure (ϕ, Φ) on (B, O). Let $H : IX \rightarrow X$ be a filler for the commutative diagram

$$\begin{tikzcd} X \arrow[r, "\mathrm{id}_X"] \arrow[d, "i_0"'] & X \arrow[d, "p"] \\ IX \arrow[r, "h"'] & B \end{tikzcd}$$

, where $h = \Phi \circ Ip$. Define a Strøm structure (ψ, Ψ) on (X, X_O) by $\psi = \phi \circ p$, $\Psi(x, t) = H(x, \min\{t, \psi(x)\})$.]

Application: Let $p : X \rightarrow B$ be a Hurewicz fibration. Let A be a subspace of X and suppose that the inclusion $A \rightarrow X$ is a closed cofibration. View A as an object in $\mathbf{TOP}/B$ with projection $p_A = p|A$—then the inclusion $W_{p_A} \rightarrow W_p$ is a closed cofibration.

EXAMPLE Let (X, x_0) be a pointed space. Assume: The inclusion $\{x_0\} \rightarrow X$ is a closed cofibration—then Proposition 11 implies that the inclusion $j : \Omega X \rightarrow \Theta X$ is a closed cofibration. Call θ the continuous function $\Gamma\Omega X \rightarrow \Theta X$ that sends $[\sigma, t]$ to σ_t, where $\sigma_t(T) = \sigma(tT)$. The arrow $i : \begin{cases} \Omega X \rightarrow \Gamma\Omega X \\ \sigma \rightarrow [\sigma, 1] \end{cases}$ is a closed cofibration and $\theta \circ i = j$. Consider the commutative diagram

$$\begin{tikzcd} \Omega X \arrow[r, "i"] \arrow[d, equal] & \Gamma\Omega X \arrow[d, "\theta"] \\ \Omega X \arrow[r, "j"'] & \Theta X \end{tikzcd}.$$

Because $\Gamma\Omega X$ and ΘX are contractible, it follows from §3, Proposition 14 that the arrow $(\mathrm{id}_{\Omega X}, \theta)$ is a homotopy equivalence in $\mathbf{TOP}(\rightarrow)$.

LEMMA Let $\phi \in C(Y, [0,1]) : A = \phi^{-1}(0)$ is a strong deformation retract of Y. Suppose that $p : X \rightarrow B$ is a Hurewicz fibration—then every commutative diagram

$$\begin{tikzcd} A \arrow[r, "g"] \arrow[d, "i"'] & X \arrow[d, "p"] \\ Y \arrow[r, "f"'] & B \end{tikzcd}$$

has a filler $F : Y \rightarrow X$.

[Fix a retraction $r : Y \rightarrow A$ and a homotopy $\Phi : IY \rightarrow Y$ between $i \circ r$ and id_Y rel A. Define a homotopy $h : IY \rightarrow Y$ by $h(y,t) = \begin{cases} \Phi(y, t/\phi(y)) & (t < \phi(y)) \\ \Phi(y, 1) & (t \geq \phi(y)) \end{cases}$. Since p is a Hurewicz fibration, there exists a homotopy $H : IY \rightarrow X$ such that $g \circ r = H \circ i_0$ and $p \circ H = f \circ h$. Take for $F : Y \rightarrow X$ the continuous function $y \rightarrow H(y, \phi(y))$.]

[Note: The hypotheses on A are realized when the inclusion $i : A \rightarrow Y$ is both a homotopy equivalence and a closed cofibration (cf. §3, Proposition 5).]

FACT Let $i : A \rightarrow Y$ be a continuous function with a closed image—then i is both a homotopy equivalence and a closed cofibration iff every commutative diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d, "i"'] & X \arrow[d, "p"] \\ Y \arrow[r] & B \end{tikzcd},$$

where p is a Hurewicz fibration, has a filler $Y \rightarrow X$.

[First take $X = PB$, $p = p_0$ to see that i is a closed cofibration. Next, identify A with $i(A)$ and produce a retraction $r : Y \rightarrow A$ from a filler for

$$\begin{tikzcd} A \arrow[r, "\mathrm{id}_A"] \arrow[d, "i"'] & A \arrow[d] \\ Y \arrow[r] & * \end{tikzcd}.$$

Finally, consider

$$\begin{tikzcd} A \arrow[r, "j"] \arrow[d, "i"'] & PY \arrow[d, "\Pi"] \\ Y \arrow[r, "\rho"'] & Y \times Y \end{tikzcd},$$

where $\rho(y) = (y, r(y))$ (Π as on p. 4–10).]

FACT Let $p : X \to B$ be a continuous function—then p is a Hurewicz fibration iff every commutative diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d, "i"'] & X \arrow[d, "p"] \\ Y \arrow[r] & B \end{tikzcd}$$

where i is both a homotopy equivalence and a closed cofibration, has a filler $Y \to X$.

FACT Let

$$\begin{tikzcd} X_0 \arrow[d] & X_1 \arrow[l] \arrow[d] & \cdots \arrow[l] \\ Y_0 & Y_1 \arrow[l] & \cdots \arrow[l] \end{tikzcd}$$

be a commutative ladder of topological spaces. Assume: $\forall\ n$, the horizontal arrows $\begin{cases} X_n \leftarrow X_{n+1} \\ Y_n \leftarrow Y_{n+1} \end{cases}$ are Hurewicz fibrations and the vertical arrows $\phi_n : X_n \to Y_n$ are homotopy equivalences—then the induced map $\phi : \lim X_n \to \lim Y_n$ is a homotopy equivalence.

[The mapping cylinder is a functor **TOP**$(\to) \to$ **TOP**, so there is an arrow $\pi_n : M_{\phi_{n+1}} \to M_{\phi_n}$. Use §3, Proposition 17 to construct a commutative triangle

$$\begin{tikzcd} X_0 \arrow[r, "\mathrm{id}"] \arrow[d, "i"'] & X_0 \\ M_{\phi_0} \arrow[ur, "r_0"'] & \end{tikzcd}$$

. The lemma then provides a filler $r_1 : M_{\phi_1} \to X_1$ for

$$\begin{tikzcd} X_1 \arrow[r, "\mathrm{id}"] \arrow[d, "i"'] & X_1 \arrow[d] \\ M_{\phi_1} \arrow[r, "r_0 \circ \pi_0"'] & X_0 \end{tikzcd}$$

, hence, by induction, a filler $r_{n+1} : M_{\phi_{n+1}} \to X_{n+1}$ for

$$\begin{tikzcd} X_{n+1} \arrow[r, "\mathrm{id}"] \arrow[d, "i"'] & X_{n+1} \arrow[d] \\ M_{\phi_{n+1}} \arrow[r, "r_n \circ \pi_n"'] & X_n \end{tikzcd}$$

. Give the composite $Y_n \xrightarrow{j} M_{\phi_n} \xrightarrow{r_n} X_n$ a name, say ψ_n, and take limits to get a left homotopy inverse ψ for ϕ.]

PROPOSITION 12 Let A be a closed subspace of Y and assume that the inclusion $A \to Y$ is a cofibration. Suppose that $p : X \to B$ is a Hurewicz fibration—then every commutative diagram

$$\begin{tikzcd} i_0 Y \cup IA \arrow[r, "F"] \arrow[d] & X \arrow[d, "p"] \\ IY \arrow[r, "h"'] & B \end{tikzcd}$$

has a filler $H : IY \to X$.

[Quote the lemma: $i_0 Y \cup IA$ is a strong deformation retract of IY (cf. p. 3–6) and $i_0 Y \cup IA$ is a zero set in IY.]

Application: Let $p : X \to B$ be a Hurewicz fibration, where B is a LCH space. Suppose that the inclusion $O \to B$ is a closed cofibration—then the arrow of restriction $\mathrm{sec}_B(X) \to \mathrm{sec}_O(X_O)$ is a Hurewicz fibration.

EXAMPLE (Vertical Homotopies) Let $p : X \to B$ be a Hurewicz fibration. Suppose that s', $s'' \in \sec_B(X)$ are homotopic—then s', s'' are vertically homotopic.

[Take any homotopy $H : IB \to X$ between s' and s''. Define $G : IB \to X$ by $G(b,t) = \begin{cases} H(b,2t) & (0 \leq t \leq 1/2) \\ s'' \circ p \circ H(b, 2-2t) & (1/2 \leq t \leq 1) \end{cases}$. Since $p \circ G(b,t) = p \circ G(b, 1-t)$, it follows that $p \circ G$ is homotopic $\operatorname{rel} B \times \{0,1\}$ to the projection $B \times [0,1] \to B$.]

LEMMA Let A be a closed subspace of Y and assume that the inclusion $A \to Y$ is a cofibration. Suppose that $p : X \to B$ is a Hurewicz fibration. Let $F : i_0 Y \cup IA \to X$ be a continuous function such that $\forall\, a \in A : p \circ F(a,t) = p \circ F(a,0)$ $(0 \leq t \leq 1)$—then there exists a continuous function $H : IY \to X$ which extends F such that $\forall\, y \in Y$: $p \circ H(y,t) = p \circ H(y,0)$ $(0 \leq t \leq 1)$.

[Choose $\phi \in C(Y,[0,1]) : A = \phi^{-1}(0)$ and fix a retraction $r : IY \to i_0 Y \cup IA$. Put $f = p \circ F \circ r$. Define $G \in C(IY, PB)$ as follows: $G(y,t)(T) =$ (i) $f(y, (t\phi(y) - T(2 - \phi(y)))/\phi(y))$ $(0 \leq T \leq t\phi(y)/2$ & $\phi(y) \neq 0)$; (ii) $f(y,t)$ $(0 \leq T \leq t\phi(y)/2$ & $\phi(y) = 0)$; (iii) $f(y, t\phi(y) - T)$ $(t\phi(y)/2 \leq T \leq t\phi(y))$; (iv) $f(y,0)$ $(t\phi(y) \leq T \leq 1)$. Take a lifting function $\Lambda : W_p \to PX$ and set $H(y,t) = \Lambda(F \circ r(y,t),\ G(y,t))(t\phi(y))$.]

LIFTING PRINCIPLE Let $p : X \to B$ be a Hurewicz fibration. Let A be a subspace of X and suppose that the inclusion $A \to X$ is a closed cofibration. View A as an object in **TOP**$/B$ with projection $p_A = p|A$ and assume that $p_A : A \to B$ is a Hurewicz fibration. Let $\Lambda_A : W_{p_A} \to PA$ be a lifting function—then there exists a lifting function $\Lambda_X : W_p \to PX$ such that $\Lambda_X | W_{p_A} = \Lambda_A$.

[The inclusion $W_{p_A} \to W_p$ is a closed cofibration (cf. p. 4–16). Therefore the inclusion $i_0 W_p \cup IW_{p_A} \to IW_p$ is a closed cofibration (cf. p. 3–6 or §3, Proposition 7). Fix a lifting function $\Lambda : W_p \to PX$. Define a continuous function $F : i_0 IW_p \cup I(i_0 W_p \cup IW_{p_A}) \to X$ by $F((x,\tau),t,T) =$ (i) $\Lambda(x,\tau)(t)$ $(T = 0$ & $(x,\tau) \in W_p)$; (ii) x $(t = 0$ & $(x,\tau) \in W_p)$; (iii) $\Lambda_A(a,\tau)(t)$ $(0 \leq t \leq T$ & $(a,\tau) \in W_{p_A})$; (iv) $\Lambda(\Lambda_A(a,\tau)(T), \tau * T)(t - T)$ $(T \leq t \leq 1$ & $(a,\tau) \in W_{p_A})$. Here, $\tau * T(t) = \begin{cases} \tau(t+T) & (t \leq 1-T) \\ \tau(1) & (t \geq 1-T) \end{cases}$. Apply the lemma to get a continuous function $H : I^2 W_p \to X$ which extends F such that $\forall\, ((x,\tau),t) \in IW_p : p \circ H((x,\tau),t,T) = p \circ H((x,\tau),t,0)$. Put $\Lambda_X(x,\tau)(t) = H((x,\tau),t,1)$—then $\Lambda_X : W_p \to PX$ is a lifting function that restricts to Λ_A.]

PROPOSITION 13 Let X be in **TOP**$/B$. Suppose that $X = A_1 \cup A_2$, where $\begin{cases} A_1 \\ A_2 \end{cases}$ are closed and the inclusions $A_0 = A_1 \cap A_2 \to \begin{cases} A_1 \\ A_2 \end{cases}$ are cofibrations. Assume: $\begin{cases} p_1 = p_{A_1} : A_1 \to B \\ p_2 = p_{A_2} : A_2 \to B \end{cases}$ & $p_0 = p_{A_0} : A_0 \to B$ are Hurewicz fibrations—then $p : X \to B$ is a Hurewicz fibration.

[Choose a lifting function $\Lambda_0 : W_{p_0} \to PA_0$. Use the lifting principle to secure lifting functions $\begin{cases} \Lambda_1 : W_{p_1} \to PA_1 \\ \Lambda_2 : W_{p_2} \to PA_2 \end{cases}$ such that $\begin{cases} \Lambda_1 | W_{p_0} = \Lambda_0 \\ \Lambda_2 | W_{p_0} = \Lambda_0 \end{cases}$. Define a lifting function $\Lambda : W_p \to PX$ by $\Lambda(x,\tau) = \begin{cases} \Lambda_1(x,\tau) & ((x,\tau) \in W_{p_1}) \\ \Lambda_2(x,\tau) & ((x,\tau) \in W_{p_2}) \end{cases}$ and cite Proposition 8.]

FACT (Mayer-Vietoris Condition) Suppose that $B = B_1 \cup B_2$, where $\begin{cases} B_1 \\ B_2 \end{cases}$ are closed and the inclusions $B_0 = B_1 \cap B_2 \to \begin{cases} B_1 \\ B_2 \end{cases}$ are cofibrations. Let $\begin{cases} X_1 \to B_1 \\ X_2 \to B_2 \end{cases}$ be Hurewicz fibrations. Assume: $\begin{cases} X_1 | B_0 \\ X_2 | B_0 \end{cases}$ have the same fiber homotopy type—then there exists a Hurewicz fibration $X \to B$ such that $\begin{cases} X_1 \ \& \ X | B_1 \\ X_2 \ \& \ X | B_2 \end{cases}$ have the same fiber homotopy type.

FACT Let

$$\begin{tikzcd}
X_0 \arrow[r, "p_0"] \arrow[d] & B_0 \arrow[d] & Y_0 \arrow[l, "q_0"'] \arrow[d] \\
X \arrow[r, "p"'] & B & Y \arrow[l, "q"]
\end{tikzcd}$$

be a commutative diagram in which the vertical arrows are inclusions and closed cofibrations. Assume that the projections $\begin{cases} p_0 \\ p \end{cases}$ are Hurewicz fibrations—then the induced map $X_0 \times_{B_0} Y_0 \to X \times_B Y$ is a closed cofibration.

[The inclusion $p^{-1}(B_0) \to X$ is a closed cofibration (cf. Proposition 11). Since X_0 is contained in $p^{-1}(B_0)$ and since the inclusion $X_0 \to X$ is a closed cofibration, the inclusion $X_0 \to p^{-1}(B_0)$ is a closed cofibration (cf. §3, Proposition 9). Proposition 13 then implies that the arrow $i_0 p^{-1}(B_0) \cup IX_0 \to B_0$ is a Hurewicz fibration. Consequently (cf. Proposition 12), the commutative diagram

$$\begin{tikzcd}
i_0 p^{-1}(B_0) \cup IX_0 \arrow[r, "\mathrm{id}"] \arrow[d] & i_0 p^{-1}(B_0) \cup IX_0 \arrow[d] \\
Ip^{-1}(B_0) \arrow[r] & B_0
\end{tikzcd}$$

has a filler $r : Ip^{-1}(B_0) \to i_0 p^{-1}(B_0) \cup IX_0$. Therefore the inclusion $X_0 \times_{B_0} Y_0 \to p^{-1}(B_0) \times_B Y_0$ is a closed cofibration. On the other hand, the projection $X \times_B Y \to Y$ is a Hurewicz fibration (cf. Proposition 4) and the inclusion $Y_0 \to Y$ is a closed cofibration, so the inclusion $p^{-1}(B_0) \times_B Y_0 \to X \times_B Y$ is a closed cofibration (cf. Proposition 11).]

Application: Consider the 2-sink $X \xrightarrow{p} B \xleftarrow{q} Y$, where $p : X \to B$ is a Hurewicz fibration. Assume: The inclusions $\Delta_X \to X \times X$, $\Delta_B \to B \times B$, $\Delta_Y \to Y \times Y$ are closed cofibrations—then the diagonal embedding $X \times_B Y \to (X \times_B Y) \times (X \times_B Y)$ is a closed cofibration.

Let $X \xrightarrow{p} B \xleftarrow{q} Y$ be a 2-sink—then the fiber join $X *_B Y$ is the double mapping cylinder of the 2-source $X \xleftarrow{\xi} X \times_B Y \xrightarrow{\eta} Y$. The fiber homotopy type of $X *_B Y$ depends only on the fiber homotopy types of X and Y. There is a projection $X *_B Y \rightarrow B$ and the fiber over b is $X_b * Y_b$. Examples: (1) The fiber join of $X \xrightarrow{p} B \leftarrow B \times \{0\}$ is $\Gamma_B X$, the fiber cone of X; (2) The fiber join of $X \xrightarrow{p} B \leftarrow B \times \{0,1\}$ is $\Sigma_B X$, the fiber suspension of X; (3) The fiber join of $B \times T_1 \rightarrow B \leftarrow B \times T_2$ is $B \times (T_1 * T_2)$; (4) The fiber join of $\{b_0\} \rightarrow B \xleftarrow{p} X$ is the mapping cone C_{b_0} of the inclusion $X_{b_0} \rightarrow X$.

Let X be in **TOP**$/B$—then $\Gamma_B X$ can be identified with the mapping cylinder M_p and $\Sigma_B X$ can be identified with the double mapping cylinder $M_{p,p}$.

LEMMA Let $f \in C_B(X,Y)$. Suppose that $\begin{cases} p : X \rightarrow B \\ q : Y \rightarrow B \end{cases}$ are Hurewicz fibrations—then the projection $\pi : M_f \rightarrow B$ is a Hurewicz fibration.

[Fix lifting functions $\begin{cases} \Lambda_X : W_p \rightarrow PX \\ \Lambda_Y : W_q \rightarrow PY \end{cases}$. Define a lifting function $\Lambda : W_\pi \rightarrow PM_f$ as follows: Given $((x,t),\tau) \in IX \times_B PB$, put

$$\Lambda((x,t),\tau)(T) = \begin{cases} (\Lambda_X(x,\tau)(T), (t-1/2)(1+T) + (1-T)/2) & (1/2 \leq t \leq 1) \\ (\Lambda_X(x,\tau)(T), t - T/2) & (0 \leq t \leq 1/2 \ \& \ T \leq 2t) \\ \Lambda_Y(f(\Lambda_X(x,\tau)(2t)), \tau_{2t})(T-2t) & (0 \leq t \leq 1/2 \ \& \ T \geq 2t), \end{cases}$$

where $\tau_{2t}(T) = \tau(\min\{2t+T,1\})$, and given $(y,\tau) \in Y \times_B PB$, put $\Lambda(y,\tau) = \Lambda_Y(y,\tau)$.]

PROPOSITION 14 Suppose that $\begin{cases} p : X \rightarrow B \\ q : Y \rightarrow B \end{cases}$ are Hurewicz fibrations—then the projection $X *_B Y \rightarrow B$ is a Hurewicz fibration.

[Consider the pushout square

$$\begin{tikzcd} X \times_B Y \arrow[r] \arrow[d] & M_\eta \arrow[d] \\ M_\xi \arrow[r] & X *_B Y \end{tikzcd}$$

(cf. p. 3–23). Here, the arrows $X \times_B Y \rightarrow \begin{cases} M_\eta \\ M_\xi \end{cases} \rightarrow X *_B Y$ are closed cofibrations and the projections $X \times_B Y \rightarrow B$, $\begin{cases} M_\eta \\ M_\xi \end{cases} \rightarrow B$ are Hurewicz fibrations. That the projection $X *_B Y \rightarrow B$ is a Hurewicz fibration is therefore a consequence of Proposition 13.]

Application: Let $p : X \rightarrow B$ be a Hurewicz fibration—then the projections $\begin{cases} \Gamma_B X \rightarrow B \\ \Sigma_B X \rightarrow B \end{cases}$ are Hurewicz fibrations.

Let $X \xrightarrow{p} B \xleftarrow{q} Y$ be a 2-sink, where p is a Hurewicz fibration. There is a commutative diagram

$$\begin{tikzcd} X \arrow[r, "p"] \arrow[d, equal] & B \arrow[d, equal] & Y \arrow[l, "q"'] \arrow[d, "\gamma"] \\ X \arrow[r, "p"'] & B & W_q \arrow[l] \end{tikzcd}$$

and γ is a homotopy equivalence, thus the induced map $X \times_B Y \to X \times_B W_q$ is a homotopy equivalence (cf. p. 4–25). Consideration of

$$\begin{tikzcd} X \arrow[d, equal] & X \times_B Y \arrow[l] \arrow[r] \arrow[d] & Y \arrow[d, equal] \\ X & X \times_B W_q \arrow[l] \arrow[r] & Y \end{tikzcd}$$

then leads to a homotopy equivalence $X *_B Y \to X *_B W_q$ (cf. p. 3–24). Example: $\forall\ b_0 \in B$, $X *_B \Theta B$ and C_{b_0} have the same homotopy type.

Assume in addition that q is a closed cofibration and define P by the pushout square

$$\begin{tikzcd} X \times_B Y \arrow[r, "\eta"] \arrow[d, "\xi"'] & Y \arrow[d] \\ X \arrow[r] & P \end{tikzcd}$$

—then Proposition 11 implies that ξ is a closed cofibration. Therefore the arrow $X *_B Y \to P$ of §3, Proposition 18 is a homotopy equivalence. Example: $\forall\ b_0 \in B$ such that the inclusion $\{b_0\} \to B$ is a closed cofibration, $\Theta B *_B \Theta B$ and $\Theta B/\Omega B$ have the same homotopy type.

PROPOSITION 15 Suppose that $\begin{cases} p : X \to B \\ q : Y \to B \end{cases}$ are Hurewicz fibrations. Let $\phi \in C_B(X, Y)$. Assume that ϕ is a homotopy equivalence—then ϕ is a homotopy equivalence in $\mathbf{TOP}/B$.

[This is the analog of §3, Proposition 13. It is a special case of Proposition 16 below.]

Application: Let $p : X \to B$ be a homotopy equivalence—then W_p is fiberwise contractible.

[Write $p = q \circ \gamma$: p and γ are homotopy equivalences, thus so is q.]

[Note: Similar reasoning leads to another proof of Proposition 9.]

EXAMPLE Let $p : X \to B$ be a Hurewicz fibration. View PX as an object in $\mathbf{TOP}/W_p$ with projection $\lambda : PX \to W_p$—then PX is fiberwise contractible.

FACT Let $p : X \to B$ be a continuous function—then p is both a homotopy equivalence and a Hurewicz fibration iff every commutative diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d, "i"'] & X \arrow[d, "p"] \\ Y \arrow[r] & B \end{tikzcd}$$

, where i is a closed cofibration, has a filler $Y \to X$.

[To discuss the necessity, use Proposition 12, noting that X is fiberwise contractible, hence $\exists\ s \in \sec_B(X) : s \circ p \underset{B}{\simeq} \mathrm{id}_X$.]

Application: Let
$$\begin{tikzcd} X' \arrow[r] \arrow[d, "p'"'] & X \arrow[d, "p"] \\ B' \arrow[r] & B \end{tikzcd}$$
be a pullback square. Suppose that p is a Hurewicz fibration and a homotopy equivalence—then p' is a Hurewicz fibration and a homotopy equivalence.

FACT Let $i : A \rightarrow Y$ be a continuous function—then i is a closed cofibration iff every commutative diagram
$$\begin{tikzcd} A \arrow[r] \arrow[d, "i"'] & X \arrow[d, "p"] \\ Y \arrow[r] & B \end{tikzcd}$$
, where p is both a homotopy equivalence and a Hurewicz fibration, has a filler $Y \rightarrow X$.

[To establish the sufficiency, first consider
$$\begin{tikzcd} A \arrow[r] \arrow[d, "i"'] & PX \arrow[d, "p_0"] \\ Y \arrow[r] & X \end{tikzcd}$$
to see that i is a cofibration. Taking i to be an inclusion, put $X = IA \cup Y\times]0,1]$—then the restriction to X of the Hurewicz fibration $IY \rightarrow Y$ is a Hurewicz fibration (cf. p. 4–12), call it p. Since p is also a homotopy equivalence, the commutative diagram
$$\begin{tikzcd} A \arrow[r] \arrow[d, "i"'] & X \arrow[d, "p"] \\ Y \arrow[r, equal] & Y \end{tikzcd}$$
has a filler $f : Y \rightarrow X$ $(a \rightarrow (a,0)\ (a \in A))$, therefore A is a zero set in Y, thus is closed.]

FACT Let $X \xrightarrow{p} B \xleftarrow{q} Y$ be a 2-sink, where $p : X \rightarrow B$ is a Hurewicz fibration. Denote by W_* the mapping track of the projection $X *_B Y \rightarrow B$—then $X *_B W_q$ and W_* have the same fiber homotopy type.

LEMMA Suppose that $\xi \in C_B(X, E)$ is a fiberwise Hurewicz fibration. Let $f \in C(X,X) : \xi \circ f = \xi$ & $f \underset{B}{\simeq} \mathrm{id}_X$—then $\exists\ g \in C(X,X) : \xi \circ g = \xi$ & $f \circ g \underset{E}{\simeq} \mathrm{id}_X$.

[Let $H : IX \rightarrow X$ be a fiber homotopy with $H \circ i_0 = f$ and $H \circ i_1 = \mathrm{id}_X$; let $G : IX \rightarrow X$ be a fiber homotopy with $G \circ i_0 = \mathrm{id}_X$ and $\xi \circ G = \xi \circ H$. Define $F : IX \rightarrow X$ by $F(x,t) = \begin{cases} f \circ G(x, 1-2t) & (0 \leq t \leq 1/2) \\ H(x, 2t-1) & (1/2 \leq t \leq 1) \end{cases}$ and put

$$k((x,t),T) = \begin{cases} \xi \circ G(x, 1-2t(1-T)) & (0 \leq t \leq 1/2) \\ \xi \circ H(x, 1-2(1-t)(1-T)) & (1/2 \leq t \leq 1) \end{cases}$$

to get a fiber homotopy $k : I^2X \rightarrow E$ with $\xi \circ F = k \circ i_0$. Choose a fiber homotopy $K : I^2X \rightarrow X$ such that $F = K \circ i_0$ and $\xi \circ K = k$. Write $K_{(t,T)} : X \rightarrow X$ for the function $x \rightarrow K((x,t),T)$. Obviously, $K_{(0,0)} \simeq K_{(0,1)} \simeq K_{(1,1)} \simeq K_{(1,0)}$, all fiber homotopies being over E. Set $g = G \circ i_1$—then $f \circ g = F \circ i_0 = K_{(0,0)} \underset{E}{\simeq} K_{(1,0)} = F \circ i_1 = \mathrm{id}_X$.]

[Note: Take $B = *$, $E = B$, $\xi = p$, so $p : X \rightarrow B$ is a Hurewicz fibration—then the lemma asserts that $\forall\ f \in C_B(X,X)$, with $f \simeq \mathrm{id}_X$, $\exists\ g \in C_B(X,X) : f \circ g \underset{B}{\simeq} \mathrm{id}_X$.]

PROPOSITION 16 Suppose that $\begin{cases}\xi \in C_B(X,E)\\ \eta \in C_B(Y,E)\end{cases}$ are fiberwise Hurewicz fibrations. Let $\phi \in C(X,Y) : \eta \circ \phi = \xi$. Assume that ϕ is a homotopy equivalence in **TOP**$/B$—then ϕ is a homotopy equivalence in **TOP**$/E$.

[Since ξ is a fiberwise Hurewicz fibration, there exists a fiber homotopy inverse $\psi : Y \rightarrow X$ for ϕ with $\xi \circ \psi = \eta$, thus, from the lemma, $\exists\ \psi' \in C(Y,Y) : \eta \circ \psi' = \eta$ & $\phi \circ \psi \circ \psi' \underset{E}{\simeq} \mathrm{id}_Y$. This says that $\phi' = \psi \circ \psi'$ is a homotopy right inverse for ϕ over E. Repeat the argument with ϕ replaced by ϕ' to conclude that ϕ' has a homotopy right inverse ϕ'' over E, hence that ϕ' is a homotopy equivalence in **TOP**$/E$ or still, that ϕ is a homotopy equivalence in **TOP**$/E$.]

[Note: To recover Proposition 15, take $B = *$, $E = B$, $\xi = p$, and $\eta = q$.]

PROPOSITION 17 Suppose given a commutative diagram

$$\begin{tikzcd} X \arrow[r, "p"] \arrow[d, "\phi"'] & B \arrow[d, "\psi"] \\ Y \arrow[r, "q"'] & A \end{tikzcd}$$

in which $\begin{cases} p \\ q \end{cases}$ are Hurewicz fibrations and $\begin{cases} \phi \\ \psi \end{cases}$ are homotopy equivalences—then (ϕ, ψ) is a homotopy equivalence in **TOP**$(\rightarrow)$.

[This is the analog of §3, Proposition 14.]

Let $X \xrightarrow{f} Z \xleftarrow{g} Y$ be a 2-sink—then the double mapping track $W_{f,g}$ of f, g is defined by the pullback square

$$\begin{tikzcd} W_{f,g} \arrow[r] \arrow[d] & PZ \arrow[d, "p_0"', "p_1"] \\ X \times Y \arrow[r, "f \times g"'] & Z \times Z \end{tikzcd}$$

. The homotopy type of $W_{f,g}$ depends only on the homotopy classes of f and g and $W_{f,g}$ is homeomorphic to $W_{g,f}$. There are Hurewicz fibrations $\begin{cases} p : W_{f,g} \rightarrow X \\ q : W_{f,g} \rightarrow Y \end{cases}$. The diagram

$$\begin{tikzcd} W_{f,g} \arrow[r, "q"] \arrow[d, "p"'] & Y \arrow[d, "g"] \\ X \arrow[r, "f"'] & Z \end{tikzcd}$$

is homotopy commutative and if the diagram

$$\begin{tikzcd} W \arrow[r, "\eta"] \arrow[d, "\xi"'] & Y \arrow[d, "g"] \\ X \arrow[r, "f"'] & Z \end{tikzcd}$$

is homotopy commutative, then there exists a $\phi : W \rightarrow W_{f,g}$ such that $\begin{cases} \xi = p \circ \phi \\ \eta = q \circ \phi \end{cases}$.

[Note: The commutative diagram

$$\begin{tikzcd} W_{f,g} \arrow[r] \arrow[d] & Y \arrow[d, "g"] \\ W_f \arrow[r, "q"'] & Z \end{tikzcd}$$

is a pullback square $(f = q \circ s)$.]

FACT Let $X \xrightarrow{f} Z \xleftarrow{g} Y$ be a 2-sink—then the assignment $(x, y, \tau) \rightarrow \tau(1/2)$ defines a Hurewicz fibration $W_{f,g} \rightarrow Z$.

[Let $\begin{cases} W_f^+ = \{(x,\tau) : f(x) = \tau(0), \tau \in C([0,1/2], Z)\} \\ W_g^- = \{(y,\tau) : g(y) = \tau(1), \tau \in C([1/2,1], Z)\} \end{cases}$. The projections $\begin{cases} W_f^+ \rightarrow Z \\ (x,\tau) \rightarrow \tau(1/2) \end{cases}$, $\begin{cases} W_g^- \rightarrow Z \\ (y,\tau) \rightarrow \tau(1/2) \end{cases}$ are Hurewicz fibrations and the commutative diagram

$$\begin{tikzcd} W_{f,g} \arrow[r] \arrow[d] & W_g^- \arrow[d] \\ W_f^+ \arrow[r] & Z \end{tikzcd}$$

is a pullback square.]

Every 2-sink $X \xrightarrow{f} Z \xleftarrow{g} Y$ determines a pullback square

$$\begin{tikzcd} P \arrow[r, "\eta"] \arrow[d, "\xi"'] & Y \arrow[d, "g"] \\ X \arrow[r, "f"'] & Z \end{tikzcd}$$

and there is an arrow $\phi : P \rightarrow W_{f,g}$ characterized by the conditions $\begin{cases} \xi = p \circ \phi \\ \eta = q \circ \phi \end{cases}$ & $P \xrightarrow{\phi} W_{f,g} \rightarrow PZ = \begin{cases} j \circ f \circ \xi \\ \quad \| \\ j \circ g \circ \eta \end{cases}$.

PROPOSITION 18 If f is a Hurewicz fibration, then $\phi : P \rightarrow W_{f,g}$ is a homotopy equivalence in **TOP**$/Y$.

[Use Proposition 9 and the fact that the pullback of a fiber homotopy equivalence is a fiber homotopy equivalence.]

Application: Let $p : X \rightarrow B$ be a Hurewicz fibration. Suppose that $\begin{cases} \Phi_1' \\ \Phi_2' \end{cases} \in C(B', B)$ are homotopic—then $\begin{cases} X_1' \\ X_2' \end{cases}$ have the same homotopy type over B'.

For example, under the assumption that $p : X \rightarrow B$ is a Hurewicz fibration, if $\Phi' : B' \rightarrow B$ is homotopic to the constant map $B' \rightarrow b_0$, then X' is fiber homotopy equivalent to $B' \times X_{b_0}$.

FACT Suppose that $p : X \rightarrow B$ is a Hurewicz fibration. Let $\Phi' : B' \rightarrow B$ be a homotopy equivalence—then the arrow $X' \rightarrow X$ is a homotopy equivalence.

Denote by $|\mathrm{id}, \Delta|_{\mathbf{TOP}}$ the comma category corresponding to the identity functor id on **TOP** × **TOP** and the diagonal functor $\Delta : \mathbf{TOP} \rightarrow \mathbf{TOP} \times \mathbf{TOP}$. So, an object in $|\mathrm{id}, \Delta|_{\mathbf{TOP}}$ is a 2-sink $X \xrightarrow{f} Z \xleftarrow{g} Y$ and a morphism of 2-sinks is a commutative diagram

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d] & Z \arrow[d] & Y \arrow[l, "g"'] \arrow[d] \\ X' \arrow[r, "f'"'] & Z' & Y' \arrow[l, "g'"] \end{tikzcd}$$

. The double mapping

track is a functor $|\mathrm{id}, \Delta|_{\mathbf{TOP}} \to \mathbf{TOP}$. It has a left adjoint $\mathbf{TOP} \to |\mathrm{id}, \Delta|_{\mathbf{TOP}}$, viz. the functor that sends X to the 2-sink $X \xrightarrow{i_0} IX \xleftarrow{i_1} X$.

FACT Let
$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d] & Z \arrow[d] & Y \arrow[l, "g"'] \arrow[d] \\ X' \arrow[r, "f'"'] & Z' & Y' \arrow[l, "g'"] \end{tikzcd}$$
be a commutative diagram in which the vertical arrows are homotopy equivalences—then the arrow $W_{f,g} \to W_{f',g'}$ is a homotopy equivalence.

Application: Suppose that $\begin{cases} p : X \to B \\ p' : X' \to B' \end{cases}$ are Hurewicz fibrations. Let $\begin{cases} g : Y \to B \\ g' : Y' \to B' \end{cases}$ be continuous functions. Assume that the diagram
$$\begin{tikzcd} X \arrow[r, "p"] \arrow[d] & B \arrow[d] & Y \arrow[l, "g"'] \arrow[d] \\ X' \arrow[r, "p'"'] & B' & Y' \arrow[l, "g'"] \end{tikzcd}$$
commutes and that the vertical arrows are homotopy equivalences—then the induced map $X \times_B Y \to X' \times_{B'} Y'$ is a homotopy equivalence.

EXAMPLE Suppose given a commutative diagram
$$\begin{tikzcd} X \arrow[r, "p"] \arrow[d, "\phi"'] & B \arrow[d, "\psi"] \\ Y \arrow[r, "q"'] & A \end{tikzcd}$$
in which $\begin{cases} p \\ q \end{cases}$ are Hurewicz fibrations and $\begin{cases} \phi \\ \psi \end{cases}$ are homotopy equivalences—then $\forall\ b \in B$, the induced map $X_b \to Y_{\psi(b)}$ is a homotopy equivalence.

[Note: Let $f : X \to Y$ be a homotopy equivalence, fix $x_0 \in X$ and put $y_0 = f(x_0)$, form the commutative diagram
$$\begin{tikzcd} \Theta X \arrow[r, "p_1"] \arrow[d] & X \arrow[d] & \{x_0\} \arrow[l] \arrow[d] \\ \Theta Y \arrow[r, "p_1"'] & Y & \{y_0\} \arrow[l] \end{tikzcd}$$
, and conclude that the arrow $\Omega X \to \Omega Y$ is a homotopy equivalence.]

Given a 2-sink $X \xrightarrow{p} B \xleftarrow{q} Y$, let $X \square_B Y$ be the double mapping cylinder of the 2-source $X \leftarrow W_{p,q} \to Y$. It is an object in $\mathbf{TOP}/B$ with projection $\begin{cases} x \to p(x) \\ y \to q(y) \end{cases}$, $((x, y, \tau), t) \to \tau(t)$.

FACT There is a homotopy equivalence $X \square_B Y \xrightarrow{\phi} W_p *_B W_q$.

[Define ϕ by $\begin{cases} \phi(x) = \gamma(x) \\ \phi(y) = \gamma(y) \end{cases}$ & $\phi((x, y, \tau), t) = ((x, \tau_t), (y, \overline{\tau}_t), t)$, where $\tau_t(T) = \tau(tT)$ and $\overline{\tau}_t(T) = \tau(tT + 1 - T)$.]

[Note: More is true if $p : X \to B$ is a Hurewicz fibration: $X \square_B Y$ and $X *_B Y$ have the same homotopy type. Indeed, $W_p *_B W_q$ has the same fiber homotopy type as $X *_B W_q$ which in turn has the same homotopy type as $X *_B Y$ (cf. p. 4–20 ff.).]

Application: $\forall\ b_0 \in B$, $\Sigma\Omega B$ and $\Theta B *_B \Theta B$ have the same homotopy type.

[Note: The suspension is taken in **TOP**, not $\mathbf{TOP}_*$.]

Given $f \in C_B(X,Y)$, let W be the subspace of $X \times PY$ consisting of the pairs $(x,\tau) : f(x) = \tau(0)$ and $p(x) = q(\tau(t))$ $(0 \leq t \leq 1)$—then W is in $\mathbf{TOP}/Y$ with projection $(x,\tau) \rightarrow \tau(1)$ and is fiberwise contractible if f is a fiber homotopy equivalence (cf. Proposition 16).

[Note: W is an object in $\mathbf{TOP}/B$ with projection $(x,\tau) \rightarrow p(x)$. Viewed as an object in $\mathbf{TOP}/Y$, its projection $(x,\tau) \rightarrow \tau(1)$ is therefore a morphism in $\mathbf{TOP}/B$ and as such, is a fiberwise Hurewicz fibration.]

LEMMA f admits a right fiber homotopy inverse iff $\text{sec}_Y(W) \neq \emptyset$.

PROPOSITION 19 Let $f \in C_B(X,Y)$. Suppose that there exists a numerable covering $\mathcal{O} = \{O_i : i \in I\}$ of B such that $\forall\ i$, $f_{O_i} : X_{O_i} \rightarrow Y_{O_i}$ is a fiber homotopy equivalence—then f is a fiber homotopy equivalence.

[It need only be shown that $\text{sec}_Y(W) \neq \emptyset$. For then, by the lemma, f has a right fiber homotopy inverse g and, repeating the argument, g has a right fiber homotopy inverse h, which means that g is a fiber homotopy equivalence, thus so is f. This said, work with $f_{O_i} \in C_{O_i}(X_{O_i}, Y_{O_i})$ and, as above, form $W_{O_i} \subset X_{O_i} \times PY_{O_i}$. Obviously, $W|Y_{O_i} = W_{O_i}$. The assumption that f_{O_i} is a fiber homotopy equivalence implies that W_{O_i} is fiberwise contractible, hence has the SEP. But $\{Y_{O_i} : i \in I\}$ is a numerable covering of Y. Therefore, on the basis of the section extension theorem, W has the SEP. In particular: $\text{sec}_Y(W) \neq \emptyset$.]

Application: Let X be in $\mathbf{TOP}/B$. Suppose that there exists a numerable covering $\mathcal{O} = \{O_i : i \in I\}$ of B such that $\forall\ i$, X_{O_i} is fiberwise contractible—then X is fiberwise contractible.

PROPOSITION 20 Let $\begin{cases} p : X \rightarrow B \\ q : Y \rightarrow B \end{cases}$ be Hurewicz fibrations, where B is numerably contractible. Suppose that $f \in C_B(X,Y)$ has the property that $f_b : X_b \rightarrow Y_b$ is a homotopy equivalence at one point b in each path component of B—then $f : X \rightarrow Y$ is a fiber homotopy equivalence.

[Fix a numerable covering $\mathcal{O} = \{O_i : i \in I\}$ of B for which the inclusions $O_i \rightarrow B$ are inessential, say homotopic to $O_i \rightarrow b_i$, where $f_{b_i} : X_{b_i} \rightarrow Y_{b_i}$ is a homotopy equivalence—then $\forall\ i$, $f_{O_i} : X_{O_i} \rightarrow Y_{O_i}$ is a fiber homotopy equivalence (cf. p. 4–24), so Proposition 19 is applicable.]

EXAMPLE Take $B = \{0\} \cup \{1/n : n = 1,2,\ldots\}$, $T = B \cup \{n : n = 1,2,\ldots\}$, and put $X = B \times T$. Observe that B is not numerably contractible. Let $k = 1,2,\ldots,\infty$, $l = 0,1,2,\ldots$, and define

$f \in C_B(X,X)$ as follows: (i) $f(1/k,l) = (1/k,l)$ $(l < k)$, $(1/k,1/k)$ $(l = k \neq 1)$, $(1/k,l-1)$ $(l > k)$; (ii) $f(1/k,1/l) = (1/k,1/l)$ $(0 < l < k)$, $(1/k,1/(l+1))$ $(l \geq k)$—then f is bijective and $\forall\, b \in B$, $f_b : X_b \rightarrow X_b$ is a homeomorphism $(X_b = \{b\} \times T)$. Nevertheless, f is not a fiber homotopy equivalence. For if it were, then f would have to be a homeomorphism, an impossibility (f^{-1} is not continuous at $(0,0)$).

EXAMPLE (Delooping Homotopy Equivalences) Suppose that $\begin{cases} X \\ Y \end{cases}$ are path connected and numerably contractible. Let $f : X \rightarrow Y$ be a continuous function. Fix $x_0 \in X$ and put $y_0 = f(x_0)$—then $f : X \rightarrow Y$ is a homotopy equivalence iff $\Omega f : \Omega X \rightarrow \Omega Y$ is a homotopy equivalence. In fact, the necessity is true without any restriction on X or Y (cf. p. 4–25). Turning to the sufficiency, write $f = q \circ s$, where $q : W_f \rightarrow Y$. Since s is a homotopy equivalence, one need only deal with q. Form the pullback square

$$\begin{tikzcd} X \times_Y \Theta Y \arrow[r] \arrow[d] & \Theta Y \arrow[d, "p_1"] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

The map $\begin{cases} \Theta X \rightarrow X \times_Y \Theta Y \\ \sigma \rightarrow (\sigma(1), f \circ \sigma) \end{cases}$ is a morphism in **TOP**$/X$ which, when restricted to the fibers over x_0, is Ωf, thus is a fiber homotopy equivalence (cf. Proposition 20). In particular: $X \times_Y \Theta Y$ is contractible. Consider now the commutative triangle

$$\begin{tikzcd} W_f \arrow[rr] \arrow[dr, "q"'] & & PY \arrow[dl, "p_1"] \\ & Y & \end{tikzcd}$$

The fiber of p_1 over y_0 is contractible; on the other hand, the fiber of q over y_0 is homeomorphic to $X \times_Y \Theta Y$ (parameter reversal). The arrow $W_f \rightarrow PY$ is therefore a homotopy equivalence (cf. Proposition 20). But p_1 is a homotopy equivalence, hence so is q.

EXAMPLE (H Groups) In any H group (= cogroup object in **HTOP**$_*$), the operations of left and right translation are homotopy equivalences (so all path components have the same homotopy type). Conversely, let (X, x_0) be a nondegenerate homotopy associative H space with the property that the operations of left and right translation are homotopy equivalences. Assume: X is numerably contractible—then X admits a homotopy inverse, thus is an H group. To see this, consider the shearing map sh : $\begin{cases} X \times X & \rightarrow & X \times X \\ (x,y) & \rightarrow & (x,xy) \end{cases}$. Agreeing to view $X \times X$ as an object in **TOP**$/X$ via the first projection, Proposition 20 implies that sh is a homotopy equivalence over X. Therefore sh is a homotopy equivalence or still, sh is a pointed homotopy equivalence, $(X \times X, (x_0, x_0))$ being nondegenerate (cf. p. 3–35). Consequently, X is an H group.

[Note: If (X, x_0) is a homotopy associative H space and if $\pi_0(X)$ is a group, then the operations of left and right translation are homotopy equivalences.]

Example: Let K be a compact ANR. Denote by $HE(K)$ the subspace of $C(K,K)$ (compact open topology) consisting of the homotopy equivalences—then $HE(K)$ is open in $C(K,K)$, hence is an ANR (cf. §6, Proposition 6). In particular: $(HE(K), \mathrm{id}_K)$ is wellpointed (cf. p. 6–14) and numerably contractible (cf. p. 3–13). Because $HE(K)$ is a topological semigroup with unit under composition and $\pi_0(HE(K))$ is a group, it follows that $HE(K)$ is an H group.

EXAMPLE (Small Skeletons) In algebraic topology, it is often necessary to determine whether a given category has a small skeleton. For instance, if B is a connected, locally path connected, locally simply connected space, then the full subcategory of $\mathbf{TOP}/B$ whose objects are the covering projections $X \rightarrow B$ has a small skeleton. Here is a less apparent example. Fix a nonempty topological space F. Given a numerably contractible topological space B, let $\mathbf{FIB}_{B,F}$ be the category whose objects are the Hurewicz fibrations $X \rightarrow B$ such that $\forall\, b \in B$, X_b has the homotopy type of F, and whose morphisms $X \rightarrow Y$ are the fiber homotopy classes $[f] : X \rightarrow Y$. The functor $\mathbf{FIB}_{B,F} \rightarrow \mathbf{FIB}_{B',F}$ determined by a homotopy equivalence $\Phi' : B' \rightarrow B$ induces a bijection $\text{Ob}\,\overline{\mathbf{FIB}}_{B,F} \rightarrow \text{Ob}\,\overline{\mathbf{FIB}}_{B',F}$, hence $\mathbf{FIB}_{B,F}$ has a small skeleton iff this is the case of $\mathbf{FIB}_{B',F}$.

Claim: Consider a 2-source $B_1 \xleftarrow{\phi_1} B_0 \xrightarrow{\phi_2} B_2$, where B_0, $\begin{cases} B_1 \\ B_2 \end{cases}$ are numerably contractible. Suppose that $\mathbf{FIB}_{B_0,F}$, $\begin{cases} \mathbf{FIB}_{B_1,F} \\ \mathbf{FIB}_{B_2,F} \end{cases}$ have small skeletons—then $\mathbf{FIB}_{M_{\phi_1,\phi_2},F}$ has a small skeleton.

[Observing that the double mapping cylinder M_{ϕ_1,ϕ_2} is numerably contractible, write $\begin{cases} \phi_1 = r_1 \circ i_1 \\ \phi_2 = r_2 \circ i_2 \end{cases}$, where $\begin{cases} r_1 \\ r_2 \end{cases}$ are homotopy equivalences and $\begin{cases} i_1 \\ i_2 \end{cases}$ are closed cofibrations (cf. §3, Proposition 16). There is a commutative diagram

$$\begin{tikzcd} M_{i_1} \arrow[d, "r_1"'] & B_0 \arrow[l, "i_1"'] \arrow[r, "i_2"] \arrow[d, equal] & M_{i_2} \arrow[d, "r_2"] \\ B_1 & B_0 \arrow[l, "\phi_1"] \arrow[r, "\phi_2"'] & B_2 \end{tikzcd}$$

and the arrow $M_{i_1,i_2} \rightarrow M_{\phi_1,\phi_2}$ is a homotopy equivalence (cf. p. 3–24). Thus one can assume that $\begin{cases} \phi_1 \\ \phi_2 \end{cases}$ are closed cofibrations. But then if B is defined by the pushout square

$$\begin{tikzcd} B_0 \arrow[r, "\phi_2"] \arrow[d, "\phi_1"'] & B_2 \arrow[d] \\ B_1 \arrow[r] & B \end{tikzcd}$$

, the arrow $M_{\phi_1,\phi_2} \rightarrow B$ is a homotopy equivalence (cf. §3, Proposition 18). So, with $B_0 = B_1 \cap B_2$, $\begin{cases} B_1 \subset B \\ B_2 \subset B \end{cases}$, take an X in $\mathbf{FIB}_{B,F}$ and put $X_0 = X|B_0$, $\begin{cases} X_1 = X|B_1 \\ X_2 = X|B_2 \end{cases}$ to get a commutative diagram

$$\begin{tikzcd} X_1 \arrow[d] & X_0 \arrow[l, "\psi_1"'] \arrow[r, "\psi_2"] \arrow[d] & X_2 \arrow[d] \\ B_1 & B_0 \arrow[l, "\phi_1"] \arrow[r, "\phi_2"'] & B_2 \end{tikzcd}$$

in which $\begin{cases} \psi_1 \\ \psi_2 \end{cases}$ are closed cofibrations (cf. Proposition 11). In the skeletons of $\mathbf{FIB}_{B_0,F}$, $\begin{cases} \mathbf{FIB}_{B_1,F} \\ \mathbf{FIB}_{B_2,F} \end{cases}$, choose objects Y_0, $\begin{cases} Y_1 \\ Y_2 \end{cases}$ and fiber homotopy equivalences $f_0 : Y_0 \rightarrow X_0$, $\begin{cases} f_1 : Y_1 \rightarrow X_1 \\ f_2 : Y_2 \rightarrow X_2 \end{cases}$: $\begin{cases} p_1 \circ f_1 = q_1 \\ p_2 \circ f_2 = q_2 \end{cases}$ (obvious notation). Let $\begin{cases} g_1 : X_1 \rightarrow Y_1 \\ g_2 : X_2 \rightarrow Y_2 \end{cases}$ be a fiber homotopy inverse for $\begin{cases} f_1 \\ f_2 \end{cases}$. Set $\begin{cases} F_1 = g_1 \circ \psi_1 \circ f_0 \\ F_2 = g_2 \circ \psi_2 \circ f_0 \end{cases}$: $\begin{cases} f_1 \circ F_1 \simeq \psi_1 \circ f_0 \\ f_2 \circ F_2 \simeq \psi_2 \circ f_0 \end{cases}$. Write $\begin{cases} F_1 = \Psi_1 \circ l_1 \\ F_2 = \Psi_2 \circ l_2 \end{cases}$, where $\begin{cases}\Psi_1\\ \Psi_2\end{cases}$ are Hurewicz fibrations and homotopy equivalences and $\begin{cases} l_1\\ l_2\end{cases}$ are closed cofibrations (cf. p. 4–12), say $\begin{cases} l_1 : Y_0 \to \overline{Y}_1 \ \ \& \ \ \Psi_1 : \overline{Y}_1 \to Y_1\\ l_2 : Y_0 \to \overline{Y}_2 \ \ \& \ \ \Psi_2 : \overline{Y}_2 \to Y_2\end{cases}$. Here: $\begin{cases}\overline{Y}_1\\ \overline{Y}_2\end{cases}$ is an object in $\begin{cases}\mathbf{TOP}/B_1\\ \mathbf{TOP}/B_2\end{cases}$ with projection $\begin{cases} q_1\circ\Psi_1\\ q_2\circ\Psi_2\end{cases}$ and $\begin{cases} f_1\circ\Psi_1 : \overline{Y}_1 \to X_1\\ f_2\circ\Psi_2 : \overline{Y}_2 \to X_2\end{cases}$ is a fiber homotopy equivalence (cf. Proposition 15). Change $\begin{cases} f_1\circ\Psi_1\\ f_2\circ\Psi_2\end{cases}$ by a homotopy over $\begin{cases} B_1\\ B_2\end{cases}$ into a map $\begin{cases} G_1\\ G_2\end{cases}$ such that $\begin{cases} G_1\circ l_1 = \psi_1\circ f_0\\ G_2\circ l_2 = \psi_2\circ f_0\end{cases}$. Form the pushout square

$$\begin{tikzcd} Y_0 \arrow[r, "l_2"] \arrow[d, "l_1"'] & \overline{Y}_2 \arrow[d] \\ \overline{Y}_1 \arrow[r] & Y \end{tikzcd}$$

—then Y is in $\mathbf{TOP}/B$ and there is a fiber homotopy equivalence $f : Y \to X$, i.e., this process picks up all the isomorphism classes in $\mathbf{FIB}_{B,F}$.]

Example: Let B be a CW complex—then B is numerably contractible (cf. p. 3–13) and $\mathbf{FIB}_{B,F}$ has a small skeleton. In fact, $B = \operatorname{colim} B^{(n)}$, so by induction, $\mathbf{FIB}_{B^{(n)},F}$ has a small skeleton $\forall\, n$. On the other hand, B and $\operatorname{tel} B$ have the same homotopy type (cf. p. 3–12) and $\operatorname{tel} B$ is a double mapping cylinder calculated on the $B^{(n)}$ (cf. p. 3–23).

FACT Let X be in $\mathbf{TOP}/B$. Suppose that $\mathcal{U} = \{U_i : i \in I\}$ is a numerable covering of X such that for every nonempty finite subset $F \subset I$, the restriction of p to $\bigcap\limits_{i\in F} U_i$ is a Hurewicz fibration—then $p : X \to B$ is a Hurewicz fibration.

[Equip I with a well ordering $<$ and use the Segal-Stasheff construction to produce a lifting function $\Lambda : W_p \to PX$. Compare this result with Proposition 13 when $I = \{1,2\}$.]

The property of being a Hurewicz fibration is not a fiber homotopy type invariant, i.e., if X and Y have the same fiber homotopy type and if $p : X \to B$ is a Hurewicz fibration, then $q : Y \to B$ need not be a Hurewicz fibration. Example: Take $X = [0,1] \times [0,1]$, $Y = ([0,1] \times \{0\}) \cup (\{0\} \times [0,1])$, $B = [0,1]$, and let p, q be the vertical projections—then X and Y are fiberwise contractible and $p : X \to B$ is a Hurewicz fibration but $q : Y \to B$ is not a Hurewicz fibration. This difficulty can be circumvented by introducing still another notion of "fibration".

Let X be in $\mathbf{TOP}/B$. Let Y be in $\mathbf{TOP}$—then the projection $p : X \to B$ is said to have the HLP w.r.t. Y up to homotopy if given continuous functions $\begin{cases} F : Y \to X\\ h : IY \to B\end{cases}$ such that $p \circ F = h \circ i_0$, there is a continuous function $H : IY \to X$ such that $F \underset{B}{\simeq} H \circ i_0$ and $p \circ H = h$.

[Note: To interpret the condition $F \underset{B}{\simeq} H \circ i_0$, view Y as an object in $\mathbf{TOP}/B$ with projection $p \circ F$.]

LEMMA The projection $p : X \to B$ has the HLP w.r.t. Y up to homotopy iff given continuous functions $\begin{cases} F : Y \rightarrow X \\ h : IY \rightarrow B \end{cases}$ such that $p \circ F = h \circ i_t$ $(0 \leq t \leq 1/2)$, there is a continuous function $H : IY \rightarrow X$ such that $F = H \circ i_0$ and $p \circ H = h$.

Let X be in $\mathbf{TOP}/B$—then $p : X \rightarrow B$ is said to be a Dold fibration if it has the HLP w.r.t. Y up to homotopy for every Y in **TOP**. Obviously, Hurewicz $\Rightarrow$ Dold, but Dold $\not\Rightarrow$ Serre and Serre $\not\Rightarrow$ Dold. The pullback of a Dold fibration is a Dold fibration and the local-global principle remains valid.

PROPOSITION 21 Let X, Y be in $\mathbf{TOP}/B$ and suppose that $q : Y \rightarrow B$ is a Dold fibration. Assume: $\exists \begin{cases} f \in C_B(X,Y) \\ g \in C_B(Y,X) \end{cases} : g \circ f \underset{B}{\simeq} \mathrm{id}_X$—then $p : X \rightarrow B$ is a Dold fibration.

[Fix a topological space E and continuous functions $\begin{cases} \Phi : E \rightarrow X \\ \psi : IE \rightarrow B \end{cases}$ such that $p \circ \Phi = \psi \circ i_0$. Since $q \circ f = p$, $\exists$ $G : IE \rightarrow Y$ with $f \circ \Phi \underset{B}{\simeq} G \circ i_0$ and $q \circ G = \psi$. Put $\Psi = g \circ G : \Phi \underset{B}{\simeq} g \circ f \circ \Phi \underset{B}{\simeq} \Psi \circ i_0$ & $p \circ \Psi = p \circ g \circ G = q \circ G = \psi$.]

The property of being a Dold fibration is therefore a fiber homotopy type invariant. Example: Take $X = ([0,1] \times \{0\}) \cup (\{0\} \times [0,1])$, $B = [0,1]$, and let p be the vertical projection—then $p : X \rightarrow B$ is a Dold fibration but not a Hurewicz fibration (nor is p an open map (cf. p. 4–15)).

EXAMPLE Define $f : [-1,1] \rightarrow [-1,1]$ by $f(x) = 2|x| - 1$. Put $X = I[-1,1]/\sim$, where $(x, 0) \sim (f(x), 1)$, and let $p : X \rightarrow \mathbf{S}^1$ be the projection—then p is an open map and a Dold fibration but not a Hurewicz fibration.

FACT Suppose that B is numerably contractible, so B admits a numerable covering $\{O\}$ for which each inclusion $O \rightarrow B$ is inessential. Let X be in $\mathbf{TOP}/B$—then the projection $p : X \rightarrow B$ is a Dold fibration iff $\forall$ O there exists a topological space T_O and a fiber homotopy equivalence $X_O \rightarrow O \times T_O$ over O.

The homotopy theory of Hurewicz fibrations carries over to Dold fibrations. The proofs are only slightly more complicated. Specifically: Propositions 15, 17, 18, and 20 are true if "Hurewicz" is replaced by "Dold".

PROPOSITION 22 Let X be in $\mathbf{TOP}/B$—then X is fiberwise contractible iff $p : X \rightarrow B$ is a Dold fibration and a homotopy equivalence.

[The necessity is a consequence of Proposition 21 and the sufficiency is a consequence of Proposition 15.]

PROPOSITION 23 Let X be in $\mathbf{TOP}/B$—then $p : X \to B$ is a Dold fibration iff $\gamma : X \to W_p$ is a fiber homotopy equivalence.

[Bearing in mind that $q : W_p \to B$ is a Hurewicz fibration, the reasoning is the same as that used in the proof of Proposition 22.]

Application: The fibers of a Dold fibration over a path connected base have the same homotopy type.

[Note: Take $X = ([0,1] \times \{0,1\}) \cup (\{0\} \times [0,1])$, $B = [0,1]$, and let p be the vertical projection—then $p : X \to B$ is not a Dold fibration.]

EXAMPLE Let $\left\{\begin{array}{l} p : X \to B \\ q : Y \to B \end{array}\right.$ be Hurewicz fibrations—then the projection $X \square_B Y \to B$ is a Dold fibration, hence $X *_B Y$ and $X \square_B Y$ have the same fiber homotopy type.

EXAMPLE Let X be a topological space. Fix a numerable covering $\mathcal{U} = \{U_i : i \in I\}$ of X—then, in the notation of p. 3–25, the projection $p_{\mathcal{U}} : B\mathcal{U} \to X$ is a Dold fibration (for $B\mathcal{U}$, as an object in $\mathbf{TOP}/X$, is fiberwise contractible).

Notation: Given $b_0 \in B$, put $B_0 = B - \{b_0\}$ and for X, Y in $\mathbf{TOP}/B$, write X_0, Y_0 in place of X_{b_0}, Y_{b_0}.

FACT (Expansion Principle) Let X be in $\mathbf{TOP}/B$. Suppose that $p_{B_0} : X_{B_0} \to B_0$ is a Dold fibration and b_0 has a halo $O \subset B$ contractible to b_0, with $O - \{b_0\}$ numerably contractible. Assume: $r : X_O \to X_0$ is a homotopy equivalence which $\forall\, b \in O$ induces a homotopy equivalence $r_b : X_b \to X_0$—then there exists a Y in $\mathbf{TOP}/B$ and an embedding $X \to Y$ over B such that $q : Y \to B$ is a Dold fibration and $\left\{\begin{array}{l} X \\ X_0 \end{array}\right.$ is a strong deformation retract of $\left\{\begin{array}{l} Y \\ Y_0 \end{array}\right.$.

[The commutative diagram
$$\begin{tikzcd} X_0' \arrow[r] \arrow[d] & X_0 \arrow[d] \\ O \arrow[r] & b_0 \end{tikzcd}$$
($X_0' = O \times X_0$) is a pullback square. Since $O \to b_0$ is a homotopy equivalence, $X_0' \to X_0$ is a homotopy equivalence (cf. p. 4–24), thus the arrow $r' : X_O \to X_0'$ defined by $x \to (p(x), r(x))$ is a homotopy equivalence. Let Y be the double mapping cylinder of the 2-source $X \leftarrow X_O \xrightarrow{r'} X_0'$: Y is in $\mathbf{TOP}/B$ and there is an embedding $X \to Y$ over B. It is a closed cofibration. Y_O is the mapping cylinder of r', so X_O is a strong deformation retract of Y_O (cf. §3, Proposition 17). Therefore X is a strong deformation retract of Y (cf. §3, Proposition 3). Similar remarks apply to X_0 and Y_0. Finally, to see that q is a Dold fibration, note that $\{O, B_0\}$ is a numerable covering of B. Accordingly, taking into account the local-global principle, it is enough to verify that $q_O : Y_O \to O$ and $q_{B_0} : Y_{B_0} \to B_0$ are Dold fibrations. Consider, e.g., the latter. The hypotheses on r, in conjunction

with Proposition 20, imply that the embedding $X_{B_0} \to Y_{B_0}$ is a fiber homotopy equivalence. But p_{B_0} is a Dold fibration, hence the same holds for q_{B_0}.]

Let $f : X \to Y$ be a pointed continuous function—then the mapping fiber E_f of f is defined by the pullback square

$$\begin{tikzcd} E_f \arrow[r] \arrow[d] & W_f \arrow[d, "q"] \\ \{x_0\} \arrow[r, "f"'] & Y \end{tikzcd}$$

, i.e., E_f is the double mapping track of the 2-sink $X \xrightarrow{f} Y \leftarrow \{y_0\}$. Example: The mapping fiber E_0 of $0 : X \to Y$ is $X \times \Omega Y$.

EXAMPLE Let $f : X \to Y$ be a pointed continuous function. Assume: f is a Hurewicz fibration. Denote by C_{y_0} the mapping cone of the inclusion $X_{y_0} \to X$—then the mapping fiber of $C_{y_0} \to Y$ has the same homotopy type as $X_{y_0} * \Omega Y$ (cf. p. 4–20 ff.).

FACT Let $X \xrightarrow{p} B \xleftarrow{q} Y$ be a 2-sink. Denote by $W_{\square}$ the mapping track of the projection $X \square_B Y \to B$—then $W_p *_B W_q$ and $W_{\square}$ have the same fiber homotopy type.

Application: The mapping fiber of the projection $X \square_B Y \to B$ has the same homotopy type as $E_p * E_q$.

Let $f : X \to Y$ be a pointed continuous function—then W_f and E_f are pointed spaces, the base point in either case being $(x_0, j(y_0))$. The pointed homotopy type of W_f or E_f depends only on the pointed homotopy class of f. The projection $q : W_f \to Y$ is a pointed Hurewicz fibration and the restriction π of the projection $p : W_f \to X$ to E_f is a pointed Hurewicz fibration with $\pi^{-1}(x_0) = \Omega Y$. By construction, $f \circ \pi$ is nullhomotopic and for any $g : Z \to X$ with $f \circ g$ nullhomotopic, there is a $\phi : Z \to E_f$ such that $g = \pi \circ \phi$.

When is a pointed continuous function which is a Hurewicz fibration actually a pointed Hurewicz fibration? Regularity, suitably localized, is what is relevant. Thus let $p : X \to B$ be a Hurewicz fibration taking x_0 to b_0. Assume: $\exists$ a lifting function Λ such that $\Lambda(x_0, j(b_0)) = j(x_0)$—then p is a pointed Hurewicz fibration.

[Note: For this, it is sufficient that $\{b_0\}$ be a zero set in B, any Hurewicz fibration $p : X \to B$ automatically becoming a pointed Hurewicz fibration $\forall\, x_0 \in X_{b_0}$ (argue as on p. 4–14). The condition is satisfied if the inclusion $\{b_0\} \to B$ is a closed cofibration.]

LEMMA Let X, Y, Z be pointed spaces; let $\begin{cases} f : X \to Z \\ g : Y \to Z \end{cases}$ be pointed continuous functions—then the projections $\begin{cases} W_{f,g} \to X \\ W_{f,g} \to Y \end{cases}$ & $W_{f,g} \to X \times Y$ are pointed Hurewicz fibrations, the base point of $W_{f,g}$ being the triple $(x_0, y_0, j(z_0))$.

[To deal with $p : W_{f,g} \to X$, define a lifting function $\Lambda : W_p \to PW_{f,g}$ by $\Lambda((x,y,\tau),\sigma)(t) = (\sigma(t), y, \tau_t)$, where

$$\tau_t(T) = \left\{\begin{matrix} f \circ \sigma(t - 2T) & (0 \leq T \leq t/2) \\ \tau\left(\dfrac{2T - t}{2 - t}\right) & (t/2 \leq T \leq 1) \end{matrix}\right. .$$

Obviously, $\Lambda((x_0, y_0, j(x_0)), j(x_0)) = j(x_0, y_0, j(x_0))$, so $p : W_{f,g} \to X$ is a pointed Hurewicz fibration.]

PROPOSITION 24 Consider the pullback square

$$\begin{tikzcd} X' \arrow[r] \arrow[d] & X \arrow[d, "p"] \\ B' \arrow[r, "\Phi'"'] & B \end{tikzcd}$$

where p is a Hurewicz fibration. Suppose that $\left\{\begin{matrix} X \\ B \end{matrix}\right.$ & B' are wellpointed, that the inclusions $\left\{\begin{matrix} \{x_0\} \to X \\ \{b_0\} \to B \end{matrix}\right.$ & $\{b_0'\} \to B'$ are closed, and that $p(x_0) = b_0 = \Phi'(b_0')$. Put $x_0' = (b_0', x_0)$—then the inclusion $\{x_0'\} \to X'$ is a closed cofibration.

[The arrow $X_{b_0} \to X$ is a closed cofibration (cf. Proposition 11). Therefore the composite $X'_{b_0'} \to X' \to X$ is a closed cofibration. On the other hand, the composite $\{x_0'\} \to X'_{b_0'} \to X' \to X$ is a closed cofibration. Therefore the inclusion $\{x_0'\} \to X'_{b_0'}$ is a closed cofibration (cf. §3, Proposition 9). But the arrow $X'_{b_0'} \to X'$ is a closed cofibration (cf. Proposition 11), thus the inclusion $\{x_0'\} \to X'$ is a closed cofibration.]

Application: Let $f : X \to Y$ be a pointed continuous function. Assume: $\left\{\begin{matrix} X \\ Y \end{matrix}\right.$ are wellpointed with closed base points—then W_f and E_f are wellpointed with closed base points.

[PY is wellpointed with a closed base point (cf. §3, Proposition 6).]

FACT Let $f : X \to Y$ be a pointed continuous function. Suppose that $\phi : X' \to X$ ($\psi : Y \to Y'$) is a pointed homotopy equivalence—then the arrow $E_{f \circ \phi} \to E_f$ ($E_f \to E_{\psi \circ f}$) is a pointed homotopy equivalence.

Application: Let X be wellpointed with $\{x_0\} \subset X$ closed—then the mapping fiber of the diagonal embedding $X \to X \times X$ has the same pointed homotopy type as ΩX.

[The embedding $j : X \to PX$ is a pointed homotopy equivalence and $\Pi : \left\{\begin{matrix} PX \to X \times X \\ \sigma \to (\sigma(0), \sigma(1)) \end{matrix}\right.$ is a pointed Hurewicz fibration.]

EXAMPLE Let $\left\{\begin{matrix} X \\ Y \end{matrix}\right.$ be wellpointed with $\left\{\begin{matrix} \{x_0\} \subset X \\ \{y_0\} \subset Y \end{matrix}\right.$ closed.

(1) The mapping fiber of the inclusion $X \vee Y \to X \times Y$ has the same pointed homotopy type as $\Omega X * \Omega Y$.

(2) The mapping fiber of the projection $X \vee Y \to Y$ has the same pointed homotopy type as $X \times \Omega Y/\{x_0\} \times \Omega Y$.

[In both situations, replace Θ by $\Gamma\Omega$ as on p. 4–16.]

FACT Let $\begin{cases} f : X \to Y \\ g : Y \to Z \end{cases}$ be pointed continuous functions—then there is a homotopy equivalence $E_{g\circ f} \to W$, where W is the double mapping track of the 2-sink $X \xrightarrow{f} Y \xleftarrow{\pi} E_g$.

[Consider the diagram
$$\begin{tikzcd}
E_{g\circ f} \arrow[r] \arrow[d] & E_g \arrow[r] \arrow[d] & * \arrow[d] \\
X \arrow[r] & Y \arrow[r] & Z
\end{tikzcd}$$
.]

Let $f : X \to Y$ be a pointed continuous function, E_f its mapping fiber.

LEMMA If f is a pointed Hurewicz fibration, then the embedding $X_{y_0} \to E_f$ is a pointed homotopy equivalence.

In general, there is a pointed Hurewicz fibration $\pi : E_f \to X$ and an embedding $\Omega Y \to E_f$. Iterate to get a pointed Hurewicz fibration $\pi' : E_\pi \to E_f$—then the triangle
$$\begin{tikzcd}
E_\pi \arrow[r] & E_f \\
\Omega Y \arrow[u] \arrow[ur] &
\end{tikzcd}$$
commutes and by the lemma, the vertical arrow is a pointed homotopy equivalence. Iterate again to get a pointed Hurewicz fibration $\pi'' : E_{\pi'} \to E_\pi$—then the triangle
$$\begin{tikzcd}
E_{\pi'} \arrow[r] & E_\pi \\
\Omega X \arrow[u] \arrow[ur] &
\end{tikzcd}$$
commutes and by the lemma, the vertical arrow is a pointed homotopy equivalence. Example: Given pointed spaces $\begin{cases} X \\ Y \end{cases}$, let $X\flat Y$ be the mapping fiber of the inclusion $f : X \vee Y \to X \times Y$—then in **HTOP**$_*$, $E_\pi \approx \Omega(X \times Y)$ and $E_{\pi'} \approx \Omega(X \vee Y)$.

LEMMA Let $\begin{cases} X \\ Y \end{cases}$ be wellpointed with $\begin{cases} \{x_0\} \subset X \\ \{y_0\} \subset Y \end{cases}$ closed. Denote by S the subspace of $X * Y$ consisting of the $\begin{cases} [x, y_0, t] \\ [x_0, y, t] \end{cases}$—then $X * Y/S = \Sigma(X\#Y)$ and the projection $X * Y \to X * Y/S$ is a pointed homotopy equivalence.

[Note: The base point of $X * Y$ is $[x_0, y_0, 1/2]$ and Σ is the pointed suspension.]

Application: Let $\begin{cases} X \\ Y \end{cases}$ be wellpointed with $\begin{cases} \{x_0\} \subset X \\ \{y_0\} \subset Y \end{cases}$ closed—then $X\flat Y$ has the same pointed homotopy type as $\Sigma(\Omega X \# \Omega Y)$.

EXAMPLE Suppose that X and Y are nondegenerate—then the Puppe formula says that in **HTOP**$_*$, $\Sigma(\Omega X \times \Omega Y) \approx \Sigma\Omega X \vee \Sigma\Omega Y \vee \Sigma(\Omega X \# \Omega Y)$, and by the above, $\Sigma(\Omega X \# \Omega Y) \approx X\flat Y$.

EXAMPLE (The Flat Product) In contrast to the smash product $\#$ (or its modification $\overline{\#}$), the flat product $\flat$ does not possess the properties that one might expect to hold by analogy. Specifically, for nondegenerate spaces, it is generally false that in $\mathbf{HTOP}_*$: (1) $(X \flat Y) \flat Z \approx X \flat (Y \flat Z)$; (2) $(X \times Y) \flat Z \approx (X \flat Z) \times (Y \flat Z)$; (3) $\Omega(X \flat Y) \approx \Omega X \flat Y$. Counterexamples: (1) Take $X = Y = \mathbf{P}^\infty(\mathbf{C})$, $Z = \mathbf{P}^\infty(\mathbf{H})$; (2) Take $X = Y = Z = \mathbf{P}^\infty(\mathbf{C})$; (3) Take $X = Y = \mathbf{P}^\infty(\mathbf{C})$. Look, e.g., at (1). Using the fact that $\Omega\mathbf{P}^\infty(\mathbf{C}) \approx \mathbf{S}^1$, $\Omega\mathbf{P}^\infty(\mathbf{H}) \approx \mathbf{S}^3$, compute: $\mathbf{P}^\infty(\mathbf{C}) \flat \mathbf{P}^\infty(\mathbf{C}) \approx \Omega\mathbf{P}^\infty(\mathbf{C}) * \Omega\mathbf{P}^\infty(\mathbf{C}) \approx \mathbf{S}^1 * \mathbf{S}^1 \approx \mathbf{S}^3$ & $\mathbf{S}^3 \flat \mathbf{P}^\infty(\mathbf{H}) \approx \Omega\mathbf{S}^3 * \mathbf{S}^3 \approx \Sigma(\Omega\mathbf{S}^3 \# \mathbf{S}^3) \approx \Sigma\Omega\mathbf{S}^3 \# \mathbf{S}^3 \approx \Sigma\Omega\mathbf{S}^3 \# \Sigma^3\mathbf{S}^0 \approx \Sigma^4\Omega\mathbf{S}^3 \# \mathbf{S}^0 \approx \Sigma^4\Omega\mathbf{S}^3 \Rightarrow (\mathbf{P}^\infty(\mathbf{C}) \flat \mathbf{P}^\infty(\mathbf{C})) \flat \mathbf{P}^\infty(\mathbf{H}) \approx \Sigma^4\Omega\mathbf{S}^3$. Similarly, $\mathbf{P}^\infty(\mathbf{C}) \flat (\mathbf{P}^\infty(\mathbf{C}) \flat \mathbf{P}^\infty(\mathbf{H})) \approx \Sigma^2\Omega\mathbf{S}^5$. The singular homology functor $H_8(—;\mathbf{Z})$ distinguishes these spaces: $H_8(\Sigma^4\Omega\mathbf{S}^3;\mathbf{Z}) \approx \mathbf{Z}$, $H_8(\Sigma^2\Omega\mathbf{S}^5;\mathbf{Z}) = 0$.

Let $f : X \to Y$ be a pointed continuous function—then the mapping fiber sequence associated with f is given by $\cdots \to \Omega^2 Y \to \Omega E_f \to \Omega X \to \Omega Y \to E_f \to X \xrightarrow{f} Y$. Example: When $f = 0$, this sequence becomes $\cdots \to \Omega^2 Y \to \Omega X \times \Omega^2 Y \to \Omega X \to \Omega Y \to X \times \Omega Y \to X \xrightarrow{0} Y$.

[Note: If the diagram

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d] & Y \arrow[d] \\ X' \arrow[r, "f'"'] & Y' \end{tikzcd}$$

commutes in $\mathbf{HTOP}_*$ and if the vertical arrows are pointed homotopy equivalences, then the mapping fiber sequences of f and f' are connected by a commutative ladder in $\mathbf{HTOP}_*$, all of whose vertical arrows are pointed homotopy equivalences.]

FACT Let $f : X \to Y$ be a pointed Hurewicz fibration. Assume: The inclusion $X_{y_0} \to X$ is nullhomotopic—then ΩY has the same pointed homotopy type as $X_{y_0} \times \Omega X$.

[For $\pi : E_f \to X$ is nullhomotopic, thus in $\mathbf{HTOP}_*$: $E_\pi \approx E_f \times \Omega X \Rightarrow \Omega Y \approx X_{y_0} \times \Omega X$.]

REPLICATION THEOREM Let $f : X \to Y$ be a pointed continuous function—then for any pointed space Z, there is an exact sequence

$$\cdots \to [Z, \Omega X] \to [Z, \Omega Y] \to [Z, E_f] \to [Z, X] \to [Z, Y]$$

in $\mathbf{SET}_*$.

If $f : X \to Y$ is a pointed Dold fibration or if $f : X \to Y$ is a Dold fibration and Z is nondegenerate, then in the replication theorem one can replace E_f by X_{y_0} (cf. p. 3–18). This replacement can also be made if $f : X \to Y$ is a Serre fibration provided that Z is a CW complex (cf. infra). In particular, when $f : X \to Y$ is either a Dold fibration or a Serre fibration, there is an exact sequence

$$\cdots \to \pi_2(Y) \to \pi_1(X_{y_0}) \to \pi_1(X) \to \pi_1(Y) \to \pi_0(X_{y_0}) \to \pi_0(X) \to \pi_0(Y).$$

LEMMA Let $f : X \to Y$ be a pointed continuous function. Assume: f is a Serre fibration—then for every pointed CW complex Z, the arrow $[Z, X_{y_0}] \to [Z, E_f]$ is a pointed bijection.

[Proposition 12 is true for Serre fibrations if the "cofibration data" is restricted to CW complexes.]

Examples: Suppose that $f : X \to Y$ is either a Dold fibration or a Serre fibration, where $\begin{cases} X \neq \emptyset \\ Y \neq \emptyset \end{cases}$. (1) If X_{y_0} is simply connected, then $\forall\ x_0 \in X_{y_0}$, $\pi_1(X, x_0) \approx \pi_1(Y, y_0)$; (2) If X is simply connected, then $\forall\ y_0 \in f(X)$, there is a bijection $\pi_1(Y, y_0) \to \pi_0(X_{y_0})$; (3) If X is path connected and if Y is simply connected, then $\forall\ y_0 \in Y$, $\pi_0(X_{y_0}) = *$; (4) If Y is path connected and X_{y_0} is path connected, then X is path connected.

LEMMA Let $f : X \to Y$ be a Hurewicz fibration. Fix $y_0 \in f(X)$ & $x_0 \in X_{y_0}$ and let (Z, z_0) be wellpointed with $\{z_0\} \subset Z$ closed—then there is a left action $\pi_1(X, x_0) \times [Z, z_0; X_{y_0}, x_0] \to [Z, z_0; X_{y_0}, x_0]$.

[Represent $\alpha \in \pi_1(X, x_0)$ by a loop $\sigma \in \Omega(X, x_0)$. Given $\phi : (Z, z_0) \to (X_{y_0}, x_0)$, consider the commutative diagram

$$\begin{tikzcd} i_0 Z \cup I\{z_0\} \arrow[r, "F"] \arrow[d] & X \arrow[d, "f"] \\ IZ \arrow[r, "h"'] & Y \end{tikzcd}$$

, where $F(z, t) = \begin{cases} (i \circ \phi)(z) & (t = 0) \\ \sigma(t) & (z = z_0) \end{cases}$ (i the inclusion $X_{y_0} \to X$) and $h(z, t) = (f \circ \sigma)(t)$. Proposition 12 says that this diagram has a filler $H : IZ \to X$. Put $\psi(z) = H(z, 1)$ to get a pointed continuous function $\psi : (Z, z_0) \to (X_{y_0}, x_0)$. Definition: $\alpha \cdot [\phi] = [\psi]$.]

[Note: There is a left action $\pi_1(X, x_0) \times [Z, z_0; X, x_0] \to [Z, z_0; X, x_0]$ and a left action $\pi_1(X_{y_0}, x_0) \times [Z, z_0; X_{y_0}, x_0] \to [Z, z_0; X_{y_0}, x_0]$ (cf. p. 3–18). The arrow $[Z, z_0; X_{y_0}, x_0] \to [Z, z_0; X, x_0]$ induced by the inclusion $X_{y_0} \to X$ is a morphism of $\pi_1(X, x_0)$-sets and the operation of $\pi_1(X_{y_0}, x_0)$ on $[Z, z_0; X_{y_0}, x_0]$ coincides with that defined via the homomorphism $\pi_1(X_{y_0}, x_0) \to \pi_1(X, x_0)$.]

EXAMPLE Let $f : X \to Y$ be a Hurewicz fibration. Fix $y_0 \in f(X)$ & $x_0 \in X_{y_0}$ and $n \geq 1$—then there is a left action $\pi_1(X, x_0) \times \pi_n(X, x_0) \to \pi_n(X, x_0)$, a left action $\pi_1(X, x_0) \times \pi_n(Y, y_0) \to \pi_n(Y, y_0)$, and a left action $\pi_1(X, x_0) \times \pi_n(X_{y_0}, x_0) \to \pi_n(X_{y_0}, x_0)$. All the homomorphisms in the exact sequence

$$\cdots \to \pi_{n+1}(Y, y_0) \to \pi_n(X_{y_0}, x_0) \to \pi_n(X, x_0) \to \pi_n(Y, y_0) \to \cdots$$

are $\pi_1(X, x_0)$-homomorphisms.

[Note: Suppose that X_{y_0} is path connected—then there is a left action $\pi_1(Y, y_0) \times \pi_n^*(X_{y_0}, x_0) \to \pi_n^*(X_{y_0}, x_0)$, where $\pi_n^*(X_{y_0}, x_0)$ is $\pi_n(X_{y_0}, x_0)$ modulo the (normal) subgroup generated by the $\alpha \cdot \xi - \xi$ ($\alpha \in \pi_1(X_{y_0}, x_0)$, $\xi \in \pi_n(X_{y_0}, x_0)$).]

EXAMPLE Let $f : X \to Y$ be a Hurewicz fibration. Fix $y_0 \in f(X)$ & $x_0 \in X_{y_0}$—then $\pi_1(Y, y_0)$ operates to the left on $\pi_0(X_{y_0})$ and the orbits are the fibers of the arrow $\pi_0(X_{y_0}) \to \pi_0(X)$.

FACT Let $f : X \to Y$ be a Hurewicz fibration. Fix $y_0 \in f(X)$ & $x_0 \in X_{y_0}$—then $\forall\ n \geq 1$, $\pi_1(X_{y_0}, x_0)$ operates trivially on $\ker(\pi_n(X_{y_0}, x_0) \to \pi_n(X, x_0))$.

EXAMPLE (Mayer-Vietoris Sequence) Let X, Y, Z be pointed spaces; let $\begin{cases} f : X \to Z \\ g : Y \to Z \end{cases}$ be pointed continuous functions—then the projection $W_{f,g} \to X \times Y$ is a pointed Hurewicz fibration (cf. p. 4–32) and there is a long exact sequence $\cdots \to \pi_{n+1}(Z) \to \pi_n(W_{f,g}) \to \pi_n(X) \times \pi_n(Y) \to \pi_n(Z) \to \cdots \to \pi_2(Z) \to \pi_1(W_{f,g}) \to \pi_1(X) \times \pi_1(Y) \to \pi_1(Z) \to \pi_0(W_{f,g}) \to \pi_0(X \times Y)$.

[Note: It follows that if X and Y are path connected and if every $\gamma \in \pi_1(Z)$ has the form $\gamma = f_*(\alpha) \cdot g_*(\beta)$ $(\alpha \in \pi_1(X), \beta \in \pi_1(Y))$, then $W_{f,g}$ is path connected.]

If $f : X \to Y$ is either a Dold fibration or a Serre fibration, then the homotopy groups of X and Y are related to those of the fibers by a long exact sequence. As for the homology groups, there is still a connection but it is intricate and best expressed in terms of a spectral sequence.

[Note: In the simplest case, viz. that of a projection $Y \times T \to Y$, the Künneth formula computes the homology of $Y \times T$ in terms of the homology of Y and the homology of T.]

EXAMPLE Let $f : X \to Y$ be a Hurewicz fibration, where X is nonempty and Y is path connected. Fix $y_0 \in Y$—then $\forall\, q \geq 1$, the projection $(X, X_{y_0}) \to (Y, y_0)$ induces a bijection $\pi_q(X, X_{y_0}) \to \pi_q(Y, y_0)$. The analog of this in homology is false. Consider, e.g., the Hopf map $\mathbf{S}^{2n+1} \to \mathbf{P}^n(\mathbf{C})$ with fiber $\mathbf{S}^1$: $H_q(\mathbf{S}^{2n+1}, \mathbf{S}^1) = 0$ $(2 < q \leq 2n)$ & $H_{2q}(\mathbf{P}^n(\mathbf{C})) \approx Z$ $(1 < q \leq n)$. However, a partial result holds in that if X_{y_0} is n-connected and Y is m-connected, then the arrow $H_q(X, X_{y_0}) \to H_q(Y, y_0)$ induced by the projection $(X, X_{y_0}) \to (Y, y_0)$ is bijective for $1 \leq q < n + m + 2$ and surjective for $q = n + m + 2$. Consequently, under these conditions, there is an exact sequence

$$\begin{gathered} H_{n+m+1}(X_{y_0}) \to H_{n+m+1}(X) \to H_{n+m+1}(Y) \to H_{n+m}(X_{y_0}) \to \cdots \\ \to H_2(Y) \to H_1(X_{y_0}) \to H_1(X) \to H_1(Y). \end{gathered}$$

[One can assume that the inclusion $\{y_0\} \to Y$ is a closed cofibration (pass to a CW resolution $K \to Y$), hence that the inclusion $X_{y_0} \to X$ is a closed cofibration (cf. Proposition 11). The mapping cone of the latter is path connected and the mapping fiber of $C_{y_0} \to Y$ has the same homotopy type as $X_{y_0} * \Omega Y$ (cf. p. 4–32), which is $(n + m + 1)$-connected (cf. p. 3–40). Thus the arrow $C_{y_0} \to Y$ is an $(n + m + 2)$-equivalence, so the Whitehead theorem implies that the induced map $H_q(C_{y_0}) \to H_q(Y)$ is bijective for $0 \leq q < n + m + 2$ and surjective for $q = n + m + 2$. But the projection $C_{y_0} \to X/X_{y_0}$ is a homotopy equivalence (cf. p. 3–24) and $H_q(X, X_{y_0}) \approx H_q(X/X_{y_0}, *)$ (cf. p. 3–8).]

Application: Suppose that X is $(n + 1)$-connected—then $H_q(X) \approx H_{q-1}(\Omega X)$ $(2 \leq q \leq 2n + 2)$.

[Note: It is a corollary that if X is nondegenerate and n-connected, then the arrow of adjunction $e : X \to \Omega\Sigma X$ induces an isomorphism $H_q(X) \to H_q(\Omega\Sigma X)$ for $0 \leq q \leq 2n + 1$. Therefore, by the Whitehead theorem, the suspension homomorphism $\pi_q(X) \to \pi_{q+1}(\Sigma X)$ is bijective for $0 \leq q \leq 2n$ and surjective for $q = 2n + 1$ (Freudenthal).]

Let X be a topological space, $\sin X$ its singular set—then $\sin X$ can be regarded as a category:

$$\begin{tikzcd} \Delta^m \arrow[rr, "\Delta^\alpha"] \arrow[dr] & & \Delta^n \arrow[dl] \\ & X & \end{tikzcd}$$

($\alpha \in \mathrm{Mor}([m],[n])$). The objects of $[(\sin X)^{\mathrm{OP}}, \mathbf{AB}]$ are called coefficient systems on X. Given a coefficient system $\mathcal{G}$, the singular homology $H_*(X;\mathcal{G})$ of X with coefficients in $\mathcal{G}$ is by definition the homology of the chain complex

$$\bigoplus_{\sigma_0 \in \sin_0 X} \mathcal{G}\sigma_0 \overset{\partial}{\leftarrow} \bigoplus_{\sigma_1 \in \sin_1 X} \mathcal{G}\sigma_1 \overset{\partial}{\leftarrow} \bigoplus_{\sigma_2 \in \sin_2 X} \mathcal{G}\sigma_2 \overset{\partial}{\leftarrow} \cdots,$$

where $\partial = \sum_0^n (-1)^i \bigoplus_{\sigma_n \in \sin_n X} \mathcal{G}d_i$.

[Note: To interpret $\mathcal{G}d_i$, recall that there are arrows $d_i : \sin_n X \to \sin_{n-1} X$ corresponding to the face operators $\delta_i : [n-1] \to [n]$ $(0 \leq i \leq n)$. So, $\forall\ \sigma \in \sin_n X$, $\mathcal{G}d_i : \mathcal{G}(\Delta^n \xrightarrow{\sigma} X) \to \mathcal{G}(\Delta^{n-1} \xrightarrow{d_i\sigma} X)$.]

Example: Fix an abelian group G and define $\mathcal{G}_G$ by $\begin{cases} \mathcal{G}_G\sigma = G \\ \mathcal{G}_G\Delta^\alpha = \mathrm{id}_G \end{cases}$ —then $H_*(X;\mathcal{G}_G) = H_*(X;G)$, the singular homology of X with coefficients in G.

A coefficient system $\mathcal{G}$ is said to be locally constant provided that $\forall\ \alpha$, $\mathcal{G}\Delta^\alpha$ is invertible. $\mathbf{LCCS}_X$ is the full subcategory of $[(\sin X)^{\mathrm{OP}}, \mathbf{AB}]$ whose objects are the locally constant coefficient systems on X.

[Note: A coefficient system $\mathcal{G}$ is said to be constant if for some abelian group G, $\mathcal{G}$ is isomorphic to $\mathcal{G}_G$.]

Suppose that X is locally path connected and locally simply connected—then the category of locally constant coefficient systems on X is equivalent to the category of locally constant sheaves of abelian groups on X.

PROPOSITION 25 $\mathbf{LCCS}_X$ is equivalent to $[(\Pi X)^{\mathrm{OP}}, \mathbf{AB}]$.

[We shall define a functor $\mathcal{G} \to \mathcal{G}_\Pi$ from $\mathbf{LCCS}_X$ to $[(\Pi X)^{\mathrm{OP}}, \mathbf{AB}]$ and a functor $\mathcal{G} \to \mathcal{G}_{\sin}$ from $[(\Pi X)^{\mathrm{OP}}, \mathbf{AB}]$ to $\mathbf{LCCS}_X$ such that $\begin{cases} (\mathcal{G}_\Pi)_{\sin} \approx \mathcal{G} \\ (\mathcal{G}_{\sin})_\Pi \approx \mathcal{G} \end{cases}$.

Definition of $\mathcal{G}_\Pi$: Given $x \in X$, put $\mathcal{G}_\Pi x = \mathcal{G}\sigma_x$, where $\sigma_x \in \sin_0 X$ with $\sigma_x(\Delta^0) = x$. Given a morphism $[\sigma] : x \to y$, put $\mathcal{G}_\Pi[\sigma] = (\mathcal{G}d_1) \circ (\mathcal{G}d_0)^{-1}$, where $\sigma \in \sin_1 X$ with $\begin{cases} d_1\sigma = x \\ d_0\sigma = y \end{cases}$. In other words, $\mathcal{G}_\Pi[\sigma]$ is the composite $\mathcal{G}y \to \mathcal{G}\sigma \to \mathcal{G}x$. Note that $\mathcal{G}_\Pi[\sigma]$ is welldefined. Indeed, if $\begin{cases} \sigma' \\ \sigma'' \end{cases} \in \sin_1 X$ with $\begin{cases} d_1\sigma' = x = d_1\sigma'' \\ d_0\sigma' = y = d_0\sigma'' \end{cases}$ and $[\sigma'] = [\sigma'']$, then there exists a $\tau \in \sin_2 X$ such that $\begin{cases} d_1\tau = \sigma' \\ d_2\tau = \sigma'' \end{cases}$ and $s_0d_0\sigma' = d_0\tau = s_0d_0\sigma''$.

Definition of $\mathcal{G}_{\text{sin}}$: Given $\sigma \in \sin_n X$, put $\mathcal{G}_{\text{sin}}\sigma = \mathcal{G}(e_n\sigma(\Delta^0))$, where $e_n : \sin_n X \to \sin_0 X$ is the arrow associated with the vertex operator $\epsilon_n : [0] \to [n]$ that sends 0 to n. Given a morphism

$$\begin{tikzcd} \Delta^m \arrow[rr, "\Delta^\alpha"] \arrow[dr, "\tau"'] & & \Delta^n \arrow[dl, "\sigma"] \\ & X & \end{tikzcd}$$

, put $\mathcal{G}_{\text{sin}}\Delta^\alpha = \mathcal{G}(\sigma \circ \Delta^\theta)$, where $\theta : [1] \to [n]$ is defined by $\begin{cases} \theta(0) = \alpha(m) \\ \theta(1) = n \end{cases}$: $\sigma \circ \Delta^\theta$ is a path in X which begins at $e_m\tau(\Delta^0)$ and ends at $e_n\sigma(\Delta^0)$.]

Because of this result, one can always pass back and forth between locally constant coefficient systems on X and cofunctors $\Pi X \to \mathbf{AB}$. The advantage of dealing with the latter is that in practice a direct description is sometimes available. For example, fix $n \geq 2$ and assign to each $x \in X$ the homotopy group $\pi_n(X, x)$—then every morphism $[\sigma] : x \to y$ determines an isomorphism $\pi_n(X, y) \to \pi_n(X, x)$ and there is a cofunctor $\pi_n X : \Pi X \to \mathbf{AB}$.

[Note: Suppose that $\mathcal{G}$ is in $[(\Pi X)^{\text{OP}}, \mathbf{AB}]$—then $\forall\ x_0 \in X$, the fundamental group $\pi_1(X, x_0)$ operates to the right on $\mathcal{G}x_0 : \mathcal{G}x_0 \times \pi_1(X, x_0) \to \mathcal{G}x_0$. Conversely, if X is path connected and if G_0 is an abelian group on which $\pi_1(X, x_0)$ operates to the right, then there exists a $\mathcal{G}$ in $[(\Pi X)^{\text{OP}}, \mathbf{AB}]$, unique up to isomorphism, with $\mathcal{G}x_0 = G_0$ and inducing the given operation of $\pi_1(X, x_0)$ on G_0.]

Application: On a simply connected space, every locally constant coefficient system is isomorphic to a constant coefficient system.

EXAMPLE Let $f : X \to Y$ be a Hurewicz fibration—then $\forall\ q \geq 0$, there is a cofunctor $\mathcal{H}_q(f) : \Pi Y \to \mathbf{AB}$ that assigns to each $y \in Y$ the singular homology group $H_q(X_y)$ of the fiber X_y. Thus let $[\tau] : y_0 \to y_1$ be a morphism. Case 1: $\begin{cases} y_0 \\ y_1 \end{cases} \notin f(X)$. In this situation, X_{y_0} & X_{y_1} are empty, hence $H_q(X_{y_0}) = 0 = H_q(X_{y_1})$. Definition: $\mathcal{H}_q(f)[\tau]$ is the zero morphism. Case 2: $\begin{cases} y_0 \\ y_1 \end{cases} \in f(X)$. Fix a homotopy $\Lambda : IX_{y_0} \to X$ such that $\begin{cases} f \circ \Lambda(x, t) = \tau(t) \\ \Lambda(x, 0) = x \end{cases}$—then the arrow $\begin{cases} X_{y_0} \to X_{y_1} \\ x \to \Lambda(x, 1) \end{cases}$ is a homotopy equivalence. Definition: $\mathcal{H}_q(f)[\tau]$ is the inverse of the induced isomorphism $H_q(X_{y_0}) \to H_q(X_{y_1})$ (it is independent of the choices).

LEMMA Suppose that X is path connected. Given a locally constant coefficient system $\mathcal{G}$, fix $x_0 \in X$, put $G_0 = \mathcal{G}x_0$, and let H_0 be the subgroup of G_0 generated by the $g - g \cdot \alpha$ $(g \in G_0, \alpha \in \pi_1(X, x_0))$—then $H_0(X; \mathcal{G}) \approx G_0/H_0$.

Let $f : X \rightarrow Y$ and $f' : X' \rightarrow Y'$ be a pair of continuous functions. Call $\mathcal{H}om(f', f)$ the simplicial set specified by taking for $\mathcal{H}om(f', f)_n$ the set of all $\begin{cases} u \in C(\Delta^n \times X', X) \\ v \in C(\Delta^n \times Y', Y) \end{cases}$ such that the diagram

$$\begin{tikzcd} \Delta^n \times X' \arrow[r, "u"] \arrow[d, "\mathrm{id}\times f'"'] & X \arrow[d, "f"] \\ \Delta^n \times Y' \arrow[r, "v"'] & Y \end{tikzcd}$$

commutes and define $\begin{cases} d_i \\ s_i \end{cases}$ in the obvious way.

Now specialize, putting $Y' = \Delta^0$, so $f' : X' \rightarrow \Delta^0$ is the constant map, and write $\mathcal{H}om(X', f)$ in place of $\mathcal{H}om(f', f)$. In succession, let $X' = \Delta^0, \Delta^1, \ldots$ to obtain a sequence of simplicial sets and simplicial maps:

$$\mathcal{H}om(\Delta^0, f) \leftleftarrows \mathcal{H}om(\Delta^1, f) \overset{\leftarrow}{\leftleftarrows} \mathcal{H}om(\Delta^2, f) \cdots .$$

Here, the arrows come from the face operators $[0] \rightrightarrows [1], [1] \overset{\rightarrow}{\rightrightarrows} [2], \cdots$. This data generates a double chain complex $K_{\bullet\bullet} = \{K_{n,m} : n \geq 0, m \geq 0\}$ of abelian groups if we write $K_{n,m} = F_{\mathrm{ab}}(\mathcal{H}om(\Delta^n, f)_m)$ and define $\begin{cases} \partial_I : K_{n,m} \rightarrow K_{n-1,m} \\ \partial_{II} : K_{n,m} \rightarrow K_{n,m-1} \end{cases}$ as follows.

(∂_I) The arrows $\mathcal{H}om(\Delta^n, f)_m \underset{\rightarrow}{\overset{\rightarrow}{\vdots}} \mathcal{H}om(\Delta^{n-1}, f)_m$ lead to arrows $K_{n,m} \underset{\rightarrow}{\overset{\rightarrow}{\vdots}} K_{n-1,m}$. Take for ∂_I their alternating sum multiplied by $(-1)^m$.

(∂_{II}) The arrows $\mathcal{H}om(\Delta^n, f)_m \underset{\rightarrow}{\overset{\rightarrow}{\vdots}} \mathcal{H}om(\Delta^n, f)_{m-1}$ lead to arrows $K_{n,m} \underset{\rightarrow}{\overset{\rightarrow}{\vdots}} K_{n,m-1}$. Take for ∂_{II} their alternating sum.

One can check that $\partial_I \circ \partial_I = 0 = \partial_{II} \circ \partial_{II}$ and $\partial_I \circ \partial_{II} + \partial_{II} \circ \partial_I = 0$. Form the total chain complex $K_\bullet = \{K_p\} : K_p = \bigoplus\limits_{n+m=p} K_{n,m}$, where $\partial = \partial_I + \partial_{II}$—then there are first quadrant spectral sequences

$$\begin{cases} {}_IE^2_{p,q} = {}_IH_p({}_{II}H_q(K_{\bullet\bullet})) \Rightarrow H_{p+q}(K_\bullet) \\ {}_{II}E^2_{p,q} = {}_{II}H_p({}_IH_q(K_{\bullet\bullet})) \Rightarrow H_{p+q}(K_\bullet) \end{cases}.$$

LEMMA $\quad {}_IE^2_{p,q} \approx \begin{cases} H_q(X) & (p = 0) \\ 0 & (p > 0) \end{cases}.$

[From the definitions, $\sin X = \mathcal{H}om(\Delta^0, f)$. On the other hand, each projection $\Delta^n \rightarrow \Delta^0$ is a homotopy equivalence and induces an arrow $\sin X \rightarrow \mathcal{H}om(\Delta^n, f)$. Since there are $n + 1$ commutative diagrams

$$\begin{tikzcd} \sin X \arrow[r, "\mathrm{id}"] \arrow[d] & \sin X \arrow[d] \\ \mathcal{H}om(\Delta^n, f) \arrow[r] & \mathcal{H}om(\Delta^{n-1}, f) \end{tikzcd}$$

, passing to homology per ∂_{II} gives

$$\begin{matrix} H_q(X) & \xleftarrow{0} & H_q(X) & \xleftarrow{\mathrm{id}} & H_q(X) & \xleftarrow{0} & \cdots \\ (p = 0) & & (p = 1) & & (p = 2) & & \end{matrix}.]$$

Thus the first spectral sequence $_IE$ collapses and $H_*(K_\bullet) \approx H_*(X)$. To explicate the second spectral sequence $_{II}E$, given $\tau \in \sin_n Y$, let X_τ be the fiber over τ of the induced map $\sin_n X \rightarrow \sin_n Y$, i.e., $X_\tau = \{\sigma : f \circ \sigma = \tau\}$. View X_τ as a subspace of $\sin_n X$ (compact open topology). Put $\mathcal{H}_q(f)\tau = H_q(X_\tau)$ and $\forall\ \alpha$, let $\mathcal{H}_q(f)\Delta^\alpha$ be the homomorphism on homology defined by the arrow $X_\tau \rightarrow X_{\tau\circ\Delta^\alpha}$—then $\mathcal{H}_q(f)$ is in $[(\sin Y)^{\text{OP}}, \textbf{AB}]$ or still, is a coefficient system on Y.

[Note: $\forall\ y \in Y$, $\mathcal{H}_q(f)\tau_y = H_q(X_y)$, where $\tau_y \in \sin_0 Y$ with $\tau_y(\Delta^0) = y$.]

LEMMA $\quad {}_{II}E^2_{p,q} \approx H_p(Y; \mathcal{H}_q(f))$.

[$_IH_q(K_{\bullet\bullet})$ can be identified with the chain complex on which the homology of $\mathcal{H}_q(f)$ is computed.]

PROPOSITION 26 Suppose that $f : X \rightarrow Y$ is a Hurewicz fibration—then $\mathcal{H}_q(f)$ is locally constant.

[Fix $\alpha \in \text{Mor}([m],[n])$—then α determines arrows $\begin{cases} C(\Delta^n, X) \rightarrow C(\Delta^m, X) \\ C(\Delta^n, Y) \rightarrow C(\Delta^m, Y) \end{cases}$ and there is a commutative diagram

$$\begin{tikzcd} C(\Delta^n, X) \arrow[r, "f_*"] \arrow[d] & C(\Delta^n, Y) \arrow[d] \\ C(\Delta^m, X) \arrow[r, "f_*"'] & C(\Delta^m, Y) \end{tikzcd}$$

. According to Proposition 5, the horizontal arrows are Hurewicz fibrations. But the vertical arrows are homotopy equivalences, thus $\forall\ \tau \in C(\Delta^n, Y)$ the induced map $X_\tau \rightarrow X_{\tau\circ\Delta^\alpha}$ is a homotopy equivalence (cf. p. 4–25), so $\mathcal{H}_q(f)\Delta^\alpha : H_q(X_\tau) \rightarrow H_q(X_{\tau\circ\Delta^\alpha})$ is an isomorphism.]

[Note: Retaining the assumption that $f : X \rightarrow Y$ is a Hurewicz fibration, one may apply the procedure figuring in the proof of Proposition 25 to the locally constant coefficient system $\mathcal{H}_q(f)$. The result is the cofunctor $\mathcal{H}_q(f) : \Pi Y \rightarrow \textbf{AB}$ defined in the example on p. 4–39.]

Proposition 26 is also true if $f : X \rightarrow Y$ is either a Dold fibration or a Serre fibration.

Consider first the case when f is Dold—then Proposition 5 still holds and the validity of the relevant homotopy theory has already been mentioned (cf. p. 4–30). As for the case when f is Serre, note that the arrow $C(\Delta^n, X) \rightarrow C(\Delta^n, Y)$ is again Serre (as can be seen from the proof of Proposition 5). Therefore, thanks to the Whitehead theorem, the lemma below suffices to complete the argument.

LEMMA Suppose given a commutative diagram

$$\begin{tikzcd} X \arrow[r, "p"] \arrow[d, "\phi"'] & B \arrow[d, "\psi"] \\ Y \arrow[r, "q"'] & A \end{tikzcd}$$

in which $\begin{cases} p \\ q \end{cases}$ are Serre fibrations and $\begin{cases} \phi \\ \psi \end{cases}$ are weak homotopy equivalences—then $\forall\ b \in B$, the induced map $X_b \rightarrow Y_{\psi(b)}$ is a weak homotopy equivalence.

[If X_b is empty, then so is $Y_{\psi(b)}$ and the assertion is trivial. Otherwise, let $a = \psi(b)$ and apply the five lemma to the commutative diagram

$$\begin{tikzcd}
\cdots \arrow[r] & \pi_{q+1}(B) \arrow[r] \arrow[d] & \pi_q(X_b) \arrow[r] \arrow[d] & \pi_q(X) \arrow[r] \arrow[d] & \pi_q(B) \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & \pi_{q+1}(A) \arrow[r] & \pi_q(Y_a) \arrow[r] & \pi_q(Y) \arrow[r] & \pi_q(A) \arrow[r] & \cdots
\end{tikzcd},$$

with the usual caveat at the π_0 and π_1 level.]

The coefficient system $\mathcal{H}_q(f)$ is defined in terms of the integral singular homology of the fibers. Embelish the notation and denote it by $\mathcal{H}_q(f;\mathbf{Z})$. One may then replace $\mathbf{Z}$ by any abelian group $G : \mathcal{H}_q(f;G)$, a coefficient system which is locally constant if $f : X \to Y$ is either a Dold fibration or a Serre fibration.

FIBRATION SPECTRAL SEQUENCE Let $f : X \to Y$ be either a Dold fibration or a Serre fibration—then for any abelian group G, there is a first quadrant spectral sequence $E = \{E^r_{p,q}, d^r\}$ such that $E^2_{p,q} \approx H_p(Y;\mathcal{H}_q(f;G)) \Rightarrow H_{p+q}(X;G)$ and $\forall\ n, H_n(X;G)$ admits an increasing filtration

$$0 = H_{-1,n+1} \subset H_{0,n} \subset \cdots \subset H_{n-1,1} \subset H_{n,0} = H_n(X;G)$$

by subgroups $H_{i,n-i}$, where $E^\infty_{p,q} \approx H_{p,q}/H_{p-1,q+1}$.

[Note: The fibration spectral sequence is natural, i.e., if the diagram

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d] & Y \arrow[d] \\
X' \arrow[r, "f'"'] & Y'
\end{tikzcd}$$

commutes, then there is a morphism $\mu : E \to E'$ of spectral sequences such that $\mu^2_{p,q}$ coincides with the homomorphism $H_p(Y;\mathcal{H}_q(f;G)) \to H_p(Y';\mathcal{H}_q(f';G))$ induced by the arrow $\mathcal{H}_q(f;G) \to \mathcal{H}_q(f';G)$.]

WANG HOMOLOGY SEQUENCE Take $Y = \mathbf{S}^{n+1}$ $(n \geq 1)$ and let $f : X \to Y$ be a Hurewicz fibration with path connected fibers X_y—then there is an exact sequence

$$\cdots \to H_q(X) \to H_{q-n-1}(X_y) \to H_{q-1}(X_y) \to H_{q-1}(X) \to \cdots .$$

EXAMPLE Suppose that $n \geq 1$—then $H_{kn}(\Omega\mathbf{S}^{n+1}) \approx \mathbf{Z}\,(k = 0, 1, \ldots)$, while $H_q(\Omega\mathbf{S}^{n+1}) = 0$ otherwise. Moreover, the Pontryagin ring $H_*(\Omega\mathbf{S}^{n+1})$ is isomorphic to $\mathbf{Z}[t]$, where t generates $H_n(\Omega\mathbf{S}^{n+1})$.

As formulated, the fibration spectral sequence applies to singular homology. There is also a companion result in singular cohomology (with additional multiplicative structure when the coefficient group G is a commutative ring).

WANG COHOMOLOGY SEQUENCE Take $Y = \mathbf{S}^{n+1}$ ($n \geq 1$) and let $f : X \to Y$ be a Hurewicz fibration with path connected fibers X_y—then there is an exact sequence

$$\cdots \to H^q(X) \to H^q(X_y) \xrightarrow{\theta} H^{q-n}(X_y) \to H^{q+1}(X) \to \cdots .$$

[Note: In the graded ring $H^*(X_y)$, $\theta(\alpha \cdot \beta) = \theta(\alpha) \cdot \beta + (-1)^{n|\alpha|}\alpha \cdot \theta(\beta)$.]

EXAMPLE Suppose that $n \geq 1$—then $\theta : H^{kn}(\Omega\mathbf{S}^{n+1}) \to H^{(k-1)n}(\Omega\mathbf{S}^{n+1})$ ($k \geq 1$) is an isomorphism and $H^0(\Omega\mathbf{S}^{n+1})$ is the infinite cyclic group generated by 1. Put $\alpha_0 = 1$ and define α_k ($k \geq 1$) inductively through the relation $\theta(\alpha_k) = \alpha_{k-1}$. Case 1: n even. One has $k!\alpha_k = \alpha_1^k$, therefore $H^*(\Omega\mathbf{S}^{n+1})$ is the divided polynomial algebra generated by $\alpha_1, \alpha_2, \ldots$. Case 2: n odd. One has $\alpha_1^2 = 0$, $\alpha_1\alpha_{2k} = \alpha_{2k+1}, \alpha_1\alpha_{2k+1} = 0$, and $\alpha_2^k = k!\alpha_{2k}$, thus α_1 generates an exterior algebra isomorphic to $H^*(\mathbf{S}^n)$ and $\alpha_2, \alpha_4, \ldots$ generate a divided polynomial algebra isomorphic to $H^*(\Omega\mathbf{S}^{2n+1})$, so $H^*(\Omega\mathbf{S}^{n+1}) \approx H^*(\mathbf{S}^n) \otimes H^*(\Omega\mathbf{S}^{2n+1})$.

In what follows, we shall assume that X is nonempty and Y is path connected.

[Note: If f is Dold, then the X_y have the same homotopy type (cf. p. 4–31), while if f is Serre, then the X_y have the same weak homotopy type (cf. Proposition 31).]

(ED_H) Let $e_H : E^\infty_{p,0} \to E^2_{p,0}$ be the edge homomorphism on the horizontal axis. The arrow of augmentation $H_0(X_y; G) \to G$ is independent of y, so there is a homomorphism $H_p(Y; \mathcal{H}_0(f; G)) \to H_p(Y; G)$. The composite $H_p(X; G) \to H_{p,0}/H_{p-1,1} \approx E^\infty_{p,0} \xrightarrow{e_H} E^2_{p,0} \approx H_p(Y; \mathcal{H}_0(f; G)) \to H_p(Y; G)$ is the homomorphism on homology induced by $f : X \to Y$.

(ED_V) Let $e_V : E^2_{0,q} \to E^\infty_{0,q}$ be the edge homomorphism on the vertical axis. Fix $y \in Y$—then there is an arrow $H_q(X_y; G) \to H_0(Y; \mathcal{H}_q(f; G))$. The composite $H_q(X_y; G) \to H_0(Y; \mathcal{H}_q(f; G)) \approx E^2_{0,q} \xrightarrow{e_V} E^\infty_{0,q} \to H_q(X; G)$ is the homomorphism on homology induced by the inclusion $X_y \to X$.

Keeping to the preceding hypotheses, $f : X \to Y$ is said to be G-orientable provided that the X_y are path connected and $\forall\ q, \mathcal{H}_q(f; G)$ is constant, so $\forall\ y$ the right action $H_q(X_y; G) \times \pi_1(Y, y) \to H_q(X_y; G)$ is trivial.

[Note: If $f : X \to Y$ is G-orientable, then by the universal coefficient theorem, $E^2_{p,q} \approx H_p(Y; H_q(X_y; G)) \approx H_p(Y) \otimes H_q(X_y; G) \oplus \text{Tor}(H_{p-1}(Y), H_q(X_y; G))$.]

EXAMPLE Let $f : X \to Y$ be G-orientable. Assume: $H_i(X_{y_0}; G) = 0$ $(0 < i \leq n)$ and $H_j(Y; \mathbf{Z}) = 0$ $(0 < j \leq m)$—then there is an exact sequence

$$\begin{aligned} H_{n+m+1}(X_{y_0}; G) \to H_{n+m+1}(X; G) \to H_{n+m+1}(Y; G) \to H_{n+m}(X_{y_0}; G) \to \cdots \\ \to H_2(Y; G) \to H_1(X_{y_0}; G) \to H_1(X; G) \to H_1(Y; G). \end{aligned}$$

[For $2 \leq r < n + m + 2$, combine the exact sequence

$$0 \to E^{\infty}_{r,0} \to E^{r}_{r,0} \xrightarrow{d^r} E^{r}_{0,r-1} \to E^{\infty}_{0,r-1} \to 0$$

with the exact sequence

$$0 \to E^{\infty}_{0,r} \to H_r(X; G) \to E^{\infty}_{r,0} \to 0,$$

observing that $H_r(Y; G) \approx E^2_{r,0} \approx E^r_{r,0}$ and $H_{r-1}(X_{y_0}; G) \approx E^2_{0,r-1} \approx E^r_{0,r-1}$, the arrow $H_r(Y; G) \to H_{r-1}(X_{y_0}; G)$ being the transgression.]

[Note: The above assumptions are less stringent than those imposed earlier in the case $G = \mathbf{Z}$ (cf. p. 4–37).]

EXAMPLE Let $f : X \to Y$ be Λ-orientable, where Λ is a principal ideal domain—then the arrow $H_*(X; \Lambda) \to H_*(Y; \Lambda)$ is an isomorphism iff $\forall\, q > 0$, $H_q(X_{y_0}; \Lambda) = 0$ and the arrow $H_*(X_{y_0}; \Lambda) \to H_*(X; \Lambda)$ is an isomorphism iff $\forall\, q > 0$, $H_q(Y; \Lambda) = 0$.

[Note: The formulation is necessarily asymmetric (take Y simply connected and consider $\Theta Y \to Y$).]

FACT Suppose that $f : X \to Y$ is $\mathbf{Z}$-orientable—then any two of the following conditions imply the third: (1) $\forall\, p, H_p(Y)$ is finitely generated; (2) $\forall\, q, H_q(X_{y_0})$ is finitely generated; (3) $\forall\, n, H_n(X)$ is finitely generated.

FACT Suppose that $f : X \to Y$ is $\mathbf{Z}$-orientable—then any two of the following conditions imply the third: (1) $\forall\, p > 0, H_p(Y)$ is finite; (2) $\forall\, q > 0, H_q(X_{y_0})$ is finite; (3) $\forall\, n > 0, H_n(X)$ is finite.

Given pointed spaces $\begin{cases} X \\ Y \end{cases}$, the mapping fiber sequence associated with the inclusion $f : X \vee Y \to X \times Y$ reads: $\cdots \to \Omega(X \vee Y) \to \Omega(X \times Y) \to X \flat Y \to X \vee Y \to X \times Y$.

[Note: The homology of $\Omega(X \vee Y)$ can be calculated in terms of the homology of ΩX and ΩY (Aguade-Castellet[†]).]

LEMMA The arrow $F : \Omega(X \times Y) \to X \flat Y$ is nullhomotopic.

[†] *Collect. Math.* **29** (1978), 3–6; see also Dula-Katz, *Pacific J. Math.* **86** (1980), 451–461.

[Put $\begin{cases} \overline{\Omega}X = \{\sigma : \sigma([1/2,1]) = x_0\} \\ \underline{\Omega}Y = \{\tau : \tau([0,1/2]) = y_0\} \end{cases}$ —then the inclusions $\begin{cases} \overline{\Omega}X \to \Omega X \\ \underline{\Omega}Y \to \Omega Y \end{cases}$ are pointed homotopy equivalences, hence the same holds for their product: $\overline{\Omega}X \times \underline{\Omega}Y \to \Omega X \times \Omega Y = \Omega(X \times Y)$. Use two parameter reversals to see that the composite $\overline{\Omega}X \times \underline{\Omega}Y \to \Theta(X \vee Y) \to X \flat Y$ is equal to the composite $\overline{\Omega}X \times \underline{\Omega}Y \to \Omega(X \times Y) \xrightarrow{F} X \flat Y$, from which $F \simeq 0$.]

GANEA-NOMURA FORMULA Suppose that X and Y are nondegenerate—then in $\mathbf{HTOP}_*$, $\Omega(X \vee Y) \approx \Omega X \times \Omega Y \times \Omega\Sigma(\Omega X \# \Omega Y)$.

[The mapping fiber of $0 : \Omega(X \times Y) \to X \flat Y$ is $\Omega(X \times Y) \times \Omega(X \flat Y)$ and by the lemma, $E_F \approx \Omega(X \times Y) \times \Omega(X \flat Y)$. Employing the notation of p. 4–34, there is a commutative triangle

$$\begin{tikzcd} & E_\pi \arrow[r, "\pi'"] & X \flat Y \\ \Omega(X \times Y) \arrow[ur] \arrow[urr, "F"'] & & \end{tikzcd}$$

. The vertical arrow is a pointed homotopy equivalence, thus $E_{\pi'} \approx E_F$ or still, $\Omega(X \vee Y) \approx \Omega(X \times Y) \times \Omega(X \flat Y) \approx \Omega X \times \Omega Y \times \Omega\Sigma(\Omega X \# \Omega Y)$ (cf. p. 4–34).]

Given pointed spaces $\begin{cases} X \\ Y \end{cases}$, the mapping fiber sequence associated with the projection $f : X \vee Y \to Y$ reads: $\cdots \to \Omega(X \vee Y) \to \Omega Y \to E_f \to X \vee Y \to Y$.

LEMMA The arrow $F : \Omega Y \to E_f$ is nullhomotopic.

[Define $g : Y \to X \vee Y$ by $g(y) = (x_0, y)$, so $f \circ g = \mathrm{id}_Y$. Let Z be any pointed space—then in view of the replication theorem, there is an exact sequence $[Z, \Omega(X \vee Y)] \to [Z, \Omega Y] \to [Z, E_f]$. Since Ωf has a right inverse, the arrow $[Z, \Omega(X \vee Y)] \to [Z, \Omega Y]$ is surjective. This means that the arrow $[Z, \Omega Y] \to [Z, E_f]$ is the zero map, therefore F is nullhomotopic.]

GRAY-NOMURA FORMULA Suppose that X and Y are nondegenerate—then in $\mathbf{HTOP}_*$, $\Omega(X \vee Y) \approx \Omega Y \times \Omega(X \times \Omega Y / \{x_0\} \times \Omega Y)$.

[Argue as in the proof of the Ganea-Nomura formula (E_f is determined on p. 4–34).]

PROPOSITION 27 Let X, Y be pointed spaces—then $\Sigma X \times Y / \{x_0\} \times Y$ has the same pointed homotopy type as $\Sigma X \vee (\Sigma X \# Y)$.

[$\Sigma X \times Y / \{x_0\} \times Y \approx \Sigma X \# Y_+ \approx \Sigma X \# (\mathbf{S}^0 \vee Y) \approx X \# \Sigma(\mathbf{S}^0 \vee Y) \approx X \# (\mathbf{S}^1 \vee \Sigma Y) \approx (X \# \mathbf{S}^1) \vee (X \# \Sigma Y) \approx \Sigma X \vee (\Sigma X \# Y)$.]

[Note: Recall that in $\mathbf{HTOP}_*$, $\Sigma(X \# Y) \approx \Sigma X \# Y \approx X \# \Sigma Y$ for arbitrary pointed X and Y (cf. p. 3–33).]

So, if X is the pointed suspension of a nondegenerate space, then the Gray-Nomura formula can be simplified: $\Omega(X \vee Y) \approx \Omega Y \times \Omega(X \vee (X \# \Omega Y))$. Consequently, for all nondegenerate X and Y,

$$\Omega\Sigma(X \vee Y) \approx \begin{cases} \Omega\Sigma X \times \Omega\Sigma(Y \vee (Y \# \Omega\Sigma X)) \\ \Omega\Sigma Y \times \Omega\Sigma(X \vee (X \# \Omega\Sigma Y)) \end{cases}.$$

Suppose that $\begin{cases} X \\ Y \end{cases}$, Z are wellpointed with $\begin{cases} \{x_0\} \subset X \\ \{y_0\} \subset Y \end{cases}$, $\{z_0\} \subset Z$ closed. Let $f : X \to Y$ be a pointed continuous function, C_f its pointed mapping cone. Let $p : Z \to C_f$ be a pointed continuous function, Z_0 its fiber over the base point. Assume: p is a Hurewicz fibration—then p is a pointed Hurewicz fibration. Form the pullback square

$$\begin{tikzcd} P \arrow[r] \arrow[d] & Z \arrow[d, "p"] \\ Y \arrow[r, "j"'] & C_f \end{tikzcd}$$

. Since $j \circ f \simeq 0$, there is a commutative triangle

$$\begin{tikzcd} & Z \arrow[d, "p"] \\ X \arrow[ur, "k"] \arrow[r, "j \circ f"'] & C_f \end{tikzcd}$$

and an induced map $e : X \to P$.

FACT The pointed mapping cone of the arrow $C_e \to Z$ has the pointed homotopy type of $X * Z_0$.

EXAMPLE Let X be wellpointed with $\{x_0\} \subset X$ closed. The pointed mapping cone of $X \to *$ is ΣX, the pointed suspension of X. Consider the pullback square

$$\begin{tikzcd} \Omega\Sigma X \arrow[r] \arrow[d] & \Theta\Sigma X \arrow[d, "p_1"] \\ * \arrow[r] & \Sigma X \end{tikzcd}$$

. Here, $e : X \to \Omega\Sigma X$ is the arrow of adjunction and the pointed mapping cone of $C_e \to \Theta\Sigma X$ has the same pointed homotopy type as $C_e \to *$, thus in **HTOP**$_*$, $\Sigma C_e \approx X * \Omega\Sigma X$.

Given a pointed space X, the pointed mapping cone sequence associated with the arrow of adjunction $e : X \to \Omega\Sigma X$ reads: $X \xrightarrow{e} \Omega\Sigma X \to C_e \to \Sigma X \to \Sigma\Omega\Sigma X \to \cdots$.

PROPOSITION 28 Let X be nondegenerate—then $\Sigma\Omega\Sigma X$ has the same pointed homotopy type as $\Sigma X \vee \Sigma(X \# \Omega\Sigma X)$.

[Because the evaluation map $r : \Sigma\Omega\Sigma X \to \Sigma X$ exhibits ΣX as a retract of $\Sigma\Omega\Sigma X$, the replication theorem of §3 implies that the arrow $F : C_e \to \Sigma X$ is nullhomotopic, hence $C_F \approx \Sigma X \vee \Sigma C_e$. Reverting to the notation of p. 3–32, there is a commutative triangle

$$\begin{tikzcd} C_e \arrow[r, "j'"] \arrow[dr, "F"'] & C_j \arrow[d] \\ & \Sigma X \end{tikzcd}$$

in which the vertical arrow is a pointed homotopy equivalence. Accordingly, $C_{j'} \approx C_F \Rightarrow \Sigma\Omega\Sigma X \approx \Sigma X \vee \Sigma C_e \approx \Sigma X \vee \Sigma(X \# \Omega\Sigma X)$, the last step by the preceding example.]

Assume: X and Y are nondegenerate. Put $X^{[0]} = \mathbf{S}^0$, $X^{[n]} = X\#\cdots\#X$ (n factors). Starting from the formula $\Omega\Sigma(X \vee Y) \approx \Omega\Sigma X \times \Omega\Sigma(Y \vee (Y\#\Omega\Sigma X))$, successive application of Proposition 28 gives:

$$\Omega\Sigma(X \vee Y) \approx \Omega\Sigma X \times \Omega\Sigma(\bigvee_0^N Y\#X^{[n]} \vee (Y\#X^{[N]}\#\Omega\Sigma X)).$$

FACT Let $\begin{cases} X \\ Y \end{cases}$ be nondegenerate and path connected—then $\forall\, q > 0$, $\pi_q(\Sigma X \vee \Sigma Y) \approx \pi_q(\Sigma X) \oplus \pi_q(\Sigma(\bigvee_0^\infty Y\#X^{[n]}))$.

[By the above, $\pi_q(\Sigma X \vee \Sigma Y)$ is isomorphic to

$$\pi_q(\Sigma X) \oplus \pi_q(\Sigma(\bigvee_0^N Y\#X^{[n]} \vee (Y\#X^{[N]}\#\Omega\Sigma X))).$$

Since $\Sigma(Y\#X^{[N]}\#\Omega\Sigma X)$ is $(N+2)$-connected (cf. p. 3–40), it follows that $\forall\, q \leq N+2 : \pi_q(\Sigma X \vee \Sigma Y) \approx \pi_q(\Sigma X) \oplus \pi_q(\Sigma(\bigvee_0^N Y\#X^{[n]}))$. But $\Sigma(\bigvee_{n>N} Y\#X^{[n]})$ is also $(N+2)$-connected. Therefore, $\forall\, q > 0$: $\pi_q(\Sigma X \vee \Sigma Y) \approx \pi_q(\Sigma X) \oplus \pi_q(\Sigma(\bigvee_0^\infty Y\#X^{[n]}))$.]

A continuous function $f : X \to Y$ is said to be an n-equivalence ($n \geq 1$) provided that f induces a one-to-one correspondence between the path components of $\begin{cases} X \\ Y \end{cases}$ and $\forall\, x_0 \in X$, $f_* : \pi_q(X, x_0) \to \pi_q(Y, f(x_0))$ is bijective for $1 \leq q < n$ and surjective for $q = n$. Example: A pair (X, A) is n-connected iff the inclusion $A \to X$ is an n-equivalence.

[Note: f is an n-equivalence iff the pair $(M_f, i(X))$ is n-connected.]

FACT Let $X \xrightarrow{p} B \xleftarrow{q} Y$ be a 2-sink. Suppose that $\begin{cases} p \text{ is an } n\text{-equivalence} \\ q \text{ is an } m\text{-equivalence} \end{cases}$—then the projection $X \,\square_B\, Y \to B$ is an $(n+m+1)$-equivalence.

[There is an arrow $X \,\square_B\, Y \xrightarrow{\phi} W_p *_B W_q$ that commutes with the projections and is a homotopy equivalence (cf. p. 4–25), thus one can assume that $\begin{cases} p \\ q \end{cases}$ are Hurewicz fibrations and work instead with $X *_B Y$ (the connectivity of the join is given on p. 3–40).]

A continuous function $f : X \to Y$ is said to be a weak homotopy equivalence if f is an n-equivalence $\forall\, n \geq 1$. Example: Consider the coreflector $k : \mathbf{TOP} \to \mathbf{CG}$—then for every topological space X, the identity map $kX \to X$ is a weak homotopy equivalence.

[Note: When X and Y are path connected, f is a weak homotopy equivalence provided that at some $x_0 \in X$, $f_* : \pi_q(X, x_0) \to \pi_q(Y, f(x_0))$ is bijective $\forall\, q \geq 1$.]

Example: Let $$\begin{tikzcd} X \arrow[r, "f"] \arrow[d] & Z \arrow[d] & Y \arrow[l, "g"'] \arrow[d] \\ X' \arrow[r, "f'"'] & Z' & Y' \arrow[l, "g'"] \end{tikzcd}$$ be a commutative diagram in which the vertical arrows are weak homotopy equivalences—then the arrow $W_{f,g} \rightarrow W_{f',g'}$ is a weak homotopy equivalence.

[Compare Mayer-Vietoris sequences (use an ad hoc argument to establish that $\pi_0(W_{f,g}) \approx \pi_0(W_{f',g'})$).]

Example: Let $\begin{cases} X^0 \subset X^1 \subset \cdots \\ Y^0 \subset Y^1 \subset \cdots \end{cases}$ be expanding sequences of topological spaces. Assume: $\forall\ n$, the inclusions $\begin{cases} X^n \rightarrow X^{n+1} \\ Y^n \rightarrow Y^{n+1} \end{cases}$ are closed cofibrations. Suppose given a sequence of continuous functions $\phi^n : X^n \rightarrow Y^n$ such that $\forall\ n$, the diagram $$\begin{tikzcd} X^n \arrow[r] \arrow[d, "\phi^n"'] & X^{n+1} \arrow[d, "\phi^{n+1}"] \\ Y^n \arrow[r] & Y^{n+1} \end{tikzcd}$$ commutes—then $\phi^\infty : X^\infty \rightarrow Y^\infty$ is a weak homotopy equivalence if this is the case of the ϕ^n.

[Consider the commutative diagram $$\begin{tikzcd} \text{tel}\, X^\infty \arrow[r] \arrow[d, "\text{tel}\, \phi"'] & X^\infty \arrow[d, "\phi^\infty"] \\ \text{tel}\, Y^\infty \arrow[r] & Y^\infty \end{tikzcd}$$ (cf. p. 3–12). Since the horizontal arrows are homotopy equivalences, it suffices to prove that tel ϕ is a weak homotopy equivalence. To see this, recall that there are projections $\begin{cases} \text{tel}\, X^\infty \rightarrow [0, \infty[\\ \text{tel}\, Y^\infty \rightarrow [0, \infty[\end{cases}$, thus a compact subset of $\begin{cases} \text{tel}\, X^\infty \\ \text{tel}\, Y^\infty \end{cases}$ must lie in $\begin{cases} \text{tel}_n\, X^\infty \\ \text{tel}_n\, Y^\infty \end{cases}$ ($\exists\ n >> 0$). But $\forall\ n$, the arrow $\text{tel}_n\, X^\infty \rightarrow \text{tel}_n\, Y^\infty$ is a weak homotopy equivalence.]

[Note: Here is a variant. Let $\begin{cases} X^0 \subset X^1 \subset \cdots \\ Y^0 \subset Y^1 \subset \cdots \end{cases}$ be expanding sequences of topological spaces. Assume: $\forall\ n$, $\begin{cases} X^n \\ Y^n \end{cases}$ is T_1. Suppose given a sequence of continuous functions $\phi^n : X^n \rightarrow Y^n$ such that $\forall\ n$, the diagram $$\begin{tikzcd} X^n \arrow[r] \arrow[d, "\phi^n"'] & X^{n+1} \arrow[d, "\phi^{n+1}"] \\ Y^n \arrow[r] & Y^{n+1} \end{tikzcd}$$ commutes—then $\phi^\infty : X^\infty \rightarrow Y^\infty$ is a weak homotopy equivalence if this is the case of the ϕ^n.]

EXAMPLE Given pointed spaces X and Y, let $X \bowtie Y$ be the double mapping track of the 2-sink $X \rightarrow X \vee Y \leftarrow Y$. The projection $X \bowtie Y \rightarrow X \times Y$ is a pointed Hurewicz fibration. Its fiber over (x_0, y_0) is $\Omega(X \vee Y)$ and the composite $\Omega(X \flat Y) \rightarrow \Omega(X \vee Y) \rightarrow X \bowtie Y$ defines a weak homotopy equivalence $\Omega(X \flat Y) \rightarrow X \bowtie Y$.

Assume: X and Y are nondegenerate—then the argument used to establish that

$$\Omega\Sigma(X\vee Y)\approx\Omega\Sigma X\times\Omega\Sigma(\bigvee_{0}^{N}Y\#X^{[n]}\vee(Y\#X^{[N]}\#\Omega\Sigma X))$$

does not explicitly produce a pointed homotopy equivalence between either side but such precision is possible. Let $\begin{cases}\iota_{\Sigma X}\\ \iota_{\Sigma Y}\end{cases}$ be the inclusions $\begin{cases}\Sigma X\to\Sigma X\vee\Sigma Y\\ \Sigma Y\to\Sigma X\vee\Sigma Y\end{cases}$. With $w_0=\iota_{\Sigma Y}$, inductively define $w_1=[w_0,\iota_{\Sigma X}],\ldots,w_n=[w_{n-1},\iota_{\Sigma X}]$, the bracket being the Whitehead product, so $w_1:\Sigma(Y\#X)\to\Sigma X\vee\Sigma Y,\ldots,w_n:\Sigma(Y\#X^{[n]})\to\Sigma X\vee\Sigma Y$. Write $\Omega(\iota_{\Sigma X})+\Omega(\bigvee_{0}^{N}w_n\vee[w_N,\iota_{\Sigma X}\circ r])$ for the composite

$$\Omega\Sigma X\times\Omega\Sigma(\bigvee_{0}^{N}Y\#X^{[n]}\vee(Y\#X^{[N]}\#\Omega\Sigma X))\to\Omega\Sigma(X\vee Y)\times\Omega\Sigma(X\vee Y)\xrightarrow{+}\Omega\Sigma(X\vee Y).$$

Then Spencer[†] has shown that $\Omega(\iota_{\Sigma X})+\Omega(\bigvee_{0}^{N}w_n\vee[w_N,\iota_{\Sigma X}\circ r])$ is a pointed homotopy equivalence.

EXAMPLE Let $\begin{cases}X\\ Y\end{cases}$ be nondegenerate and path connected—then the map

$$\Omega(\iota_{\Sigma X})+\Omega(\bigvee_{0}^{\infty}w_n):\Omega\Sigma X\times\Omega\Sigma(\bigvee_{0}^{\infty}Y\#X^{[n]})\to\Omega\Sigma(X\vee Y)$$

is a weak homotopy equivalence.

Let L be the free Lie algebra over $\mathbf{Z}$ on two generators t_1,t_2. The basic commutators of weight one are t_1 and t_2. Put $e(t_1)=0$, $e(t_2)=0$. Proceeding inductively, suppose that the basic commutators of weight less than n have been defined and ordered as $t_1,\ldots,t_p$ and that a function e from $\{1,\ldots,p\}$ to the nonnegative integers has been defined: $\forall\, i, e(i)<i$. Take for the basic commutators of weight n the $[t_i,t_j]$, where weight t_i+ weight $t_j=n$ and $e(i)\leq j<i$. Order these commutators in any way and label them $t_{p+1},\ldots,t_{p+q}$. Complete the construction by setting $e([t_i,t_j])=j$. Let B be the set of basic commutators thus obtained—then B is an additive basis for L, the Hall basis.

EXAMPLE (Hilton-Milnor Formula) Let $\begin{cases}X\\ Y\end{cases}$ be nondegenerate and path connected. Put $\begin{cases}Z(t_1)=X\\ Z(t_2)=Y\end{cases}$ and let $\begin{cases}\zeta_1:\Sigma Z(t_1)\to\Sigma X\vee\Sigma Y\\ \zeta_2:\Sigma Z(t_2)\to\Sigma X\vee\Sigma Y\end{cases}$ be the inclusions. For $t\in B$ of weight $n>1$, write uniquely $t=[t_i,t_j]$, where weight t_i+ weight $t_j=n$. Via recursion on the weight, put $Z(t)=Z(t_i)\#Z(t_j)$ and let $\zeta_t:\Sigma Z(t)\to\Sigma X\vee\Sigma Y$ be the Whitehead product $[\zeta_i,\zeta_j]$, where $\begin{cases}\zeta_i:\Sigma Z(t_i)\to\Sigma X\vee\Sigma Y\\ \zeta_j:\Sigma Z(t_j)\to\Sigma X\vee\Sigma Y\end{cases}$. The ζ_t combine to define a continuous function $\zeta=\sum_{t\in B}\Omega\zeta_t$ from $(w)\prod_{t\in B}\Omega\Sigma Z(t)$ (cf. p. 1–36) to $\Omega\Sigma(X\vee Y)$. Claim: ζ is a weak homotopy equivalence. To see this, attach to each $N=1,2,\ldots$, a "remainder" $R_N=\bigvee_{\substack{i\geq N\\ e(i)<N}}Z(t_i)$. Applying the preceding example to $\Omega\Sigma(Z(t_N)\vee\bigvee_{\substack{i>N\\ e(i)<N}}Z(t_i))$, it follows that the map

$$\sum_{i=1}^{N}\Omega\zeta_i+\Omega(\bigvee_{\substack{i>N\\ e(i)\leq N}}\zeta_i):\prod_{i=1}^{N}\Omega\Sigma Z(t_i)\times\Omega\Sigma(R_{N+1})\to\Omega\Sigma(X\vee Y)$$

[†] *J. London Math. Soc.* **4** (1971), 291–303.

is a weak homotopy equivalence. To finish, let $N \to \infty$ (justified, since the connectivity of R_{N+1} tends to ∞ with N).

[Note: The isomorphism $\zeta_* : \underset{t \in B}{\oplus} \pi_*(\Omega\Sigma Z(t)) \to \pi_*(\Omega\Sigma(X \vee Y))$ depends on the choice of the Hall basis B. Consult Goerss[†] for an intrinsic description.]

A nonempty path connected topological space X is said to be homotopically trivial if X is n-connected for all n, i.e., provided that $\forall\, q > 0$, $\pi_q(X) = 0$. Example: A contractible space is homotopically trivial.

Example: Let $X \xrightarrow{f} Z \xleftarrow{g} Y$ be a 2-sink. Assume: X & Z are homotopically trivial—then the arrow $W_{f,g} \to Y$ is a weak homotopy equivalence.

EXAMPLE A homotopy equivalence is a weak homotopy equivalence but the converse is false.

(1) (The Wedge of the Broom) Consider the subspace X of $\mathbf{R}^2$ consisting of the line segments joining $(0,1)$ to $(0,0)$ & $(1/n, 0)$ $(n = 1, 2, \ldots)$—then X is contractible, thus it and its base point $(0,0)$ have the same homotopy type. But in the pointed homotopy category, $(X,(0,0))$ and $(\{(0,0)\},(0,0))$ are not equivalent. Consider $X \vee X$, the subspace of $\mathbf{R}^2$ consisting of the line segments joining $\begin{cases} (0,1) \text{ to } (0,0) \ \&\ (1/n,0) \\ (0,-1) \text{ to } (0,0) \ \&\ (-1/n,0) \end{cases}$ $(n = 1, 2, \ldots)$—then $X \vee X$ is path connected and homotopically trivial. However, $X \vee X$ is not contractible, so the map that sends $X \vee X$ to $(0,0)$ is a weak homotopy equivalence but not a homotopy equivalence.

(2) (The Warsaw Circle) Consider the subspace X of $\mathbf{R}^2$ consisting of the union of $\{(x,y) : \begin{cases} x = 0, -2 \le y \le 1 \\ 0 \le x \le 1, y = -2 \\ x = 1, -2 \le y \le 0 \end{cases}$ and $\{(x,y) : 0 < x \le 1, y = \sin(2\pi/x)\}$—then X is path connected and homotopically trivial. However, X is not contractible, so the map that sends X to $(0,0)$ is a weak homotopy equivalence but not a homotopy equivalence.

FACT Let $p : X \to B$ be a Hurewicz fibration, where X and B are path connected and X is nonempty. Suppose that $[p]$ is both a monomorphism and an epimorphism in **HTOP**—then p is a weak homotopy equivalence.

A continuous function $f : (X, A) \to (Y, B)$ is said to be a relative n-equivalence $(n \ge 1)$ provided that the sequence $* \to \pi_0(X, A) \to \pi_0(Y, B)$ is exact and $\forall\, x_0 \in A$, $f_* : \pi_q(X, A, x_0) \to \pi_q(Y, B, f(x_0))$ is bijective for $1 \le q < n$ and surjective for $q = n$.

PROPOSITION 29 Suppose that $\begin{cases} X_1 \\ X_2 \end{cases}$ & $\begin{cases} Y_1 \\ Y_2 \end{cases}$ are open subspaces of $\begin{cases} X \\ Y \end{cases}$ with

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[†] *Quart. J. Math.* 44 (1993), 43–85.

$\begin{cases} X = X_1 \cup X_2 \\ Y = Y_1 \cup Y_2 \end{cases}$. Let $f : X \rightarrow Y$ be a continuous function such that $\begin{cases} X_1 = f^{-1}(Y_1) \\ X_2 = f^{-1}(Y_2) \end{cases}$. Fix $n \geq 1$. Assume: $f : (X_i, X_1 \cap X_2) \rightarrow (Y_i, Y_1 \cap Y_2)$ is a relative n-equivalence ($i = 1, 2$)—then $f : (X, X_i) \rightarrow (Y, Y_i)$ is a relative n-equivalence ($i = 1, 2$).

[This is the content of the result on p. 3–46.]

A continuous function $f : (X, A) \rightarrow (Y, B)$ is said to be a relative weak homotopy equivalence if f is a relative n-equivalence $\forall\ n \geq 1$. Example: Let $p : X \rightarrow B$ be a Serre fibration, where B is path connected and X is nonempty—then $\forall\ b \in B$, the arrow $(X, X_b) \rightarrow (B, b)$ is a relative weak homotopy equivalence.

LEMMA Let $f : (X, A) \rightarrow (Y, B)$ be a continuous function. Assume: $f : A \rightarrow B$ and $f : X \rightarrow Y$ are weak homotopy equivalences—then $f : (X, A) \rightarrow (Y, B)$ is a relative weak homotopy equivalence.

PROPOSITION 30 Suppose that $\begin{cases} X_1 \\ X_2 \end{cases}$ & $\begin{cases} Y_1 \\ Y_2 \end{cases}$ are open subspaces of $\begin{cases} X \\ Y \end{cases}$ with $\begin{cases} X = X_1 \cup X_2 \\ Y = Y_1 \cup Y_2 \end{cases}$. Let $f : X \rightarrow Y$ be a continuous function such that $\begin{cases} X_1 = f^{-1}(Y_1) \\ X_2 = f^{-1}(Y_2) \end{cases}$. Assume: $\begin{cases} f : X_1 \rightarrow Y_1 \\ f : X_2 \rightarrow Y_2 \end{cases}$ & $f : X_1 \cap X_2 \rightarrow Y_1 \cap Y_2$ are weak homotopy equivalences—then $f : X \rightarrow Y$ is a weak homotopy equivalence.

[The lemma implies that $f : (X_i, X_1 \cap X_2) \rightarrow (Y_i, Y_1 \cap Y_2)$ is a relative weak homotopy equivalence ($i = 1, 2$). Therefore, on the basis of Proposition 29, $f : (X, X_i) \rightarrow (Y, Y_i)$ is a relative weak homotopy equivalence ($i = 1, 2$). Since a given $x \in X$ belongs to at least one of the X_i, this suffices (modulo low dimensional details).]

Application: Let

$$\begin{tikzcd} X \arrow[d] & Z \arrow[l, "f"'] \arrow[r, "g"] \arrow[d] & Y \arrow[d] \\ X' & Z' \arrow[l, "f'"] \arrow[r, "g'"'] & Y' \end{tikzcd}$$

be a commutative diagram in which the vertical arrows are weak homotopy equivalences—then the arrow $M_{f,g} \rightarrow M_{f',g'}$ is a weak homotopy equivalence.

[Note: If in addition $\begin{cases} f \\ f' \end{cases}$ are closed cofibrations, then the arrow $X \sqcup_g Y \rightarrow X' \sqcup_{g'} Y'$ is a weak homotopy equivalence (cf. §3, Proposition 18).]

FACT Let $\begin{cases} X \\ Y \end{cases}$ be topological spaces and let $f : X \rightarrow Y$ be a continuous function. Assume: $\mathcal{V} = \{V\}$ is an open covering of Y which is closed under finite intersections such that $\forall\ V \in \mathcal{V}$, $f : f^{-1}(V) \rightarrow V$ is a weak homotopy equivalence—then $f : X \rightarrow Y$ is a weak homotopy equivalence.

[Use Zorn on the collection of subspaces B of Y that have the following properties: B is a union of elements of $\mathcal{V}$, $f : f^{-1}(B) \to B$ is a weak homotopy equivalence, and $\forall\, V \in \mathcal{V}$, $f : f^{-1}(B \cap V) \to B \cap V$ is a weak homotopy equivalence. Order this collection by inclusion and fix a maximal element B_0. Claim: $B_0 = Y$. If not, choose $V \in \mathcal{V} : V \not\subset B_0$ and consider $B_0 \cup V$.]

SUBLEMMA Let $f \in C(X,Y)$ and suppose given continuous functions $\begin{cases} \phi : \mathbf{S}^{n-1} \to X \\ \psi : \mathbf{D}^n \to Y \end{cases}$ with $f \circ \phi = \psi | \mathbf{S}^{n-1}$—then there exists a neighborhood U of $\mathbf{S}^{n-1}$ in $\mathbf{D}^n$ and continuous functions $\begin{cases} \overline{\phi} : U \to X \\ \overline{\psi} : \mathbf{D}^n \to Y \end{cases}$ such that $\overline{\phi} | \mathbf{S}^{n-1} = \phi$ and $f \circ \overline{\phi} = \overline{\psi} | U$, where $\psi \simeq \overline{\psi} \operatorname{rel} \mathbf{S}^{n-1}$.

[Let $U = \{x : 1/2 < \|x\| \leq 1\}$ and put $\overline{\phi}(x) = \phi(x/\|x\|)$ $(x \in U)$. Write $v(x) = \begin{cases} x & (\|x\| \leq 1) \\ x/\|x\| & (\|x\| \geq 1) \end{cases}$. Define $H : I\mathbf{D}^n \to Y$ by $H(x,t) = \psi(v((1+t)x))$ and take $\overline{\psi} = H \circ i_1$.]

LEMMA Suppose that $\begin{cases} X_1 \\ X_2 \end{cases}$ & $\begin{cases} Y_1 \\ Y_2 \end{cases}$ are subspaces of $\begin{cases} X \\ Y \end{cases}$ with $\begin{cases} X = \operatorname{int} X_1 \cup \operatorname{int} X_2 \\ Y = \operatorname{int} Y_1 \cup \operatorname{int} Y_2 \end{cases}$. Let $f : X \to Y$ be a continuous function such that $\begin{cases} f(X_1) \subset Y_1 \\ f(X_2) \subset Y_2 \end{cases}$. Assume: $\begin{cases} f : X_1 \to Y_1 \\ f : X_2 \to Y_2 \end{cases}$ & $f : X_1 \cap X_2 \to Y_1 \cap Y_2$ are weak homotopy equivalences—then $f : X \to Y$ is a weak homotopy equivalence.

[In the notation employed at the end of §3, given continuous functions $\begin{cases} \phi : \dot{I}^q \to X \\ \psi : I^q \to Y \end{cases}$ such that $f \circ \phi = \psi | \dot{I}^q$, it is enough to find a continuous function $\Phi : I^q \to X$ such that $\Phi | \dot{I}^q = \phi$ and $f \circ \Phi \simeq \psi \operatorname{rel} \dot{I}^q$. This can be done by a subdivision argument. The trick is to consider $\begin{cases} \phi^{-1}(X - \operatorname{int} X_1) \cup \overline{\psi^{-1}(Y - Y_1)} \\ \phi^{-1}(X - \operatorname{int} X_2) \cup \overline{\psi^{-1}(Y - Y_2)} \end{cases}$. These sets are closed. However, they need not be disjoint and the point of the sublemma is to provide an escape for this difficulty.]

EXAMPLE In the usual topology, take $Y = \mathbf{R}$, $Y_1 = \mathbf{Q}$, $Y_2 = \mathbf{P}$; in the discrete topology, take $X = \mathbf{R}$, $X_1 = \mathbf{Q}$, $X_2 = \mathbf{P}$—then the identity map $X \to Y$ is not a weak homotopy equivalence, yet the restrictions $\begin{cases} X_1 \to Y_1 \\ X_2 \to Y_2 \end{cases}$, $X_1 \cap X_2 \to Y_1 \cap Y_2$ are weak homotopy equivalences.

FACT Let $\begin{cases} X \\ Y \end{cases}$ be topological spaces and let $f : X \to Y$ be a continuous function. Suppose that $\begin{cases} \mathcal{U} = \{U_i : i \in I\} \\ \mathcal{V} = \{V_i : i \in I\} \end{cases}$ are open coverings of $\begin{cases} X \\ Y \end{cases}$ such that $\forall\, i : f(U_i) \subset V_i$. Assume: For every nonempty finite subset $F \subset I$, the induced map $\bigcap\limits_{i \in F} U_i \to \bigcap\limits_{i \in F} V_i$ is a weak homotopy equivalence—then f is a weak homotopy equivalence.

Topological spaces $\begin{cases} X \\ Y \end{cases}$ are said to have the same weak homotopy type if there exists a topological space Z and weak homotopy equivalences $\begin{cases} f : Z \to X \\ g : Z \to Y \end{cases}$. The relation of having the same weak homotopy type is an equivalence relation.

[Note: One can always replace Z by a CW resolution $K \to Z$, hence $\left\{\begin{matrix} X \\ Y \end{matrix}\right.$ have the same weak homotopy type iff they admit CW resolutions $\left\{\begin{matrix} K \to X \\ K \to Y \end{matrix}\right.$ by the same CW complex K.]

Transitivity is the only issue. For this, let X_1, X_2, X_3 be topological spaces, let K, L be CW complexes, and consider the diagram

$$\begin{tikzcd} & K \arrow[dl, "f_1"'] \arrow[dr, "f_2"] & & L \arrow[dl, "g_2"'] \arrow[dr, "g_3"] & \\ X_1 & & X_2 & & X_3 \end{tikzcd}$$

, where $\left\{\begin{matrix} f_1 \\ f_2 \end{matrix}\right.$, $\left\{\begin{matrix} g_2 \\ g_3 \end{matrix}\right.$ are weak homotopy equivalences. Since (K, f_2) and (L, g_2) are both CW resolutions of X_2, there is a homotopy equivalence $\phi : K \to L$ such that $f_2 \simeq g_2 \circ \phi$ (cf. p. 5–18). Thus $g_3 \circ \phi : K \to X_3$ is a weak homotopy equivalence, so X_1 and X_3 have the same weak homotopy type.

EXAMPLE Two aspherical spaces having isomorphic fundamental groups have the same weak homotopy type.

[Note: A path connected topological space X is said to be aspherical provided that $\forall\, q > 1$, $\pi_q(X) = 0$. Example: If X is path connected and metrizable with $\dim X = 1$, then X is aspherical.]

Let X be in $\mathbf{TOP}/B$. Assume that the projection $p : X \to B$ is surjective—then p is said to be a quasifibration if $\forall\, b \in B$, the arrow $(X, X_b) \to (B, b)$ is a relative weak homotopy equivalence. If $p : X \to B$ is a quasifibration, then $\forall\, b_0 \in B, \forall\, x_0 \in X_{b_0}$, there is an exact sequence

$$\cdots \to \pi_2(B) \to \pi_1(X_{b_0}) \to \pi_1(X) \to \pi_1(B) \to \pi_0(X_{b_0}) \to \pi_0(X) \to \pi_0(B).$$

LEMMA Let $p : X \to B$ be a Serre fibration. Suppose that B is path connected and X is nonempty—then p is a quasifibration.

EXAMPLE Take $X = ([-1,0] \times \{1\}) \cup (\{0\} \times [0,1]) \cup ([0,1] \times \{0\})$, $B = [-1,1]$, and let p be the vertical projection—then p is a quasifibration (X and B are contractible, as are all the fibers) but p is neither a Serre fibration nor a Dold fibration.

[Note: The pullback of a Serre fibration is a Serre fibration, i.e., Proposition 4 is valid with "Hurewicz" replaced by "Serre". This fails for quasifibrations. Let $B' = [0,1]$ and define $\Phi' : B' \to B$ by $\Phi(t) = \begin{cases} t\sin(1/t) & (t > 0) \\ 0 & (t = 0) \end{cases}$—then the projection $p' : X' \to B'$ is not a quasifibration (consider π_0).]

PROPOSITION 31 Let $p : X \to B$ be a quasifibration, where B is path connected—then the fibers of p have the same weak homotopy type.

[Using the mapping track W_p, factor p as $q \circ \gamma$ and note that $\forall\ b \in B$, γ induces a weak homotopy equivalence $X_b \rightarrow q^{-1}(b)$. But $q : W_p \rightarrow B$ is a Hurewicz fibration and since B is path connected, the fibers of q have the same homotopy type (cf. p. 4–13).]

EXAMPLE Let $B = [0,1]^n$ $(n \geq 1)$. Put $X = B \times B - \Delta_B$ and let p be the vertical projection—then p is not a quasifibration (cf. p. 4–8).

LEMMA Let $p : X \rightarrow B$ be a continuous function. Suppose that $O \subset B$ and $p_O : X_O \rightarrow O$ is a quasifibration—then the arrow $(X, X_O) \rightarrow (B, O)$ is a relative weak homotopy equivalence iff $\forall\ b \in O$, the arrow $(X, X_b) \rightarrow (B, b)$ is a relative weak homotopy equivalence.

PROPOSITION 32 Let X be in **TOP**$/B$. Suppose that $\begin{cases} O_1 \\ O_2 \end{cases}$ are open subspaces of B with $B = O_1 \cup O_2$. Assume: $\begin{cases} p_{O_1} : X_{O_1} \rightarrow O_1 \\ p_{O_2} : X_{O_2} \rightarrow O_2 \end{cases}$ & $p_{O_1 \cap O_2} : X_{O_1 \cap O_2} \rightarrow O_1 \cap O_2$ are quasifibrations—then $p : X \rightarrow B$ is a quasifibration.

[From the lemma, the arrows $(X_{O_i}, X_{O_1 \cap O_2}) \rightarrow (O_i, O_1 \cap O_2)$ are relative weak homotopy equivalences $(i = 1, 2)$. Therefore the arrow $(X, X_{O_i}) \rightarrow (B, O_i)$ is a relative weak homotopy equivalence $(i = 1, 2)$ (cf. Proposition 29). Since p is clearly surjective, another appeal to the lemma completes the proof.]

Application: Let X be in **TOP**$/B$. Suppose that $\mathcal{O} = \{O_i : i \in I\}$ is an open covering of B which is closed under finite intersections. Assume: $\forall\ i$, $p_{O_i} : X_{O_i} \rightarrow O_i$ is a quasifibration—then $p : X \rightarrow B$ is a quasifibration.

[The argument is the same as that indicated on p. 4–52 for weak homotopy equivalences.]

[Note: This is the local-global principle for quasifibrations. Here, numerability is irrelevant.]

EXAMPLE Let X be $\mathbf{R}^2$ equipped with the following topology: Basic neighborhoods of (x, y), where $\begin{cases} x \leq 0 \\ x \geq 1 \end{cases}$ & $-\infty < y < \infty$ or $\begin{cases} 0 < x < 1 \ \&\ y > 0 \\ 0 < x < 1 \ \&\ y < 0 \end{cases}$, are the usual neighborhoods but the basic neighborhoods of $(x, 0)$, where $0 < x < 1$, are the open semicircles centered at $(x, 0)$ of radius $< \min\{x, 1-x\}$ that lie in the closed upper half plane. Take $B = \mathbf{R}^2$ (usual topology)—then the identity map $p : X \rightarrow B$ is not a quasifibration (since $\pi_1(B) = 0$, $\pi_1(X) \neq 0$ and the fibers are points). Put $\begin{cases} O_1 = \{(x, y) : x > 0\} \\ O_2 = \{(x, y) : x < 1\} \end{cases}$: $\begin{cases} O_1 \\ O_2 \end{cases}$ are open subspaces of B with $B = O_1 \cup O_2$. Moreover, $\begin{cases} X_{O_1} \\ X_{O_2} \end{cases}$ are contractible, thus $\begin{cases} p_{O_1} : X_{O_1} \rightarrow O_1 \\ p_{O_2} : X_{O_2} \rightarrow O_2 \end{cases}$ are quasifibrations. However, $p_{O_1 \cap O_2} : X_{O_1 \cap O_2} \rightarrow O_1 \cap O_2$ is not a quasifibration.

FACT **Let $p : X \to B$ be a surjective continuous function, where $B = \operatorname{colim} B^n$ is T_1. Assume: $\forall\, n$, $p^{-1}(B^n) \to B^n$ is a quasifibration—then p is a quasifibration.**

Let A be a subspace of X, $i : A \to X$ the inclusion.

(WDR) A is said to be a weak deformation retract of X if there is a homotopy $H : IX \to X$ such that $H \circ i_0 = \mathrm{id}_X$, $H \circ i_t(A) \subset A$ $(0 \leq t \leq 1)$, and $H \circ i_1(X) \subset A$.

[Note: Define $r : X \to A$ by $i \circ r = H \circ i_1$—then $i \circ r \simeq \mathrm{id}_X$ and $r \circ i \simeq \mathrm{id}_A$.]

A strong deformation retract is a weak deformation retract. The comb is a weak deformation retract of $[0,1]^2$ (consider the homotopy $H((x,y),t) = (x,(1-t)y)$) but the comb is not a retract of $[0,1]^2$.

[Note: A pointed space (X, x_0) is contractible to x_0 in **TOP**$_*$ iff $\{x_0\}$ is a weak (or strong) deformation retract of X. The broom with base point $(0,0)$ is an example of a pointed space which is contractible in **TOP** but not in **TOP**$_*$. Therefore a deformation retract need not be a weak deformation retract.]

On a subspace A of X such that the inclusion $A \to X$ is a cofibration, "strong" = "weak".

PROPOSITION 33 Let $p : X \to B$ be a surjective continuous function. Suppose that O is a subspace of B for which $p_O : X_O \to O$ is a quasifibration and $\begin{cases} O \\ X_O \end{cases}$ is a weak deformation retract of $\begin{cases} B \\ X \end{cases}$, say $\begin{cases} \rho : B \to O \\ r : X \to X_O \end{cases}$. Assume: $p \circ r = \rho \circ p$ and $\forall\, b \in B$, $r|X_b$ is a weak homotopy equivalence $X_b \to X_{\rho(b)}$—then $p : X \to B$ is a quasifibration.

[Given $b \in B$, $r : (X, X_b) \to (X_O, X_{\rho(b)})$, as a map of pairs, is a relative weak homotopy equivalence and, by assumption, the diagram

$$\begin{tikzcd}
(X, X_b) \arrow[r] \arrow[d] & (X_O, X_{\rho(b)}) \arrow[d] \\
(B, b) \arrow[r] & (O, \rho(b))
\end{tikzcd}$$

commutes.]

Application: Let

$$\begin{tikzcd}
X \arrow[d] & Z \arrow[l, "f"'] \arrow[r, "g"] \arrow[d] & Y \arrow[d] \\
X' & Z' \arrow[l, "f'"] \arrow[r, "g'"'] & Y'
\end{tikzcd}$$

be a commutative diagram in which the vertical arrows are quasifibrations. Assume: $\forall\, z' \in Z'$, $\begin{cases} f|Z_{z'} \\ g|Z_{z'} \end{cases}$ is a weak homotopy equivalence $\begin{cases} Z_{z'} \to X_{f'(z')} \\ Z_{z'} \to Y_{g'(z')} \end{cases}$—then the arrow $M_{f,g} \to M_{f',g'}$ is a quasifibration.

PROPOSITION 34 Let
$$\begin{tikzcd} X \arrow[d] & Z \arrow[l, "f"'] \arrow[r, "g"] \arrow[d] & Y \arrow[d] \\ X' & Z' \arrow[l, "f'"] \arrow[r, "g'"'] & Y' \end{tikzcd}$$
be a commutative diagram in which the left vertical arrow is a surjective Hurewicz fibration and the right vertical arrow is a quasifibration. Assume:
$$\begin{tikzcd} X \arrow[d] & Z \arrow[l, "f"'] \arrow[d] \\ X' & Z' \arrow[l, "f'"] \end{tikzcd}$$
is a pullback square, $\left\{\begin{array}{l} f \\ f' \end{array}\right.$ are closed cofibrations, and $\forall\ z' \in Z'$, $g|Z_{z'}$ is a weak homotopy equivalence $Z_{z'} \rightarrow Y_{g'(z')}$—then the induced map $X \sqcup_g Y \rightarrow X' \sqcup_{g'} Y'$ is a quasifibration.

[Consider the commutative diagram
$$\begin{tikzcd} M_{f,g} \arrow[r, "\mu"] \arrow[d, "\phi"'] & M_{f',g'} \arrow[d, "\phi'"] \\ X \sqcup_g Y \arrow[r, "\nu"'] & X' \sqcup_{g'} Y' \end{tikzcd}$$
. Since $\left\{\begin{array}{l} f \\ f' \end{array}\right.$ are cofibrations, $\left\{\begin{array}{l} \phi \\ \phi' \end{array}\right.$ are homotopy equivalences (cf. §3, Proposition 18) and, by the above, μ is a quasifibration. Thus it need only be shown that $\forall\ m' \in M_{f',g'}$, the arrow $\mu^{-1}(m') \rightarrow \nu^{-1}(\phi'(m'))$ is a weak homotopy equivalence, which can be done by examining cases.]

The conclusion of Proposition 34 cannot be strengthened to "Hurewicz fibration". To see this, take $X = [-1,0] \times [0,1]$, $Y = [0,2] \times [0,2]$, $Z = \{0\} \times [0,1]$, $X' = [-1,0]$, $Y' = [0,2]$, $Z' = \{0\}$, let $\left\{\begin{array}{l} f : Z \rightarrow X \\ g : Z \rightarrow Y \end{array}\right.$, $\left\{\begin{array}{l} f' : Z' \rightarrow X' \\ g' : Z' \rightarrow Y' \end{array}\right.$ be the inclusions, and let $X \rightarrow X'$, $Z \rightarrow Z'$, $Y \rightarrow Y'$ be the vertical projections—then $X \sqcup_g Y = X \cup Y$, $X' \sqcup_{g'} Y' = X' \cup Y'$, and the induced map $X \cup Y \rightarrow X' \cup Y'$ is the vertical projection. But it is not a Hurewicz fibration since it fails to have the slicing structure property (cf. p. 4–14).

EXAMPLE (Cone Construction) Fix nonempty topological spaces X, Y and let $\phi : X \times Y \rightarrow Y$ be a continuous function. Define E by the pushout square
$$\begin{tikzcd} X \times Y \arrow[r, "\phi"] \arrow[d] & Y \arrow[d] \\ \Gamma X \times Y \arrow[r] & E \end{tikzcd}$$
. Assume: $\forall\ x \in X$, $\phi_x : \{x\} \times Y \rightarrow Y$ is a weak homotopy equivalence. Consider the commutative diagram
$$\begin{tikzcd} \Gamma X \times Y \arrow[d] & X \times Y \arrow[l] \arrow[r, "\phi"] \arrow[d] & Y \arrow[d] \\ \Gamma X & X \arrow[l] \arrow[r] & * \end{tikzcd}$$
. Since the arrows $X \rightarrow \Gamma X$, $X \times Y \rightarrow \Gamma X \times Y$ are closed cofibrations, all the hypotheses of Proposition 34 are met. Therefore the induced map $E \rightarrow \Sigma X$ is a quasifibration.

[Note: The same construction can be made in the pointed category provided that (X, x_0) is wellpointed with $\{x_0\} \subset X$ closed.]

EXAMPLE (Dold-Lashof Construction) Let G be a topological semigroup with unit in which the operations of left and right translation are homotopy equivalences. Let $p : X \rightarrow B$ be a quasifibration. Assume: There is a right action $\begin{cases} X \times G \rightarrow X \\ (x, g) \rightarrow x \cdot g \end{cases}$ such that $p(x \cdot g) = p(x)$ and the arrow $\begin{cases} G \rightarrow X_{p(x)} \\ g \rightarrow x \cdot g \end{cases}$ is a weak homotopy equivalence. Define $\overline{X}$ by the pushout square

$$\begin{tikzcd} X \times G \arrow[r] \arrow[d] & X \arrow[d] \\ \Gamma X \times G \arrow[r] & \overline{X} \end{tikzcd}$$

and put $\overline{B} = C_p$. Since the diagram

$$\begin{tikzcd} \Gamma X \times G \arrow[d] & X \times G \arrow[l] \arrow[r] \arrow[d] & X \arrow[d] \\ \Gamma X & X \arrow[l] \arrow[r] & B \end{tikzcd}$$

commutes, Proposition 34 implies that $\overline{p} : \overline{X} \rightarrow \overline{B}$ is a quasifibration. Represent a generic point of $\overline{X}\,(\overline{B})$ by the symbol $[x, t, g]$ ($[x, t]$) (with the obvious understanding at $t = 0$ or $t = 1$), so $\overline{p}[x, t, g] = [x, t]$. The assignment $\begin{cases} \overline{X} \times G \rightarrow \overline{X} \\ ([x, t, g], h) \rightarrow [x, t, gh] \end{cases}$ is unambiguous and satisfies the algebraic conditions for a right action of G on $\overline{X}$ but it is not necessarily continuous. The resolution is to place a smaller topology on $\overline{X}$. Let $t : \overline{X} \rightarrow [0, 1]$ be the function $[x, t, g] \rightarrow t$; let $x : t^{-1}(]0, 1[) \rightarrow X$ be the function $[x, t, g] \rightarrow x$; let $g : t^{-1}([0, 1[) \rightarrow G$ be the function $[x, t, g] \rightarrow g$; let $x \cdot g : t^{-1}(]0, 1]) \rightarrow X$ be the function $[x, t, g] \rightarrow x \cdot g$. Definition: The coordinate topology on $\overline{X}$ is the initial topology determined by $t, x, g, x \cdot g$. The injection $\begin{cases} X \rightarrow \overline{X} \\ x \rightarrow [x, 1, e] \end{cases}$ is an embedding, as is the injection $\begin{cases} G \rightarrow \overline{X} \\ g \rightarrow [x, t, g] \end{cases}$ $(t \neq 0, 1)$. Moreover, G acts continuously and $\forall\, \overline{x} \in \overline{X}$, the arrow $\begin{cases} G \rightarrow \overline{X}_{\overline{p}(\overline{x})} \\ g \rightarrow \overline{x} \cdot g \end{cases}$ is a weak homotopy equivalence. Now equip $\overline{B}$ with its coordinate topology (cf. p. 3–3)—then $\overline{p} : \overline{X} \rightarrow \overline{B}$ is continuous and remains a quasifibration (apply Propositions 32 and 33 to $\begin{cases} O_1 = \{[x, t] : 0 < t \leq 1\} \\ O_2 = \{[x, t] : 0 \leq t < 1\} \end{cases}$). In other words, $(\overline{X}, \overline{B})$ satisfies the same conditions as (X, B) and there is a commutative diagram

$$\begin{tikzcd} X \arrow[r] \arrow[d] & \overline{X} \arrow[d] \\ B \arrow[r] & \overline{B} \end{tikzcd},$$

where $X \rightarrow \overline{X}$ is inessential (consider $H : \begin{cases} IX \rightarrow \overline{X} \\ (x, t) \rightarrow [x, t, e] \end{cases}$).

Example: Let G be a topological group—then $\overline{G}$ (coordinate topology) is homeomorphic to $G *_c G$ (coarse join).

Let G be a topological group, X a topological space. Suppose that X is a right G-space: $\begin{cases} X \times G \rightarrow X \\ (x, g) \rightarrow x \cdot g \end{cases}$—then the projection $X \rightarrow X/G$ is an open map and X/G is Hausdorff iff $X \times_{X/G} X$ is closed in $X \times X$. The continuous function $\theta : X \times G \rightarrow X \times_{X/G} X$ defined by $(x, g) \rightarrow (x, x \cdot g)$ is surjective. It is injective iff the action is free, i.e., iff $\forall\, x \in X$, the stabilizer $G_x = \{g : x \cdot g = x\}$ of x in G is trivial. A free right G-space X is said to be principal provided that θ is a homeomorphism or still, that the division function $d : \begin{cases} X \times_{X/G} X \to G \\ (x, x \cdot g) \to g \end{cases}$ is continuous.

Let X be in **TOP**$/B$—then X is said to be a principal G-space over B if X is a principal G-space, B is a trivial G-space, the projection $p : X \to B$ is open, surjective, and equivariant, and G operates transitively on the fibers. There is a commutative triangle

$$\begin{tikzcd} & X \arrow[dl] \arrow[d] \\ X/G \arrow[r] & B \end{tikzcd}$$

and the arrow $X/G \to B$ is a homeomorphism. $\mathbf{PRIN}_{B,G}$ is the category whose objects are the principal G-spaces over B and whose morphisms are the equivariant continuous functions over B. If $\Phi' \in C(B', B)$, then for every X in $\mathbf{PRIN}_{B,G}$ there is a pullback square

$$\begin{tikzcd} X' \arrow[r, "f'"] \arrow[d] & X \arrow[d] \\ B' \arrow[r, "\Phi'"'] & B \end{tikzcd}$$

with $X' = B' \times_B X$ in $\mathbf{PRIN}_{B',G}$ and f' equivariant.

LEMMA Every morphism in $\mathbf{PRIN}_{B,G}$ is an isomorphism.

[Note: The objects in $\mathbf{PRIN}_{B,G}$ which are isomorphic to $B \times G$ (product topology) are said to be trivial. It follows from the lemma that the trivial objects are precisely those that admit a section.]

Application: Let $\begin{cases} X' \\ X \end{cases}$ be in $\begin{cases} \mathbf{PRIN}_{B',G} \\ \mathbf{PRIN}_{B,G} \end{cases}$; let $f' \in C(X', X), \Phi' \in C(B', B)$. Assume: f' is equivariant and $p \circ f' = \Phi' \circ p'$—then the commutative diagram

$$\begin{tikzcd} X' \arrow[r, "f'"] \arrow[d] & X \arrow[d] \\ B' \arrow[r, "\Phi'"'] & B \end{tikzcd}$$

is a pullback square.

[Compare this diagram with the pullback square defining the fiber product.]

Let X be in **TOP**$/B$—then X is said to be a G-bundle over B if X is a free right G-space, B is a trivial G-space, the projection $p : X \to B$ is open, surjective, and equivariant, and there exists an open covering $\mathcal{O} = \{O_i : i \in I\}$ of B such that $\forall\, i$, X_{O_i} is equivariantly homeomorphic to $O_i \times G$ over O_i. Since the division function is necessarily continuous and G operates transitively on the fibers, X is a principal G-space over B. If $\mathcal{O}$ can be chosen numerable, then X is said to be a numerable G-bundle over B (a condition that is automatic when B is a paracompact Hausdorff space, e.g., a CW complex). $\mathbf{BUN}_{B,G}$ is the full subcategory of $\mathbf{PRIN}_{B,G}$ whose objects are the numerable G-bundles over B. Each X in $\mathbf{BUN}_{B,G}$ is numerably locally trivial with fiber G and the local-global principle implies that the projection $X \to B$ is a Hurewicz fibration. There is a functor $I : \mathbf{BUN}_{B,G} \to \mathbf{BUN}_{IB,G}$ that sends $p : X \to B$ to $Ip : IX \to IB$, where $(x,t)\cdot g = (x \cdot g, t)$.

EXAMPLE A G-bundle over B need not be numerable. For instance, take $G = \mathbf{R}$—then every object in $\mathbf{BUN}_{B,\mathbf{R}}$ admits a section ($\mathbf{R}$ being contractible), hence is trivial. Let now X be the subset of $\mathbf{R}^3$ defined by the equation $x_1x_3+x_2^2 = 1$ and let $\mathbf{R}$ act on X via $(x_1,x_2,x_3)\cdot t = (x_1, x_2+tx_1, x_3-2tx_2-t^2x_1)$. X is an $\mathbf{R}$-bundle over $X/\mathbf{R}$ but it is not numerable. For if it were, then there would exist a section $X/\mathbf{R} \to X$, an impossibility since $X/\mathbf{R}$ is not Hausdorff.

FACT Suppose that X is a G-bundle over B—then the projection $p : X \to B$ is a Serre fibration (cf. p. 4–11) which is $\mathbf{Z}$-orientable if B and G are path connected.

Let $\left\{\begin{array}{l} X' \\ X \end{array}\right.$ be in $\left\{\begin{array}{l} \mathbf{BUN}_{B',G} \\ \mathbf{BUN}_{B,G} \end{array}\right.$. Write $X' \times_G X$ for the orbit space $(X' \times X)/G$—then there is a commutative diagram

$$\begin{tikzcd} X' \times X \arrow[r] \arrow[d] & X' \arrow[d] \\ X' \times_G X \arrow[r] & B' \end{tikzcd}$$

which is a pullback square. As an object in $\mathbf{TOP}/B'$, $X' \times_G X$ is numerably locally trivial with fiber X so, e.g., has the SEP if X is contractible. The $s' \in \sec_{B'}(X' \times_G X)$ correspond bijectively to the equivariant $f' \in C(X',X)$. As an object in $\mathbf{TOP}/B' \times B$, $X' \times_G X$ is numerably locally trivial with fiber G. Given $\Phi' \in C(B',B)$, there exists an equivariant $f' \in C(X',X)$ rendering the diagram

$$\begin{tikzcd} X' \arrow[r, "f'"] \arrow[d] & X \arrow[d] \\ B' \arrow[r, "\Phi'"'] & B \end{tikzcd}$$

commutative iff the arrow $\left\{\begin{array}{l} B' \to B' \times B \\ b' \to (b', \Phi'(b')) \end{array}\right.$ admits a lifting

$$\begin{tikzcd} & X' \times_G X \arrow[d] \\ B' \arrow[r] \arrow[ur, dashed] & B' \times B \end{tikzcd}.$$

COVERING HOMOTOPY THEOREM Let $\left\{\begin{array}{l} X' \\ X \end{array}\right.$ be in $\left\{\begin{array}{l} \mathbf{BUN}_{B',G} \\ \mathbf{BUN}_{B,G} \end{array}\right.$. Suppose that $f' : X' \to X$ is an equivariant continuous function and $h : IB' \to B$ is a homotopy with $p \circ f' = h \circ i_0 \circ p'$—then there exists an equivariant homotopy $H : IX' \to X$ such that $H \circ i_0 = f'$ and for which the diagram

$$\begin{tikzcd} IX' \arrow[r, "H"] \arrow[d] & X \arrow[d] \\ IB' \arrow[r, "h"'] & B \end{tikzcd}$$

commutes.

[Take $\Phi' = h \circ i_0$ to get a lifting

$$\begin{tikzcd} & X' \times_G X \arrow[d] \\ B' \arrow[r] \arrow[ur, dashed] & B' \times B \end{tikzcd}$$

and a commutative diagram

$$\begin{tikzcd} B' \arrow[r] \arrow[d, "i_0"'] & IX' \times_G X \arrow[d] \\ IB' \arrow[r] & IB' \times B \end{tikzcd}$$

. The projection $IX' \times_G X \to IB' \times B$ is a Hurewicz fibration, thus the diagram has a filler $IB' \to IX' \times_G X$ and this guarantees the existence of H.]

Application: Let X be in $\mathbf{BUN}_{B,G}$. Suppose that $\begin{cases} \Phi_1' \\ \Phi_2' \end{cases} \in C(B', B)$ are homotopic—then $\begin{cases} X_1' \\ X_2' \end{cases}$ are isomorphic in $\mathbf{BUN}_{B',G}$.

FACT The functor $I : \mathbf{BUN}_{B,G} \to \mathbf{BUN}_{IB,G}$ has a representative image.

The relation "isomorphic to" is an equivalence relation on $\text{Ob}\,\mathbf{BUN}_{B,G}$. Call $k_G B$ the "class" of equivalence classes arising therefrom—then $k_G B$ is a "set" (see below). Since for any $\Phi' \in C(B', B)$ and each X in $\mathbf{BUN}_{B,G}$, the isomorphism class $[X']$ of X' in $\mathbf{BUN}_{B',G}$ depends only on the homotopy class $[\Phi']$ of Φ', k_G is a cofunctor $\mathbf{HTOP} \to \mathbf{SET}$. A topological space B_G is said to be a classifying space for G if B_G represents k_G, i.e., if there exists a natural isomorphism $\Xi : [—, B_G] \to k_G$, an X_G in $\Xi_{B_G}(\text{id}_{B_G})$ being a universal numerable G-bundle over B_G. From the definitions, $\forall\ \Phi \in C(B, B_G)$, $\Xi_B[\Phi] = [X]$, where X is defined by the pullback square

$$\begin{tikzcd} X \arrow[r] \arrow[d] & X_G \arrow[d] \\ B \arrow[r, "\Phi"'] & B_G \end{tikzcd}$$

and Φ is the classifying map.

(UN) Assume that $\begin{cases} \Xi' : [—, B_G'] \to k_G \\ \Xi'' : [—, B_G''] \to k_G \end{cases}$ are natural isomorphisms—then there exist mutually inverse homotopy equivalences $\begin{cases} \Phi' : B_G' \to B_G'' \\ \Phi'' : B_G'' \to B_G' \end{cases}$ such that $\begin{cases} k_G[\Phi']([X_G'']) = [X_G'] \\ k_G[\Phi'']([X_G']) = [X_G''] \end{cases}$.

Recall that the members of a class are sets, therefore $k_G B$ is not a class but rather a conglomerate. Still, $\mathbf{BUN}_{B,G}$ has a small skeleton $\overline{\mathbf{BUN}}_{B,G}$. Indeed, any X in $\mathbf{BUN}_{B,G}$ is isomorphic to $B \times G$. Here, the topology on $B \times G$ depends on X and is in general not the product topology but the action is the same $((b, g) \cdot h = (b, gh))$. Thus one can modify the definition of k_G and instead take for $k_G B$ the set $\text{Ob}\,\overline{\mathbf{BUN}}_{B,G}$.

PROPOSITION 35 Suppose that there exists a B_G in $\mathbf{TOP}$ and an X_G in $\mathbf{BUN}_{B_G,G}$ such that X_G is contractible—then k_G is representable.

[Define a natural transformation $\Xi : [—, B_G] \to k_G$ by assigning to a given homotopy class $[\Phi]$ ($\Phi \in C(B, B_G)$) the isomorphism class $[X]$ of the numerable G-bundle X over B

defined by the pullback square
$$\begin{tikzcd} X \arrow[r] \arrow[d] & X_G \arrow[d] \\ B \arrow[r, "\Phi"'] & B_G \end{tikzcd}$$
. The claim is that $\forall\, B, \Xi_B : [B, B_G] \rightarrow k_G B$ is bijective.

Surjectivity: Take any X in $\mathbf{BUN}_{B,G}$ and form $X \times_G X_G$. Since X_G is contractible, $X \times_G X_G$ has the SEP, thus $\sec_B(X \times_G X_G)$ is nonempty, so there exists an equivariant $f \in C(X, X_G)$. Determine $\Phi \in C(B, B_G)$ from the commutative diagram
$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d] & X_G \arrow[d] \\ B \arrow[r, "\Phi"'] & B_G \end{tikzcd}$$
— then $\Xi_B[\Phi] = [X]$.

Injectivity: Let Φ', $\Phi'' \in C(B, B_G)$ and assume that $\Xi_B[\Phi'] = \Xi_B[\Phi'']$, say $[X'] = [X'']$, where
$$\begin{tikzcd} X' \arrow[rr, "\phi"] \arrow[dr] & & X'' \arrow[dl] \\ & B & \end{tikzcd}$$
, with ϕ equivariant. There are pullback squares
$$\begin{tikzcd} X' \arrow[r, "f'"] \arrow[d] & X_G \arrow[d] \\ B \arrow[r, "\Phi'"'] & B_G \end{tikzcd}$$
,
$$\begin{tikzcd} X'' \arrow[r, "f''"] \arrow[d] & X_G \arrow[d] \\ B \arrow[r, "\Phi''"'] & B_G \end{tikzcd}$$
. Put $B_0 = B \times ([0, 1/2[\cup]1/2, 1])$ and define $H_0 : IX'|B_0 \rightarrow X_G$ by
$$H_0(x', t) = \begin{cases} f'(x') & (t < 1/2) \\ f'' \circ \phi(x') & (t > 1/2) \end{cases}$$
: H_0 is equivariant, hence corresponds to a section s_0 of $(IX' \times_G X_G)|B_0$. Since B_0 is a halo of $i_0 B \cup i_1 B$ in IB and since $IX' \times_G X_G$ has the SEP, $\exists\, s \in \sec_{IB}(IX' \times_G X_G) : s|B \times (\{0\} \cup \{1\}) = s_0|B \times (\{0\} \cup \{1\})$. Translated, this means that there exists an equivariant homotopy $H : IX' \rightarrow X_G$. Determine $h : IB \rightarrow B_G$ from the commutative diagram
$$\begin{tikzcd} IX' \arrow[r, "H"] \arrow[d] & X_G \arrow[d] \\ IB \arrow[r, "h"'] & B_G \end{tikzcd}$$
—then $\begin{cases} h \circ i_0 = \Phi' \\ h \circ i_1 = \Phi'' \end{cases} \Rightarrow [\Phi'] = [\Phi''].]$

The converse of Proposition 35 is also true: In order that k_G be representable, it is necessary that X_G be contractible. Thus let X_G^∞ be the numerable G-bundle over B_G^∞ produced by the Milnor construction—then X_G^∞ is contractible, so Ξ^∞ is a natural isomorphism. As the same holds for Ξ by assumption, there are pullback squares
$$\begin{tikzcd} X_G \arrow[r, "f"] \arrow[d] & X_G^\infty \arrow[d] \\ B_G \arrow[r, "\Phi"'] & B_G^\infty \end{tikzcd}$$
,
$$\begin{tikzcd} X_G^\infty \arrow[r, "f^\infty"] \arrow[d] & X_G \arrow[d] \\ B_G^\infty \arrow[r, "\Phi^\infty"'] & B_G \end{tikzcd}$$
and $\Phi^\infty \circ \Phi \simeq \mathrm{id}_{B_G}$. Owing to the covering homotopy theorem, $f^\infty \circ f$ is equivariantly homotopic to an isomorphism
$$\begin{tikzcd} X_G \arrow[rr, "\phi"] \arrow[dr] & & X_G \arrow[dl] \\ & B_G & \end{tikzcd}$$
. But ϕ is

necessarily inessential, X_G^∞ being contractible.

EXAMPLE Let E be an infinite dimensional Hilbert space—then its general linear group $\mathbf{GL}(E)$ is contractible (cf. p. 6–10). Any compact Lie group G can be embedded as a closed subgroup of $\mathbf{GL}(E)$. So, if $X_G = \mathbf{GL}(E)$, $B_G = \mathbf{GL}(E)/G$, then B_G is a classifying space for G and X_G is universal.

[B_G is a paracompact Hausdorff space. Local triviality of X_G is a consequence of a generality due to Gleason, viz: Suppose that G is a compact Lie group and X is a Hausdorff principal G-space which is completely regular—then X, as an object in $\mathbf{TOP}/B$ $(B = X/G)$, is a G-bundle.]

EXAMPLE Let G be a noncompact connected semisimple Lie group with finite center, $K \subset G$ a maximal compact subgroup. The coset space $K\backslash G$ is contractible, being diffeomorphic to some $\mathbf{R}^n$. Let Γ be a discrete subgroup of G. Assume: Γ is cocompact and torsion free—then Γ operates on $K\backslash G$ by right translation and $K\backslash G$ is a numerable Γ-bundle over $K\backslash G/\Gamma$. So, if $X_\Gamma = K\backslash G$, $B_\Gamma = K\backslash G/\Gamma$, then B_Γ is a classifying space for Γ and X_Γ is universal.

[Note: B_Γ is a compact riemannian manifold. Its universal covering space is X_Γ, thus B_Γ is aspherical and of homotopy type $(\Gamma, 1)$.]

MILNOR CONSTRUCTION Let G be a topological group. Consider the subset of $([0,1] \times G)^\omega$ made up of the strings $\{(t_i, g_i)\}$ for which $\sum_i t_i = 1$ & $\#\{i : t_i \neq 0\} < \omega$. Write $\{(t_i', g_i')\} \sim \{(t_i'', g_i'')\}$ iff $\forall\, i, t_i' = t_i''$ and at those i such that $t_i' = t_i''$ is positive, $g_i' = g_i''$. Call X_G^∞ the resulting set of equivalence classes. Define coordinate functions t_i and g_i by $t_i : \begin{cases} X_G^\infty \to [0,1] \\ x \to t_i(x) \end{cases}$ and $g_i : \begin{cases} t_i^{-1}(]0,1]) \to G \\ x \to g_i(x) \end{cases}$, where $x = [(t_i(x), g_i(x))]$. The Milnor topology on X_G^∞ is the initial topology determined by the t_i and g_i. Thus topologized, X_G^∞ is a right G-space: $\begin{cases} X_G^\infty \times G \to X_G^\infty \\ (x, g) \to x \cdot g \end{cases}$. Here, $t_i(x \cdot g) = t_i(x)$ and $g_i(x \cdot g) = g_i(x)g$. Let B_G^∞ be the orbit space X_G^∞/G.

[Note: Put $X_G^0 = G$, $X_G^n = G *_c \cdots *_c G$, the $(n+1)$-fold coarse join of G with itself. One can identify X_G^n with $\{x : \forall\, i \geq n+1, t_i(x) = 0\}$. Each X_G^n is a zero set in X_G^∞ and there is an equivariant embedding $X_G^n \to X_G^{n+1}$. So, $X_G^0 \subset X_G^1 \subset \cdots$ is an expanding sequence of topological spaces and the colimit in $\mathbf{TOP}$ associated with this data is X_G^∞ equipped with the final topology determined by the inclusions $X_G^n \to X_G^\infty$. The colimit topology is finer than the Milnor topology and in general, there is no guarantee that the G-action $(x, g) \to x \cdot g$ remains continuous.]

(M) X_G^∞ is a numerable G-bundle over B_G^∞.

[It is clear that X_G^∞ is a principal G-space. Write O_i for the image of $t_i^{-1}(]0,1])$ under the projection $X_G^\infty \to B_G^\infty$—then $\{O_i\}$ is a countable cozero set covering of B_G^∞, hence is numerable (cf. p. 1–25). On the other hand, $\forall\, i, \sec_{O_i}(X_G^\infty | O_i)$ is nonempty. To see this, define a continuous fiber preserving function $f_i : X_G^\infty|O_i \to X_G^\infty|O_i$ by $f_i(x) = x \cdot g_i(x)^{-1}$: $\forall\, g \in G$, $f_i(x \cdot g) = f_i(x)$. Consequently, f_i drops to a section $s_i : O_i \to X_G^\infty|O_i$, therefore $X_G^\infty|O_i$ is trivial.]

(D) X_G^∞ is contractible.

[Let Δ_G^∞ be the subset of X_G^∞ consisting of those x such that $g_i(x) = e$ if $t_i(x) > 0$— then Δ_G^∞ is contractible, so one need only construct a homotopy $H : IX_G^\infty \to X_G^\infty$ such that $H \circ i_0 = \mathrm{id}_{X_G^\infty}$ and $H \circ i_1(X_G^\infty) \subset \Delta_G^\infty$. Put $U_k = \tau_k^{-1}(]0,1])$ and $A_k = \tau_k^{-1}(1)$, where $\tau_k = \sum\limits_{i \leq k} t_i$. Define $H'_k : IU_k \to U_k$ by

$$t_i(H'_k(x,t)) = \begin{cases} \dfrac{t + (1-t)\tau_k(x)}{\tau_k(x)}\, t_i(x) & (i \leq k) \\ (1-t)t_i(x) & (i > k) \end{cases}$$

and $g_i(H'_k(x,t)) = g_i(x)$ when $t_i(H'_k(x,t)) > 0$. Note that $H'_k(x,0) = x$, $H'_k(x,1) \in A_k$, and $x \in \Delta_G^\infty \Rightarrow H'_k(x,t) \in \Delta_G^\infty$ $(0 \leq t \leq 1)$. Define $H''_k : IA_k \to A_{k+1}$ by

$$t_i(H''_k(x,t)) = \begin{cases} (1-t)t_i(x) & (i \leq k) \\ t & (i = k+1) \\ 0 & (i > k+1) \end{cases}$$

and $g_i(H''_k(x,t)) = \begin{cases} g_i(x) & (i \leq k) \\ e & (i = k+1) \end{cases}$ when $t_i(H''_k(x,t)) > 0$. Note that $H''_k(x,0) = x$, $H''_k(x,1) \in \Delta_G^\infty$, and $x \in \Delta_G^\infty \Rightarrow H''_k(x,t) \in \Delta_G^\infty$ $(0 \leq t \leq 1)$. Combine $\begin{cases} H'_k \\ H''_k \end{cases}$ and obtain a homotopy $H_k : IU_k \to U_{k+1}$ such that $H_k(x,0) = x$, $H_k(x,1) \in \Delta_G^\infty$, and $x \in \Delta_G^\infty \Rightarrow H_k(x,t) \in \Delta_G^\infty$ $(0 \leq t \leq 1)$. Proceeding recursively, write $G_1 = H_1$ and

$$G_{k+1}(x,t) = \begin{cases} G_k(x,t) & (2/3 \leq \tau_k(x) \leq 1) \\ H_{k+1}(G_k(x,t), 2t(2 - 3\tau_k(x))) & (1/2 \leq \tau_k(x) \leq 2/3) \\ H_{k+1}(G_k(x, 2t(3\tau_k(x) - 1)), t) & (1/3 \leq \tau_k(x) \leq 1/2) \\ H_{k+1}(x,t) & (0 \leq \tau_k(x) \leq 1/3) \end{cases}$$

to get a sequence of homotopies $G_k : IU_k \to U_{k+1}$ such that $G_{k+1}|I\tau_k^{-1}(]2/3,1]) = G_k|I\tau_k^{-1}(]2/3,1])$ and $G_k(x,0) = x$, $G_k(x,1) \in \Delta_G^\infty$. Take for H the homotopy $IX_G^\infty \to X_G^\infty$ that agrees on $I\tau_k^{-1}(]2/3,1])$ with G_k.]

[Note: The argument shows that Δ_G^∞ is a weak deformation retract of X_G^∞.]

FACT (Borel Construction) Let X be in $\mathbf{BUN}_{B,G}$. There is a pullback square

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d] & X_G^\infty \arrow[d] \\ B \arrow[r, "\Phi"'] & B_G^\infty \end{tikzcd}$$

and since f is equivariant, the continuous function $\begin{cases} X \to X \times X_G^\infty \\ x \to (x, f(x)) \end{cases}$ induces a map $B \to X \times_G X_G^\infty$, which is a homotopy equivalence (cf. p. 3–25).

FACT Let $\alpha : G \rightarrow K$ be a continuous homomorphism—then α determines a continuous function $f_\alpha : X_G^\infty \rightarrow X_K^\infty$ such that $f_\alpha(x \cdot g) = f_\alpha(x) \cdot \alpha(g)$. There is a commutative diagram

$$\begin{tikzcd} X_G^\infty \arrow[r, "f_\alpha"] \arrow[d] & X_K^\infty \arrow[d] \\ B_G^\infty \arrow[r, "\Phi_\alpha"'] & B_K^\infty \end{tikzcd}$$

and Φ_α is a homotopy equivalence iff α is a homotopy equivalence.

CLASSIFICATION THEOREM For any topological group G, the functor k_G is representable.

[This follows from Proposition 35 and the Milnor construction.]

The isomorphism classes of numerable G-bundles over B are therefore in a one-to-one correspondence with the elements of $[B, B_G^\infty]$. By comparison, recall that on general grounds the isomorphism classes of G-bundles over B are in a one-to-one correspondence with the elements of the cohomology set $H^1(B; \mathbf{G})$ ($\mathbf{G}$ the sheaf of G-valued continuous functions on B).

LEMMA Suppose that G is metrizable—then the Milnor topology on X_G^∞ is metrizable.

[Fix a metric d_G on $G : d_G \leq 1$. Define a metric d on X_G^∞ by

$$d(x,y) = \sum_i \min\{t_i(x), t_i(y)\} d_G(g_i(x), g_i(y)) + (1 - \sum_i \min\{t_i(x), t_i(y)\}).$$

To check the triangle inequality, consider $\frac{1}{2}|t_i(x) - t_i(y)| + \min\{t_i(x), t_i(y)\} d_G(g_i(x), g_i(y))$ and distinguish two cases: $t_i(z) \geq \min\{t_i(x), t_i(y)\}$ & $t_i(z) < \min\{t_i(x), t_i(y)\}$. In the metric topology, the coordinate functions are continuous, thus the metric topology is finer than the Milnor topology. To go the other way, let $\{x_n\}$ be a net in X_G^∞ such that $x_n \rightarrow x$ in the Milnor topology. Claim: $x_n \rightarrow x$ in the metric topology. Fix $\epsilon > 0$. Since $\sum_i t_i(x) = 1$, $\exists\, N : \sum_1^N t_i(x) > 1 - \frac{\epsilon}{4}$. Choose $n_0 : \forall\, n \geq n_0$ & $1 \leq i \leq N$, $|t_i(x_n) - t_i(x)| < \frac{\epsilon}{4N}$ and $t_i(x) > 0 \Rightarrow t_i(x_n) > 0$ with $d_G(g_i(x_n), g_i(x)) < \frac{\epsilon}{4N}$, from which

$$d(x_n, x) \leq \sum_1^N \min\{t_i(x_n), t_i(x)\} d_G(g_i(x_n), g_i(x)) + (1 - \sum_1^N \min\{t_i(x_n), t_i(x)\}) \leq \frac{\epsilon}{4} + 1 - (1 - \frac{\epsilon}{2}) < \epsilon.]$$

[Note: B_G^∞ is also metrizable. For this, it need only be shown that B_G^∞ is locally metrizable and paracompact (cf. p. 1–19). Local metrizability follows from the fact that $X_G^\infty | O_i$ is homeomorphic to $O_i \times G$. Since a metrizable space is paracompact and since $\{O_i\}$ is numerable, B_G^∞ admits a neighborhood finite closed covering by paracompact subspaces, hence is a paracompact Hausdorff space (cf. p. 5–4).]

EXAMPLE X_G^∞ in the colimit topology is contractible. This is because $\forall\, n$, the inclusion $X_G^n \rightarrow X_G^{n+1}$ is a cofibration (cf. p. 3–4) and inessential, thus the result on p. 3–20 can be applied. Consequently, if the underlying topology on G is locally compact and Hausdorff (e.g., if G is Lie), then $\operatorname{colim}(X_G^n \times G) = (\operatorname{colim} X_G^n) \times G$, so X_G^∞ in the colimit topology is a right G-space. As such, it is a numerable G-bundle

over B_G^∞, which is therefore a classifying space for G (cf. Proposition 35). While the topology on B_G^∞ arising in this fashion is finer than that produced by the Milnor construction, it has the advantage of being "computable". For example, let G be $\mathbf{S}^0$, $\mathbf{S}^1$, or $\mathbf{S}^3$, the multiplicative group of elements of norm one in $\mathbf{R}$, $\mathbf{C}$, or $\mathbf{H}$—then $X_G^n = \mathbf{S}^n$, $\mathbf{S}^{2n+1}$, or $\mathbf{S}^{4n+3}$, hence $X_G^\infty = \mathbf{S}^\infty$ and factoring in the action, $B_G^\infty = \mathbf{P}^\infty(\mathbf{R}), \mathbf{P}^\infty(\mathbf{C})$, or $\mathbf{P}^\infty(\mathbf{H})$. As a colimit of the $\mathbf{S}^n$, $\mathbf{S}^\infty$ is not first countable. However, the three topologies on its underlying set coming from the Milnor construction are metrizable, in particular first countable.

[Note: Here is another model for X_G and B_G when $G = \mathbf{S}^0, \mathbf{S}^1$, or $\mathbf{S}^3$. Take an infinite dimensional Banach space E over $\mathbf{R}$, $\mathbf{C}$, or $\mathbf{H}$ and let S be its unit sphere—then S is an AR (cf. p. 6–13), hence contractible (cf. p. 6–14), so $X_G = S$ is universal and $B_G = S/G$ is classifying.]

Let G be a compact Lie group—then Notbohm[†] has shown that the homotopy type of B_G^∞ determines the Lie group isomorphism class of G.

Consider G as a pointed space with base point e. Let $x_G^\infty = [(1,e),(0,e),\ldots]$ be the base point in X_G^∞, $b_G^\infty = x_G^\infty \cdot G$ the base point in B_G^∞—then $\forall\ q \geq 0$, $\pi_q(G) \approx \pi_{q+1}(B_G^\infty)$. Choose a homotopy $H : IX_G^\infty \rightarrow X_G^\infty$ such that $\begin{cases} H(x,0) = x_G^\infty \\ H(x,1) = x \end{cases}$. Taking adjoints and projecting leads to a map $X_G^\infty \rightarrow \Theta B_G^\infty$. The triangle

$$\begin{tikzcd} X_G^\infty \arrow[rr] \arrow[dr] & & \Theta B_G^\infty \arrow[dl, "p_1"] \\ & B_G^\infty & \end{tikzcd}$$

commutes, thus there is an arrow $G \rightarrow \Omega B_G^\infty$.

PROPOSITION 36 The arrow $G \rightarrow \Omega B_G^\infty$ is a homotopy equivalence.

[The map $X_G^\infty \rightarrow \Theta B_G^\infty$ is a homotopy equivalence (by contractibility). But the projections $X_G^\infty \rightarrow B_G^\infty, \Theta B_G^\infty \xrightarrow{p_1} B_G^\infty$ are Hurewicz fibrations. Therefore the map $X_G^\infty \rightarrow \Theta B_G^\infty$ is a fiber homotopy equivalence (cf. Proposition 15).]

EXAMPLE **Take $B = \mathbf{S}^n$ $(n \geq 1)$—then $k_G\mathbf{S}^n \approx [\mathbf{S}^n, B_G^\infty] \approx \pi_1(B_G^\infty, b_G^\infty)\backslash[\mathbf{S}^n, s_n; B_G^\infty, b_G^\infty] \approx \pi_1(B_G^\infty, b_G^\infty)\backslash\pi_n(B_G^\infty, b_G^\infty) \approx \pi_0(G,e)\backslash\pi_{n-1}(G,e)$, i.e., in brief: $k_G\mathbf{S}^n \approx \pi_0(G)\backslash\pi_{n-1}(G)$.**

LEMMA **Suppose that G is an ANR—then X_G^∞ and B_G^∞ are ANRs (cf. p. 6–45) and the arrow $G \rightarrow \Omega B_G^\infty$ is a pointed homotopy equivalence.**

[Being ANRs, (G,e) & $\begin{cases} (X_G^\infty, x_G^\infty) \\ (B_G^\infty, b_G^\infty) \end{cases}$ are wellpointed (cf. p. 6–14). Therefore X_G^∞ is contractible to x_G^∞ in $\mathbf{TOP}_*$ and the arrow $G \rightarrow \Omega B_G^\infty$ is a pointed map. But $(\Omega B_G^\infty, j(b_G^\infty))$ is wellpointed (cf. p.

[†] *J. London Math. Soc.* **52** (1995), 185–198.

3-17) (actually ΩB_G^∞ is an ANR (cf. §6, Proposition 7)), so the arrow $G \to \Omega B_G^\infty$ is a pointed homotopy equivalence (cf. p. 3-19).]

EXAMPLE Let G be a Lie group—then G is an ANR (cf. p. 6-28). Consider $k_G \Sigma B$, where (B, b_0) is nondegenerate and ΣB is the pointed suspension. Thus $k_G \Sigma B \approx [\Sigma B, B_G^\infty] \approx \pi_1(B_G^\infty, b_G^\infty) \backslash [B, b_0; \Omega B_G^\infty, j(b_G^\infty)] \approx \pi_0(G, e) \backslash [B, b_0; G, e]$, which, when G is path connected, simplifies to $[B, b_0; G, e]$ or still, $[B, G]$ (the action of $\pi_1(G, e)$ on $[B, b_0; G, e]$ is trivial).

[Note: Suppose that G is an arbitrary path connected topological group—then again $k_G \Sigma B \approx [B, b_0; \Omega B_G^\infty, j(b_G^\infty)]$. However, ΩB_G^∞ is a path connected H group, hence $[B, b_0; \Omega B_G^\infty, j(b_G^\infty)] \approx [B, \Omega B_G^\infty]$ and, by Proposition 36, $[B, \Omega B_G^\infty] \approx [B, G]$.]

§ 4

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§5. VERTEX SCHEMES AND CW COMPLEXES

Vertex schemes and CW complexes pervade algebraic topology. What follows is an account of their basic properties. All the relevant facts will be stated with precision but I shall only provide proofs for those that are not readily available in the standard treatments.

A *vertex scheme* K is a pair (V, Σ) consisting of a set $V = \{v\}$ and a subset $\Sigma = \{\sigma\} \subset 2^V$ subject to: (1) $\forall\, \sigma : \sigma \neq \emptyset$ & $\#(\sigma) < \omega$; (2) $\forall\, \sigma : \emptyset \neq \tau \subset \sigma \Rightarrow \tau \in \Sigma$; (3) $\forall\, v : \{v\} \in \Sigma$. The elements v of V are called the *vertexes* of K and the elements σ of Σ are called the *simplexes* of K, the nonempty $\tau \subset \sigma$ being termed the *faces* of σ. A *vertex map* $f : K_1 = (V_1, \Sigma_1) \to K_2 = (V_2, \Sigma_2)$ is a function $f : V_1 \to V_2$ such that $\forall\, \sigma_1 \in \Sigma_1, f(\sigma_1) \in \Sigma_2$. **VSCH** is the category whose objects are the vertex schemes and whose morphisms are the vertex maps.

EXAMPLE Let X be a set; let $\mathcal{S} = \{S\}$ be a collection of subsets of X—then the *nerve* of $\mathcal{S}$, written $N(\mathcal{S})$, is the vertex scheme whose vertexes are the nonempty elements of $\mathcal{S}$ and whose simplexes are the nonempty finite subsets of $\mathcal{S}$ with nonempty intersection.

Let $K = (V, \Sigma)$ be a vertex scheme. If $\#(\Sigma) < \omega$ $(\leq \omega)$, then K is said to be *finite* (*countable*). If $\forall\, v, \#\{\sigma : v \in \sigma\} < \omega$, then K is said to be *locally finite*. A *subscheme* of K is a vertex scheme $K' = (V', \Sigma')$ such that $\begin{cases} V' \subset V \\ \Sigma' \subset \Sigma \end{cases}$. An *$n$-simplex* is a simplex of cardinality $n + 1$ $(n \geq 0)$. The *n-skeleton* of K is the subscheme $K^{(n)} = (V^{(n)}, \Sigma^{(n)})$ of K defined by putting $V^{(n)} = V$ and letting $\Sigma^{(n)} \subset \Sigma$ be the set of m-simplexes of K with $m \leq n$. The *combinatorial dimension* of K, written $\dim K$, is -1 if K is empty, otherwise is n if K contains an n-simplex but no $(n+1)$-simplex and is ∞ if K contains n-simplexes for all $n \geq 0$. If K is finite, then $\dim K$ is finite. The converse is trivially false.

EXAMPLE In the plane, take $V = \{(0,0)\} \cup \{(1, 1/n) : n \geq 1\}$. Let $K = (V, \Sigma)$ be any vertex scheme having for its 1-simplexes the sets $\sigma_n = \{(0,0), (1, 1/n)\}$ $(n \geq 1)$—then K is not locally finite.

Given a vertex scheme $K = (V, \Sigma)$, let $|K|$ be the set of all functions $\phi : V \to [0,1]$ such that $\phi^{-1}(]0,1]) \in \Sigma$ & $\sum\limits_v \phi(v) = 1$. Assign to each σ the sets $\begin{cases} \langle\sigma\rangle = \{\phi \in |K| : \phi^{-1}(]0,1]) = \sigma\} \\ |\sigma| = \{\phi \in |K| : \phi^{-1}(]0,1]) \subset \sigma\} \end{cases}$. So, $\forall\, \sigma : \langle\sigma\rangle \subset |\sigma|$ and $|K| = \bigcup\limits_\sigma \langle\sigma\rangle$, a disjoint union. Traditionally, there are two ways to topologize $|K|$.

(WT) If σ is an n-simplex, then $|\sigma|$ can be viewed as a compact Hausdorff space: $|\sigma| \leftrightarrow \Delta^n$. This said, the *Whitehead topology* on $|K|$ is the final topology determined by the inclusions $|\sigma| \to |K|$. $|K|$ is a perfectly normal paracompact Hausdorff space. Moreover, $|K|$ is $\begin{cases} \text{compact} \\ \text{locally compact} \end{cases}$ iff K is $\begin{cases} \text{finite} \\ \text{locally finite} \end{cases}$.

(BT) There is a map $\begin{cases} V \to [0,1]^{|K|} \\ v \to b_v : b_v(\phi) = \phi(v) \end{cases}$. The b_v are called the barycentric coordinates, the initial topology on $|K|$ determined by them being the barycentric topology, a topology that is actually metrizable: $d(\phi, \psi) = \sum\limits_v |b_v(\phi) - b_v(\psi)|$.

To keep things straight, denote by $|K|_b$ the set $|K|$ equipped with the barycentric topology—then the identity map $i : |K| \to |K|_b$ is continuous, thus the Whitehead topology is finer than the barycentric topology. The two agree iff K is locally finite.

[Note: A vertex map $f : K_1 = (V_1, \Sigma_1) \to K_2 = (V_2, \Sigma_2)$ induces a map $|f| : \begin{cases} |K_1| \to |K_2| \\ \phi_1 \to \phi_2 \end{cases}$, where $\phi_2(v_2) = \sum\limits_{f(v_1)=v_2} \phi_1(v_1)$. Topologically, $|f|$ is continuous in either the Whitehead topology or the barycentric topology. Consequently, there are two functors from **VSCH** to **TOP**, connected by the obvious natural transformation.]

EXAMPLE Let E be a vector space over $\mathbf{R}$. Let V be a basis for E; let Σ be the set of nonempty finite subsets of V. Call $K(E)$ the associated vertex scheme. Equip E with the finite topology—then $|K(E)|$ can be identified with the convex hull of V in E. But $|K(E)|$ and $|K(E)|_b$ are homeomorphic iff E is finite dimensional.

[Note: Let $K = (V, \Sigma)$ be a vertex scheme. Take for E the free $\mathbf{R}$-module on V, equipped with the finite topology—then $|K|$ can be embedded in $|K(E)|$.]

PROPOSITION 1 The identity map $i : |K| \to |K|_b$ is a homotopy equivalence.

[The collection $\{b_v^{-1}(]0,1])\}$ is an open covering of $|K|_b$, hence has a precise neighborhood finite open refinement $\{U_v\}$. Choose a partition of unity $\{\kappa_v\}$ on $|K|_b$ subordinate to $\{U_v\}$. Let $j : |K|_b \to |K|$ be the map that sends ψ to the function $\begin{cases} V \to [0,1] \\ v \to \kappa_v(\psi) \end{cases}$. Consider the homotopies $\begin{cases} H : I|K| \to |K| \\ G : I|K|_b \to |K|_b \end{cases}$ defined by $\begin{cases} H(\phi, t) = t\phi + (1-t) j \circ i(\phi) \\ G(\psi, t) = t\psi + (1-t) i \circ j(\psi) \end{cases}$.]

Let X be a topological space—then two continuous functions $\begin{cases} f : X \to |K| \\ g : X \to |K| \end{cases}$ are said to be contiguous if $\forall\, x \in X\ \exists\, \sigma \in \Sigma : \{f(x), g(x)\} \subset |\sigma|$.

FACT Suppose that $\begin{cases} f : X \to |K| \\ g : X \to |K| \end{cases}$ are contiguous—then $f \simeq g$.

[Define a homotopy $H : IX \to |K|_b$ between $i \circ f$ and $i \circ g$ by writing $b_v(H(x,t)) = (1-t) b_v(f(x)) + t b_v(g(x))$ and apply Proposition 1.]

EXAMPLE Let X be a topological space; let $\mathcal{U} = \{U\}$ be a numerable open covering of X—then a $\mathcal{U}$-map is a continuous function $f : X \rightarrow |N(\mathcal{U})|$ such that $\forall\ U \in \mathcal{U} : (b_U \circ f)^{-1}(]0,1]) \subset U$. Every partition of unity on X subordinate to $\mathcal{U}$ defines a $\mathcal{U}$-map and any two $\mathcal{U}$-maps are contiguous, hence homotopic.

FACT Let X be a topological space. Suppose that $\begin{cases} f : X \rightarrow |K| \\ g : X \rightarrow |K| \end{cases}$ are two continuous functions such that $\forall\ x \in X\ \exists\ v \in V : \{f(x), g(x)\} \subset b_v^{-1}(]0,1])$—then $f \simeq g$.

ADJUNCTION THEOREM Let K and L' be vertex schemes. Let K' be a subscheme of K and let $f : K' \rightarrow L'$ be a vertex map—then there exists a vertex scheme L containing L' as a subscheme and a homeomorphism $|K| \sqcup_{|f|} |L'| \rightarrow |L|$ whose restriction to $|L'|$ is the identity map.

A topological space X is said to be a polyhedron if there exists a vertex scheme K and a homeomorphism $f : |K| \rightarrow X$ ($|K|$ in the Whitehead topology). The ordered pair (K, f) is called a triangulation of X. Put $f_v = b_v \circ f^{-1}$—then the collection $\mathcal{T}_K = \{f_v^{-1}(]0,1])\}$ is a numerable open covering of X and Whitehead's[†] "Theorem 35" says: For any open covering $\mathcal{U}$ of X, there exists a triangulation (K, f) of X such that $\mathcal{T}_K$ refines $\mathcal{U}$.

Every polyhedron is a perfectly normal paracompact Hausdorff space. A polyhedron is metrizable iff it is locally compact. Every open subset of a polyhedron is a polyhedron.

Let X be a topological space—then a closure preserving closed covering $\mathcal{A} = \{A_j : j \in J\}$ of X is said to be absolute if for every subset $I \subset J$, the subspace $X_I = \bigcup\limits_i A_i$ has the final topology with respect to the inclusions $A_i \rightarrow X_I$. Example: Every neighborhood finite closed covering of X is absolute.

[Note: Let K be a vertex scheme—then $\{|\sigma|\}$ is an absolute closure preserving closed covering of $|K|$ but, in general, is only a closure preserving closed covering of $|K|_b$.]

EXAMPLE Take $X = [0,1]$, put $X_1 = [0,1]$, $X_n = \{0\} \cup [1/n, 1]$ $(n > 1)$—then $\{X_n\}$ is a closure preserving closed covering of X but $\{X_n\}$ is not absolute since $X = \bigcup\limits_{n>1} X_n$ does not have the final topology with respect to the inclusions $X_n \rightarrow X$ $(n > 1)$.

LEMMA Let $\mathcal{A} = \{A_j : j \in J\}$ be an absolute closure preserving closed covering of X—then for any compact Hausdorff space K, $\mathcal{A} \times K = \{A_j \times K : j \in J\}$ is an absolute closure preserving closed covering of $X \times K$.

[†] *Proc. London Math. Soc.* **45** (1939), 243–327.

FACT If X is a topological space and if $\mathcal{A} = \{A_j : j \in J\}$ is an absolute closure preserving closed covering of X such that each A_j is a normal (normal and countably paracompact, perfectly normal, collectionwise normal, paracompact) Hausdorff space, then X is a normal (normal and countably paracompact, perfectly normal, collectionwise normal, paracompact) Hausdorff space.

[In every case, X is T_1. And: T_1+ normal $\Rightarrow$ Hausdorff.

(Normal) Let A be a closed subset of X, take an $f \in C(A, [0,1])$, and let $\mathcal{F}$ be the set of continuous functions F that are extensions of f and have domains of the form $A \cup X_I$, where $X_I = \bigcup\limits_i A_i$ $(I \subset J)$. Order $\mathcal{F}$ by writing $F' \leq F''$ iff F'' is an extension of F'. Every chain in $\mathcal{F}$ has an upper bound, so by Zorn, $\mathcal{F}$ has a maximal element F_0. But the domain of F_0 is necessarily all of X and $F_0|A = f$.

(Normal and Countably Paracompact) First recall that a normal Hausdorff space is countably paracompact iff its product with $[0,1]$ is normal. Since $\mathcal{A} \times [0,1] = \{A_j \times [0,1] : j \in J\}$ is an absolute closure preserving closed covering of $X \times [0,1]$, it follows that $X \times [0,1]$ is normal, thus X is countably paracompact.

(Perfectly Normal) Fix a closed subset A of X. To prove that A is a zero set in X, equip J with a well ordering $<$. Given $j \in J$, put $X(j) = \bigcup\limits_{i \leq j} A_i$. Inductively construct continuous functions $f_j : X(j) \rightarrow [0,1]$ such that $f_{j''}|X(j') = f_{j'}$ if $j' < j''$ and $Z(f_j) = A \cap X(j)$.

(Collectionwise Normal) Let A be a closed subset of X, E any Banach space—then it suffices to show that every $f \in C(A, E)$ admits an extension $F \in C(X, E)$ (cf. p. 6–37). This can be done by imitating the argument used to establish normality.

(Paracompact) Tamano's theorem says that a normal Hausdorff space X is paracompact iff $X \times \beta X$ is normal, which enables one to proceed as in the proof of countable paracompactness.]

EXAMPLE The ordinal space $[0, \Omega[$ is not paracompact but $\{[0, \alpha] : \alpha < \Omega\}$ is a covering of $[0, \Omega[$ by compact Hausdorff spaces and $[0, \Omega[$ has the final topology with respect to the inclusions $[0, \alpha] \rightarrow [0, \Omega[$.

FACT Let X be a topological space; let $\mathcal{A} = \{A_j : j \in J\}$ be an absolute closure preserving closed covering of X. Suppose that each A_j can be embedded as a closed subspace of a polyhedron—then X can be embedded as a closed subspace of a polyhedron.

[For every j there is a vertex scheme K_j, a vector space E_j over $\mathbf{R}$, and a closed embedding $f_j : A_j \rightarrow |K_j|$ $(\subset E_j)$. Write E for the direct sum of the E_j and give E the finite topology. Let E_I stand for the direct sum of the E_i $(i \in I)$ and put $K_I = K(E_I)$—then $|K_I| \subset |K(E)|$. Here, as above, I is a subset of J. Consider the set $\mathcal{P}$ of all pairs (I, f_I), where $f_I : X_I \rightarrow |K_I|$ is a closed embedding. Order $\mathcal{P}$ by stipulating that $(I', f_{I'}) \leq (I'', f_{I''})$ iff $I' \subset I''$ and (1) $f_{I''}|X_{I'} = f_{I'}$ & (2) $f_{I''}(X_{I''} - X_{I'}) \cap |K_{I'}| = \emptyset$. Every chain in $\mathcal{P}$ has an upper bound, so by Zorn, $\mathcal{P}$ has a maximal element (I_0, f_{I_0}). Verify that $X_{I_0} = X$.]

Application: Let X be a paracompact Hausdorff space. Suppose that X admits a covering $\mathcal{U}$ by open sets U, each of which is homeomorphic to a closed subspace of a polyhedron—then X is homeomorphic to a closed subspace of a polyhedron.

The embedding theorem of dimension theory implies that every second countable compact Hausdorff space of finite topological dimension can be embedded in some euclidean space (cf. p. 19–28). It therefore follows that if a topological space X has an absolute closure preserving closed covering made up of metrizable compacta of finite topological dimension, then X can be embedded as a closed subspace of a polyhedron. This setup is realized, e.g., by the CW complexes (cf. p. 5–12).

The product $X \times Y$ of polyhedrons X and Y need not be a polyhedron (cf. p. 5–14), although this will be the case if one of the factors is locally compact.

FACT Let X and Y be polyhedrons—then $X \times Y$ has the homotopy type of a polyhedron.

[Consider a product $|K| \times |L|$. Since $\begin{cases} |K| \ \&\ |K|_b \\ |L| \ \&\ |L|_b \end{cases}$ have the same homotopy type, it need only be shown that $|K|_b \times |L|_b$ has the homotopy type of a polyhedron. Let $\begin{cases} \mathcal{U} \\ \mathcal{V} \end{cases}$ be the cozero set covering of $\begin{cases} |K|_b \\ |L|_b \end{cases}$ associated with the barycentric coordinates—then $\begin{cases} K \\ L \end{cases}$ can be identified with the corresponding nerve $\begin{cases} N(\mathcal{U}) \\ N(\mathcal{V}) \end{cases}$. Put $\mathcal{U} \times \mathcal{V} = \{U \times V : U \in \mathcal{U}, V \in \mathcal{V}\}$. Claim: There is a homotopy equivalence $|N(\mathcal{U} \times \mathcal{V})|_b \rightarrow |N(\mathcal{U})|_b \times |N(\mathcal{V})|_b$. Indeed, the projections $\begin{cases} \mathcal{U} \times \mathcal{V} \rightarrow \mathcal{U} & (U \times V \rightarrow U) \\ \mathcal{U} \times \mathcal{V} \rightarrow \mathcal{V} & (U \times V \rightarrow V) \end{cases}$ define vertex maps $\begin{cases} p_{\mathcal{U}} : N(\mathcal{U} \times \mathcal{V}) \rightarrow N(\mathcal{U}) \\ p_{\mathcal{V}} : N(\mathcal{U} \times \mathcal{V}) \rightarrow N(\mathcal{V}) \end{cases}$, from which $p : |N(\mathcal{U} \times \mathcal{V})|_b \rightarrow |N(\mathcal{U})|_b \times |N(\mathcal{V})|_b$, where $p = |p_{\mathcal{U}}| \times |p_{\mathcal{V}}|$. A homotopy inverse $q : |N(\mathcal{U})|_b \times |N(\mathcal{V})|_b \rightarrow |N(\mathcal{U} \times \mathcal{V})|_b$ to p is given in terms of barycentric coordinates by $b_{U \times V}(q(\phi, \psi)) = b_U(\phi) b_V(\psi)$.]

Let X be a topological space; let A be a closed subspace of X—then X is said to be obtained from A by attaching n-cells if there exists an indexed collection of continuous functions $f_i : \mathbf{S}^{n-1} \rightarrow A$ such that X is homeomorphic to the adjunction space $(\coprod_i \mathbf{D}^n) \sqcup_f A$ $(f = \coprod_i f_i)$. When this is so, $X - A$ is homeomorphic to $\coprod_i (\mathbf{D}^n - \mathbf{S}^{n-1}) = \coprod_i \mathbf{B}^n$, a decomposition that displays its path components as a collection of n-cells.

EXAMPLE Put $s_n = (1, 0, \ldots, 0) \in \mathbf{R}^{n+1}$ $(n \geq 1)$. Let I be a set indexing a collection of copies of the pointed space $(\mathbf{S}^n, s_n)$—then the wedge $\bigvee_I \mathbf{S}^n$ is a pointed space with basepoint $*$. Since the quotient $\mathbf{D}^n / \mathbf{S}^{n-1}$ can be identified with $\mathbf{S}^n$, $\bigvee_I \mathbf{S}^n$ is obtained from $*$ by attaching n-cells.

Let X be a topological space—then a CW structure on X is a sequence $X^{(0)}, X^{(1)}, \ldots$ of closed subspaces $X^{(n)} : \begin{cases} X = \bigcup\limits_0^\infty X^{(n)} \\ X^{(n)} \subset X^{(n+1)} \end{cases}$ and subject to:

(CW_1) $X^{(0)}$ is discrete.

(CW_2) $X^{(n)}$ is obtained from $X^{(n-1)}$ by attaching n-cells ($n > 0$).

(CW_3) X has the final topology determined by the inclusions $X^{(n)} \rightarrow X$.

A CW complex is a topological space X equipped with a CW structure. Just as a polyhedron may have more than one triangulation, a CW complex may have more than one CW structure. Every CW complex is a perfectly normal paracompact Hausdorff space.

[Note: Let K be a vertex scheme. Consider $|K|$ (Whitehead topology)—then $|K^{(0)}|$ is discrete and $|K^{(n)}|$ is obtained from $|K^{(n-1)}|$ by attaching n-cells $(n > 0) : |\sigma| - \langle\sigma\rangle \rightarrow |K^{(n-1)}|$, σ an n-simplex. Since $|K|$ has the final topology determined by the inclusions $|K^{(n)}| \rightarrow |K|$, it follows that the sequence $\{|K^{(n)}|\}$ is a CW structure on $|K|$.]

CW is the full subcategory of **TOP** whose objects are the CW complexes and **HCW** is the associated homotopy category.

EXAMPLE Equip $\mathbf{R}^\infty$ with the finite topology. Let $\mathbf{S}^\infty = \bigcup\limits_0^\infty \mathbf{S}^n$ and give it the induced topology or, what amounts to the same, the final topology determined by the inclusions $\mathbf{S}^n \rightarrow \mathbf{S}^\infty$. The sequence $\{\mathbf{S}^n\}$ is a CW structure on $\mathbf{S}^\infty$. Indeed, $\mathbf{S}^n$ is obtained from $\mathbf{S}^{n-1}$ by attaching two n-cells ($n > 0$) (seal the upper and lower hemispheres at the equator). On the other hand, $\mathbf{R}^n$ is not obtained from $\mathbf{R}^{n-1}$ by attaching n-cells. Therefore the sequence $\{\mathbf{R}^n\}$ is not a CW structure on $\mathbf{R}^\infty$. But $\mathbf{R}^\infty$ is obviously a polyhedron. A less apparent aspect is this. Put $s_\infty = (1, 0, \ldots)$—then it can be shown that $\mathbf{S}^\infty$ and $\mathbf{S}^\infty - \{s_\infty\}$ are homeomorphic. Since stereographic projection from s_∞ defines a homeomorphism $\mathbf{S}^\infty - \{s_\infty\} \rightarrow \mathbf{R}^\infty$, the conclusion is that $\mathbf{S}^\infty$ and $\mathbf{R}^\infty$ are actually homeomorphic.

[Note: The sequence $\{\mathbf{D}^n\}$ is not a CW structure for $\mathbf{D}^\infty = \bigcup\limits_0^\infty \mathbf{D}^n$. However, $\mathbf{D}^n \cup \mathbf{S}^n$ can be obtained from $\mathbf{D}^{n-1} \cup \mathbf{S}^{n-1}$ by attaching four n-cells ($n > 0$), so the sequence $\{\mathbf{D}^n \cup \mathbf{S}^n\}$ is a CW structure for $\mathbf{D}^\infty$.]

Let X be a CW complex with CW structure $\{X^{(n)}\} : X^{(n)}$ is the n-skeleton of X. The inclusion $X^{(n)} \rightarrow X$ is a closed cofibration (cf. p. 3–5) and $\forall\, n \geq 1$, the pair $(X, X^{(n)})$ is n-connected. Put $\mathcal{E}_0 = X^{(0)}$ and denote by $\mathcal{E}_n$ the set of path components of $X^{(n)} - X^{(n-1)}$ ($n > 0$). Let $\mathcal{E} = \bigcup\limits_0^\infty \mathcal{E}_n$—then an element e of $\mathcal{E}$ is said to be a cell in X, e being termed an n-cell if $e \in \mathcal{E}_n$. Set theoretically, X is the disjoint union of its cells. On the basis of the definitions, for every $e \in \mathcal{E}_n$, there exists a continuous function $\Phi_e : \mathbf{D}^n \rightarrow e \cup X^{(n-1)}$, the characteristic map of e, such that $\Phi_e|\mathbf{B}^n$ is an embedding and (i) $\Phi_e(\mathbf{B}^n) = e$; (ii) $\Phi_e(\mathbf{S}^{n-1}) \subset X^{(n-1)}$; (iii) $\Phi_e(\mathbf{D}^n) = \bar{e}$. X has the final topology determined by the Φ_e.

A subspace $A \subset X$ is called a subcomplex if there exists a subset $\mathcal{E}_A \subset \mathcal{E} : A = \cup\mathcal{E}_A$ & $\forall\ e \in \mathcal{E}_A \cap \mathcal{E}_n$, $\Phi_e(\mathbf{D}^n) \subset A$. A subcomplex A of X is itself a CW complex with CW structure $\{A^{(n)} = A \cap X^{(n)}\}$. The inclusion $A \rightarrow X$ is a closed cofibration and for every open $U \supset A$ there exists an open $V \supset A$ with $V \subset U$ such that A is a strong deformation retract of V. If $\mathcal{E}' \subset \mathcal{E}$, then $\cup\mathcal{E}'$ is a subcomplex iff $\cup\mathcal{E}'$ is closed. Arbitrary unions and intersections of subcomplexes are subcomplexes. In general, the $\bar{e}$ are not subcomplexes, although this will be the case if all the characteristic maps are embeddings. The combinatorial dimension of X, written $\dim X$, is -1 if X is empty, otherwise is the smallest value of n such that $X = X^{(n)}$ (or ∞ if there is no such n). It is a fact that $\dim X$ is equal to the topological dimension of X (cf. p. 19–21), therefore is independent of the CW structure.

Let X be a CW complex—then the collection $\overline{\mathcal{E}} = \{\bar{e} : e \in \mathcal{E}\}$ is a closed covering of X and X has the final topology determined by the inclusions $\bar{e} \rightarrow X$ but $\overline{\mathcal{E}}$ need not be closure preserving.

EXAMPLE (Simplicial Sets) Let X be a simplicial set—then its geometric realization $|X|$ is a CW complex with CW structure $\{|X^{(n)}|\}$. In fact, $|X^{(0)}|$ is discrete and, using the notation of p. 0–18, the commutative diagram

$$\begin{tikzcd}
X_n^{\#} \cdot \dot{\Delta}[n] \arrow[r] \arrow[d] & X^{(n-1)} \arrow[d] \\
X_n^{\#} \cdot \Delta[n] \arrow[r] & X^{(n)}
\end{tikzcd}$$

is a pushout square in **SISET**. Since the geometric realization functor $|?|$ is a left adjoint, it preserves colimits. Therefore the commutative diagram

$$\begin{tikzcd}
X_n^{\#} \cdot \dot{\Delta}^n \arrow[r] \arrow[d] & {|X^{(n-1)}|} \arrow[d] \\
X_n^{\#} \cdot \Delta^n \arrow[r] & {|X^{(n)}|}
\end{tikzcd}$$

is a pushout square in **TOP**, which means that $|X^{(n)}|$ is obtained from $|X^{(n-1)}|$ by attaching n-cells ($n > 0$). Moreover, $X = \operatorname{colim} X^{(n)} \Rightarrow |X| = \operatorname{colim} |X^{(n)}|$, so $|X|$ has the final topology determined by the inclusions $|X^{(n)}| \rightarrow |X|$. Denoting now by G the identity component of the homeomorphism group of $[0,1]$, there is a left action $G \times |X| \rightarrow |X|$ and the orbits of G are the cells of $|X|$.

[Note: If Y is a simplicial subset of X, then $|Y|$ is a subcomplex of $|X|$, thus the inclusion $|Y| \rightarrow |X|$ is a closed cofibration.]

It is true but not obvious that if X is a simplicial set, then $|X|$ is actually a polyhedron (cf. p. 13–11).

A CW pair is a pair (X, A), where X is a CW complex and $A \subset X$ is a subcomplex. $\mathbf{CW}^2$ is the full subcategory of $\mathbf{TOP}^2$ whose objects are the CW pairs and $\mathbf{HCW}^2$ is the associated homotopy category.

A pointed CW complex is a pair (X, x_0), where X is a CW complex and $x_0 \in X^{(0)}$. **CW**$_*$ is the full subcategory of **TOP**$_*$ whose objects are the pointed CW complexes and **HCW**$_*$ is the associated homotopy category.

[Note: If (X, x_0) is a pointed CW complex, then $\forall\ q \geq 1$, $\pi_q(X, x_0) \approx \text{colim}\ \pi_q(X^{(n)}, x_0)$.]

Let X be a CW complex—then $\forall\ x_0 \in X$, the inclusion $\{x_0\} \to X$ is a cofibration (cf. p. 3–17), thus (X, x_0) is wellpointed. Of course, a given x_0 need not be in $X^{(0)}$ but there always exists some CW structure on X having x_0 as a 0-cell.

Let X be a topological space, $A \subset X$ a closed subspace—then a relative CW structure on (X, A) is a sequence $(X, A)^{(0)}, (X, A)^{(1)}, \ldots$ of closed subspaces $(X, A)^{(n)}$: $\begin{cases} X = \bigcup\limits_0^\infty (X, A)^{(n)} \\ (X, A)^{(n)} \subset (X, A)^{(n+1)} \end{cases}$ and subject to:

(RCW$_1$) $(X, A)^{(0)}$ is obtained from A by attaching 0-cells.

(RCW$_2$) $(X, A)^{(n)}$ is obtained from $(X, A)^{(n-1)}$ by attaching n-cells $(n > 0)$.

(RCW$_3$) X has the final topology determined by the inclusions $(X, A)^{(n)} \to X$.

[Note: $(X, A)^{(0)}$ is the coproduct of A and a discrete space, so when $A = \emptyset$ the definition reduces to that of a CW structure.]

A relative CW complex is a topological space X and a closed subspace A equipped with a relative CW structure.

[Note: If (X, A) is a relative CW complex, then the inclusion $A \to X$ is a closed cofibration and X/A is a CW complex. On the other hand, if X is a CW complex and if $A \subset X$ is a subcomplex, then (X, A) is a relative CW complex.]

Example: Suppose that (X, A) is a relative CW complex—then (IX, IA) is a relative CW complex, where $(IX, IA)^{(n)} = i_0(X, A)^{(n)} \cup (I(X, A)^{(n-1)} \cup IA) \cup i_1(X, A)^{(n)}$.

Let (X, A) be a relative CW complex with relative CW structure $\{(X, A)^{(n)}\} : (X, A)^{(n)}$ is the n-skeleton of X relative to A. The inclusion $(X, A)^{(n)} \to X$ is a closed cofibration (cf. p. 3–5) and $\forall\ n \geq 1$, the pair $(X, (X, A)^{(n)})$ is n-connected. The relative combinatorial dimension of (X, A), written $\dim(X, A)$, is -1 if X is empty, otherwise is the smallest value of n such that $X = (X, A)^{(n)}$ (or ∞ if there is no such n). Obviously, $\dim(X, A) = \dim(X/A)$ provided that X is nonempty.

LEMMA Let (X, A) be a relative CW complex—then for every compact subset $K \subset X$ there exists an index n such that $K \subset (X, A)^{(n)}$.

[Consider the image of K under the projection $X \to X/A$, bearing in mind that X/A is a CW complex.]

Application: Let (X, A, x_0) be a pointed pair. Assume: (X, A) is a relative CW complex—then $\forall\ q \geq 1$, $\pi_q(X, x_0) \approx \operatorname{colim} \pi_q((X, A)^{(n)}, x_0)$.

HOPF EXTENSION THEOREM Let (X, A) be a relative CW complex with $\dim(X, A) \leq n + 1$ $(n \geq 1)$. Suppose that $f \in C(A, \mathbf{S}^n)$—then $\exists\ F \in C(X, \mathbf{S}^n) : F|A = f$ iff $f^*(H^n(\mathbf{S}^n)) \subset i^*(H^n(X))$, $i : A \rightarrow X$ the inclusion.

HOPF CLASSIFICATION THEOREM Let (X, A) be a relative CW complex with $\dim(X, A) \leq n$ $(n \geq 1)$. Fix a generator $\iota \in H^n(\mathbf{S}^n, s_n; \mathbf{Z})$—then the assignment $[f] \rightarrow f^*\iota$ defines a bijection $[X, A; \mathbf{S}^n, s_n] \rightarrow H^n(X, A; \mathbf{Z})$.

EXAMPLE The unit tangent bundle of $\mathbf{S}^{2n}$ can be identified with the Stiefel manifold $\mathbf{V}_{2n+1,2}$. It is $(2n - 2)$-connected with euclidean dimension $4n - 1$. One has $H_q(\mathbf{V}_{2n+1,2}) \approx \mathbf{Z}$ $(q = 0, 4n - 1)$, $H_{2n-1}(\mathbf{V}_{2n+1,2}) \approx \mathbf{Z}/2\mathbf{Z}$, and $H_q(\mathbf{V}_{2n+1,2}) = 0$ otherwise. By the Hopf classification theorem, $[\mathbf{V}_{2n+1,2}, \mathbf{S}^{4n-1}] \approx H^{4n-1}(\mathbf{V}_{2n+1,2})$, so there is a map $f : \mathbf{V}_{2n+1,2} \rightarrow \mathbf{S}^{4n-1}$ such that f^* induces an isomorphism $H^{4n-1}(\mathbf{S}^{4n-1}) \rightarrow H^{4n-1}(\mathbf{V}_{2n+1,2})$. Consequently, under f_*, $H_*(\mathbf{V}_{2n+1,2}; \mathbf{Q}) \approx H_*(\mathbf{S}^{4n-1}; \mathbf{Q})$, thus the mapping fiber E_f of f is rationally acyclic, i.e., $\widetilde{H}_*(E_f; \mathbf{Q}) = 0$ (cf. p. 4–44).

Let $\begin{cases} X \\ Y \end{cases}$ be CW complexes with CW structures $\begin{cases} \{X^{(n)}\} \\ \{Y^{(n)}\} \end{cases}$—then a skeletal map is a continuous function $f : X \rightarrow Y$ such that $\forall\ n : f(X^{(n)}) \subset Y^{(n)}$.

[Note: A CW complex is filtered by its skeletons, so the term "skeletal map" is just the name used for "filtered map" in the CW context.]

EXAMPLE (Simplicial Sets) If $f : X \rightarrow Y$ is a simplicial map, then $|f| : |X| \rightarrow |Y|$ is a skeletal map and transforms cells of $|X|$ onto cells of $|Y|$.

SKELETAL APPROXIMATION THEOREM Let X and Y be CW complexes. Suppose that A is a subcomplex of X—then for any continuous function $f : X \rightarrow Y$ such that $f|A$ is skeletal there exists a skeletal map $g : X \rightarrow Y$ such that $f|A = g|A$ and $f \simeq g \operatorname{rel} A$.

[Note: In particular, every continuous function $f : X \rightarrow Y$ is homotopic to a skeletal map $g : X \rightarrow Y$.]

Let $\begin{cases} (X, A) \\ (Y, B) \end{cases}$ be relative CW complexes with relative CW structures $\begin{cases} \{(X, A)^{(n)}\} \\ \{(Y, B)^{(n)}\} \end{cases}$—then a relative skeletal map is a continuous function $f : (X, A) \rightarrow (Y, B)$ such that $\forall\ n : f((X, A)^{(n)}) \subset (Y, B)^{(n)}$.

RELATIVE SKELETAL APPROXIMATION THEOREM Let (X, A) and (Y, B) be relative CW complexes—then every continuous function $f : (X, A) \to (Y, B)$ is homotopic rel A to a relative skeletal map $g : (X, A) \to (Y, B)$.

Here is a summary of the main topological properties of CW complexes.

(TCW_1) Every CW complex is compactly generated.

(TCW_2) Every CW complex is stratifiable, hence is hereditarily paracompact.

(TCW_3) Every CW complex is uniformly locally contractible, therefore locally contractible.

(TCW_4) Every CW complex is numerably contractible.

(TCW_5) Every CW complex is locally path connected.

(TCW_6) Every CW complex is the coproduct of its path components and these are subcomplexes.

(TCW_7) Every connected CW complex is path connected.

(TCW_8) Every connected CW complex has a universal covering space.

[Note: If X is a connected CW complex with CW structure $\{X^{(n)}\}$ and if $p : \widetilde{X} \to X$ is a covering projection, then the sequence $\{\widetilde{X}^{(n)} = p^{-1}(X^{(n)})\}$ is a CW structure on $\widetilde{X}$ with respect to which p is skeletal.]

If (X, A) is a relative CW complex, then certain topological properties of A are automatically transmitted to X. For example, if A is in **CG**, **Δ-CG**, or **CGH**, then the same holds for X. Analogous remarks apply to a Hausdorff A which is normal, perfectly normal, paracompact, etc.

(F) A CW complex X is said to be finite if $\#(\mathcal{E}) < \omega$. Every finite CW complex is compact and conversely. A compact subset of a CW complex is contained in a finite subcomplex.

(C) A CW complex X is said to be countable if $\#(\mathcal{E}) \leq \omega$. A CW complex is countable iff it does not contain an uncountable discrete set. Every countable CW complex is Lindelöf and conversely.

[Note: The homotopy groups of a countable connected CW complex are countable.]

(LF) A CW complex X is said to be locally finite if each $x \in X$ has a neighborhood U such that U is contained in a finite subcomplex of X. Every locally finite CW complex is locally compact and conversely. Every locally finite CW complex is metrizable and conversely. A locally finite connected CW complex is countable.

What spaces carry a CW structure? There is no known characterization but the foregoing conditions impose a priori limitations. For example, a nonmetrizable LCH space cannot be equipped with a CW structure. On the other hand, the Cantor set and the Hilbert cube are metrizable compact Hausdorff spaces but neither supports a CW structure.

[Note: Every compact differentiable manifold can be triangulated but examples are known of compact topological manifolds that cannot be triangulated, i.e., that are not polyhedrons (Davis-Januszkiewicz[†]).]

EXAMPLE (The Sorgenfrey Line) Topologize $X = \mathbf{R}$ by choosing for the basic neighborhoods of a given x all sets of the form $[x, y[$ $(x < y)$. In this topology, the line is a perfectly normal paracompact Hausdorff space but it is not locally compact. While not second countable, X is first countable (and separable), therefore is compactly generated. However, X is not locally connected, thus carries no CW structure.

[Note: The square of the Sorgenfrey line is not normal (apply Jones' lemma).]

EXAMPLE (The Niemytzki Plane) Let X be the closed upper half plane in $\mathbf{R}^2$. Topologize X as follows: The basic neighborhoods of (x, y) $(y > 0)$ are as usual but the basic neighborhoods of $(x, 0)$ are the $\{(x, 0)\} \cup B$, where B is an open disk in the upper half plane with horizontal tangent at $(x, 0)$. X is a compactly generated CRH space. In addition, X is Moore, hence is perfect. And X is connected, locally path connected, and even contractible (consider the homotopy $H((x, y), t) = \begin{cases} (x, y) + t(0, 1) & (0 \leq t \leq 1/2) \\ t(0, 1) + 2(1 - t)(x, y) & (1/2 \leq t \leq 1) \end{cases}$). However, X is not normal, thus carries no CW structure.

[Note: X is neither countably paracompact nor metacompact but is countably metacompact.]

EXAMPLE An open subset of a polyhedron is a polyhedron but an open subset of a CW complex need not be a CW complex. To see this, fix an enumeration $\{q_n\}$ of $\mathbf{Q} \cap]0, 1[$. Consider the CW complex X defined as follows: $X^{(0)} = \{0, 1\}, X^{(1)} = [0, 1] \begin{cases} 0 \rightarrow 0 \\ 1 \rightarrow 1 \end{cases}$ and at each point q_n attach a 2-cell by taking for $f_n : \mathbf{S}^1 \rightarrow X^{(1)}$ the constant map $f_n = q_n$. Choose a point $x_n \in e_n$ $(\in \mathcal{E}_2)$ and put $A = \{x_n\}$—then A is closed and $U = X - A$ carries no CW structure.

[Otherwise: (a) $[0, 1] \subset U^{(1)}$; (b) $\forall\, n, U^{(1)} \cap e_n \neq \emptyset$; (c) $\forall\, n, q_n \in U^{(0)}$.]

PROPOSITION 2 Every CW complex has the homotopy type of a polyhedron.

[Let X be a CW complex with CW structure $\{X^{(n)}\} : X = \operatorname{colim} X^{(n)}$. Taking into account §3, Proposition 15, it will be enough to construct a sequence of vertex schemes $K_{(n)}$ such that $\forall\, n$, $K_{(n-1)}$ is a subscheme of $K_{(n)}$ and a sequence of homotopy equivalences $\phi_n : X^{(n)} \rightarrow |K_{(n)}|$ such that $\forall\, n$, $\phi_n | X^{(n-1)} = \phi_{n-1}$. Proceeding by induction, make the obvious choices when $n = 0$ and then assume that $K_{(0)}, \ldots, K_{(n-1)}$ and $\phi_0, \ldots, \phi_{n-1}$ have been defined. At level n there is an index set I_n and a pushout square

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[†] *J. Differential Geom.* **34** (1991), 347–388.

$$\begin{tikzcd} I_n \cdot \dot{\Delta}^n \arrow[r, "f"] \arrow[d] & X^{(n-1)} \arrow[d] \\ I_n \cdot \Delta^n \arrow[r] & X^{(n)} \end{tikzcd}$$

($f = \coprod_i f_i$). Given $i \in I_n$, use the simplicial approximation theorem to produce a vertex scheme K_i and a vertex map $g_i : K_i \rightarrow K_{(n-1)}$ with $|K_i| = \dot{\Delta}^n$ and $|g_i| \simeq \phi_{n-1} \circ f_i$. Combine the K_i and put $|g| = \coprod_i |g_i|$. The adjunction theorem implies that there exists a vertex scheme $K_{(n)}$ containing $K_{(n-1)}$ as a subscheme and a homeomorphism $I_n \cdot \Delta^n \sqcup_{|g|} |K_{(n-1)}| \rightarrow |K_{(n)}|$ whose restriction to $|K_{(n-1)}|$ is the identity map. The triangle

$$\begin{tikzcd} I_n \cdot \dot{\Delta}^n \arrow[r, "f"] \arrow[dr, "|g|"'] & X^{(n-1)} \arrow[d, "\phi_{n-1}"] \\ & |K_{(n-1)}| \end{tikzcd}$$

is homotopy commutative: $|g| \simeq \phi_{n-1} \circ f$. Since ϕ_{n-1} is a homotopy equivalence, one can find a homotopy equivalence $\phi_n : I_n \cdot \Delta^n \sqcup_f X^{(n-1)} \rightarrow I_n \cdot \Delta^n \sqcup_{|g|} |K_{(n-1)}|$ such that $\phi_n | X^{(n-1)} = \phi_{n-1}$ (cf. p. 3–24), which completes the induction.]

[Note: Similar methods lead to the expected analogs in $\mathbf{CW}^2$ or $\mathbf{CW}_*$. Consider, e.g., a CW pair (X, A) with relative CW structure $\{(X, A)^{(n)}\} : (X, A)^{(n)} = X^{(n)} \cup A$. Choose a vertex scheme L and a homotopy equivalence $\phi : A \rightarrow |L|$—then there is a vertex scheme $K_{(0)}$ containing L as a subscheme and a homotopy equivalence of pairs $((X, A)^{(0)}, A) \rightarrow (|K_{(0)}|, |L|)$ so, arguing as above, there is a vertex scheme K containing L as a subscheme and a homotopy equivalence $\Phi : X \rightarrow |K|$ such that $\Phi | A = \phi$. Conclusion: In $\mathbf{HTOP}^2, (X, A) \approx (|K|, |L|)$ (cf. §3, Proposition 14).]

PROPOSITION 3 Let X be a CW complex. Assume: (i) X is finite (countable) or (ii) $\dim X \leq n$—then there exists a vertex scheme K such that X has the homotopy type of $|K|$, where (i) K is finite (countable) or (ii) $\dim K \leq n$.

[This is implicit in the proof of the preceding proposition.]

Let X be a CW complex; let $\mathcal{A}$ be the collection of finite subcomplexes of X—then $\mathcal{A}$ is an absolute closure preserving closed covering of X. Since every finite subcomplex of X is a second countable compact Hausdorff space of finite topological dimension, it follows that X can be embedded as a closed subspace of a polyhedron (cf. p. 5–5).

FACT Every CW complex is the retract of a polyhedron, hence every open subset of a CW complex is the retract of a polyhedron.

EXAMPLE Every polyhedron is a CW complex but there exist CW complexes that cannot be triangulated. Thus let $f(t) = t \sin(\pi/2t)$ $(0 < t \leq 1)$ and set $f(0) = 0$. Denote by m the absolute minimum of f on $[0,1]$ (so $-1 < m < 0$). Take for X the image of the square $[0,1] \times [0,1]$ under the map $(u,v) \rightarrow (u, uv, f(v))$. The following subspaces constitute a CW structure on X:

$$X^{(0)} = \{(0,0,0), (1,0,0), (0,0,1), (1,1,1), (0,0,m)\},$$

$$X^{(1)} = \{(u,0,0) : 0 \leq u \leq 1\} \cup \{(u,u,1) : 0 \leq u \leq 1\} \cup \begin{cases} \{(0,0,v) : m \leq v \leq 0\} \\ \{(0,0,v) : 0 \leq v \leq 1\} \end{cases} \cup \{(1,v,f(v)) : 0 \leq v \leq 1\},$$

and $X^{(2)} = X$. Using the fact that f has a sequence $\{M_n\}$ of relative maxima: $M_1 > M_2 > \cdots (1 > M_1)$, look at the $(0,0,M_n)$ and deduce that X is not a polyhedron.

FACT Let X be a CW complex. Suppose that all the characteristic maps are embeddings—then X is a polyhedron.

There are two other issues.

(Products) Let $\begin{cases} X \\ Y \end{cases}$ be CW complexes with CW structures $\begin{cases} \{X^{(n)}\} \\ \{Y^{(n)}\} \end{cases}$. Put $(X \times_k Y)^{(n)} = \bigcup\limits_{p+q=n} X^{(p)} \times_k Y^{(q)}$. Consider $X \times_k Y$—then the sequence $\{(X \times_k Y)^{(n)}\}$ satisfies CW_1, CW_2, and CW_3 above, meaning that it is a CW structure on $X \times_k Y$. When can "$\times_k$" be replaced by "$\times$"? Useful sufficient conditions to ensure this are that one of the factors be locally finite or that both of the factors be countable (necessary conditions have been discussed by Tanaka[†]).

EXAMPLE (Dowker's Product) Suppose that X and Y are CW complexes—then the product $X \times Y$ need not be compactly generated, hence, when this happens, $X \times Y$ is not a CW complex. Here is an illustration. Definition of X: Put $X^{(0)} = \mathbf{N}^{\mathbf{N}} \cup \{0\}$ (discrete topology), let $f_s : \{0,1\} \rightarrow X^{(0)}$ be the map $\begin{cases} 0 \rightarrow 0 \\ 1 \rightarrow s \end{cases}$ ($s \in \mathbf{N}^{\mathbf{N}}$), write $X^{(1)}$ for the space thereby obtained from $X^{(0)}$ by attaching 1-cells, and take $X = X^{(0)} \cup X^{(1)}$. Definition of Y: Put $Y^{(0)} = \mathbf{N} \cup \{0\}$ (discrete topology), let $f_n : \{0,1\} \rightarrow Y^{(0)}$ be the map $\begin{cases} 0 \rightarrow 0 \\ 1 \rightarrow n \end{cases}$ ($n \in \mathbf{N}$), write $Y^{(1)}$ for the space thereby obtained from $Y^{(0)}$ by attaching 1-cells, and take $Y = Y^{(0)} \cup Y^{(1)}$. Let Φ_s (Φ_n) be the characteristic map of the 1-cell corresponding to $s \in \mathbf{N}^{\mathbf{N}}$ ($n \in \mathbf{N}$). Consider the following subset of $X \times Y : K = \{(\Phi_s(1/s_n), \Phi_n(1/s_n)) : (s,n) \in \mathbf{N}^{\mathbf{N}} \times \mathbf{N}\}$. Evidently K is a closed subset of $X \times_k Y$. But K is not a closed subset of $X \times Y$. For if it were, $X \times Y - K$ would be open and since the point $(0,0) \in X \times Y - K$, there would be a basic neighborhood $U \times V : (0,0) \in U \times V \subset X \times Y - K$. Given $s \in \mathbf{N}^{\mathbf{N}}$, $\exists$ a real number $a_s : 0 < a_s \leq 1$ such that $U \supset \{\Phi_s(p) : p < a_s\}$ and given $n \in \mathbf{N}$, $\exists$ a real number $b_n : 0 < b_n \leq 1$ such that $V \supset \{\Phi_n(q) : q < b_n\}$. Define $\overline{s} \in \mathbf{N}^{\mathbf{N}}$ by $\overline{s}_n = 1 + [\max\{n, 1/b_n\}]$ (so $\overline{s}_n > n$ & $\overline{s}_n > 1/b_n$); define $\overline{n} \in \mathbf{N}$ by $\overline{n} = 1 + [1/a_{\overline{s}}]$ (so

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[†] *Proc. Amer. Math. Soc.* **86** (1982), 503–507.

$\overline{n} > 1/a_{\overline{s}}$)—then the pair $(\Phi_{\overline{s}}(1/\overline{s}_{\overline{n}}), \Phi_{\overline{n}}(1/\overline{s}_{\overline{n}}))$ is in both $U \times V$ and K. Contradiction. Incidentally, one can show that the projections $\begin{cases} X \times_k Y \to X \\ X \times_k Y \to Y \end{cases}$ are not Hurewicz fibrations (although, of course, they are CG fibrations).

[Note: This construction has an obvious interpretation in terms of cones. Observe too that X and Y are polyhedrons. Corollary: The square of a polyhedron need not be a polyhedron.]

FACT Every countable CW complex has the homotopy type of a locally finite countable CW complex.

[Let X be a countable CW complex. Fix an enumeration $\{e_k\}$ of its cells. Given e_k, denote by $X(e_k)$ the intersection of all subcomplexes of X containing e_k—then $X(e_k)$ is a finite subcomplex of X. Put $X^n = \bigcup\limits_0^n X(e_k)$: $X^0 \subset X^1 \subset \cdots$ is an expanding sequence of topological spaces with $X^\infty = X$. The telescope $\operatorname{tel} X^\infty$ of X^∞ has the same homotopy type as $X^\infty = X$ (cf. p. 3–12) and is a CW complex. In fact, $\operatorname{tel} X^\infty$ is the subcomplex of $X \times_k [0, \infty[= X \times [0, \infty[$ made up of the cells $e \times \{n\}$, $e \times]n, n+1[$, where e is a cell of X^m $(m \leq n)$, a description which makes it clear that $\operatorname{tel} X^\infty$ is locally finite.]

[Note: Suppose that X is a locally finite countable CW complex—then there exists a sequence of finite subcomplexes X_n such that $\forall\, n, X_n \subset \operatorname{int} X_{n+1}$, with $X = \bigcup\limits_n X_n$.]

(Adjunctions) Let $\begin{cases} X \\ Y \end{cases}$ be CW complexes with CW structures $\begin{cases} \{X^{(n)}\} \\ \{Y^{(n)}\} \end{cases}$. Suppose that A is a subcomplex of X. Let $f : A \to Y$ be a skeletal map—then the adjunction space $X \sqcup_f Y$ is a CW complex, the CW structure being $\{X^{(n)} \sqcup_{f^{(n)}} Y^{(n)}\}(f^{(n)} = f|A^{(n)})$. Examples: (1) If X is a CW complex and if $A \subset X$ is a subcomplex, then the quotient X/A is a CW complex; (2) If X is a CW complex, then its cone ΓX and its suspension ΣX are CW complexes; (3) If X and Y are CW complexes and if $f : X \to Y$ is a skeletal map, then the mapping cylinder M_f of f is a CW complex, containing both X and Y as embedded subcomplexes; (4) If X and Y are CW complexes and if $f : X \to Y$ is a skeletal map, then the mapping cone C_f of f is a CW complex containing Y as an embedded subcomplex.

[Note: There are also pointed analogs of these results. For example, if $\begin{cases} (X, x_0) \\ (Y, y_0) \end{cases}$ are pointed CW complexes, then the smash product $X \#_k Y$ is a pointed CW complex.]

Let X and Y be CW complexes. Let A be a subcomplex of X and let $f : A \to Y$ be a continuous function—then $X \sqcup_f Y$ has the homotopy type of a CW complex. Proof: By the skeletal approximation theorem, there exists a skeletal map $g : A \to Y$ such that $f \simeq g$, so $X \sqcup_f Y$ has the same homotopy type as $X \sqcup_g Y$ (cf. p. 3–24).

FACT A CW complex is path connected iff its 1-skeleton is path connected.

EXAMPLE (Trees) Let X be a nonempty connected CW complex—then a tree in X is a nonempty simply connected subcomplex T of X with $\dim T \leq 1$. Every tree in X is contractible and contained in a maximal tree. A tree is maximal iff it contains $X^{(0)}$. If T is a maximal tree in X, then X/T is a connected CW complex with exactly one 0-cell and the projection $X \rightarrow X/T$ is a homotopy equivalence (cf. p. 3–24).

WHE CRITERION Let $\begin{cases} X \\ Y \end{cases}$ be topological spaces, $f : X \rightarrow Y$ a continuous function—then f is a weak homotopy equivalence if for any finite CW pair (K, L) and any diagram

$$\begin{tikzcd} L \arrow[r, "\phi"] \arrow[d] & X \arrow[d, "f"] \\ K \arrow[r, "\psi"'] & Y \end{tikzcd}$$

, where $f \circ \phi = \psi|L$, there exists a $\Phi : K \rightarrow X$ such that $\Phi|L = \phi$ and $f \circ \Phi \simeq \psi \,\mathrm{rel}\, L$.

[Indeed, diagrams of the form

$$\begin{tikzcd} s_n \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ \mathbf{S}^n \arrow[r] & Y \end{tikzcd}, \quad \begin{tikzcd} \mathbf{S}^n \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ \mathbf{D}^{n+1} \arrow[r] & Y \end{tikzcd}$$

evidently suffice.]

LEMMA Suppose that $f : X \rightarrow Y$ is an n-equivalence—then in any diagram

$$\begin{tikzcd} \mathbf{S}^{n-1} \arrow[r, "\phi"] \arrow[d] & X \arrow[d, "f"] \\ \mathbf{D}^n \arrow[r, "\psi"'] & Y \end{tikzcd}$$

, where $f \circ \phi \simeq \psi$ on $\mathbf{S}^{n-1}$ by $h : I\mathbf{S}^{n-1} \rightarrow Y$, there exists a $\Phi : \mathbf{D}^n \rightarrow X$ such that $\Phi|\mathbf{S}^{n-1} = \phi$ and an $H : I\mathbf{D}^n \rightarrow Y$ such that $H|I\mathbf{S}^{n-1} = h$ and $f \circ \Phi \simeq \psi$ on $\mathbf{D}^n$ by H.

HOMOTOPY EXTENSION LIFTING PROPERTY Suppose that $f : X \rightarrow Y$ is a weak homotopy equivalence. Let (K, L) be a relative CW complex—then in any diagram

$$\begin{tikzcd} L \arrow[r, "\phi"] \arrow[d] & X \arrow[d, "f"] \\ K \arrow[r, "\psi"'] & Y \end{tikzcd}$$

, where $f \circ \phi \simeq \psi$ on L by $h : IL \rightarrow Y$, there exists a $\Phi : K \rightarrow X$ such that $\Phi|L = \phi$ and an $H : IK \rightarrow Y$ such that $H|IL = h$ and $f \circ \Phi \simeq \psi$ on K by H.

Application: Let $f : X \rightarrow Y$ be a weak homotopy equivalence—then for any CW complex K, the arrow $f_* : [K, X] \rightarrow [K, Y]$ is bijective.

[To see that f_* is surjective (injective), apply the homotopy extension lifting property to $(K, \emptyset)$ $((IK, i_0K \cup i_1K))$.]

[Note: The condition is also characteristic. Thus first take $K = *$ and reduce to when $\begin{cases} X \\ Y \end{cases}$ are path connected. Next, take $K = \bigvee_I \mathbf{S}^1$ (I a suitable index set) to get that $\forall\, x \in X$, $f_* : \pi_1(X, x) \rightarrow \pi_1(Y, f(x))$ is surjective. Finish by taking $K = \mathbf{S}^n$ (cf. p. 3–18).]

EXAMPLE Let $\begin{cases} (X, x_0) \\ (Y, y_0) \end{cases}$ be pointed connected CW complexes. Suppose that $f \in C(X, x_0; Y, y_0)$ has the property that $\forall\, n > 1$, $f_* : \pi_n(X, x_0) \rightarrow \pi_n(Y, y_0)$ is bijective—then for any pointed simply connected CW complex (K, k_0), the arrow $f_* : [K, k_0; X, x_0] \rightarrow [K, k_0; Y, y_0]$ is bijective.

FACT Let $p : X \rightarrow B$ be a continuous function—then p is both a weak homotopy equivalence and a Serre fibration iff for any relative CW complex (K, L) and any diagram

$$\begin{tikzcd} L \arrow[r, "\phi"] \arrow[d] & X \arrow[d, "p"] \\ K \arrow[r, "\psi"'] & B \end{tikzcd}$$

, where $p \circ \phi = \psi | L$, there exists a $\Phi : K \rightarrow X$ such that $\Phi | L = \phi$ and $p \circ \Phi = \psi$.

[Note: The characterization can be simplified: A continuous function $p : X \rightarrow B$ is both a weak homotopy equivalence and a Serre fibration iff every commutative diagram

$$\begin{tikzcd} \mathbf{S}^{n-1} \arrow[r] \arrow[d] & X \arrow[d] \\ \mathbf{D}^n \arrow[r] & B \end{tikzcd} \quad (n \geq 0)$$

admits a filler $\mathbf{D}^n \rightarrow X$.]

Application: Let

$$\begin{tikzcd} X' \arrow[r] \arrow[d, "p'"'] & X \arrow[d, "p"] \\ B' \arrow[r] & B \end{tikzcd}$$

be a pullback square. Suppose that p is a Serre fibration and a weak homotopy equivalence—then p' is a Serre fibration and a weak homotopy equivalence.

A continuous function $f : (X, A) \rightarrow (Y, B)$ is said to be a weak homotopy equivalence of pairs provided that $f : X \rightarrow Y$ and $f : A \rightarrow B$ are weak homotopy equivalences.

[Note: A weak homotopy equivalence of pairs is a relative weak homotopy equivalence (cf. p. 4–51) but not conversely.]

Application: Let $f : (X, A) \rightarrow (Y, B)$ be a weak homotopy equivalence of pairs—then for any CW pair (K, L), the arrow $f_* : [K, L; X, A] \rightarrow [K, L; Y, B]$ is bijective.

[Note: The condition is also characteristic. For $[K, \emptyset; X, A] \approx [K, \emptyset; Y, B] \Rightarrow [K, X] \approx [K, Y]$ and $[IK, i_0 K; X, A] \approx [IK, i_0 K; Y, B] \Rightarrow [K, A] \approx [K, B]$.]

REALIZATION THEOREM Suppose that X and Y are CW complexes. Let $f : X \rightarrow Y$ be a weak homotopy equivalence—then f is a homotopy equivalence.

[Note: It is a corollary that the result remains true when X and Y have the homotopy type of CW complexes.]

Application: A connected CW complex is contractible iff it is homotopically trivial.

EXAMPLE Let X and Y be CW complexes—then the identity map $X \times_k Y \rightarrow X \times Y$ is a homotopy equivalence.

[A priori, the identity map $X \times_k Y \rightarrow X \times Y$ is a weak homotopy equivalence. However, X and Y each have the homotopy type of a polyhedron (cf. Proposition 2), thus the same holds for their product $X \times Y$ (cf. p. 5–5).]

EXAMPLE (H Groups) Let (X, x_0) be a nondegenerate homotopy associative H space. Assume: X is path connected—then the shearing map sh : $\begin{cases} X \times X & \rightarrow & X \times X \\ (x, y) & \rightarrow & (x, xy) \end{cases}$ is a weak homotopy equivalence, thus X is an H group if X carries a CW structure (cf. p. 4–27).

The pointed version of the realization theorem says that if $\begin{cases} X \\ Y \end{cases}$ are CW complexes and if $f : X \rightarrow Y$ is a weak homotopy equivalence, then f is a pointed homotopy equivalence for any choice of $\begin{cases} x_0 \in X \\ y_0 \in Y \end{cases}$ with $f(x_0) = y_0$. Proof: By the realization theorem, f is a homotopy equivalence, so f is actually a pointed homotopy equivalence, $\begin{cases} (X, x_0) \\ (Y, y_0) \end{cases}$ being wellpointed (cf. p. 3–19).

RELATIVE REALIZATION THEOREM Suppose that (X, A) and (Y, B) are CW pairs. Let $f : (X, A) \rightarrow (Y, B)$ be a weak homotopy equivalence of pairs—then f is a homotopy equivalence of pairs.

[Note: This result need not be true if one merely assumes that f is a relative weak homotopy equivalence. Example: Take X path connected, fix a point $a_0 \in A$, and consider the projection $(X \times A, a_0 \times A) \rightarrow (X, a_0)$. It is a relative weak homotopy equivalence but the induced map on relative singular homology is not necessarily an isomorphism.]

The relative realization theorem is a consequence of the following assertion. Suppose that (X, A) and (Y, B) are relative CW complexes. Let $f : (X, A) \rightarrow (Y, B)$ be a weak homotopy equivalence of pairs with $f|A : A \rightarrow B$ a homotopy equivalence—then f is a homotopy equivalence of pairs.

EXAMPLE Let (K, L) be a relative CW complex. Assume: The inclusion $L \rightarrow K$ is a weak homotopy equivalence—then the inclusion $L \rightarrow K$ is a homotopy equivalence. Proof: Consider the arrow $(L, L) \rightarrow (K, L)$.

PROPOSITION 4 Let (Y, B) and (Y', B') be pairs and let $h : (Y, B) \rightarrow (Y', B')$ be a continuous function; let (X, A) and (X', A') be CW pairs and let $f : (X, A) \rightarrow (Y, B)$

& $f' : (X',A') \to (Y',B')$ be continuous functions. Assume: f' is a weak homotopy equivalence of pairs—then there exists a continuous function $g : (X,A) \to (X',A')$, unique up to homotopy of pairs, such that the diagram

$$\begin{tikzcd} (X,A) \arrow[r, "g"] \arrow[d, "f"'] & (X',A') \arrow[d, "f'"] \\ (Y,B) \arrow[r, "h"'] & (Y',B') \end{tikzcd}$$

commutes up to homotopy of pairs.

[The arrow $f'_* : [X,A;X',A'] \to [X,A;Y',B']$ is bijective.]

Given a topological space X, a CW resolution for X is an ordered pair (K,f), where K is a CW complex and $f : K \to X$ is a weak homotopy equivalence. The homotopy type of a CW resolution is unique. Proof: Let $f : K \to X$ & $f' : K' \to X$ be CW resolutions of X—then by Proposition 4, there exists a continuous function $g : K \to K'$ such that the diagram

$$\begin{tikzcd} K \arrow[r, "g"] \arrow[d, "f"'] & K' \arrow[d, "f'"] \\ X \arrow[r, equal] & X \end{tikzcd}$$

is homotopy commutative: $f \simeq f' \circ g$. Therefore g is a weak homotopy equivalence, hence is a homotopy equivalence (via the realization theorem).

RESOLUTION THEOREM Every topological space X admits a CW resolution $f : K \to X$.

[Note: If X is path connected (n-connected), then one can choose K path connected with $K^{(0)}$ ($K^{(n)}$) a singleton.]

Application: Suppose that X is homotopically trivial—then for any CW complex K, the elements of $C(K,X)$ are inessential.

Given a pair (X,A), a relative CW resolution for (X,A) is an ordered pair $((K,L),f)$, where (K,L) is a CW pair and $f : (K,L) \to (X,A)$ is a weak homotopy equivalence of pairs. A relative CW resolution is unique up to homotopy of pairs (cf. Proposition 4).

RELATIVE RESOLUTION THEOREM Every pair (X,A) admits a relative CW resolution $f : (K,L) \to (X,A)$.

[Fix CW resolutions $\begin{cases} \phi : L \to A \\ \psi : K \to X \end{cases}$ and let $i : A \to X$ be the inclusion. Using Proposition 4, choose a $g : L \to K$ such that $\psi \circ g \simeq i \circ \phi$. Owing to the skeletal approximation theorem, one can assume that g is skeletal, thus its mapping cylinder M_g is a CW complex containing L and K as embedded subcomplexes. If $r : M_g \to K$ is the usual retraction, then r is a homotopy equivalence and $\psi \circ r|L \simeq i \circ \phi$. Since the inclusion $L \to M_g$ is a cofibration, $\psi \circ r$ is homotopic to a map $f : M_g \to X$ such that $f|L = i \circ \phi$. Change the notation to conclude the proof.]

[Note: If (X, A) is n-connected, then one can choose K with $K^{(n)} \subset L$.]

It follows from the proof of the relative resolution theorem that given (X, A) and a CW resolution $g : L \to A$, there exists a relative CW resolution $f : (K, L) \to (X, A)$ extending g.

Let X and Y be topological spaces—then X is said to be dominated in homotopy by Y if there exist continuous functions $\begin{cases} f : X \to Y \\ g : Y \to X \end{cases}$ such that $g \circ f \simeq \mathrm{id}_X$. Example: A topological space is contractible iff it is dominated in homotopy by a one point space.

[Note: Let $f : X \to Y$ be a continuous function, M_f its mapping cylinder—then f admits a left homotopy inverse $g : Y \to X$ iff $i(X)$ is a retract of M_f. By comparison, f is a homotopy equivalence iff $i(X)$ is a strong deformation retract of M_f (cf. §3, Proposition 17).]

EXAMPLE Let X be a topological space which is dominated in homotopy by a compact connected n-manifold Y. Assume: $H^n(X; \mathbf{Z}_2) \neq 0$—then Kwasik[†] has shown that X and Y have the same homotopy type.

FACT If X is dominated in homotopy by a CW complex, then the path components of X are open.

DOMINATION THEOREM Let X be a topological space—then X has the homotopy type of a CW complex iff X is dominated in homotopy by a CW complex.

[Suppose that X is dominated in homotopy by a CW complex Y : $\begin{cases} f : X \to Y \\ g : Y \to X \end{cases}$ & $g \circ f \simeq \mathrm{id}_X$. Fix a CW resolution $h : K \to X$. Using Proposition 4, choose continuous functions $\begin{cases} f' : K \to Y \\ g' : Y \to K \end{cases}$ such that the diagram

$$\begin{tikzcd} K \arrow[r, "f'"] \arrow[d, "h"'] & Y \arrow[r, "g'"] \arrow[d, equal] & K \arrow[d, "h"] \\ X \arrow[r, "f"'] & Y \arrow[r, "g"'] & X \end{tikzcd}$$

is homotopy commutative. Claim: h is a homotopy equivalence with homotopy inverse $g' \circ f$. In fact: $(g \circ f) \circ h \simeq g \circ f' \simeq h \circ (g' \circ f')$ & $(g \circ f) \circ h \simeq h \circ \mathrm{id}_K \Rightarrow g' \circ f' \simeq \mathrm{id}_K$ (cf. Proposition 4), so $(g' \circ f) \circ h \simeq g' \circ f' \simeq \mathrm{id}_K$ & $h \circ (g' \circ f) \simeq g \circ f \simeq \mathrm{id}_X$.]

Application: Every retract of a CW complex has the homotopy type of a CW complex.

[†] *Canad. Math. Bull.* **27** (1984), 448–451.

[Note: Consequently, every open subset of a CW complex has the homotopy type of a CW complex (cf. p. 5–12).]

COUNTABLE DOMINATION THEOREM Let X be a topological space—then X has the homotopy type of a countable CW complex iff X is dominated in homotopy by a countable CW complex.

[Suppose that X is dominated in homotopy by a countable CW complex Y : $\begin{cases} f : X \to Y \\ g : Y \to X \end{cases}$ & $g \circ f \simeq \mathrm{id}_X$. Using the notation of the preceding proof, consider the image $g'(Y)$ of Y in K. Claim: $g'(Y)$ is contained in a countable subcomplex L_0 of K. Indeed, for any cell e of Y, $g'(\bar{e})$ is compact, thus is contained in a finite subcomplex of K and a countable union of finite subcomplexes is a countable subcomplex. Fix a homotopy $H : IK \to K$ between $g' \circ f \circ h$ and id_K. Since IL_0 is a countable CW complex, there exists a countable subcomplex $L_1 \subset K : H(IL_0) \subset L_1$. Iteration then gives a sequence $\{L_n\}$ of countable subcomplexes L_n of $K : \forall\, n, H(IL_n) \subset L_{n+1}$. The union $L = \bigcup\limits_n L_n$ is a countable CW complex whose homotopy type is that of X.]

Application: Every Lindelöf space having the homotopy type of a CW complex has the homotopy type of a countable CW complex.

[The subcomplex generated by a Lindelöf subspace of a CW complex is necessarily countable.]

Is it true that if X is dominated in homotopy by a finite CW complex, then X has the homotopy type of a finite CW complex? The answer is "no" in general but "yes" under certain assumptions.

Notation: Given a group G, let $\mathbf{Z}[G]$ be its integral group ring and write $\widetilde{K}_0(G)$ for the reduced Grothendieck group attached to the category of finitely generated projective $\mathbf{Z}[G]$-modules.

The following results are due to Wall[†].

OBSTRUCTION THEOREM Suppose that X is path connected and dominated in homotopy by a finite CW complex—then there exists an element $\widetilde{w}(X) \in \widetilde{K}_0(\pi_1(X))$ such that $\widetilde{w}(X) = 0$ iff X has the homotopy type of a finite CW complex.

One calls $\widetilde{w}(X)$ Wall's obstruction to finiteness. Example: If X is simply connected and dominated in homotopy by a finite CW complex, then X has the homotopy type of a finite CW complex.

[†] *Ann. of Math.* **81** (1965), 56–69.

FULFILLMENT LEMMA Let G be a finitely presented group—then given any $\alpha \in \widetilde{K}_0(G)$, there exists a connected CW complex X_α which is dominated in homotopy by a finite CW complex such that $\pi_1(X_\alpha) = G$ and $\widetilde{w}(X) = \alpha$.

Let A be a Dedekind domain, e.g., the ring of algebraic integers in an algebraic number field—then the reduced Grothendieck group of A is isomorphic to the ideal class group of A. This fact, in conjunction with the fulfillment lemma, can be used to generate examples. Thus fix a prime p, put $\omega_p = \exp(2\pi\sqrt{-1}/p)$, and consider $\mathbf{Z}[\omega_p]$, the ring of algebraic integers in $\mathbf{Q}(\omega_p)$. It is known that $\widetilde{K}_0(\mathbf{Z}/p\mathbf{Z})$ is isomorphic to the reduced Grothendieck group of $\mathbf{Z}[\omega_p]$. But the ideal class group of $\mathbf{Z}[\omega_p]$ is nontrivial for $p > 19$ (Montgomery). Moral: There exist connected CW complexes which are dominated in homotopy by a finite CW complex, yet do not have the homotopy type of a finite CW complex.

EXAMPLE Every path connected compact Hausdorff space X which is dominated in homotopy by a CW complex is automatically dominated in homotopy by a finite CW complex. Is $\widetilde{w}(X) = 0$? Every connected compact ANR (in particular, every connected compact topological manifold) has the homotopy type of a CW complex (cf. p. 6–19), thus is dominated in homotopy by a finite CW complex and one can prove that its Wall obstruction to finiteness must vanish, so such an X does have the homotopy type of a finite CW complex. Still, some restriction on X is necessary. This is because Ferry[†] has shown that any Hausdorff space which is dominated in homotopy by a second countable compact Hausdorff space must itself have the homotopy type of a second countable compact Hausdorff space and since there exist connected CW complexes with a nonzero Wall obstruction to finiteness, it follows that there exist path connected metrizable compacta which are dominated in homotopy by a finite CW complex, yet do not have the homotopy type of a finite CW complex.

EXAMPLE Suppose that X is path connected and dominated in homotopy by a finite CW complex—then Gersten[‡] has shown that for any connected CW complex K of zero Euler characteristic, the product $X \times K$ has the homotopy type of a finite CW complex, i.e., multiplication by K kills Wall's obstruction to finiteness. For example, one can take $K = \mathbf{S}^{2n+1}$. In particular: $X \times \mathbf{S}^1$ is homotopy equivalent to a finite CW complex Y, say $f : X \times \mathbf{S}^1 \to Y$. Since X is homotopy equivalent to $X \times \mathbf{R}$ and $X \times \mathbf{R}$ is the covering space of $X \times \mathbf{S}^1$ determined by $\pi_1(X) \subset \pi_1(X \times \mathbf{S}^1)$, it follows that X is homotopy equivalent to the covering space $\widetilde{Y}$ of Y determined by the subgroup $f_*(\pi_1(X))$ of $\pi_1(Y)$. Conclusion: X has the homotopy type of a finite dimensional CW complex.

A (pointed) topological space is said to be a (pointed) CW space if it has the (pointed) homotopy type of a (pointed) CW complex. **CWSP** (**CWSP**$_*$) is the full subcategory of **TOP** (**TOP**$_*$) whose objects are the CW spaces (pointed CW spaces) and **HCWSP** (**HCWSP**$_*$) is the associated homotopy category. Example: Suppose that (X, A) is a relative CW complex, where A is a CW space—then X is a CW space.

[†] *Topology* **19** (1980), 101–110; see also *SLN* **870** (1981), 1–5 and 73–81.

[‡] *Amer. J. Math.* **88** (1966), 337–346; see also Kwasik, *Comment. Math. Helv.* **58** (1983), 503–508.

[Note: If (X, x_0) is a pointed CW space, then (X, x_0) is nondegenerate (cf. p. 3–35).]

Every CW space is numerably contractible (cf. p. 3–13). Every connected CW space is path connected. Every totally disconnected CW space is discrete. Every homotopically trivial CW space is contractible (cf. p. 5–17).

[Note: A CW space need not be locally path connected.]

The product $X \times Y$ of CW spaces $\begin{cases} X \\ Y \end{cases}$ is a CW space. Proof: There exist CW complexes $\begin{cases} K \\ L \end{cases}$ such that in **HTOP**, $\begin{cases} X \approx K \\ Y \approx L \end{cases} \Rightarrow X \times Y \approx K \times L \approx K \times_k L$ (cf. p. 5–17) and $K \times_k L$ is a CW complex.

A CW space need not be compactly generated. Example: Suppose that X is not in **CG**—then ΓX is not in **CG** but ΓX is a CW space. However, for any CW space X, the identity map $kX \rightarrow X$ is a homotopy equivalence.

PROPOSITION 5 Let X be a connected CW space—then X has a simply connected covering space $\widetilde{X}$ which is universal. Moreover, every simply connected covering space of X is homeomorphic over X to $\widetilde{X}$.

[Fix a CW complex K and a homotopy equivalence $\phi : X \rightarrow K$. Let $\widetilde{K}$ be a universal covering space of K and define $\widetilde{X}$ by the pullback square
$$\begin{tikzcd} \widetilde{X} \arrow[r, "\widetilde{\phi}"] \arrow[d] & \widetilde{K} \arrow[d] \\ X \arrow[r, "\phi"'] & K \end{tikzcd}$$
. Since the covering projection $\widetilde{K} \rightarrow K$ is a Hurewicz fibration (cf. p. 4–7), $\widetilde{\phi}$ is a homotopy equivalence (cf. p. 4–24), so $\widetilde{X}$ is a simply connected covering space of X. To see that $\widetilde{X}$ is universal, let $\widetilde{X}'$ be some other connected covering space of X—then the claim is that there is an arrow $\widetilde{X} \xrightarrow{f} \widetilde{X}'$ and a commutative triangle
$$\begin{tikzcd} \widetilde{X} \arrow[rr, "f"] \arrow[dr] & & \widetilde{X}' \arrow[dl] \\ & X & \end{tikzcd}$$
. For this, form the pullback square
$$\begin{tikzcd} \widetilde{K}' \arrow[r, "\widetilde{\psi}"] \arrow[d] & \widetilde{X}' \arrow[d] \\ K \arrow[r, "\psi"'] & X \end{tikzcd}$$
, ψ a homotopy inverse for ϕ. Due to the universality of $\widetilde{K}$, there is an arrow $\widetilde{K} \xrightarrow{g} \widetilde{K}'$ and a commutative triangle
$$\begin{tikzcd} \widetilde{K} \arrow[rr, "g"] \arrow[dr] & & \widetilde{K}' \arrow[dl] \\ & K & \end{tikzcd}$$
. Consider the diagram

$$\begin{tikzcd}
\widetilde{X} \arrow[r, "\widetilde{\phi}"] \arrow[d, "p"'] & \widetilde{K} \arrow[r, "g"] \arrow[d] & \widetilde{K}' \arrow[r, "\widetilde{\psi}"] \arrow[d] & \widetilde{X}' \arrow[d, "p'"] \\
X \arrow[r, "\phi"'] & K \arrow[r, equal] & K \arrow[r, "\psi"'] & X
\end{tikzcd}$$

From the definitions, $p' \circ \widetilde{\psi} \circ g \circ \widetilde{\phi} = \psi \circ \phi \circ p \simeq p$, thus $\exists\, f \in C_X(\widetilde{X}, \widetilde{X}') : f \simeq \widetilde{\psi} \circ g \circ \widetilde{\phi}$. Finally, if $\widetilde{X}'$ is simply connected, then $\widetilde{K}'$ is simply connected and one can assume that g is a homeomorphism. Therefore f is a fiber homotopy equivalence (cf. §4, Proposition 15). Because the fibers are discrete, it follows that f is also an open bijection, hence is a homeomorphism.]

EXAMPLE The Cantor set is not a CW space. The topologist's sine curve $C = A \cup B$, where $\begin{cases} A = \{(0, y) : -1 \leq y \leq 1\} \\ B = \{(x, \sin(2\pi/x)) : 0 < x \leq 1\} \end{cases}$, is not a CW space. The wedge of the broom is not a CW space but the broom, being contractible, is a CW space, although it carries no CW structure. The product $\prod\limits_1^\infty \mathbf{S}^n$ is not a CW space.

FACT Suppose that X is a connected CW space. Assume: $\pi_1(X)$ is finite and $\forall\, q > 1$, $\pi_q(X)$ is finitely generated—then there exists a homotopy equivalence $f : K \to X$, where K is a CW complex such that $\forall\, n, K^{(n)}$ is finite.

Dydak[†] has shown that the full subcategory of $\mathbf{HCWSP}_*$ whose objects are the pointed connected CW spaces is balanced.

Every open subset of a CW complex is a CW space (cf. p. 5–20). Every open subset of a metrizable topological manifold is a CW space (cf. p. 6–28).

PROPOSITION 6 Let U be an open subset of a normed linear space E—then U is a CW space.

[Fix a countable neighborhood basis at zero in E consisting of convex balanced sets U_n such that $U_{n+1} \subset U_n$. Assuming that U is nonempty, for each $x \in U$, there exists an index $n_{(x)} : x + 2U_{n(x)} \subset U$. Since U is paracompact, the open covering $\{x + U_{n(x)} : x \in U\}$ has a neighborhood finite open refinement $\mathcal{O} = \{O\}$. So, $\forall\, O \in \mathcal{O}\ \exists\, x_O \in U : O \subset x_O + U_{n(O)}$ $(n(O) = n(x_O))$. Let $\{\kappa_O : O \in \mathcal{O}\}$ be a partition of unity on U subordinate to $\mathcal{O}$.

[†] *Proc. Amer. Math. Soc.* **116** (1992), 1171–1173; see also Dyer-Roitberg, *Topology Appl.* **46** (1992), 119–124.

Consider $N(\mathcal{O})$, the nerve of $\mathcal{O}$. If $\{O_1, \ldots, O_k\}$ is a simplex of $N(\mathcal{O})$ and if $n(O_1) \leq \cdots \leq n(O_k)$, then the convex hull of $\{x_{O_1}, \ldots, x_{O_k}\}$ is contained in $x_{O_1} + 2U_{n(O_1)} \subset U$. Define continuous functions $\begin{cases} f : U \to |N(\mathcal{O})| \\ g : |N(\mathcal{O})| \to U \end{cases}$ by $\begin{cases} f(x) = \sum\limits_{O} \kappa_O(x)\chi_O \\ g(\phi) = \sum\limits_{O} \phi(O)x_O \end{cases}$ and put $H(x,t) = tx + (1-t)\sum\limits_{O} \kappa_O(x)x_O$ to get a homotopy $H : IU \to U$ between $g \circ f$ and id_U. This shows that U is dominated in homotopy by $|N(\mathcal{O})|$, hence, by the domination theorem, has the homotopy type of a CW complex.]

[Note: If E is second countable, then U has the homotopy type of a countable CW complex. Reason: Every open covering of a second countable metrizable space has a countable star finite refinement (cf. p. 1–25).]

FACT Let E be a normed linear space. Suppose that E_0 is a dense linear subspace of E. Equip E_0 with the finite topology—then for every open subset U of E, the inclusion $U \cap E_0 \to U$ is a weak homotopy equivalence.

FACT Let E be a normed linear space. Suppose that $E^0 \subset E^1 \subset \cdots$ is an increasing sequence of finite dimensional linear subspaces of E whose union is dense in E. Given an open subset U of E, put $U^n = U \cap E^n$—then $U^0 \subset U^1 \subset \cdots$ is an expanding sequence of topological spaces and the inclusion $U^\infty \to U$ is a homotopy equivalence.

PROPOSITION 7 Let $A \to X$ be a closed cofibration and let $f : A \to Y$ be a continuous function. Assume: A, X, and Y are CW spaces—then $X \sqcup_f Y$ is a CW space.

[There is a CW pair (K, L) and a commutative diagram

$$\begin{tikzcd} K \arrow[d] & L \arrow[l] \arrow[r, "g"] \arrow[d] & Y \arrow[d, equal] \\ X & A \arrow[l] \arrow[r, "f"'] & Y \end{tikzcd},$$

where the vertical arrows are homotopy equivalences and g is the composite. Accordingly, $K \sqcup_g Y \approx X \sqcup_f Y$ in **HTOP** (cf. p. 3–24 ff.) and $K \sqcup_g Y$ is a CW space (cf. p. 5–14).]

Application: Let $X \xleftarrow{f} Z \xrightarrow{g} Y$ be a 2-source. Assume: X, Y, and Z are CW spaces—then $M_{f,g}$ is a CW space.

[Note: One can establish an analogous result for the double mapping track of a 2-sink in **CWSP** (cf. §6, Proposition 8). For example, given a nonempty CW space X, $\forall\, x_0 \in X$, $\Omega(X, x_0)$ is a CW space (consider the 2-sink $* \to X \leftarrow *$).]

EXAMPLE Suppose that X and Y are CW spaces—then their join $X * Y$ is a CW space.

[Note: The double mapping cylinder of $X \leftarrow X \times Y \rightarrow Y$ defines the join. If X and Y are CW complexes, then $X * Y$ is a CW complex provided that $X \times Y = X \times_k Y$. Otherwise, consider $X *_k Y$, the double mapping cylinder of $X \leftarrow X \times_k Y \rightarrow Y$.]

LEMMA Let $X^0 \subset X^1 \subset \cdots$ be an expanding sequence of topological spaces. Assume: $\forall\, n, X^n$ is a CW complex containing X^{n-1} as a subcomplex—then X^∞ is a CW complex containing X^n as a subcomplex.

EXAMPLE (The Mapping Telescope) Let $\begin{cases} (\mathbf{X},\mathbf{f}) \\ (\mathbf{Y},\mathbf{g}) \end{cases}$ be objects in **FIL(TOP)**. Suppose that $\phi : (\mathbf{X},\mathbf{f}) \rightarrow (\mathbf{Y},\mathbf{g})$ is a homotopy morphism, i.e., $\forall\, n$, the diagram

$$\begin{tikzcd} X_n \arrow[r, "f_n"] \arrow[d, "\phi_n"'] & X_{n+1} \arrow[d, "\phi_{n+1}"] \\ Y_n \arrow[r, "g_n"'] & Y_{n+1} \end{tikzcd}$$

is homotopy commutative—then there is an arrow $\mathrm{tel}\,\phi : \mathrm{tel}(\mathbf{X},\mathbf{f}) \rightarrow \mathrm{tel}(\mathbf{Y},\mathbf{g})$ such that $\forall\, n$, the diagram

$$\begin{tikzcd} X_n \arrow[d] & \mathrm{tel}_n(\mathbf{X},\mathbf{f}) \arrow[l] \arrow[r] \arrow[d] & \mathrm{tel}(\mathbf{X},\mathbf{f}) \arrow[d] \\ Y_n & \mathrm{tel}_n(\mathbf{Y},\mathbf{g}) \arrow[l] \arrow[r] & \mathrm{tel}(\mathbf{Y},\mathbf{g}) \end{tikzcd}$$

is homotopy commutative and $\mathrm{tel}\,\phi$ is a homotopy equivalence if each ϕ_n is a homotopy equivalence. Thanks to the skeletal approximation theorem and the lemma, it then follows that for any object $(\mathbf{X},\mathbf{f})$ in **FIL(CW)**, there exists another object $(\mathbf{X},\mathbf{g})$ in **FIL(CW)** such that $\mathrm{tel}(\mathbf{X},\mathbf{f})$ and $\mathrm{tel}(\mathbf{X},\mathbf{g})$ have the same homotopy type and $\mathrm{tel}(\mathbf{X},\mathbf{g})$ is a CW complex.

[The mapping telescope is a double mapping cylinder (cf. p. 3–23). Use the fact that a homotopy morphism of 2-sources, i.e., a homotopy commutative diagram

$$\begin{tikzcd} X \arrow[d] & Z \arrow[l, "f"'] \arrow[r, "g"] \arrow[d] & Y \arrow[d] \\ X' & Z' \arrow[l, "f'"] \arrow[r, "g'"'] & Y' \end{tikzcd}$$

, gives rise to an arrow $M_{f,g} \rightarrow M_{f',g'}$ which is a homotopy equivalence if this is the case of the vertical arrows (cf. p. 3–24).]

PROPOSITION 8 Let $X^0 \subset X^1 \subset \cdots$ be an expanding sequence of topological spaces. Assume: $\forall\, n$, X^n is a CW space and the inclusion $X^n \rightarrow X^{n+1}$ is a cofibration—then X^∞ is a CW space.

[There is a commutative ladder

$$\begin{tikzcd} K^0 \arrow[r] \arrow[d] & K^1 \arrow[r] \arrow[d] & \cdots \\ X^0 \arrow[r] & X^1 \arrow[r] & \cdots \end{tikzcd}$$

, where the vertical arrows $K^n \rightarrow X^n$ are homotopy equivalences and $K^0 \subset K^1 \subset \cdots$ is an expanding sequence of CW complexes such that $\forall\, n$, (K^n, K^{n-1}) is a CW pair. The induced map $K^\infty \rightarrow X^\infty$ is a homotopy equivalence (cf. §3, Proposition 15) and, by the lemma, K^∞ is a CW complex.]

Application: Let $(\mathbf{X}, \mathbf{f})$ be an object in **FIL(TOP)**. Assume: $\forall\ n, X_n$ is a CW space—then $\operatorname{tel}(\mathbf{X}, \mathbf{f})$ is a CW space.

FACT Let X be a topological space. Suppose that $\mathcal{U} = \{U_i : i \in I\}$ is a numerable covering of X with the property that for every nonempty finite subset $F \subset I$, $\bigcap\limits_{i \in F} U_i$ is a CW space—then X is a CW space.

[In the notation of the Segal-Stasheff construction, show that $B\mathcal{U}$ is a CW space.]

Application: Let X be a topological space. Suppose that $\mathcal{U} = \{U_i : i \in I\}$ is a numerable covering of X with the property that for every nonempty finite subset $F \subset I$, $\bigcap\limits_{i \in F} U_i$ is either empty or contractible—then X is a CW space.

[Note: One can be more precise: X and $|N(\mathcal{U})|$ have the same homotopy type. Example: Every paracompact open subset of a locally convex topological vector space is a CW space (cf. Proposition 6).]

EXAMPLE Let X be the Cantor set. In ΣX, let U_1 be the image of $X \times [0, 2/3[$ and let U_2 be the image of $X \times]1/3, 1]$—then $\{U_1, U_2\}$ is a numerable open covering of ΣX. Both U_1 and U_2 are contractible, hence are CW spaces. But ΣX is not a CW space. In this connection, observe that $U_1 \cap U_2$ has the same homotopy type as X, thus is not a CW space.

A sequence of groups π_n $(n \geq 1)$ is said to be a homotopy system if $\forall\ n > 1 : \pi_n$ is abelian and there is a left action $\pi_1 \times \pi_n \rightarrow \pi_n$.

HOMOTOPY SYSTEM THEOREM Let $\{\pi_n : n \geq 1\}$ be a homotopy system—then there exists a pointed connected CW complex (X, x_0) and $\forall\ n \geq 1$, an isomorphism $\pi_n(X, x_0) \rightarrow \pi_n$ such that the action of $\pi_1(X, x_0)$ on $\pi_n(X, x_0)$ corresponds to the action of π_1 on π_n.

[Note: One can take X locally finite if all the π_n are countable.]

Let π be a group and let n be an integer ≥ 1, where π is abelian if $n > 1$—then a pointed path connected space (X, x_0) is said to have homotopy type (π, n) if $\pi_n(X, x_0)$ is isomorphic to π and $\pi_q(X, x_0) = 0$ $(q \neq n)$. An Eilenberg-MacLane space of type (π, n) is a pointed connected CW space (X, x_0) of homotopy type (π, n). Notation: $(X, x_0) = (K(\pi, n), k_{\pi,n})$. Two spaces of homotopy type (π, n) have the same weak homotopy type and two Eilenberg-MacLane spaces of type (π, n) have the same pointed homotopy type. Every Eilenberg-MacLane space is nondegenerate, therefore the same is true of its loop space which, moreover, is a pointed CW space (cf. p. 6–24). Example: $\Omega K(\pi, n+1) = K(\pi, n)$, π abelian.

EXAMPLE A model for $K(G,1)$, G a discrete topological group, is B_G^∞ (cf. p. 6–25).

Upon specializing the homotopy system theorem, it follows that for every π, $(K(\pi,n), k_{\pi,n})$ exists as a pointed CW complex. If in addition π is abelian, then $(K(\pi,n), k_{\pi,n})$ carries the structure of a homotopy commutative H group, unique up to homotopy, and the assignment $(X,A) \rightarrow [X,A;K(\pi,n),k_{\pi,n}]$ defines a cofunctor $\mathbf{TOP}^2 \rightarrow \mathbf{AB}$.

EXAMPLE A model for $K(\mathbf{Z}^n,1)$ is $\mathbf{T}^n$.

[Note: Suppose that X is a homotopy commutative H space with the pointed homotopy type of a finite connected CW complex—then Hubbuck[†] has shown that in $\mathbf{HTOP}_*$, $X \approx \mathbf{T}^n$ for some $n \geq 0$.]

EXAMPLE A model for $K(\mathbf{Z}/n\mathbf{Z},1)$ is the orbit space $\mathbf{S}^\infty/\Gamma$, where Γ is the subgroup of $\mathbf{S}^1$ generated by a primitive n^{th} root of unity.

[Note: Recall that $\mathbf{S}^\infty$ is contractible (cf. p. 3–20).]

EXAMPLE A model for $K(\mathbf{Q},1)$ is the pointed mapping telescope of the sequence $\mathbf{S}^1 \rightarrow \mathbf{S}^1 \rightarrow \cdots$, the kth map having degree k.

[Note: Shelah[‡] has shown that if X is a compact metrizable space which is path connected and locally path connected, then $\pi_1(X)$ cannot be isomorphic to $\mathbf{Q}$.]

The homotopy type of $\prod\limits_{q=1}^{N} K(\mathbf{Z},2q)$ or $\prod\limits_{q=1}^{N} K(\mathbf{Z}/n\mathbf{Z},2q)$ admits an interpretation in terms of the theory of algebraic cycles (Lawson[‖]).

$(\pi,1)$ Suppose that (X,x_0) has homotopy type $(\pi,1)$—then for any pointed connected CW complex (K,k_0), the assignment $[f] \rightarrow f_*$ defines a bijection $[K,k_0;X,x_0] \rightarrow \text{Hom}(\pi_1(K,k_0),\pi_1(X,x_0))$. Since (K,k_0) is wellpointed, the orbit space $\pi_1(X,x_0)\backslash[K,k_0;X,x_0]$ can be identified with $[K,X]$ (cf. p. 3–18), thus there is a bijection $[K,X] \rightarrow \pi_1(X,x_0)\backslash\text{Hom}(\pi_1(K,k_0),\pi_1(X,x_0))$, the set of conjugacy classes of homomorphisms $\pi_1(K,k_0) \rightarrow \pi_1(X,x_0)$. If π is abelian, then $\text{Hom}(\pi_1(K,k_0),\pi_1(X,x_0)) \approx \text{Hom}(H_1(K,k_0),\pi_1(X,x_0)) \approx H^1(K,k_0;\pi_1(X,x_0))$ and the forgetful function $[K,k_0;X,x_0] \rightarrow [K,X]$ is bijective.

Example: Fix a pointed connected CW complex (K,k_0)—then the functor $\mathbf{GR} \rightarrow \mathbf{SET}$ that sends π to $[K,k_0;K(\pi,1),k_{\pi,1}]$ is represented by $\pi_1(K,k_0)$.

[†] *Topology* **8** (1969), 119–126.

[‡] *Proc. Amer. Math. Soc.* **103** (1988), 627–632.

[‖] *Ann. of Math.* **129** (1989), 253–291.

EXAMPLE Take $X = K(\pi, 1)$, $x_0 = k_{\pi,n}$ and realize (X, x_0) as a pointed CW complex. Assume: X is locally finite and finite dimensional. Write $HE(X, x_0)$ $(HE(X))$ for the space of homotopy equivalences of (X, x_0) (X) equipped with the compact open topology—then $\pi_0(HE(X, x_0))$ $(\pi_0(HE(X)))$ is the isomorphism group of (X, x_0) (X) viewed as an object in **HTOP**$_*$ (**HTOP**). By the above, $\pi_0(HE(X, x_0)) \approx \operatorname{Aut}\pi$ $(\pi_0(HE(X)) \approx \operatorname{Out}\pi)$. The evaluation $\begin{cases} HE(X) \to X \\ f \to f(x_0) \end{cases}$ is a Hurewicz fibration (cf. §4, Proposition 6) and its fiber over x_0 is $HE(X, x_0)$. With id_X as the base point, one has $\pi_q(HE(X, x_0), \mathrm{id}_X) = 0$ $(q > 0)$, $\pi_q(HE(X), \mathrm{id}_X) = 0$ $(q > 1)$, and $\pi_1(HE(X), \mathrm{id}_X) \approx \operatorname{Cen}\pi$, the center of π. The homotopy sequence of the evaluation thus reduces to $1 \to \pi_1(HE(X), \mathrm{id}_X) \to \pi_1(X, x_0) \to \pi_0(HE(X, x_0), \mathrm{id}_X) \to \pi_0(HE(X), \mathrm{id}_X) \to 1$, i.e., to $1 \to \operatorname{Cen}\pi \to \pi \to \operatorname{Aut}\pi \to \operatorname{Out}\pi \to 1$.

EXAMPLE Let $p : X \to B$ be a Hurewicz fibration, where $B = K(G, 1)$. Suppose that $\forall\, b \in B$, X_b is a $K(\pi, 1)$ (π abelian)—then the only nontrivial part of the homotopy sequence for p is the short exact sequence $1 \to \pi \to \pi_1(X) \to G \to 1$. Therefore $\pi_1(X)$ is an extension of π by G and X is a $K(\pi_1(X), 1)$ (cf. §6, Proposition 11). Algebraically, there is a left action $G \times \pi \to \pi$ and geometrically, there is a left action $G \times \pi \to \pi$. These two actions are identical.

EXAMPLE Consider a 2-source $\pi' \leftarrow G \rightarrow \pi''$ in **GR**, where the arrows are monomorphisms. Define π by the pushout square

$$\begin{tikzcd} G \arrow[r] \arrow[d] & \pi'' \arrow[d] \\ \pi' \arrow[r] & \pi \end{tikzcd}$$

, i.e., $\pi = \pi' *_G \pi''$—then there exists a pointed CW complex $X = K(\pi, 1)$ and pointed subcomplexes $\begin{cases} X' = K(\pi', 1) \\ X'' = K(\pi'', 1) \end{cases}$, $Y = K(G, 1)$ such that $X = X' \cup X''$ and $Y = X' \cap X''$.

EXAMPLE Let X and Y be connected CW complexes. Suppose that $f : X \to Y$ is a continuous function such that for every finite connected CW complex K, the induced map $[K, X] \to [K, Y]$ is bijective—then f is a homotopy equivalence iff $\forall\, x \in X$, $f_* : \pi_1(X, x) \to \pi_1(Y, f(x))$ is surjective (cf. p. 3–18) but this condition is not automatic. To construct an example, let S_∞ be the subgroup of the symmetric group of $\mathbf{N}$ consisting of those permutations that have finite support. Each injection $\iota : \mathbf{N} \to \mathbf{N}$ determines a homomorphism $\iota_\infty : S_\infty \to S_\infty$, viz. $\begin{cases} \iota_\infty(\sigma) | (\mathbf{N} - \iota(\mathbf{N})) = \mathrm{id} \\ \iota_\infty(\sigma) | \iota(\mathbf{N}) = \iota \circ \sigma \circ \iota^{-1} \end{cases}$, and on any finite product, $\prod \iota_\infty : S_\infty \backslash \prod S_\infty \to S_\infty \backslash \prod S_\infty$ is bijective. Here the action of S_∞ on $\prod S_\infty$ is by conjugation. Choose $\phi : K(S_\infty, 1) \to K(S_\infty, 1)$ such that $\phi_* = \iota_\infty$ on S_∞—then for every finite connected CW complex K, the induced map $[K, K(S_\infty, 1)] \to [K, K(S_\infty, 1)]$ is bijective (consider first a finite wedge of circles). However, ϕ is not a homotopy equivalence unless ι is surjective.

[Note: There are various conditions on $\pi_1(X)$ (or $\pi_1(Y)$) which guarantee that f_* is surjective (under the given assumptions). For example, any of the following will do: (1) $\pi_1(X)$ (or $\pi_1(Y)$) nilpotent; (2) $\pi_1(X)$ (or $\pi_1(Y)$) finitely generated; (3) $\pi_1(X)$ (or $\pi_1(Y)$) free.]

EXAMPLE Let π be a group—then $K(\pi,1)$ can be realized by a path connected metrizable topological manifold (cf. p. 6–28) iff π is countable and has finite cohomological dimension (Johnson[†]).

[Note: Under these circumstances, the cohomological dimension of π cannot exceed the euclidean dimension of $K(\pi,1)$, there being equality iff $K(\pi,1)$ is compact.]

EXAMPLE The homotopy type of an aspherical compact topological manifold is completely determined by its fundamental group. Question: If X and Y are aspherical compact topological manifolds and if $\pi_1(X) \approx \pi_1(Y)$, is it then true that X and Y are homeomorphic? Borel has conjectured that the answer is "yes". To get an idea of the difficulty of this problem, a positive resolution easily leads to a proof of the Poincaré conjecture (modulo a result of Milnor). Additional information and references can be found in Farrell-Jones[‡].

(π,n) Suppose that (X,x_0) has homotopy type (π,n), where π is abelian. Let $\iota \in H^n(X,x_0;\pi_n(X,x_0))$ be the fundamental class—then for any pointed connected CW complex (K,k_0), the assignment $[f] \rightarrow f^*\iota$ defines a bijection $[K,k_0;X,x_0] \rightarrow H^n(K,k_0;\pi_n(X,x_0))$.

Assuming that π' and π'' are abelian, $[K(\pi',n),k_{\pi',n};K(\pi'',n),k_{\pi'',n}] \approx [K(\pi',n),K(\pi'',n)] \approx \text{Hom}(\pi',\pi'')$. Example: Suppose that $0 \rightarrow \pi' \rightarrow \pi \rightarrow \pi'' \rightarrow 0$ is a short exact sequence of abelian groups—then (1) The mapping fiber of the arrow $K(\pi,n) \rightarrow K(\pi'',n)$ is a $K(\pi',n)$; (2) The mapping fiber of the arrow $K(\pi',n+1) \rightarrow K(\pi,n+1)$ is a $K(\pi'',n)$; (3) The mapping fiber of the arrow $K(\pi'',n) \rightarrow K(\pi',n+1)$ is a $K(\pi,n)$.

[Note: $\mathbf{CWSP}_*$ is closed under the formation of mapping fibers (cf. §6, Proposition 8).]

EXAMPLE A model for $K(\mathbf{Z},2)$ is $\mathbf{P}^\infty(\mathbf{C})$. Fix $n > 1$ and choose a map $\mathbf{P}^\infty(\mathbf{C}) \rightarrow K(\mathbf{Z},2n)$ representing a generator of $H^{2n}(\mathbf{P}^\infty(\mathbf{C});\mathbf{Z}) \approx \mathbf{Z}$. Put $Y = \mathbf{P}^\infty(\mathbf{C})$ and define X by the pullback square

$$\begin{tikzcd} X \arrow[r] \arrow[d] & \Theta K(\mathbf{Z},2n) \arrow[d] \\ Y \arrow[r] & K(\mathbf{Z},2n) \end{tikzcd}$$

. The fiber X_{y_0} is a $K(\mathbf{Z},2n-1)$. Since $2n-1 \geq 3$, there is an isomorphism $\pi_{2n-1}(X_{y_0}) \approx \pi_{2n-1}(X)$ but the corresponding arrow in homology $H_{2n-1}(X_{y_0}) \rightarrow H_{2n-1}(X)$ is not even one-to-one.

Let (X,A) be a relative CW complex—then for any abelian group π, there is a bijection $[X,A;K(\pi,n),k_{\pi,n}] \rightarrow H^n(X,A;\pi)$ which, in fact, is an isomorphism of abelian groups, natural in (X, A). This applies in particular when $A = \emptyset$, thus there is an isomorphism $[X, K(\pi, n)] \to H^n(X; \pi)$ of abelian groups, natural in X. So, on **HCW** the cofunctor $H^n(—; \pi)$ is representable by $K(\pi, n)$. But on **HTOP** itself, this is no longer true in that the relation $[X, K(\pi, n)] \approx H^n(X; \pi)$ can fail if X is not a CW complex.

[†] *Proc. Camb. Phil. Soc.* **70** (1971), 387–393.

[‡] *CBMS Regional Conference* **75** (1990), 1–54; see also Conner-Raymond, *Bull. Amer. Math. Soc.* **83** (1977), 36–85.

EXAMPLE Let X be the Warsaw circle and take $\pi = \mathbf{Z}$—then $H^1(X, \mathbf{Z}) = 0$, while $[X, K(\mathbf{Z}, 1)] \approx \mathbf{Z}$ or still, $[X, K(\mathbf{Z}, 1)] \approx \check{H}^1(X; \mathbf{Z})$.

In general, for an arbitrary abelian group π and an arbitrary pair (X, A), there is a natural isomorphism $[X, A; K(\pi, n), k_{\pi,n}] \to \check{H}(X, A; \pi)$ (cf. p. 20–1). Moral: It is Čech cohomology rather than singular cohomology that is the representable theory.

Suppose that (X, x_0) is a pointed connected CW complex. Equip $C(X, K(\pi, n))$ with the compact open topology—then $[X, K(\pi, n)] = \pi_0(C(X, K(\pi, n)))$, X being a compactly generated Hausdorff space. Because the forgetful function $[X, x_0; K(\pi, n), k_{\pi,n}] \to [X, K(\pi, n)]$ is surjective, every path component of $C(X, K(\pi, n))$ contains a pointed map $f_0 : f_0(x_0) = k_{\pi,n}$.

EXAMPLE Let (X, x_0) be a pointed connected CW complex. Assume: X is locally finite—then for any abelian group π, $\pi_q(C(X, K(\pi, n)), f_0) \approx \begin{cases} H^{n-q}(X; \pi) & (1 \leq q \leq n) \\ 0 & (q > n) \end{cases}$.

[Since $K(\pi, n)$ is an H group, all the path components of $C(X, K(\pi, n))$ have the same homotopy type. Let f_0 be the constant map $X \to k_{\pi,n}$, $C_0(X, K(\pi, n))$ its path component. To compute $\pi_q(C_0(X, K(\pi, n)), f_0)$, consider the Hurewicz fibration $C_0(X, K(\pi, n)) \to K(\pi, n)$ which sends f to $f(x_0)$ (cf. §4, Proposition 6), bearing in mind that $\pi_1(C_0(X, K(\pi, n)), f_0)$ is abelian.]

[Note: Suppose in addition that X is finite—then $C(X, K(\pi, n))$ (compact open topology) is a CW space (cf. p. 6–23) and there is a decomposition $H^n(C(X, K(\pi, n)) \times X; \pi) \approx \bigoplus_{q=0}^{n} H^q(C(X, K(\pi, n)); H^{n-q}(X; \pi))$. Let $\text{ev} : C(X, K(\pi, n)) \times X \to K(\pi, n)$ be the evaluation. Take the fundamental class $\iota \in H^n(K(\pi, n); \pi)$ and write $\text{ev}^* \iota = \bigoplus_{q=0}^{n} \mu_q$, where $\mu_q \in H^q(C(X, K(\pi, n)); H^{n-q}(X; \pi))$. Let $[f_q] \in [C(X, K(\pi, n)), K(H^{n-q}(X; \pi), q)]$ correspond to μ_q (conventionally, $K(H^n(X; \pi), 0)$ is $H^n(X; \pi)$ (discrete topology)). The f_q determine an arrow $C(X, K(\pi, n)) \to \prod_{q=0}^{n} K(H^{n-q}(X; \pi), q)$. It is a weak homotopy equivalence, hence, by the realization theorem, a homotopy equivalence.]

EXAMPLE Let (X, x_0) be a pointed connected CW complex. Assume: X is locally finite and finite dimensional—then for any group π, $\pi_q(C(X, K(\pi, 1)), f_0) \approx \begin{cases} \text{Cen}(\pi, f_0) & (q = 1) \\ 0 & (q > 1) \end{cases}$. Here, $\text{Cen}(\pi, f_0)$ is the centralizer of $(f_0)_*(\pi_1(X, x_0))$ in $\pi_1(K(\pi, 1), k_{\pi,1}) \approx \pi$. Special case: Suppose that (X, x_0) is aspherical, let $\pi = \pi_1(X, x_0)$, take $f_0 = \text{id}_X$, and conclude that the path component of the identity in

$C(X,X)$ has homotopy type $(\text{Cen}\,\pi, 1)$, $\text{Cen}\,\pi$ the center of π. Example: $\text{Cen}\,\pi$ is trivial if X is a compact connected riemannian manifold whose sectional curvatures are < 0.

[Reduce to when $X^{(0)} = \{x_0\}$ (cf. p. 5–15), observe that $\pi_q(C(X, K(\pi,1)), f_0) \approx \pi_q(C(X^{(1)}, K(\pi,1)), f_0|X^{(1)})$, and use the fact that $X^{(1)}$ is a wedge of circles.]

[Note: It can happen that π is finitely generated but $\text{Cen}(\pi, f_0)$ is infinitely generated even if $X = \mathbf{S}^1$ (Hansen[†]).]

A compactly generated group is a group G equipped with a compactly generated topology in which inversion $G \to G$ is continuous and multiplication $G \times_k G \to G$ is continuous. Since multiplication is not required to be continuous on $G \times G$ (product topology), a compactly generated group is not necessarily a topological group, although this will be the case if G is a LCH space or if G is first countable. Example: Let G be a simplicial group—then its geometric realization $|G|$ is a compactly generated group (cf. p. 13–2).

[Note: If G is a topological group, then kG is a compactly generated group but kG need not be a topological group (cf. p. 1–36). A compactly generated group is T_0 iff it is Δ-separated. Therefore any Δ-separated compactly generated group which is not Hausdorff cannot be a topological group.]

Suppose that π is abelian—then it is always possible to realize $K(\pi, n)$ as a pointed CW complex carrying the structure of an abelian compactly generated group on which $\text{Aut}\,\pi$ operates to the right by base point preserving skeletal homeomorphisms such that $\forall\ \phi \in \text{Aut}\,\pi$, there is a commutative square

$$\begin{tikzcd}
\pi_n(K(\pi,n)) \arrow[r, phantom, "\approx"] \arrow[d, "\phi_*"'] & \pi \arrow[d, "\phi"] \\
\pi_n(K(\pi,n)) \arrow[r, phantom, "\approx"] & \pi
\end{tikzcd}$$

(Adem-Milgram[‡]) $(0 = k_{\pi,n})$. With this understanding, let G be a group, assume that π is a right G-module, and denote by $\chi : G \to \text{Aut}\,\pi$ the associated homomorphism. Calling $\widetilde{K}(G,1)$ the universal covering space of $K(G,1)$, form the product $\widetilde{K}(G,1) \times K(\pi,n)$, and write $K(\pi,n;\chi)$ for the orbit space $(\widetilde{K}(G,1) \times K(\pi,n))/G$. As an object in $\mathbf{TOP}/K(G,1)$, $K(\pi,n;\chi)$ is locally trivial with fiber $K(\pi,n)$, thus the projection $p_\chi : K(\pi,n;\chi) \to K(G,1)$ is a Hurewicz fibration (local-global principle) and $K(\pi,n;\chi)$ is a CW space (cf. §6, Proposition 11). The inclusion $\widetilde{K}(G,1) \times \{0\} \to \widetilde{K}(G,1) \times K(\pi,n)$ defines a section $s_\chi : K(G,1) \to K(\pi,n;\chi)$, so $K(\pi,n;\chi)$ is an object in $\mathbf{TOP}(K(G,1))$ (cf. p. 0–3). Example: Take $G = \text{Aut}\,\pi :$ $\begin{cases} \pi \times \text{Aut}\,\pi \to \pi \\ (\alpha, \phi) \to \phi^{-1}(\alpha) \end{cases}$ —then the associated homomorphism $\text{Aut}\,\pi \to \text{Aut}\,\pi$ is $\text{id}_{\text{Aut}\,\pi} \equiv \chi_\pi$.

[†] *Compositio Math.* 28 (1974), 33–36.

[‡] *Cohomology of Finite Groups*, Springer Verlag (1994), 51.

[Note: Given G, consider the trivial action $\pi \times G \to \pi$, where $\chi : \begin{cases} G \to \operatorname{Aut}\pi \\ g \to \mathrm{id}_\pi \end{cases}$. In this case, $K(\pi, n; \chi)$ reduces to the product $K(G,1) \times K(\pi, n)$.]

Example: Take $\pi = \mathbf{Z}$, $G = \mathbf{Z}/2\mathbf{Z}$ and let $\chi : G \to \operatorname{Aut}\pi$ be the nontrivial homomorphism—then $K(\mathbf{Z}, 2; \chi)$ "is" $B_{\mathbf{O}(2)}$.

EXAMPLE The homotopy sequence for p_χ breaks up into a collection of split short exact sequences $0 \to \pi_q(K(\pi,n)) \to \pi_q(K(\pi,n;\chi)) \to \pi_q(K(G,1)) \to 0$. Case 1: $n \geq 2$. Here, $\pi_q(K(\pi,n;\chi)) \approx \begin{cases} \pi & (q = n) \\ G & (q = 1) \end{cases}$ and $\pi_q(K(\pi,n;\chi)) = 0$ otherwise. The algebraic right action $\pi \times G \to \pi$ corresponds to an algebraic left action $G \times \pi \to \pi$ and this is the same as the geometric left action $G \times \pi \to \pi$. Case 2: $n = 1$. In this situation, $\pi_1(K(\pi,n;\chi))$ is a split extension of π by G and the higher homotopy groups are trivial. If $\Theta_{s,p}K(\pi,n;\chi)$ is the subspace of $PK(\pi,n;\chi)$ made up of those σ such that $\sigma(0) \in s_\chi(K(G,1))$ and $p_\chi(\sigma(t)) = p_\chi(\sigma(0))$ $(0 \leq t \leq 1)$, then the projection $\Theta_{s,p}K(\pi,n;\chi) \to K(\pi,n;\chi)$ sending σ to $\sigma(1)$ is a Hurewicz fibration whose fiber over the base point is $\Omega K(\pi,n)$. Specialize and take $G = \operatorname{Aut}\pi$ (so $\chi = \chi_\pi$). Let B be a connected CW complex. The "class" of fiber homotopy classes of Hurewicz fibrations $X \to B$ with fiber $K(\pi,n)$ is a "set" (cf. p. 4–28 ff.). As such, it is in a one-to-one correspondence with the set of homotopy classes $[B, K(\pi, n+1; \chi_\pi)] : [X] \leftrightarrow [\Phi], \Phi : B \to K(\pi, n+1; \chi_\pi)$ the classifying map, where X is defined by the pullback square

$$\begin{tikzcd} X \arrow[r] \arrow[d] & \Theta_{s,p}K(\pi, n+1; \chi_\pi) \arrow[d] \\ B \arrow[r] & K(\pi, n+1; \chi_\pi) \end{tikzcd}$$

. For example, if X is a connected CW space with two nonzero homotopy groups $\pi_1(X) = G$ and $\pi_n(X) = \pi$ $(n > 1)$, then the geometry furnishes a right action $\pi \times G \to \pi$ and an associated homomorphism $\chi : G \to \operatorname{Aut}\pi$. To construct X up to homotopy, fix a map $f : X \to K(G,1)$ which induces the identity on G, pass to the mapping track W_f, and consider the Hurewicz fibration $W_f \to K(G,1)$. There is an arrow $\Phi : K(G,1) \to K(\pi, n+1; \chi_\pi)$ such that $\chi = \Phi_* : G \to \operatorname{Aut}\pi$ and $[W_f] \leftrightarrow [\Phi]$.

[Note: Suppose that B is a pointed simply connected CW complex—then the set of fiber homotopy classes of Hurewicz fibrations $X \to B$ with fiber $K(\pi, n)$ is in a one-to-one correspondence with $\operatorname{Aut}\pi \backslash H^{n+1}(B;\pi)$. Proof: The set of homotopy classes $[B, K(\pi, n+1; \chi_\pi)]$ can be identified with the set of pointed homotopy classes $[B, K(\pi, n+1; \chi_\pi)]$ mod $\pi_1(K(\pi, n+1; \chi_\pi))$, i.e., with the set of pointed homotopy classes $[B, K(\pi, n+1; \chi_\pi)]$ mod $\operatorname{Aut}\pi$, i.e., with the set of pointed homotopy classes $[B, K(\pi, n+1)]$ mod $\operatorname{Aut}\pi$ (cf. p. 5–16), i.e., with $\operatorname{Aut}\pi \backslash H^{n+1}(B;\pi)$. Translated, this means that in the simply connected case, one can use

$$\begin{tikzcd} \Theta K(\pi, n+1) \arrow[d] \\ K(\pi, n+1) \end{tikzcd}$$

to carry out the classification but then it is also necessary to build in the action of $\operatorname{Aut}\pi$.]

EXAMPLE Let G be a group; let $\begin{cases} \chi' : G \to \operatorname{Aut}\pi' \\ \chi'' : G \to \operatorname{Aut}\pi'' \end{cases}$ be homomorphisms, where $\begin{cases} \pi' \\ \pi'' \end{cases}$ are abelian—then $[K(\pi', n+1; \chi'),\ K(\pi'', n+1; \chi'')]_G \approx \operatorname{Hom}_G(\pi', \pi'')$, $[\quad , \quad]_G$ standing for homotopy in

$\mathbf{TOP}(K(G,1))$.

Notation: Given X in $\mathbf{TOP}/B$ and $\phi \in C(E,B)$, let $\text{lif}_\phi(E,X)$ be the set of liftings $\Phi : E \rightarrow X$ of ϕ. Relative to a choice of base points $b_0 \in B$, $x_0 \in X_{b_0}$, and $e_0 \in E$, where $\phi(e_0) = b_0$, let $\text{lif}_\phi(E,e_0;X,x_0)$ be the subset of $\text{lif}_\phi(E,X)$ consisting of those Φ such that $\Phi(e_0) = x_0$. Write $[E,X]_\phi$ for the set of fiber homotopy classes in $\text{lif}_\phi(E,X)$ and $[E,e_0;X,x_0]_\phi$ for the set of pointed fiber homotopy classes in $\text{lif}_\phi(E,e_0;X,x_0)$.

LEMMA If (B,b_0), (E,e_0) are wellpointed with $\{b_0\} \subset B$, $\{e_0\} \subset E$ closed, then the fundamental group $\pi_1(X_{b_0},x_0)$ operates to the left on $[E,e_0;X,x_0]_\phi$ and the forgetful function $[E,e_0;X,x_0]_\phi \rightarrow [E,X]_\phi$ passes to the quotient to define an injection $\pi_1(X_{b_0},x_0)\backslash[E,e_0;X,x_0]_\phi \rightarrow [E,X]_\phi$ which, when X_{b_0} is path connected, is a bijection.

Let G and π be groups. Given $\chi \in \text{Hom}(G, \text{Aut}\,\pi)$, denote by $\text{Hom}_\chi(G,\pi)$ the set of crossed homomorphisms per χ, so $f : G \rightarrow \pi$ is in $\text{Hom}_\chi(G,\pi)$ iff $f(g'g'') = f(g')(\chi(g')f(g''))$. There is a left action $\pi \times \text{Hom}_\chi(G,\pi) \rightarrow \text{Hom}_\chi(G,\pi)$, viz. $(\alpha \cdot f)(g) = \alpha f(g)(\chi(g)\alpha^{-1})$.

[Note: The elements of $\text{Hom}_\chi(G,\pi)$ correspond bijectively to the sections $s : G \rightarrow \pi \rtimes_\chi G$, where $\pi \rtimes_\chi G$ is the semidirect product (cf. p. 5–56).]

EXAMPLE Suppose that B is a connected CW complex. Fix a group π and a Hurewicz fibration $p : X \rightarrow B$ with fiber $K(\pi,1)$. Assume: $\text{sec}_B(X) \neq \emptyset$, say $s \in \text{sec}_B(X)$. Choose $b_0 \in B$ and put $x_0 = s(b_0)$. Let (E,e_0) be a pointed connected CW complex, $\phi : E \rightarrow B$ a pointed continuous function. There is a split short exact sequence $1 \rightarrow \pi_1(X_{b_0},x_0) \rightarrow \pi_1(X,x_0) \rightarrow \pi_1(B,b_0) \rightarrow 1$, from which a left action of $G = \pi_1(E,e_0)$ on $\pi = \pi_1(X_{b_0},x_0)$ or still, a homomorphism $\chi : G \rightarrow \text{Aut}\,\pi$, $\chi(g)$ thus being conjugation by $(s \circ \phi)_*(g)$. Attach to $\Phi \in \text{lif}_\phi(E,e_0;X,x_0)$ an element $f_\Phi \in \text{Hom}_\chi(G,\pi)$ via the prescription $f_\Phi(g) = \Phi_*(g)(s \circ \phi)_*(g)^{-1}$—then the assignment $\Phi \rightarrow f_\Phi$ induces a bijection $[E,e_0;X,x_0]_\phi \rightarrow \text{Hom}_\chi(G,\pi)$, so $[E,X]_\phi \approx \pi\backslash[E,e_0;X,x_0]_\phi \approx \pi\backslash\text{Hom}_\chi(G,\pi)$.

[Note: The considerations on p. 5–27 are recovered by taking $B = *$ and $X = K(\pi,1)$.]

(Locally Constant Coefficients) Let (X,x_0) be a pointed connected CW complex. Assume given a homomorphism $\chi_{\mathcal{G}} : \pi_1(X,x_0) \rightarrow G$ and a homomorphism $\chi : G \rightarrow \text{Aut}\,\pi$, where π is abelian. Let $\mathcal{G} : \Pi X \rightarrow \mathbf{AB}$ be the cofunctor determined by the composite $\chi \circ \chi_{\mathcal{G}}$ (cf. p. 4–39). Choose a pointed continuous function $f_{\mathcal{G}} : X \rightarrow K(G,1)$ corresponding to $\chi_{\mathcal{G}}$ and put $k_{\pi,n;\chi} = s_\chi(k_{G,1})$—then $[X,x_0;K(\pi,n;\chi),k_{\pi,n;\chi}]_{f_{\mathcal{G}}} \approx H^n(X,x_0;\mathcal{G})$. So, if $n = 1$, $H^1(X,x_0;\mathcal{G}) \approx \text{Hom}_{\chi\circ\chi_{\mathcal{G}}}(\pi_1(X,x_0),\pi)$ (see the preceding example) $\Rightarrow$ $H^1(X;\mathcal{G}) \approx \pi\backslash H^1(X,x_0;\mathcal{G}) \approx \pi\backslash\text{Hom}_{\chi\circ\chi_{\mathcal{G}}}(\pi_1(X,x_0),\pi) \approx [X,K(\pi,1;\chi)]_{f_{\mathcal{G}}}$ but if $n > 1$, $H^n(X,x_0;\mathcal{G}) \approx H^n(X;\mathcal{G}) \approx [X,K(\pi,n;\chi)]_{f_{\mathcal{G}}}$.

[Note: The cohomology of any cofunctor $\mathcal{G} : \Pi X \to \mathbf{AB}$ fits into this scheme. Simply take $\pi = \mathcal{G}x_0$, $G = \operatorname{Aut}\pi$, $\chi = \chi_\pi$, and let $\chi_\mathcal{G} : \pi_1(X, x_0) \to \operatorname{Aut}\pi$ be the homomorphism derived from the right action $\pi \times \pi_1(X, x_0) \to \pi$ (of course, $H^0(X;\mathcal{G})$ is $\operatorname{fix}_{\chi_\mathcal{G}}(\pi)$, the subgroup of π whose elements are fixed by $\chi_\mathcal{G}$). When $\chi_\mathcal{G}$ is trivial, one can choose $f_\mathcal{G}$ as the map to the base point of $K(\operatorname{Aut}\pi, 1)$ and recover the fact that $[X, K(\pi, n)] \approx H^n(X;\pi)$.]

LEMMA Fix a set of representatives f_i for $[X, x_0; K(G,1), k_{G,1}]$—then $[X, x_0; K(\pi, n;\chi), k_{\pi,n;\chi}]$ is in a one-to-one correspondence with the union $\bigcup_i [X, x_0; K(\pi, n;\chi), k_{\pi,n;\chi}]_{f_i}$ (which is necessarily disjoint).

Application: There is a one-to-one correspondence between the set of pointed homotopy classes of pointed continuous functions $f : X \to K(\pi, n;\chi)$ such that $\pi_1(f) = \chi_\mathcal{G}$ and the elements of $H^n(X;\mathcal{G})$ $(n > 1)$.

FACT Let $\begin{cases} (X, x_0) \\ (Y, y_0) \end{cases}$ be pointed connected CW complexes; let $f \in C(X, x_0; Y, y_0)$. Assume given a homomorphism $\chi_\mathcal{G} : \pi_1(Y, y_0) \to G$ and a homomorphism $\chi : G \to \operatorname{Aut}\pi$. Put $\chi_{f^*\mathcal{G}} = \chi_\mathcal{G} \circ \pi_1(f)$ and suppose that $f^* : [Y, y_0; K(\pi, n;\chi), k_{\pi,n;\chi}] \to [X, x_0; K(\pi, n;\chi), k_{\pi,n;\chi}]$ is bijective—then $H^n(Y;\mathcal{G}) \approx H^n(X; f^*\mathcal{G})$.

The singular homology and cohomology groups of an Eilenberg-MacLane space of type (π, n) with coefficients in G depend only on (π, n) and G. Notation: $H_q(\pi, n; G), H^q(\pi, n; G)$ (or $H_q(\pi, n), H^q(\pi, n)$ if $G = \mathbf{Z}$). Example: $H_n(\pi, n) \approx \pi/[\pi, \pi]$.

[Note: There are isomorphisms $H_*\pi \approx H_*(\pi, 1)$ $(H^*\pi \approx H^*(\pi, 1))$, where $H_*\pi$ $(H^*\pi)$ is the homology (cohomology) of π. In general, if G is a right π-module and if $\mathcal{G}$ is the locally constant coefficient system on $K(\pi, 1)$ associated with G, then $H_*(\pi; G)$ $(H^*(\pi; G))$ is isomorphic to $H_*(K(\pi, 1);\mathcal{G})$ $(H^*(K(\pi, 1);\mathcal{G}))$.]

EXAMPLE If π is abelian, then $\forall\ n \geq 2, H_{n+1}(\pi, n) = 0$ but this can fail if $n = 1$ since, e.g., $H_2(\mathbf{Z}/2\mathbf{Z} \oplus \mathbf{Z}/2\mathbf{Z}, 1) \approx H_1(\mathbf{Z}/2\mathbf{Z}, 1) \otimes H_1(\mathbf{Z}/2\mathbf{Z}, 1) \approx \mathbf{Z}/2\mathbf{Z}$. When does $H_2(\pi, 1)$ vanish? To formulate the answer, let $0 \to \pi_{\text{tor}} \to \pi \to \Pi \to 0$ be the short exact sequence in which π_{tor} is the torsion subgroup of π and denote by $\pi_{\text{tor}}(p)$ the p-primary component of π_{tor}—then Varadarajan[†] has shown that $H_2(\pi, 1) = 0$ iff $\operatorname{rank}\Pi \leq 1$ plus $\forall\ p : (p_1)\ \pi_{\text{tor}}(p) \otimes \Pi = 0$ & $(p_2)\ \pi_{\text{tor}}(p)$ is the direct sum of a divisible group and a cyclic group. Example: Assume that π is finite—then $H_2(\pi, 1) = 0$ iff π is cyclic. Other examples include $\pi = \mathbf{Z}$, $\pi = \mathbf{Q}$, and $\pi = \mathbf{Z}/p^\infty\mathbf{Z}$ (the p-primary component of $\mathbf{Q}/\mathbf{Z}$).

[†] *Ann. of Math.* **84** (1966), 368–371.

EXAMPLE Let (X, x_0) be a pointed path connected space. Denote by $\text{hur}_n(X)$ the image in $H_n(X)$ of $\pi_n(X)$ under the Hurewicz homomorphism.

$(\pi, 1)$ Set $\pi = \pi_1(X)$ and assume that $\pi_q(X) = 0$ for $1 < q < n$—then $H_q(X) \approx H_q(\pi, 1)$ $(q < n)$ and $H_n(X)/\text{hur}_n(X) \approx H_n(\pi, 1)$.

[Note: In particular, there is an exact sequence $\pi_2(X) \rightarrow H_2(X) \rightarrow H_2(\pi, 1) \rightarrow 0$.]

(π, n) Set $\pi = \pi_n(X)$ $(n > 1)$ and assume that $\pi_q(X) = 0$ for $1 \leq q < n$ & $\pi_q(X) = 0$ for $n < q < N$—then $H_q(X) \approx H_q(\pi, n)$ $(q < N)$ and $H_N(X)/\text{hur}_N(X) \approx H_N(\pi, n)$.

[Note: Take $N = n + 1$ to see that under the stated conditions the Hurewicz homomorphism $\pi_{n+1}(X) \rightarrow H_{n+1}(X)$ is surjective.]

EXAMPLE Let π be a finitely generated (finite) abelian group—then $\forall\, q \geq 1$, $H_q(\pi, n)$ is finitely generated (finite). The $H_q(\pi, 1)$ are handled by computation. Simply note that $H_q(\mathbf{Z}, 1) = \begin{cases} \mathbf{Z} & (q = 1) \\ 0 & (q > 1) \end{cases}$ & $H_q(\mathbf{Z}/k\mathbf{Z}, 1) = \begin{cases} \mathbf{Z}/k\mathbf{Z} & (q \text{ odd}) \\ 0 & (q \text{ even}) \end{cases}$ and use the Künneth formula. To pass inductively from n to $n+1$, apply the generalities on p. 4–44 to the $\mathbf{Z}$-orientable Hurewicz fibration $\Theta K(\pi, n+1) \rightarrow K(\pi, n+1)$. One can, of course, say much more. Indeed, Cartan[†] has explicitly calculated the $H_q(\pi, n; G)$, $H^q(\pi, n; G)$ for any finitely generated abelian G. However, there are occasions when a qualitative description suffices. To illustrate, recall that $H^*(\mathbf{Z}, n; \mathbf{Q})$ is an exterior algebra on one generator of degree n if n is odd and a polynomial algebra on one generator of degree n if n is even. Therefore, if n is odd, then $H_q(\mathbf{Z}, n; \mathbf{Q}) = \mathbf{Q}$ for $q = 0$ & $q = n$ with $H_q(\mathbf{Z}, n; \mathbf{Q}) = 0$ otherwise and if n is even, then $H_q(\mathbf{Z}, n; \mathbf{Q}) = \mathbf{Q}$ for $q = kn$ $(k = 0, 1, \ldots)$ with $H_q(\mathbf{Z}, n; \mathbf{Q}) = 0$ otherwise. So, by the above, if n is odd, then $H_q(\mathbf{Z}, n)$ is finite for $q \neq 0$ & $q \neq n$ and if n is even, then $H_q(\mathbf{Z}, n)$ is finite unless $q = kn$ $(k = 0, 1, \ldots)$, $H_{kn}(\mathbf{Z}, n)$ being the direct sum of a finite group and an infinite cyclic group.

EXAMPLE If π' and π'' are finitely generated abelian groups and if $\mathbf{F}$ is a field, then the algebra $H^*(\pi' \oplus \pi'', n; \mathbf{F})$ is isomorphic to the tensor product over $\mathbf{F}$ of the algebras $H^*(\pi', n; \mathbf{F})$ and $H^*(\pi'', n; \mathbf{F})$. Specialize and take $\mathbf{F} = \mathbf{F}_2$—then for π a finitely generated abelian group, the determination of $H^*(\pi, n; \mathbf{F}_2)$ reduces to the determination of $H^*(\pi, n; \mathbf{F}_2)$ when $\pi = \mathbf{Z}/2^k\mathbf{Z}$, $\pi = \mathbf{Z}/p^l\mathbf{Z}$ ($p =$ odd prime), or $\pi = \mathbf{Z}$. The second possibility is easily dispensed with: $H^q(\mathbf{Z}/p^l\mathbf{Z}, n; \mathbf{F}_2) = 0$ $\forall\, q > 0$, so $H^*(\mathbf{Z}/p^l\mathbf{Z}, n; \mathbf{F}_2) = \mathbf{F}_2$. The outcome in the other cases involves the Steenrod squares Sq^i and their iterates Sq^I. To review the definitions, a sequence $I = (i_1, \ldots, i_r)$ of positive integers is termed admissible provided that $i_1 \geq 2i_2, \ldots, i_{r-1} \geq 2i_r$, its excess $e(I)$ being the difference $(i_1 - 2i_2) + \cdots + (i_{r-1} - 2i_r) + i_r$. Sq^I is the composite $Sq^{i_1} \circ \cdots \circ Sq^{i_r}$ ($Sq^I = \text{id}$ if $e(I) = 0$).

[†] *Collected Works*, vol. III, Springer Verlag (1979), 1300–1394; see also Moore, *Astérisque* **32–33** (1976), 173–212.

($\pi = \mathbf{Z}/2^k\mathbf{Z}$) Let u_n be the unique nonzero element of $H^n(\mathbf{Z}/2^k\mathbf{Z}, n; \mathbf{F}_2)$.

(k=1) $H^*(\mathbf{Z}/2\mathbf{Z}, 1; \mathbf{F}_2) = \mathbf{F}_2[u_1]$, the polynomial algebra with generator u_1. For $n > 1$, $H^*(\mathbf{Z}/2\mathbf{Z}, n; \mathbf{F}_2) = \mathbf{F}_2[(Sq^I u_n)]$, the polynomial algebra with generators the $Sq^I u_n$, where I runs through all admissible sequences of excess $e(I) < n$.

(k>1) $H^*(\mathbf{Z}/2^k\mathbf{Z}, 1; \mathbf{F}_2) = \bigwedge(u_1) \otimes \mathbf{F}_2[v_2]$, the tensor product of the exterior algebra with generator u_1 and the polynomial algebra with generator v_2. Here, v_2 is the image of the fundamental class under the Bockstein operator $H^1(\mathbf{Z}/2^k\mathbf{Z}, 1; \mathbf{F}_2) \rightarrow H^2(\mathbf{Z}/2^k\mathbf{Z}, 1; \mathbf{F}_2)$ corresponding to the exact sequence $0 \rightarrow \mathbf{Z}/2\mathbf{Z} \rightarrow \mathbf{Z}/2^{k+1}\mathbf{Z} \rightarrow \mathbf{Z}/2^k\mathbf{Z} \rightarrow 0$. Using this, extend the definition and let v_n be the image of the fundamental class under the Bockstein operator $H^n(\mathbf{Z}/2^k\mathbf{Z}, n; \mathbf{Z}/2^k\mathbf{Z}) \rightarrow H^{n+1}(\mathbf{Z}/2^k\mathbf{Z}, n; \mathbf{F}_2)$. Write $\overline{Sq}^I u_n = Sq^I u_n$ if $i_r > 1$ and $\overline{Sq}^I u_n = Sq^{i_1} \circ \cdots \circ Sq^{i_{r-1}} v_n$ if $i_r = 1$—then for $n > 1$, $H^*(\mathbf{Z}/2^k\mathbf{Z}, n; \mathbf{F}_2) = \mathbf{F}_2[(\overline{Sq}^I u_n)]$, the polynomial algebra with generators the $\overline{Sq}^I u_n$, where I runs through all admissible sequences of excess $e(I) < n$.

($\pi = \mathbf{Z}$) Let u_n be the unique nonzero element of $H^n(\mathbf{Z}, n; \mathbf{F}_2)$—then $H^*(\mathbf{Z}, 1; \mathbf{F}_2) = \bigwedge(u_1)$, the exterior algebra with generator u_1, and for $n > 1$, $H^*(\mathbf{Z}, n; \mathbf{F}_2) = \mathbf{F}_2[(Sq^I u_n)]$, the polynomial algebra with generators the $Sq^I u_n$, where I runs through all admissible sequences of excess $e(I) < n$ and $i_r > 1$.

Let π be a finitely generated abelian group—then, as vector spaces over $\mathbf{F}_2$, the $H^q(\pi, n; \mathbf{F}_2)$ are finite dimensional, so it makes sense to consider the associated Poincaré series: $P(\pi, n; t) = \sum\limits_{q=0}^{\infty} \dim(H^q(\pi, n; \mathbf{F}_2)) \cdot t^q$. Obviously, $P(\pi' \oplus \pi'', n; t) = P(\pi', n; t) \cdot P(\pi'', n; t)$. Examples: (1) $P(\mathbf{Z}/2\mathbf{Z}, 1; t) = \sum\limits_0^{\infty} t^q$; (2) $P(\mathbf{Z}, 1; t) = 1 + t$.

(PS$_1$) $P(\pi, n; t)$ converges in the interval $0 \leq t < 1$.

[It suffices to treat the cases $\pi = \mathbf{Z}/2^k\mathbf{Z}$, $\pi = \mathbf{Z}/p^l\mathbf{Z}$ (p = odd prime), $\pi = \mathbf{Z}$. The second case is trivial: $P(\mathbf{Z}/p^l\mathbf{Z}, n; t) = 1$.

($\pi = \mathbf{Z}/2^k\mathbf{Z}$) In view of what has been said above, $H^*(\mathbf{Z}/2^k\mathbf{Z}, n; \mathbf{F}_2)$ and $H^*(\mathbf{Z}/2\mathbf{Z}, n; \mathbf{F}_2)$ are isomorphic as vector spaces over $\mathbf{F}_2$, thus one need only examine the situation when $k = 1$ and $n > 1$. Given an admissible I, let $|I| = i_1 + \cdots + i_r$ ($\Rightarrow e(I) = 2i_1 - |I|$)—then $P(\mathbf{Z}/2\mathbf{Z}, n; t) = \prod\limits_{e(I)<n} \frac{1}{1 - t^{n+|I|}}$. Since the number of admissible I with $e(I) < n$ such that $n + |I| = N$ is equal to the number of decompositions of N of the form $N = 1 + 2^{h_1} + \cdots + 2^{h_{n-1}}$, where $0 \leq h_1 \leq \cdots \leq h_{n-1}$, it follows that

$$P(\mathbf{Z}/2\mathbf{Z}, n; t) = \prod_{0 \leq h_1 \leq \cdots \leq h_{n-1}} \frac{1}{1 - t^{1+2^{h_1}+\cdots+2^{h_{n-1}}}}.$$

The associated series $\sum\limits_{0 \leq h_1 \leq \cdots \leq h_{n-1}} t^{1+2^{h_1}+\cdots+2^{h_{n-1}}}$ is convergent if $0 \leq t < 1$.

($\pi = \mathbf{Z}$) Assuming that $n > 1$, the extra condition $i_r > 1$ is incorporated by the requirement $h_{n-1} = h_{n-2}$. Consequently, $P(\mathbf{Z}, n; t) = P(\mathbf{Z}/2\mathbf{Z}, n-1; t)/P(\mathbf{Z}, n-1; t)$ or still,

$$P(\mathbf{Z}, n; t) = \frac{P(\mathbf{Z}/2\mathbf{Z}, n-1; t) \cdot P(\mathbf{Z}/2\mathbf{Z}, n-3; t) \cdots}{P(\mathbf{Z}/2\mathbf{Z}, n-2; t) \cdot P(\mathbf{Z}/2\mathbf{Z}, n-4; t) \cdots}$$

via iteration of the data.]

Put $\Phi(\pi, n; x) = \log_2 P(\pi, n; 1 - 2^{-x})$ $(0 \leq x < \infty)$.

(PS$_2$) Suppose that π is the direct sum of μ cyclic groups of order a power of 2, a finite group of odd order, and ν cyclic groups of infinite order—then: (i) $\mu \geq 1 \Rightarrow \Phi(\pi, n; x) \sim \frac{\mu x^n}{n!}$; (ii) $\mu = 0$ & $\nu \geq 1 \Rightarrow \Phi(\pi, n; x) \sim \frac{\nu x^{n-1}}{(n-1)!}$; (iii) $\mu = 0$ & $\nu = 0 \Rightarrow \Phi(\pi, n; x) = 0$.

[The essential point is the asymptotic relation $\Phi(\mathbf{Z}/2\mathbf{Z}, n; x) \sim \frac{x^n}{n!}$, everything else being a corollary. Observe first that $P(\mathbf{Z}/2\mathbf{Z}, 1; t) = \frac{1}{1-t} \Rightarrow \Phi(\mathbf{Z}/2\mathbf{Z}, 1; x) = x$. Proceeding by induction on n, introduce the abbreviations $P_n(t) = P(\mathbf{Z}/2\mathbf{Z}, n; t)$, $\Phi_n(x) = \Phi(\mathbf{Z}/2\mathbf{Z}, n; x)$, and the auxiliary functions $Q_n(t) = \prod\limits_{0 \leq h_1 \leq \cdots \leq h_{n-1}} \frac{1}{1 - t^{2^{h_1} + \cdots + 2^{h_{n-1}}}}$, $\Psi_n(x) = \log_2 Q_n(1 - 2^{-x})$—then $Q_n(t)/P_{n-1}(t) \leq P_n(t) \leq Q_n(t)$ $(0 \leq t < 1) \Rightarrow \Psi_n(x) - \Phi_{n-1}(x) \leq \Phi_n(x) \leq \Psi_n(x)$ $(0 \leq x < \infty)$. Because $\Phi_{n-1}(x) \sim \frac{x^{n-1}}{(n-1)!}$ (induction hypothesis), one need only show that $\Psi_n(x) \sim \frac{x^n}{n!}$. But from the definitions, $Q_n(t)/P_{n-1}(t) = Q_n(t^2)$, hence $\Psi_n(x) = \Phi_{n-1}(x) + \Psi_n(x - 1 - \log_2(1 - 2^{-x-1}))$. So, $\forall\, \epsilon > 0$, $\exists\, x_\epsilon > 0 : \forall\, x > x_\epsilon$,

$$\Psi_n(x-1) + \frac{(1-\epsilon)}{(n-1)!}x^{n-1} \leq \Psi_n(x) \leq \Psi_n(x - 1 + \epsilon) + \frac{(1+\epsilon)}{(n-1)!}x^{n-1}.$$

Claim: Given A and $n \geq 1$, there exists a polynomial $F_n(x)$ of degree n with leading term $\frac{Ax^n}{n!}$ such that $F_n(x) = F_n(x-1) + \frac{Ax^{n-1}}{(n-1)!}$.

[Use induction on n: Put $F_1(x) = Ax$ and consider $F_n(x) = \frac{Ax^n}{n!} + \sum\limits_{k=2}^{n} \frac{(-1)^k}{k!} F_{n-k+1}(x)$.]

Claim: Let $f \in C([0, \infty[)$. Assume: $f(x) \leq f(x-1) + \frac{Ax^{n-1}}{(n-1)!}$ $\left(f(x) \geq f(x-1) + \frac{Ax^{n-1}}{(n-1)!}\right)$—then there exists a constant C' (C'') such that $f(x) \leq F_n(x) + C'$ $(f(x) \geq F_n(x) + C'')$.

[Let $C' = \max\{f(x) - F_n(x) : 0 \leq x \leq 1\} : f(x) \leq F_n(x) + C'$ $(0 \leq x \leq 1)$ and by induction on $N : N \leq x \leq N+1 \Rightarrow f(x) \leq f(x-1) + \frac{Ax^{n-1}}{(n-1)!} \leq F_n(x-1) + C' + \frac{Ax^{n-1}}{(n-1)!} = F_n(x) + C'$.]

These generalities allow one to say that $\forall\, \epsilon > 0$, there exist polynomials R'_ϵ and R''_ϵ of degree $< n : \forall\, x >> 0$,

$$(1-\epsilon)\frac{x^n}{n!} + R'_\epsilon(x) \leq \Psi_n(x) \leq \left(\frac{1+\epsilon}{1-\epsilon}\right)\frac{x^n}{n!} + R''_\epsilon(x).$$

Since ϵ is arbitrary, this means that $\Psi_n(x) \sim \frac{x^n}{n!}$.]

LEMMA Suppose that A is path connected—then $\forall\, n \geq 1$ there exists a path connected space $X \supset A$ which is obtained from A by attaching $(n+1)$-cells such that $\pi_n(X) = 0$ and, under the inclusion $A \to X$, $\pi_q(A) \approx \pi_q(X)$ $(q < n)$.

[Let $\{\alpha\}$ be a set of generators for $\pi_n(A)$. Represent α by $f_\alpha : \mathbf{S}^n \to A$ and put $X = (\coprod\limits_\alpha \mathbf{D}^{n+1}) \sqcup_f A$ $(f = \coprod\limits_\alpha f_\alpha)$.]

Let X be a pointed path connected space. Fix $n \geq 0$—then an nth Postnikov approximate to X is a pointed path connected space $X[n] \supset X$, where $(X[n], X)$ is a relative CW complex whose cells in $X[n] - X$ have dimension $> n+1$, such that $\pi_q(X[n]) = 0$ $(q > n)$ and, under the inclusion $X \rightarrow X[n]$, $\pi_q(X) \approx \pi_q(X[n])$ $(q \leq n)$.

[Note: $X[0]$ is homotopically trivial and $X[1]$ has homotopy type $(\pi_1(X), 1)$.]

PROPOSITION 9 Every pointed path connected space X admits an nth Postnikov approximate $X[n]$.

[Using the lemma, construct a sequence $X = X_0 \subset X_1 \subset \cdots$ of pointed path connected spaces X_k such that $\forall\ k > 0$, X_k is obtained from X_{k-1} by attaching $(n+k+1)$-cells, $\pi_{n+k}(X_k) = 0$, and, under the inclusion $X_{k-1} \rightarrow X_k$, $\pi_q(X_{k-1}) \approx \pi_q(X_k)$ $(q < n+k)$. Consider $X[n] = \operatorname{colim} X_k$.]

[Note: If X is a pointed connected CW space, then the $X[n]$ are pointed connected CW spaces.]

EXAMPLE Let π be a group and let n be an integer ≥ 1, where π is abelian if $n > 1$—then a pointed connected CW space X is said to be a Moore space of type (π, n) provided that $\pi_n(X)$ is isomorphic to π and $\begin{cases} \pi_q(X) = 0 & (q < n) \\ H_q(X) = 0 & (q > n) \end{cases}$. Notation: $X = M(\pi, n)$. If $n = 1$, then $M(\pi, n)$ exists iff $H_2(\pi, 1) = 0$ but if $n > 1$, then $M(\pi, n)$ always exists. If $n = 1$ and $H_2(\pi, 1) = 0$, then the pointed homotopy type of $M(\pi, 1)$ is not necessarily unique (e.g., when $\pi = \mathbf{Z}$) but if $n > 1$, then the pointed homotopy type of $M(\pi, n)$ is unique. In any event, $M(\pi, n)[n] = K(\pi, n)$.

FACT Suppose that X is a pointed path connected space. Fix $n \geq 1$—then there exists a pointed n-connected space $\widetilde{X}_n$ in $\mathbf{TOP}/X$ such that the projection $\widetilde{X}_n \rightarrow X$ is a pointed Hurewicz fibration and induces an isomorphism $\pi_q(\widetilde{X}_n) \rightarrow \pi_q(X)$ $\forall\ q > n$.

[Consider the mapping fiber of the inclusion $X \rightarrow X[n]$.]

EXAMPLE Take $X = \mathbf{S}^3$—then the fibers of the projection $\widetilde{X}_3 \rightarrow X$ have homotopy type $(\mathbf{Z}, 2)$ and $\forall\ q \geq 1$, $H_q(\widetilde{X}_3) = \begin{cases} 0 & (q \text{ odd}) \\ \mathbf{Z}/(q/2)\mathbf{Z} & (q \text{ even}) \end{cases}$.

[Use the Wang cohomology sequence and the fact that $H^*(\mathbf{Z}, 2)$ is the polynomial algebra over $\mathbf{Z}$ generated by an element of degree 2.]

[Note: Given a prime p, let $\mathcal{C}$ be the class of finite abelian groups with order prime to p—then from the above, $H_n(\widetilde{X}_3) \in \mathcal{C}$ $(0 < n < 2p)$, so by the mod $\mathcal{C}$ Hurewicz theorem, $\pi_n(\widetilde{X}_3) \in \mathcal{C}$ $(0 < n < 2p)$ and the Hurewicz homomorphism $\pi_{2p}(\widetilde{X}_3) \rightarrow H_{2p}(\widetilde{X}_3)$ is $\mathcal{C}$-bijective. Therefore the p-primary component of $\pi_n(\mathbf{S}^3)$ is 0 if $n < 2p$ and is $\mathbf{Z}/p\mathbf{Z}$ if $n = 2p$.]

Put $W_1 = \widetilde{X}_1$. Let W_2 be the mapping fiber of the inclusion $\widetilde{X}_1 \to \widetilde{X}_1[2]$—then the mapping fiber of the projection $W_2 \to W_1$ has homotopy type $(\pi_2(X), 1)$. Iterate: The result is a sequence of pointed Hurewicz fibrations $W_n \to W_{n-1}$, where the mapping fiber has homotopy type $(\pi_n(X), n-1)$ and W_n is n-connected with $\pi_q(W_n) \approx \pi_q(X)$ $(\forall\ q > n)$. The diagram $X = W_0 \leftarrow W_1 \leftarrow \cdots$ is called "the" Whitehead tower of X.

[Note: If X is a pointed connected CW space, then the W_n are pointed connected CW spaces and the mapping fiber of the projection $W_n \to W_{n-1}$ is a $K(\pi_n(X), n-1)$.]

EXAMPLE Let X be a pointed simply connected CW complex which is finite and noncontractible. Assume: $\exists\ i > 0$ such that $H_i(X; \mathbf{F}_2) \neq 0$—then $\pi_q(X)$ contains a subgroup isomorphic to $\mathbf{Z}$ or $\mathbf{Z}/2\mathbf{Z}$ for infinitely many q.

[Because the $H_q(X)$ are finitely generated $\forall\ q$, the same is true of the $\pi_q(X)$ (cf. p. 5–44). The set of positive integers n such that $\pi_n(X) \otimes \mathbf{Z}/2\mathbf{Z} \neq 0$ is nonempty. To get a contradiction, suppose that there is a largest such N. Working with the Whitehead tower of X, let $P_n(t) = \sum_{q=0}^{\infty} \dim(H^q(W_n; \mathbf{F}_2)) \cdot t^q$, the mod 2 Poincaré series of $H^*(W_n; \mathbf{F}_2)$ (meaningful, the $H^q(W_n; \mathbf{F}_2)$ being finite dimensional over $\mathbf{F}_2$). In particular: $P_N(t) = 1$, $P_{N-1}(t) = P(\pi_N(X), N; t)$, $P_1(t) = P_X(t)$, the Poincaré series of $H^*(X; \mathbf{F}_2)$. On general grounds, there is a majorization $P_n(t) \prec P_{n-1}(t) \cdot P(\pi_n(X), n-1; t)$, where the symbol $\prec$ means that each coefficient of the formal power series on the left is $\leq$ the corresponding coefficient of the formal power series on the right. So, starting with $n = N - 1$ and multiplying out, one finds that $P(\pi_N(X), N; t) \prec P_X(t) \cdot \prod_{1<i<N} P(\pi_i(X), i-1; t)$. Since $P_X(t)$ is a polynomial, hence is bounded on $[0,1], \exists\ C > 0 : P(\pi_N(X), N; t) \leq C \cdot \prod_{1<i<N} P(\pi_i(X), i-1; t)$ or still, in the notation of p. 5–37, $\Phi(\pi_N(X), N; x) \leq \log_2 C + \sum_{1<i<N} \Phi(\pi_i(X), i-1; x)$ $(0 \leq x < \infty)$. Comparing the asymptotics of either side leads to an immediate contradiction (cf. p. 5–37).]

[Note: This analysis is due to Serre[†]. It has been extended to all odd primes by Umeda[‡]. Accordingly, if X is a pointed simply connected CW complex which is finite and noncontractible, then $\pi_q(X)$ is nonzero for infinitely many q. Proof: If $\forall\ p \in \mathbf{\Pi}$ & $\forall\ i > 0$, $H_i(X; \mathbf{F}_p) = 0$, then the arrow $X \to *$ is a homology equivalence (cf. p. 8–8), thus by the Whitehead theorem, X is contractible.]

LEMMA Let (X, A, x_0) be a pointed pair. Assume: (X, A) is a relative CW complex whose cells in $X - A$ have dimension $> n + 1$. Suppose that (Y, y_0) is a pointed space such that $\pi_q(Y, y_0) = 0\ \forall\ q > n$—then every pointed continuous function $f : A \to Y$ has a pointed continuous extension $F : X \to Y$.

[†] *Comment. Math. Helv.* **27** (1953), 198–232.

[‡] *Proc. Japan Acad.* **35** (1959), 563–566; see also McGibbon-Neisendorfer, *Comment. Math. Helv.* **59** (1984), 253–257.

It follows from the lemma that if X and Y are pointed path connected spaces and if $f : X \to Y$ is a pointed continuous function, then for $m \leq n$ there exists a pointed continuous function $f_{n,m} : X[n] \to Y[m]$ rendering the diagram

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d] & Y \arrow[d] \\ X[n] \arrow[r, "f_{n,m}"'] & Y[m] \end{tikzcd}$$

commutative, any two such being homotopic $\text{rel}\,X$. Proof: Let $F : X \to Y[m]$ be the composite $X \xrightarrow{f} Y \to Y[m]$. To establish the existence of $f_{n,m}$, consider any filler for

$$\begin{tikzcd} X[n] \arrow[dr, dashed] & \\ X \arrow[u] \arrow[r, "F"'] & Y[m] \end{tikzcd}$$

and to establish the uniqueness of $f_{n,m}$ $\text{rel}\,X$, take two extensions $f'_{n,m}$ & $f''_{n,m}$, define $\Phi : i_0X[n] \cup IX \cup i_1X[n] \to Y[m]$ by $\begin{cases} \Phi(x,0) = f'_{n,m}(x) \\ \Phi(x,1) = f''_{n,m}(x) \end{cases}$, $\Phi(x,t) = F(x)$, and consider any filler for

$$\begin{tikzcd} IX[n] \arrow[dr, dashed] & \\ i_0X[n] \cup IX \cup i_1X[n] \arrow[u] \arrow[r, "\Phi"'] & Y[m] \end{tikzcd}.$$

Application: Let $X'[n]$ and $X''[n]$ be n^{th} Postnikov approximates to X—then in $\mathbf{HTOP}^2$, $(X'[n], X) \approx (X''[n], X)$.

EXAMPLE Let X and Y be pointed connected CW spaces—then it can happen that $X[n]$ and $Y[n]$ have the same pointed homotopy type for all n, yet X and Y are not homotopy equivalent. To construct an example, let K be a pointed simply connected CW complex. Assume: K is finite and noncontractible. Put $X = (w)\prod\limits_0^\infty K[n]$, $Y = X \times K$—then $\forall\, n$, $X[n] \approx Y[n]$ in $\mathbf{HTOP}_*$. However, it is not true that $X \approx Y$ in $\mathbf{HTOP}$. For if so, K would be dominated in homotopy by X or still, by $K[0] \times \cdots \times K[n]$ $(\exists\, n)$, thus $\forall\, q$, $\pi_q(K)$ would be a direct summand of $\pi_q(K[0] \times \cdots \times K[n])$. But this is impossible: The $\pi_q(K)$ are nonzero for infinitely many q (cf. p. 5–39).

[Note: This subject has its theoretical aspects as well (McGibbon-Møller[†]).]

Let X be a pointed path connected space. Given a sequence $X[0], X[1], \ldots$ of Postnikov approximates to X, $\forall\, n \geq 1$ there is a pointed continuous function $f_n : X[n] \to X[n-1]$ such that the triangle

$$\begin{tikzcd} & X \arrow[dl] \arrow[dr] & \\ X[n] \arrow[rr, "f_n"'] & & X[n-1] \end{tikzcd}$$

commutes. Put $P_0X = X[0]$, let

[†] *Topology* **31** (1992), 177–201; see also Dror-Dwyer-Kan, *Proc. Amer. Math. Soc.* **74** (1979), 183–186.

s_0 be the identity map, and denote by P_1X the mapping track of f_1:

$$\begin{tikzcd} X[1] \arrow[r, "f_1"] \arrow[d, "s_1"'] & X[0] \arrow[d, "s_0"] \\ P_1X \arrow[r, "p_1"'] & P_0X \end{tikzcd}.$$

Recall that s_1 is a pointed homotopy equivalence, while p_1 is the usual pointed Hurewicz fibration associated with this setup. Repeat the procedure, taking for P_2X the mapping track of $s_1 \circ f_2$:

$$\begin{tikzcd} X[2] \arrow[r, "f_2"] \arrow[d, "s_2"'] & X[1] \arrow[d, "s_1"] \\ P_2X \arrow[r, "p_2"'] & P_1X \end{tikzcd}.$$

The upshot is that the f_n can be converted to pointed Hurewicz fibrations p_n, where at each stage there is a commutative triangle

$$\begin{tikzcd} & X \arrow[dl] \arrow[dr] & \\ P_nX \arrow[rr, "p_n"'] & & P_{n-1}X \end{tikzcd}.$$

The diagram $P_0X \leftarrow P_1X \leftarrow \cdots$ of pointed Hurewicz fibrations is called "the" Postnikov tower of X. Obviously, $\pi_q(P_nX) = 0$ $(q > n)$, $\pi_q(X) \approx \pi_q(P_nX)$ $(q \leq n)$, and $\pi_q(P_nX) \approx \pi_q(P_{n-1}X)$ $(q \neq n)$. Therefore the mapping fiber of p_n has homotopy type $(\pi_n(X), n)$.

[Note: If X is a pointed connected CW space, then the P_nX are pointed connected CW spaces, so the mapping fiber of p_n is a $K(\pi_n(X), n)$.]

EXAMPLE Let X be a pointed path connected space. Fix $n > 1$—then $\pi_n(X)$ defines a locally constant coefficient system on $P_{n-1}X$ and there is an exact sequence

$$H_{n+2}(P_nX) \rightarrow H_{n+2}(P_{n-1}X) \rightarrow H_1(P_{n-1}X; \pi_n(X)) \rightarrow H_{n+1}(P_nX) \rightarrow H_{n+1}(P_{n-1}X)$$

$$\rightarrow H_0(P_{n-1}X; \pi_n(X)) \rightarrow H_n(P_nX) \rightarrow H_n(P_{n-1}X) \rightarrow 0.]$$

[Work with the fibration spectral sequence of $p_n : P_nX \rightarrow P_{n-1}X$, noting that $E^r_{p,q} = 0$ if $0 < q < n$ or $q = n + 1$.]

A nonempty path connected topological space X is said to be abelian if $\pi_1(X)$ is abelian and if $\forall\ n > 1$, $\pi_1(X)$ operates trivially on $\pi_n(X)$. Every simply connected space is abelian as is every path connected H space or every path connected compactly generated semigroup with unit (obvious definition).

[Note: If X is abelian, then $\forall\ x_0 \in X$, the forgetful function $[\mathbf{S}^n, s_n; X, x_0] \rightarrow [\mathbf{S}^n, X]$ is bijective (cf. p. 3–18).]

EXAMPLE $\mathbf{P}^n(\mathbf{R})$ is abelian iff n is odd.

Let X be a pointed connected CW space. Assume: X is abelian. There is a commutative triangle

$$\begin{tikzcd} & X \arrow[dl] \arrow[dr] & \\ X[n+1] \arrow[rr, "f_{n+1}"'] & & X[n] \end{tikzcd}$$

and an embedding $IX \to M_{f_{n+1}}$. Define $\widehat{X}[n]$ by the pushout square

$$\begin{tikzcd} IX \arrow[r, "p"] \arrow[d] & X \arrow[d] \\ M_{f_{n+1}} \arrow[r] & \widehat{X}[n] \end{tikzcd}$$

—then $\widehat{X}[n]$ contains $X[n]$ as a strong deformation retract, hence $\pi_q(\widehat{X}[n]) \approx \pi_q(X[n])$ $(q \geq 1)$. Using the exact sequence

$$\cdots \to \pi_{q+1}(X[n+1]) \to \pi_{q+1}(\widehat{X}[n]) \to \pi_{q+1}(\widehat{X}[n], X[n+1]) \to \pi_q(X[n+1]) \to \pi_q(\widehat{X}[n]) \to \cdots,$$

one finds that $\pi_q(\widehat{X}[n], X[n+1]) = 0$ $(q \neq n+2)$ and $\pi_{n+2}(\widehat{X}[n], X[n+1]) \approx \pi_{n+1}(X[n+1])$ $\approx \pi_{n+1}(X)$. Thus the relative Hurewicz homomorphism hur : $\pi_{n+2}(\widehat{X}[n], X[n+1])$ $\to H_{n+2}(\widehat{X}[n], X[n+1])$ is bijective, so the composite $\kappa_{n+2} : H_{n+2}(\widehat{X}[n], X[n+1])$ $\xrightarrow{\mathrm{hur}^{-1}} \pi_{n+2}(\widehat{X}[n], X[n+1]) \to \pi_{n+1}(X)$ is an isomorphism. Since $H_{n+1}(\widehat{X}[n], X[n+1]) = 0$, the universal coefficient theorem implies that $H^{n+2}(\widehat{X}[n], X[n+1]; \pi_{n+1}(X))$ can be identified with $\mathrm{Hom}(H_{n+2}(\widehat{X}[n], X[n+1]); \pi_{n+1}(X))$, therefore κ_{n+2} corresponds to a cohomology class in $H^{n+2}(\widehat{X}[n], X[n+1]; \pi_{n+1}(X))$ whose image $\mathrm{k}^{n+2} (= \mathrm{k}^{n+2}(X))$ in $H^{n+2}(X[n]; \pi_{n+1}(X))$ is the Postnikov invariant of X in dimension $n+2$. Put $K_{n+2} = K(\pi_{n+1}(X), n+2)$, let $k_{n+2} : X[n] \to K_{n+2}$ be the arrow associated with k^{n+2}, and define $W[n+1]$ by the pullback square

$$\begin{tikzcd} W[n+1] \arrow[r] \arrow[d] & \Theta K_{n+2} \arrow[d] \\ X[n] \arrow[r, "k_{n+2}"'] & K_{n+2} \end{tikzcd}$$

—then $W[n+1]$ is a CW space (cf. §6, Proposition 9) and there is a lifting

$$\begin{tikzcd} & W[n+1] \arrow[d] \\ X[n+1] \arrow[ur, "\Lambda_{n+1}"] \arrow[r, "f_{n+1}"'] & X[n] \end{tikzcd}$$

of f_{n+1} which is a weak homotopy equivalence or still, a homotopy equivalence (realization theorem). The restriction of Λ_{n+1} to X is an embedding and $\Lambda_{n+1} : (X[n+1], X) \to (W[n+1], X)$ is a homotopy equivalence of pairs.

[Note: Λ_{n+1} is constructed by considering a specific factorization of k_{n+2} as a composite $X[n] \to \widehat{X}[n]/X[n+1] \to K_{n+2}$ (k_{n+2} is determined only up to homotopy).]

INVARIANCE THEOREM Let $\begin{cases} X \\ Y \end{cases}$ be pointed CW spaces. Assume: $\begin{cases} X \\ Y \end{cases}$ are abelian. Suppose that $\phi : X \to Y$ is a pointed continuous function. Fix pointed ϕ_n :

$X[n] \rightarrow Y[n]$ such that the diagram
$$\begin{tikzcd} X \arrow[r, "\phi"] \arrow[d] & Y \arrow[d] \\ X[n] \arrow[r, "\phi_n"'] & Y[n] \end{tikzcd}$$
commutes—then $\forall\, n$, $\phi_n^* \mathbf{k}^{n+2}(Y) = \phi_{\text{co}} \mathbf{k}^{n+2}(X)$ in $H^{n+2}(X[n]; \pi_{n+1}(Y))$.

[Note: Here, ϕ_{co} is the coefficient group homomorphism $H^{n+2}(X[n]; \pi_{n+1}(X)) \rightarrow H^{n+2}(X[n]; \pi_{n+1}(Y))$.]

NULLITY THEOREM Let X be a pointed CW space. Assume: X is abelian—then $\mathbf{k}^{n+1} = 0$ iff the Hurewicz homomorphism $\pi_n(X) \rightarrow H_n(X)$ is split injective.

EXAMPLE Suppose that $\mathbf{k}^{n+1} = 0$—then $W[n]$ is fiber homotopy equivalent to $X[n-1] \times K(\pi_n(X), n)$ (cf. p. 4–24), hence $X[n] \approx X[n-1] \times K(\pi_n(X), n)$. Therefore X has the same pointed homotopy type as the weak product $(w) \prod_1^\infty K(\pi_n(X), n)$ provided that the Hurewicz homomorphism $\pi_n(X) \rightarrow H_n(X)$ is split injective for all n. This condition can be realized. In fact, Puppe[†] has shown that if G is a path connected abelian compactly generated semigroup with unit, then $\forall\, n$, the Hurewicz homomorphism $\pi_n(G) \rightarrow H_n(G)$ is split injective, thus $G \approx (w) \prod_1^\infty K(\pi_n(G), n)$ when G is in addition a CW space.

[Note: Analogous remarks apply if G is a path connected abelian topological semigroup with unit. Reason: The identity map $kG \rightarrow G$ is a weak homotopy equivalence.]

ABELIAN OBSTRUCTION THEOREM Let (X, A) be a relative CW complex; let Y be a pointed abelian CW space. Suppose that $\forall\, n > 0$, $H^{n+1}(X, A; \pi_n(Y)) = 0$—then every $f \in C(A, Y)$ admits an extension $F \in C(X, Y)$, any two such being homotopic rel A provided that $\forall\, n > 0$, $H^n(X, A; \pi_n(Y)) = 0$.

EXAMPLE Let (X, x_0) be a pointed CW complex; let (Y, y_0) be a pointed simply connected CW complex. Assume: $\forall\, n > 0$, $H^n(X; \pi_n(Y)) = 0$—then $[X, x_0; Y, y_0] = *$.

[In fact, $H^n(X, x_0; \pi_n(Y, y_0)) \approx H^n(X; \pi_n(Y)) = 0 \Rightarrow [X, x_0; Y, y_0] = *(\Rightarrow [X, Y] = *$ (cf. p. 3–18)).]

PROPOSITION 10 Let X be a pointed abelian CW space. Assume: The $H_q(X)$ are finitely generated $\forall\, q$—then $\forall\, n$, the $H_q(X[n])$ are finitely generated $\forall\, q$.

[The assertion is trivial if $n = 0$. Next, $X[1]$ is a $K(\pi_1(X), 1)$, hence $\pi_1(X) \approx H_1(X)$, which is finitely generated. For $q > 1$, $H_q(X[1]) \approx H_q(\pi_1(X), 1)$ and these too are finitely generated (cf. p. 5–35). Proceeding by induction, suppose that the $H_q(X[n])$ are finitely generated $\forall\ q$—then the $H_q(X[n], X)$ are finitely generated $\forall\ q$. In particular, $H_{n+2}(X[n], X)$ is finitely generated. Since $\pi_{n+1}(X[n]) = \pi_{n+2}(X[n]) = 0$, the arrow $\pi_{n+2}(X[n], X) \to \pi_{n+1}(X)$ is an isomorphism. But X is abelian, so from the relative Hurewicz theorem, $\pi_{n+2}(X[n], X) \approx H_{n+2}(X[n], X)$. Therefore $\pi_{n+1}(X)$ is finitely generated. Consider now the mapping track W_{n+2} of $k_{n+2} : X[n] \to K_{n+2}$. The fiber of the $\mathbf{Z}$-orientable Hurewicz fibration $W_{n+2} \to K_{n+2}$ over the base point is homeomorphic to $W[n+1]$ (parameter reversal). The $H_q(K_{n+2}) = H_q(\pi_{n+1}(X), n+2)$ are finitely generated $\forall\ q$ (cf. p. 5–35), as are the $H_q(W_{n+2})$ (induction hypothesis), thus the $H_q(W[n+1])$ are finitely generated $\forall\ q$ (cf. p. 4–44). Because $X[n+1]$ and $W[n+1]$ have the same homotopy type, this completes the passage from n to $n+1$.]

[†] *Math. Zeit.* **68** (1958), 367–421.

Application: Let X be a pointed abelian CW space. Assume: The $H_q(X)$ are finitely generated $\forall\ q$—then the $\pi_q(X)$ are finitely generated $\forall\ q$.

[Note: This result need not be true for a nonabelian X. Example: Take $X = \mathbf{S}^1 \vee \mathbf{S}^2$—then the $H_q(X)$ are finitely generated $\forall\ q$ and $\pi_1(X) \approx \mathbf{Z}$. On the other hand, $\pi_2(X) \approx H_2(\widetilde{X})$, $\widetilde{X}$ the universal covering space of X, i.e., the real line with a copy of $\mathbf{S}^2$ attached at each integral point. Therefore $\pi_2(X)$ is free abelian on countably many generators.]

PROPOSITION 11 Let X be a pointed abelian CW space. Assume: The $H_q(X)$ are finite $\forall\ q > 0$—then $\forall\ n$, the $H_q(X[n])$ are finite $\forall\ q > 0$.

Application: Let X be a pointed abelian CW space. Assume: The $H_q(X)$ are finite $\forall\ q > 0$—then the $\pi_q(X)$ are finite $\forall\ q > 0$.

EXAMPLE (Homotopy Groups of Spheres) The $\pi_q(\mathbf{S}^{2n+1})$ of the odd dimensional sphere are finite for $q > 2n+1$ and the $\pi_q(\mathbf{S}^{2n})$ of the even dimensional sphere are finite for $q > 2n$ except that $\pi_{4n-1}(\mathbf{S}^{2n})$ is the direct sum of $\mathbf{Z}$ and a finite group. Here are the details.

($2n+1$) Fix a map $f : \mathbf{S}^{2n+1} \to K(\mathbf{Z}, 2n+1)$ classifying a generator of $H^{2n+1}(\mathbf{S}^{2n+1})$—then f_* induces an isomorphism $H_*(S^{2n+1}; \mathbf{Q}) \to H_*(K(\mathbf{Z}, 2n+1); \mathbf{Q})$ (cf. p. 5–35), so $\forall\ q > 0$, $H_q(E_f; \mathbf{Q}) = 0$ (cf. p. 4–44). Accordingly, $\forall\ q > 0$, $H_q(E_f)$ is finite (being finitely generated). Therefore all the homotopy groups of E_f are finite. But $\pi_q(E_f) \approx \pi_q(\mathbf{S}^{2n+1})$ if $q > 2n+1$.

($2n$) The even dimensional case requires a double application of the odd dimensional case. First, consider the Stiefel manifold $\mathbf{V}_{2n+1,2}$ and the map $f : \mathbf{V}_{2n+1,2} \to \mathbf{S}^{4n-1}$ defined on p. 5–9. As noted there, $\forall\ q > 0$, $H_q(E_f; \mathbf{Q}) = 0$, hence the $\pi_q(E_f)$ are finite and this means that the $\pi_q(\mathbf{V}_{2n+1,2})$ are finite save for $\pi_{4n-1}(\mathbf{V}_{2n+1,2})$ which is the direct sum of $\mathbf{Z}$ and a finite group. Second, examine the homotopy sequence of the Hurewicz fibration $\mathbf{V}_{2n+1,2} \to \mathbf{S}^{2n}$, noting that its fiber is $\mathbf{S}^{2n-1}$.

Given a category $\mathbf{C}$, the tower category $\mathbf{TOW(C)}$ of $\mathbf{C}$ is the functor category $[[\mathbf{N}]^{\mathrm{OP}}, \mathbf{C}]$. Example: The Postnikov tower of a pointed path connected space is an object in $\mathbf{TOW(TOP_*)}$.

Take $\mathbf{C} = \mathbf{AB}$—then an object $(\mathbf{G}, \mathbf{f})$ in $\mathbf{TOW(AB)}$ is a sequence $\{G_n, f_n : G_{n+1} \to G_n\}$, where G_n is an abelian group and $f_n : G_{n+1} \to G_n$ is a homomorphism, a morphism $\phi : (\mathbf{G}', \mathbf{f}') \to (\mathbf{G}'', \mathbf{f}'')$ in $\mathbf{TOW(AB)}$ being a sequence $\{\phi_n\}$, where $\phi_n : G'_n \to G''_n$ is a homomorphism and $\phi_n \circ f'_n = f''_n \circ \phi_{n+1}$. $\mathbf{TOW(AB)}$ is an abelian category. As such, it has enough injectives.

[Note: Equip $[\mathbf{N}]$ with the topology determined by $\leq$, i.e., regard $[\mathbf{N}]$ as an A space—then $\mathbf{TOW(AB)}$ is equivalent to the category of sheaves of abelian groups on $[\mathbf{N}]$.]

The functor $\lim : \mathbf{TOW(AB)} \to \mathbf{AB}$ that sends $\mathbf{G}$ to $\lim \mathbf{G}$ is left exact (being a right adjoint) but it need not be exact. The right derived functors $\lim^i$ of $\lim$ live only in dimensions 0 and 1, i.e., the $\lim^i$ $(i > 1)$ necessarily vanish. To compute $\lim^1 \mathbf{G}$, form $G = \prod\limits_n G_n$ and define $d : G \to G$ by $d(x_0, x_1, \ldots) = (x_0 - f_0(x_1), x_1 - f_1(x_2), \ldots)$—then $\ker d = \lim \mathbf{G}$ and $\operatorname{coker} d = \lim^1 \mathbf{G}$. Example: Suppose that $\forall\ n$, G_n is finite—then $\lim^1 \mathbf{G} = 0$.

[Note: Translated to sheaves, $\lim^i$ corresponds to the ith right derived functor of the global section functor.]

The fact that the $\lim^i$ $(i > 1)$ vanish is peculiar to the case at hand. Indeed, if $(I, \leq)$ is a directed set and if $\mathbf{I}$ is the associated filtered category, then for a suitable choice of I, one can exhibit a $\mathbf{G}$ in $[\mathbf{I}^{\mathrm{OP}}, \mathbf{AB}]$ such that $\lim^i \mathbf{G} \neq 0\ \forall\ i > 0$ (Jensen[†]).

EXAMPLE Let $\mu \neq \nu$ be relatively prime natural numbers > 1. Define $\mathbf{G}(\mu)$ in $\mathbf{TOW(AB)}$ by $G(\mu)_n = \mathbf{Z}\ \forall\ n$ & $\begin{cases} G(\mu)_{n+1} \to G(\mu)_n \\ 1 \to \mu \end{cases}$ and $\phi \in \operatorname{Mor}(\mathbf{G}(\mu), \mathbf{G}(\mu))$ by $\phi_n(1) = \nu$—then the cokernel of ϕ is isomorphic to the constant tower on $[\mathbf{N}]$ with value $\mathbf{Z}/\nu\mathbf{Z}$. Applying $\lim$ to the exact sequence $0 \to \mathbf{G}(\mu) \xrightarrow{\phi} \mathbf{G}(\mu) \to \operatorname{coker}\phi \to 0$ and noting that $\lim \mathbf{G}(\mu) = 0$, one obtains a sequence $0 \to 0 \to 0 \to \mathbf{Z}/\nu\mathbf{Z} \to 0$ which is not exact. On the other hand, the sequence $0 \to \mathbf{Z}/\nu\mathbf{Z} \to \lim^1 \mathbf{G}(\mu) \xrightarrow{\lim^1 \phi} \mathbf{G}(\mu) \to 0$ is exact, so $\lim^1 \mathbf{G}(\mu)$ contains a copy of $\mathbf{Z}/\nu\mathbf{Z}\ \forall\ \nu : (\mu, \nu) = 1$.

To extend the applicability of the preceding considerations, replace $\mathbf{AB}$ by $\mathbf{GR}$. Again, there is a functor $\lim : \mathbf{TOW(GR)} \to \mathbf{GR}$ that sends $\mathbf{G}$ to $\lim \mathbf{G}$. As for $\lim^1 \mathbf{G}$, it is the quotient $\prod\limits_n G_n/\sim$, where $\begin{cases} x' &= \{x'_n\} \\ x'' &= \{x''_n\} \end{cases}$ are equivalent iff $\exists\ x = \{x_n\}$ such that

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[†] *SLN* **254** (1972), 51–52.

$\forall\, n : x''_n = x_n x'_n f_n(x_{n+1}^{-1})$. While not necessarily a group, $\lim^1 \mathbf{G}$ is a pointed set with base point the equivalence class of $\{e_n\}$ and it is clear that $\lim^1 : \mathbf{TOW(GR)} \rightarrow \mathbf{SET_*}$ is a functor.

[Note: Put $X = \prod\limits_n G_n$—then the assignment $((g_0, g_1, \ldots), (x_0, x_1, \ldots)) \rightarrow (g_0 x_0 f_0(g_1^{-1}), g_1 x_1 f_1(g_2^{-1}), \ldots)$ defines a left action of the group $\prod\limits_n G_n$ on the pointed set X. The stabilizer of the base point is $\lim \mathbf{G}$ and the orbit space $\prod\limits_n G_n \backslash X$ is $\lim^1 \mathbf{G}$. For the definition and properties of $\lim^1$ "in general", consult Bousfield-Kan[†].]

LEMMA Let $* \rightarrow \mathbf{G}' \rightarrow \mathbf{G} \rightarrow \mathbf{G}'' \rightarrow *$ be an exact sequence in $\mathbf{TOW(GR)}$—then there is a natural exact sequence of groups and pointed sets

$$* \rightarrow \lim \mathbf{G}' \rightarrow \lim \mathbf{G} \rightarrow \lim \mathbf{G}'' \rightarrow \lim^1 \mathbf{G}' \rightarrow \lim^1 \mathbf{G} \rightarrow \lim^1 \mathbf{G}'' \rightarrow *.$$

[Note: Specifically, the assumption is that $\forall\, n$, the sequence $* \rightarrow G'_n \rightarrow G_n \rightarrow G''_n \rightarrow *$ is exact in $\mathbf{GR}$.]

EXAMPLE Suppose that $\{G_n\}$ is a tower of finitely generated abelian groups—then $\lim^1 G_n$ is isomorphic to a group of the form $\mathrm{Ext}(G, \mathbf{Z})$, where G is countable and torsion free. To see this, write G'_n for the torsion subgroup of G_n and call G''_n the quotient G_n/G'_n. Since each G'_n is finite, $\lim^1 G'_n = * \Rightarrow \lim^1 G_n \approx \lim^1 G''_n$. Assume, therefore, that the G_n are torsion free. Let $K_n = \bigoplus\limits_{i \leq n} G_i = G_n \oplus K_{n-1}$ and define $K_n \rightarrow K_{n-1}$ by $G_n \rightarrow G_{n-1} \rightarrow K_{n-1}$ on the first factor and by the identity on the second factor. So, $\forall\, n$, $K_n \rightarrow K_{n-1}$ is surjective, thus the sequence $0 \rightarrow \lim G_n \rightarrow \lim K_n \rightarrow \lim K_n/G_n \rightarrow \lim^1 G_n \rightarrow 0$ is exact. Because G_n, K_n, and K_n/G_n are free abelian, the sequence $0 \rightarrow \mathrm{Hom}(K_n/G_n, \mathbf{Z}) \rightarrow \mathrm{Hom}(K_n, \mathbf{Z}) \rightarrow \mathrm{Hom}(G_n, \mathbf{Z}) \rightarrow 0$ is exact $\Rightarrow$ the sequence $0 \rightarrow \mathrm{colim}\ \mathrm{Hom}(K_n/G_n, \mathbf{Z}) \rightarrow \mathrm{colim}\ \mathrm{Hom}(K_n, \mathbf{Z}) \rightarrow \mathrm{colim}\ \mathrm{Hom}(G_n, \mathbf{Z}) \rightarrow 0$ is exact $\Rightarrow$ the sequence $0 \rightarrow \mathrm{Hom}(\mathrm{colim}\ \mathrm{Hom}(G_n, \mathbf{Z}), \mathbf{Z}) \rightarrow \mathrm{Hom}(\mathrm{colim}\ \mathrm{Hom}(K_n, \mathbf{Z}), \mathbf{Z}) \rightarrow \mathrm{Hom}(\mathrm{colim}\ \mathrm{Hom}(K_n/G_n, \mathbf{Z}), \mathbf{Z}) \rightarrow \mathrm{Ext}(\mathrm{colim}\ \mathrm{Hom}(G_n, \mathbf{Z}), \mathbf{Z}) \rightarrow \mathrm{Ext}(\mathrm{colim}\ \mathrm{Hom}(K_n, \mathbf{Z}), \mathbf{Z})$ is exact $\Rightarrow$ the sequence $0 \rightarrow \lim G_n \rightarrow \lim K_n \rightarrow \lim K_n/G_n \rightarrow \mathrm{Ext}(\mathrm{colim}\ \mathrm{Hom}(G_n, \mathbf{Z}), \mathbf{Z}) \rightarrow 0$ is exact (for $\mathrm{colim}\ \mathrm{Hom}(K_n, \mathbf{Z}) \approx \bigoplus\limits_n \mathrm{Hom}(G_n, \mathbf{Z})$, which is free). Consequently, $\lim^1 G_n \approx \mathrm{Ext}(\mathrm{colim}\ \mathrm{Hom}(G_n, \mathbf{Z}), \mathbf{Z})$, where $\mathrm{colim}\ \mathrm{Hom}(G_n, \mathbf{Z})$ is countable and torsion free.

[Note: It follows that $\lim^1 G_n$ is divisible, hence if $\lim^1 G_n \neq *$, then on general grounds, there exist cardinals α and $\gamma(p)$ $(p \in \mathbf{\Pi}) : \lim^1 G_n \approx \alpha \cdot \mathbf{Q} \oplus \bigoplus\limits_p \gamma(p) \cdot (\mathbf{Z}/p^\infty \mathbf{Z})$. But here one can say more, viz. $\alpha = 2^\omega$ and $\forall\, p$, $\gamma(p)$ is finite or 2^ω.]

[†] *SLN* **304** (1972), 305–308.

Huber-Warfield[†] have shown that an abelian G is isomorphic to a $\lim^1 \mathbf{G}$ for some $\mathbf{G}$ in **TOW(AB)** iff $\operatorname{Ext}(\mathbf{Q}, G) = 0$.

When is $\lim^1 \mathbf{G} = *$? An obvious sufficient condition is that the $f_n : G_{n+1} \to G_n$ be surjective for every n. More generally, $\mathbf{G}$ is said to be Mittag-Leffler if $\forall\, n\ \exists\, n' \geq n :$ $\forall\, n'' \geq n'$, $\operatorname{im}(G_{n'} \to G_n) = \operatorname{im}(G_{n''} \to G_n)$.

MITTAG-LEFFLER CRITERION Suppose that $\mathbf{G}$ is Mittag-Leffler—then $\lim^1 \mathbf{G} = *$.

[Note: There is a partial converse, viz. if $\lim^1 \mathbf{G} = *$ and if the G_n are countable, then $\mathbf{G}$ is Mittag-Leffler (Dydak-Segal[‡]).]

EXAMPLE Fix a sequence $\mu_0 < \mu_1 \cdots$ of natural numbers ($\mu_0 > 1$). Put $G_n = \prod\limits_{k \geq n} \mathbf{Z}/\mu_k\mathbf{Z}$ and let $G_{n+1} \to G_n$ be the inclusion—then $\mathbf{G}$ is not Mittag-Leffler, yet $\lim^1 \mathbf{G} = *$.

FACT Assume: $\lim^1 \mathbf{G} \neq *$ and the G_n are countable—then $\lim^1 \mathbf{G}$ is uncountable.

EXAMPLE Let X be a CW complex. Suppose that $X_0 \subset X_1 \subset \cdots$ is an expanding sequence of subcomplexes of X such that $X = \bigcup\limits_n X_n$. Fix a cofunctor $\mathcal{G} : \Pi X \to \mathbf{AB}$ and put $\mathcal{G}_n = \mathcal{G}|X_n$—then $\forall\, q \geq 1$, there is an exact sequence $0 \to \lim^1 H^{q-1}(X_n; \mathcal{G}_n) \to H^q(X; \mathcal{G}) \to \lim H^q(X_n; \mathcal{G}_n) \to 0$ of abelian groups (Whitehead[‖]). To illustrate, take $X = K(\mathbf{Q}, 1)$ (realized as on p. 5–27) and let $\mathcal{G} : \Pi X \to \mathbf{AB}$ be the cofunctor corresponding to the usual action of $\mathbf{Q}$ on $\mathbf{Q}[\mathbf{Q}]$ (cf. p. 4–39). This data generates a short exact sequence $0 \to \lim^1 H^1(\mathbf{Z}; \mathbf{Q}[\mathbf{Q}]) \to H^2(\mathbf{Q}; \mathbf{Q}[\mathbf{Q}]) \to \lim H^2(\mathbf{Z}; \mathbf{Q}[\mathbf{Q}]) \to 0$. The tower $H^1(\mathbf{Z}; \mathbf{Q}[\mathbf{Q}]) \leftarrow H^1(\mathbf{Z}; \mathbf{Q}[\mathbf{Q}]) \leftarrow \cdots$ is not Mittag-Leffler but $H^1(\mathbf{Z}; \mathbf{Q}[\mathbf{Q}])$ is countable, therefore $\lim^1 H^1(\mathbf{Z}; \mathbf{Q}[\mathbf{Q}])$ is uncountable. In particular: $H^2(\mathbf{Q}; \mathbf{Q}[\mathbf{Q}]) \neq 0$.

FACT Let $\{G_n\}$ be a tower of nilpotent groups. Assume: $\forall\, n$, $\#(G_n) \leq \omega$—then $\lim^1 G_n = *$ iff $\lim^1 G_n/[G_n, G_n] = *$.

[For as noted above, in the presence of countability, $\lim^1 G_n/[G_n, G_n] = * \Rightarrow \{G_n/[G_n, G_n]\}$ is Mittag-Leffler.]

PROPOSITION 12 Let $\begin{cases} \{X_n\} \\ \{Y_n\} \end{cases}$ be two sequences of pointed spaces. Suppose given pointed continuous functions $\begin{cases} \phi_n : X_n \to X_{n+1} \\ \psi_n : Y_{n+1} \to Y_n \end{cases}$. Assume: The ϕ_n are closed cofibrations and the ψ_n are pointed Hurewicz fibrations—then there is an exact sequence

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[†] *Arch. Math.* 33 (1979), 430–436.

[‡] *SLN* 688 (1978), 78–80.

[‖] *Elements of Homotopy Theory*, Springer Verlag (1978), 273–274.

$$* \to \lim{}^1[X_n, \Omega Y_n] \xrightarrow{\iota} [\operatorname{colim} X_n, \lim Y_n] \to \lim[X_n, Y_n] \to *$$

in **SET**$_*$ and ι is an injection.

[Write $X_\infty = \operatorname{colim} X_n$ & $Y_\infty = \lim Y_n$. Embedded in the data are arrows $\begin{cases} \Phi_n : X_n \to X_\infty \\ \Psi_n : Y_\infty \to Y_n \end{cases}$ with $\begin{cases} \Phi_{n+1} \circ \phi_n = \Phi_n \\ \psi_n \circ \Psi_{n+1} = \Psi_n \end{cases}$ and $\forall$ n, an arrow $[X_{n+1}, Y_{n+1}] \to [X_n, Y_n]$, viz. $[f] \to [\psi_n \circ f \circ \phi_n]$.

Define $\xi_n : [X_\infty, Y_\infty] \to [X_n, Y_n]$ by $\xi_n([f]) = [\Psi_n \circ f \circ \Phi_n]$. Because the collection $\{\xi_n : [X_\infty, Y_\infty] \to [X_n, Y_n]\}$ is a natural source, there exists a unique pointed map $\xi_\infty : [X_\infty, Y_\infty] \to \lim[X_n, Y_n]$ such that $\forall$ n, the triangle

$$\begin{tikzcd} {[X_\infty, Y_\infty]} \arrow[r, "\xi_\infty"] \arrow[dr, "\xi_n"'] & \lim[X_n, Y_n] \arrow[d] \\ & {[X_n, Y_n]} \end{tikzcd}$$

commutes. To prove that ξ_∞ is surjective, take $\{[f_n]\} \in \lim[X_n, Y_n]$—then $\forall$ n, $\psi_n \circ f_{n+1} \circ \phi_n \simeq f_n$. Set $\overline{f}_0 = f_0$ and, proceeding inductively, assume that $\overline{f}_1 \in [f_1], \ldots, \overline{f}_n \in [f_n]$ have been found with $\psi_{k-1} \circ \overline{f}_k \circ \phi_{k-1} = \overline{f}_{k-1}$ $(1 \leq k \leq n)$. Choose a pointed homotopy $h_n : IX_n \to Y_n : \begin{cases} h_n \circ i_0 = \psi_n \circ f_{n+1} \circ \phi_n \\ h_n \circ i_1 = \overline{f}_n \end{cases}$. Since ψ_n is a pointed Hurewicz fibration, the commutative diagram

$$\begin{tikzcd} X_n \arrow[r, "f_{n+1} \circ \phi_n"] \arrow[d, "i_0"'] & Y_{n+1} \arrow[d, "\psi_n"] \\ IX_n \arrow[r, "h_n"'] & Y_n \end{tikzcd}$$

admits a pointed filler $H_n : IX_n \to Y_{n+1}$. Fix a retraction $r_n : IX_{n+1} \to i_0 X_{n+1} \cup I\phi_n(X_n)$ (cf. §3, Proposition 1) and specify a pointed continuous function $F_{n+1} : i_0 X_{n+1} \cup I\phi_n(X_n) \to Y_{n+1}$ by the prescription $\begin{cases} F_{n+1}(x_{n+1}, 0) = f_{n+1}(x_{n+1}) \\ F_{n+1}(\phi_n(x_n), t) = H_n(x_n, t) \end{cases}$. Put $\overline{h}_n = \psi_n \circ F_{n+1} \circ r_n$ to get a commutative diagram

$$\begin{tikzcd} i_0 X_{n+1} \cup I\phi_n(X_n) \arrow[r, "F_{n+1}"] \arrow[d] & Y_{n+1} \arrow[d, "\psi_n"] \\ IX_{n+1} \arrow[r, "\overline{h}_n"'] & Y_n \end{tikzcd}$$

. Bearing in mind that ϕ_n is a closed cofibration, this diagram has a pointed filler $\overline{H}_{n+1} : IX_{n+1} \to Y_{n+1}$ (cf. §4, Proposition 12). Finally, to push the induction forward, let $\overline{f}_{n+1} = \overline{H}_{n+1} \circ i_1$. Conclusion: There exists a pointed continuous function $\overline{f}_\infty : X_\infty \to Y_\infty$ such that $\xi_\infty([\overline{f}_\infty]) = \{[f_n]\}$, i.e., ξ_∞ is surjective.

As for the kernel of ξ_∞, it consists of those $[f] : \forall$ n, $\Psi_n \circ f \circ \Phi_n$ is nullhomotopic. Thus there are pointed homotopies $\Xi_n : IX_n \to Y_n$ such that $\Xi_n \circ i_0 = 0_n$ & $\Xi_n \circ i_1 = \Psi_n \circ f \circ \Phi_n$ with $\psi_n \circ \Xi_{n+1} \circ I\phi_n \circ i_0 = 0_n$ & $\psi_n \circ \Xi_{n+1} \circ I\phi_n \circ i_1 = \Psi_n \circ f \circ \Phi_n$, where 0_n is the zero morphism $X_n \to Y_n$. To define $\eta_\infty : \ker \xi_\infty \to \lim{}^1[X_n, \Omega Y_n]$, let $\sigma_{n,f} : X_n \to \Omega Y_n$ be the pointed continuous function given by

$$\sigma_{n,f}(x_n, t) = \begin{cases} \Xi_n(x_n, 2t) & (0 \leq t \leq 1/2) \\ \psi_n \circ \Xi_{n+1}(\phi_n(x_n), 2 - 2t) & (1/2 \leq t \leq 1) \end{cases}.$$

The $\sigma_{n,f}$ determine a string in $\prod_n [X_n, \Omega Y_n]$ or still, an element of $\lim^1[X_n, \Omega Y_n]$, call it $[\sigma_f]$. Definition: $\eta_\infty([f]) = [\sigma_f]$. One can check that η_∞ does not depend on the choice of the Ξ_n and is independent of the choice of $f \in [f]$. Claim: η_∞ is bijective. To verify, e.g., injectivity, suppose that $\eta_\infty([f']) = \eta_\infty([f''])$—then there exists a string $\{[\sigma_n]\} \in \prod_n [X_n, \Omega Y_n] : \forall\ n$,

$$\begin{cases} \sigma_n(x_n, 3t) & (0 \leq t \leq 1/3) \\ \sigma_{n,f'}(x_n, 3t-1) & (1/3 \leq t \leq 2/3) \\ \psi_n \circ \sigma_{n+1}(\phi_n(x_n), 3-3t) & (2/3 \leq t \leq 1) \end{cases}$$

represents $\sigma_{n,f''}$. In addition, the formulas

$$\begin{cases} \Xi'_n(x_n, 1-3t) & (0 \leq t \leq 1/3) \\ \sigma_n(x_n, 2-3t) & (1/3 \leq t \leq 2/3) \\ \Xi''_n(x_n, 3t-2) & (2/3 \leq t \leq 1) \end{cases}$$

define a pointed homotopy $H_n : IX_n \to Y_n$ having the property that $H_n \circ i_0 = \Psi_n \circ f' \circ \Phi_n$ & $H_n \circ i_1 = \Psi_n \circ f'' \circ \Phi_n$. Arguing as before, construct pointed homotopies $\overline{H}_n : IX_n \to Y_n$ such that $\overline{H}_n \circ i_0 = \Psi_n \circ f' \circ \Phi_n$ & $\overline{H}_n \circ i_1 = \Psi_n \circ f'' \circ \Phi_n$ with $\psi_n \circ \overline{H}_{n+1} \circ I\phi_n = \overline{H}_n$. The $\overline{H}_n$ combine and induce a pointed homotopy $\overline{H}_\infty : IX_\infty \to Y_\infty$ between f' and f'', i.e., η_∞ is injective.]

Application: Let $\{X_n\}$ be a sequence of pointed spaces. Suppose given pointed continuous functions $\phi_n : X_n \to X_{n+1}$ such that $\forall\ n$, ϕ_n is a closed cofibration—then for any pointed space Y, there is an exact sequence

$$* \to \lim{}^1[\Sigma X_n, Y] \xrightarrow{\iota} [\operatorname{colim} X_n, Y] \to \lim[X_n, Y] \to *$$

in $\mathbf{SET}_*$ and ι is an injection.

EXAMPLE Fix an abelian group π. Let (X, x_0) be a pointed CW complex. Suppose that $x_0 \in X_0 \subset X_1 \subset \cdots$ is an expanding sequence of subcomplexes of X such that $X = \bigcup_n X_n$—then $\forall\ q \geq 1$, there is an exact sequence $0 \to \lim^1 \widetilde{H}^{q-1}(X_n; \pi) \to \widetilde{H}^q(X; \pi) \to \lim \widetilde{H}^q(X_n; \pi) \to 0$ of abelian groups. Example: $\forall\ q \geq 1$, $H^q(\mathbf{Z}/p^\infty\mathbf{Z}, n) \approx \lim H^q(\mathbf{Z}/p^k\mathbf{Z}, n)$.

[In the above, substitute $Y = K(\pi, q)$.]

LEMMA Let X be a pointed finite CW complex. Let K be a pointed connected CW complex. Assume: The homotopy groups of K are finite—then the pointed set $[X, K]$ is finite.

[This result is contained in obstruction theory but one can also give a direct inductive proof.]

EXAMPLE Let (X, x_0) be a pointed CW complex. Suppose that $x_0 \in X_0 \subset X_1 \subset \cdots$ is an expanding sequence of finite subcomplexes of X such that $X = \bigcup\limits_n X_n$. Let K be a pointed connected CW complex. Assume: The homotopy groups of K are finite—then the natural map $\pi_X : [X, K] \rightarrow \lim[X_n, K]$ is bijective. In fact, surjectivity is automatic, so injectivity is what's at issue. For this, consider the natural map $\pi_{IX} : [IX, K] \rightarrow \lim[i_0 X \cup IX_n \cup i_1 X, K]$ and the obvious arrows $i_0, i_1 : \lim[i_0 X \cup IX_n \cup i_1 X, K] \rightarrow [X, K]$. Since $i_0 \circ \pi_{IX} = i_1 \circ \pi_{IX}$ and since π_{IX} is surjective, $i_0 = i_1$. That π_X is injective is thus a consequence of the following claim.

Claim: If $\pi_X([f_0]) = \pi_X([f_1])$, then there exists $[F] \in \lim[i_0 X \cup IX_n \cup i_1 X, K] : \begin{cases} [f_0] = i_0([F]) \\ [f_1] = i_1([F]) \end{cases}$.

[Let $i_0^n, i_1^n : [i_0 X \cup IX_n \cup i_1 X, K] \rightarrow [X, K]$ be the obvious arrows. For each n, there is at least one $[F_n] \in [i_0 X \cup IX_n \cup i_1 X, K] : \begin{cases} [f_0] = i_0^n([F_n]) \\ [f_1] = i_1^n([F_n]) \end{cases}$. Denote by I_n the subset of $[i_0 X \cup IX_n \cup i_1 X, K]$ consisting of all such $[F_n]$—then, from the lemma, I_n is finite, hence $\lim I_n \neq \emptyset$.]

[Note: The ΣX_n are finite CW complexes, therefore the $[\Sigma X_n, K]$ are finite groups, so $\lim^1[\Sigma X_n, K] = *$. But this only means that the kernel of π_X is $[0]$.]

Application: Let $\{Y_n\}$ be a sequence of pointed spaces. Suppose given pointed continuous functions $\psi_n : Y_{n+1} \rightarrow Y_n$ such that $\forall\, n$, ψ_n is a pointed Hurewicz fibration—then for any pointed space X, there is an exact sequence

$$* \rightarrow \lim{}^1[X, \Omega Y_n] \xrightarrow{\iota} [X, \lim Y_n] \rightarrow \lim[X, Y_n] \rightarrow *$$

in $\mathbf{SET}_*$ and ι is an injection.

[Note: The exact sequence $* \rightarrow \lim^1 \pi_{q+1}(Y_n) \xrightarrow{\iota} \pi_q(\lim Y_n) \rightarrow \lim \pi_q(Y_n) \rightarrow *$ of pointed sets is a special case (take $X = \mathbf{S}^q$).]

EXAMPLE For each n, put $Y_n = \mathbf{S}^1$ and let $\psi_n : Y_{n+1} \rightarrow Y_n$ be the squaring map $\begin{cases} \mathbf{S}^1 \rightarrow \mathbf{S}^1 \\ s \rightarrow s^2 \end{cases}$ —then $\lim \pi_1(Y_n) = 0$ but $\lim^1 \pi_1(Y_n) \approx \widehat{\mathbf{Z}}_2/\mathbf{Z}$, the 2-adic integers mod $\mathbf{Z}$.

EXAMPLE Let $\boldsymbol{\pi} = \{\pi_n\}$ be a tower of abelian groups. Assume: $\boldsymbol{\pi}$ is Mittag-Leffler—then $\forall\, q \geq 1$, $K(\lim \boldsymbol{\pi}, q) = \lim K(\pi_n, q)$, so for any pointed CW complex (X, x_0), there is an exact sequence $0 \rightarrow \lim^1 \widetilde{H}^{q-1}(X; \pi_n) \rightarrow \widetilde{H}^q(X; \lim \boldsymbol{\pi}) \rightarrow \lim \widetilde{H}^q(X; \pi_n) \rightarrow 0$ of abelian groups.

Given a pointed path connected space X, let $P_\infty X = \lim P_n X$—then $\forall\, q \geq 0$, $\pi_q(P_\infty X) \approx \lim \pi_q(P_n X) \approx \pi_q(P_q X)$. Proof: The relevant $\lim^1$ term vanishes.

PROPOSITION 13 The canonical arrow $X \rightarrow P_\infty X$ is a weak homotopy equivalence.

[For each n, there is an inclusion $X \rightarrow X[n]$, a projection $P_\infty X \rightarrow P_n X$, and a pointed homotopy equivalence $X[n] \rightarrow P_n X$. Consider the associated commutative diagram

$$\begin{tikzcd} X \arrow[r] \arrow[d] & P_\infty X \arrow[d] \\ X[n] \arrow[r] & P_n X \end{tikzcd}$$

, recalling that $\pi_n(X) \approx \pi_n(X[n])$.]

FACT Let $\{X_n, f_n : X_{n+1} \rightarrow X_n\}$ be a tower in **TOP**. Assume: The X_n are CW spaces and the f_n are Hurewicz fibrations—then $\lim X_n$ is a CW space iff all but finitely many of the f_n are homotopy equivalences.

[Necessity: If infinitely many of the f_n are not homotopy equivalences, then $\lim X_n$ is not numerably contractible.

Sufficiency: If all of the f_n are homotopy equivalences, then X_0 and $\lim X_n$ have the same homotopy type (cf. p. 4–17).]

Application: Suppose that X is a pointed connected CW space—then the canonical arrow $X \rightarrow P_\infty X$ is a homotopy equivalence iff X has finitely many nontrivial homotopy groups.

WHITEHEAD THEOREM Suppose that X and Y are path connected topological spaces.

(1) Let $f : X \rightarrow Y$ be an n-equivalence—then $f_* : H_q(X) \rightarrow H_q(Y)$ is bijective for $1 \leq q < n$ and surjective for $q = n$.

(2) Suppose in addition that X and Y are simply connected. Let $f : X \rightarrow Y$ be a continuous function such that $f_* : H_q(X) \rightarrow H_q(Y)$ is bijective for $1 \leq q < n$ and surjective for $q = n$—then f is an n-equivalence.

[The condition on f_* amounts to requiring that $H_q(M_f, i(X)) = 0$ for $q \leq n$, thus the result follows from the relative Hurewicz theorem.]

EXAMPLE Let X be a pointed connected CW space—then the inclusion $X \rightarrow X[n]$ is an $(n+1)$-equivalence, hence there are bijections $H_q(X) \approx H_q(X[n])$ $(q \leq n)$ and a surjection $H_{n+1}(X) \rightarrow H_{n+1}(X[n])$. So, if X is abelian and if the $\pi_q(X)$ are finitely generated $\forall$ q, then the $H_q(X)$ are finitely generated $\forall$ q (cf. p. 5–44).

EXAMPLE (Suspension Theorem) Suppose that X is nondegenerate and n-connected. Let K be a pointed CW complex—then the suspension map $[K, X] \rightarrow [\Sigma K, \Sigma X]$ is bijective if $\dim K \leq 2n$ and surjective if $\dim K \leq 2n+1$. In fact the arrow of adjunction $e : X \rightarrow \Omega\Sigma X$ induces an isomorphism $H_q(X) \rightarrow H_q(\Omega\Sigma X)$ for $0 \leq q \leq 2n+1$ (cf. p. 4–37), therefore by the Whitehead theorem e is a $(2n+1)$-equivalence. So, if $\dim K$ is finite and if $n \geq 2 + \dim K$, then $[\Sigma^n K, \Sigma^n X] \approx [\Sigma^{n+1} K, \Sigma^{n+1} X]$.

A continuous function $f : X \to Y$ is said to be a homology equivalence if $\forall\ n \geq 0$, $f_* : H_n(X) \to H_n(Y)$ is an isomorphism. Example: Consider the coreflector k : **TOP** $\to$ **CG**—then for every topological space X, the identity map $kX \to X$ is a homology equivalence.

EXAMPLE A homology equivalence $f : X \to Y$ need not be a weak homotopy equivalence. One can take, e.g., X to be Poincaré's homology 3-sphere $\mathbf{S}^3/\mathbf{SL}(2,5)$ and $Y = \mathbf{S}^3$. There is a homology equivalence $f : X \to Y$ obtained by collapsing the 2-skeleton of X to a point which, though, is not a weak homotopy equivalence, the fundamental group of X being $\mathbf{SL}(2,5)$. Eight different descriptions of X have been examined by Kirby-Scharlemann[†].

WHITEHEAD THEOREM (bis) Suppose that X and Y are path connected topological spaces.

(1) Let $f : X \to Y$ be a weak homotopy equivalence—then f is a homology equivalence.

[Note: It is a corollary that in general a weak homotopy equivalence is a homology equivalence.]

(2) Suppose in addition that X and Y are simply connected. Let $f : X \to Y$ be a homology equivalence—then f is a weak homotopy equivalence.

Consequently, if X and Y are simply connected topological spaces that are dominated in homotopy by CW complexes, then a continuous function $f : X \to Y$ is a homotopy equivalence iff it is a homology equivalence.

The following familiar remarks serve to place this result in perspective.

(1) There exist path connected topological spaces X and Y such that $\forall\ n : \pi_n(X)$ is isomorphic to $\pi_n(Y)$ but $\exists\ n : H_n(X)$ is not isomorphic to $H_n(Y)$.

(2) There exist simply connected topological spaces X and Y such that $\forall\ n : H_n(X)$ is isomorphic to $H_n(Y)$ but $\exists\ n : \pi_n(X)$ is not isomorphic to $\pi_n(Y)$.

(3) There exist path connected topological spaces X and Y admitting a homology equivalence $f : X \to Y$ with the property that $f_* : \pi_1(X) \to \pi_1(Y)$ is an isomorphism, yet f is not a weak homotopy equivalence.

[Note: Recall too that there exist topological spaces X and Y such that $\forall\ n : H_n(X)$ is isomorphic to $H_n(Y)$ and $\forall\ n : \pi_n(X, x_0)$ is isomorphic to $\pi_n(Y, y_0)$ ($\forall\ x_0 \in X, \forall\ y_0 \in Y$), yet X and Y do not have the same homotopy type. Example: $\begin{cases} X = \{0\} \cup \{1/n : n \geq 1\} \\ Y = \{0\} \cup \{n : n \geq 1\} \end{cases}$.]

[†] In: *Geometric Topology*, J. Cantrell (ed.), Academic Press (1979), 113–146.

EXAMPLE There exists a sequence $X_1, X_2, \ldots$ of simply connected CW complexes X_n having isomorphic integral singular cohomology rings such that $\forall\ n' \neq n''$, the homotopy types of $X_{n'}$ & $X_{n''}$ are distinct (Body-Douglas[†]).

EXAMPLE Let X be a pointed connected CW space—then ΣX is contractible iff $H_1(\pi, 1) = 0 = H_2(\pi, 1)$ $(\pi = \pi_1(X))$ and $H_q(X) = 0$ $(q \geq 2)$.

EXAMPLE (Stable Splitting) Let G be a finite abelian group—then there exist positive integers T and t such that $\Sigma^T K(G, 1)$ has the pointed homotopy type of a wedge $X_1 \vee \cdots \vee X_t$, where the X_i are pointed simply connected CW spaces. For let $G = G(p_1) \oplus \cdots \oplus G(p_n)$ be the primary decomposition of G. Since the arrow $K(G(p_1), 1) \vee \cdots \vee K(G(p_n), 1) \rightarrow K(G(p_1), 1) \times \cdots \times K(G(p_n), 1) = K(G, 1)$ is a homology equivalence, its suspension is a pointed homotopy equivalence, thus one can assume that G is p-primary, say $G = \mathbf{Z}/p^{e_1}\mathbf{Z} \oplus \cdots \oplus \mathbf{Z}/p^{e_r}\mathbf{Z}$, so $K(G, 1) = \prod_1^r K(\mathbf{Z}/p^{e_i}\mathbf{Z}, 1)$. Accordingly, thanks to the Puppe formula and the fact that $\Sigma(X \# Y) \approx \Sigma X \# Y \approx X \# \Sigma Y$, it suffices to consider $K(\mathbf{Z}/p^e\mathbf{Z}, 1)$.

Claim: There exist pointed simply connected CW spaces $X_1, \ldots, X_{p-1}$ and a pointed homotopy equivalence $\Sigma K(\mathbf{Z}/p^e\mathbf{Z}, 1) \rightarrow X_1 \vee \cdots \vee X_{p-1}$.

[A generator of the multiplicative group of units in $\mathbf{Z}/p\mathbf{Z}$ defines a pointed homotopy equivalence $K(\mathbf{Z}/p^e\mathbf{Z}, 1) \rightarrow K(\mathbf{Z}/p^e\mathbf{Z}, 1)$.]

The rather restrictive assumption that $\begin{cases} \pi_1(X) = 0 \\ \pi_1(Y) = 0 \end{cases}$ is not necessary in order to guarantee that a homology equivalence $f : X \rightarrow Y$ is a weak homotopy equivalence. For example, $\begin{cases} X \\ Y \end{cases}$ abelian will do and in fact one can get away with considerably less.

Notation: Given a group G, let $\mathbf{Z}[G]$ be its integral group ring and $I[G] \subset \mathbf{Z}[G]$ the augmentation ideal. Given a G-module M, let M_G be its group of coinvariants, i.e., the quotient $M/I[G] \cdot M$ or still, $H_0(G; M)$.

[Note: In this context, "G-module" means left G-module. If K is a normal subgroup of G, then the action of G on M induces an action of G/K on M_K and $M_G \approx (M_K)_{G/K}$.]

FUNDAMENTAL EXACT SEQUENCE Fix a G-module M. Let K be a normal subgroup of G—then there is an exact sequence

$$H_2(G; M) \rightarrow H_2(G/K; M_K) \rightarrow H_1(K; M)_{G/K} \rightarrow H_1(G; M) \rightarrow H_1(G/K; M_K) \rightarrow 0.$$

[The LHS spectral sequence reads: $E^2_{p,q} \approx H_p(G/K; H_q(K; M)) \Rightarrow H_{p+q}(G; M)$. Explicate the associated five term exact sequence $H_2(G; M) \rightarrow E^2_{2,0} \xrightarrow{d^2} E^2_{0,1} \rightarrow H_1(G; M) \rightarrow E^2_{1,0} \rightarrow 0$.]

[†] *Topology* **13** (1974), 209–214.

Application: Let K be a normal subgroup of G—then there is an exact sequence $H_2(G) \to H_2(G/K) \to K/[G,K] \to H_1(G) \to H_1(G/K) \to 0$.

[Specialize the fundamental exact sequence and take $M = \mathbf{Z}$ (trivial G-action). Observe that the arrows $\begin{cases} H_1(G) \to H_1(G/K) \\ H_2(G) \to H_2(G/K) \end{cases}$ are induced by the projection $G \to G/K$.]

Using a superscript to denote the "invariants" functor, the fundamental exact sequence in cohomology is $0 \to H^1(G/K; M^K) \to H^1(G;M) \to H^1(K;M)^{G/K} \to H^2(G/K;M^K) \to H^2(G;M)$.

Notation: Given a group G, let $\Gamma^0(G) \supset \Gamma^1(G) \supset \cdots$ be its descending central series, so $\Gamma^{i+1}(G) = [G, \Gamma^i(G)]$. In particular: $\Gamma^0(G) = G$, $\Gamma^1(G) = [G,G]$ and G is nilpotent if there exists a $d : \Gamma^d(G) = \{1\}$, the smallest such d being its degree of nilpotency: $\operatorname{nil} G$.

FACT Let G be a nilpotent group—then G is finitely generated iff $G/[G,G]$ is finitely generated.

EXAMPLE Let G be a nilpotent group—then G is finitely generated iff $\forall\, q \geq 1$, $H_q(G)$ is finitely generated. For suppose that G is finitely generated. Case 1: $\operatorname{nil} G \leq 1$. In this situation, G is abelian and the assertion is true (cf. p. 5–35). Case 2: $\operatorname{nil} G > 1$. Argue by induction, using the LHS spectral sequence $E^2_{p,q} \approx H_p(G/\Gamma^i(G); H_q(\Gamma^i(G)/\Gamma^{i+1}(G))) \Rightarrow H_{p+q}(G/\Gamma^{i+1}(G))$. To discuss the converse, note that $H_1(G) \approx G/[G,G]$ and quote the preceding result.

It is false in general that a subgroup of a finitely generated group is finitely generated. Example: Let G be the free group on two symbols and consider $[G,G]$.

FACT Suppose that G is a finitely generated nilpotent group—then every subgroup of G is finitely generated.

FACT Suppose that G is a finitely generated nilpotent group—then G is finitely presented.

[The class of finitely presented groups is closed with respect to the formation of extensions.]

Notation: Given a group G, G_{tor} is its subset of elements of finite order.

[Note: G_{tor} need not be a subgroup of G (consider $G = \mathbf{Z}/2\mathbf{Z} * \mathbf{Z}/2\mathbf{Z}$) but will be if G is nilpotent (since $\operatorname{nil} G \leq d$ and $y^m = e \Rightarrow (xy)^{m^d} = x^{m^d}$).]

FACT Suppose that G is a finitely generated nilpotent group. Assume: G is torsion—then G is finite.

Application: If G is a finitely generated nilpotent group, then G_{tor} is a finite nilpotent normal subgroup.

PROPOSITION 14 Let $f : G \to K$ be a homomorphism of groups. Assume: (i) $f_* : H_1(G) \to H_1(K)$ is bijective and (ii) $f_* : H_2(G) \to H_2(K)$ is surjective—then $\forall\ i \geq 0$, the induced map $G/\Gamma^i(G) \to K/\Gamma^i(K)$ is an isomorphism.

[The assertion is trivial if $i = 0$ and holds by assumption if $i = 1$. Fix $i > 1$ and proceed by induction. There is a commutative diagram

$$\begin{tikzcd}
H_2(G) \arrow[r] \arrow[d] & H_2(G/\Gamma^i(G)) \arrow[r] \arrow[d] & \Gamma^i(G)/\Gamma^{i+1}(G) \arrow[r] \arrow[d] & H_1(G) \arrow[r] \arrow[d] & H_1(G/\Gamma^i(G)) \arrow[r] \arrow[d] & 0 \arrow[d] \\
H_2(K) \arrow[r] & H_2(K/\Gamma^i(K)) \arrow[r] & \Gamma^i(K)/\Gamma^{i+1}(K) \arrow[r] & H_1(K) \arrow[r] & H_1(K/\Gamma^i(K)) \arrow[r] & 0
\end{tikzcd}$$

with exact rows, hence, by the five lemma, $\Gamma^i(G)/\Gamma^{i+1}(G) \approx \Gamma^i(K)/\Gamma^{i+1}(K)$. But then from

$$\begin{tikzcd}
1 \arrow[r] & \Gamma^i(G)/\Gamma^{i+1}(G) \arrow[r] \arrow[d] & G/\Gamma^{i+1}(G) \arrow[r] \arrow[d] & G/\Gamma^i(G) \arrow[r] \arrow[d] & 1 \\
1 \arrow[r] & \Gamma^i(K)/\Gamma^{i+1}(K) \arrow[r] & K/\Gamma^{i+1}(K) \arrow[r] & K/\Gamma^i(K) \arrow[r] & 1
\end{tikzcd},$$

one concludes that $G/\Gamma^{i+1}(G) \approx K/\Gamma^{i+1}(K)$.]

Application: Let $f : G \to K$ be a homomorphism of nilpotent groups. Assume: (i) $f_* : H_1(G) \to H_1(K)$ is bijective and (ii) $f_* : H_2(G) \to H_2(K)$ is surjective—then f is an isomorphism.

Let G and π be groups. Suppose that G operates on π, i.e., suppose given a homomorphism $\chi : G \to \operatorname{Aut}\pi$. Put $\Gamma_\chi^0(\pi) = \pi$ and, via recursion, write $\Gamma_\chi^{i+1}(\pi)$ for the subgroup of π generated by the $\alpha(\chi(g)\alpha_i)\alpha^{-1}\alpha_i^{-1}(\alpha \in \pi, \alpha_i \in \Gamma_\chi^i(\pi))$, where $g \in G$—then $\Gamma_\chi^i(\pi)$ is a G-stable normal subgroup of π containing $\Gamma_\chi^{i+1}(\pi)$. The quotient $\Gamma_\chi^i(\pi)/\Gamma_\chi^{i+1}(\pi)$ is abelian and the induced action of G is trivial. One says that G operates nilpotently on π or that π is χ-nilpotent if there exists a $d : \Gamma_\chi^d(\pi) = \{1\}$, the smallest such d being its degree of nilpotency: $\operatorname{nil}_\chi \pi$. Example: Take $G = \pi$ and let $\chi : \pi \to \operatorname{Aut}\pi$ be the representation of π by inner automorphisms—then π is χ-nilpotent iff π is nilpotent.

[Note: From the definitions, for any χ, $\Gamma^i(\pi) \subset \Gamma_\chi^i(\pi)$, thus if π is χ-nilpotent, then π must be nilpotent.]

Let Π and π be groups, where $\Pi \subset \operatorname{Aut}\pi$. Suppose that $\pi = \pi_0 \supset \pi_1 \supset \cdots \supset \pi_d = \{1\}$ is a finite filtration of π by Π-stable normal subgroups such that Π operates trivially on the π_i/π_{i+1}—then there is a lemma in group theory that says Π must be nilpotent (Suzuki[†]). So, given $\chi : G \to \operatorname{Aut}\pi, \operatorname{im}\chi$ is nilpotent provided that π is χ-nilpotent.

[†] *Group Theory*, vol. II, Springer Verlag (1986), 19–20.

FACT Given a homomorphism $\chi : G \to \text{Aut}\,\pi$, consider the semidirect product $\pi \rtimes_\chi G$, i.e., the set of all ordered pairs $(\alpha, g) \in \pi \times G$ with law of composition $(\alpha', g')(\alpha'', g'') = (\alpha'(\chi(g')\alpha''), g'g'')$—then $\pi \rtimes_\chi G$ is nilpotent iff π is χ-nilpotent and G is nilpotent.

EXAMPLE Every finite p-group is nilpotent. Since the semidirect product of two finite p-groups is a finite p-group, it follows that if G and π are finite p-groups and if G operates on π, then G actually operates nilpotently on π.

FACT Suppose that G operates on π—then G operates nilpotently on π iff π is nilpotent and G operates nilpotently on $\pi/[\pi, \pi]$.

EXAMPLE Let $1 \to G' \to G \to G'' \to 1$ be a short exact sequence of groups. Obviously: G nilpotent $\Rightarrow \begin{cases} G' \\ G'' \end{cases}$ nilpotent. The converse is false (consider $A_3 \subset S_3$). However, there is a characterization: G is nilpotent iff $\begin{cases} G' \\ G'' \end{cases}$ are nilpotent and the action of G'' on $G'/[G', G']$ is nilpotent.

Example: Suppose that $\pi = M$ is a G-module. Since M is abelian, it is nilpotent but it needn't be χ-nilpotent. In fact, $\Gamma^i_\chi(M) = (I[G])^i \cdot M$, therefore M is χ-nilpotent iff $(I[G])^d \cdot M = 0$ for some d. When this is so, M is referred to as a nilpotent G-module.

EXAMPLE Let π be a nilpotent G-module. Fix $n \geq 1$—then $\forall\, q \geq 0$, $H_q(\pi, n)$ is a nilpotent G-module.

[G operates nilpotently on the $\Gamma^i_\chi(\pi)$ and $\forall\, i$, there is a short exact sequence $0 \to \Gamma^{i+1}_\chi(\pi) \to \Gamma^i_\chi(\pi) \to \Gamma^i_\chi(\pi)/\Gamma^{i+1}_\chi(\pi) \to 0$ of G-modules, the action of G on $\Gamma^i_\chi(\pi)/\Gamma^{i+1}_\chi(\pi)$ being trivial. The mapping fiber of the arrow $K(\Gamma^i_\chi(\pi), n) \to K(\Gamma^i_\chi(\pi)/\Gamma^{i+1}_\chi(\pi), n)$ is a $K(\Gamma^{i+1}_\chi(\pi), n)$. Consider the associated fibration spectral sequence, noting that by induction, G operates nilpotently on $E^2_{p,q}$.]

FACT Suppose that G is a finitely generated nilpotent group. Let M be a nilpotent G-module.

(1) If M is finitely generated, then $\forall\, q \geq 0$, $H_q(G; M)$ is finitely generated.

(2) If M is not finitely generated, then $H_0(G; M)$ is not finitely generated.

A nonempty path connected topological space X is said to be nilpotent if $\pi_1(X)$ is nilpotent and if $\forall\, n > 1$, $\pi_1(X)$ operates nilpotently on $\pi_n(X)$. Examples: (1) Every abelian topological space is nilpotent; (2) Every path connected topological space whose homotopy groups are finite p-groups is nilpotent (cf. supra); (3) Take for X the Klein bottle—then $\pi_1(X)$ is not a nilpotent group; (4) Take for X the real projective plane—then $\pi_1(X) \approx \mathbf{Z}/2\mathbf{Z}$, $\pi_2(X) \approx \mathbf{Z}$ and the action of $\pi_1(X)$ on $\pi_2(X)$ is the inversion $n \to -n$, thus $\pi_1(X)$ does not operate nilpotently on $\pi_2(X)$; (5) Take for X the torus $\mathbf{S}^1 \times \mathbf{S}^1$—then X is nilpotent but its 1-skeleton $X^{(1)} = \mathbf{S}^1 \vee \mathbf{S}^1$ is not nilpotent.

EXAMPLE Let G be a topological group with base point e and denote by G_0 the path component of e—then $\pi_0(G) = G/G_0$ can be identified with $\pi_1(B_G^\infty)$ and $\pi_n(G) = \pi_n(G_0)$ can be identified with $\pi_{n+1}(B_G^\infty)$ (cf. p. 4–65). These identifications are compatible in that the homomorphism $\chi_n : \pi_0(G) \rightarrow \operatorname{Aut} \pi_n(G_0)$ arising from the operation of G on itself by inner automorphisms corresponds to the action of $\pi_1(B_G^\infty)$ on $\pi_{n+1}(B_G^\infty)$. Accordingly, B_G^∞ is a nilpotent topological space iff $\pi_0(G)$ is a nilpotent group and $\forall\ n \geq 1$, $\pi_n(G_0)$ is χ_n-nilpotent or still, $\forall\ n \geq 1$, the semidirect product $\pi_n(G_0) \rtimes_{\chi_n} \pi_0(G)$ is nilpotent (cf. p. 5–56). The forgetful function $[\mathbf{S}^n, s_n; G_0, e] \rightarrow [\mathbf{S}^n, G_0]$ is bijective, hence $[\mathbf{S}^n, G_0] \approx \pi_n(G_0)$. In addition, $[\mathbf{S}^n, G]$ is isomorphic to $\pi_n(G_0) \rtimes_{\chi_n} \pi_0(G)$. To see this, let $f : \mathbf{S}^n \rightarrow G$ be a continuous function. Choose $g_f \in G : f(\mathbf{S}^n) \subset G_0 g_f$, put $f_0 = f \cdot g_f^{-1}$ and consider the assignment $[f] \rightarrow ([f_0], g_f G_0)$. It therefore follows that B_G^∞ is a nilpotent topological space iff $\forall\ n \geq 1$, $[\mathbf{S}^n, G]$ is a nilpotent group. Example: $B_{\mathbf{O}(2n+1)}^\infty$ is nilpotent but $B_{\mathbf{O}(2n)}^\infty$ is not nilpotent.

[Note: Here is another illustration. The higher homotopy groups of a connected nilpotent Lie group are trivial. So, if G is an arbitrary nilpotent Lie group, then B_G^∞ is a nilpotent topological space.]

FACT Let G be a topological group. Assume: $\forall\ n \geq 1$, $[\mathbf{S}^n, G]$ is a nilpotent group—then for any finite CW complex K, $[K, G]$ is a nilpotent group.

[Take K connected and argue by induction on the number of cells.]

EXAMPLE Let X be a nilpotent CW space—then Mislin[†] has shown that X is dominated in homotopy by a finite CW complex iff the $H_q(X)$ are finitely generated $\forall\ q$ and there exists $q_0 : \forall\ q > q_0$, $H_q(X) = 0$. Moreover, under these conditions, Wall's obstruction to finiteness is zero provided that $\pi_1(X)$ is infinite but this can fail if $\pi_1(X)$ is finite (Mislin[‡]).

DROR'S WHITEHEAD THEOREM Suppose that X and Y are nilpotent topological spaces. Let $f : X \rightarrow Y$ be a homology equivalence—then f is a weak homotopy equivalence.

[To prove that f is a weak homotopy equivalence amounts to proving that for every n, the pair $(M_f, i(X))$ is n-connected, where, a priori, $H_*(M_f, i(X)) = 0$. Consider the commutative diagram

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d] & Y \arrow[d] \\ X[1] \arrow[r, "f_{1,1}"'] & Y[1] \end{tikzcd}$$

. Since the vertical arrows are 2-equivalences, $f_{1,1}$

[†] *Ann. of Math.* **103** (1976), 547–556.

[‡] *Topology* **14** (1975), 311–317.

induces a bijection $H_1(X[1]) \to H_1(Y[1])$ and a surjection $H_2(X[1]) \to H_2(Y[1])$. But $\left\{\begin{matrix} X[1] \\ Y[1] \end{matrix}\right.$ has homotopy type $\left\{\begin{matrix} (\pi_1(X), 1) \\ (\pi_1(Y), 1) \end{matrix}\right.$ and $\left\{\begin{matrix} \pi_1(X) \\ \pi_1(Y) \end{matrix}\right.$ are nilpotent groups, thus $f_* : \pi_1(X) \to \pi_1(Y)$ is an isomorphism (cf. p. 5–55) and so $(M_f, i(X))$ is 1-connected. Noting that here $\pi_2(M_f, i(X))$ is abelian, fix $n > 1$ and assume inductively that $\pi_q(M_f, i(X)) = 0$ for $q < n$—then, from the relative Hurewicz theorem, $\pi_n(M_f, i(X))_{\pi_1(X)} = 0$, i.e., $\pi_n(M_f, i(X)) = I[\pi_1(X)] \cdot \pi_n(M_f, i(X))$. On the other hand, there is an exact sequence $\pi_n(M_f) \to \pi_n(M_f, i(X)) \to \pi_{n-1}(i(X))$ of $\pi_1(X)$-modules. Because the flanking terms are, by hypothesis, nilpotent $\pi_1(X)$-modules, the same must be true of $\pi_n(M_f, i(X))$. Conclusion: $\pi_n(M_f, i(X)) = 0$.]

PROPOSITION 15 Let $f : X \to Y$ be a Hurewicz fibration, where X and Y are path connected. Assume: X is nilpotent—then $\forall\, y_0 \in Y$, the path components of X_{y_0} are nilpotent.

[Fix $x_0 \in X_{y_0}$ and take X_{y_0} path connected. The homomorphisms in the homotopy sequence

$$\cdots \to \pi_{n+1}(Y, y_0) \to \pi_n(X_{y_0}, x_0) \to \pi_n(X, x_0) \to \pi_n(Y, y_0) \to \cdots$$

of f are $\pi_1(X, x_0)$-homomorphisms (cf. p. 4–36). Of course, $\pi_1(X, x_0)$ operates on $\pi_n(Y, y_0)$ through f_* and if $i : X_{y_0} \to X$ is the inclusion, then $\alpha \cdot \xi = (i_*\alpha) \cdot \xi$ $(\alpha \in \pi_1(X_{y_0}, x_0), \xi \in \pi_n(X_{y_0}, x_0))$. Since the base points will play no further role, drop them from the notation.

$(n = 1)$ To see that $\pi_1(X_{y_0})$ is nilpotent, consider the short exact sequence associated with the exact sequence $\pi_2(Y) \xrightarrow{\partial} \pi_1(X_{y_0}) \xrightarrow{i_*} \pi_1(X)$, noting that $\operatorname{im}\partial$ is contained in the center of $\pi_1(X_{y_0})$.

$(n > 1)$ There is an exact sequence $\pi_{n+1}(Y) \xrightarrow{\partial} \pi_n(X_{y_0}) \xrightarrow{i_*} \pi_n(X)$ and by assumption, $\exists\, d : (I[\pi_1(X)])^d \cdot \pi_n(X) = 0$. Claim: $(I[\pi_1(X_{y_0})])^{d+1} \cdot \pi_n(X_{y_0}) = 0$. For let $\alpha \in (I[\pi_1(X_{y_0})])^d$, $\xi \in \pi_n(X_{y_0}) : i_*(\alpha \cdot \xi) = i_*\alpha \cdot i_*\xi = 0 \Rightarrow \alpha \cdot \xi = \partial\eta$ $(\eta \in \pi_{n+1}(Y))$. And: $\forall\, \beta \in \pi_1(X_{y_0})$, $(i_*\beta - 1) \cdot \eta = (f_*i_*\beta - 1) \cdot \eta = 0$, so $0 = \partial((i_*\beta - 1) \cdot \eta) = (i_*\beta - 1) \cdot \partial\eta = ((\beta - 1)\alpha) \cdot \xi$. Hence the claim.]

Application: Let X and Y be pointed path connected spaces. Assume: X is nilpotent—then for every pointed continuous function $f : X \to Y$, the path components of the mapping fiber E_f of f are nilpotent.

EXAMPLE Let (K, k_0) be a pointed connected CW complex. Assume: K is finite—then for any pointed path connected space (X, x_0), the path components of $C(K, k_0; X, x_0)$ are nilpotent. In particular, the fundamental group of the path component of the constant map $K \to x_0$ is nilpotent, thus $[K, k_0; \Omega X, j(x_0)]$ is a nilpotent group. Observe that the base points play a role here: $[\mathbf{S}^1, \Omega\mathbf{P}^2(\mathbf{R})]$ is a group but it is not nilpotent.

FACT Let $f : X \to B$ be a Hurewicz fibration. Given $\Phi' \in C(B', B)$, define X' by the pullback square

$$\begin{tikzcd} X' \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ B' \arrow[r, "\Phi'"'] & B \end{tikzcd}$$

. Assume: $\begin{cases} X \\ B \end{cases}$ & B' are nilpotent—then the path components of X' are nilpotent.

[Work with the Mayer-Vietoris sequence (cf. p. 4–37).]

EXAMPLE The preceding result implies that nilpotency behaves well with respect to pullbacks but the situation for pushouts is not as satisfactory since nilpotency is not ordinarily inherited (consider $\mathbf{S}^1 \vee \mathbf{S}^2$). For example, suppose that $f : X \to Y$ is a continuous function, where X and Y are nonempty path connected CW spaces. Assume: Y is nilpotent—then Rao[†] has shown that the mapping cone C_f of f is nilpotent iff one of the following conditions is satisfied: (i) $f_* : \pi_1(X) \to \pi_1(Y)$ is surjective; (ii) $\forall\, q > 0$, $H_q(X) = 0$; (iii) $\exists$ a prime p such that $\pi_1(C_f)$ is a finite p-group and $\forall\, q > 0$, $H_q(X)$ is a p-group of finite exponent. Example: If $f : X \to Y$ is a closed cofibration, then under (i), (ii), or (iii), $Y/f(X)$ is nilpotent (cf. p. 3–24). Moreover, under (ii), the projection $Y \to Y/f(X)$ is a homology equivalence (cf. p. 3–8), hence by Dror's Whitehead theorem is a homotopy equivalence.

Let $\begin{cases} X \\ Y \end{cases}$ be pointed connected CW spaces. Suppose that $f : X \to Y$ is a pointed continuous function—then f is said to admit a principal refinement of order n if f can be written as a composite $X \xrightarrow{\Lambda} W_N \xrightarrow{q_N} W_{N-1} \to \cdots \to W_1 \xrightarrow{q_1} W_0 = Y$, where Λ is a pointed homotopy equivalence and each $q_i : W_i \to W_{i-1}$ is a pointed Hurewicz fibration for which there is an abelian group π_i and a pointed continuous function $\Phi_{i-1} : W_{i-1} \to K(\pi_i, n+1)$ such that the diagram

$$\begin{tikzcd} W_i \arrow[r] \arrow[d, "q_i"'] & \Theta K(\pi_i, n+1) \arrow[d] \\ W_{i-1} \arrow[r, "\Phi_{i-1}"'] & K(\pi_i, n+1) \end{tikzcd}$$

is a pullback square.

[Note: W_i is a pointed connected CW space homeomorphic to $E_{\Phi_{i-1}}$ (parameter reversal).]

Example: If X is a pointed abelian CW space, then $\forall\, n$, the arrow $f_n : X[n] \to X[n-1]$ admits a principal refinement of order n:

$$\begin{tikzcd} & W[n] \arrow[d] \\ X[n] \arrow[ru] \arrow[r] & X[n-1] \end{tikzcd}$$

(cf. p. 5–42), with $N = 1$.

[†] *Proc. Amer. Math. Soc.* **87** (1983), 335–341.

EXAMPLE (Central Extensions) Let π and G be groups, where π is abelian—then the isomorphism classes of central extensions $1 \rightarrow \pi \rightarrow \Pi \rightarrow G \rightarrow 1$ of π by G are in a one-to-one correspondence with the elements of $H^2(G, 1; \pi)$ or still, with the elements of $[K(G,1), K(\pi, 2)]$. Therefore G is nilpotent iff the constant map $K(G,1) \rightarrow *$ admits a principal refinement of order 1.

[Any nilpotent G generates a finite sequence of central extensions $1 \rightarrow \Gamma^i(G)/\Gamma^{i+1}(G) \rightarrow G/\Gamma^{i+1}(G) \rightarrow G/\Gamma^i(G) \rightarrow 1$.]

Let X be a pointed connected CW space—then, in view of the preceding example, the arrow $f_1 : X[1] \rightarrow X[0]$ admits a principal refinement of order 1 iff $\pi_1(X)$ is nilpotent.

PROPOSITION 16 Let X be a pointed connected CW space. Fix $n > 1$—then the arrow $f_n : X[n] \rightarrow X[n-1]$ admits a principal refinement of order n iff $\pi_1(X)$ operates nilpotently on $\pi_n(X)$.

[Necessity: Suppose that f_n factors as a composite $X[n] \xrightarrow{\Lambda} W_N \xrightarrow{q_N} W_{N-1} \rightarrow \cdots \rightarrow W_1 \xrightarrow{q_1} W_0 = X[n-1]$, where Λ and the q_i are as in the definition. Obviously, $\pi_1(X) \approx \pi_1(W_i)$ for all i. Since $\pi_n(W_0) = \pi_n(X[n-1]) = 0$, $\pi_1(X)$ operates nilpotently on $\pi_n(W_0)$. Claim: $\pi_1(X)$ operates nilpotently on $\pi_n(W_1)$. Thus let $\overline{W_0}$ be the mapping track of Φ_0 and define $\overline{W_1}$ by the pullback square
$$\begin{tikzcd} \overline{W_1} \arrow[r] \arrow[d] & \Theta K(\pi_1, n+1) \arrow[d] \\ \overline{W_0} \arrow[r] & K(\pi_1, n+1) \end{tikzcd}$$
—then there is a pointed homotopy equivalence $W_1 \rightarrow \overline{W_1}$ and, from the proof of the "$n > 1$" part of Proposition 15, $\pi_1(X)$ operates nilpotently on $\pi_n(\overline{W_1})$. Iterate to conclude that $\pi_1(X)$ operates nilpotently on $\pi_n(W_N) \approx \pi_n(X)$.

Sufficiency: One can copy the argument employed in the abelian case to construct the Postnikov invariant (cf. p. 5–42). At the first stage, the only difference is that after replacing n by $n-1$, the coefficient group for cohomology is not $\pi_n(X)$ but $\pi_n(X)_{\pi_1(X)} = H_0(\pi_1(X); \pi_n(X))$. Because the initial lifting
$$\begin{tikzcd} & W_1 \arrow[d, "q_1"] \\ X[n] \arrow[ur, "\Lambda_1"] \arrow[r, "f_n"'] & X[n-1] \end{tikzcd}$$
of f_n is a pointed homotopy equivalence iff $I[\pi_1(X)] \cdot \pi_n(X) = 0$, it is in general necessary to repeat the procedure, which will then terminate after finitely many steps.]

Application: Let X be a pointed connected CW space—then X is nilpotent iff $\forall\, n$, the arrow $f_n : X[n] \rightarrow X[n-1]$ admits a principal refinement of order n.

[Note: If X is nilpotent and if $\chi_n : \pi_1(X) \rightarrow \operatorname{Aut} \pi_n(X)$ is the homomorphism corresponding to the action of $\pi_1(X)$ on $\pi_n(X)$, then a choice for the abelian groups figuring in the principal refinement of the arrow $X[n] \rightarrow X[n-1]$ are the $\Gamma^i_{\chi_n}(\pi_n(X))/\Gamma^{i+1}_{\chi_n}(\pi_n(X))$.]

EXAMPLE Let K be a finite CW complex—then for any pointed nilpotent CW space X, the path components of $C(K, X)$ are nilpotent.

[Bearing in mind §4, Proposition 5, use Proposition 15 and induction to show that $\forall\ n$, the path components of $C(K, X[n])$ are nilpotent.]

EXAMPLE Let (K, k_0) be a pointed CW complex. Assume: K is finite—then for any pointed nilpotent CW space (X, x_0), the path components of $C(K, k_0; X, x_0)$ are nilpotent. Indeed, $C(K, k_0; X, x_0)$ $= C(K_0, k_0; X, x_0) \times C(K_1, X) \times \cdots \times C(K_n, X)$, where $K_0, K_1, \ldots, K_n$ are the path components of K and $k_0 \in K_0$.

NILPOTENT OBSTRUCTION THEOREM Let (X, A) be a relative CW complex; let Y be a pointed nilpotent CW space. Suppose that $\forall\ n > 0\ \&\ \forall\ i \geq 0$, $H^{n+1}(X, A; \Gamma^i_{\chi_n}(\pi_n(Y))/\Gamma^{i+1}_{\chi_n}(\pi_n(Y))) =$ 0—then every $f \in C(A, Y)$ admits an extension $F \in C(X, Y)$, any two such being homotopic rel A provided that $\forall\ n > 0\ \&\ \forall\ i \geq 0$, $H^n(X, A; \Gamma^i_{\chi_n}(\pi_n(Y))/\Gamma^{i+1}_{\chi_n}(\pi_n(Y))) = 0$.

PROPOSITION 17 Let X be a pointed connected CW space, $\widetilde{X}$ its universal covering space. Assume: $\pi_1(X)$ is nilpotent—then X is nilpotent iff $\forall\ n \geq 1$, $\pi_1(X)$ operates nilpotently on $H_n(\widetilde{X})$.

[$\widetilde{X}$ exists and is a pointed connected CW space (cf. Proposition 5).

Necessity: Consider the Postnikov tower of $\widetilde{X}$, so $\widetilde{p}_n : P_n\widetilde{X} \to P_{n-1}\widetilde{X}$. Suppose inductively that $\pi_1(X)$ operates nilpotently on the homology of $P_{n-1}\widetilde{X}$. Since X is nilpotent, the $H_q(\pi_n(X), n)$ are nilpotent $\pi_1(X)$-modules (cf. p. 5–56), i.e., $\pi_1(X)$ operates nilpotently on the homology of the mapping fiber of $\widetilde{p}_n$. Therefore, by the universal coefficient theorem, the $E^2_{p,q} \approx H_p(P_{n-1}\widetilde{X}; H_q(\pi_n(X), n))$ in the fibration spectral sequence of $\widetilde{p}_n$ are nilpotent $\pi_1(X)$-modules, thus the same is true of the $H_i(P_n\widetilde{X})$. But the arrow $\widetilde{X} \to P_n\widetilde{X}$ induces an isomorphism of $\pi_1(X)$-modules $H_i(\widetilde{X}) \to H_i(P_n\widetilde{X})$ for $i \leq n$.

Sufficiency: Introduce the Whitehead tower of $\widetilde{X}$ and argue as above.]

PROPOSITION 18 Let X be a pointed connected CW space. Assume: X is nilpotent—then the $\pi_q(X)$ are finitely generated $\forall\ q$ iff the $H_q(X)$ are finitely generated $\forall\ q$.

[Suppose that the $\pi_q(X)$ are finitely generated $\forall\ q$—then, $\widetilde{X}$ being simply connected, hence abelian, the $H_q(\widetilde{X})$ are finitely generated $\forall\ q$ (cf. p. 5–51). On the other hand, according to Proposition 17, $\pi_1(X)$ operates nilpotently on the $H_q(\widetilde{X})$. Consequently, the $H_p(\pi_1(X); H_q(\widetilde{X}))$ are finitely generated (cf. p. 5–56). However, these terms are precisely the $E^2_{p,q}$ in the spectral sequence of the covering projection $\widetilde{X} \to X$ (see below), so $\forall\ i$, $H_i(X)$ is finitely generated.

Suppose that the $H_q(X)$ are finitely generated $\forall\ q$—then, since $\pi_1(X)/[\pi_1(X), \pi_1(X)] \approx H_1(X)$, the nilpotent group $\pi_1(X)$ is finitely generated (cf. p. 5–54). As for the $\pi_q(X)$ $(q > 1)$, their finite generation will follow if it can be shown that the $H_q(\widetilde{X})$ are finitely generated (cf. p. 5–44). Proceeding by contradiction, fix an i_0 such that $H_{i_0}(\widetilde{X})$ is not finitely generated and take i_0 minimal. The $E^2_{p,q} \approx H_p(\pi_1(X); H_q(\widetilde{X}))$ are finitely generated if $q < i_0$ but $E^2_{0,i_0} \approx H_0(\pi_1(X); H_{i_0}(\widetilde{X}))$ is not finitely generated (cf. p. 5–56), thus E^∞_{0,i_0} is not finitely generated. Therefore $H_{i_0}(X)$ contains a subgroup which is not finitely generated.]

[Note: A finitely generated nilpotent group is finitely presented and its integral group ring is (left and right) noetherian. This said, it then follows that under the equivalent conditions of the proposition, X necessarily has the pointed homotopy type of a pointed CW complex with a finite n-skeleton $\forall\ n$ (Wall[†]).]

The spectral sequence $E^2_{p,q} \approx H_p(\pi_1(X); H_q(\widetilde{X})) \Rightarrow H_{p+q}(X)$ of the covering projection $\widetilde{X} \rightarrow X$ is an instance of a fibration spectral sequence. In fact, consider the inclusion $i : X \rightarrow X[1] = K(\pi_1(X), 1)$ and pass to its mapping track $W_i \rightarrow K(\pi_1(X), 1)$—then E_i has the same pointed homotopy type as $\widetilde{X}$. Moreover, $H_p(\pi_1(X); H_q(\widetilde{X})) \approx H_p(K(\pi_1(X), 1); \mathcal{H}_q(\widetilde{X}))$, where $\mathcal{H}_q(\widetilde{X})$ is the locally constant coefficient system on $K(\pi_1(X), 1)$ determined by $H_q(\widetilde{X})$ (cf. p. 5–34).

FACT Suppose that $\begin{cases} X \\ Y \end{cases}$ are pointed connected CW spaces. Let $f : X \rightarrow Y$ be a pointed Hurewicz fibration with $\pi_0(X_{y_0}) = *$—then $\pi_1(X)$ operates nilpotently on the $\pi_q(X_{y_0})\ \forall\ q$ iff X_{y_0} is nilpotent and $\pi_1(Y)$ operates nilpotently on the $H_q(X_{y_0})\ \forall\ q$.

EXAMPLE Suppose that $\begin{cases} X \\ Y \end{cases}$ are pointed connected CW spaces. Let $f : X \rightarrow Y$ be a pointed Hurewicz fibration with $\pi_0(X_{y_0}) = *$—then any two of the following conditions imply the third and the third implies that X_{y_0} is nilpotent: (i) X is nilpotent; (ii) Y is nilpotent; (iii) $\pi_1(X)$ operates nilpotently on the $\pi_q(X_{y_0})\ \forall\ q$. Assume now that $\pi_1(Y)$ operates nilpotently on the $H_q(X_{y_0})\ \forall\ q$. Claim: X is nilpotent iff both Y and X_{y_0} are nilpotent. For X nilpotent $\Rightarrow X_{y_0}$ nilpotent (cf. Proposition 15) $\Rightarrow \pi_1(X)$ operates nilpotently on the $\pi_q(X_{y_0})\ \forall\ q \Rightarrow Y$ nilpotent, and conversely.

HILTON-ROITBERG[‡] COMPARISON THEOREM Suppose that $\begin{cases} X \\ Y \end{cases}$ & $\begin{cases} X' \\ Y' \end{cases}$ are pointed connected CW spaces. Let $f : X \rightarrow Y$ and $f' : X' \rightarrow Y'$ be pointed Hurewicz fibrations such that E_f

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[†] *Ann. of Math.* **81** (1965), 56–69.

[‡] *Quart. J. Math.* **27** (1976), 433–444; see also Schön, *Quart. J. Math.* **32** (1981), 235–237.

and $E_{f'}$ are path connected and $\begin{cases} \pi_1(Y) \\ \pi_1(Y') \end{cases}$ operates nilpotently on the $\begin{cases} H_q(E_f) \\ H_q(E_{f'}) \end{cases}$ $\forall\ q$. Suppose there is a commutative diagram

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d] & Y \arrow[d] \\ X' \arrow[r, "f'"'] & Y' \end{tikzcd}$$

, where $\pi_1(Y) \approx \pi_1(Y')$ or $\pi_1(Y)$ & $\pi_1(Y')$ are nilpotent—then, assuming that all isomorphisms are induced, any two of the following conditions imply the third: (1) $\forall\ p, H_p(Y) \approx H_p(Y')$; (2) $\forall\ q, H_q(E_f) \approx H_q(E_{f'})$; (3) $\forall\ n, H_n(X) \approx H_n(X')$.

A nonempty path connected topological space X is said to be acyclic provided that $\forall\ q > 0$, $H_q(X) = 0$. So: X acyclic $\Rightarrow \pi = [\pi, \pi]$ and $H_1(\pi, 1) = 0 = H_2(\pi, 1)$ (cf. p. 5–35), where $\pi = \pi_1(X)$. Example: Every nilpotent acyclic space is homotopically trivial (quote Dror's Whitehead theorem).

EXAMPLE (Acyclic Groups) A group G is said to be acyclic if $\forall\ n > 0$, $H_n(G) = 0$ or, equivalently, if $K(G, 1)$ is an acyclic space. Nontrivial finite groups are never acyclic (Swan[†]). However, there are plenty of concretely defined infinite acyclic groups. A list of examples has been compiled by Harpe-McDuff[‡]. They include: (1) The symmetric group on an infinite set; (2) The group of invertible linear transformations of an infinite dimensional vector space; (3) The group of invertible bounded linear transformations of an infinite dimensional Hilbert space; (4) The automorphism group of the measure algebra of the unit interval; (5) The group of compactly supported homeomorphisms of $\mathbf{R}^n$.

FACT Let G be a group which is the colimit of subgroups G_n ($n \in \mathbf{N}$) with the property that $\forall\ n$, there exists a nontrivial $g_n \in G_{n+1}$ and a homomorphism $\phi_n : G_n \to \mathrm{Cen}_{G_{n+1}}(G_n)$ such that $\forall\ g \in G_n$, $g = [g_n, \phi_n(g)]$—then G is acyclic.

[It suffices to work with coefficients in an arbitrary field $\mathbf{k}$. Since $H_*(G; \mathbf{k}) \approx \operatorname{colim} H_*(G_n; \mathbf{k})$, one need only show that $\forall\ n \geq 1$ & $\forall\ N \geq 1$, the morphism $H_q(G_n; \mathbf{k}) \to H_q(G_{n+N}; \mathbf{k})$ induced by the inclusion $G_n \to G_{n+N}$ is trivial when $1 \leq q < 2^N$. For this, fix n and use induction on N. Recall that conjugation induces the identity on homology and apply the Künneth formula.]

[Note: It is clear that ϕ_n is injective ($\Rightarrow g_n \in G_{n+1} - G_n$). Observe too that it is not necessary to assume that $\phi_n(G_n)$ is contained in the centralizer of G_n in G_{n+1} as this is implied by the other condition. Proof: $\forall\ g, h \in G_n : [g_n, \phi_n(gh)] = [g_n, \phi_n(g)] \cdot [\phi_n(g), [g_n, \phi_n(h)]] \cdot [g_n, \phi_n(h)] \Rightarrow gh = g[\phi_n(g), h]h \Rightarrow e = [\phi_n(g), h]$.]

[†] *Proc. Amer. Math. Soc.* **11** (1960), 885–887.

[‡] *Comment. Math. Helv.* **58** (1983), 48–71; see also Berrick, In: *Group Theory*, K. Cheng and Y. Leong (ed.), Walter de Gruyter (1989), 253–266.

EXAMPLE Let $H_c(\mathbf{Q})$ be the set of bijections of $\mathbf{Q}$ that are the identity outside some finite interval. Given a group G, let $F_c(\mathbf{Q}, G)$ be the set of functions $\mathbf{Q} \rightarrow G$ that send all elements outside some finite interval to the identity. Both $H_c(\mathbf{Q})$ and $F_c(\mathbf{Q}, G)$ are groups and there is a homomorphism $\chi : H_c(\mathbf{Q}) \rightarrow \operatorname{Aut} F_c(\mathbf{Q}, G)$, viz. $\chi(\beta)\alpha(q) = \alpha(\beta^{-1}(q))$. The cone of G is the associated semidirect product: $\Gamma G = F_c(\mathbf{Q}, G) \rtimes_\chi H_c(\mathbf{Q})$. The assignment $\begin{cases} G \rightarrow \Gamma G \\ g \rightarrow \alpha_g \end{cases} : \alpha_g(q) = \begin{cases} g & (q = 0) \\ e & (q \neq 0) \end{cases}$ is a monomorphism of groups and ΓG is acyclic.

[Let $\Gamma G_n = \{(\alpha, \beta) : \operatorname{spt} \alpha \cup \operatorname{spt} \beta \subset [-n, n]\}$ and construct a homomorphism $\phi_n : \Gamma G_n \rightarrow \operatorname{Cen}_{\Gamma G_{n+1}}(\Gamma G_n)$ in terms of a bijection $\beta_n \in H_c(\mathbf{Q})$: $\operatorname{spt} \beta_n \subset [-n-1, n+1]$ & $\forall\, k : \beta_n^k[-n, n] \cap [-n, n] = \emptyset$.]

FACT Every group can be embedded in an acyclic simple group.

[By the above, every group can be embedded in an acyclic group. On the other hand, every group can be embedded in a simple group (Robinson[†]). So given G, there is a sequence $G \subset G_1 \subset G_2 \subset \cdots$, where G_n is acyclic if n is odd and simple if n is even. Consider $\bigcup\limits_n G_n$.]

Recall that a group G is said to be perfect if $G = [G, G]$. Examples: (1) Every acyclic group is perfect; (2) Every nonabelian simple group is perfect.

[Note: The fundamental group of an acyclic space is perfect.]

The homomorphic image of a perfect group is perfect. Therefore, if G is perfect and π is nilpotent, then G operates nilpotently on π iff G operates trivially on π (cf. p. 5–55). Proof: A perfect nilpotent group is trivial.

Every group G has a unique maximal perfect subgroup G_{per}, the perfect radical of G. The automorphisms of G stabilize G_{per}, thus G_{per} is normal.

(P_1) Let $f : G \rightarrow K$ be a homomorphism of groups—then $f(G_{\text{per}}) \subset K_{\text{per}}$.

(P_2) Let $f : G \rightarrow K$ be a homomorphism of groups, where $K_{\text{per}} = \{1\}$—then $G_{\text{per}} \subset \ker f$.

FACT A locally free group is acyclic iff it is perfect.

[Note: A group is said to be locally free if its finitely generated subgroups are free.]

LEMMA Let $f : G \rightarrow K$ be an epimorphism of groups. Put $N = \ker f$—then $f(G_{\text{per}}) = K_{\text{per}}$ provided that $\exists\, n : N^{(n)} \subset G_{\text{per}}$.

[Note: $N^{(n)}$ is the nth derived group of $N : N^{(0)} = N, N^{(i+1)} = [N^{(i)}, N^{(i)}]$. Obviously, $N^{(0)} \subset G_{\text{per}}$ if N is perfect and $N^{(1)} \subset G_{\text{per}}$ if N is central.]

[†] *Finiteness Conditions and Generalized Soluble Groups*, vol. I, Springer Verlag (1972), 144.

Application: Let N be a perfect normal subgroup of G—then the perfect radical of G/N is the quotient G_{per}/N, hence the perfect radical of G/N is trivial iff $N = G_{\text{per}}$.

EXAMPLE Let A be a ring with unit. Agreeing to employ the usual notation of algebraic K-theory, denote by $\mathbf{GL}(A)$ the infinite general linear group of A and write $\mathbf{E}(A)$ for the subgroup of $\mathbf{GL}(A)$ consisting of the elementary matrices—then, according to the Whitehead lemma, $\mathbf{E}(A) = [\mathbf{E}(A), \mathbf{E}(A)] = [\mathbf{GL}(A), \mathbf{GL}(A)]$, thus $\mathbf{E}(A)$ is the perfect radical of $\mathbf{GL}(A)$. Let now $\mathbf{ST}(A)$ be the Steinberg group of $A : \mathbf{ST}(A)$ is perfect and there is an epimorphism $\mathbf{ST}(A) \to \mathbf{E}(A)$ of groups whose kernel is the center of $\mathbf{ST}(A)$.

[Note: On occasion, it is necessary to consider rings which may not have a unit (pseudorings). Given a pseudoring A, let $\overline{A}$ be the set of all functions $X : \mathbf{N} \times \mathbf{N} \to A$ such that $\#\{(i,j) : X_{ij} \neq 0\} < \omega$—then $\overline{A}$ is again a pseudoring (matrix operations). The law of composition $X \star Y = X + Y + X \times Y$ equips $\overline{A}$ with the structure of a semigroup with unit. Definition: $\overline{\mathbf{GL}}(A)$ is the group of units of $(\overline{A}, \star)$. Therefore, using obvious notation, $\overline{\mathbf{E}}(A) = [\overline{\mathbf{E}}(A), \overline{\mathbf{E}}(A)] = [\overline{\mathbf{GL}}(A), \overline{\mathbf{GL}}(A)]$. Every bijection $\phi : \mathbf{N} \to \mathbf{N} \times \mathbf{N}$ defines an isomorphism of pseudorings: $\overline{\overline{\mathbf{A}}} \approx \overline{A}$, hence $\overline{\mathbf{GL}}(\overline{A}) \approx \overline{\mathbf{GL}}(A)$. In the event that A has a unit, the assignment $\begin{cases} \overline{\mathbf{GL}}(A) \to \mathbf{GL}(A) \\ X \to X + I \end{cases}$ is an isomorphism of groups ($\Rightarrow \overline{\mathbf{GL}}(\overline{A}) \approx \mathbf{GL}(A)$).]

EXAMPLE (Universal Central Extensions) Let G be a group—then a central extension $1 \to N \to U \to G \to 1$ is said to be universal if for any other central extension $1 \to \pi \to \Pi \to G \to 1$ there is a unique homomorphism
$$\begin{tikzcd} U \arrow[rr] \arrow[dr] & & \Pi \arrow[dl] \\ & G & \end{tikzcd}$$
over G. A central extension $1 \to N \to U \to G \to 1$ is universal iff $H_1(U) = 0 = H_2(U)$. On the other hand, a universal central extension $1 \to N \to U \to G \to 1$ exists iff G is perfect. To identify N in terms of G, use a portion of the fundamental exact sequence: $H_2(U) \to H_2(G) \to N/[U,N] \to H_1(U)$ or still, $0 \to H_2(G) \to N/[U,N] \to 0 \Rightarrow H_2(G) \approx N$. Example: Take $G = \mathbf{E}(A)$—then $H_1(\mathbf{ST}(A)) = 0 = H_2(\mathbf{ST}(A))$ and there is a universal central extension $1 \to H_2(\mathbf{E}(A)) \to \mathbf{ST}(A) \to \mathbf{E}(A) \to 1$.

EXAMPLE Let **ACYGR** be the full subcategory of **GR** whose objects are the acyclic groups—then Berrick[†] has defined a functor $\alpha : \mathbf{AB} \to \mathbf{ACYGR}$ such that $\forall\, G$, the center of αG is naturally isomorphic to G. The quotient $\beta G = \alpha G / \text{Cen}\, G$ is a perfect group and the central extension $1 \to G \to \alpha G \to \beta G \to 1$ is universal, so $G \approx H_2(\beta G)$.

[Note: By contrast, the cone construction defines a functor $\Gamma : \mathbf{GR} \to \mathbf{ACYGR}$.]

FACT Let $\begin{cases} G_1 \\ G_2 \end{cases}$ be groups—then the perfect radical of $G_1 \times G_2$ is $(G_1)_{\text{per}} \times (G_2)_{\text{per}}$.

[†] *J. Pure Appl. Algebra* **44** (1987), 35–43.

FACT Let $\begin{cases} G_1 \\ G_2 \end{cases}$ be groups with trivial perfect radicals—then the perfect radical of their free product $G_1 * G_2$ is trivial.

[A theorem of Kurosch says that any subgroup G of $G_1 * G_2$ has the form $F * (\underset{i}{\ast} G_i)$, where F is a free group and $\forall\ i$, G_i is isomorphic to a subgroup of either G_1 or G_2. Put $X = K(F,1) \vee \bigvee_i K(G_i, 1)$: $\pi_1(X) \approx G$. If G is perfect, then $0 = H_1(X) \approx H_1(F) \oplus \bigoplus_i H_1(G_i)$, and it follows that F and the G_i are perfect, hence trivial.]

Let $\begin{cases} X \\ Y \end{cases}$ be pointed connected CW spaces. Suppose that $f : X \to Y$ is a pointed continuous function—then f is said to be acyclic if its mapping fiber E_f is acyclic. For this, it is therefore necessary that $\pi_0(E_f) = *$.

[Note: Using the mapping cylinder M_f, write $f = r \circ i$ (cf. p. 3–21)—then $(M_f, i(x_0))$ is nondegenerate, thus $r : M_f \to Y$ is a pointed homotopy equivalence (cf. p. 3–35) which implies that the arrow $E_i \to E_{r \circ i} = E_f$ is a pointed homotopy equivalence (cf. p. 4–33). Conclusion: $f : X \to Y$ is acyclic iff $i : X \to M_f$ is acyclic.]

Observation: Suppose that $f : X \to Y$ is acyclic—then $f_* : \pi_1(X) \to \pi_1(Y)$ is surjective and its kernel is a perfect normal subgroup of $\pi_1(X)$.

[Inspect the exact sequence $\pi_2(Y) \to \pi_1(E_f) \to \pi_1(X) \to \pi_1(Y) \to \pi_0(E_f)$.]

PROPOSITION 19 Let $\begin{cases} X \\ Y \end{cases}$ be pointed connected CW spaces, $f : X \to Y$ a pointed continuous function—then f is a pointed homotopy equivalence iff f is acyclic and $f_* : \pi_1(X) \to \pi_1(Y)$ is an isomorphism.

[The necessity is clear. As for the sufficiency, the arrow $\pi_2(Y) \to \pi_1(E_f)$ is surjective, hence $\pi_1(E_f)$ is both abelian and perfect. But this means that $\pi_1(E_f)$ must be trivial, so, being a pointed connected CW space, E_f is contractible.]

Let P be a set of primes. Fix an abelian group G—then G is said to be P-primary if $\forall\ g \in G$, $\exists\ F \subset P\ (\#(F) < \omega)\ \&\ n \in \mathbf{N} : (\prod_{p \in F} p)^n g = 0\ (\prod_{\emptyset} = 1)$ and G is said to be uniquely P-divisible if $\forall\ g \in G, \forall\ p \in P, \exists!\ h \in G : ph = g$.

[Note: If P is empty, then the only P-primary abelian group is the trivial group and every abelian group is uniquely P-divisible.]

LEMMA Let $\mathcal{C}$ be a class of abelian groups containing 0. Assume: $\mathcal{C}$ is closed under the formation of direct sums and five term exact sequences, i.e., for any exact sequence $G_1 \to G_2 \to G_3 \to G_4 \to G_5$ of abelian groups: $\begin{cases} G_1, G_2 \\ G_4, G_5 \end{cases} \in \mathcal{C} \Rightarrow G_3 \in \mathcal{C}$—then there exists a set of primes P such that $\mathcal{C}$ is either the class of P-primary abelian groups or the class of uniquely P-divisible abelian groups.

[The hypotheses imply that $\mathcal{C}$ is colimit closed. Given a set P of primes, it follows that if $\mathbf{Z}/p\mathbf{Z} \in \mathcal{C}\ \forall\ p \in P$, then every P-primary abelian group is in $\mathcal{C}$ or if $\mathbf{Q} \in \mathcal{C}$ and $\mathbf{Z}/p\mathbf{Z} \in \mathcal{C}\ \forall\ p \notin P$, then every uniquely P-divisible abelian group is in $\mathcal{C}$. On the other hand, if some $G \in \mathcal{C}$ is not uniquely P-divisible, then $\mathbf{Z}/p\mathbf{Z} \in \mathcal{C}$ (consider $G \xrightarrow{p} G$) and if some $G \in \mathcal{C}$ is not torsion, then $\mathbf{Q} \in \mathcal{C}$ (consider $\mathbf{Q} \otimes G = \text{colim}(\cdots \rightarrow G \xrightarrow{n} G \rightarrow \cdots)$). To summarize: (1) If $\mathbf{Q} \notin \mathcal{C}$ and if $\mathbf{Z}/p\mathbf{Z} \in \mathcal{C}$ exactly for $p \in P$, then $\mathcal{C}$ consists of the P-primary abelian groups; (2) If $\mathbf{Q} \in \mathcal{C}$ and if $\mathbf{Z}/p\mathbf{Z} \in \mathcal{C}$ exactly for $p \notin P$, then $\mathcal{C}$ consists of the uniquely P-divisible abelian groups.]

Application: Fix abelian groups $\begin{cases} A \\ B \end{cases}$—then $A \otimes B = 0 = \text{Tor}(A, B)$ iff there exists a set P of primes such that one of the groups is P-primary and the other is uniquely P-divisible.

[Supposing that $A \otimes B = 0 = \text{Tor}(A, B)$, the class of abelian groups G for which $G \otimes B = 0 = \text{Tor}(G, B)$ satisfies the assumptions of the lemma.]

EXAMPLE Given a 2-sink $X \xrightarrow{p} B \xleftarrow{q} Y$, where $\begin{cases} X \\ Y \end{cases}$ & B are pointed connected CW spaces, form $X \square_B Y$ (cf. p. 4–25). Let $r : X \square_B Y \rightarrow B$ be the projection—then the following conditions are equivalent: (i) r is a pointed homotopy equivalence; (ii) E_r is acyclic; (iii) $\exists\ P$ such that one of $\begin{cases} \widetilde{H}_*(E_p) = \bigoplus\limits_i \widetilde{H}_i(E_p) \\ \widetilde{H}_*(E_q) = \bigoplus\limits_j \widetilde{H}_j(E_q) \end{cases}$ is P-primary and the other is uniquely P-divisible. To see this, recall that $E_r \approx E_p * E_q$ (cf. p. 4–32) and, on general grounds, $\widetilde{H}_{k+1}(E_p * E_q) \approx \bigoplus\limits_{i+j=k} \widetilde{H}_i(E_p) \otimes \widetilde{H}_j(E_q) \oplus \bigoplus\limits_{i+j=k-1} \text{Tor}(\widetilde{H}_i(E_p), \widetilde{H}_j(E_q))$. In particular: E_r acyclic $\Rightarrow 0 = \widetilde{H}_1(E_r) = \widetilde{H}_0(E_p) \otimes \widetilde{H}_0(E_q)$, so at least one of E_p and E_q is path connected, thus $E_p * E_q$ is simply connected (cf. p. 3–40) or still, E_r is contractible and r is a pointed homotopy equivalence. Therefore (i) and (ii) are equivalent. To check (ii) $\Leftrightarrow$ (iii), use the algebra developed above.

EXAMPLE Let $\begin{cases} X \\ Y \end{cases}$ be pointed connected CW spaces, $f : X \rightarrow Y$ a pointed continuous function. Denote by C_π the mapping cone of the pointed Hurewicz fibration $\pi : E_f \rightarrow X$—then, specializing the preceding example, the projection $C_\pi \rightarrow Y$ is a pointed homotopy equivalence iff $\exists\ P$ such that one of $\begin{cases} \widetilde{H}_*(E_f) = \bigoplus\limits_i \widetilde{H}_i(E_f) \\ \widetilde{H}_*(\Omega Y) = \bigoplus\limits_j \widetilde{H}_j(\Omega Y) \end{cases}$ is P-primary and the other is uniquely P-divisible. To illustrate the situation when P is the set of all primes, consider the short exact sequence $0 \rightarrow \mathbf{Z} \rightarrow \mathbf{Q} \rightarrow \mathbf{Q}/\mathbf{Z} \rightarrow 0$—then the mapping fiber of the arrow $K(\mathbf{Z}, n+1) \rightarrow K(\mathbf{Q}, n+1)$ is a $K(\mathbf{Q}/\mathbf{Z}, n)$ (cf. p. 5–29). Furthermore, $\Omega K(\mathbf{Q}, n+1) = K(\mathbf{Q}, n)$ and $\widetilde{H}_*(\mathbf{Q}, n)$ is a uniquely divisible abelian group (being a vector space over $\mathbf{Q}$), while $\widetilde{H}_*(\mathbf{Q}/\mathbf{Z}, n)$ is a torsion abelian group (cf. p. 7–9). When $P = \emptyset$, there are two possibilities: (1) $\widetilde{H}_*(E_f) = 0$; (2) $\widetilde{H}_*(\Omega Y) = 0$. In the first case, f is acyclic and in the second case, Y is contractible and $\pi : E_f \rightarrow X$ is a pointed homotopy equivalence. Consequently, if $\pi_1(Y) \neq 0$, then f is acyclic iff the projection $C_\pi \rightarrow Y$ is a pointed homotopy equivalence.

[Note: A priori, C_π is calculated in **TOP** but is viewed as an object in $\mathbf{TOP}_*$. As such, it has the same pointed homotopy type as the pointed mapping cone of π.]

FACT Suppose that $f : X \to Y$ is acyclic. Let Z be any pointed space—then the arrow $[Y, Z] \to [X, Z]$ is injective.

[The orbits of the action of $[\Sigma E_f, Z]$ on $[C_\pi, Z]$ are the fibers of the arrow $[C_\pi, Z] \to [X, Z]$ (cf. p. 3–33). But ΣE_f is contractible in $\mathbf{TOP}_*$, hence $[\Sigma E_f, Z]$ is the trivial group and, as noted above, one can replace C_π by Y.]

PROPOSITION 20 Let $\left\{\begin{matrix} X \\ Y \end{matrix}\right.$ be pointed connected CW spaces. Suppose that $f : X \to Y$ is a pointed continuous function with $\pi_0(E_f) = *$—then f is acyclic iff f is a homology equivalence and $\pi_1(Y)$ operates nilpotently on the $H_q(E_f)$ $\forall\, q$.

[Consider the commutative diagram

$$\begin{tikzcd} W_f \arrow[r] \arrow[d] & Y \arrow[d, equal] \\ Y \arrow[r, equal] & Y \end{tikzcd}$$

and apply the Hilton-Roitberg comparison theorem.]

EXAMPLE Take $X = \mathbf{S}^3/\mathbf{SL}(2,5), Y = \mathbf{S}^3$—then the arrow $X \to Y$ is an acyclic map (cf. p. 5–52).

FACT Let $\left\{\begin{matrix} X \\ Y \end{matrix}\right.$ be pointed connected CW spaces, $f : X \to Y$ a pointed continuous function. Denote by C_f its mapping cone—then f acyclic $\Rightarrow$ C_f contractible and C_f contractible $\Rightarrow$ f acyclic provided that $\pi_1(Y) = 0$.

[If C_f is contractible and Y is simply connected, then f is a homology equivalence (cf. p. 3–22) and $\pi_1(Y)$ operates trivially on the $H_q(E_f)$ $\forall\, q$, so Proposition 20 can be cited.]

FACT Let $\left\{\begin{matrix} X \\ Y \end{matrix}\right.$ be pointed connected CW spaces, $f : X \to Y$ a pointed continuous function. Assume: X is acyclic and $f_* : \pi_1(X) \to \pi_1(Y)$ is trivial—then f is nullhomotopic.

[Take X to be a pointed connected CW complex, consider a lifting $\widetilde{f} : X \to \widetilde{Y}$ of f, and show that $\widetilde{Y} \to C_{\widetilde{f}}$ is an acyclic map.]

[Note: It is a corollary that if X is acyclic and $\operatorname{Hom}(\pi_1(X, x_0), \pi_1(Y, y_0)) = *$, then $C(X, x_0; Y, y_0)$ is homotopically trivial.]

Application: Let X & $\left\{\begin{matrix} Y \\ Y' \end{matrix}\right.$ be pointed connected CW spaces. Suppose that $f : X \to Y$ & $f' : X \to Y'$ are pointed continuous functions with f acyclic—then there exists a pointed continuous function $g : Y \to Y'$ such that $g \circ f \simeq f'$ iff $\ker \pi_1(f) \subset \ker \pi_1(f')$.

[Note: Up to pointed homotopy, g is unique.]

PROPOSITION 21 Let $\left\{\begin{matrix} X \\ Y \end{matrix}\right.$ be pointed connected CW spaces. Suppose that $f : X \to Y$ is a pointed continuous function with $\pi_0(E_f) = *$—then f is a pointed homotopy equivalence iff f is a homology equivalence and $\pi_1(X)$ operates nilpotently on the $\pi_q(E_f)$ $\forall$ q.

[The stated condition on $\pi_1(X)$ implies that $\pi_1(Y)$ operates nilpotently on the $H_q(E_f)$ $\forall$ q (cf. p. 5–62), thus, by Proposition 20, E_f is acyclic. But E_f is also nilpotent. Therefore E_f is contractible and $f : X \to Y$ is a pointed homotopy equivalence.]

It will be convenient to insert here a technical addendum to the fibration spectral sequence.

Notation: A continuous function $f : X \to Y$ induces a functor $f^* : \mathbf{LCCS}_Y \to \mathbf{LCCS}_X$ or still, a functor $f^* : [(\Pi Y)^{\mathrm{OP}}, \mathbf{AB}] \to [(\Pi X)^{\mathrm{OP}}, \mathbf{AB}]$ (cf. §4, Proposition 25). If X is a subspace of Y and f is the inclusion, one writes $\mathcal{G}|X$ instead of $f^*\mathcal{G}$.

Let $f : X \to Y$ be a Hurewicz fibration, where $\left\{\begin{matrix} X \\ Y \end{matrix}\right.$ and the X_y are path connected. Fix a cofunctor $\mathcal{G} : \Pi Y \to \mathbf{AB}$—then $\forall$ $y \in Y$, the projection $X_y \to Y$ is inessential, hence $f^*\mathcal{G}|X_y$ is constant. So, $\forall$ $q \geq 0$, there is a cofunctor $\mathcal{H}_q(f;\mathcal{G}) : \Pi Y \to \mathbf{AB}$ that assigns to each $y \in Y$ the singular homology group $H_q(X_y; f^*\mathcal{G}|X_y)$ and the fibration spectral sequence assumes the form $E^2_{p,q} \approx H_p(Y; \mathcal{H}_q(f;\mathcal{G})) \Rightarrow H_{p+q}(X; f^*\mathcal{G})$.

[Note: A morphism $[\tau] : y_0 \to y_1$ determines a homotopy equivalence $X_{y_0} \to X_{y_1}$ (cf. p. 4–39) and an isomorphism $\mathcal{G}[\tau] : \mathcal{G}y_1 \to \mathcal{G}y_0$, thus $\mathcal{H}_q(f;\mathcal{G})[\tau]$ is the composite $H_q(X_{y_1}; \mathcal{G}y_1) \to H_q(X_{y_0}; \mathcal{G}y_1) \to H_q(X_{y_0}; \mathcal{G}y_0)$.]

PROPOSITION 22 Let $\left\{\begin{matrix} X \\ Y \end{matrix}\right.$ be pointed connected CW spaces, $f : X \to Y$ a pointed continuous function—then f is acyclic iff for every locally constant coefficient system $\mathcal{G}$ on Y, the induced map $f_* : H_*(X; f^*\mathcal{G}) \to H_*(Y; \mathcal{G})$ is an isomorphism.

[Upon passing to the mapping track, one can assume that f is a pointed Hurewicz fibration.

Necessity: $\forall$ $y \in Y$, X_y is acyclic, thus from the universal coefficient theorem, $\forall$ $q > 0$, $H_q(X_y; f^*\mathcal{G}|X_y) = 0$. Accordingly, the edge homomorphism $e_H : E^\infty_{p,0} \to E^2_{p,0}$ is an isomorphism, so $\forall$ $p \geq 0$, $H_p(X; f^*\mathcal{G}) \approx H_p(Y; \mathcal{G})$.

Sufficiency: The integral group ring $\mathbf{Z}[\pi_1(Y)]$ is a right $\pi_1(Y)$-module. Viewed as a locally constant coefficient system on Y, its homology is that of $\widetilde{Y}$. Form the pullback square

$$\begin{tikzcd} X \times_Y \widetilde{Y} \arrow[r, "f'"] \arrow[d] & \widetilde{Y} \arrow[d] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

—then $H_*(X \times_Y \widetilde{Y}) \approx H_*(X; f^*(\mathbf{Z}[\pi_1(Y)]))$ and $f'_* : H_*(X \times_Y$

$\widetilde{Y}) \to H_*(\widetilde{Y})$ is the composite $H_*(X \times_Y \widetilde{Y}) \to H_*(X; f^*(\mathbf{Z}[\pi_1(Y)])) \xrightarrow{f_*} H_*(Y; \mathbf{Z}[\pi_1(Y)]) \to H_*(\widetilde{Y})$. By hypothesis, f_* is an isomorphism, hence f'_* is too. Since $\widetilde{Y}$ is simply connected, $E_{f'}$ is path connected. Consider the commutative diagram

$$\begin{tikzcd} X \times_Y \widetilde{Y} \arrow[r, "f'"] \arrow[d, "f'"'] & \widetilde{Y} \arrow[d, "\text{id}"] \\ \widetilde{Y} \arrow[r, "\text{id}"'] & \widetilde{Y} \end{tikzcd}.$$

Owing to the Hilton-Roitberg comparison theorem, the projection $E_{f'} \to *$ is a homology equivalence. Therefore E_f is acyclic.]

Application: Let X, Y, Z be pointed connected CW spaces. Suppose that $\left\{\begin{array}{l} f : X \to Y \\ g : Y \to Z \end{array}\right.$ are pointed continuous functions. Assume: f is acyclic—then g is acyclic iff $g \circ f$ is acyclic.

FACT Let $X \xleftarrow{f} Z \xrightarrow{g} Y$ be a pointed 2-source, where $\left\{\begin{array}{l} X \\ Y \end{array}\right.$ & Z are pointed connected CW spaces. Consider the pushout square

$$\begin{tikzcd} Z \arrow[r, "g"] \arrow[d, "f"'] & Y \arrow[d, "\eta"] \\ X \arrow[r, "\xi"'] & P \end{tikzcd}.$$

Assume: f is a cofibration—then f (or g) acyclic $\Rightarrow$ η (or ξ) acyclic.

PLUS CONSTRUCTION Fix a pointed connected CW space X. Let N be a perfect normal subgroup of $\pi_1(X)$—then there exists a pointed connected CW space X_N^+ and an acyclic map $f_N^+ : X \to X_N^+$ such that $\ker \pi_1(f_N^+) = N$ ($\Rightarrow \pi_1(X_N^+) \approx \pi_1(X)/N$). Moreover, the pointed homotopy type of X_N^+ is unique, i.e., if $g_N^+ : X \to Y_N^+$ is acyclic and if $\ker \pi_1(g_N^+) = N$, then there is a pointed homotopy equivalence $\phi : X_N^+ \to Y_N^+$ such that $\phi \circ f_N^+ \simeq g_N^+$.

[Existence: We shall first deal with the case when $N = \pi_1(X)$. Thus let $\{\alpha\}$ be a set of generators for $\pi_1(X)$. Represent α by $f_\alpha : \mathbf{S}^1 \to X$ and put $X_1 = (\coprod_\alpha \mathbf{D}^2) \sqcup_f X$ ($f = \coprod f_\alpha$) to obtain a relative CW complex (X_1, X) with $\pi_1(X_1) = 0$ (cf. p. 5-37). Consider the exact sequence $H_2(X_1) \to H_2(X_1, X) \to H_1(X)$: (a) $\pi_2(X_1) \approx H_2(X_1)$; (b) $H_2(X_1, X)$ is free abelian on generators w_α, say; (c) $H_1(X) = 0$. Given α, choose a continuous function $g_\alpha : \mathbf{S}^2 \to X_1$ such that the homotopy class $[g_\alpha]$ maps to w_α under the composite $\pi_2(X_1) \to H_2(X_1) \to H_2(X_1, X)$. Put $X_N^+ = (\coprod_\alpha \mathbf{D}^3) \sqcup_g X_1$ ($g = \coprod_\alpha g_\alpha$)—then the pair (X_N^+, X_1) is a relative CW complex with $\pi_1(X_N^+) = 0$. The inclusion $X \to X_N^+$ is a closed cofibration. In addition, it is a homology equivalence (for $H_*(X_N^+, X) = 0$), hence is an acyclic map (cf. Proposition 20). Turning to the general case, let $\widetilde{X}_N$ be the covering space of X corresponding to N (so $\pi_1(\widetilde{X}_N) \approx N$). Apply the foregoing procedure to $\widetilde{X}_N$ to get an acyclic closed cofibration $\widetilde{f}_N^+ : \widetilde{X}_N \to \widetilde{X}_N^+$, where $\widetilde{X}_N^+$ is simply connected. Define

X_N^+ by the pushout square
$$\begin{tikzcd}\widetilde{X}_N \arrow[r, "\widetilde{f}_N^+"] \arrow[d] & \widetilde{X}_N^+ \arrow[d] \\ X \arrow[r, "f_N^+"'] & X_N^+\end{tikzcd}$$
. Thanks to Proposition 7, X_N^+ is a pointed connected CW space. And: f_N^+ is an acyclic closed cofibration (cf. p. 5–70). Finally, the Van Kampen theorem implies that $\pi_1(X_N^+) \approx \pi_1(X)/N$.

Uniqueness: Since $N = \begin{cases} \ker \pi_1(f_N^+) \\ \ker \pi_1(g_N^+) \end{cases}$, there exists a pointed continuous function $\phi : X_N^+ \to Y_N^+$ such that $\phi \circ f_N^+ \simeq g_N^+$ (cf. p. 5–68). But $\begin{cases} f_N^+ \\ g_N^+ \end{cases}$ acyclic $\Rightarrow \phi$ acyclic and $\phi_* : \pi_1(X_N^+) \to \pi_1(Y_N^+)$ is necessarily an isomorphism. Therefore ϕ is a pointed homotopy equivalence (cf. Proposition 19).

[Note: X_N^+ is called the plus construction with respect to N. Like an Eilenberg-MacLane space, X_N^+ is really a pointed homotopy type, thus, while a given representative may have a certain property, it need not be true that all representatives do. As for ϕ, if f_N^+ is an acyclic closed cofibration and if g_N^+ is another such, then matters can be arranged so that there is commutativity on the nose: $\phi \circ f_N^+ = g_N^+$. This in turn means that ϕ is a homotopy equivalence in $X\backslash\mathbf{TOP}$ (cf. §3, Proposition 13).]

One can interpret X_N^+ as a representing object of the functor on the homotopy category of pointed connected CW spaces which assigns to each Y the set of all $[f] \in [X,Y] : \ker \pi_1(f) \supset N$.

Different notation is used when $N = \pi_1(X)_{\text{per}}$, the perfect radical of $\pi_1(X)$: X_N^+ is replaced by X^+ and $f_N^+ : X \to X_N^+$ is replaced by $i^+ : X \to X^+$. Example: X acyclic $\Rightarrow X^+$ contractible.

[Note: The perfect radical of $\pi_1(X^+)$ is trivial (cf. p. 5–65).]

Examples: Let $\begin{cases} X \\ Y \end{cases}$ be pointed connected CW spaces—then (1) $X^+ \times Y^+$ is a model for $(X \times Y)^+$; (2) $X^+ \vee Y^+$ is a model for $(X \vee Y)^+$; (3) $X^+ \# Y^+$ is a model for $(X \# Y)^+$.

EXAMPLE (Homology Spheres) Fix $n > 1$. Suppose that X is a pointed connected CW space such that $\widetilde{H}_q(X) = \begin{cases} \mathbf{Z} \ (q = n) \\ 0 \ (q \neq n) \end{cases}$—then $\pi_1(X)$ is perfect and X^+ has the same pointed homotopy type as $\mathbf{S}^n$.

FACT Let X be a pointed connected CW space—then for any pointed acyclic CW space Z, the arrow $[Z, E_{i^+}] \to [Z, X]$ is bijective.

[Note: The central extension $1 \to \operatorname{im} \pi_2(X^+) \to \pi_1(E_{i^+}) \to \pi_1(X)_{\text{per}} \to 1$ is universal.]

Convention: Henceforth it will be assumed that $i^+ : X \to X^+$ is an acyclic closed cofibration.

LEMMA Let $\begin{cases} X \\ Y \end{cases}$ be pointed connected CW spaces. Suppose that $f : X \to Y$ is a pointed continuous function—then there is a pointed continuous function $f^+ : X^+ \to Y^+$ rendering the diagram

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d] & Y \arrow[d] \\ X^+ \arrow[r, "f^+"'] & Y^+ \end{tikzcd}$$

commutative, f^+ being unique up to pointed homotopy.

Application: Let $\begin{cases} X \\ Y \end{cases}$ be pointed connected CW spaces. Assume: X and Y have the same pointed homotopy type—then X^+ and Y^+ have the same pointed homotopy type.

PROPOSITION 23 Let X be a pointed connected CW space. Denote by $\widetilde{X}_N$ the covering space of X corresponding to N, where N is a normal subgroup of $\pi_1(X)$ containing $\pi_1(X)_{\text{per}}$—then $\widetilde{X}_N^+$ has the same pointed homotopy type as the covering space of X^+ corresponding to the normal subgroup $N/\pi_1(X)_{\text{per}}$ of $\pi_1(X^+) \approx \pi_1(X)/\pi_1(X)_{\text{per}}$.

[The pointed homotopy type of $\widetilde{X}_N$ can be calculated as the mapping fiber of the composite $X \to X[1] = K(\pi_1(X), 1) \to K(\pi_1(X)/N, 1)$. This arrow factors through X^+ and $\pi_1(X)/N \approx (\pi_1(X)/\pi_1(X)_{\text{per}})/(N/\pi_1(X)_{\text{per}})$.]

Notation: Given a group G, put $BG = K(G, 1)$.

EXAMPLE BG_{per} is the covering space of BG corresponding to G_{per}. There is an arrow $BG_{\text{per}}^+ \to BG^+$ and BG_{per}^+ "is" the universal covering space of BG^+.

EXAMPLE Let A be a ring with unit—then the fundamental group of the mapping fiber of $B\mathbf{GL}(A) \to B\mathbf{GL}(A)^+$ is isomorphic to $\mathbf{ST}(A)$.

PROPOSITION 24 Let $\begin{cases} X \\ Y \end{cases}$ be pointed connected CW spaces. Suppose that $f : X \to Y$ is a pointed continuous function with $\pi_0(E_f) = *$—then $\pi_0(E_{f^+}) = *$ and the perfect radical of $\pi_1(E_{f^+})$ is trivial.

[Note: It follows that there is a commutative triangle

$$\begin{tikzcd} E_f \arrow[r] \arrow[d] & E_f^+ \arrow[dl] \\ E_{f^+} & \end{tikzcd}$$

.]

FACT Let $\left\{\begin{matrix} X \\ Y \end{matrix}\right.$ be pointed connected CW spaces. Suppose that $f : X \to Y$ is a pointed continuous function with $\pi_0(E_f) = *$—then the arrow $E_f^+ \to E_{f^+}$ is a pointed homotopy equivalence if $\pi_1(Y)_{\text{per}}$ is trivial or if E_f^+ is nilpotent and $\pi_1(Y)_{\text{per}}$ operates nilpotently on the $H_q(E_f)$ $\forall\, q$.

[Note: $\pi_1(Y)_{\text{per}}$ operates nilpotently on the $H_q(E_f)$ $\forall\, q$ iff $\pi_1(Y)_{\text{per}}$ operates trivially on the $H_q(E_f)$ $\forall\, q$ (cf. p. 5–64).]

EXAMPLE (Central Extensions) Let π and G be groups, where π is abelian. Consider a central extension $1 \to \pi \to \Pi \to G \to 1$—then $B\pi$ can be identified with the mapping fiber of the arrow $B\Pi^+ \to BG^+$.

[Since π is abelian, $B\pi = B\pi^+$ and G $(= \pi_1(BG))$ operates trivially on π, hence operates trivially on the $H_q(B\pi)$ $\forall\, q$.]

EXAMPLE Let G be an abelian group—then there is a universal central extension $1 \to G \to \alpha G \to \beta G \to 1$ (cf. p. 5–65). Specializing the preceding example, the mapping fiber of the arrow $K(\alpha G, 1)^+ \to K(\beta G, 1)^+$ is a $K(G, 1)$ and $K(\beta G, 1)^+$ is a $K(G, 2)$.

[Recall that αG is acyclic, thus $K(\alpha G, 1)^+$ is contractible.]

PROPOSITION 25 Let $\left\{\begin{matrix} X \\ Y \end{matrix}\right.$ be pointed connected CW spaces. Suppose that $f : X \to Y$ is a pointed continuous function for which the normal closure of $f_*(\pi_1(X)_{\text{per}})$ is $\pi_1(Y)_{\text{per}}$—then the adjunction space $X^+ \sqcup_f Y$ represents Y^+.

[Since $i^+ : X \to X^+$ is an acyclic closed cofibration, the same is true of the inclusion $Y \to X^+ \sqcup_f Y$ (cf. p. 5–70). On the other hand, by Van Kampen, the fundamental group of $X^+ \sqcup_f Y$ is isomorphic to $\pi_1(Y)$ modulo the normal closure of $f_*(\pi_1(X)_{\text{per}})$, i.e., to $\pi_1(Y)/\pi_1(Y)_{\text{per}}$.]

EXAMPLE (Algebraic K-Theory) Let A be a ring with unit—then by definition, $K_0(A)$ is the Grothendieck group attached to the category of finitely generated projective A-modules and for $n \geq 1$, $K_n(A)$ is taken to be the homotopy group $\pi_n(B\mathbf{GL}(A)^+)$. While it is immediate that K_0 is a functor from **RG** to **AB**, the plus construction requires some choices, so to guarantee that K_n is a functor one has to fix the data. Thus first construct $B\mathbf{GL}(\mathbf{Z})^+$. This done, define $B\mathbf{GL}(A)^+$ by the pushout square

$$\begin{tikzcd} B\mathbf{GL}(\mathbf{Z}) \arrow[r] \arrow[d] & B\mathbf{GL}(A) \arrow[d] \\ B\mathbf{GL}(\mathbf{Z})^+ \arrow[r] & B\mathbf{GL}(A)^+ \end{tikzcd}$$

. Here, Proposition 25 comes in (the normal closure of $\text{im}(\mathbf{E}(\mathbf{Z}) \to \mathbf{E}(A))$ is $\mathbf{E}(A)$). Observe that the K_n preserve products: $K_n(A' \times A'') \approx K_n(A') \times K_n(A'')$.

$(n = 1)$ $K_1(A) = \pi_1(B\mathbf{GL}(A)^+) \approx \pi_1(B\mathbf{GL}(A))/\pi_1(B\mathbf{GL}(A))_{\text{per}} \approx \mathbf{GL}(A)/[\mathbf{GL}(A), \mathbf{GL}(A)] = H_1(\mathbf{GL}(A))$.

$(n = 2)$ $K_2(A) = \pi_2(B\mathbf{GL}(A)^+) \approx \pi_2(B\mathbf{E}(A)^+) \approx H_2(B\mathbf{E}(A)^+) \approx H_2(B\mathbf{E}(A)) = H_2(\mathbf{E}(A))$.

[Note: The central extension $1 \rightarrow K_2(A) \rightarrow \mathbf{ST}(A) \rightarrow \mathbf{E}(A) \rightarrow 1$ is universal (cf. p. 5–65) and $BK_2(A)$ can be identified with the mapping fiber of the arrow $B\,\mathbf{ST}(A)^+ \rightarrow B\mathbf{E}(A)^+$.]

$(n = 3)$ $K_3(A) = \pi_3(B\mathbf{GL}(A)^+) \approx \pi_3(B\mathbf{E}(A)^+) \approx \pi_3(B\,\mathbf{ST}(A)^+) \approx H_3(B\,\mathbf{ST}(A)^+) \approx H_3(B\,\mathbf{ST}(A)) = H_3(\mathbf{ST}(A))$.

There is no known homological interpretation of K_4 and beyond.

EXAMPLE (Relative Algebraic K-Theory) Let A be a ring with unit, $I \subset A$ a two sided ideal. Write $\widehat{\mathbf{GL}}(A/I)$ for the image of $\mathbf{GL}(A)$ in $\mathbf{GL}(A/I)$—then $\widehat{\mathbf{GL}}(A/I) \supset \mathbf{E}(A/I)$, thus $\widehat{\mathbf{GL}}(A/I)$ is normal and $G_{A,I} = \mathbf{GL}(A/I)/\widehat{\mathbf{GL}}(A/I)$ is abelian. Since $B\widehat{\mathbf{GL}}(A/I)^+$ can be identified with the mapping fiber of the arrow $B\mathbf{GL}(A/I)^+ \rightarrow BG^+_{A,I} (= BG_{A,I})$ (cf. p. 5–73), it follows that $\pi_n(B\widehat{\mathbf{GL}}(A/I)^+) \approx \pi_n(B\mathbf{GL}(A/I)^+)$ $(n > 1)$ but $\pi_1(B\widehat{\mathbf{GL}}(A/I)^+) \approx \operatorname{im}(K_1(A) \rightarrow K_1(A/I))$ and there is a short exact sequence $0 \rightarrow \pi_1(B\widehat{\mathbf{GL}}(A/I)^+) \rightarrow K_1(A/I) \rightarrow G_{A,I} \rightarrow 0$. If $\mathcal{K}(A,I)$ is the mapping fiber of the arrow $B\mathbf{GL}(A)^+ \rightarrow B\widehat{\mathbf{GL}}(A/I)^+$, then $\mathcal{K}(A,I)$ is path connected, so letting $K_n(A,I) = \pi_n(\mathcal{K}(A,I))$ $(n \geq 1)$, one obtains a functorial long exact sequence $\cdots \rightarrow K_{n+1}(A/I) \rightarrow K_n(A,I) \rightarrow K_n(A) \rightarrow K_n(A/I) \rightarrow \cdots \rightarrow K_1(A,I) \rightarrow K_1(A) \rightarrow K_1(A/I)$.

PROPOSITION 26 Let X be a pointed connected CW space. Put $\pi = \pi_1(X)$ and denote by $\widetilde{X}_{\text{per}}$ the mapping fiber of the composite $X \rightarrow K(\pi,1) \rightarrow K(\pi/\pi_{\text{per}},1)$. Assume: π/π_{per} is nilpotent and π/π_{per} operates nilpotently on the $H_q(\widetilde{X}_{\text{per}})$ $\forall$ q—then X^+ is nilpotent.

[Since $(\pi/\pi_{\text{per}})_{\text{per}}$ is trivial (cf. p. 5–65), $\widetilde{X}^+_{\text{per}}$ can be identified with the mapping fiber of the composite $X^+ \rightarrow K(\pi,1)^+ \rightarrow K(\pi/\pi_{\text{per}},1)^+$ (cf. p. 5–73). By construction, $\widetilde{X}^+_{\text{per}}$ is simply connected (cf. Proposition 23), hence nilpotent. But $K(\pi/\pi_{\text{per}},1)^+ = K(\pi/\pi_{\text{per}},1)$ is also nilpotent. Therefore, bearing in mind that the inclusion $\widetilde{X}_{\text{per}} \rightarrow \widetilde{X}^+_{\text{per}}$ is a homology equivalence, it follows that X^+ is nilpotent (cf. p. 5–62).]

FACT Let G be a group. Fix $\phi \in \operatorname{Aut} G$. Assume: Given $g_1, \ldots, g_n \in G$, $\exists$ $g \in G : \phi(g_i) = g g_i g^{-1}$ $(1 \leq i \leq n)$—then $\phi_* : H_*(G) \rightarrow H_*(G)$ is the identity.

Application: Let G be a group. Let K be a normal subgroup of G which is the colimit of subgroups K_n $(n \in \mathbf{N})$ such that $\forall$ n, $G = K \cdot \operatorname{Cen}_G(K_n)$—then G operates trivially on $H_*(K)$.

EXAMPLE Let A be a ring with unit—then $B\mathbf{GL}(A)^+$ is nilpotent. To see this, consider the short exact sequence $1 \rightarrow \mathbf{E}(A) \rightarrow \mathbf{GL}(A) \rightarrow \mathbf{GL}(A)/\mathbf{E}(A) \rightarrow 1$. Here, $\mathbf{E}(A) = \mathbf{GL}(A)_{\text{per}}$ and $B\mathbf{E}(A)$ is the mapping fiber of the arrow $B\mathbf{GL}(A) \rightarrow K(\mathbf{GL}(A)/\mathbf{E}(A),1)$. The quotient $\mathbf{GL}(A)/\mathbf{E}(A)$ is abelian, hence nilpotent. On the other hand, if $\mathbf{E}(n,A)$ is the subgroup of $\mathbf{GL}(n,A)$ consisting of the elementary matrices, then $\mathbf{E}(A) = \operatorname{colim} \mathbf{E}(n,A)$ and $\forall$ n, $\mathbf{GL}(A) = \mathbf{E}(A) \cdot \operatorname{Cen}_{\mathbf{GL}(A)}(\mathbf{E}(n,A))$, so $\mathbf{GL}(A)$ operates trivially on $H_*(\mathbf{E}(A))$. That $B\mathbf{GL}(A)^+$ is nilpotent is therefore a consequence of Proposition 26.

[Note: More is true. Thus define a homomorphism $\oplus : \mathbf{GL}(A) \times \mathbf{GL}(A) \rightarrow \mathbf{GL}(A)$ by $(X,Y) \rightarrow X \oplus Y$, where $(X \oplus Y)_{ij} = \begin{cases} x_{kl} & (i = 2k-1, j = 2l-1) \\ y_{kl} & (i = 2k, j = 2l) \end{cases}$ & 0 otherwise—then Loday† has shown that the composite $B\mathbf{GL}(A)^+ \times B\mathbf{GL}(A)^+ \rightarrow B(\mathbf{GL}(A) \times \mathbf{GL}(A))^+ \rightarrow B\mathbf{GL}(A)^+$ serves to equip $B\mathbf{GL}(A)^+$ with the structure of a homotopy commutative H group. In particular: $B\mathbf{GL}(A)^+$ is abelian.]

EXAMPLE Let A be a ring with unit. Write $\mathbf{UT}(A)$ for the ring of upper triangular 2-by-2 matrices with entries in A—then the projection $p : \mathbf{UT}(A) \rightarrow A \times A$ ($p \begin{pmatrix} a_1 & a \\ 0 & a_2 \end{pmatrix} = (a_1, a_2)$) induces an epimorphism $p : \mathbf{GL}(\mathbf{UT}(A)) \rightarrow \mathbf{GL}(A \times A)$. Its kernel is not perfect, therefore $Bp : B\mathbf{GL}(\mathbf{UT}(A)) \rightarrow B\mathbf{GL}(A \times A)$ is not acyclic. Nevertheless, Bp is a homology equivalence. Consider now the commutative diagram

$$\begin{tikzcd} B\mathbf{GL}(\mathbf{UT}(A)) \arrow[r] \arrow[d, "Bp"'] & B\mathbf{GL}(\mathbf{UT}(A))^+ \arrow[d, "Bp^+"] \\ B\mathbf{GL}(A \times A) \arrow[r] & B\mathbf{GL}(A \times A)^+ \end{tikzcd}$$

. Since the horizontal arrows are homology equivalences, Bp^+ is a pointed homotopy equivalence, so $\forall\, n \geq 1$, $K_n(\mathbf{UT}(A)) \approx K_n(A) \times K_n(A)$.

[Note: Bp^+ is acyclic (cf. Proposition 19), thus the composite $B\mathbf{GL}(\mathbf{UT}(A)) \xrightarrow{Bp} B\mathbf{GL}(A \times A) \rightarrow B\mathbf{GL}(A \times A)^+$ is acyclic even though Bp is not.]

FACT Let G be a group. Assume:

($\oplus$) There is a homomorphism $\oplus : G \times G \rightarrow G$ such that for any finite set $\{g_1, \ldots, g_n\} \subset G$, $\exists \begin{cases} u \\ v \end{cases} \in G : \begin{cases} u(g_i \oplus e)u^{-1} = g_i \\ v(e \oplus g_i)v^{-1} = g_i \end{cases} (i = 1, \ldots, n)$.

($\wp$) There is a homomorphism $\wp : G \rightarrow G$ such that for any finite set $\{g_1, \ldots, g_n\} \subset G$, $\exists\, \rho \in G : \rho(g_i \oplus \wp g_i)\rho^{-1} = g_i$ $(i = 1, \ldots, n)$.

Then G is acyclic.

[Fix a field of coefficients $\mathbf{k}$. Let $\Delta : G \rightarrow G \times G$ be the diagonal map—then $\wp$ and $\oplus \circ (\mathrm{id} \times \wp) \circ \Delta$ operate in the same way on homology. Since $H_1(G; \mathbf{k}) = 0$, one can take $n > 1$ and assume inductively that $H_q(G; \mathbf{k}) = 0$ $(0 < q < n)$. Let $x \in H_n(G; \mathbf{k}) : \wp_*(x) = (\oplus \circ (\mathrm{id} \times \wp) \circ \Delta)_*(x) = \oplus_*(x \otimes 1 + 1 \otimes \wp_*(x)) = x + \wp_*(x) \Rightarrow x = 0$.]

EXAMPLE (Delooping Algebraic K-Theory) Let A be a ring with unit. Denote by ΓA the set of all functions $X : \mathbf{N} \times \mathbf{N} \rightarrow A$ such that $\forall\, i, \#\{j : X_{ij} \neq 0\} < \omega$ and $\forall\, j, \#\{i : X_{ij} \neq 0\} < \omega$—then ΓA is a ring with unit containing $\overline{A}$ as a two sided ideal. ΓA is called the cone of A and the quotient $\Sigma A = \Gamma A / \overline{A}$ is called the suspension of A. Define a homomorphism $\oplus : \Gamma A \times \Gamma A \rightarrow \Gamma A$ by $(X,Y) \rightarrow X \oplus Y$, where $(X \oplus Y)_{ij} = \begin{cases} x_{kl} & (i = 2k-1, j = 2l-1) \\ y_{kl} & (i = 2k, j = 2l) \end{cases}$ & 0 otherwise and define a homomorphism $\wp : \Gamma A \rightarrow \Gamma A$ by $\wp(X)_{ij} = X_{mn}$ if $\begin{cases} i = 2^k(2m-1) \\ j = 2^k(2n-1) \end{cases}$ for some k, m, n & 0 otherwise. Evidently, $X \oplus \wp X = \wp X$ for all

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† *Ann. Sci. École Norm. Sup.* **9** (1976), 309–377.

$X \in \Gamma A$ and $\begin{cases} \oplus \\ \wp \end{cases}$ induce homomorphisms $\oplus : \mathbf{GL}(\Gamma A) \times \mathbf{GL}(\Gamma A) \to \mathbf{GL}(\Gamma A)$ & $\wp : \mathbf{GL}(\Gamma A) \to \mathbf{GL}(\Gamma A)$ satisfying the preceding assumptions. Therefore $\mathbf{GL}(\Gamma A)$ is acyclic, so $\mathbf{GL}(\Gamma A) = \mathbf{E}(\Gamma A)$. Taking into account the exact sequences $1 \to \overline{\mathbf{GL}}(\overline{A}) \to \overline{\mathbf{GL}}(\Gamma A) \to \overline{\mathbf{GL}}(\Sigma A), \overline{\mathbf{E}}(\Gamma A) \to \overline{\mathbf{E}}(\Sigma A) \to 1$, it follows that there is an exact sequence $1 \to \mathbf{GL}(A) \to \mathbf{GL}(\Gamma A) \to \mathbf{E}(\Sigma A) \to 1$. The mapping fiber of the arrow $B\mathbf{GL}(\Gamma A)^+ \to B\mathbf{E}(\Sigma A)^+$ is $B\mathbf{GL}(A)^+$. Since $B\mathbf{GL}(\Gamma A)^+$ is contractible, this means that in $\mathbf{HTOP}_*$, $B\mathbf{GL}(A)^+ \approx \Omega B\mathbf{E}(\Sigma A)^+$. Consequently, $\forall\ n \geq 1, K_n(A) = \pi_n(B\mathbf{GL}(A)^+) \approx \pi_n(\Omega B\mathbf{E}(\Sigma A)^+) \approx \pi_{n+1}(B\mathbf{E}(\Sigma A)^+) \approx \pi_{n+1}(B\mathbf{GL}(\Sigma A)^+) = K_{n+1}(\Sigma A)$. It is also true that $K_0(A) \approx K_1(\Sigma A)$ (Farrell-Wagoner[†]). Let $\Omega_0 B\mathbf{GL}(\Sigma A)^+$ be the path component of $\Omega B\mathbf{GL}(\Sigma A)^+$ containing the constant loop—then in $\mathbf{HTOP}_*$, $\Omega B\mathbf{E}(\Sigma A)^+ \approx \Omega_0 B\mathbf{GL}(\Sigma A)^+$ (cf. p. 5–72). But $\pi_1(B\mathbf{GL}(\Sigma A)^+) = K_1(\Sigma A)$, hence $K_0(A) \times B\mathbf{GL}(A)^+ \approx \Omega B\mathbf{GL}(\Sigma A)^+$.

[Note: Additional information can be found in Wagoner[‡]. There it is shown that by fixing the data, the pointed homotopy equivalence $K_0(A) \times B\mathbf{GL}(A)^+ \approx \Omega B\mathbf{GL}(\Sigma A)^+$ can be made natural, i.e., if $f : A' \to A''$ is a morphism of rings, then the diagram

$$\begin{tikzcd} K_0(A') \times B\mathbf{GL}(A')^+ \arrow[r, phantom, "\approx"] \arrow[d] & \Omega B\mathbf{GL}(\Sigma A')^+ \arrow[d] \\ K_0(A'') \times B\mathbf{GL}(A'')^+ \arrow[r, phantom, "\approx"] & \Omega B\mathbf{GL}(\Sigma A'')^+ \end{tikzcd}$$

is pointed homotopy commutative.]

EXAMPLE Let A be a ring with unit—then $\Sigma\mathbf{UT}(A) \approx \mathbf{UT}(\Sigma A) \Rightarrow K_0(\mathbf{UT}(A)) \approx K_1(\Sigma\mathbf{UT}(A)) \approx K_1(\mathbf{UT}(\Sigma A)) \approx K_1(\Sigma A) \times K_1(\Sigma A) \approx K_0(A) \times K_0(A)$.

KAN-THURSTON THEOREM Let X be a pointed connected CW space—then there exists a group G_X and an acyclic map $\kappa_X : K(G_X, 1) \to X$.

[Because of Proposition 2, one can take for X a pointed connected CW complex with all characteristic maps embeddings. Moreover, it will be enough to deal with finite X, the transition to infinite X being straightforward (given the naturality built into the argument). Since $\dim X \leq 1 \Rightarrow X$ aspherical, we shall assume that $\dim X > 1$ and proceed by induction on $\#(\mathcal{E})$, supposing that the construction has been carried out in such a way that if X_0 is a connected subcomplex of X, then $K(G_{X_0}, 1) = \kappa_X^{-1}(X_0)$ and $G_{X_0} \to G_X$ is injective. To execute the inductive step, consider the pushout square

$$\begin{tikzcd} \mathbf{S}^{n-1} \arrow[r] \arrow[d] & X \arrow[d] \\ \mathbf{D}^n \arrow[r] & Y \end{tikzcd}$$

$(n \geq 2)$, where the horizontal arrows are embeddings and $\begin{cases} X_0 = \operatorname{im}(\mathbf{S}^{n-1} \to X) \\ Y_0 = \operatorname{im}(\mathbf{D}^n \to Y) \end{cases}$ are connected

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[†] *Comment. Math. Helv.* **47** (1972), 474–501.

[‡] *Topology* **11** (1972), 349–370.

subcomplexes of $\begin{cases} X \\ Y \end{cases}$, so
$$\begin{tikzcd} X_0 \arrow[r] \arrow[d] & X \arrow[d] \\ Y_0 \arrow[r] & Y \end{tikzcd}$$
is a pushout square. Recalling that there is a monomorphism $G_{X_0} \to \Gamma G_{X_0}$ of groups (cf. p. 5–64), define G_Y by the pushout square
$$\begin{tikzcd} G_{X_0} \arrow[r] \arrow[d] & G_X \arrow[d] \\ \Gamma G_{X_0} \arrow[r] & G_Y \end{tikzcd}$$
and realize $K(G_Y, 1)$ by the pushout square
$$\begin{tikzcd} K(G_{X_0}, 1) \arrow[r] \arrow[d] & K(G_X, 1) \arrow[d] \\ K(\Gamma G_{X_0}, 1) \arrow[r] & K(G_Y, 1) \end{tikzcd}$$
(cf. p. 5–28). Extend $\kappa_X : K(G_X, 1) \to X$ to $\kappa_Y : K(G_Y, 1) \to Y$ in the obvious way (thus $\kappa_Y K(\Gamma G_{X_0}, 1) \subset Y_0$ and the diagram
$$\begin{tikzcd} K(G_X, 1) \arrow[r, "\kappa_X"] \arrow[d] & X \arrow[d] \\ K(G_Y, 1) \arrow[r, "\kappa_Y"'] & Y \end{tikzcd}$$
commutes). The induction hypothesis implies that κ_X and κ_{X_0} are acyclic. In addition, $K(\Gamma G_{X_0}, 1)$ is an acyclic space and Y_0 is contractible, hence $\kappa_Y | K(\Gamma G_{X_0}, 1)$ is acyclic (cf. Proposition 20). Therefore, by comparing Mayer-Vietoris sequences and applying the five lemma, it follows that κ_Y is acyclic (cf. Proposition 22). Finally, the condition on connected subcomplexes passes on to Y.]

[Note: Put $N = \ker \pi_1(\kappa_X)$—then X is a model for $K(G_X, 1)_N^+$.]

Application: Every nonempty path connected topological space has the homology of a $K(G, 1)$.

EXAMPLE Suppose given two sequences π_n $(n \geq 2)$ & G_q $(q \geq 1)$ of abelian groups—then there exists a pointed connected CW space Z such that $\forall\, n \geq 2 : \pi_n(Z) \approx \pi_n$ & $\forall\, q \geq 1 : H_q(Z) \approx G_q$. Thus choose $X : \pi_{n+1}(X) \approx \pi_n$ $(n \geq 2)$ (homotopy system theorem) and put $Y = \bigvee_1^\infty M(G_q, q)$ (cf. p. 5–38): $H_q(Y) \approx G_q$ $(q \geq 1)$. Using Kan-Thurston, form $\begin{cases} \kappa_X : K(G_X, 1) \to X \\ \kappa_Y : K(G_Y, 1) \to Y \end{cases}$ and consider $Z = E_{\kappa_X} \times K(G_Y, 1)$, the mapping fiber of the arrow $K(G_X \times G_Y, 1) = K(G_X, 1) \times K(G_Y, 1) \to X$. Example: If G_q $(q \geq 1)$ is any sequence of abelian groups, then there exists a group G such that $\forall\, q \geq 1 : H_q(G) \approx G_q$.

[Note: Z also has the property that $\pi_1(Z)$ operates trivially on $\pi_n(Z)$ $\forall\, n \geq 2$.]

The homotopy categories of algebraic topology are not complete (or cocomplete), a circumstance that precludes application of the representable functor theorem and the general adjoint functor theorem (or their duals). However, there is still a certain amount of structure. For instance, consider **HTOP**. It has products and the double mapping track furnishes weak pullbacks. Therefore **HTOP** is weakly complete, i.e., every diagram $\Delta : \mathbf{I} \to \mathbf{HTOP}$ has a weak limit (meaning: "existence without uniqueness"). **HTOP** is also weakly cocomplete. In fact, **HTOP** has coproducts, while weak pushouts are furnished also weakly cocomplete. In fact, **HTOP** has coproducts, while weak pushouts are furnished by the double mapping cylinder. Example: Let $(\mathbf{X},\mathbf{f})$ be an object in **FIL(HTOP)**—then $\text{tel}(\mathbf{X},\mathbf{f})$ is a weak colimit of $(\mathbf{X},\mathbf{f})$.

[Note: The discussion of $\mathbf{HTOP}_*$ is analogous. Example: Let $f : X \rightarrow Y$ be a pointed continuous function, C_f its pointed mapping cone—then C_f is a weak cokernel of $[f]$.]

EXAMPLE For each n, put $Y_n = \mathbf{S}^3$ and let $[\psi_n] : Y_{n+1} \rightarrow Y_n$ be the homotopy class of maps of degree 2—then $Y = \lim Y_n$ does not exist in **HTOP**. To see this, assume the contrary, thus $\forall\ X$, $[X,Y] \approx \lim[X,Y_n]$, so, in particular, Y must be 3-connected. Form the adjunction space $\mathbf{D}^3 \sqcup_f \mathbf{S}^2$, where $f : \mathbf{S}^2 \rightarrow \mathbf{S}^2$ is skeletal of degree 3. Since $\dim(\mathbf{D}^3 \sqcup_f \mathbf{S}^2) \leq 3$, of necessity $[\mathbf{D}^3 \sqcup_f \mathbf{S}^2, Y] = *$. But according to the Hopf classification theorem, $[\mathbf{D}^3 \sqcup_f \mathbf{S}^2, \mathbf{S}^3] \approx H^3(\mathbf{D}^3 \sqcup_f \mathbf{S}^2; \mathbf{Z})$, which is $\mathbf{Z}/3\mathbf{Z}$, and in the limit, $[\mathbf{D}^3 \sqcup_f \mathbf{S}^2, Y] \approx \mathbf{Z}/3\mathbf{Z}$.

EXAMPLE Working in $\mathbf{HTOP}_*$, let $f : X \rightarrow Y$ be a pointed Hurewicz fibration, where X and Y are path connected. Suppose that $K = \ker[f]$ exists, say $[\kappa] : K \rightarrow X$. If π is the projection $E_f \rightarrow X$, then $f \circ \pi \simeq 0$, so there exists a pointed continuous function $\phi : E_f \rightarrow K$ such that $\kappa \circ \phi \simeq \pi$ and, by construction, $f \circ \kappa \simeq 0$, so there exists a pointed continuous function $\psi : K \rightarrow E_f$ such that $\kappa \simeq \pi \circ \psi$. Thus $\kappa \circ \phi \circ \psi \simeq \kappa \Rightarrow \phi \circ \psi \simeq \text{id}_K, [\kappa]$ being a monomorphism in $\mathbf{HTOP}_*$. Take now $X = \mathbf{SO}(3)$, $Y = \mathbf{SO}(3)/\mathbf{SO}(2)$, and let $f : X \rightarrow Y$ be the canonical map—then $\pi_1(E_f) \approx \mathbf{Z}$, $\pi_1(K) \approx \mathbf{Z}/2\mathbf{Z}$ and $\mathbf{Z}/2\mathbf{Z}$ is not a direct summand of $\mathbf{Z}$.

[Note: Similar examples show that cokernels do not exist in $\mathbf{HTOP}_*$.]

Let **C** be a category with products and weak pullbacks—then every diagram in **C** has a weak limit. Any functor $F : \mathbf{C} \rightarrow \mathbf{SET}$ that preserves products and weak pullbacks necessarily preserves weak limits.

PROPOSITION 27 Let **C** be a category with products and weak pullbacks. Assume: $\text{Ob}\,\mathbf{C}$ contains a set $\mathcal{U} = \{U\}$ with the following properties.

$(\mathcal{U}_1)$ A morphism $f : X \rightarrow Y$ is an isomorphism provided that $\forall\ U \in \mathcal{U}$, the arrow $\text{Mor}\,(Y,U) \rightarrow \text{Mor}\,(X,U)$ is bijective.

$(\mathcal{U}_2)$ Each object $(\mathbf{X},\mathbf{f})$ in **TOW(C)** has a weak limit X_∞ such that $\forall\ U \in \mathcal{U}$, the arrow $\text{colim}\,\text{Mor}\,(X_n,U) \rightarrow \text{Mor}\,(X_\infty,U)$ is bijective.

Then a functor $F : \mathbf{C} \rightarrow \mathbf{SET}$ is representable iff it preserves products and weak pullbacks.

[The condition is certainly necessary. As for the sufficiency, introduce the comma category $|*,F|$. Recall that an object of $|*,F|$ is a pair (x,X) $(x \in FX, X \in \text{Ob}\,\mathbf{C})$, while a morphism $(x,X) \rightarrow (y,Y)$ is an arrow $f : X \rightarrow Y$ such that $(Ff)x = y$. The and F is representable iff $|*, F|$ has an initial object. Let $\mathcal{U}_F$ be the subset of $\mathrm{Ob}|*, F|$ consisting of the pairs (u, U) $(u \in FU, U \in \mathcal{U})$.

Claim: $\forall\ (x, X) \in \mathrm{Ob}|*, F|\ \exists\ (\overline{x}, \overline{X}) \in \mathrm{Ob}|*, F|$ and a morphism $(\overline{x}, \overline{X}) \to (x, X)$ such that $\forall\ (u, U) \in \mathcal{U}_F$ there is a unique morphism $(\overline{x}, \overline{X}) \to (u, U)$.

[Define an object $(\mathbf{X}, \mathbf{f})$ in $\mathbf{TOW}(|*, F|)$ by setting $(x_0, X_0) = (x, X) \times \prod(u, U)$ and inductively choose $(x_{n+1}, X_{n+1}) \to (x_n, X_n)$ to equalize all pairs of morphisms $(x_n, X_n) \rightrightarrows (u, U)$ $((u, U) \in \mathcal{U}_F)$. Any weak limit of $(\mathbf{X}, \mathbf{f})$ created via $\mathcal{U}_2$ is a candidate for $(\overline{x}, \overline{X})$.]

The existence of an initial object in $|*, F|$ is then a consequence of observing that for all (x, X) & (y, Y): (i) Every morphism $(\overline{x}, \overline{X}) \to (\overline{y}, \overline{Y})$ is an isomorphism (apply the claim and $\mathcal{U}_1$); (ii) There is at least one morphism $(\overline{x}, \overline{X}) \to (y, Y)$ (the composite $\overline{(\overline{x}, \overline{X}) \times (y, Y)} \to (\overline{x}, \overline{X}) \times (y, Y) \to (\overline{x}, \overline{X})$ is an isomorphism); (iii) There is at most one morphism $(\overline{x}, \overline{X}) \to (y, Y)$ (form the equalizer (z, Z) of $(\overline{x}, \overline{X}) \rightrightarrows (y, Y)$ and consider the composite $(\overline{z}, \overline{Z}) \to (z, Z) \to (\overline{x}, \overline{X})$).]

[Note: Proposition 27 can also be formulated in terms of a category $\mathbf{C}$ that has coproducts and weak pushouts together with a set $\mathcal{U} = \{U\}$ of objects satisfying the following conditions.

($\mathcal{U}_1$) A morphism $f : X \to Y$ is an isomorphism provided that $\forall\ U \in \mathcal{U}$, the arrow $\mathrm{Mor}(U, X) \to \mathrm{Mor}(U, Y)$ is bijective.

($\mathcal{U}_2$) Each object $(\mathbf{X}, \mathbf{f})$ in $\mathbf{FIL}(\mathbf{C})$ has a weak colimit X_∞ such that $\forall\ U \in \mathcal{U}$, the arrow $\mathrm{colim}\,\mathrm{Mor}(U, X_n) \to \mathrm{Mor}(U, X_\infty)$ is bijective.

Under these hypotheses, the conclusion is that a cofunctor $F : \mathbf{C} \to \mathbf{SET}$ is representable iff it converts coproducts into products and weak pushouts into weak pullbacks.]

EXAMPLE Let $\mathbf{C}$ be a category with coproducts and weak pushouts whose representable cofunctors are precisely those that convert coproducts into products and weak pushouts into weak pullbacks. Suppose that $\mathbf{T} = (T, m, \epsilon)$ is an idempotent triple in $\mathbf{C}$ and let $S \subset \mathrm{Mor}\,\mathbf{C}$ be the class consisting of those f such that Tf is an isomorphism—then (1) S admits a calculus of left fractions; (2) S is saturated; (3) S satisfies the solution set condition; (4) S is coproduct closed, i.e., $s_i : X_i \to Y_i$ in $S\ \forall\ i \in I \Rightarrow \coprod_i s_i : \coprod_i X_i \to \coprod_i Y_i$ in S. Conversely, any class $S \subset \mathrm{Mor}\,\mathbf{C}$ with properties (1)–(4) is generated by an idempotent triple, thus $S^\perp$ is the object class of a reflective subcategory of $\mathbf{C}$.

[The functor $L_S : \mathbf{C} \to S^{-1}\mathbf{C}$ preserves coproducts and weak pushouts. So, for fixed $Y \in \mathrm{Ob}\, S^{-1}\mathbf{C}$, $\mathrm{Mor}(L_S—, Y)$ is a cofunctor $\mathbf{C} \to \mathbf{SET}$ which converts coproducts into products and weak pushouts into weak pullbacks, hence is representable: $\mathrm{Mor}(L_S X, Y) \approx \mathrm{Mor}(X, Y_S)$. Use the assignment $Y \to Y_S$ to define a functor $S^{-1}\mathbf{C} \to \mathbf{C}$ and take for T the composite $\mathbf{C} \to S^{-1}\mathbf{C} \to \mathbf{C}$. Let $\epsilon_X \in \mathrm{Mor}(X, TX)$ correspond to $\mathrm{id}_{L_S X}$ under the bijection $\mathrm{Mor}(L_S X, L_S X) \approx \mathrm{Mor}(X, TX)$—then $\epsilon : \mathrm{id}_{\mathbf{C}} \to T$ is a natural transformation, $\epsilon T = T\epsilon$ is a natural isomorphism, and Tf is an isomorphism iff $f \in S$.]

Notation: **CONCW$_*$** is the full subcategory of **CW$_*$** whose objects are the pointed connected CW complexes and **HCONCW$_*$** is the associated homotopy category.

LEMMA **HCONCW$_*$** has coproducts and weak pushouts.

[If $X \xleftarrow{f} Z \xrightarrow{g} Y$ is a 2-source in **CONCW$_*$**, then using the skeletal approximation theorem, one can always arrange that $M_{f,g}$ remains in **CONCW$_*$**.]

BROWN REPRESENTABILITY THEOREM A cofunctor $F : \mathbf{HCONCW}_* \rightarrow \mathbf{SET}$ is representable iff it converts coproducts into products and weak pushouts into weak pullbacks.

[Take for $\mathcal{U}$ the set $\{(\mathbf{S}^n, s_n) : n \in \mathbf{N}\}$—then $\mathcal{U}_1$ holds since in **CONCW$_*$** a pointed continuous function $f : X \rightarrow Y$ is a pointed homotopy equivalence iff it is a weak homotopy equivalence (cf. p. 5–17) and $\mathcal{U}_2$ holds since one can take for a weak colimit of an object $(\mathbf{X}, \mathbf{f})$ in $\mathbf{FIL}(\mathbf{HCONCW}_*)$ the pointed mapping telescope constructed using pointed skeletal maps (cf. p. 5–25).]

[Note: Since F converts coproducts into products, F takes an initial object to a terminal object: $F* = *$ and $X \rightarrow * \Rightarrow * = F* \rightarrow FX$, thus FX has a natural base point.]

Spelled out, here are the conditions on F figuring in the Brown representability theorem.

(Wedge Condition) For any collection $\{X_i : i \in I\}$ in **CONCW$_*$**, $F(\bigvee_i X_i) \approx \prod_i FX_i$.

(Mayer-Vietoris Condition) For any weak pushout square

$$\begin{tikzcd} Z \arrow[r, "g"] \arrow[d, "f"'] & Y \arrow[d, "\eta"] \\ X \arrow[r, "\xi"'] & P \end{tikzcd}$$

in **HCONCW$_*$**,

$$\begin{tikzcd} FP \arrow[r, "F\eta"] \arrow[d, "F\xi"'] & FY \arrow[d, "Fg"] \\ FX \arrow[r, "Ff"'] & FZ \end{tikzcd}$$

is a weak pullback square in **SET**, so $\forall \begin{cases} x \in FX \\ y \in FY \end{cases}$: $(Ff)x = (Fg)y$, $\exists\, p \in FP : \begin{cases} (F\xi)p = x \\ (F\eta)p = y \end{cases}$.

[Note: It is not necessary to make the verification for an arbitrary weak pushout square. In fact, it is sufficient to consider pointed double mapping cylinders calculated relative to skeletal maps, thus it is actually enough to consider diagrams of the form

$$\begin{tikzcd} C \arrow[r] \arrow[d] & B \arrow[d] \\ A \arrow[r] & X \end{tikzcd}$$

, where X is a pointed connected CW complex and $\begin{cases} A \\ B \end{cases}$ & C are pointed connected subcomplexes such that $X = A \cup B, C = A \cap B$.]

$$\begin{tikzcd} C \arrow[r] \arrow[d] & B \arrow[d] \\ A \arrow[r] & X \end{tikzcd}$$

, where X is a pointed connected CW complex and $\left\{\begin{matrix} A \\ B \end{matrix}\right.$ & C are pointed connected subcomplexes such that $X = A \cup B, C = A \cap B$.]

Examples: (1) Fix a pointed path connected space (X, x_0)—then $[—; X, x_0]$ is a cofunctor on **HCONCW**$_*$ satisfying the wedge and Mayer-Vietoris conditions, hence there exists a pointed connected CW complex (K, k_0) and a natural isomorphism $\Xi : [—; K, k_0] \rightarrow [—; X, x_0]$, each $f \in \Xi_{K,k_0}([\mathrm{id}_K])$ being a weak homotopy equivalence $K \rightarrow X$, thus the Brown representability theorem implies the resolution theorem; (2) Fix $n \in \mathbf{N}$ and an abelian group π—then the cofunctor $H^n(—; \pi)$ (singular cohomology) satisfies the wedge and Mayer-Vietoris conditions, hence there exists a pointed connected CW complex $(K(\pi, n), k_{\pi,n})$ and a natural isomorphism $\Xi : [—; K(\pi, n), k_{\pi,n}] \rightarrow H^n(—; \pi)$, thus the Brown representability theorem implies the existence of Eilenberg-MacLane spaces of type (π, n) (π abelian); (3) Fix a group π—then the cofunctor that assigns to a pointed connected CW complex (K, k_0) the set of homomorphisms $\pi_1(K, k_0) \rightarrow \pi$ satisfies the wedge and Mayer-Vietoris conditions, hence there exists a pointed connected CW complex $(K(\pi, 1), k_{\pi,1})$ and a natural isomorphism $\Xi : [—; K(\pi, 1), k_{\pi,1}] \rightarrow \mathrm{Hom}(\pi_1 —; \pi)$, thus the Brown representability theorem implies the existence of Eilenberg-MacLane spaces of type $(\pi, 1)$ (π arbitrary).

[Note: Both **HCW**$_*$ and **HCW** have coproducts and weak pushouts but Brown representability can fail. Indeed, Matveev[†] has given an example of a nonrepresentable cofunctor F : **HCW**$_* \rightarrow$ **SET** which converts coproducts into products and weak pushouts into weak pullbacks and Heller[‡] has given an example of a nonrepresentable cofunctor F : **HCW** $\rightarrow$ **SET** which converts coproducts into products and weak pushouts into weak pullbacks.]

EXAMPLE Let U : **GR** $\rightarrow$ **SET** be the forgetful functor.

(**HCW**$_*$) Suppose that F : **HCW**$_* \rightarrow$ **GR** is a cofunctor such that $U \circ F$ converts coproducts into products and weak pushouts into weak pullbacks—then $U \circ F$ is representable.

[Represent the composite **HCONCW**$_* \rightarrow$ **HCW**$_* \rightarrow$ **GR** $\rightarrow$ **SET** by K. Put $G = F\mathbf{S}^0$ and equip it with the discrete topology.

Claim: For any X in **CONCW**$_*$, $U \circ F(X_+) \approx [X_+, K \times G]$.

[There is a split short exact sequence $1 \rightarrow FX \rightarrow FX_+ \rightarrow F\mathbf{S}^0 \rightarrow 1$, hence $U \circ F(X_+) \approx U \circ F(X) \times$

[†] *Math. Notes* **39** (1986), 471–474.

[‡] *J. London Math. Soc.* **23** (1981), 551–562.

$G \approx [X,K] \times G$ or, reinstating the base points: $U \circ F(X_+) \approx [X,x_0;K,k_0] \times G$. And: $[X,x_0;K,k_0] \approx [X,K] \Rightarrow [X,x_0;K,k_0] \times G \approx [X,K] \times G \approx [X,K] \times [X,G] \approx [X,K \times G] \approx [X_+,K \times G]$.]

Given (X,x_0) in $\mathbf{CW}_*$, let X_{i_0}, X_i ($i \in I$) be its set of path components, where $x_0 \in X_{i_0}$—then $X = X_{i_0} \vee \bigvee_i X_{i+}$, so $U \circ F(X) \approx U \circ F(X_{i_0}) \times \prod_i U \circ F(X_{i+}) \approx [X_{i_0},K] \times \prod_i [X_{i+},K \times G] \approx [X_{i_0},K \times G] \times \prod_i [X_{i+},K \times G] \approx [X,K \times G]$.]

(HCW) Suppose that $F : \mathbf{HCW} \rightarrow \mathbf{GR}$ is a cofunctor such that $U \circ F$ converts coproducts into products and weak pushouts into weak pullbacks—then $U \circ F$ is representable.

[Let F_* be the composite $\mathbf{HCONCW}_* \rightarrow \mathbf{HCONCW} \rightarrow \mathbf{HCW} \rightarrow \mathbf{GR} \rightarrow \mathbf{SET}$.

Claim: If $F* = *$, then F_* is representable.

[The assumption on F implies that $FA = *$ for any discrete topological space A. To check that F_* satisfies the wedge condition, put $X = \coprod_i X_i$ and let $A \subset X$ be the set made up of the base points $x_i \in X_i$—then $F(X/A) \approx FX$. But $X/A = \bigvee_i X_i \Rightarrow F_*(\bigvee_i X_i) \approx U \circ F(X) \approx \prod_i F_* X_i$. As F_* necessarily satisfies the Mayer-Vietoris condition, F_* is representable: $[—,K_*] \approx F_*$.]

Claim: If $F* = *$, then $U \circ F$ is representable.

[If X is in $\mathbf{CW}$ and if $X = \coprod_i X_i$ is its decomposition into path components, then $U \circ F(X) \approx \prod_i U \circ F(X_i) \approx \prod_i F_* X_i \approx \prod_i [X_i,K_*] \approx [\coprod_i X_i,K_*] \approx [X,K_*]$.]

Given X in $\mathbf{CW}$, view $\pi_0(X)$ as a discrete topological space—then $U \circ F \circ \pi_0$ is represented by $F*$ (discrete topology). On the other hand, F is the semidirect product of $F \circ \pi_0$ and the kernel F_0 of $F \rightarrow F \circ \pi_0$ induced by the embedding $\pi_0(X) \rightarrow X$. Moreover, $U \circ F \approx U \circ F_0 \times U \circ F \circ \pi_0$ and $F_0 * = * \Rightarrow U \circ F_0$ is representable.]

Given a small, full subcategory $\mathbf{C}_0$ of $\mathbf{HCW}_*$, denote by $\overline{\mathbf{C}}_0$ the full subcategory of $\mathbf{HCW}_*$ whose objects are those Y such that $g : Y \rightarrow Z$ is an isomorphism (= pointed homotopy equivalence) if $g_* : [X_0,Y] \rightarrow [X_0,Z]$ is bijective for all $X_0 \in \operatorname{Ob} \mathbf{C}_0$.

FACT Suppose that $F : \mathbf{HCW}_* \rightarrow \mathbf{SET}$ is a cofunctor which converts coproducts into products and weak pushouts into weak pullbacks—then there exists an object X_F in $\mathbf{HCW}_*$ and a natural transformation $\Xi : [—,X_F] \rightarrow F$ such that $\forall\, X_0 \in \operatorname{Ob} \mathbf{C}_0$, $\Xi_{X_0} : [X_0,X_F] \rightarrow FX_0$ is bijective.

FACT Suppose that $F : \mathbf{HCW}_* \rightarrow \mathbf{SET}$ is a cofunctor which converts coproducts into products and weak pushouts into weak pullbacks—then F is representable if for some $\mathbf{C}_0, X_F \in \operatorname{Ob} \overline{\mathbf{C}}_0$.

[With Ξ as above, put $x_F = \Xi_{X_F}([\mathrm{id}_{X_F}])$, so that $\forall\, X \in \operatorname{Ob} \mathbf{HCW}_*$, $\Xi_X([f]) = F[f]x_F$ ($[f] \in [X,X_F]$).

Surjectivity: Given $X \in \operatorname{Ob} \mathbf{HCW}_*$, call $\mathbf{C}_0'$ the full subcategory of $\mathbf{HCW}_*$ obtained by adding X and X_F to $\mathbf{C}_0$. Determine X_F' and $\Xi' : [—,X_F'] \rightarrow F$ accordingly. In particular, $\Xi'_{X_F} : [X_F,X_F'] \rightarrow FX_F$ is surjective, thus $\exists\, [f] \in [X_F,X_F'] : x_F = F[f]x_F'$. From the definitions, $\forall\, X_0 \in \operatorname{Ob} \mathbf{C}_0$, $f_* : [X_0,X_F] \rightarrow$

$\Xi_Z : [Z, X_F] \to FZ$ is surjective, $\exists\ g : Z \to X_F$ such that $\Xi_Z([g]) = z$, hence $x_F = F[g \circ f]x_F$. From the definitions, $\forall\ X_0 \in \text{Ob}\,\mathbf{C}_0$, $(g \circ f)_* : [X_0, X_F] \to [X_0, X_F]$ is bijective. Therefore $g \circ f$ is an isomorphism. Finally, $f \circ u \simeq f \circ v \Rightarrow g \circ f \circ u \simeq g \circ f \circ v \Rightarrow u \simeq v$.]

Application: Let $\mathbf{C}_0$ be the full subcategory of $\mathbf{HCW}_*$ consisting of the $(\mathbf{S}^n, s_n)$ $(n \geq 0)$, so $\overline{\mathbf{C}}_0 = \mathbf{HCONCW}_*$—then a cofunctor $F : \mathbf{HCW}_* \to \mathbf{SET}$ which converts coproducts into products and weak pushouts into weak pullbacks is representable provided that $\#(F\mathbf{S}^0) = 1$.

[In fact, $\pi_0(X_F) = [\mathbf{S}^0, X_F] = F\mathbf{S}^0$, thus X_F is connected.]

EXAMPLE Fix a nonempty topological space F. Given a CW complex B, let $k_F B$ be the set $\text{Ob}\,\overline{\mathbf{FIB}}_{B,F}$, where $\overline{\mathbf{FIB}}_{B,F}$ is the skeleton of $\mathbf{FIB}_{B,F}$ (cf. p. 4–28)—then k_F is a cofunctor $\mathbf{HCW} \to \mathbf{SET}$ which converts coproducts into products and weak pushouts into weak pullbacks (cf. p. 4–19). However, k_F is not automatically representable since Brown representability can fail in $\mathbf{HCW}$. To get around this difficulty, one employs a subterfuge. Thus given a pointed CW complex (B, b_0), let $\mathbf{FIB}_{B,F;*}$ be the category whose objects are the pairs (p, i), where $p : X \to B$ is a Hurewicz fibration such that $\forall\ b \in B$, X_b has the homotopy type of F and $i : F \to p^{-1}(b_0)$ is a homotopy equivalence, and whose morphisms $(p, i) \to (q, j)$ are the fiber homotopy classes $[f] : X \to Y$ and the homotopy classes $[\phi] : F \to F$ such that $f_{b_0} \circ i \simeq j \circ \phi$. As in the unpointed case, $\mathbf{FIB}_{B,F;*}$ has a small skeleton and there is a cofunctor $k_{F;*} : \mathbf{HCW}_* \to \mathbf{SET}$ which converts coproducts into products and weak pushouts into weak pullbacks. Since $\#(k_{F;*}\mathbf{S}^0) = 1$, it follows from the above that $k_{F;*}$ is representable: $[-; B_F, b_F] \approx k_{F;*}$, (B_F, b_F) a pointed connected CW complex. If now B is a CW complex, then the functor $\mathbf{FIB}_{B,F} \to \mathbf{FIB}_{B_+,F;*}$ that assigns to $p : X \to B$ the pair $(p \amalg c, \text{id}_F)$ $(c : F \to *)$ induces a bijection $\text{Ob}\,\overline{\mathbf{FIB}}_{B,F} \to \text{Ob}\,\overline{\mathbf{FIB}}_{B_+,F;*}$, so $k_F B \approx k_{F;*}B_+ \approx [B_+, *; B_F, b_F] \approx [B, B_F]$, i.e., B_F represents k_F. Example: Take $F = K(\pi, n)$ (π abelian)—then B_F has the same pointed homotopy type as $K(\pi, n+1; \chi_\pi)$ (cf. p. 5–32) ($K(\pi, n+1; \chi_\pi)$ is not necessarily a CW complex).

Example: Consider the Hurewicz fibration $p_1 : \Theta\mathbf{S}^n \to \mathbf{S}^n$ $(n \geq 2)$. Let $i : \Omega\mathbf{S}^n \to \Omega\mathbf{S}^n$ be the identity and $\iota : \Omega\mathbf{S}^n \to \Omega\mathbf{S}^n$ the inversion—then the pairs (p_1, i) and (p_1, ι) are not isomorphic in $\mathbf{FIB}_{\mathbf{S}^n, \Omega\mathbf{S}^n;*}$.

Let G be a topological group—then in the notation of p. 4–60, the restriction $k_G|\mathbf{HCW}$ is a cofunctor $\mathbf{HCW} \to \mathbf{SET}$ which converts coproducts into products and weak pushouts into weak pullbacks. To ensure that it is representable, one can introduce the pointed analog of $\mathbf{BUN}_{B,G}$, say $\mathbf{BUN}_{B,G;*}$, and proceed as above. The upshot is that the classifying space B_G is now a CW complex but this need not be true of the universal space X_G. To clarify the situation, consider the pullback square

$$\begin{tikzcd} X_G \arrow[r] \arrow[d] & X_G^\infty \arrow[d] \\ B_G \arrow[r] & B_G^\infty \end{tikzcd}$$

. Since

for any CW complex B, $[B, B_G] \approx k_G B \approx [B, B_G^\infty]$, the arrow $B_G \to B_G^\infty$ is a weak homotopy equivalence (cf. p. 5–15 ff.). Therefore the arrow $X_G \to X_G^\infty$ is a weak homotopy equivalence, so X_G is homotopically trivial (X_G^∞ being contractible).

LEMMA X_G is contractible iff G is a CW space.

[Necessity: For then X_G is a CW space and because the fibers of the Hurewicz fibration $X_G \to B_G$ are homeomorphic to G, it follows that G is a CW space (cf. p. 6–25).

Sufficiency: Due to §6, Proposition 11, X_G is a CW space. But a homotopically trivial CW space is contractible.]

Moral: When G is a CW space, k_G can be represented by a CW complex (cf. §4, Proposition 35).

[Note: Under these conditions, B_G and B_G^∞ have the same homotopy type (representing objects are isomorphic), thus B_G^∞ is a CW space (see p. 6–25 for another argument).]

Notation: $\mathbf{FCONCW}_*$ is the full subcategory of $\mathbf{CONCW}_*$ whose objects are the pointed finite connected CW complexes and $\mathbf{HFCONCW}_*$ is the associated homotopy category.

[Note: Any skeleton $\overline{\mathbf{HFCONCW}}_*$ of $\mathbf{HFCONCW}_*$ is countable (cf. p. 6–28).]

A cofunctor $F : \mathbf{HFCONCW}_* \to \mathbf{SET}$ is said to be representable in the large if there exists a pointed connected CW complex X and a natural isomorphism $[-, X] \to F$.

[Note: In this context, $[-, X]$ stands for the restriction to $\mathbf{HFCONCW}_*$ of the representable cofunctor determined by X. Observe that in general it is meaningless to consider FX.]

Example: The restriction to $\mathbf{HFCONCW}_*$ of any cofunctor $\mathbf{HCONCW}_* \to \mathbf{SET}$ satisfying the wedge and Mayer-Vietoris conditions is representable in the large.

Let $F : \mathbf{HCONCW}_* \to \mathbf{SET}$ be a cofunctor.

(Finite Mayer-Vietoris Condition) For any weak pushout square
$$\begin{tikzcd} Z \arrow[r, "g"] \arrow[d, "f"'] & Y \arrow[d, "\eta"] \\ X \arrow[r, "\xi"'] & P \end{tikzcd}$$
in $\mathbf{HCONCW}_*$, where Z is finite,
$$\begin{tikzcd} FP \arrow[r, "F\eta"] \arrow[d, "F\xi"'] & FY \arrow[d, "Fg"] \\ FX \arrow[r, "Ff"'] & FZ \end{tikzcd}$$
is a weak pullback square in $\mathbf{SET}$, so $\forall \begin{cases} x \in FX \\ y \in FY \end{cases} : (Ff)x = (Fg)y$, $\exists\, p \in FP : \begin{cases} (F\xi)p = x \\ (F\eta)p = y \end{cases}$.

(Limit Condition) For any pointed connected CW complex X and for any collection $\{X_i : i \in I\}$ of pointed connected subcomplexes of X such that $X = \operatorname{colim} X_i$, where I is directed and the X_i are ordered by inclusion, the arrow $FX \to \lim FX_i$ is bijective.

SUBLEMMA Let $F : \mathbf{HCONCW}_* \to \mathbf{SET}$ be a cofunctor satisfying the wedge and finite Mayer-Vietoris conditions. Fix an X in $\mathbf{CONCW}_*$ and choose $x \in FX$. Suppose that $X \xleftarrow{f} K \xrightarrow{g} X$ is a pointed 2-source, where K is in $\mathbf{FCONCW}_*$ and $\begin{cases} f \\ g \end{cases}$ are skeletal with $(Ff)x = (Fg)x$—then there is a Y in $\mathbf{CONCW}_*$ containing X as an embedded pointed subcomplex, say $i : X \to Y$, such that $i \circ f \simeq i \circ g$ and a $y \in FY$ such that $(Fi)y = x$.

[Consider the weak pushout square
$$\begin{tikzcd} K \vee K \arrow[r, "f \vee g"] \arrow[d, "\nabla_K"'] & X \arrow[d] \\ K \arrow[r] & Y \end{tikzcd}$$
, where Y is the pointed double mapping cylinder of the folding map ∇_K and the wedge $f \vee g$. By construction, Y is a pointed weak coequalizer of $\begin{cases} f \\ g \end{cases}$ and the existence of $y \in FY$ follows from the assumptions.]

LEMMA Let $F : \mathbf{HCONCW}_* \to \mathbf{SET}$ be a cofunctor satisfying the wedge, finite Mayer-Vietoris, and limit conditions. Fix an X in $\mathbf{CONCW}_*$ and choose $x \in FX$—then there is a Y in $\mathbf{CONCW}_*$ containing X as an embedded pointed subcomplex, say $i : X \to Y$, such that $i \circ f \simeq i \circ g$ for any pointed 2-source $X \xleftarrow{f} K \xrightarrow{g} X$, where K is in $\mathbf{FCONCW}_*$ and $\begin{cases} f \\ g \end{cases}$ are skeletal with $(Ff)x = (Fg)x$, and a $y \in FY$ such that $(Fi)y = x$.

[Since it is enough to let K run over the objects in $\overline{\mathbf{HFCONCW}_*}$, one need only deal with a set $\{X \xleftarrow{f_s} K_s \xrightarrow{g_s} X : s \in S\}$ of pointed 2-sources. Given any $T \subset S$, proceed as in the proof of the sublemma and form the weak pushout square
$$\begin{tikzcd} \bigvee_t (K_t \vee K_t) \arrow[r] \arrow[d] & X \arrow[d, "i_T"] \\ \bigvee_t K_t \arrow[r] & Y_T \end{tikzcd}$$
, so for $T' \subset T''$ there is a commutative triangle
$$\begin{tikzcd} & X \arrow[dl] \arrow[dr] & \\ Y_{T'} \arrow[rr, "j"'] & & Y_{T''} \end{tikzcd}$$
. Consider the set $\mathcal{T}$ of pairs (T, y_T) $(y_T \in FY_T) : (Fi_T)y_T = x$. Order $\mathcal{T}$ by writing $(T', y_{T'}) \le (T'', y_{T''})$ iff $T' \subset T''$ and $(Fj)y_{T''} = y_{T'}$—then the limit condition implies that every chain in $\mathcal{T}$ has an upper bound, thus $\mathcal{T}$ has a maximal element (T_0, y_{T_0}) (Zorn). Thanks to the sublemma, $T_0 = S$, therefore one can take $Y = Y_S, y = y_S$.]

PROPOSITION 28 Let $F : \mathbf{HCONCW}_* \to \mathbf{SET}$ be a cofunctor satisfying the wedge, finite Mayer-Vietoris, and limit conditions—then the restriction of F to $\mathbf{HFCONCW}_*$ is representable in the large.

[Put $X^0 = \bigvee\limits_{K,k} K$, where K runs over the objects in $\overline{\mathbf{HFCONCW}_*}$ and for each K, k runs over FK. Using the wedge condition, choose $x^0 \in FX^0$ such that the associated natural transformation $\Xi^0 : [-, X^0] \rightarrow F$ has the property that $\Xi_K^0 : [K, X^0] \rightarrow FK$ is surjective for all K. Per the lemma, construct $X^0 \subset X^1$ & $x^1 \in FX^1$ and continue by induction to obtain an expanding sequence $X^0 \subset X^1 \subset \cdots$ of topological spaces and elements $x^0 \in FX^0, x^1 \in FX^1, \ldots$ such that $\forall\ n$, X^n is a pointed connected CW complex containing X^{n-1} as a pointed subcomplex and $x^n \rightarrow x^{n-1}$ under $X^{n-1} \rightarrow X^n$. Put $X = X^\infty$—then X is a pointed connected CW complex containing X^n as a pointed subcomplex (cf. p. 5–25). Let $x \in FX$ be the element corresponding to $\{x^n\}$ via the limit condition and let $\Xi : [-, X] \rightarrow F$ be the associated natural transformation. That Ξ_K is surjective for all K is automatic. But Ξ_K is also injective for all $K : \Xi_K([f]) = \Xi_K([g])$, i.e., $(Ff)x = (Fg)x$ (f, g skeletal) $\Rightarrow (Ff)x^n = (Fg)x^n$ $(\exists\ n) \Rightarrow i \circ f \simeq i \circ g$ $(i : X^n \rightarrow X^{n+1})$.]

Given a cofunctor $F : \mathbf{HFCONCW}_* \rightarrow \mathbf{SET}$, for X in $\mathbf{CONCW}_*$, let $\overline{F}X = \lim FX_k$, where X_k runs over the pointed finite connected subcomplexes of X ordered by inclusion—then $\overline{F}$ is the object function of a cofunctor $\mathbf{HCONCW}_* \rightarrow \mathbf{SET}$ whose restriction to $\mathbf{HFCONCW}_*$ is (naturally isomorphic to) F. On the basis of the definitions, $\overline{F}$ satisfies the limit condition. Moreover, $\overline{F}$ satisfies the wedge condition provided that F converts finite coproducts into finite products so, in order to conclude that F is representable in the large, it need only be shown that $\overline{F}$ satisfies the finite Mayer-Vietoris condition (cf. Proposition 28). Assume, therefore, that F converts weak pushouts into weak pullbacks. Consider a diagram

$$\begin{tikzcd} C \arrow[r] \arrow[d] & B \arrow[d] \\ A \arrow[r] & X \end{tikzcd}$$

, where X is a pointed connected CW complex and $\left\{\begin{matrix} A \\ B \end{matrix}\right.$ & C are pointed connected subcomplexes such that $X = A \cup B$, $C = A \cap B$ with C finite. To prove that

$$\begin{tikzcd} \overline{F}X \arrow[r] \arrow[d] & \overline{F}B \arrow[d] \\ \overline{F}A \arrow[r] & \overline{F}C \end{tikzcd}$$

is a weak pullback square, let $\left\{\begin{matrix} K_i \\ L_j \end{matrix}\right.$ run over the pointed finite connected subcomplexes of $\left\{\begin{matrix} A \\ B \end{matrix}\right.$ which contain C and using obvious notation, let $\left\{\begin{matrix} \overline{a} \in \overline{F}A \\ \overline{b} \in \overline{F}B \end{matrix}\right. : \overline{a}|C = \overline{b}|C$—then the question is whether there exists $\overline{x} \in \overline{F}X : \left\{\begin{matrix} \overline{x}|A = \overline{a} \\ \overline{x}|B = \overline{b} \end{matrix}\right.$. For this, note first that $\left\{\begin{matrix} \overline{F}A = \lim FK_i \\ \overline{F}B = \lim FL_j \end{matrix}\right.$ and $\overline{F}X = \lim FX_{ij}$ $(X_{ij} = K_i \cup L_j)$. Represent

$\begin{cases}\overline{a}\\ \overline{b}\end{cases}$ by $\begin{cases}\{a_i\} & (a_i \in FK_i)\\ \{b_j\} & (b_j \in FL_j)\end{cases}$ and let S_{ij} be the set of $x_{ij} \in FX_{ij}$: $\begin{cases} x_{ij}|K_i = a_i\\ x_{ij}|L_j = b_j\end{cases}$. Since S_{ij} is nonempty and $\lim S_{ij}$ is a subset of $\lim FX_{ij}$, it suffices to prove that $\lim S_{ij}$ is nonempty as any $\overline{x} \in \lim S_{ij}$ will work. However, this is a subtle point that has been resolved only by placing restrictions on the range of F.

EXAMPLE Let U : **CPTHAUS** $\rightarrow$ **SET** be the forgetful functor. Suppose that F : **HFCONCW**$_*$ $\rightarrow$ **CPTHAUS** is a cofunctor such that $U \circ F$ converts finite coproducts into finite products and weak pushouts into weak pullbacks—then $U \circ F$ is representable in the large. In fact, if T_{ij} is the subspace of FX_{ij} such that $UT_{ij} = S_{ij}$, then T_{ij} is closed and $\lim T_{ij}$ is calculated over a cofiltered category, hence $\lim T_{ij}$ is a nonempty compact Hausdorff space. But U preserves limits, therefore $\lim S_{ij} = U(\lim T_{ij})$ is also nonempty.

[Note: More is true: $\overline{U \circ F}$ satisfies the Mayer-Vietoris condition, hence is representable. Example: If Y is a pointed connected CW complex whose homotopy groups are finite, then for every pointed finite connected CW complex X, $[X, Y]$ is finite (cf. p. 5–49), thus is a compact Hausdorff space (discrete topology) and so $\overline{[-, Y]}$ is representable.]

REPLICATION THEOREM Let $f : K \rightarrow L$ be a pointed skeletal map, where $\begin{cases} K\\ L\end{cases}$ are in **FCONCW**$_*$—then for any cofunctor F : **HFCONCW**$_*$ $\rightarrow$ **SET** which converts finite coproducts into finite products and weak pushouts into weak pullbacks, there is an exact sequence

$$\cdots \rightarrow F\Sigma L \rightarrow F\Sigma K \rightarrow FC_f \rightarrow FL \rightarrow FK$$

in **SET**$_*$.

[Note: F takes (abelian) cogroup objects to (abelian) group objects, so all the arrows to the left of $F\Sigma K$ are homomorphisms of groups. In addition, $F\Sigma K$ operates to the left on FC_f and the orbits are the fibers of the arrow $FC_f \rightarrow FL$ (cf. p. 3–33).]

Application: There is an exact sequence

$$F\Sigma K_i \times F\Sigma L_j \rightarrow F\Sigma C \rightarrow FX_{ij} \rightarrow FK_i \times FL_j$$

in **SET**$_*$.

[The pointed mapping cone of the arrow $K_i \vee L_j \rightarrow X_{ij}$ has the same pointed homotopy type as ΣC.]

Let $(I, \leq)$ be a nonempty directed set, **I** the associated filtered category. Suppose that $\Delta : \mathbf{I}^{\mathrm{OP}} \rightarrow$ **SET** is a diagram, where $\forall\, i \in \mathrm{Ob}\,\mathbf{I}$, $\Delta_i \neq \emptyset$ and $\forall\, \delta \in \mathrm{Mor}\,\mathbf{I}$, $\Delta\delta$ is surjective.

In I, write $i \sim j$ iff there exists a bijective map $f : \Delta_i \to \Delta_j$ and a k with $\left\{\begin{array}{l} i \\ j \end{array}\right. \leq k$ such that the triangle

$$\begin{tikzcd} & \Delta_k \arrow[dl] \arrow[dr] & \\ \Delta_i \arrow[rr, "f"'] & & \Delta_j \end{tikzcd}$$

commutes.

LEMMA If $\#(I/{\sim}) \leq \omega$, then $\lim \Delta$ is nonempty.

ADAMS REPRESENTABILITY THEOREM Let $U : \mathbf{GR} \to \mathbf{SET}$ be the forgetful functor. Suppose that $F : \mathbf{HFCONCW}_* \to \mathbf{GR}$ is a cofunctor such that $U \circ F$ converts finite coproducts into finite products and weak pushouts into weak pullbacks—then $U \circ F$ is representable in the large.

[The arrow $S_{i'j'} \to S_{ij}$ is surjective if $\left\{\begin{array}{l} K_i \subset K_{i'} \\ L_j \subset L_{j'} \end{array}\right.$. This is because $F\Sigma C$ acts transitively to the left on $\left\{\begin{array}{l} S_{i'j'} \\ S_{ij} \end{array}\right.$ and $S_{i'j'} \to S_{ij}$ is equivariant. Claim: $\#(\{ij\}/{\sim}) \leq \omega$. For one can check that $ij \sim i'j'$ iff $F\Sigma K_i \times F\Sigma L_j \to F\Sigma C$ & $F\Sigma K_{i'} \times F\Sigma L_{j'} \to F\Sigma C$ have the same image, of which there are at most a countable number of possibilities. The lemma thus implies that $\lim S_{ij}$ is nonempty.]

Working in $\mathbf{CONCW}_*$, two pointed continuous functions $f, g : X \to Y$ are said to be prehomotopic if for any pointed finite connected CW complex K and any pointed continuous function $\phi : K \to X$, $f \circ \phi \simeq g \circ \phi$. Homotopic maps are prehomotopic but the converse is false since, e.g., there are phantom maps that are not nullhomotopic (see below).

Notation: $\mathbf{PREHCONCW}_*$ is the quotient category of $\mathbf{CONCW}_*$ defined by the congruence of prehomotopy, $[X, Y]_{\text{pre}}$ being the set of morphisms from X to Y.

If $F : \mathbf{HFCONCW}_* \to \mathbf{SET}$ is a cofunctor, then $\overline{F}$ can be viewed as a cofunctor $\mathbf{PREHCONCW}_* \to \mathbf{SET}$. Given X in $\mathbf{CONCW}_*$, there is a bijection $\text{Nat}([—, X]_{\text{pre}}, \overline{F}) \to \overline{F}X$ (Yoneda). On the other hand, there is a bijection $\text{Nat}([—, X], F) \to \overline{F}X$, viz. $\Xi \to \{\Xi_{X_k}([i_k])\}$, $i_k : X_k \to X$ the inclusion. Example: Take $F = [—, X]$, so $\overline{[X, X]} = \lim [X_k, X]$, and put $\iota_X = \{[i_k]\}$—then $\text{id}_{[—, X]} \leftrightarrow \iota_X$.

PROPOSITION 29 Let Y be in $\mathbf{CONCW}_*$. Assume: $\overline{[—, Y]}$ satisfies the finite Mayer-Vietoris condition—then for all X in $\mathbf{CONCW}_*$, the natural map $[X, Y]_{\text{pre}} \to \lim [X_k, Y]$ is bijective.

[Injectivity is immediate. Turning to surjectivity, note that by definition $\lim [X_k, Y] = \overline{[X, Y]}$. Fix $x_0 \in \overline{[X, Y]}$ and let $y_0 = \iota_Y$ $(\in \overline{[Y, Y]})$. Put $Z_0 = X \vee Y$ and write $z_0 = (x_0, y_0) \in \overline{[Z_0, Y]} \approx \overline{[X, Y]} \times \overline{[Y, Y]}$. Imitating the argument used in the proof of Proposition

28, construct a Z in **CONCW$_*$** containing Z_0 as an embedded pointed subcomplex and an element $z \in \overline{[Z,Y]}$ which restricts to z_0 such that the associated natural transformation $[K,Z] \to [K,Y]$ is a bijection for all K. Specialize and take $K = \mathbf{S}^n$ ($n \in \mathbf{N}$) to see that the inclusion $j : Y \to Z$ is a pointed homotopy equivalence (realization theorem) and then compose the inclusion $i : X \to Z$ with a homotopy inverse for j to get a pointed continuous function $f_0 : X \to Y$ whose prehomotopy class is sent to x_0.]

FACT If Y is a pointed connected CW complex whose homotopy groups are countable, then $\overline{[-,Y]}$ satisfies the finite Mayer-Vietoris condition.

[Note: Under this assumption on Y, it follows that for all X in **CONCW$_*$**, the natural map $[X,Y] \to \lim [X_k, Y]$ is surjective (and even bijective provided that the homotopy groups of Y are finite (cf. p. 5–50 & p. 5–87)).]

PROPOSITION 30 Suppose that F : **HFCONCW$_*$** $\to$ **SET** is a cofunctor which converts finite coproducts into finite products and weak pushouts into weak pullbacks. Assume: $\overline{F}$ satisfies the finite Mayer-Vietoris condition—then the cofunctor $\overline{F}$: **PREHCONCW$_*$** $\to$ **SET** is representable.

[By Proposition 28, there is an X in **CONCW$_*$** and a natural isomorphism $\Xi : [-,X] \to F$. Repeating the reasoning used in the proof of Proposition 29, one finds that the extension $\overline{\Xi} : [-,X]_{\text{pre}} \to \overline{F}$ is a natural isomorphism as well.]

PROPOSITION 31 Suppose that F, F' : **HFCONCW$_*$** $\to$ **SET** are cofunctors which convert finite coproducts into finite products and weak pushouts into weak pullbacks. Assume: $\overline{F}$ and $\overline{F}'$ satisfy the finite Mayer-Vietoris condition. Fix natural isomorphisms $\Xi : [-,X] \to F$, $\Xi' : [-,X'] \to F'$, where X, X' are pointed connected CW complexes. Let $T : F \to F'$ be a natural transformation—then there is a pointed continuous function $f : X \to X'$, unique up to prehomotopy, such that the diagram

$$\begin{tikzcd} {[K,X]} \arrow[r, "f_*"] \arrow[d, "\Xi_K"'] & {[K,X']} \arrow[d, "\Xi'_K"] \\ FK \arrow[r, "T_K"'] & F'K \end{tikzcd}$$

commutes for all K.

[Note: If $F = F'$ and T is the identity, then $f : X \to X'$ is a pointed homotopy equivalence.]

PROPOSITION 32 Any representing object in the Adams representability theorem is a group object in **PREHCONCW$_*$** and all such have the same pointed homotopy type.

FACT Let F : **HFCONCW$_*$** $\to$ **SET** be a cofunctor which converts finite coproducts into finite

FACT Let $F : \mathbf{HFCONCW}_* \to \mathbf{SET}$ be a cofunctor which converts finite coproducts into finite products and weak pushouts into weak pullbacks. Assume: $\forall\ K$, $\#(FK) \leq \omega$—then F is representable in the large.

[Note: It is unknown whether the cardinality assumption can be dropped.]

Given pointed connected CW complexes $\begin{cases} X \\ Y \end{cases}$, a pointed continuous function $f : X \to Y$ is said to be a phantom map if it is prehomotopic to 0. Let $\text{Ph}(X, Y)$ be the set of pointed homotopy classes of phantom maps from X to Y—then there is an exact sequence

$$* \to \text{Ph}(X,Y) \to [X,Y] \to \lim [X_k, Y]$$

in $\mathbf{SET}_*$. Of course, $[0] \in \text{Ph}(X,Y)$ but $\#(\text{Ph}(X,Y)) > 1$ is perfectly possible. Example: Take $X = K(\mathbf{Q},3), Y = K(\mathbf{Z},4)$ ($\Rightarrow [X,Y] \approx H^4(\mathbf{Q},3) \approx \text{Ext}(\mathbf{Q},\mathbf{Z}) \approx \mathbf{R}$), realize X as the pointed mapping telescope of the sequence $\mathbf{S}^3 \to \mathbf{S}^3 \to \cdots$, the kth map having degree k, and note that up to homotopy, every $\phi : K \to X$ factors through $\mathbf{S}^3 (\Rightarrow \text{Ph}(X,Y) = [X,Y])$.

Is the arrow $[X,Y] \to \lim [X_k, Y]$ always surjective? While the answer is "yes" under various assumptions on X or Y, what happens in general has yet to be decided.

[Note: By contrast, there is a bijection $\text{Ph}(X,Y) \to \lim^1 [\Sigma X_k, Y]$ of pointed sets (Gray-McGibbon[†]).]

EXAMPLE Meier[‡] has shown that $\text{Ph}(K(\mathbf{Z},n), \mathbf{S}^{n+1}) \approx \text{Ext}(\mathbf{Q},\mathbf{Z})$ for all positive even n. Special case: $\text{Ph}(\mathbf{P}^\infty(\mathbf{C}), \mathbf{S}^3) \approx \text{Ext}(\mathbf{Q},\mathbf{Z})$.

[Note: Suppose that G is an abelian group which is countable and torsion free—then $\exists\ X\ \&\ Y$: $\text{Ph}(X,Y) \approx \text{Ext}(G,\mathbf{Z})$ (Roitberg[‖]).]

EXAMPLE (Universal Phantom Maps) Let X be a pointed connected CW complex. Assume: X has a finite number of cells in each dimension—then it is clear that $f : X \to Y$ is a phantom map iff $\forall\, n > 0$, $f|X^{(n)}$ is nullhomotopic. Denote by $\text{tel}^+\ X$ the pointed telescope of X which starts at $X^{(1)}$ rather than $X^{(0)}$. Recall that the projection $p : \text{tel}^+\ X \to X$ is a pointed homotopy equivalence (cf. p. 3–12). Now collapse each integral joint of $\text{tel}^+\ X$ to a point, i.e., mod out by $\bigvee\limits_{n>0} X^{(n)}$. The resulting quotient can be identified with $\bigvee\limits_{n>0} \Sigma X^{(n)}$ and the arrow $\Theta : \text{tel}^+\ X \to \bigvee\limits_{n>0} \Sigma X^{(n)}$ is a phantom map. It is universal in the sense that if $f : X \to Y$ is a phantom map and if $\overline{f} = f \circ p$, then there is a pointed continuous function $F : \bigvee\limits_{n>0} \Sigma X^{(n)} \to Y$ such that $\overline{f} \simeq F \circ \Theta$. This is because the inclusion $i : \bigvee\limits_{n>0} X^{(n)} \to \text{tel}^+ X$ is a closed cofibration, hence $C_i \approx \bigvee\limits_{n>0} \Sigma X^{(n)}$ (cf. p. 3–24). Corollary: All phantom maps out of X are nullhomotopic iff Θ is nullhomotopic.

[†] *Topology* **32** (1993), 371–394.

[‡] *Quart. J. Math.* **29** (1978), 469–481.

[‖] *Topology Appl.* **59** (1994), 261–271.

[Note: Here is an application. Suppose that $\left\{\begin{matrix} X \\ Y \end{matrix}\right.$ are pointed connected CW complexes with a finite number of cells in each dimension. Claim: If $f : X \to Y$ and $g : Y \to Z$ are phantom maps, then $g \circ f : X \to Z$ is nullhomotopic. To see this, observe that the composite $\bigvee\limits_{n>0} \Sigma X^{(n)} \xrightarrow{F} Y \xrightarrow{p^{-1}} \text{tel}^+ Y \xrightarrow{\Theta} \bigvee\limits_{n>0} \Sigma Y^{(n)}$ is a phantom map. Accordingly, its restriction to each $\Sigma X^{(n)}$ is nullhomotopic, so actually $\Theta \circ p^{-1} \circ F \simeq 0$. Therefore $g \circ f \simeq (\overline{g} \circ p^{-1}) \circ (\overline{f} \circ p^{-1}) \simeq (G \circ \Theta \circ p^{-1}) \circ (F \circ \Theta \circ p^{-1}) \simeq G \circ (\Theta \circ p^{-1} \circ F) \circ \Theta \circ p^{-1} \simeq 0$.]

§ 5

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§6. ABSOLUTE NEIGHBORHOOD RETRACTS

From the point of view of homotopy theory, the central result of this § is the CW-ANR theorem which says that a topological space has the homotopy type of a CW complex iff it has the homotopy type of an ANR. But absolute neighborhood retracts also have a life of their own. For example, their theory is an essential component of infinite dimensional topology.

Consider a pair (X, A), i.e., a topological space X and a subspace $A \subset X$. Let Y be a topological space. Suppose given a continuous function $f : A \rightarrow Y$—then the extension question is: Does there exist a continuous function $F : X \rightarrow Y$ such that $F|A = f$? While this is a complex multifaceted issue, there is an evident connection with the theory of retracts. For if we take $Y = A$, then the existence of a continuous extension $r : X \rightarrow A$ of the identity map id_A amounts to saying that A is a retract of X. Every retract of a Hausdorff space X is necessarily closed in X. On the other hand, if A is closed in X, then with no assumptions on X, a continuous function $f : A \rightarrow Y$ has a continuous extension $F : X \rightarrow Y$ iff Y is a retract of the adjunction space $X \sqcup_f Y$. The opposite end of the spectrum is when A is dense in X. In this case, one can be quite specific and we shall start with it.

Let (X, A) be a pair with A dense in X. Write τ_X and τ_A for the corresponding topologies. Define a map ex $: \tau_A \rightarrow \tau_X$ by $\text{ex}(O) = X - \overline{A - O}$, the bar denoting closure in X—then $\text{ex}(O) \cap A = O$ and $\text{ex}(O) = \bigcup\{U : U \in \tau_X \ \& \ U \cap A = O\}$. Obviously, $\begin{cases} \text{ex}(\emptyset) = \emptyset \\ \text{ex}(A) = X \end{cases}$ and $\forall\, O, P \in \tau_A : \text{ex}(O \cap P) = \text{ex}(O) \cap \text{ex}(P)$. Put $\text{ex}(\mathcal{O}) = \{\text{ex}(O) : O \in \mathcal{O}\}(\mathcal{O} \subset \tau_A)$.

PROPOSITION 1 Let A be a dense subspace of a topological space X; let Y be a regular Hausdorff space—then a given $f \in C(A, Y)$ admits a continuous extension $F \in C(X, Y)$ iff $X = \cup\text{ex}(f^{-1}(\mathcal{V}))$ for every open covering $\mathcal{V}$ of Y.

[The condition is clearly necessary. As for the sufficiency, suppose that $X \neq \emptyset$ and $\#(Y) > 1$. Call $\begin{cases} \tau_X \\ \tau_Y \end{cases}$ the topologies on X and Y.

(F^*) Define a map $F^* : \tau_Y \rightarrow \tau_X$ by

$$F^*(V) = \bigcup\{\text{ex}(f^{-1}(V')) : V' \in \tau_Y \ \& \ \overline{V'} \subset V\}.$$

Note that $\begin{cases} F^*(\emptyset) = \emptyset \\ F^*(Y) = X \end{cases}$ and $\forall\, V_1, V_2 \in \tau_Y : F^*(V_1 \cap V_2) = F^*(V_1) \cap F^*(V_2)$. Let $\{V_j\} \subset \tau_Y$—then $F^*(\bigcup_j V_j) \supset \bigcup_j F^*(V_j)$ and in fact equality prevails. To see this, write

$\bigcup_j V_j = \cup\mathcal{V}$, where $\mathcal{V}$ is the set of all $V \in \tau_Y : \overline{V} \subset V_j$ $(\exists\, j)$. Take a $V' \in \tau_Y : \overline{V'} \subset \bigcup_j V_j$. Since $Y = (Y - \overline{V'})\bigcup(\cup\mathcal{V})$, $X = \text{ex}(f^{-1}(Y - \overline{V'}))\bigcup(\cup\text{ex}(f^{-1}(\mathcal{V})))$. But $\emptyset = \text{ex}(f^{-1}(V')) \cap \text{ex}(f^{-1}(Y - \overline{V'})) \Rightarrow \text{ex}(f^{-1}(V')) \subset \cup\text{ex}(f^{-1}(\mathcal{V})) \subset \bigcup_j F^*(V_j)$, from which it follows that $F^*(\bigcup_j V_j) \subset \bigcup_j F^*(V_j)$.

(F_*) Define a map $F_* : \tau_X \to \tau_Y$ by

$$F_*(U) = \bigcup\{V : V \in \tau_Y \ \&\ F^*(V) \subset U\}.$$

Note that $\forall\, U \in \tau_X$ and $\forall\, V \in \tau_Y : V \subset F_*(U) \Leftrightarrow F^*(V) \subset U$. Indeed, F^* respects arbitrary unions. We claim now that $\forall\, x \in X\ \exists\, y \in Y : F_*(X - \overline{\{x\}}) = Y - \{y\}$. Let $F_*(X - \overline{\{x\}}) = Y - B_x$. Case 1: $B_x = \emptyset$. Here, $X = F^*(Y) \subset X - \overline{\{x\}}$, an impossibility. Case 2: $\#(B_x) > 1$. Choose $y_1, y_2 \in B_x : y_1 \neq y_2$. Choose V_1, $V_2 \in \tau_Y : V_1 \cap V_2 = \emptyset$ & $\begin{cases} y_1 \in V_1 \\ y_2 \in V_2 \end{cases}$—then $V_1 \cap V_2 \subset F_*(X - \overline{\{x\}}) \Rightarrow F^*(V_1 \cap V_2) \subset X - \overline{\{x\}}$, i.e., $F^*(V_1) \cap F^*(V_2) \subset X - \overline{\{x\}}$, thus either $F^*(V_1)$ or $F^*(V_2)$ is contained in $X - \overline{\{x\}}$ and so either V_1 or V_2 is contained in $F_*(X - \overline{\{x\}}) = Y - B_x$, a contradiction.

(F) Define a map $F : X \to Y$ by stipulating that $F(x) = y$ iff $F_*(X - \overline{\{x\}}) = Y - \{y\}$. The definitions imply that $\begin{cases} F^{-1}(V) = F^*(V) \\ F^{-1}(V) \cap A = f^{-1}(V) \end{cases}$ $(V \in \tau_Y)$, therefore $F \in C(X, Y)$ and $F|A = f$.]

Retain the assumption that A is dense in X and Y is regular Hausdorff. Assign to each $x \in X$ the collection $\mathcal{U}(x)$ of all its neighborhoods—then a continuous function $f : A \to Y$ has a continuous extension $F : X \to Y$ iff $\forall\, x$ the filter base $f(\mathcal{U}(x) \cap A)$ converges. The nontrivial part of this assertion is a simple consequence of the preceding result. For suppose that for some open covering $\mathcal{V}$ of $Y : X \neq \cup\text{ex}(f^{-1}(\mathcal{V}))$. Choose $x \in X : x \notin \cup\text{ex}(f^{-1}(\mathcal{V}))$, so $\forall\, U \in \mathcal{U}(x)$ and $\forall\, V \in \mathcal{V} : U \cap A \not\subset f^{-1}(V)$ or still, $f(U \cap A) \not\subset V$. But $f(\mathcal{U}(x) \cap A)$ converges to $y \in Y$. Accordingly, there is (i) $V_0 \in \mathcal{V} : y \in V_0$ and (ii) $U_0 \in \mathcal{U}(x) : f(U_0 \cap A) \subset V_0$. Contradiction.

Here are two other applications.

(C) Suppose that Y is compact Hausdorff—then a continuous function $f : A \to Y$ has a continuous extension $F : X \to Y$ iff for every finite open covering $\mathcal{V}$ of Y there exists a finite open covering $\mathcal{U}$ of X such that $\mathcal{U} \cap A$ is a refinement of $f^{-1}(\mathcal{V})$.

In this statement, one can replace "compact" by "Lindelöf" if "finite" is replaced by "countable". More is true: It suffices to assume that Y is merely $\mathbf{R}$-compact (recall that every Lindelöf regular Hausdorff space is $\mathbf{R}$-compact).

(R-C) Suppose that Y is $\mathbf{R}$-compact—then a continuous function $f : A \to Y$ has a continuous extension $F : X \to Y$ iff for every countable open covering $\mathcal{V}$ of Y there exists a countable open covering $\mathcal{U}$ of X such that $\mathcal{U} \cap A$ is a refinement of $f^{-1}(\mathcal{V})$.

[There is a closed embedding $Y \to \prod \mathbf{R}$. Postcompose f with a generic projection $\prod \mathbf{R} \to \mathbf{R}$ and extend it to X. Form the associated diagonal map $F : X \to \prod \mathbf{R}$—then F is continuous and $F|A = f$ (viewed as a map $A \to \prod \mathbf{R}$). Conclude by remarking that $F(X) = F(\overline{A}) \subset \overline{F(A)} \subset \overline{Y} = Y$.]

[Note: The $\mathbf{R}$-compactness of Y is essential. Consider $X = [0, \Omega]$, $A = Y = [0, \Omega[$, and let $f = \mathrm{id}_A$ (Y is not $\mathbf{R}$-compact, being countably compact but not compact).]

EXAMPLE The proposition can fail if the assumption "Y regular Hausdorff" is weakened to "Y Hausdorff". Let X be the set of nonnegative real numbers. Put $D = \{1/n : n = 1, 2, \ldots\}$—then the collection of all sets of the form $U \cup (V - D)$, where U and V are open in the usual topology on X, is also a topology, call the resulting space Y. Observe that Y is Hausdorff but not regular. Let $A = X - D$ and define $f \in C(A, Y)$ by $f(x) = x$. It is clear that there is no $F \in C(X, Y) : F|A = f$, yet for every open covering $\mathcal{V}$ of Y, $X = \cup \mathrm{ex}\,(f^{-1}(\mathcal{V}))$.

FACT Let A be a dense subspace of a topological space X; let Y be a regular Hausdorff space—then a given $f \in C(A, Y)$ admits a continuous extension $F \in C(X, Y)$ iff $\forall\, x \in X - A\ \exists\, f_x \in C(A \cup \{x\}, Y)$: $f_x|A = f$.

Let X and Y be topological spaces.

(EP) A subspace $A \subset X$ is said to have the extension property with respect to Y (EP w.r.t. Y) if $\forall\, f \in C(A, Y)\ \exists\, F \in C(X, Y)$: $F|A = f$.

(NEP) A subspace $A \subset X$ is said to have the neighborhood extension property with respect to Y (NEP w.r.t. Y) if $\forall\, f \in C(A, Y)\ \exists \begin{cases} U \supset A \\ F \in C(U, Y) \end{cases}$ (U open): $F|A = f$.

[Note: In this terminology, A is a retract (neighborhood retract) of X iff A has the EP (NEP) w.r.t. Y for every Y.]

Two related special cases of importance are when $Y = \mathbf{R}$ or $Y = [0, 1]$. If A has the EP w.r.t. $\mathbf{R}$, then A has the EP w.r.t. $[0, 1]$. Reason: If $f \in C(A, [0, 1])$ and if $F \in C(X, \mathbf{R})$ is a continuous extension of f, then $\min\{1, \max\{0, F\}\}$ is a continuous extension of f with range a subset of $[0, 1]$. The converse is trivially false. Example: Let X be CRH space—then X, as a subspace of βX, has the EP w.r.t. $[0, 1]$ but X has the EP w.r.t. $\mathbf{R}$ iff X is pseudocompact (of course in general X, as a subspace of υX, has the EP w.r.t. $\mathbf{R}$). Bear in mind that a CRH space is compact iff it is both $\mathbf{R}$-compact and pseudocompact.

[Note: Suppose that X is Hausdorff—then X is normal iff every closed subspace has the EP w.r.t. $\mathbf{R}$ (or, equivalently, $[0,1]$).]

Suppose that X is a CRH space. Let A be a subspace of X.

(β) If A has the EP w.r.t. $[0,1]$, then the closure of A in βX is βA and conversely.

(υ) If A has the EP w.r.t. $\mathbf{R}$, then the closure of A in υX is υA and conversely provided that X is in addition normal.

[Note: The Niemytzki plane is a nonnormal hereditarily $\mathbf{R}$-compact space, so the unconditional converse is false.]

Two subsets A and B of a topological space X are said to be completely separated in X if $\exists\ \phi \in C(X,[0,1])$: $\begin{cases} \phi|A = 0 \\ \phi|B = 1 \end{cases}$. For this to be the case, it is necessary and sufficient that A and B are contained in disjoint zero sets. Example: Suppose that X is a CRH space—then any two disjoint closed subsets of X, one of which is compact, are completely separated in X (no compactness assumption being necessary if X is in addition normal).

[Note: It is enough to find a function $f \in C(X)$: $\begin{cases} f|A \leq 0 \\ f|B \geq 1 \end{cases}$. Reason: Take $\phi = \min\{1, \max\{0, f\}\}$. Moreover, 0 and 1 can be replaced by any real numbers r and s with $r < s$.]

PROPOSITION 2 Let $A \subset X$—then A has the EP w.r.t. $[0,1]$ iff any two completely separated subsets of A are completely separated in X.

[Assume that A has the stated property. Fix an $f \in C(A,[0,1])$. To construct an extension $F \in C(X,[0,1])$ of f, we shall first define by recursion two sequences $\{f_n\}$ and $\{g_n\}$ subject to: $f_n \in BC(A)$ & $\|f_n\| \leq 3r_n$ and $g_n \in BC(X)$ & $\|g_n\| \leq r_n$, where $r_n = (1/2)(2/3)^n$ (so $\sum\limits_1^\infty r_n = 1$). Set $f_1 = f$. Given f_n, let $\begin{cases} S_n^- = \{x \in A : f_n(x) \leq -r_n\} \\ S_n^+ = \{x \in A : f_n(x) \geq r_n\} \end{cases}$. Since $\begin{cases} S_n^- \\ S_n^+ \end{cases}$ are completely separated in A, they are, by hypothesis, completely separated in X. Choose $g_n \in BC(X)$: $\begin{cases} g_n|S_n^- = -r_n \\ g_n|S_n^+ = r_n \end{cases}$ & $\|g_n\| \leq r_n$. Push the recursion forward by setting $f_{n+1} = f_n - g_n|A$. The series $\sum\limits_1^\infty g_n$ is uniformly convergent on X, thus its sum G is a continuous function on $X : G|A = f$. Take $F = \max\{0, G\}$.]

Application: Suppose that X is a CRH space—then any compact subset of X has the EP w.r.t. $[0,1]$ (cf. p. 2–14).

FACT Let $A \subset X$; let $f \in BC(A)$—then $\exists\ F \in BC(X) : F|A = f$ iff $\forall\ a, b \in \mathbf{R} : a < b$, the sets $\begin{cases} f^{-1}(]-\infty, a]) \\ f^{-1}([b, +\infty[) \end{cases}$ are completely separated in X.

PROPOSITION 3 Let $A \subset X$—then A has the EP w.r.t. $\mathbf{R}$ iff A has the EP w.r.t. $[0, 1]$ and is completely separated from any zero set in X disjoint from it.

[Necessity: Let Z be a zero set in X disjoint from $A : Z = Z(g)$, where $g \in C(X, [0, 1])$. Put $f = (1/g)|A$. Choose $h \in C(X) : h|A = f$. Consider gh.

Sufficiency: Fix an $f \in C(A)$. Because $\arctan \circ f \in C(A, [-\pi/2, \pi/2])$, it has an extension $G \in C(X, [-\pi/2, \pi/2])$. Let $B = G^{-1}(\pm\pi/2)$—then B is a zero set in X disjoint from A, so there exists $\phi \in C(X, [0, 1]) : \begin{cases} \phi|A = 1 \\ \phi|B = 0 \end{cases}$. Put $F = \tan(\phi G)$: $F \in C(X)$ & $F|A = f$.]

Consequently, every zero set in X that has the EP w.r.t. $[0, 1]$ actually has the EP w.r.t. $\mathbf{R}$. On the other hand, a zero set in X need not have the EP w.r.t. $[0, 1]$. Examples: (1) Take for X the Isbell-Mrówka space $\Psi(\mathbf{N})$—then $A = \mathcal{S}$ is a zero set in X but $\mathcal{S}$ does not have the EP w.r.t. $[0, 1]$; (2) Take for X the Niemytzki plane—then $A = \{(x, y) : y = 0\}$ is a zero set in X but A does not have the EP w.r.t. $[0, 1]$.

EXAMPLE (Katětov Space) As a subspace of $\mathbf{R}$, $\mathbf{N}$ has the EP w.r.t. $[0, 1]$, so the closure of $\mathbf{N}$ in $\beta\mathbf{R}$ is $\beta\mathbf{N}$. Let $X = \beta\mathbf{R} - (\beta\mathbf{N} - \mathbf{N})$—then $\beta X = \beta\mathbf{R}$ and X is a LCH space which is actually pseudocompact (an unbounded continuous function on X would be unbounded on a closed subset of $\mathbf{R}$ disjoint from $\mathbf{N}$). However, X is not countably compact, thus is not normal (cf. §1, Proposition 5). As a subspace of X, $\mathbf{N}$ has the EP w.r.t. $[0, 1]$ but does not have the EP w.r.t. $\mathbf{R}$.

[Note: $\mathbf{N}$ is a closed G_δ but is not a zero set in X.]

A subspace $A \subset X$ is said to be $\mathcal{Z}$-embedded in X if every zero set in A is the intersection of A with a zero set in X. Example: Any cozero set in X is $\mathcal{Z}$-embedded in X. If A has the EP w.r.t. $[0, 1]$, then A is $\mathcal{Z}$-embedded in X (but not conversely), so, e.g., any retract of X is $\mathcal{Z}$-embedded in X. Examples: Suppose that X is Hausdorff—then (1) Every subspace of a perfectly normal X is $\mathcal{Z}$-embedded in X; (2) Every F_σ-subspace of a normal X is $\mathcal{Z}$-embedded in X; (3) Every Lindelöf subspace of a completely regular X is $\mathcal{Z}$-embedded in X.

FACT Let $A \subset X$—then A has the EP w.r.t. $\mathbf{R}$ iff A is $\mathcal{Z}$-embedded in X and is completely separated from any zero set in X disjoint from it.

[Note: It is a corollary that if A is a zero set in X, then A has the EP w.r.t. $\mathbf{R}$ iff A is $\mathcal{Z}$-embedded in X. Both the Isbell-Mrówka space and the Niemytzki plane contain zero sets that are not $\mathcal{Z}$-embedded.]

Application: Suppose that X is a Hausdorff space—then X is normal iff every closed subset of X is $\mathcal{Z}$-embedded in X.

PROPOSITION 4 Let $A \subset X$—then A has the EP w.r.t. $[0,1]$ ($\mathbf{R}$) iff for every finite (countable) numerable open covering $\mathcal{O}$ of A there exists a finite (countable) numerable open covering $\mathcal{U}$ of X such that $\mathcal{U} \cap A$ is a refinement of $\mathcal{O}$.

[The proof of necessity is similar to but simpler than the proof of sufficiency so we shall deal just with it, assuming only that there exists a numerable open covering $\mathcal{U}$ of X such that $\mathcal{U} \cap A$ is a refinement of $\mathcal{O}$, thereby omitting the cardinality assumption on $\mathcal{U}$.

([0,1]) Let $\left\{\begin{array}{l} S' \\ S'' \end{array}\right.$ be two completely separated subsets of A; let $\left\{\begin{array}{l} Z' \\ Z'' \end{array}\right.$ be two disjoint zero sets in A : $\left\{\begin{array}{l} S' \subset Z' \\ S'' \subset Z'' \end{array}\right.$. Let $\mathcal{O} = \{A - Z', A - Z''\}$. Take $\mathcal{U}$ per $\mathcal{O}$ and choose a neighborhood finite cozero set covering $\mathcal{V}$ of X such that $\mathcal{V}$ is a star refinement of $\mathcal{U}$ (cf. §1, Proposition 13). Put $\left\{\begin{array}{l} W' = X - \bigcup\{V \in \mathcal{V} : V \cap Z' = \emptyset\} \\ W'' = X - \bigcup\{V \in \mathcal{V} : V \cap Z'' = \emptyset\} \end{array}\right.$—then $\left\{\begin{array}{l} W' \\ W'' \end{array}\right.$ are disjoint zero sets in X : $\left\{\begin{array}{l} Z' \subset W' \\ Z'' \subset W'' \end{array}\right.$. Therefore S' and S'' are completely separated in X, thus, by Proposition 2, A has the EP w.r.t. $[0,1]$.

($\mathbf{R}$) Let Z be a zero set in $X : A \cap Z = \emptyset$, say $Z = Z(f)$, where $f \in C(X,[0,1])$. The collection $\mathcal{O} = \{f^{-1}(]1/n,1]) \cap A\}$ is a countable cozero set covering of A, hence is numerable (cf. p. 1–25). Take $\mathcal{U}$ per $\mathcal{O}$ and choose a neighborhood finite cozero set covering $\mathcal{V} = \{V_j : j \in J\}$ of X and a zero set covering $\mathcal{Z} = \{Z_j : j \in J\}$ of X such that $\mathcal{V}$ is a refinement of $\mathcal{U}$ with $Z_j \subset V_j$ $(\forall\, j)$ (cf. p. 1–25). Given j, $\exists\, n_j : Z_j \cap A \subset f^{-1}(]1/n_j,1]) \cap A$. Put $W = \bigcup\limits_j Z_j \cap f^{-1}([1/n_j,1])$—then W is a zero set in X containing A and disjoint from Z, so A and Z are completely separated in X. Since the first part of the proof implies that A necessarily has the EP w.r.t. $[0,1]$, it follows from Proposition 3 that A has the EP w.r.t. $\mathbf{R}$.]

FACT Let $A \subset X$—then A is $\mathcal{Z}$-embedded in X iff for every finite numerable open covering $\mathcal{O}$ of A there exists a cozero set U containing A and a finite numerable open covering $\mathcal{U}$ of U such that $\mathcal{U} \cap A$ is a refinement of $\mathcal{O}$.

LEMMA Let (X,d) be a metric space; let A be a nonempty closed proper subspace of X—then there exists a subset $\{a_i : i \in I\}$ of A and a neighborhood finite open covering $\{U_i : i \in I\}$ of $X - A$ such that $\forall\, i : x \in U_i \Rightarrow d(x,a_i) \leq 2d(x,A)$.

[Assign to each $x \in X - A$ the open ball B_x of radius $d(x,A)/4$. The collection $\{B_x : x \in X - A\}$ is an open covering of $X - A$, thus by paracompactness has a neighborhood finite open refinement $\{U_i : i \in I\}$. Each U_i determines a point $x_i \in X - A : U_i \subset B_{x_i}$, from which a point $a_i \in A : d(x_i, a_i) \leq (5/4)d(x_i, A)$. Obviously, $\forall\ x \in U_i : d(x, a_i) \leq (3/2)d(x_i, A)$ and $d(x_i, A) \leq (4/3)d(x, A)$.]

DUGUNDJI EXTENSION THEOREM Let (X, d) be a metric space; let A be a closed subspace of X. Let E be a locally convex topological vector space. Equip $\begin{cases} C(A, E) \\ C(X, E) \end{cases}$ with the compact open topology—then there exists a linear embedding $\text{ext} : C(A, E) \rightarrow C(X, E)$ such that $\forall\ f \in C(A, E)$, $\text{ext}(f)|A = f$ and the range of $\text{ext}(f)$ is contained in the convex hull of the range of f.

[Assume that A is nonempty, proper and, using the notation of the lemma, choose a partition of unity $\{\kappa_i : i \in I\}$ on $X - A$ subordinate to $\{U_i : i \in I\}$. Given $f \in C(A, E)$, let

$$\text{ext}(f)(x) = \begin{cases} f(x) & (x \in A) \\ \sum\limits_i \kappa_i(x) f(a_i) & (x \in X - A). \end{cases}$$

Then $\text{ext}(f)|A = f$ and it is clear that $\text{ext}(f)(X)$ is contained in the convex hull of $f(A)$. The continuity of $\text{ext}(f)$ is built in at the points of $X - A$. As for the points of A, fix $a_0 \in A$ and let N be a balanced convex neighborhood of zero in E. Choose a $\delta > 0 : d(a, a_0) \leq \delta \Rightarrow f(a) - f(a_0) \in N (a \in A)$. Suppose that $\begin{cases} x \in X - A \\ d(x, a_0) < \delta/3 \end{cases}$. If $\kappa_i(x) > 0$, then, from the lemma, $d(x, a_i) \leq 2d(x, A)$, hence $d(a_i, a_0) \leq 3d(x, a_0) < \delta$. Consequently,

$$\text{ext}(f)(x) - \text{ext}(f)(a_0) = \sum_i \kappa_i(x)(f(a_i) - f(a_0)) \in \sum_i \kappa_i(x) N \subset N.$$

Therefore $\text{ext}(f) \in C(X, E)$. By construction, ext is linear and one-to-one, so the only remaining issue is its continuity. Take a nonempty compact subset K of X and let $O(K, N) = \{F \in C(X, E) : F(K) \subset N\}$. Put $K_A = K \cap A \cup \{a_i \in A : K \cap U_i \neq \emptyset\}$. Let $O(K_A, N) = \{f \in C(A, E) : f(K_A) \subset N\}$. Plainly, $f \in O(K_A, N) \Rightarrow \text{ext}(f) \in O(K, N)$. Claim: K_A is compact. To see this, let $\{x_n\}$ be a sequence in K_A. Since $K \cap A$ is compact, we can suppose that $\{x_n\}$ has no subsequence in $K \cap A$, thus without loss of generality, $x_n = a_{i_n}$ for some $i_n : K \cap U_{i_n} \neq \emptyset$. Pick $y_n \in K \cap U_{i_n}$ and assume that $y_n \rightarrow y \in K$. Case 1: $y \in K \cap A$. Here, $d(x_n, y) = d(a_{i_n}, y) \leq 3d(y_n, y) \rightarrow 0$. Case 2: $y \in K \cap (X - A)$. There is a neighborhood of y that meets finitely many of the U_i and once y_n is in this neighborhood, the index i_n is constrained to a certain finite subset of I, which means that $\{x_n\}$ has a constant subsequence.]

[Note: Suppose that E is a normed linear space—then the image of $\text{ext}|BC(A, E)$ is contained in $BC(X, E)$ and, per the uniform topology, $\text{ext} : BC(A, E) \rightarrow BC(X, E)$ is a linear isometric embedding: $\forall\ f \in BC(A, E)$, $\|f\| = \|\text{ext}(f)\|$.]

In passing, observe that if the a_i are chosen from some given dense subset $A_0 \subset A$, then the range of $\text{ext}(f)$ is contained in the union of $f(A)$ and the convex hull of $f(A_0)$.

The Dugundji extension theorem has many applications. To mention one, it is a key ingredient in the proof of a theorem of Milyutin to the effect that if X and Y are uncountable metrizable compact Hausdorff spaces, then $C(X)$ and $C(Y)$ are linearly homeomorphic (Pełczyński[†]). Extensions to the case of noncompact X and Y have been given by Etcheberry[‡].

[Note: The Banach-Stone theorem states that if X and Y are compact Hausdorff spaces, then X and Y are homeomorphic provided that the Banach spaces $C(X)$ and $C(Y)$ are isometrically isomorphic (Behrends[‖]).]

Is Dugundji's extension theorem true for an arbitrary topological vector space E? In other words, can the "locally convex" supposition on E be dropped? The answer is "no", even if E is a linear metric space (cf. p. 6–12).

[Note: A topological vector space E is said to be a linear metric space if it is metrizable. Every linear metric space E admits a translation invariant metric (Kakutani) but E need not be normable.]

Let X be a CRH space; let A be a nonempty closed subspace of X. Let E be a locally convex topological vector space (normed linear space)—then a linear operator $T : C(A,E) \to C(X,E)$ $(T : BC(A,E) \to BC(X,E))$ continuous for the compact open topology (uniform topology) is said to be a linear extension operator if for all f in $C(A,E)$ $(BC(A,E))$: $Tf|A = f$. Write $\text{LEO}(X,A;E)$ $(\text{LEO}_b(X,A;E))$ for the set of linear extension operators associated with $C(A,E)$ $(BC(A,E))$. Assuming that X is metrizable, the Dugundji extension theorem asserts: $\forall\, A, C(A,E)$ $(BC(A,E))$ possesses a linear extension operator (and even more in that the "same" operator works for both). Question: What conditions on X or A serve to ensure that $\text{LEO}(X,A;E)$ $(\text{LEO}_b(X,A;E))$ is not empty?

EXAMPLE (The Michael Line) Take the set $\mathbf{R}$ and topologize it by isolating the points of $\mathbf{P}$, leaving the points of $\mathbf{Q}$ with their usual neighborhoods. The resulting space X is Hausdorff and hereditarily paracompact but not locally compact. And $A = \mathbf{Q}$ is a closed subspace of X which, however, is not a G_δ in X. Let $E = C(\mathbf{P})$, $\mathbf{P}$ in its usual topology—then E is a locally convex topological vector space (compact open topology). Claim: $\text{LEO}(X,A;E)$ is empty. For this, it suffices to exhibit an $f \in C(A,E)$

[†] *Dissertationes Math.* **58** (1968), 1–92; see also Semadeni, *Banach Spaces of Continuous Functions*, PWN (1971), 379.

[‡] *Studia Math.* **53** (1975), 103–127; see also Hess, *SLN* **991** (1983), 103–110.

[‖] *SLN* **736** (1979), 138–140.

that cannot be extended to an $F \in C(X, E)$. If $\mathbf{P}$ has its usual topology, then the continuous function $\begin{cases} A \times \mathbf{P} \rightarrow \mathbf{R} \\ (x, y) \rightarrow 1/(y - x) \end{cases}$ has no continuous extension $X \times \mathbf{P} \rightarrow \mathbf{R}$ (thus $X \times \mathbf{P}$ is not normal). Defining $f \in C(A, E)$ by $f(x)(y) = 1/(y - x)$, it follows that f has no extension $F \in C(X, E)$.

A Hausdorff space X is said to be submetrizable if its topology contains a metrizable topology. Examples: (1) The Michael line is submetrizable and normal but not perfect; (2) The Niemytzki plane is submetrizable and perfect but not normal.

FACT Let X be a submetrizable CRH space. Suppose that A is a nonempty closed subspace of X with a compact frontier—then $\forall\, E, \mathrm{LEO}(X, A; E)$ $(\mathrm{LEO}_b(X, A; E))$ is not empty.

[Note: In view of the preceding example, the hypothesis on A is not superfluous.]

When $E = \mathbf{R}$, denote by $\mathrm{LEO}(X, A)$ $(\mathrm{LEO}_b(X, A))$ the set of linear extension operators for $C(A)$ $(BC(A))$.

EXAMPLE $\mathrm{LEO}_b(X, A)$ can be empty, even if X is a compact Hausdorff space. For a case in point, take $X = \beta\mathbf{N}$ & $A = \beta\mathbf{N} - \mathbf{N}$. Claim: $\mathrm{LEO}(X, A)$ $(= \mathrm{LEO}_b(X, A))$ is empty. Suppose not and let $T : C(A) \rightarrow C(X)$ be a linear extension operator. Fix an uncountable collection $\mathcal{U} = \{U_i : i \in I\}$ of nonempty pairwise disjoint open subsets of A. Pick an $a_i \in U_i$ and choose an $f_i \in C(A, [0, 1])$: $\begin{cases} f_i(a_i) = 1 \\ f_i|(A - U_i) = 0 \end{cases}$. Let $O_i = \{x \in X : Tf_i(x) > 1/2\}$. Since X is separable, there exists an uncountable subset I_0 of I and a point $x_0 \in X : x_0 \in \bigcap_{i \in I_0} O_i$. Let n be some integer $> \|T\|$. Select distinct indices i_k $(k = 1, \ldots, 2n)$ in I_0. Put $f = \sum_1^{2n} f_{i_k}$, so $\|f\| = 1$. A contradiction then results by writing

$$n = n\|f\| \geq \|Tf\| \geq Tf(x_0) = \sum_1^{2n} Tf_{i_k}(x_0) > 2n \cdot \frac{1}{2} = n.$$

[Note: Let X be a compact Hausdorff space; let A be a nonempty closed subspace of X. Set $\rho(X, A) = \inf\{\|T\| : T \in \mathrm{LEO}(X, A)\}$ (where $\rho(X, A) = \infty$ if $\mathrm{LEO}(X, A)$ is empty). Of course, $\rho(X, A) \geq 1$ and Benyamini[†] has shown that $\forall\, r : 1 \leq r < \infty$, there exists a pair $(X, A) : \rho(X, A) = r$.]

The space X figuring in the preceding example is not perfect (no point of $\beta\mathbf{N} - \mathbf{N}$ is a G_δ in $\beta\mathbf{N}$). Can one get a positive result if perfection is assumed? The answer is "no". Indeed, van Douwen[‡] has constructed an example of a CRH space X that is simultaneously perfect and paracompact, yet contains a nonempty closed subspace A for which $\mathrm{LEO}_b(X, A) = \emptyset$.

[†] *Israel J. Math.* **16** (1973), 258–262.

[‡] *General Topology Appl.* **5** (1975), 297–319.

The assumption that $\text{LEO}_b(X, A)$ is not empty $\forall$ A has implications for the topology of X. To quantify the situation, given $r : 1 \leq r < \infty$, let b_r be the condition: $\forall$ A, $\{T \in \text{LEO}_b(X, A)$: $\|T\| \leq r\} \neq \emptyset$. Claim: If b_r is in force, then for any discrete collection $\mathcal{A} = \{A_i : i \in I\}$ of nonempty closed subsets of X there is a collection $\mathcal{U} = \{U_i : i \in I\}$ of open subsets of X such that (1) $A_i \subset U_i$ & $i \neq j \Rightarrow U_i \cap A_j = \emptyset$ and (2) $\text{ord}(\mathcal{U}) \leq [r]$. Thus put $A = \cup\mathcal{A}$, let $\chi_i : A \to [0,1]$ be the characteristic function of A_i, choose $T \in \text{LEO}_b(X, A) : \|T\| \leq r$, and consider $\mathcal{U} = \{U_i : i \in I\}$, where $U_i = \{x \in X : T\chi_i(x) > r/[r] + 1\}$. Example: Suppose that X satisfies b_r for some $r < 2$—then X is collectionwise normal.

[Note: Let X be the Michael line—then one can show that X satisfies b_1, yet $\text{LEO}(X, A) = \emptyset$ if $A = \mathbf{Q}$.]

FACT Let X be a Moore space. Assume: X satisfies b_r for some r—then X is normal and metacompact.

Let X be a nonempty topological space—then an equiconnecting structure on X is a continuous function $\lambda : IX^2 \to X$ such that $\forall$ $x, y \in X$ and $\forall$ $t \in [0,1]$: $\begin{cases} \lambda(x,y,0) = x \\ \lambda(x,y,1) = y \end{cases}$ & $\lambda(x,x,t) = x$. A subset $A \subset X$ for which $\lambda(IA^2) \subset A$ is called λ-convex. In order that X have an equiconnecting structure, it is necessary that X be both contractible and locally contractible but these conditions are not sufficient as can be seen by considering Borsuk's cone (cf. p. 6–15). Example: Suppose that X is a contractible topological group. Let $H : IX \to X$ be a homotopy contracting X to its unit element e—then the prescription $\lambda(x,y,t) = H(e,t)^{-1}H(xy^{-1},t)y$ defines an equiconnecting structure on X. In particular, if X is a topological vector space, then $H(x,t) = (1-t)x$ will do.

[Note: Let E be an infinite dimensional Banach space. Consider $\mathbf{GL}(E)$, the group of invertible bounded linear transformations $T : E \to E$. Equip $\mathbf{GL}(E)$ with the topology induced by the operator norm—then $\mathbf{GL}(E)$ is a topological group and, being an open subset of a Banach space, has the homotopy type of a CW complex (cf. §5, Proposition 6). If E is actually a Hilbert space, then $\mathbf{GL}(E)$ is contractible (Kuiper[†]) but this need not be true in general (even if E is reflexive), although it is the case of certain specific spaces, e.g., $C([0,1])$ or $L^p([0,1])$ $(1 \leq p \leq \infty)$. See Mityagin[‡] for proofs and other remarks.]

FACT A nonempty topological space X has an equiconnecting structure iff the diagonal Δ_X is a strong deformation retract of $X \times X$.

[Necessity: Given λ, consider the homotopy $H : IX^2 \to X^2$ defined by $H((x,y),t) = (\lambda(x,y,t), y)$.

[†] *Topology* **3** (1965), 19–30.

[‡] *Russian Math. Surveys* **25** (1970), 59–103.

Sufficiency: Given H, consider the equiconnecting structure $\lambda : IX^2 \rightarrow X$ defined by

$$\lambda(x,y,t) = \begin{cases} p_1(H((x,y),2t)) & (0 \leq t \leq 1/2) \\ p_2(H((x,y),2-2t)) & (1/2 \leq t \leq 1) \end{cases},$$

where p_1 and p_2 are the projections onto the first and second factors.]

FACT Suppose that X is a nonempty topological space for which the inclusion $\Delta_X \rightarrow X \times X$ is a cofibration—then X has an equiconnecting structure iff X is contractible.

[Choose a homotopy $H : IX \rightarrow X$ contracting X to x_0 : $\begin{cases} H(x,0) = x \\ H(x,1) = x_0 \end{cases}$ and then define $\Lambda : IX^2 \rightarrow X^2$ by $\Lambda((x,y),t) = (H(x,t), H(y,t))$ to see that Δ_X is a weak deformation retract of $X \times X$.]

A nonempty topological space X is said to be locally convex if it admits an equiconnecting structure λ such that every $x \in X$ has a neighborhood basis comprised of λ-convex sets. The convex subsets of a locally convex topological vector space are therefore locally convex, where $\lambda(x,y,t) = (1-t)x + ty$. On the other hand, the long ray L^+ is not locally convex.

EXAMPLE Let $K = (V, \Sigma)$ be a vertex scheme. Suppose that K is full, i.e., if $F \subset V$ is finite and nonempty, then $F \in \Sigma$. Claim: $|K|$ is locally convex. Thus fix a point $* \in V$. Let $\phi \in |K|$—then $\phi = \sum\limits_{v \neq *} b_v(\phi)\chi_v + (1 - \sum\limits_{v \neq *} b_v(\phi))\chi_*$. Here, χ_v (χ_*) is the characteristic function of $\{v\}$ $(\{*\})$. Define $\beta : |K| \times |K| \rightarrow |K|$ by $\beta(\phi,\psi) = \sum\limits_{v \neq *} \beta(\phi,\psi)_v \chi_v + (1 - \sum\limits_{v \neq *} \beta(\phi,\psi)_v)\chi_*$, where $\beta(\phi,\psi)_v = \min\{b_v(\phi), b_v(\psi)\}$. The assignment

$$\lambda(\phi,\psi,t) = \begin{cases} (1-2t)\phi + 2t\beta(\phi,\psi) & (0 \leq t \leq 1/2) \\ (2-2t)\beta(\phi,\psi) + (2t-1)\psi & (1/2 \leq t \leq 1) \end{cases}$$

is an equiconnecting structure on $|K|$ relative to which $|K|$ is locally convex.

FACT Let $A \subset X$, where X is metrizable and A is closed—then A has the EP w.r.t. any locally convex topological space.

PLACEMENT LEMMA Every metric space (X,d) can be isometrically embedded as a closed subspace of a normed linear space E, where $\text{wt}\, E = \omega\text{wt}\, X$.

[Denote by Σ the collection of all nonempty finite subsets of X. Give Σ the discrete topology. Fix a point $x_0 \in X$. Attach to each $x \in X$ a function $f_x : \begin{cases} \Sigma \rightarrow \mathbf{R} \\ \sigma \rightarrow d(x,\sigma) - d(x_0,\sigma) \end{cases}$ —then $f_x \in BC(\Sigma)$ and the assignment $\iota : \begin{cases} X \rightarrow BC(\Sigma) \\ x \rightarrow f_x \end{cases}$ is an isometric embedding. Note that $f_{x_0} \equiv 0$. Let E be the linear span of $\iota(X)$ in $BC(\Sigma)$. To see that $\iota(X)$ is closed in E, take a $\phi \in E - \iota(X)$, say $\phi = \sum\limits_{0}^{n} r_i f_{x_i}$ (r_i real), put $\sigma = \{x_0, \ldots, x_n\}$ and choose δ

positive and less than $(1/2)\min_i \|\phi - f_{x_i}\|$. Claim: No element of $\iota(X)$ can be within δ of ϕ. Suppose not, so $\exists\ x \in X : \|\phi - f_x\| < \delta$. Since ι is an isometry,

$$d(x, x_i) = \|f_{x_i} - f_x\| \geq \|\phi - f_{x_i}\| - \|\phi - f_x\| > 2\delta - \delta = \delta,$$

from which $\|\phi - f_x\| \geq |\phi(\sigma) - f_x(\sigma)| = d(x, \sigma) \geq \delta$, a contradiction. There remains the assertion on the weights. For this, let D be a dense subset of $\iota(X)$ of cardinality $\leq \kappa : f_{x_0} \in D$—then the linear span of D is dense in E and contains a dense subset of cardinality $\leq \omega\kappa$.]

[Note: One can obviously arrange that E is complete provided this is the case of (X, d).]

FACT Every CRH space X can be embedded as a closed subspace of a locally convex topological vector space E.

Let Y be a nonempty metrizable space.

(AR) Y is said to be an absolute retract (AR) if under any closed embedding $Y \rightarrow Z$ into a metrizable space Z, the image of Y is a retract of Z.

(ANR) Y is said to be an absolute neighborhood retract (ANR) if under any closed embedding $Y \rightarrow Z$ into a metrizable space Z, the image of Y is a neighborhood retract of Z.

[Note: There is no map from a nonempty set to the empty set, thus $\emptyset$ cannot be an AR, but there is a map from the empty set to the empty set, so we shall extend the terminology and agree that $\emptyset$ is an ANR.]

PROPOSITION 5 Let Y be a nonempty metrizable space—then Y is an AR (ANR) iff for every pair (X, A), where X is metrizable and $A \subset X$ is closed, A has the EP (NEP) w.r.t. Y.

[The indirect assertion is obvious. Turning to the direct assertion, in view of the placement lemma, Y can be realized as a closed subspace of a normed linear space E. Assuming that Y is an AR, fix a retraction $r : E \rightarrow Y$. If now $f : A \rightarrow Y$ is a continuous function, then by the Dugundji extension theorem, $\exists\ F \in C(X, E) : F|A = f$. Consider $r \circ F$.]

EXAMPLE Cauty[†] has given an example of a linear metric space E which is not an absolute retract. So, for this E, the Dugundji extension theorem must fail.

[†] *Fund. Math.* **146** (1994), 85–99.

[Note: Therefore a metrizable space that has an equiconnecting structure need not be an AR.]

A countable product of nonempty metrizable spaces is an AR iff all the factors are ARs. Example: $[0,1]^n, \mathbf{R}^n, [0,1]^\omega$, and $\mathbf{R}^\omega$ are absolute retracts. A countable product of nonempty metrizable spaces is an ANR iff all the factors are ANRs and all but finitely many of the factors are ARs. Example: $\mathbf{S}^n$ and $\mathbf{T}^n$ are absolute neighborhood retracts but $\left\{\begin{matrix} \mathbf{S}^n \times \mathbf{S}^n \times \cdots \\ \mathbf{T}^n \times \mathbf{T}^n \times \cdots \end{matrix}\right.$ (ω factors) are not absolute neighborhood retracts.

Every retract (neighborhood retract) of an AR (ANR) is an AR (ANR). An open subspace of an ANR is an ANR.

EXAMPLE Let E be a normed linear space—then every nonempty convex subset of E is an AR and every open subset of E is an ANR. Assume in addition that E is infinite dimensional. Let S be the unit sphere in E—then S is an AR. To establish this, it need only be shown that S is a retract of D, the closed unit ball in E. Fix a proper dense linear subspace $E_0 \subset E$ (the kernel of a discontinuous linear functional on E will do). In the notation of the Dugundji extension theorem, work with the pair (D,S), picking the points defining ext in $S \cap E_0$, and let $f = \mathrm{id}_S$—then there exists a continuous function $\mathrm{ext}(f) : D \rightarrow E$ such that $\mathrm{ext}(f)|S = \mathrm{id}_S$, with $\mathrm{ext}(f)(D)$ contained in $S \cup (D \cap E_0)$, a proper subset of D. Choose a point p in the interior of $D : p \notin \mathrm{ext}(f)(D)$, let $r : D - \{p\} \rightarrow S$ be the corresponding radial retraction and consider $r \circ \mathrm{ext}(f)$. Corollary: Not every continuous function $D \rightarrow D$ has a fixed point.

[Note: There is another way to argue. Klee[†] has shown that if E is an infinite dimensional normed linear space and if $K \subset E$ is compact, then E and $E - K$ are homeomorphic. In particular, $E - \{0\}$ is homeomorphic to E, thus is an AR, and so S, being a retract of $E - \{0\}$, is an AR. Matters are trivial if E is an infinite dimensional Banach space, since then E is actually homeomorphic to S.]

EXAMPLE Let Y be any set lying between $]0,1[^n$ and $[0,1]^n$—then Y is an AR. Thus let f be a closed embedding $Y \rightarrow Z$ of Y into a metrizable space Z. Call j the inclusion $Y \rightarrow [0,1]^n$, so $j \circ f^{-1} \in C(f(Y),[0,1]^n)$. Choose a $g \in C(Z,[0,1]^n) : g|f(Y) = j \circ f^{-1}$. Fix a compatible metric d on Z and define a continuous function $h : Z \rightarrow [0,1]^n \times [0,1]$ by sending z to $(g(z), \min\{1, d(z, f(Y))\})$. The range of h is therefore a subset of $i_0 Y \cup [0,1]^n \times]0,1]$. Let $r : i_0 Y \cup [0,1]^n \times]0,1] \rightarrow i_0 Y$ be the retraction determined by projecting from the point $(1/2, \ldots, 1/2, -1) \in \mathbf{R}^{n+1}$ and let $p : i_0 Y \rightarrow Y$ be the canonical map. That $f(Y)$ is a retract of Z is then seen by considering the composite $f \circ p \circ r \circ h$.

FACT Let Y be an AR; let B be a nonempty closed subspace of Y—then B is an AR iff B is a strong deformation retract of Y.

[†] *Proc. Amer. Math. Soc.* **7** (1956), 673–674.

[To see that the condition is necessary, fix a retraction $r : Y \to B$ and define a continuous function $h : i_0Y \cup IB \cup i_1Y \to Y$ by $h(y,t) = \begin{cases} y \ (y \in Y, t = 0) \\ y \ (y \in B, 0 \leq t \leq 1) \\ r(y) \ (y \in Y, t = 1) \end{cases}$. Since $i_0Y \cup IB \cup i_1Y$ is a closed subspace of IY and since B is an AR, it follows from Proposition 5 that h has a continuous extension $H : IY \to Y$.]

Let Y be an AR—then Y is homeomorphic to its diagonal Δ_Y which is therefore a strong deformation retract of $Y \times Y$ and this means that Y has an equiconnecting structure (cf. p. 6–10).

[Note: A metrizable locally convex topological space is an AR (cf. p. 6–11 and Proposition 5) but not every AR is locally convex.]

FACT Let Y be an ANR; let B be a closed subspace of Y—then B is an ANR iff the inclusion $B \to Y$ is a cofibration.

[If B is an ANR, then so is $i_0Y \cup IB$ (cf. p. 6–43 (NES_4)), thus there exists a neighborhood O of $i_0Y \cup IB$ in IY and a retraction $r : O \to i_0Y \cup IB$. Choose a neighborhood V of B in $Y : IV \subset O$ and fix $\phi \in C(Y,[0,1]) : \begin{cases} \phi|B = 1 \\ \phi|Y - V = 0 \end{cases}$. Consider the map $\begin{cases} IY \to i_0Y \cup IB \\ (y,t) \to r(y,\phi(y)t) \end{cases}$.]

Let Y be an ANR—then Y is homeomorphic to its diagonal Δ_Y, hence the inclusion $\Delta_Y \to Y \times Y$ is a cofibration. Consequently, Y is uniformly locally contractible (cf. p. 3–14) and $\forall\, y_0 \in Y$, (Y, y_0) is wellpointed (cf. p. 3–15).

[Note: It is unknown whether every metrizable uniformly locally contractible space is an ANR. Any counterexample would necessarily have infinite topological dimension (cf. infra).]

Thanks to the placement lemma and the fact that a retract of a contractible (locally contractible) space is contractible (locally contractible), every AR (ANR) is contractible (locally contractible). Both the broom and the cone over the Cantor set are contractible but, failing to be locally contractible, neither is an ANR.

LEMMA Suppose that Y is a contractible ANR—then Y is an AR.

A locally path connected topological space X is said to be locally n-connected ($n \geq 1$) provided that for any $x \in X$ and any neighborhood U of x there exists a neighborhood $V \subset U$ of x such that the arrow $\pi_q(V,x) \to \pi_q(U,x)$ induced by the inclusion $V \to U$ is the trivial map ($1 \leq q \leq n$). If X is locally n-connected for all n, then X is called locally homotopically trivial. Example: A locally contractible space is locally homotopically trivial.

EXAMPLE Working in ℓ^2, let $p_k = (r_k(2k+1), 0, \ldots)$, where $r_k = 1/2k(k+1)$ ($k = 1, 2, \ldots$), and put $p_0 = \lim\limits_k p_k$ ($= (0, 0, \ldots)$). Denote by $X_k(n)$ the set consisting of those points $x = \{x_i\} : x_i = 0$

$(i > n+1)$ and whose distance from p_k is r_k. The union $\{p_0\} \cup \bigcup_{k=1}^{\infty} X_k(n+1)$ is locally n-connected but not locally $(n+1)$-connected, while the union $\{p_0\} \cup \bigcup_{k=1}^{\infty} X_k(k)$ is locally homotopically trivial but not locally contractible.

Let Y be a nonempty metrizable space.

(LC^n) Y is locally n-connected iff for every pair (X, A), where X is metrizable and $A \subset X$ is closed with $\dim(X - A) \leq n+1$, A has the NEP w.r.t. Y.

($\text{C}^n + \text{LC}^n$) Y is n-connected and locally n-connected iff for every pair (X, A), where X is metrizable and $A \subset X$ is closed with $\dim(X - A) \leq n+1$, A has the EP w.r.t. Y.

Let Y be a nonempty metrizable space of topological dimension $\leq n$.

($\text{LC}^n + \dim \leq n$) Y is locally n-connected iff Y is locally contractible iff Y is an ANR.

($\text{C}^n + \text{LC}^n + \dim \leq n$) Y is n-connected and locally n-connected iff Y is contractible and locally contractible iff Y is an AR.

The proofs of these results can be found in Dugundji[†].

[Note: It follows that a metrizable space of finite topological dimension is uniformly locally contractible iff it is an ANR and has an equiconnecting structure iff it is an AR.]

EXAMPLE (Borsuk's Cone) There exists a contractible, locally contractible compact metrizable space that is not an ANR. Choose a sequence: $0 = t_0 < t_1 < \cdots < 1$, $\lim t_n = 1$. Inside the product $\prod_0^{\infty}[0,1]$, for $n = 1, 2, \ldots$, form $Y_n = [t_{n-1}, t_n] \times [0,1]^n \times 0 \times \cdots$, put $Y_\infty = 1 \times \prod_1^{\infty}[0,1]$, and let $Y = (\bigcup_1^{\infty} \text{fr}\, Y_n) \cup Y_\infty$—then Y is a compact connected metrizable space which we claim is locally contractible yet has nontrivial singular homology in every dimension, thus is not an ANR (cf. p. 6–20). Local contractibility at the points of $Y - Y_\infty$ being obvious, let $y_\infty = (1, y_1, \ldots) \in Y_\infty$ and fix a neighborhood U of y_∞. There is no loss of generality in assuming that U is the intersection of Y with a set $[a_0, 1] \times [a_1, b_1] \times \cdots \times [a_k, b_k] \times [0,1] \times \cdots$. Consider a neighborhood V of y_∞ that is the intersection of Y with a set $[a_0, 1] \times [a_1, b_1] \times \cdots \times [a_k, b_k] \times [a_{k+1}, b_{k+1}] \times [0,1] \times \cdots$, where $b_{k+1} - a_{k+1} < 1$. There are two cases: $\begin{cases} 1 \notin [a_{k+1}, b_{k+1}] \\ 0 \notin [a_{k+1}, b_{k+1}] \end{cases}$. As both are handled in a similar manner, suppose, e.g., that $1 \notin [a_{k+1}, b_{k+1}]$ and define a homotopy $H : IV \rightarrow U$ between the inclusion $V \rightarrow U$ and the constant map $V \rightarrow y_\infty$ by letting $H(v, t)$ be consecutively

$$\begin{cases} (v_0, v_1, \ldots, v_k, (1-3t)v_{k+1}, v_{k+2}, \ldots) \\ (3t - 1 + (2-3t)v_0, v_1, \ldots, v_k, 0, v_{k+2}, \ldots) \\ (1, y_1 - 3(1-t)(y_1 - v_1), y_2 - 3(1-t)(y_2 - v_2), \ldots). \end{cases}$$

[†] *Compositio Math.* **13** (1958), 229–246; see also Kodama, *Proc. Japan Acad. Sci.* **33** (1957), 79–83.

Here, $v = (v_0, v_1, \ldots) \in V$ and $\begin{cases} 0 \leq t \leq 1/3 \\ 1/3 \leq t \leq 2/3 \\ 2/3 \leq t \leq 1 \end{cases}$. That Y is not an ANR is seen by remarking that $\operatorname{fr} Y_n$ is a retract of Y, hence $H_n(\operatorname{fr} Y_n) \approx \mathbf{Z}$ is isomorphic to a direct summand of $H_n(Y)$. The cone ΓY of Y is a contractible, locally contractible compact metrizable space. And Y, as a closed subspace of ΓY, is a neighborhood retract of ΓY. Therefore ΓY is not an ANR. Finally, Y is not uniformly locally contractible, so ΓY does not have an equiconnecting structure.

[Note: Other, more subtle examples of this sort are known (Daverman-Walsh[†]).]

FACT Let $Y \subset \mathbf{R}^n$—then Y is a neighborhood retract of $\mathbf{R}^n$ iff Y is locally compact and locally contractible.

Haver[‡] has shown that if a locally contractible metrizable space Y can be written as a countable union of compacta of finite topological dimension, then Y is an ANR. Example: Every metrizable CW complex X is an ANR. Indeed, for this one can assume that X is connected (cf. Proposition 12). But then X, being locally finite, is necessarily countable, hence can be written as a countable union of finite subcomplexes.

Certain function spaces or automorphism groups that arise "in nature" turn out to be ARs or, equivalently, contractible ANRs. Example: Let E be an infinite dimensional Hilbert space—then $\mathbf{GL}(E)$ is contractible (cf. p. 6–10). However, $\mathbf{GL}(E)$ is an open subset of a Banach space, thus is an ANR. Conclusion: $\mathbf{GL}(E)$ is an AR.

EXAMPLE (Measurable Functions) Let Y be a nonempty metrizable space. Denote by M_Y the set of equivalence classes of Borel measurable functions $f : [0,1] \rightarrow Y$ equipped with the topology of convergence in measure—then M_Y is metrizable, a compatible metric being given by the assignment $(f,g) \rightarrow \int_0^1 d(f(x), g(x))dx$, where d is a compatible metric on Y bounded by 1. Nhu[‖] has shown that M_Y is an ANR. Claim: M_Y is contractible. To see this, fix a point $y_0 \in Y$ and consider the homotopy $H(f,t)(x) = \begin{cases} f(x) & (x > t) \\ y_0 & (x \leq t) \end{cases}$. Therefore M_Y is an AR.

[Note: Take $Y = \mathbf{R}$—then $M_{\mathbf{R}}$ is a linear metric space. But its dual $M_{\mathbf{R}}^*$ is trivial, hence $M_{\mathbf{R}}$ is not locally convex.]

EXAMPLE (Measurable Transformations) Let Γ be the set of equivalence classes of measure preserving Borel measurable bijections $\gamma : [0,1] \rightarrow [0,1]$, i.e., let Γ be the automorphism group of the measure algebra $\mathbf{A}$ of the unit interval. Equip Γ with the topology of pointwise convergence on $\mathbf{A}$—then a subbasis for the neighborhoods at a fixed $\gamma_0 \in \Gamma$ is the collection of all sets of the form $\{\gamma : |\gamma A \Delta \gamma_0 A| < \epsilon\}$ ($A \in \mathbf{A}$ & $\epsilon > 0$), Δ being symmetric difference. With respect to this topology, Γ is a first countable Hausdorff topological group, so Γ is metrizable. Nhu[†] has shown that Γ is an ANR. Claim: Γ is contractible. To see this, let B be the complement of A in $[0,1]$ and assign to each pair (A, γ) its return partition, viz. the sequence $\{\Omega_n\}$, where $\Omega_0 = B$, $\Omega_1 = A \cap \gamma^{-1} A$, and for $n \geq 2$, $\Omega_n = A \cap \gamma^{-1} B \cap \cdots \cap \gamma^{-(n-1)} B \cap \gamma^{-n} A$. Define $\gamma_A \in \Gamma$ by $\gamma_A(x) = \gamma^n(x)$ $(x \in \Omega_n)$, check that the map $\begin{cases} \mathbf{A} \times \Gamma \to \Gamma \\ (A, \gamma) \to \gamma_A \end{cases}$ is continuous, and consider the homotopy $H(t, \gamma) = \gamma_{[t,1]}$. Therefore Γ is an AR.

[†] *Michigan Math. J.* **30** (1983), 17–30.

[‡] *Proc. Amer. Math. Soc.* **40** (1973), 280–284.

[‖] *Fund. Math.* **124** (1984), 243–254.

[Note: Confining the discussion to the unit interval is not unduly restrictive since the Halmos-von Neumann theorem says that every separable, non atomic, normalized measure algebra is isomorphic to $\mathbf{A}$.]

Let X be a second countable topological manifold of euclidean dimension n. Denote by $H(X)$ the set of all homeomorphisms $X \to X$ endowed with the compact open topology—then $H(X)$ is a topological group (cf. p. 2–6). Moreover, $H(X)$ is metrizable and one can ask: Is $H(X)$ an ANR? If X is not compact, then the answer is "no" since there are examples where $H(X)$ is not even locally contractible (Edwards-Kirby[‡]). If X is compact, then $H(X)$ is locally contractible (Černavskiĭ[‖]) and there is some evidence to support a conjecture that $H(X)$ might be an ANR.

[Note: If X is not compact but is homeomorphic to the interior of a compact topological manifold with boundary, then $H(X)$ is locally contractible (Černavskiĭ(ibid.)). Example: $H(\mathbf{R}^n)$ is locally contractible.]

EXAMPLE Take $X = [0,1]$—then $H([0,1])$ is homeomorphic to $\mathbf{R}^\omega \times \{0,1\}$ (thus is an ANR). In other words, the claim is that the identity component $H_e([0,1])$ of $H([0,1])$ is homeomorphic to $\mathbf{R}^\omega$. Form the product $\prod\limits_{n=0}^{\infty} \prod\limits_{i=1}^{2^n}]0,1[_{n,i}$ and define a homeomorphism between it and $H_e([0,1])$ by assigning to a typical string $(x_{n,i})$ an order preserving homeomorphism $\phi : [0,1] \to [0,1]$ via the following procedure. Suppose that n is given and that there have been defined two sets of points

$$\begin{cases} A_n = \{0 = a(n,0) < a(n,1) < \cdots < a(n,2^n) = 1\} \\ B_n = \{0 = b(n,0) < b(n,1) < \cdots < b(n,2^n) = 1\}, \end{cases}$$

with $\phi(a(n,i)) = b(n,i)$. To extend the definition of ϕ to an order preserving bijection $A_{n+1} \to B_{n+1}$, where $\begin{cases} A_{n+1} \supset A_n \\ B_{n+1} \supset B_n \end{cases}$ and both have cardinality $2^{n+1} + 1$, distinguish two cases. Case 1: n is odd. Let α_i

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[†] *Proc. Amer. Math. Soc.* **110** (1990), 515–522.

[‡] *Ann. of Math.* **93** (1971), 63–88.

[‖] *Math. Sbornik* **8** (1969), 287–333; see also Rushing, *Topological Embeddings*, Academic Press (1973), 270–293.

be the midpoint of $[a(n,i-1),a(n,i)]$ and set $\beta_i=\phi(\alpha_i)=x_{n,i}(b(n,i)-b(n,i-1))+b(n,i-1)$. Case 2: n is even. Let β_i be the midpoint of $[b(n,i-1),b(n,i)]$ and set $\alpha_i=\phi^{-1}(\beta_i)=x_{n,i}(a(n,i)-a(n,i-1))$ $+a(n,i-1)$. Define $\begin{cases} A_{n+1}=A_n\cup\{\alpha_i : i=1,\ldots,2^n\} \\ B_{n+1}=B_n\cup\{\beta_i : i=1,\ldots,2^n\}\end{cases}$, so that in obvious notation

$$\begin{cases} A_{n+1}=\{0=a(n+1,0)<a(n+1,1)<\cdots<a(n+1,2^{n+1})=1\} \\ B_{n+1}=\{0=b(n+1,0)<b(n+1,1)<\cdots<b(n+1,2^{n+1})=1\},\end{cases}$$

with $\phi(a(n+1,i))=b(n+1,i)$. If now $\begin{cases} A=\bigcup\limits_1^\infty A_n \\ B=\bigcup\limits_1^\infty B_n\end{cases}$, then $\begin{cases} A \\ B\end{cases}$ are dense in $[0,1]$ and $\phi:A\rightarrow B$ is an order preserving bijection, hence admits an extension to an order preserving homeomorphism $\phi:[0,1]\rightarrow[0,1]$.

[Note: $H([0,1])$ and $H(]0,1[)$ are homeomorphic. In fact, the arrow of restriction $H([0,1])\rightarrow H(]0,1[)$ is continuous and has for its inverse the arrow of extension $H(]0,1[)\rightarrow H([0,1])$, which is also continuous. Corollary: $H(]0,1[)$ is an ANR. Corollary: $H(\mathbf{R})$ is an ANR.]

EXAMPLE Take $X=\mathbf{S}^1$—then $H(\mathbf{S}^1)$ is homeomorphic to $\mathbf{R}^\omega\times\mathbf{S}^1\times\{0,1\}$ (thus is an ANR). To see this, it suffices to observe that $H(\mathbf{S}^1)$ is homeomorphic to $G\times\mathbf{S}^1$, where G is the subgroup of $H(\mathbf{S}^1)$ consisting of those ϕ which fix $(1,0)$.

Therefore, if X is a compact 1-manifold, then $H(X)$ is an ANR. The same conclusion obtains if X is a compact 2-manifold (Luke-Mason[†]) but if $n>2$, then it is unknown whether $H(X)$ is an ANR.

EXAMPLE Take $X=[0,1]^\omega$, the Hilbert cube—then $H(X)$ (compact open topology) is metrizable and Ferry[‡] has shown that $H(X)$ is an ANR.

LEMMA Let $K=(V,\Sigma)$ be a vertex scheme—then $|K|_b$ is an ANR.

[There are three steps to the proof.

(I) Fix a point $*\notin V$ and put $V_*=V\cup\{*\}$. Let Σ_* be the set of all nonempty finite subsets of V_*. Call K_* the associated vertex scheme. Claim: $|K_*|_b$ is an AR. Indeed, the inclusion $|K_*|_b\rightarrow\ell^1(V_*)$ is an isometric embedding with a convex range.

(II) Let Γ_* be the subspace of $|K_*|_b$ consisting of χ_*, the characteristic function of $\{*\}$, and those $\phi\neq\chi_*:\phi^{-1}(]0,1])\cap V\in\Sigma$. Claim: Γ_* is an AR. To establish this, it suffices to exhibit a retraction $r:|K_*|_b\rightarrow\Gamma_*$. Take a $\phi\in|K_*|_b$. Case 1: $\phi=\chi_*$. There is no choice here: $r(\chi_*)=\chi_*$. Case 2: $\phi\neq\chi_*$. Suppose that $\phi^{-1}(]0,1])-\{*\}=\{v_0,\ldots,v_n\}$.

[†] *Trans. Amer. Math. Soc.* **164** (1972), 275–285.

[‡] *Ann. of Math.* **106** (1977), 101–119.

Order the vertexes v_i so that $\phi(v_0) \geq \cdots \geq \phi(v_n)$. Denote by k the maximal index: $\{v_0, \ldots, v_k\} \in \Sigma$ and define $r(\phi)$ by the following formulas:

$$\begin{cases} r(\phi)(*) = 1 - \sum\limits_{v \in V} r(\phi)(v) \\ r(\phi)(v) = 0 \ (v \in V - \{v_0, \ldots, v_k\}) \end{cases}$$

and

$$\begin{cases} k = n : r(\phi)(v_i) = \phi(v_i) \ (0 \leq i \leq k) \\ k < n : r(\phi)(v_i) = \phi(v_i) - \phi(v_{k+1}) \ (0 \leq i \leq k). \end{cases}$$

One can check that r is welldefined and continuous.

(III) Since $\Gamma_* - \{\chi_*\}$ is open in Γ_*, it is an ANR. Claim: $|K|_b$ is a retract of $\Gamma_* - \{\chi_*\}$, hence is an ANR. To see this, consider the map $\phi \to \dfrac{\phi - \phi(*)\chi_*}{1 - \phi(*)}$.]

A topological space is said to be a (finite, countable) CW space if it has the homotopy type of a (finite, countable) CW complex. The following theorems characterize these classes in terms of ANRs.

CW-ANR THEOREM Let X be a topological space—then X has the homotopy type of a CW complex iff X has the homotopy type of an ANR.

[If X has the homotopy type of a CW complex, then there exists a vertex scheme K such that X has the homotopy type of $|K|$ (cf. §5, Proposition 2) or still, the homotopy type of $|K|_b$ (cf. §5, Proposition 1) and, by the lemma, $|K|_b$ is an ANR. Conversely, if X has the homotopy type of an ANR Y, use the placement lemma to realize Y as a closed subspace of a normed linear space E. Fix an open $U \subset E : U \supset Y$ and a retraction $r : U \to Y$. Since U has the homotopy type of a CW complex (cf. §5, Proposition 6), the domination theorem implies that the same is true of Y.]

COUNTABLE CW-ANR THEOREM Let X be a topological space—then X has the homotopy type of a countable CW complex iff X has the homotopy type of a second countable ANR.

[If X has the homotopy type of a countable CW complex, then there exists a countable locally finite vertex scheme K such that X has the homotopy type of $|K|$ (cf. §5, Proposition 3 and p. 5–14). Therefore $|K| = |K|_b$ is Lindelöf, hence second countable, and, by the lemma, $|K|_b$ is an ANR. Conversely, if X has the homotopy type of a second countable ANR Y, then the "E" figuring in the preceding argument is second countable, therefore the "U" has the homotopy type of a countable CW complex (cf. §5, Proposition 6) and the countable domination theorem can be applied.]

FINITE CW-ANR THEOREM Let X be a topological space—then X has the homotopy type of a finite CW complex iff X has the homotopy type of a compact ANR.

[One direction is easy: If X has the homotopy type of a finite CW complex, then there exists a finite vertex scheme K such that X has the homotopy type of $|K| = |K|_b$ (cf. §5, Proposition 3), which, by the lemma, is an ANR. The converse, however, is difficult: Its proof depends on an application of a number of theorems from infinite dimensional topology (West[†]).]

Application: The singular homology groups of a compact ANR are finitely generated and vanish beyond a certain point and the fundamental group of a compact connected ANR is finitely presented.

According to the CW-ANR theorem, if Y is an ANR, then it and each of its open subsets has the homotopy type of a CW complex. On the other hand, it can be shown that every metrizable space with this property is an ANR (Cauty[‡]).

FACT Let Y be a nonempty metrizable space—then Y is an AR iff Y is a homotopically trivial ANR.

[A connected CW complex is homotopically trivial iff it is contractible. Quote the CW-ANR theorem.]

Let X and Y be topological spaces. Let $\mathcal{O} = \{O\}$ be an open covering of Y—then two continuous functions $\begin{cases} f : X \to Y \\ g : X \to Y \end{cases}$ are said to be $\underline{\mathcal{O}\text{-contiguous}}$ if $\forall\, x \in X\ \exists\, O \in \mathcal{O} : \{f(x), g(x)\} \subset O$.

LEMMA Suppose that Y is an ANR—then there exists an open covering $\mathcal{O} = \{O\}$ of Y such that for any topological space X : $\begin{cases} f \in C(X,Y) \\ g \in C(X,Y) \end{cases}$ $\mathcal{O}$-contiguous $\Rightarrow f \simeq g$.

[Choose a normed linear space E containing Y as a closed subspace. Fix a neighborhood U of Y in E and a retraction $r : U \to Y$. Let $\mathcal{C} = \{C\}$ be a covering of U by convex open sets. Put $\mathcal{O} = \mathcal{C} \cap Y$. Take two $\mathcal{O}$-contiguous functions f and g. Define $h : IX \to E$ by $h(x,t) = (1-t)f(x) + tg(x)$—then $h(IX) \subset U$, so $H = r \circ h$ is a homotopy $IX \to Y$ between f and g.]

Let X be a topological space, $\mathcal{U} = \{U\}$ an open covering of X. Let $K = (V, \Sigma)$ be a vertex scheme—then a function $f : |K^{(0)}| \to X$ is said to be $\underline{\text{confined}}$ by $\mathcal{U}$ if $\forall\, \sigma \in \Sigma\ \exists\, U \in \mathcal{U} : f(|\sigma| \cap |K^{(0)}|) \subset U$.

[†] *Ann. of Math.* **106** (1977), 1–18.

[‡] *Fund. Math.* **144** (1994), 11–22.

LEMMA Suppose that Y is an ANR. Let $\mathcal{O} = \{O\}$ be an open covering of Y—then there exists an open refinement $\mathcal{P} = \{P\}$ of $\mathcal{O}$ such that for every vertex scheme $K = (V, \Sigma)$ and every function $f : |K^{(0)}| \rightarrow Y$ confined by $\mathcal{P}$ there exists a continuous function $F : |K| \rightarrow Y$ such that $F|\,|K^{(0)}| = f$ and $\forall\, \sigma \in \Sigma, \forall\, P \in \mathcal{P} : f(|\sigma| \cap |K^{(0)}|) \subset P \Rightarrow \exists\, O \in \mathcal{O} : F(|\sigma|) \cup P \subset O$.

[Choose a normed linear space E containing Y as a closed subspace. Fix a neighborhood U of Y in E and a retraction $r : U \rightarrow Y$. Let $\mathcal{C} = \{C\}$ be a refinement of $r^{-1}(\mathcal{O})$ consisting of convex open sets. Put $\mathcal{P} = \mathcal{C} \cap Y$—then $\mathcal{P}$ is an open refinement of $\mathcal{O}$ which we claim has the properties in question. Thus let $K = (V, \Sigma)$ be a vertex scheme. Take a function $f : |K^{(0)}| \rightarrow Y$ confined by $\mathcal{P}$. Given $\sigma \in \Sigma$, write C_σ for the convex hull of $f(|\sigma| \cap |K^{(0)}|)$, itself a subset of some element $C \in \mathcal{C}$. Construct by induction continuous functions $\Phi_n : |K^{(n)}| \rightarrow U$ subject to: $\Phi_0 = f, \Phi_{n+1}|\,|K^{(n)}| = \Phi_n$, and $\forall\, \sigma \in \Sigma, \Phi_n(|\sigma| \cap |K^{(n)}|) \subset C_\sigma$. Here the point is that if Φ_n has been constructed and if σ is an $(n+1)$-simplex, then $|\sigma| - \langle\sigma\rangle \subset |K^{(n)}|$, therefore the restriction of Φ_n to $|\sigma| - \langle\sigma\rangle$ can be continuously extended to $|\sigma|$, C_σ being an AR. This done, define $\Phi : |K| \rightarrow U$ by $\Phi|\,|K^{(n)}| = \Phi_n$. Since each Φ_n is continuous, so is Φ. Consider $F = r \circ \Phi$.]

These lemmas can be used to prove that if Y is an ANR of topological dimension $\leq n$, then Y is dominated in homotopy by $|K|$, where K is a vertex scheme: $\dim K \leq n$, a result not directly implied by the CW-ANR theorem. In succession, let $\mathcal{O}$ be an open covering of Y per the first lemma, let $\mathcal{P}$ be an open refinement of $\mathcal{O}$ per the second lemma, and let $\mathcal{Q}$ be a neighborhood finite star refinement of $\mathcal{P}$ (cf. §1, Proposition 13)—then $\mathcal{Q}$ has a precise open refinement $\mathcal{V}$ of order $\leq n+1$ (cf. §19, Proposition 6). Obviously, $\dim N(\mathcal{V}) \leq n$, $N(\mathcal{V})$ the nerve of $\mathcal{V}$. Fix a point y_V in each $V \in N(\mathcal{V})^{(0)}$. Define $f : |N(\mathcal{V})^{(0)}| \rightarrow Y$ by $f(\chi_V) = y_V$. Claim: f is confined by $\mathcal{P}$. For suppose that $\sigma = \{V_1, \ldots, V_k\}$ is a simplex of $N(\mathcal{V})$. Since $V_1 \cap \cdots \cap V_k \neq \emptyset$ and since $\mathcal{V}$ is a star refinement of $\mathcal{P}$, there exists $P \in \mathcal{P}$: $V_1 \cup \cdots \cup V_k \subset P \Rightarrow f(|\sigma| \cap |N(\mathcal{V})^{(0)}|) \subset P$. Now take $F : |N(\mathcal{V})| \rightarrow Y$ as above and choose a $\mathcal{V}$-map $G : Y \rightarrow |N(\mathcal{V})|$ (cf. p. 5–3). One can check that $F \circ G$ and id_Y are $\mathcal{O}$-contiguous, hence homotopic.

[Note: By analogous arguments, if Y is a compact (connected) ANR of topological dimension $\leq n$, then Y is dominated in homotopy by $|K|$, where K is a vertex scheme: $\dim K \leq n$ and $|K|$ is compact (connected).]

Application: Let Y be an ANR of topological dimension $\leq n$—then the singular homology groups of Y vanish in all dimensions $> n$.

EXAMPLE Suppose that Y is a compact connected ANR: $\dim Y = 1$ & $\pi_1(Y) \neq 1$—then $\pi_1(Y)$ is finitely generated and free. Consequently, Y has the homotopy type of a finite wedge of 1-spheres.

There are two variants of the CW-ANR theorem.

(Paired Version) A CW pair is a pair (X, A), where X is a CW complex and $A \subset X$ is a subcomplex; an ANR pair is a pair (Y, B), where Y is an ANR and $B \subset Y$ is closed and an ANR. Working then in the category of pairs of topological spaces, the result is that an arbitrary object in this category has the homotopy type of a CW pair iff it has the homotopy type of an ANR pair.

(Pointed Version) A pointed CW complex is a pair (X, x_0), where X is a CW complex and $x_0 \in X^{(0)}$; a pointed ANR is a pair (Y, y_0), where Y is an ANR and $y_0 \in Y$. Working then in the category of pointed topological spaces, the result is that an arbitrary object in this category has the homotopy type of a pointed CW complex iff it has the homotopy type of a pointed ANR.

[Note: There is also a CW-ANR theorem for the category of pointed pairs of topological spaces.]

In $\mathbf{HTOP}^2$, the relevant reduction is that if (X, A) is a CW pair, then there exists a vertex scheme K and a subscheme L such that $(X, A) \approx (|K|, |L|)$, while in $\mathbf{HTOP}_*$, the relevant reduction is that if (X, x_0) is a pointed CW complex, then there exists a vertex scheme K and a vertex $v_0 \in V$ such that $(X, x_0) \approx (|K|, |v_0|)$ (cf. p. 5-12).

Convention: The function spaces encountered below carry the compact open topology.

LEMMA Let X, Y, and Z be topological spaces.

(i) Let $f \in C(X, Y)$—then the homotopy class of the precomposition arrow $f^* : C(Y, Z) \rightarrow C(X, Z)$ depends only on the homotopy class of f.

(ii) Let $g \in C(Y, Z)$—then the homotopy class of the postcomposition arrow $g_* : C(X, Y) \rightarrow C(X, Z)$ depends only on the homotopy class of g.

Application: The homotopy type of $C(X, Y)$ depends only on the homotopy types of X and Y.

[Note: By the same token, in $\mathbf{TOP}^2$ the homotopy type of $(C(X, A; Y, B), C(X, B))$ depends only on the homotopy types of (X, A) and (Y, B), whereas in $\mathbf{TOP}_*$ the homotopy type of $C(X, x_0; Y, y_0)$ depends only on the homotopy types of (X, x_0) and (Y, y_0).]

PROPOSITION 6 Let K be a nonempty compact metrizable space; let Y be a metrizable space—then $C(K, Y)$ is an ANR iff Y is an ANR.

[Necessity: Assuming that Y is nonempty, embed Y in $C(K, Y)$ via the assignment $y \rightarrow j(y)$, where $j(y)$ is the constant map $K \rightarrow y$. Fix a point $k_0 \in K$ and denote by $e_0 : C(K, Y) \rightarrow Y$ the evaluation $\phi \rightarrow \phi(k_0)$. Because $j \circ e_0$ is a retraction of $C(K, Y)$ onto $j(Y)$, it follows that if $C(K, Y)$ is an ANR, then so is Y.

Sufficiency: Let (X,A) be a pair, where X is metrizable and $A \subset X$ is closed. Let $f : A \rightarrow C(K,Y)$ be a continuous function. Define a continuous function $\phi : A \times K \rightarrow Y$ by setting $\phi(a,k) = f(a)(k)$. Since Y is an ANR, there is a neighborhood O of $A \times K$ in $X \times K$ and a continuous function $\Phi : O \rightarrow Y$ with $\Phi | A \times K = \phi$. Fix a neighborhood U of A in $X : U \times K \subset O$. Define a continuous function $F : U \rightarrow C(K,Y)$ by setting $F(u)(k) = \Phi(u,k)$. Obviously, $F|A = f$, thus $C(K,Y)$ is an ANR (cf. Proposition 5).]

Keeping to the above notation, the compactness of K implies that $\pi_0(C(K,Y)) = [K,Y]$. Assume in addition that Y is separable—then $C(K,Y)$ is separable. But $C(K,Y)$ is also an ANR, hence its path components are open. Conclusion: $\#[K,Y] \leq \omega$.

Here is another corollary. Suppose that X is a finite CW space—then, on the basis of the CW-ANR theorem, for any CW space Y, $C(X,Y)$ has the homotopy type of an ANR, hence is again a CW space.

[Note: Some assumption on X is necessary. Example: Give $\{0,1\}$ the discrete topology and consider $\{0,1\}^\omega$.]

EXAMPLE Let X be a topological space—then the free loop space ΛX of X is defined by the pullback square

$$\begin{tikzcd} \Lambda X \arrow[r] \arrow[d] & PX \arrow[d, "\Pi"] \\ X \arrow[r] & X \times X \end{tikzcd}$$

, where Π is the Hurewicz fibration $\sigma \rightarrow (\sigma(0), \sigma(1))$ and $X \rightarrow X \times X$ is the diagonal embedding. The arrow $\Lambda X \rightarrow X$ is a Hurewicz fibration and its fiber over x_0 is $\Omega(X, x_0)$, so if X is path connected, then the homotopy type of $\Omega(X, x_0)$ is independent of the choice of x_0. Since ΛX can be identified with $C(\mathbf{S}^1, X)$ (compact open topology), the free loop space of X is a CW space when X is a CW space.

[Note: Given a topological group G, define W_G^∞ by the pullback square

$$\begin{tikzcd} W_G^\infty \arrow[r] \arrow[d] & PX_G^\infty \arrow[d, "\Pi"] \\ X_G^\infty \times G \arrow[r, "\Phi"'] & X_G^\infty \times X_G^\infty \end{tikzcd}$$

, where $\Phi(x,g) = (x, x \cdot g)$—then W_G^∞ / G can be identified with ΛB_G^∞ and there is a weak homotopy equivalence $\Lambda B_G^\infty \rightarrow (X_G^\infty \times G)/G$ (the action of G on itself being by conjugation).]

EXAMPLE Suppose that X and Y are path connected CW spaces for which there exists an n such that (i) X has the homotopy type of a locally finite CW complex with a finite n-skeleton and (ii) $\pi_q(Y) = 0$ ($\forall\, q > n$)—then $C(X,Y)$ is a CW space.

[Take X to be a locally finite CW complex with a finite n-skeleton $X^{(n)}$. One can assume that n is > 0 because when $n = 0$, Y is contractible and the result is trivial. Consider the inclusion $i : X^{(n)} \rightarrow X$—then the precomposition arrow $i^* : C(X,Y) \rightarrow C(X^{(n)}, Y)$ is a Hurewicz fibration (cf. §4, Proposition 6) and, in view of the assumption on Y, its fibers are either empty or contractible. But $C(X^{(n)}, Y)$ is a CW space, thus so is $C(X, Y)$ (cf. Proposition 11).]

PROPOSITION 7 Let K be a nonempty compact metrizable space, $L \subset K$ a nonempty closed subspace; let Y be a metrizable space, $Z \subset Y$ a closed subspace. Suppose that Y is an ANR—then $C(K, L; Y, Z)$ is an ANR iff Z is an ANR.

[Assuming that Z is nonempty, one may proceed as in the proof of Proposition 6 and show that Z is homeomorphic to a retract of $C(K, L; Y, Z)$, from which the necessity. Consider now a pair (X, A), where X is metrizable and $A \subset X$ is closed. Let $f : A \rightarrow C(K, L; Y, Z)$ be a continuous function. Define a continuous function $\phi : A \times L \rightarrow Z$ by setting $\phi(a, \ell) = f(a)(\ell)$. Since Z is an ANR, there is a neighborhood O of $A \times L$ in $X \times L$ and a continuous function $\Phi : O \rightarrow Z$ with $\Phi | A \times L = \phi$. Fix a neighborhood U of A in $X : \overline{U} \times L \subset O$. Define a continuous function $\psi : A \times K \cup \overline{U} \times L \rightarrow Y$ by setting $\begin{cases} \psi(a, k) = f(a)(k) \\ \psi(u, \ell) = \Phi(u, \ell) \end{cases}$. Since Y is an ANR, there is a neighborhood P of $A \times K \cup \overline{U} \times L$ in $X \times K$ and a continuous function $\Psi : P \rightarrow Y$ with $\Psi | A \times K \cup \overline{U} \times L = \psi$. Fix a neighborhood V of A in $X : V \times K \subset P$ & $V \subset U$. Define a continuous function $F : V \rightarrow C(K, L; Y, Z)$ by setting $F(v)(k) = \Psi(v, k)$. Obviously, $F | A = f$, thus $C(K, L; Y, Z)$ is an ANR (cf. Proposition 5).]

Take, e.g., $(K, L) = (\mathbf{S}^n, s_n)$ $(s_n = (1, 0, \ldots, 0) \in \mathbf{R}^{n+1}, n \geq 1)$ and let $y_0 \in Y$—then $\pi_n(Y, y_0) = \pi_0(C(\mathbf{S}^n, s_n; Y, y_0))$. Accordingly, if Y is separable, then $\pi_n(Y, y_0)$ is countable. Example: The homotopy groups of a countable connected CW complex are countable.

LOOP SPACE THEOREM Let (X, x_0) be a pointed CW space—then the loop space $\Omega(X, x_0)$ is a pointed CW space.

[Fix a pointed ANR (Y, y_0) with the pointed homotopy type of (X, x_0) (cf. p. 6–22)—then $\Omega(Y, y_0) = C(\mathbf{S}^1, s_1; Y, y_0)$ is a pointed ANR (cf. Proposition 7), so $\Omega(X, x_0) = C(\mathbf{S}^1, s_1; X, x_0)$ is a pointed CW space.]

EXAMPLE Suppose that (X, x_0) is path connected and numerably contractible. Assume: ΩX is a CW space—then X is a CW space. Thus let $f : K \rightarrow X$ be a pointed CW resolution. Owing to the loop space theorem, ΩK is a CW space. But the arrow $\Omega f : \Omega K \rightarrow \Omega X$ is a weak homotopy equivalence and since ΩX is a CW space, it follows from the realization theorem that Ωf is a homotopy equivalence. Therefore f is a homotopy equivalence (cf. p. 4–27).

[Note: Let X be the Warsaw circle—then X is not a CW space. On the other hand, there exists a continuous bijection $\phi : [0, 1[\rightarrow X$ which is a regular Hurewicz fibration. As this implies that ϕ is a pointed

Hurewicz fibration (cf. p. 4–14), ΩX has the same pointed homotopy type as $\Omega[0,1[$ (cf. p. 4–35), hence is a CW space, so X is not numerably contractible.]

EXAMPLE (Classifying Spaces) Let G be a topological group—then B_G^∞ is path connected and numerably contractible (inspect the Milnor construction). Moreover, according to §4, Proposition 36, G and ΩB_G^∞ have the same homotopy type. Taking into account the preceding example, it follows that if G is a CW space, then the same is true of B_G^∞. Corollary: Any classifying space for G is a CW space provided that G itself is a CW space.

LEMMA Let $X \xrightarrow{f} Z \xleftarrow{g} Y$ be a 2-sink. Assume: X, Y, and Z are ANRs—then $W_{f,g}$ is an ANR.

PROPOSITION 8 Let $X \xrightarrow{f} Z \xleftarrow{g} Y$ be a 2-sink. Assume: X, Y, and Z are CW spaces—then $W_{f,g}$ is a CW space.

[Fix ANRs $\begin{cases} X' \\ Y' \end{cases}$, homotopy equivalences $\begin{cases} \phi : X' \to X \\ \psi : Y' \to Y \end{cases}$, and put $\begin{cases} f' = f \circ \phi \\ g' = g \circ \psi \end{cases}$—then there is a commutative diagram

$$\begin{tikzcd}
X' \arrow[r, "f'"] \arrow[d, "\phi"'] & Z \arrow[d, equal] & Y' \arrow[l, "g'"'] \arrow[d, "\psi"] \\
X \arrow[r, "f"'] & Z & Y \arrow[l, "g"]
\end{tikzcd}$$

thus the arrow $W_{f',g'} \to W_{f,g}$ is a homotopy equivalence (cf. p. 4–25). Choose a homotopy equivalence $\zeta : Z \to Z'$, where Z' is an ANR. There is an arrow $W_{f',g'} \to W_{\zeta \circ f', \zeta \circ g'}$ and it too is a homotopy equivalence. But from the lemma, $W_{\zeta \circ f', \zeta \circ g'}$ is an ANR.]

For a case in point, let X and Y be CW spaces—then $\forall\ f \in C(X,Y)$, W_f is a CW space, and $\forall\ f \in C(X, x_0; Y, y_0)$, E_f is a CW space.

FACT Let $p : X \to B$ be a regular Hurewicz fibration. Assume: $\exists\ b_0 \in B$ such that $\Omega(B, b_0)$ and X_{b_0} are CW spaces—then $\forall\ x_0 \in X_{b_0}$, $\Omega(X, x_0)$ is a CW space.

[By regularity, there is a lifting function $\Lambda_0 : W_p \to PX$ with the property that $\Lambda_0(x, \tau) \in j(X)$ whenever $\tau \in j(B)$. Define $f : \Omega(B, b_0) \to X_{b_0}$ by $f(\tau) = \Lambda_0(x_0, \tau)(1)$, so $f(j(b_0)) = x_0$. The mapping fiber E_f of f has the same homotopy type as $\Omega(X, x_0)$.]

PROPOSITION 9 Suppose that $p : X \to B$ is a Hurewicz fibration and let $\Phi' \in C(B', B)$. Assume: X, B, and B' are CW spaces—then $X' = B' \times_B X$ is a CW space.

[In view of the preceding proposition, this follows from §4, Proposition 18.]

Application: Let $p : X \to B$ be a Hurewicz fibration, where X and B are CW spaces—then $\forall\ b \in B$, X_b is a CW space.

[Note: Let X be a CW space. Relative to a base point, work first with $PX \xrightarrow{p_0} X$ to see that ΘX is a CW space and then consider $\Theta X \xrightarrow{p_1} X$ to see that ΩX is a CW space, thereby obtaining an unpointed variant of the loop space theorem.]

PROPOSITION 10 Suppose that $p : X \to B$ is a Hurewicz fibration and let $O \subset B$. Assume: X is an ANR, B is metrizable, and the inclusion $O \to B$ is a closed cofibration—then X_O is an ANR.

[The inclusion $X_O \to X$ is a closed cofibration (cf. §4, Proposition 11), a condition which is characteristic (cf. p. 6–14).]

Application: Let $p : X \to B$ be a Hurewicz fibration, where X and B are ANRs—then $\forall\ b \in B$, X_b is an ANR.

[Given $b \in B$, the inclusion $\{b\} \to B$ is a closed cofibration (cf. p. 6–14).]

EXAMPLE Let (Y, B, b_0) be a pointed pair. Assume: Y and B are ANRs, with $B \subset Y$ closed. Let $\Theta(Y, B)$ be the subspace of ΘY consisting of those τ such that $\tau(1) \in B$—then $\Theta(Y, B)$ is an ANR. In fact, ΘY is an ANR and there is a pullback square

$$\begin{tikzcd} \Theta(Y,B) \arrow[r] \arrow[d] & \Theta Y \arrow[d, "p_1"] \\ B \arrow[r] & Y \end{tikzcd}.$$

EXAMPLE Take $Y = \mathbf{S}^n \times \mathbf{S}^n \times \cdots$ (ω factors), $y_0 = (s_n, s_n, \ldots)$—then Y is not an ANR. Nevertheless, for every pair (X, A), where X is metrizable and $A \subset X$ is closed, A has the HEP w.r.t. Y (cf. p. 6–41). Therefore ΘY is an AR. Still, ΩY is not an ANR. Indeed, none of the fibers of the Hurewicz fibration $p_1 : \Theta Y \to Y$ is an ANR.

PROPOSITION 11 Suppose that $p : X \to B$ is a Hurewicz fibration. Assume: B is a CW space and $\forall\ b \in B, X_b$ is a CW space—then X is a CW space.

[Fix a CW resolution $f : K \to X$. Consider the Hurewicz fibration $q : W_f \to X (f = q \circ s)$. Since $s : K \to W_f$ is a homotopy equivalence, W_f is a CW space. Moreover, q is a weak homotopy equivalence and the composite $p \circ q : W_f \to B$ is a Hurewicz fibration. The fibers $(p \circ q)^{-1}(b) = q^{-1}(X_b)$ are therefore CW spaces. Comparison of the homotopy sequences of $p \circ q$ and p shows that the arrow $q_b : q^{-1}(X_b) \to X_b$ is a weak homotopy equivalence, hence a homotopy equivalence. Because B is numerably contractible (being a CW space), one can then apply §4, Proposition 20 to conclude that $q : W_f \to X$ is a homotopy equivalence.]

[Note: If $p : X \to B$ is a Hurewicz fibration and if X and the X_b are CW spaces, then it need not be true that B is a CW space (consider the Warsaw circle).]

Let $p : X \to B$ be a Hurewicz fibration, where X is metrizable and B and the X_b are ANRs. Question: Is X an ANR? While the answer is unknown in general, the following lemma implies that the answer is "yes" provided that the topological dimension of X is finite (cf. p. 6–15). Infinite dimensional results can be found in Ferry[†].

LEMMA Suppose that $p : X \to B$ is a Hurewicz fibration. Assume: B is an ANR and $\forall\ b \in B$, X_b is locally contractible—then X is locally contractible.

[Fix $x_0 \in X$, put $b_0 = p(x_0)$, and let U be any neighborhood of x_0. Since p has the slicing structure property (cf. p. 4–14), it is an open map. Accordingly, one can assume at the outset that there is a continuous function $\Phi : p(U) \to PB$ such that $\begin{cases} \Phi(b)(0) = b \\ \Phi(b)(1) = b_0 \end{cases}$ & $\Phi(b_0)(t) = b_0$ $(0 \leq t \leq 1)$. Using the local contractibility of X_{b_0}, choose a neighborhood $O_0 \subset U \cap X_{b_0}$ of x_0 in X_{b_0} and a homotopy $\phi : IO_0 \to U \cap X_{b_0}$ satisfying $\begin{cases} \phi(x, 0) = x \\ \phi(x, 1) = u_0 \end{cases}$ $(u_0 \in U \cap X_{b_0})$. Fix a neighborhood U_0 of $x_0 : U_0 \subset U$ and $O_0 = U_0 \cap X_{b_0}$. Let $\Lambda_0 : W_p \to PX$ be a lifting function with the property that $\Lambda_0(x, \tau) \in j(X)$ whenever $\tau \in j(B)$. Define $F \in C(U, PX)$ by $F(x) = \Lambda_0(x, \Phi(p(x)))$. Because $F(x_0) = j(x_0) \in \{\sigma \in PX : \sigma([0,1]) \subset U_0\}$, there is a neighborhood $V \subset U_0$ of x_0 such that $\forall\ x \in V$, $F(x)(t) \in U_0$ $(0 \leq t \leq 1)$. If now $H : IV \to U$ is the homotopy $H(x,t) = \begin{cases} F(x)(2t) & (0 \leq t \leq 1/2) \\ \phi(F(x)(1), 2t - 1) & (1/2 \leq t \leq 1) \end{cases}$, then $\begin{cases} H(x, 0) = x \\ H(x, 1) = u_0 \end{cases}$, i.e., the inclusion $V \to U$ is inessential.]

Let Y be a metrizable space. Suppose that Y admits a covering $\mathcal{V}$ by pairwise disjoint open sets V, each of which is an ANR—then Y is an ANR. To see this, assume that Y is realized as a closed subspace of a metrizable space Z. Fix a compatible metric d on Z. Given a nonempty $V \in \mathcal{V}$, put $O_V = \{z : d(z, V) < d(z, Y - V)\}$—then O_V is open in Z and $O_V \cap Y = V$. Moreover, the O_V are pairwise disjoint. By hypothesis, there exists an open subset U_V of O_V containing V and a retraction $r_V : U_V \to V$. Form $U = \bigcup\limits_V U_V$, a neighborhood of Y in Z, and define a retraction $r : U \to Y$ by $r|U_V = r_V$.

What is less apparent is that the same assertion is still true if the V are not pairwise disjoint.

LEMMA Let Y be a metrizable space. Suppose that $Y = Y_1 \cup Y_2$, where Y_1 and Y_2 are open and ANRs—then Y is an ANR.

[This is proved in a more general context on p. 6–43 (cf. NES_3).]

PROPOSITION 12 Let Y be a metrizable space. Suppose that Y admits a covering $\mathcal{V}$ by open sets V, each of which is an ANR—then Y is an ANR.

[†] *Pacific J. Math.* **75** (1978), 373–382.

[Use the domino principle (cf. p. 1–24).]

Application: Every metrizable topological manifold is an ANR, hence by the CW-ANR theorem has the homotopy type of a CW complex.

In particular, every compact topological manifold is an ANR, hence by the finite CW-ANR theorem has the homotopy type of a finite CW complex. If X and Y are finite CW complexes, then $\#[X,Y] \leq \omega$ (cf. p. 6–23). Specializing to the attaching process (and recalling that the inclusion $\mathbf{S}^{n-1} \rightarrow \mathbf{D}^n$ is a closed cofibration), it follows that the set of homotopy types of compact topological manifolds is countable.

[Note: One can even prove that the set of homeomorphism types of compact topological manifolds is countable (Cheeger-Kister[†]).]

The use of the term "set" in the above is justified by remarking that the full subcategory of **TOP** whose objects are the compact topological manifolds has a small skeleton.

EXAMPLE Let $p : X \rightarrow B$ be a covering projection. Suppose that X is metrizable and B is an ANR—then X is an ANR.

[Note: The assumption that X is metrizable is superfluous.]

EXAMPLE Let $p : X \rightarrow B$ be a Hurewicz fibration. Assume: X is an ANR and B is a path connected, numerably contractible, paracompact Hausdorff space—then B is an ANR. For let O be an open subset of B with the property that the inclusion $O \rightarrow B$ is inessential, say homotopic to $O \rightarrow b$. Since X_O is fiber homotopy equivalent to $O \times X_b$ (cf. p. 4–24), $\text{sec}_O(X_O)$ is nonempty (cf. §4, Proposition 1), so O is homeomorphic to a retract of X_O, an ANR. Therefore B is locally an ANR, hence an ANR (recall that locally metrizable + paracompact $\Rightarrow$ metrizable; cf. p. 1–19).

EXAMPLE Let X be an aspherical compact topological manifold. Assume: $\chi(X) \neq 0$—then the path component of the identity in $C(X,X)$ is contractible.

[Since $C(X,X)$ is an ANR (cf. Proposition 6), the path component of the identity in $C(X,X)$ is a $K(\text{Cen}\,\pi, 1)$ (cf. p. 5–30 ff.), where $\pi = \pi_1(X)$. On the other hand, the assumption $\chi(X) \neq 0$ implies that $\text{Cen}\,\pi$ is trivial.]

Let X and Y be metrizable spaces. Let A be a closed subspace of X and let $f : A \rightarrow Y$ be a continuous function—then Borges[‡] has shown that $X \sqcup_f Y$ is metrizable iff every point of $X \sqcup_f Y$ belongs to a compact subset of countable character, i.e., having a countable neighborhood basis in X. In particular, this condition is satisfied if $X \sqcup_f Y$ is first countable or if A is compact.

[†] *Topology* **9** (1970), 149–151.

[‡] *Proc. Amer. Math. Soc.* **24** (1970), 446–451.

[Note: In any event, $X \sqcup_f Y$ is a perfectly normal paracompact Hausdorff space (AD_5 (cf. p. 3–1)).]

LEMMA Let B be a closed subspace of a metrizable space Y such that the inclusion $B \rightarrow Y$ is a cofibration. Suppose that B and $Y - B$ are ANRs—then Y is an ANR.

[Fix a Strøm structure (ψ, Ψ) on (Y, B) and put $V = \psi^{-1}([0,1[)$. Show that V is an ANR.]

FACT Let X and Y be ANRs. Let A be a closed subspace of X and let $f : A \rightarrow Y$ be a continuous function. Suppose that A is an ANR—then $X \sqcup_f Y$ is an ANR provided that it is metrizable.

LEMMA Let B be a closed subspace of a metrizable space Y such that the inclusion $B \rightarrow Y$ is a cofibration. Suppose that B is an AR and $Y - B$ is an ANR—then Y is an AR if B is a strong deformation retract of Y.

[It follows from the previous lemma that Y is an ANR. But Y and B have the same homotopy type and B is contractible.]

FACT Let X and Y be ARs. Let A be a closed subspace of X and let $f : A \rightarrow Y$ be a continuous function. Suppose that A is an AR—then $X \sqcup_f Y$ is an AR provided that it is metrizable.

EXAMPLE Take $X = [0,1]^2$, $A = [1/4, 3/4] \times \{1/2\}$, $Y = [0,1]^3$ and let $f : A \rightarrow Y$ be a continuous surjective map—then $X \sqcup_f Y$ is a compact AR of topological dimension 3, yet it is not homeomorphic to any CW complex.

Let (X, A) be a CW pair. Is it true that A has the EP w.r.t. any locally convex topological vector space? A priori, this is not clear since CW complexes are not metrizable in general. There is, however, a class of topologically significant spaces, encompassing both the class of metrizable spaces and the class of CW complexes for which a satisfactory extension theory exists.

Let X be a Hausdorff space; let τ be the topology on X—then X is said to be stratifiable if there exists a function $\text{ST}_X : \mathbf{N} \times \tau \rightarrow \tau$, termed a stratification, such that (a) $\forall\, U \in \tau$, $\overline{\text{ST}_X(n, U)} \subset U$; (b) $\forall\, U \in \tau$, $\bigcup\limits_n \text{ST}_X(n, U) = U$; (c) $\forall\, U, V \in \tau : U \subset V \Rightarrow \text{ST}_X(n, U) \subset \text{ST}_X(n, V)$. A stratifiable space is perfectly normal and every subspace of a stratifiable space is stratifiable. A finite or countable product of stratifiable spaces is stratifiable. A stratifiable space need not be compactly generated and a compactly generated space need not be stratifiable, even if it is regular and countable (Foged[†]). Example: Every metrizable space is stratifiable. Example: The Sorgenfrey line, the Niemytzki plane, and the Michael line are not stratifiable.

[†] *Proc. Amer. Math. Soc.* **81** (1981), 337–338; see also *Proc. Amer. Math. Soc.* **92** (1984), 470–472.

[Note: Junnila[†] has shown that every topological space is the open image of a stratifiable space.]

FACT Let X be a topological space; let $\mathcal{A} = \{A_j : j \in J\}$ be an absolute closure preserving closed covering of X. Suppose that each A_j is stratifiable—then X is stratifiable.

[X is necessarily a perfectly normal Hausdorff space (cf. p. 5–4). As for stratifiability, consider the set $\mathcal{P}$ of all pairs (I, ST_I), where $I \subset J$ and ST_I is a stratification of $X_I = \bigcup_i A_i$. Order $\mathcal{P}$ by stipulating that $(I', \mathrm{ST}_{I'}) \leq (I'', \mathrm{ST}_{I''})$ iff $I' \subset I''$ and for each open subset U of $X_{I''}$:

$$\mathrm{ST}_{I''}(n, U) \cap X_{I'} = \mathrm{ST}_{I'}(n, U \cap X_{I'}) \ \& \ \overline{\mathrm{ST}_{I''}(n, U)} \cap X_{I'} = \overline{\mathrm{ST}_{I'}(n, U \cap X_{I'})}.$$

Every chain in $\mathcal{P}$ has an upper bound, so by Zorn, $\mathcal{P}$ has a maximal element (I_0, ST_{I_0}). Verify that $X_{I_0} = X$.]

Application: Every CW complex is stratifiable.

[The collection of finite subcomplexes of a CW complex X is an absolute closure preserving closed covering of X.]

Application: Let E be a vector space over $\mathbf{R}$. Equip E with the finite topology—then E is stratifiable.

[Fix a basis $\{e_i : i \in I\}$ for E. Assign to each finite subset of I the span of the corresponding e_i. The resulting collection of linear subspaces is an absolute closure preserving closed covering of E.]

FACT Suppose that X and Y are stratifiable—then the coarse join $X *_c Y$ is stratifiable.

Application: Let G be a stratifiable topological group—then $\forall\, n, X_G^n$ is stratifiable.

LEMMA Let $X = \bigcup\limits_0^\infty X_n$ be a topological space, where $X_n \subset X_{n+1}$ and X_n is stratifiable and a zero set in X, say $X_n = \phi_n^{-1}(0)$ $(\phi_n \in C(X, [0,1]))$. Suppose that there is a retraction $r_n : \phi_n^{-1}([0,1[) \to X_n$ such that $\forall\, x \in X_n - X_{n-1}$ $(X_{-1} = \emptyset)$, the sets $r_n^{-1}(U) \cap \phi_n^{-1}([0,t[)$ form a neighborhood basis of x in X (U a neighborhood of x in X_n and $0 < t \leq 1$)—then X is stratifiable.

[The assumptions imply that X is Hausdorff. To construct ST_X, fix a stratification ST_{X_n} of X_n : $\mathrm{ST}_{X_n}(k, U) \subset \mathrm{ST}_{X_n}(k+1, U)$. Given an open subset U of X, denote by $U(n,k)$ the interior of

$$\left\{x \in X_n : r_n^{-1}(x) \cap \phi_n^{-1}([0, 1/(k+1)[) \subset U\right\}$$

in X_n and for $N = 1, 2, \ldots$, put

$$\mathrm{ST}_X(N, U) = \bigcup_{n,k \leq N} r_n^{-1}(\mathrm{ST}_{X_n}(N, U(n,k))) \cap \phi_n^{-1}([0, 1/(k+2)[).]$$

[†] *Colloq. Math. Soc. János Bolyai* **23** (1980), 689–703; see also Harris, *Pacific J. Math.* **91** (1980), 95–104.

EXAMPLE (Classifying Spaces) Let G be a stratifiable topological group—then X_G^∞ and B_G^∞ are stratifiable.

[Since the X_G^n are stratifiable, the lemma can be used to establish the stratifiability of X_G^∞. As for B_G^∞, in the notation of the Milnor construction, $X_G^\infty|O_i$ is homeomorphic to $O_i \times G$, thus O_i is stratifiable and so B_G^∞ admits a neighborhood finite closed covering by stratifiable subspaces, hence is stratifiable.]

FACT Let X and Y be stratifiable. Let A be a closed subspace of X and let $f : A \to Y$ be a continuous function—then $X \sqcup_f Y$ is stratifiable.

Application: Suppose that (X, A) is a relative CW complex. Assume: A is stratifiable—then X is stratifiable.

Let X be a topological space; let $\mathcal{S}$ and $\mathcal{T}$ be collections of subsets of X—then $\mathcal{S}$ is said to be cushioned in $\mathcal{T}$ if there exists a function $\Gamma : \mathcal{S} \to \mathcal{T}$ such that $\forall\, \mathcal{S}_0 \subset \mathcal{S} : \overline{\cup\{S : S \in \mathcal{S}_0\}} \subset \cup\{\Gamma(S) : S \in \mathcal{S}_0\}$. For example, if $\mathcal{S}$ is closure preserving, then $\mathcal{S}$ is cushioned in $\overline{\mathcal{S}}$. A collection $\mathcal{S}$ which is the union of a countable number of subcollection $\mathcal{S}_n$, each of which is cushioned in $\mathcal{T}$, is said to be σ-cushioned in $\mathcal{T}$.

Michael[†] has shown that a CRH space X is paracompact iff every open covering of X has a σ-cushioned open refinement (cf. p. 1–3). This result can be used to prove that stratifiable spaces are paracompact. For suppose that $\mathcal{U} = \{U\}$ is an open covering of X. Put $\mathcal{U}_n = \{\mathrm{ST}_X(n, U) : U \in \mathcal{U}\}$. Let $\mathcal{U}_0 \subset \mathcal{U}$—then $\forall\, U \in \mathcal{U}_0$, $\mathrm{ST}_X(n, U) \subset \mathrm{ST}_X(n, \cup\mathcal{U}_0) \subset \overline{\mathrm{ST}_X(n, \cup\mathcal{U}_0)} \subset \cup\mathcal{U}_0$, from which $\overline{\cup\{\mathrm{ST}_X(n, U) : U \in \mathcal{U}_0\}} \subset \cup\mathcal{U}_0$, thus $\mathcal{U}_n$ is cushioned in $\mathcal{U}$ and so $\mathcal{U}$ has a σ-cushioned open refinement. Therefore X is paracompact. Example: A nonmetrizable Moore space is not stratifiable (Bing (cf. p. 1–18)).

[Note: Another way to argue is to show that every stratifiable space is collectionwise normal and subparacompact (cf. §1, Proposition 10 and the ensuing remark).]

Let X be a CRH space—then X is said to satisfy Arhangel'skiĭ's condition if there exists a sequence $\{\mathcal{U}_n\}$ of collections of open subsets of βX such that each $\mathcal{U}_n$ covers X and $\forall\, x \in X : \bigcap\limits_n \mathrm{st}(x, \mathcal{U}_n) \subset X$. Example: Every topologically complete CRH space X satisfies Arhangel'skiĭ's condition. In fact X is a G_δ in βX, thus $X = \bigcap\limits_1^\infty U_n$ (U_n open in βX) and so we can take $\mathcal{U}_n = \{U_n\}$. Example: Every Moore space satisfies Arhangel'skiĭ's condition.

FACT Let X be a CRH space. Suppose that X satisfies Arhangel'skiĭ's condition—then X is compactly generated.

[†] *Proc. Amer. Math. Soc.* **10** (1959), 309–314.

Let X be a CRH space—then Kullman[†] has shown that X is Moore iff X is submetacompact, has a G_δ diagonal, and satisfies Arhangel'skiĭ's condition. Since a stratifiable space is paracompact and has a perfect square, it follows that every stratifiable space satisfying Arhangel'skiĭ's condition is metrizable (Bing (cf. p. 1–18)). Consequently, a nonmetrizable stratifiable space cannot be embedded in a topologically complete stratifiable space. Example: Every stratifiable LCH space is metrizable.

A Hausdorff space X is said to satisfy Ceder's condition if X has a σ-closure preserving basis. Example: Suppose that X is metrizable—then X satisfies Ceder's condition. Reason: The Nagata-Smirnov metrization theorem says that a regular Hausdorff space X is metrizable iff X has a σ-neighborhood finite basis. On the other hand, every CW complex satisfies Ceder's condition (cf. infra) and a CW complex is not in general metrizable.

FACT Let X be a Hausdorff space. Suppose that X is the closed image of a metrizable space—then X satisfies Ceder's condition.

Any X that satisfies Ceder's condition is stratifiable. Proof: Let $\mathcal{O} = \bigcup\limits_n \mathcal{O}_n$ be a σ-closure preserving basis for X, attach to each closed $A \subset X : O(n, A) = X - \bigcup\{\overline{O} : O \in \mathcal{O}_n \ \& \ A \cap \overline{O} = \emptyset\}$ and then define $\text{ST}_X : \mathbf{N} \times \tau \rightarrow \tau$ by setting $\text{ST}_X(n, U) = X - \overline{O(n, X - U)}$.

[Note: It is unknown whether the converse holds.]

EXAMPLE (M complexes) A topological space is said to be an M_0 space if it is metrizable and, recursively, a topological space is said to be an M_{n+1} space if it is homeomorphic to an adjunction $X \sqcup_f Y$, where X is an M_0 space and Y is an M_n space. An M_∞ space is a topological space that is an M_n space for some n.

A topological space X is said to be an M complex if there exists a sequence of closed M_∞ subspaces $A_j : \begin{cases} X = \bigcup\limits_j A_j \\ A_j \subset A_{j+1} \end{cases}$ and the topology on X is the final topology determined by the inclusions $A_j \rightarrow X$. Example: Every CW complex is an M complex. Since an M complex is the quotient of a metrizable space, an M complex is necessarily compactly generated. Therefore a subspace of an M complex is an M complex iff it is compactly generated. Every M complex satisfies Ceder's condition, hence is stratifiable.

[Note: Not every CW complex is the closed image of a metrizable space.]

DUGUNDJI EXTENSION THEOREM Let X be a stratifiable space; let A be a closed subspace of X. Let E be a locally convex topological vector space. Equip $\begin{cases} C(A, E) \\ C(X, E) \end{cases}$ with the compact open topology—then there exists a linear embedding ext : $C(A,E) \rightarrow C(X,E)$ such that $\forall\, f \in C(A,E)$, $\text{ext}(f)|A = f$ and the range of $\text{ext}(f)$ is contained in the convex hull of the range of f.

[†] *Proc. Amer. Math. Soc.* **27** (1971), 154–160.

[Normalize ST_X : $\begin{cases} \text{ST}_X(n,X) = X \\ \text{ST}_X(1, X - \{x\}) = \emptyset \end{cases}$ & $\text{ST}_X(n,U) \subset \text{ST}_X(n+1,U)$. Given $x \in U$, let $n(x,U)$ be the smallest integer $n : x \in \text{ST}_X(n,U)$. Put $U(x) = \text{ST}_X(n(x,U),U) - \overline{\text{ST}_X(n(x,U), X - \{x\})}$, a neighborhood of x. Plainly, $U(x) \cap V(y) \neq \emptyset$ & $n(x,U) \leq n(y,V) \Rightarrow y \in U$. On the other hand, $\begin{cases} n(x,X) = 1 \\ X(x) = X \end{cases} \Rightarrow \{U : y \in U(x)\} \neq \emptyset$. Assuming that A is nonempty and proper, attach to each $x \in X - A : n(x) = \max\{n(a,O)(O \in \tau) : a \in A \;\&\; x \in O(a)\}$—then $n(x) < n(x, X - A)$. Since every subspace of X is stratifiable, $X - A$ is, in particular, paracompact. Thus the open covering $\{(X-A)(x) : x \in X - A\}$ has a neighborhood finite open refinement $\{U_i : i \in I\}$. Each U_i determines a point $x_i \in X - A : U_i \subset (X-A)(x_i)$, from which a point $a_i \in A$ and a neighborhood O_i of $a_i : x_i \in O_i(a_i)$ & $n(x_i) = n(a_i, O_i)$. Choose a partition of unity $\{\kappa_i : i \in I\}$ on $X - A$ subordinate to $\{U_i : i \in I\}$. Given $f \in C(A,E)$, let

$$\text{ext}(f)(x) = \begin{cases} f(x) & (x \in A) \\ \sum\limits_i \kappa_i(x) f(a_i) & (x \in X - A). \end{cases}$$

Referring back to the proof of the Dugundji extension theorem in the metrizable case and eschewing the obvious, it is apparent that there are two nontrivial claims.

Claim 1: $\text{ext}(f)$ is continuous at the points of A.

[Let $a \in A$; let N be a convex neighborhood of $f(a)$ in E. By the continuity of f, there exists a neighborhood O of a in $X : f(A \cap O) \subset N$. Assertion: $\text{ext}(f)(O(a)(a)) \subset N$. Case 1: $x \in A \cap O(a)(a)$. Here, $x \in A \cap O$ and $\text{ext}(f)(x) = f(x) \in N$. Case 2: $x \in (X - A) \cap O(a)(a)$. Take any index $i : \kappa_i(x) \neq 0 (\Rightarrow x \in U_i)$—then $\emptyset \neq U_i \cap O(a)(a) \subset (X-A)(x_i) \cap O(a) \Rightarrow x_i \in O(a) \Rightarrow n(a,O) \leq n(x_i) = n(a_i, O_i) \Rightarrow a_i \in O \Rightarrow f(a_i) \in N \Rightarrow \text{ext}(f)(x) \in N$.]

Claim 2: $\text{ext} \in \text{LEO}(X, A; E)$.

[Define a function $\phi : X \rightarrow 2^A$ by the rule $\begin{cases} \phi(a) = \{a\} & (a \in A) \\ \phi(x) = \{a_i : i \in I_x\} & (x \in X - A) \end{cases}$, I_x the set $\{i \in I : x \in \text{spt}\, \kappa_i\}$. Given a nonempty compact subset K of X, put $K_A = \bigcup\limits_{x \in K} \phi(x)$. Assertion: K_A is compact. Since the $\phi(x)$ are finite, hence compact, it will be enough to show that for every $x \in X$ and for every open subset V of A containing $\phi(x)$ there exists an open subset U of X containing x such that $\cup\phi(U) \subset V$. Case 1: $x \in X - A$. Here one need only remark that there exists a neighborhood U of x in $X - A : y \in U \Rightarrow \phi(y) \subset \phi(x)$. Case 2: $a \in A$. Let O be an open subset of X: $\phi(a) = \{a\} \subset O$. If $x \in A \cap O(a)(a)$, then $\phi(x) = \{x\} \subset O$, while if $x \in (X - A) \cap O(a)(a)$, then arguing as in the first claim, $\forall\, i \in I_x, a_i \in O$. Conclusion: $\cup\phi(O(a)(a)) \subset A \cap O$.]]

[Note: Suppose that E is a normed linear space—then the image of $\text{ext}|BC(A,E)$ is contained in $BC(X,E)$ and, per the uniform topology, ext : $BC(A,E) \rightarrow BC(X,E)$ is a linear isometric embedding: $\forall\, f \in BC(A,E)$, $\|f\| = \|\text{ext}(f)\|$.]

FACT Let $A \subset X$, where X is stratifiable and A is closed—then A has the EP w.r.t. any locally convex topological space.

Is it true that if K is a compact Hausdorff space and X is stratifiable, then $C(K, X)$ is stratifiable? The answer is "no" even if $K = [0, 1]$.

EXAMPLE Let X be the closed upper half plane in $\mathbf{R}^2$. Topologize X as follows: The basic neighborhoods of (x, y) $(y > 0)$ are as usual but the basic neighborhoods of $(x, 0)$ are the "butterflies" $N_\epsilon(x)$ $(\epsilon > 0)$, where $N_\epsilon(x)$ is the point $(x, 0)$ together with all points in the open upper half plane having distance $< \epsilon$ from $(x, 0)$ and lying beneath the union of the two rays emanating from $(x, 0)$ with slopes $\pm\epsilon$. Thus topologized, X is stratifiable (and satisfies Ceder's condition). Moreover, X is first countable and separable. But X is not second countable, so X is not metrizable. Therefore X carries no CW structure (since for a CW complex, metrizability is equivalent to first countability). Claim: $C([0, 1], X)$ is not stratifiable. To see this, assign to each $r \in \mathbf{R}$ an element $f_r \in C([0, 1], X)$ by putting $f_r(1/2) = (r, 0)$ and then laying down $[0, 1]$ symmetrically around the circle of radius 1 centered at $(r, 1)$. The set $\{f_r\}$ is a closed discrete subspace of $C([0, 1], X)$ of cardinality 2^ω. Construct a closed separable subspace of $C([0, 1], X)$ containing $\{f_r\}$ and finish by quoting Jones' lemma.

[Note: X is compactly generated (being first countable). However, $C([0, 1], X)$ is not compactly generated.]

Cauty[†] has shown that if X is a CW complex, then for any compact Hausdorff space K, $C(K, X)$ is stratifiable, hence is perfectly normal and paracompact.

Let κ be an infinite cardinal. A Hausdorff space X is said to be κ-collectionwise normal if for every discrete collection $\{A_i : i \in I\}$ of closed subsets of X with $\#(I) \leq \kappa$ there exists a pairwise disjoint collection $\{U_i : i \in I\}$ of open subsets of X such that $\forall\, i \in I : A_i \subset U_i$. So: X is collectionwise normal iff X is κ-collectionwise normal for every κ.

[Note: Recall that every paracompact Hausdorff space is collectionwise normal (cf. §1, Proposition 9).]

EXAMPLE If X is normal, then X is ω-collectionwise normal (cf. p. 1–14) and conversely.

Let κ be an infinite cardinal; let I be a set: $\#(I) = \kappa$. Assuming that $0 \notin I$, let $V = \{0\} \cup I$ and put $\Sigma = \{\{0\}, \{i\}(i \in I)\} \cup \{\{0, i\}(i \in I)\}$—then $K = (V, \Sigma)$ is a vertex scheme. Equipping I with the discrete topology, one may view $|K|$ as the cone

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[†] *Arch. Math. (Basel)* **27** (1976), 306–311; see also Guo, *Tsukuba J. Math.* **18** (1994), 505–517.

ΓI. Therefore $|K|$ is contractible, hence so is $|K|_b$ (cf. §5, Proposition 1), the latter being by definition the star space $\mathrm{S}(\kappa)$ corresponding to κ. It is clear that $\mathrm{S}(\kappa)$ is completely metrizable of weight κ. The elements of $\mathrm{S}(\kappa)$ are equivalence classes $[i,t]$ of pairs (i,t), where $(i',t') \sim (i'',t'')$ iff $t' = 0 = t''$ or $i' = i''$ & $t' = t''$. There is a continuous map $\pi_\kappa : \begin{cases} \mathrm{S}(\kappa) \to [0,1] \\ [i,t] \to t \end{cases}$ and $\forall\ i \in I$ there is an embedding $e_i : \begin{cases} [0,1] \to \mathrm{S}(\kappa) \\ t \to [i,t] \end{cases}$. The point $e_i(0)$ is independent of i and will be denoted by 0_κ.

PROPOSITION 13 Let X be a Hausdorff space—then X is κ-collectionwise normal iff every closed subspace A of X has the EP w.r.t. $\mathrm{S}(\kappa)$.

[Necessity: Fix an $f \in C(A, \mathrm{S}(\kappa))$ and let $\Phi : X \to [0,1]$ be a continuous extension of $\pi_\kappa \circ f$. Put $A_i = f^{-1}(\{[i,t] : 0 < t \leq 1\}) : \{A_i : i \in I\}$ is a discrete collection of closed subsets of $\Phi^{-1}(]0,1])$. Since $\Phi^{-1}(]0,1])$ is an F_σ, it too is κ-collectionwise normal, thus there exists a pairwise disjoint collection $\{U_i : i \in I\}$ of open subsets of X such that $\forall\ i \in I : A_i \subset U_i$. Define a continuous function $g : A \cup (X - \bigcup_i U_i) \to [0,1]$ by the conditions $\begin{cases} g|A = \pi_\kappa \circ f \\ g|X - \bigcup_i U_i = 0 \end{cases}$ and extend it to a continuous function $G : X \to [0,1]$. Set $F(x) = \begin{cases} e_i \circ G(x) & (x \in U_i) \\ 0_\kappa & (x \in X - \bigcup_i U_i) \end{cases}$—then $F \in C(X, \mathrm{S}(\kappa))$ and $F|A = f$.

Sufficiency: Let $\{A_i : i \in I\}$ be a discrete collection of closed subsets of X with $\#(I) = \kappa$. Put $A = \bigcup_i A_i$—then A is a closed subspace of X. Define $f \in C(A, \mathrm{S}(\kappa))$ piecewise: $f|A_i = [i,1]$. Extend f to $F \in C(X, \mathrm{S}(\kappa))$ and consider the collection $\{U_i : i \in I\}$, where $U_i = F^{-1}(\{[i,t] : 1/2 < t \leq 1\})$.]

Application: The star space $\mathrm{S}(\kappa)$ is an AR.

EXAMPLE Let κ be an infinite cardinal—then there exists a κ-collectionwise normal space X which is not κ^+-collectionwise normal, κ^+ the cardinal successor of κ. For this, fix a set I^+ of cardinality κ^+ and equip I^+ with the discrete topology. There is an embedding $I^+ \to \prod \mathrm{S}(\kappa)$, the terms of the product being indexed by the elements of $C(I^+, \mathrm{S}(\kappa))$. Let X be the result of retopologizing $\prod \mathrm{S}(\kappa)$ by isolating the points of $\prod \mathrm{S}(\kappa) - I^+$.

Claim: X is κ-collectionwise normal.

[Let $\{A_i : i \in I\}$ be a discrete collection of closed subsets of X with $\#(I) = \kappa$. Since $X - I^+$ is discrete, there is no loss of generality in assuming that the A_i are contained in I^+. Define a continuous function $f : \bigcup_i A_i \to \mathrm{S}(\kappa)$ by $f|A_i = [i,1]$ and then, using Proposition 13, extend f to an element $F \in C(I^+, \mathrm{S}(\kappa))$, determining a projection $p_F : \prod \mathrm{S}(\kappa) \to \mathrm{S}(\kappa)$ such that $p_F|I^+ = F$. Consider the collection $\{U_i : i \in I\}$, where $U_i = p_F^{-1}(\{[i,t] : 1/2 < t \leq 1\})$.]

Claim: X is not κ^+-collectionwise normal.

[If X were κ^+-collectionwise normal, then it would be possible to separate the points of I^+ by a collection of nonempty pairwise disjoint open subsets of X of cardinality κ^+. Taking into account how X is manufactured from $\prod \mathbf{S}(\kappa)$, one arrives at a contradiction to an obvious corollary of the Hewitt-Pondiczery theorem.]

[Note: Give $I^+ \times \{0\} \cup \bigcup\limits_1^\infty (X - I^+) \times \{1/n\}$ the topology induced by the product $X \times [0,1]$—then this space is perfectly normal and κ-collectionwise normal but is not κ^+-collectionwise normal. And: It is not a LCH space (cf. p. 1–15).]

KOWALSKY'S LEMMA Let κ be an infinite cardinal. Let Y be an AR of weight κ—then every metrizable space X of weight $\leq \kappa$ can be embedded in Y^ω.

[Let $\mathcal{U} = \bigcup\limits_n \mathcal{U}_n$ be a σ-discrete basis for $X : \mathcal{U}_n = \{U_n(i) : i \in I_n\}$, where $I = \coprod\limits_n I_n$ and $\#(I) \leq \kappa$. Write $\cup\mathcal{U}_n = \bigcup\limits_m A_{mn}, A_{mn}$ closed in X. Fix distinct points a, b which do not belong to I. Since wt $Y = \kappa$, there exists in Y a collection of nonempty pairwise disjoint open sets V_j ($j \in I \cup \{a, b\}$). Choose a point $y_j \in V_j$. Given n, define a continuous function $f_n : \cup\mathcal{U}_n \rightarrow Y$ by $f_n|\overline{U_n(i)} = y_i$ ($i \in I_n$) and extend f_n to a continuous function $F_n : X \rightarrow Y$. Given mn, define a continuous function $f_{mn} : A_{mn} \cup (X - \cup\mathcal{U}_n) \rightarrow Y$ by $\begin{cases} f_{mn}|A_{mn} = y_a \\ f_{mn}|X - \cup\mathcal{U}_n = y_b \end{cases}$ and extend f_{mn} to a continuous function $F_{mn} : X \rightarrow Y$. Let $\Phi_{mn} : X \rightarrow Y^2$ be the diagonal of F_n and F_{mn}. Let Φ be the diagonal of the Φ_{mn}, so $\Phi : X \rightarrow (Y^2)^{\omega^2} \equiv Y^\omega$—then Φ is an embedding.]

[Note: Suppose that Y is not compact—then every completely metrizable space X of weight $\leq \kappa$ can be embedded in Y^ω as a closed subspace. For X, as a subspace of Y^ω, is a G_δ (being completely metrizable), thus on elementary grounds is homeomorphic to a closed subspace of $Y^\omega \times \mathbf{R}^\omega$: Take a compatible metric d on Y^ω, represent the complement $Y^\omega - X$ as a countable union $\bigcup\limits_j B_j$ of closed subsets B_j, let $d_j : Y^\omega \rightarrow \mathbf{R}$ be the function $y \rightarrow d(y, B_j)$, and consider the graph of the diagonal of the d_j. Claim: There is a closed embedding $\mathbf{R} \rightarrow Y^\omega$. To see this, fix a closed discrete subset $\{y_n : n \in \mathbf{Z}\}$ in Y. Let $\begin{cases} S = \bigcup\limits_{-\infty}^{\infty} [2n, 2n+1] \\ T = \bigcup\limits_{-\infty}^{\infty} [2n+1, 2n+2] \end{cases}$ and define continuous functions $\begin{cases} f : S \rightarrow Y \\ g : T \rightarrow Y \end{cases}$ by $\begin{cases} f|[2n, 2n+1] = y_n \\ g|[2n+1, 2n+2] = y_n \end{cases}$. Extend $\begin{cases} f \\ g \end{cases}$ to a continuous function $\begin{cases} F : \mathbf{R} \rightarrow Y \\ G : \mathbf{R} \rightarrow Y \end{cases}$ and let $H : \mathbf{R} \rightarrow Y^2$ be the diagonal of F and G. If $\Phi : \mathbf{R} \rightarrow Y^\omega$ is any embedding, then the diagonal of Φ and H is a closed embedding $\mathbf{R} \rightarrow Y^\omega \times Y^2 \equiv Y^\omega$.]

Application: Every metrizable space X of weight $\leq \kappa$ can be embedded in $\mathbf{S}(\kappa)^\omega$.

Let κ be an infinite cardinal. Let X be a topological space—then a subspace $A \subset X$ is said to have the extension property with respect to $\mathcal{B}(\kappa)$ (EP w.r.t. $\mathcal{B}(\kappa)$) if it has the EP w.r.t. every Banach space of weight $\leq \kappa$. Since every completely metrizable AR can be realized as a closed subspace of a Banach space (cf. p. 6–12), it is clear that A has the EP w.r.t. $\mathcal{B}(\kappa)$ iff it has the EP w.r.t. every completely metrizable AR of weight $\leq \kappa$.

PROPOSITION 14 Fix a pair (X, A). Suppose that for some noncompact AR Y of weight κ, A has the EP w.r.t. Y—then A has the EP w.r.t. $\mathcal{B}(\kappa)$.

[Let E be a Banach space of weight $\leq \kappa$. Owing to Kowalsky's lemma, E can be realized as a closed subspace of Y^ω. Let $f \in C(A, E)$. By hypothesis, f has a continuous extension $F \in C(X, Y^\omega)$. Consider $r \circ F$, where $r : Y^\omega \to E$ is a retraction.]

One conclusion that can be drawn from this is that A has the EP w.r.t. $\mathbf{R}$ iff A has the EP w.r.t $\mathcal{B}(\omega)$. So: If X is a Hausdorff space, then X is normal iff every closed subspace A of X has the EP w.r.t. every separable Banach space.

Another conclusion is that A has the EP w.r.t. $\mathbf{S}(\kappa)$ iff A has the EP w.r.t. $\mathcal{B}(\kappa)$. Consequently, if X is a Hausdorff space, then X is κ-collectionwise normal iff every closed subspace A of X has the EP with respect to $\mathcal{B}(\kappa)$ (cf. Proposition 13). Corollary: A Hausdorff space X is collectionwise normal iff every closed subspace A of X has the EP w.r.t. every Banach space.

FACT Let $A \subset X$—then A has the EP w.r.t. $\mathbf{R}$ iff $IA \subset IX$ has the EP w.r.t. $[0, 1]$.

Let X be a topological space. Let $\{\mathcal{U}_n\}$ be a sequence of open coverings of X—then $\{\mathcal{U}_n\}$ is said to be a star sequence if $\forall\ n$, $\mathcal{U}_{n+1}$ is a star refinement of $\mathcal{U}_n$. By means of a standard construction from metrization theory, one can associate with a given star sequence $\{\mathcal{U}_n\}$ a continuous pseudometric δ on X such that $\delta(x, y) = 0$ iff $y \in \bigcap\limits_1^\infty \text{st}(x, \mathcal{U}_n)$, a subset $U \subset X$ being open in the topology generated by δ iff $\forall\ x \in U\ \exists\ n : \text{st}(x, \mathcal{U}_n) \subset U$. Let X_δ be the metric space obtained from X by identifying points at zero distance from one another and write $p : X \to X_\delta$ for the projection.

PROPOSITION 15 Let $A \subset X$—then A has the EP w.r.t. $\mathcal{B}(\kappa)$ iff for every numerable open covering $\mathcal{O}$ of A of cardinality $\leq \kappa$ there exists a numerable open covering $\mathcal{U}$ of X of cardinality $\leq \kappa$ such that $\mathcal{U} \cap A$ is a refinement of $\mathcal{O}$.

[Necessity: Let $\mathcal{O} = \{O_i : i \in I\}$ be a numerable open covering of A with $\#(I) \leq \kappa$. Choose a partition of unity $\{\kappa_i : i \in I\}$ on A subordinate to $\mathcal{O}$. Form the Banach space $\ell^1(I) : r = (r_i) \in \ell^1(I)$ iff $\|r\| = \sum_i |r_i| < \infty$. The assignment $\begin{cases} A \rightarrow \ell^1(I) \\ a \rightarrow (\kappa_i(a)) \end{cases}$ defines a continuous function f whose range is contained in $S^+ = \{r : \|r\| = 1\} \cap \{r : \forall\, i, r_i \geq 0\}$, a closed convex subset of $\ell^1(I)$. Therefore f has a continuous extension $F : X \rightarrow S^+$. Let p_i be the projection $\begin{cases} \ell^1(I) \rightarrow \mathbf{R} \\ r \rightarrow r_i \end{cases}$; let $\sigma_i = p_i \circ F$—then $\sigma_i|A = \kappa_i$ and $\sum_i \sigma_i(x) = 1$ $(\forall\, x \in X)$. Put $U_i = \sigma_i^{-1}(]0,1])$ and apply NU (cf. p. 1–23) to see that the collection $\mathcal{U} = \{U_i : i \in I\}$ is a numerable open covering of X of cardinality $\leq \kappa$. And by construction, $\mathcal{U} \cap A$ is a refinement of $\mathcal{O}$.

Sufficiency: Let E be a Banach space of weight $\leq \kappa$. Fix a dense subset E_0 in E of cardinality $\leq \kappa$ and let $\mathcal{E}_n$ be the open covering of E consisting of the open balls of radius $1/3^n$ centered at the points of E_0. Suppose that $f : A \rightarrow E$ is continuous—then $\forall\, n, f^{-1}(\mathcal{E}_n)$ is a numerable open covering of A of cardinality $\leq \kappa$, so there exists a star sequence $\{\mathcal{U}_n\}$ of open coverings of X of cardinality $\leq \kappa$ such that $\forall\, n$, $\mathcal{U}_n \cap A$ is a refinement of $f^{-1}(\mathcal{E}_n)$. Viewed as a map from A endowed with the topology induced by the pseudometric δ associated with $\{\mathcal{U}_n\}$, f is continuous, thus passes to the quotient to give a continuous function $f_\delta : A_\delta \rightarrow E$, where $A_\delta = p(A)$. Because f_δ is actually uniformly continuous, there exists a continuous extension $\overline{f}_\delta : \overline{A}_\delta \rightarrow E$ of f_δ to the closure $\overline{A}_\delta$ of A_δ in X_δ. Choose $F_\delta \in C(X_\delta, E) : F_\delta|\overline{A}_\delta = \overline{f}_\delta$ and consider $F = F_\delta \circ p$.]

Examples: Let X be a CRH space—then $\forall\, \kappa$ (1) Every compact subspace of X has the EP w.r.t. $\mathcal{B}(\kappa)$; (2) Every pseudocompact subspace of X which has the EP w.r.t. $[0,1]$ has the EP w.r.t. $\mathcal{B}(\kappa)$; (3) Every Lindelöf subspace of X which has the EP w.r.t. $\mathbf{R}$ has the EP w.r.t. $\mathcal{B}(\kappa)$.

Suppose that X is collectionwise normal. Let A be a closed subspace of X; let $\mathcal{O} = \{O_i : i \in I\}$ be a neighborhood finite open covering of A—then Proposition 15 implies that there exists a neighborhood finite open covering $\mathcal{U} = \{U_i : i \in I\}$ of X such that $\forall\, i \in I$, $U_i \cap A \subset O_i$. Question: Is it possible to arrange matters so that $\forall\, i \in I$, $U_i \cap A = O_i$? The answer is "no" since Rudin's Dowker space fails to admit this improvement (Przymusiński-Wage[†]) but "yes" if X is in addition countably paracompact (Katětov[‡]).

[†] *Fund. Math.* **109** (1980), 175–187.

[‡] *Colloq. Math.* **6** (1958), 145–151.

Let (X, δ) be a pseudometric space; let A be a closed subspace of X—then A has the EP w.r.t. every AR Y. Proof: Let X_δ be the metric space obtained from X by identifying points at zero distance from one another, write p for the projection $X \rightarrow X_\delta$, and put $A_\delta = p(A)$, a closed subspace of X_δ. Each $f \in C(A, Y)$ passes to the quotient to give an $f_\delta \in C(A_\delta, Y)$ for which there exists an extension $F_\delta \in C(X_\delta, Y)$. Consider $F = F_\delta \circ p$.

The weight of a pseudometric is the weight of its associated topology.

LEMMA Let $A \subset X$—then A has the EP w.r.t. $\mathcal{B}(\kappa)$ iff every continuous pseudometric on A of weight $\leq \kappa$ can be extended to a continuous pseudometric on X.

[Necessity: Let δ be a continuous pseudometric on A of weight $\leq \kappa$. Let A_δ be the metric space obtained from A by identifying points at zero distance from one another. Embed A_δ isometrically into a Banach space E of weight $\leq \kappa$—then the projection $A \rightarrow A_\delta \subset E$ has a continuous extension $\Phi : X \rightarrow E$ and the assignment $\Delta : \begin{cases} X \times X \rightarrow \mathbf{R} \\ (x', x'') \rightarrow \|\Phi(x') - \Phi(x'')\| \end{cases}$ is a continuous extension of δ.

Sufficiency: Let E be a Banach space of weight $\leq \kappa$; let $f \in C(A, E)$. Define a pseudometric δ on A by $\delta(a', a'') = \|f(a') - f(a'')\|$—then δ is continuous of weight $\leq \kappa$, hence admits a continuous extension Δ. Call $X(\Delta)$ the set X equipped with the topology determined by Δ. Let $A(\Delta)$ be the closure of A in $X(\Delta)$. Extend f continuously to a function $f(\Delta) : A(\Delta) \rightarrow E$ and note that $A(\Delta) \subset X(\Delta)$ has the EP w.r.t. E.]

FACT Let A be a zero set in X. Suppose that A has the EP w.r.t. $\mathcal{B}(\kappa)$—then A has the EP w.r.t. every AR Y of weight $\leq \kappa$.

[Choose a $\phi \in C(X, [0,1]) : A = \phi^{-1}(0)$. Fix a compatible metric d on Y. Given $f \in C(A, Y)$, define a pseudometric δ on A by $\delta(a', a'') = d(f(a'), f(a''))$. Let Δ be a continuous extension of δ to X and consider the sum of $\Delta(x', x'')$ and $|\phi(x') - \phi(x'')|$.]

Let X be a CRH space. Suppose that X is perfectly normal and collectionwise normal—then it follows that every closed subspace A of X has the EP w.r.t. every AR.

FACT Let X be a submetrizable CRH space. Suppose that $A \subset X$ has the EP with respect to every normed linear space—then A is a zero set in X.

[Note: Take for X the Michael line and let $A = \mathbf{Q}$—then X is a paracompact Hausdorff space, so A has the EP w.r.t. every Banach space. On the other hand, X is submetrizable but A is not a G_δ. Therefore A does not have the EP w.r.t. every normed linear space.]

LEMMA Fix a pair (X, A). Suppose that A has the EP w.r.t. $\mathcal{B}(\kappa)$—then every continuous function $\phi : i_0 X \cup IA \rightarrow \mathbf{S}(\kappa)$ has a continuous extension $\Phi : IX \rightarrow \mathbf{S}(\kappa)$.

[The restriction ψ of ϕ to IA determines a continuous function $A \to C([0,1],\mathrm{S}(\kappa))$. But $C([0,1],\mathrm{S}(\kappa))$ is a completely metrizable AR (cf. the proof of Proposition 6), the weight of which is $\leq \kappa$, so our assumption on A guarantees that this function has a continuous extension $X \to C([0,1],\mathrm{S}(\kappa))$, leading thereby to a continuous function $\Psi : IX \to \mathrm{S}(\kappa)$ whose restriction to IA is ψ. Choose an $f \in C(X,[0,1]) : f^{-1}(0) = \{x : \phi(x,0) = \Psi(x,0)\}$. Let F be the function $\left\{\begin{array}{l} X \to \mathrm{S}(\kappa) \\ x \to \Psi(x,f(x)) \end{array}\right.$. Because $\mathrm{S}(\kappa)$ is contractible, there is a homotopy $H : IX \to \mathrm{S}(\kappa)$ such that $\left\{\begin{array}{l} H(x,0) = \phi(x,0) \\ H(x,1) = F(x) \end{array}\right.$. Consider the function $\Phi : IX \to \mathrm{S}(\kappa)$ defined by $\Phi(x,t) = \left\{\begin{array}{ll} \Psi(x,t) & (t \geq f(x)) \\ H(x,t/f(x)) & (t < f(x)) \end{array}\right.$.]

PROPOSITION 16 Let $A \subset X$—then A has the EP w.r.t. $\mathcal{B}(\kappa)$ iff $i_0X \cup IA$, as a subspace of IX, has the EP w.r.t. every completely metrizable ANR Y of weight $\leq \kappa$.

[Necessity: Let $f : i_0X \cup IA \to Y$ be continuous. Using Kowalsky's lemma, realize Y as a closed subspace of $\mathrm{S}(\kappa)^\omega$ and let $r : O \to Y$ be a retraction (O open in $\mathrm{S}(\kappa)^\omega$). Given a projection $p : \mathrm{S}(\kappa)^\omega \to \mathrm{S}(\kappa)$, let $\phi_p = p \circ f$—then by what has been said above, ϕ_p has a continuous extension $\Phi_p : IX \to \mathrm{S}(\kappa)$. Therefore f has a continuous extension $\Phi : IX \to \mathrm{S}(\kappa)^\omega$. Set $P = \Phi^{-1}(O)$. Since P is a cozero set in IX containing IA and since the projection $IX \to X$ takes zero sets to zero sets, there is a cozero set U in X such that $A \subset U$ and $IU \subset P$. On the other hand, A has the EP w.r.t. $\mathbf{R}$, so it follows from Proposition 3 that $\exists\ \phi \in C(X,[0,1]) : \left\{\begin{array}{l} \phi|A = 1 \\ \phi|X - U = 0 \end{array}\right.$. Define $F \in C(IX,Y)$ by $F(x,t) = r(\Phi(x,\phi(x)t)) : F$ is a continuous extension of f.

Sufficiency: Let $\mathcal{O} = \{O_i : i \in I\}$ be a neighborhood finite cozero set covering of A with $\#(I) \leq \kappa$. Put

$$\mathcal{P} = \{O_i \times]1/3,1] : i \in I\} \cup \{i_0X \cup A \times [0,2/3[\}.$$

Then $\mathcal{P}$ is a neighborhood finite cozero set covering of $i_0X \cup IA$ of cardinality $\leq \kappa$, thus Proposition 15 implies that there exists a numerable open covering $\mathcal{V}$ of IX of cardinality $\leq \kappa$ such that $\mathcal{V} \cap (i_0X \cup IA)$ is a refinement of $\mathcal{P}$. Let $\mathcal{U} = \mathcal{V} \cap (i_1X) : \mathcal{U}$ is a numerable open covering of i_1X such that $\mathcal{U} \cap (i_1A)$ is a refinement of $\mathcal{P} \cap (i_1A) = i_1\mathcal{O}$. Finish by quoting Proposition 15.]

EXAMPLE Suppose that the inclusion $A \to X$ is a cofibration—then $i_0X \cup IA$ is a retract of IX (cf. §3, Proposition 1), so Proposition 16 implies that A has the EP w.r.t. every Banach space.

[Note: This applies in particular to a relative CW complex (X,A).]

Let X and Y be topological spaces.

(HEP) A subspace $A \subset X$ is said to have the homotopy extension property with respect to Y (HEP w.r.t. Y) if given continuous functions $\begin{cases} F : X \rightarrow Y \\ h : IA \rightarrow Y \end{cases}$ such that $F|A = h \circ i_0$, there is a continuous function $H : IX \rightarrow Y$ such that $F = H \circ i_0$ and $H|IA = h$.

[Note: In this terminology, the inclusion $A \rightarrow X$ is a cofibration iff A has the HEP w.r.t. Y for every Y.]

Suppose that A has the HEP w.r.t. Y. Let $\begin{cases} f \in C(A,Y) \\ g \in C(A,Y) \end{cases}$ be homotopic. Assume: f has a continuous extension $F \in C(X,Y)$—then g has a continuous extension $G \in C(X,Y)$ and $F \simeq G$. Therefore, under these circumstances, the extension question for continuous functions $A \rightarrow Y$ is a problem in the homotopy category.

If $A \subset X$ is closed and if $i_0X \cup IA$, as a subspace of IX, has the EP w.r.t. Y, then it is clear that A has the HEP w.r.t. Y. Conditions ensuring that this is so are provided by Proposition 16. Here are two illustrations.

(1) Every closed subspace A of a normal Hausdorff space X has the HEP w.r.t. every second countable completely metrizable ANR Y.

(2) Every closed subspace A of a collectionwise normal Hausdorff space X has the HEP w.r.t. every completely metrizable ANR Y.

[Note: Historically, these results were obtained by imposing in addition a countable paracompactness assumption on X. Reason: If X is a normal Hausdorff space, then the product IX is normal iff X is countably paracompact.]

If $A \subset X$ and if A has the EP w.r.t. $\mathcal{B}(\kappa)$, then A has the HEP w.r.t. every completely metrizable ANR Y of weight $\leq \kappa$. Proof: Take a pair of continuous functions $\begin{cases} F : X \rightarrow Y \\ h : IA \rightarrow Y \end{cases}$ such that $F|A = h \circ i_0$ and define $\phi : i_0X \cup IA \rightarrow Y$ by $\begin{cases} \phi(x,0) = F(x) \\ \phi(a,t) = h(a,t) \end{cases}$. In view of Proposition 16, the only issue is the continuity of ϕ. To see this, embed Y in a Banach space E of weight $\leq \kappa$. Since IA, as a subspace of IX, has the EP w.r.t. $\mathcal{B}(\kappa)$, h has a continuous extension $\overline{h} : I\overline{A} \rightarrow E$. Define $\overline{\phi} : i_0X \cup I\overline{A} \rightarrow E$ by $\begin{cases} \overline{\phi}(x,0) = F(x) \\ \overline{\phi}(\overline{a},t) = \overline{h}(\overline{a},t) \end{cases}$—then $\overline{\phi}$ is a welldefined continuous function which agrees with ϕ on $i_0X \cup IA$.

EXAMPLE The product $Y = \mathbf{S}^n \times \mathbf{S}^n \times \cdots$ (ω factors) is not an ANR. But if X is normal and $A \subset X$ is closed, then A has the HEP w.r.t. Y.

FACT Suppose that X is Hausdorff. Let A be a zero set in X.

(1) If X is normal, then A has the HEP w.r.t. every second countable ANR Y.

(2) If X is collectionwise normal, then A has the HEP w.r.t. every ANR Y.

FACT Let Y be a nonempty metrizable space. Suppose that Y is locally contractible—then Y is an ANR iff for every pair (X, A), where X is metrizable and $A \subset X$ is closed, A has the HEP w.r.t. Y.

Let $\mathcal{X}$ be a homeomorphism invariant class of normal Hausdorff spaces that is closed hereditary, i.e., if $X \in \mathcal{X}$ and if $A \subset X$ is closed, then $A \in \mathcal{X}$.

Let $\mathcal{X}$ be the class consisting of the Hausdorff spaces satisfying Ceder's condition—then it is unknown whether $\mathcal{X}$ is closed hereditary.

A nonempty topological space Y is said to be an extension space for $\mathcal{X}$ if every closed subspace of every element of $\mathcal{X}$ has the EP w.r.t. Y. Denote by $\text{ES}(\mathcal{X})$ the class of extension spaces for $\mathcal{X}$. Obviously, if $\mathcal{X}' \subset \mathcal{X}''$, then $\text{ES}(\mathcal{X}'') \subset \text{ES}(\mathcal{X}')$, so $\forall\ \mathcal{X} : \text{ES(normal)} \subset \text{ES}(\mathcal{X})$.

(ES_1) The class $\text{ES}(\mathcal{X})$ is closed under the formation of products.

(ES_2) Any retract of an extension space for $\mathcal{X}$ is in $\text{ES}(\mathcal{X})$.

(ES_3) Suppose that $Y = Y_1 \cup Y_2$, where Y_1 and Y_2 are open and $\begin{cases} Y_1 \\ Y_2 \end{cases} \in \text{ES}(\mathcal{X})$ & $Y_1 \cap Y_2 \in \text{ES}(\mathcal{X})$—then $Y \in \text{ES}(\mathcal{X})$.

(ES_4) Assume: The elements of $\mathcal{X}$ are hereditarily normal. Suppose that $Y = Y_1 \cup Y_2$, where Y_1 and Y_2 are closed and $\begin{cases} Y_1 \\ Y_2 \end{cases} \in \text{ES}(\mathcal{X})$ & $Y_1 \cap Y_2 \in \text{ES}(\mathcal{X})$—then $Y \in \text{ES}(\mathcal{X})$.

(ES_5) Suppose that $Y = Y_1 \cup Y_2$, where Y_1 and Y_2 are closed—then $Y \in \text{ES}(\mathcal{X})$ & $Y_1 \cap Y_2 \in \text{ES}(\mathcal{X}) \Rightarrow \begin{cases} Y_1 \\ Y_2 \end{cases} \in \text{ES}(\mathcal{X})$.

EXAMPLE A nonempty topological space Y is an extension space for the class of metrizable spaces iff it is an extension space for the class of M complexes.

A nonempty topological space Y is said to be a neighborhood extension space for $\mathcal{X}$ if every closed subspace of every element of $\mathcal{X}$ has the NEP w.r.t. Y. Denote by $\text{NES}(\mathcal{X})$ the class of neighborhood extension spaces for $\mathcal{X}$. Obviously, if $\mathcal{X}' \subset \mathcal{X}''$, then $\text{NES}(\mathcal{X}'') \subset \text{NES}(\mathcal{X}')$, so $\forall\ \mathcal{X} : \text{NES(normal)} \subset \text{NES}(\mathcal{X})$. Of course, $\text{ES}(\mathcal{X}) \subset \text{NES}(\mathcal{X})$. In the other direction, every contractible element of $\text{NES}(\mathcal{X})$ is in $\text{ES}(\mathcal{X})$.

[Note: It is convenient to agree that $\emptyset \in \text{NES}(\mathcal{X})$. So, if $Y \in \text{NES}(\mathcal{X})$ and if $V \subset Y$ is open, then $V \in \text{NES}(\mathcal{X})$.]

(NES_1) The class $\text{NES}(\mathcal{X})$ is closed under the formation of finite products.

(NES_2) Any neighborhood retract of a neighborhood extension space for $\mathcal{X}$ is in $\text{NES}(\mathcal{X})$.

(NES$_3$) Suppose that $Y = Y_1 \cup Y_2$, where Y_1 and Y_2 are open and $\begin{cases} Y_1 \\ Y_2 \end{cases} \in \text{NES}(\mathcal{X})$—then $Y \in \text{NES}(\mathcal{X})$.

(NES$_4$) Assume: The elements of $\mathcal{X}$ are hereditarily normal. Suppose that $Y = Y_1 \cup Y_2$, where Y_1 and Y_2 are closed and $\begin{cases} Y_1 \\ Y_2 \end{cases} \in \text{NES}(\mathcal{X})$ & $Y_1 \cap Y_2 \in \text{NES}(\mathcal{X})$—then $Y \in \text{NES}(\mathcal{X})$.

(NES$_5$) Suppose that $Y = Y_1 \cup Y_2$, where Y_1 and Y_2 are closed—then $Y \in \text{NES}(\mathcal{X})$ & $Y_1 \cap Y_2 \in \text{NES}(\mathcal{X}) \Rightarrow \begin{cases} Y_1 \\ Y_2 \end{cases} \in \text{NES}(\mathcal{X})$.

[Note: There is a slight difference between the formulation of ES$_3$, and NES$_3$. Reason: An empty intersection is permitted in NES$_3$ but not in ES$_3$ (consider $X = [0,1]$, $A = Y = \{0,1\}$).]

EXAMPLE (CW Complexes) Metrizable CW complexes are ANRs (cf. p. 6–16).

(1) Every finite CW complex is in NES(normal).

(2) Every CW complex is in NES(compact) (but it is not true that every CW complex is in NES(paracompact)).

(3) Every CW complex is in NES(stratifiable).

[First, if K is a full vertex scheme, then $|K|$ is a locally convex topological space (cf. p. 6–11), so $|K| \in \text{ES(stratifiable)}$ (cf. p. 6–34). Second, if K is a vertex scheme and if L is a subscheme, then $|L|$ is a neighborhood retract of $|K|$. Third, if X is a CW complex, then X is the retract of a polyhedron (cf. p. 5–12).]

FACT Every CW complex has the homotopy type of an ANR which is in NES(paracompact).

EXAMPLE Suppose that $X = Y \cup Z$ is metrizable. Let K and L be finite CW complexes. Assume: Every closed subspace of $\begin{cases} Y \\ Z \end{cases}$ has the EP w.r.t. $\begin{cases} K \\ L \end{cases}$—then every closed subspace of X has the EP w.r.t. $K * L$.

The "ES" arguments are similar to but simpler than the "NES" arguments. Of the latter, the most difficult is the one for NES$_3$, which runs as follows. Take an X in $\mathcal{X}$ and let $A \subset X$ be closed—then the claim is that $\forall\, f \in C(A,Y)$ there exists an open $U \supset A$ and an $F \in C(U,Y) : F|A = f$. Since X is covered by the open sets $\begin{cases} f^{-1}(Y_1) \cup (X - A) \\ f^{-1}(Y_2) \cup (X - A) \end{cases}$ and since X is normal, there exist closed sets $\begin{cases} X_1 \subset X \\ X_2 \subset X \end{cases}$ which cover X with $\begin{cases} X_1 \subset f^{-1}(Y_1) \cup (X - A) \\ X_2 \subset f^{-1}(Y_2) \cup (X - A) \end{cases}$. Put $\begin{cases} A_1 = X_1 \cap A \\ A_2 = X_2 \cap A \end{cases}$. There are now two cases, depending on whether $Y_1 \cap Y_2$ is empty or not. The second possibility is more involved than the first so we shall look only at it. Because $Y_1 \cap Y_2 \in \text{NES}(\mathcal{X})$, the restriction $f|A_1 \cap A_2$ has an extension $f_{12} \in C(O, Y_1 \cap Y_2)$, where O is some open subset of $X_1 \cap X_2$ containing $A_1 \cap A_2$. Choose an open subset P of $X_1 \cap X_2 : A_1 \cap A_2 \subset P \subset \overline{P} \subset O$. Observing that $A \cap \overline{P} = A_1 \cap A_2$, define $g \in C(A \cup \overline{P}, Y)$ by $g(x) = \left\{\begin{array}{ll} f(x) & (x \in A) \\ f_{12}(x) & (x \in \overline{P}) \end{array}\right.$. Because $\left\{\begin{array}{l} Y_1 \in \text{NES}(\mathcal{X}) \\ Y_2 \in \text{NES}(\mathcal{X}) \end{array}\right.$, the restriction $\left\{\begin{array}{l} g|A_1 \cup \overline{P} \\ g|A_2 \cup \overline{P} \end{array}\right.$ has an extension $\left\{\begin{array}{l} G_1 \in C(O_1, Y_1) \\ G_2 \in C(O_2, Y_2) \end{array}\right.$, where $\left\{\begin{array}{l} O_1 \\ O_2 \end{array}\right.$ is some open subset of $\left\{\begin{array}{l} X_1 \\ X_2 \end{array}\right.$ containing $\left\{\begin{array}{l} A_1 \cup \overline{P} \\ A_2 \cup \overline{P} \end{array}\right.$. Choose an open subset $\left\{\begin{array}{l} P_1 \text{ of } X_1 \\ P_2 \text{ of } X_2 \end{array}\right.$: $\left\{\begin{array}{l} A_1 \cup \overline{P} \subset P_1 \subset \overline{P}_1 \subset O_1 \\ A_2 \cup \overline{P} \subset P_2 \subset \overline{P}_2 \subset O_2 \end{array}\right.$ and an open subset $V \subset X : A \subset V$ & $(X_1 \cap X_2 - P) \cap \overline{V} = \emptyset$. Let $\left\{\begin{array}{l} B_1 = (\overline{P_1 - X_2} \cap \overline{V}) \cup \overline{P} \\ B_2 = (\overline{P_2 - X_1} \cap \overline{V}) \cup \overline{P} \end{array}\right.$. It is clear that $\left\{\begin{array}{l} B_1 \subset O_1 \\ B_2 \subset O_2 \end{array}\right.$, with $B_1 \cap B_2 = \overline{P}$, so the prescription $G(x) = \left\{\begin{array}{l} G_1(x)(x \in B_1) \\ G_2(x)(x \in B_2) \end{array}\right.$ is a continuous extension of f to $B_1 \cup B_2 \supset A$. The set $(P_1 - X_2) \cup (P_2 - X_1) \cup P$ is open in X. Denote by U its intersection with V and let $F = G|U$.

[Note: To reduce NES_4 to NES_3, put instead $\left\{\begin{array}{l} A_1 = f^{-1}(Y_1) \\ A_2 = f^{-1}(Y_2) \end{array}\right.$. Since $\left\{\begin{array}{l} \overline{A_1 - A_2} \cap (A_2 - A_1) = \emptyset \\ (A_1 - A_2) \cap \overline{A_2 - A_1} = \emptyset \end{array}\right.$ and since X is hereditarily normal, there exists an open set $U_0 \subset X : A_1 - A_2 \subset U_0 \subset \overline{U}_0 \subset X - (A_2 - A_1)$. Setting $\left\{\begin{array}{l} X_1 = \overline{U}_0 \cup (A_1 \cap A_2) \\ X_2 = (X - U_0) \cup (A_1 \cap A_2) \end{array}\right.$, the argument then proceeds as before.]

Why work with classes of normal Hausdorff spaces? Answer: If the class $\mathcal{X}$ contains a space that is not normal, then every nonempty Hausdorff $Y \in \text{NES}(\mathcal{X})$ is necessarily a singleton.

FACT Suppose that Y is an AR (ANR).

(1) Let $\mathcal{X}$ be the class of perfectly normal paracompact Hausdorff spaces—then $Y \in \text{ES}(\mathcal{X})$ $(\text{NES}(\mathcal{X}))$.

(2) Let $\mathcal{X}$ be the class of perfectly normal Hausdorff spaces—then $Y \in \text{ES}(\mathcal{X})$ $(\text{NES}(\mathcal{X}))$ iff Y is second countable.

[For the necessity, remark that every collection of nonempty pairwise disjoint open subsets of Y is countable. Reason: The construction on p. 6–35 ff. furnishes a perfectly normal Hausdorff space X containing an uncountable closed discrete subspace A, the points of which cannot be separated by a collection of nonempty pairwise disjoint open subsets of X.]

(3) Let $\mathcal{X}$ be the class of paracompact Hausdorff spaces—then $Y \in \text{ES}(\mathcal{X})$ $(\text{NES}(\mathcal{X}))$ iff Y is completely metrizable.

[To establish the necessity, assume, e.g., that Y is an AR. Let X be the result of retopologizing βY by isolating the points of $\beta Y - Y$. Every open covering of X has a σ-discrete open refinement, hence X is a paracompact Hausdorff space. Since Y sits inside X as a closed subspace, there is a retraction $r : X \to Y$. On the other hand, Y is metrizable, thus is Moore, so Y satisfies Arhangel'skii's condition. Fix a sequence $\{\mathcal{V}_n\}$ of collections of open subsets of βY such that each $\mathcal{V}_n$ covers Y and $\forall\ y \in Y$: $\bigcap\limits_n \text{st}(y, \mathcal{V}_n) \subset Y$. Assign to a given $V \in \mathcal{V}_n$ the open subset $P_V \subset V$ determined by intersecting V with the interior in βY of $r^{-1}(V \cap Y)$. Put $P_n = \bigcup\{P_V : V \in \mathcal{V}_n\} : P_n \supset Y$ & $Y = \bigcap\limits_n P_n$, therefore Y is topologically complete or still, is completely metrizable.]

(4) Let $\mathcal{X}$ be the class of normal Hausdorff spaces—then $Y \in \text{ES}(\mathcal{X})$ ($\text{NES}(\mathcal{X})$) iff Y is second countable and completely metrizable.

FACT Let $\mathcal{X}$ be the class consisting of the Hausdorff spaces that can be realized as a closed subspace of a product of a compact Hausdorff space and a metrizable space (the elements of $\mathcal{X}$ are precisely those paracompact Hausdorff spaces satisfying Arhangel'skii's condition)—then every AR (ANR) is in $\text{ES}(\mathcal{X})$ ($\text{NES}(\mathcal{X})$).

[Suppose that $X \in \mathcal{X}$ is closed in $K \times Z$, where K is compact Hausdorff and Z is metrizable. The projection $K \times Z \to Z$ is closed and has compact fibers, thus the same is true of its restriction p to X. Fix a closed subspace $A \subset X$. Take an AR Y of weight $\leq \kappa$ and let $f \in C(A, Y)$. Embed Y in $\mathbf{S}(\kappa)^\omega$ and apply Proposition 13 to produce a continuous extension $\phi : X \to \mathbf{S}(\kappa)^\omega$ of f. Write Φ for the diagonal of ϕ and p—then $\Phi(A)$ is closed in $\mathbf{S}(\kappa)^\omega \times p(X)$. Therefore the restriction to $\Phi(A)$ of the projection $\psi : \mathbf{S}(\kappa)^\omega \times p(X) \to \mathbf{S}(\kappa)^\omega$ has a continuous extension $\Psi : \mathbf{S}(\kappa)^\omega \times p(X) \to Y$. Put $F = \Psi \circ \Phi : F \in C(X, Y)$ & $F|A = f$.]

Application: If K is a compact Hausdorff space and if Y is an ANR, then $C(K, Y)$ is an ANR (so for any CW complex X, $C(K, X)$ is a CW space).

[Inspect the proof of Proposition 6, keeping in mind the preceding result.]

Suppose that G is a stratifiable topological group—then X_G^∞ and B_G^∞ are stratifiable (cf. p. 6–31) and Cauty[†] has shown that if G is also in NES(stratifiable), then the same holds for X_G^∞ and B_G^∞. Example: If G is an ANR, then X_G^∞ and B_G^∞ are ANRs (cf. p. 4–65).

LEMMA Let Y be a topological space. Suppose that Y admits a covering $\mathcal{V}$ by pairwise disjoint open sets V, each of which is in NES(collectionwise normal)—then Y is in NES(collectionwise normal).

[†] *Arch. Math. (Basel)* **28** (1977), 623–631.

[Let X be collectionwise normal, $A \subset X$ closed, and let $f \in C(A,Y)$. Put $A_V = f^{-1}(V)$, $f_V = f|A_V$—then there exists a neighborhood O_V of A_V in X and an $F_V \in C(O_V, V) : F_V|A_V = f_V$. Since $\{A_V\}$ is a discrete collection of closed subsets of X, there exists a pairwise disjoint collection $\{U_V\}$ of open subsets of X such that $\forall\, V : A_V \subset U_V$. Set $U = \bigcup\limits_V (O_V \cap U_V)$ and define $F : U \rightarrow Y$ by $F|O_V \cap U_V = F_V|O_V \cap U_V$ to get a continuous extension of f to U.]

Let Y be a topological space. Suppose that Y admits a numerable covering $\mathcal{V}$ by open sets V, each of which is in NES(collectionwise normal)—then, from the proof of Proposition 12, it follows that Y is in NES(collectionwise normal).

FACT Let Y be a topological space. Suppose that Y admits a covering $\mathcal{V}$ by open sets V, each of which is in NES(paracompact)—then Y is in NES(paracompact).

Application: Every topological manifold is in NES(paracompact).

[Note: This applies in particular to the Prüfer manifold, which is not metrizable and contains a closed submanifold that is not a neighborhood retract.]

Assume: $I\mathcal{X} \subset \mathcal{X}$. Let $Y \in \text{NES}(\mathcal{X})$—then for every pair (X, A), where $X \in \mathcal{X}$ and $A \subset X$ is closed, A has the HEP w.r.t. Y. Proof: $i_0 X \cup IA$, as a closed subspace of IX, has the EP w.r.t. Y.

EXAMPLE (CW Complexes) If X is stratifiable and $A \subset X$ is closed, then A has the HEP w.r.t. any CW complex.

PROPOSITION 17 Assume: $I\mathcal{X} \subset \mathcal{X}$. Let $Y \in \text{NES}(\mathcal{X})$ and suppose that Y is homotopy equivalent to a $Z \in \text{ES}(\mathcal{X})$—then $Y \in \text{ES}(\mathcal{X})$.

[Choose continuous functions $\phi : Y \rightarrow Z$, $\psi : Z \rightarrow Y$ such that $\psi \circ \phi \simeq \text{id}_Y$, $\phi \circ \psi \simeq \text{id}_Z$. Take an X in $\mathcal{X}$ and let $A \subset X$ be closed. Given $f \in C(A, Y)$, $\exists\, F \in C(X, Z) : F \circ i = \phi \circ f$, where $i : A \rightarrow X$ is the inclusion. But A has the HEP w.r.t. Y and $\psi \circ F \circ i \simeq f$, so f admits a continuous extension to X.]

FACT Suppose that X is an ANR. Let Y be a topological space such that every closed subset $A \subset X$ has the EP w.r.t. Y. Fix a weak homotopy equivalence $K \rightarrow Y$, where K is a CW complex—then every closed subset $A \subset X$ has the EP w.r.t. K.

[Owing to the CW-ANR theorem, the induced map $[X, K] \rightarrow [X, Y]$ is bijective (cf. p. 5–15). On the other hand, every closed subset $A \subset X$ has the HEP w.r.t. K (metrizable $\Rightarrow$ stratifiable).]

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§7. $\mathcal{C}$-THEORY

A classical technique in algebraic topology is to work modulo a Serre class of abelian groups. I shall review these matters here, supplying proofs of the less familiar facts.

Let $\mathcal{C} \subset \operatorname{Ob}\mathbf{AB}$ be a nonempty class of abelian groups—then $\mathcal{C}$ is said to be a Serre class provided that for any short exact sequence $0 \to G' \to G \to G'' \to 0$ in $\mathbf{AB}$, $G \in \mathcal{C}$ iff $\begin{cases} G' \\ G'' \end{cases} \in \mathcal{C}$ or, equivalently, for any exact sequence $G' \to G \to G''$ in $\mathbf{AB}$, $\begin{cases} G' \\ G'' \end{cases} \in \mathcal{C} \Rightarrow G \in \mathcal{C}$.

[Note: To show that a nonempty class $\mathcal{C} \subset \operatorname{Ob}\mathbf{AB}$ is a Serre class, it is usually simplest to check that $\mathcal{C}$ is closed under subgroups, homomorphic images, and extensions.]

Example: For any Serre class $\mathcal{C}$, the subclass $\mathcal{C}_{\text{tor}}$ of torsion groups in $\mathcal{C}$ is a Serre class.

[Note: A Serre class $\mathcal{C}$ is said to be torsion if $\mathcal{C} = \mathcal{C}_{\text{tor}}$.]

EXAMPLE (p-Primary Abelian Groups) An abelian p-group G is said to be p-primary. The rank $r(G)$ of a p-primary G is the cardinality of a maximal independent system in G. If $G[p] = \{g : pg = 0\}$, then $G[p]$ is a vector space over $\mathbf{F}_p$ and $\dim G[p] = r(G)$. The final rank $r_f(G)$ of a p-primary G is the infimum of the $r(p^n G)$ $(n \in \mathbf{N})$. Every p-primary G can be written as $G = G' \oplus G''$, where G' is bounded and $r(G'') = r_f(G'')$ (Fuchs[†]). Fix now a symbol ∞, considered to be larger than all cardinals. Given a Serre class $\mathcal{C}$ of p-primary abelian groups, let $\Phi(\mathcal{C})$ be the smallest cardinal number $> r(G)\ \forall\ G \in \mathcal{C}$ if such a number exists, otherwise put $\Phi(\mathcal{C}) = \infty$, and let $\Psi(\mathcal{C})$ be the smallest cardinal number $> r_f(G)\ \forall\ G \in \mathcal{C}$ if such a number exists, otherwise put $\Psi(\mathcal{C}) = \infty$. Obviously, $\Phi(\mathcal{C}) \geq \Psi(\mathcal{C})$, $\begin{cases} \Phi(\mathcal{C}) = 1 & \text{or} \quad \Phi(\mathcal{C}) \geq \omega \\ \Psi(\mathcal{C}) = 1 & \text{or} \quad \Psi(\mathcal{C}) \geq \omega \end{cases}$. And: $\mathcal{C}$ is precisely the class of p-primary G for which $r(G) < \Phi(\mathcal{C})$ & $r_f(G) < \Psi(\mathcal{C})$. On the other hand, suppose that $\begin{cases} \Phi \\ \Psi \end{cases}$ are cardinal numbers or ∞ with $\Phi \geq \Psi$, $\begin{cases} \Phi = 1 & \text{or} \quad \Phi \geq \omega \\ \Psi = 1 & \text{or} \quad \Psi \geq \omega \end{cases}$. Let $\mathcal{C}$ be the class of p-primary G for which $r(G) < \Phi$ & $r_f(G) < \Psi$—then $\mathcal{C}$ is a Serre class such that $\Phi(\mathcal{C}) = \Phi$ & $\Psi(\mathcal{C}) = \Psi$. Thus the conclusion is that there is a one-to-one correspondence between the conglomerate of Serre classes of p-primary abelian groups and the conglomerate of ordered pairs (Φ, Ψ), where $\begin{cases} \Phi \\ \Psi \end{cases}$ are cardinal numbers or ∞ : $\Phi \geq \Psi$, $\begin{cases} \Phi = 1 & \text{or} \quad \Phi \geq \omega \\ \Psi = 1 & \text{or} \quad \Psi \geq \omega \end{cases}$.

[Note: If $\mathcal{C}$ is a Serre class and if $\mathcal{C}(p)$ is the subclass of $\mathcal{C}$ consisting of the p-primary G in $\mathcal{C}$, then $\mathcal{C}(p)$ is a Serre class.]

[†] *Infinite Abelian Groups*, vol. I, Academic Press (1970), 152.

Notation: Given a Serre class $\mathcal{C}$, tf($\mathcal{C}$) is the subclass of $\mathcal{C}$ made up of the torsion free groups in $\mathcal{C}$.

PROPOSITION 1 Let $\mathcal{C}$ be a Serre class. Assume: tf($\mathcal{C}$) contains a group of infinite rank—then either tf($\mathcal{C}$) is the class of all torsion free abelian groups or tf($\mathcal{C}$) is the class of all torsion free abelian groups of cardinality $< \kappa$, where $\kappa > \omega$.

[Any torsion free abelian group G of infinite rank contains a free abelian group of rank $= \#(G)$.]

EXAMPLE Fix a cardinal number $\kappa > \omega$. Let $\mathcal{T}_\kappa$ be the class of torsion abelian groups of cardinality $< \kappa$; let $\mathcal{F}_\kappa$ be the class of torsion free abelian groups of cardinality $< \kappa$. Take any Serre class $\mathcal{T}$ of torsion abelian groups: $\mathcal{T} \supset \mathcal{T}_\kappa$—then the class $\mathcal{C}$ consisting of all abelian groups G which are extensions of a group in $\mathcal{T}$ by a group in $\mathcal{F}_\kappa$ is a Serre class such that $\mathcal{C}_{\text{tor}} = \mathcal{T}$ and tf($\mathcal{C}$) $= \mathcal{F}_\kappa$.

A characteristic is a sequence $\chi = \{\chi_p : p \in \mathbf{\Pi}\}$, where each χ_p is a nonnegative integer or ∞. Given characteristics $\begin{cases} \chi' \\ \chi'' \end{cases}$, write $\chi' \sim \chi''$ iff $\#\{p : \chi'_p \neq \chi''_p\} < \omega$ and $\chi'_p = \infty \Leftrightarrow \chi''_p = \infty$—then $\sim$ is an equivalence relation on the set of characteristics, an equivalence class $\mathbf{t}$ being called a type. The sum $\mathbf{t}' + \mathbf{t}''$ of types $\begin{cases} \mathbf{t}' \\ \mathbf{t}'' \end{cases}$ is the type containing the characteristic $\{\chi'_p + \chi''_p : p \in \mathbf{\Pi}\}$ and $\mathbf{t}' \leq \mathbf{t}''$ provided that $\chi'_p \leq \chi''_p$ for almost all p, $\mathbf{t}'' - \mathbf{t}'$ being the largest type $\mathbf{t}$ such that $\mathbf{t} + \mathbf{t}' \leq \mathbf{t}''$.

(Rational Groups) A nonzero abelian group G is said to be rational if it is isomorphic to a subgroup of $\mathbf{Q}$ or still, is torsion free of rank 1. Such groups can be classified. For assume that G is rational, say $G \subset \mathbf{Q}$. Take $g \in G : g \neq 0$. Given $p \in \mathbf{\Pi}$, consider the set $S_p(g)$ of nonnegative integers n such that the equation $p^n x = g$ has a solution in G. Put $\chi_p(g) = \sup S_p(g)$, the p-height of g—then $\chi(g) = \{\chi_p(g) : p \in \mathbf{\Pi}\}$ is a characteristic. Moreover, distinct nonzero elements of G determine equivalent characteristics. Definition: The type $\mathbf{t}(G)$ of G is the type of the characteristic of any nonzero element of G. Every type $\mathbf{t}$ can be realized by a rational group, i.e., $\mathbf{t} = \mathbf{t}(G)$ $(\exists\, G)$ and rational $\begin{cases} G' \\ G'' \end{cases}$ are isomorphic iff $\mathbf{t}(G') = \mathbf{t}(G'')$ (in general, G' is isomorphic to a subgroup of G'' iff $\mathbf{t}(G') \leq \mathbf{t}(G'')$).

Example: Suppose that $\mathbf{Z} \subset G \subset \mathbf{Q}$—then $G/\mathbf{Z} \approx \bigoplus_p \mathbf{Z}/p^{\chi_p}\mathbf{Z}$, $\{\chi_p : p \in \mathbf{\Pi}\}$ the characteristic of 1, and $\text{Hom}(G, G)$ is isomorphic to the subring of $\mathbf{Q}$ generated by 1 and the $p^{-1} : pG = G$.

FACT If G and K are rational, then $G \otimes K$ is rational and $\mathbf{t}(G \otimes K) = \mathbf{t}(G) + \mathbf{t}(K)$.

FACT If G and K are rational, then $\operatorname{Hom}(G, K) = 0$ if $\mathbf{t}(G) \not\leq \mathbf{t}(K)$, but is rational if $\mathbf{t}(G) \leq \mathbf{t}(K)$ with $\mathbf{t}(\operatorname{Hom}(G, K)) = \mathbf{t}(K) - \mathbf{t}(G)$.

Notation: $\mathbf{T}$ is a nonempty set of types such that (i) $\mathbf{t}_0 \in \mathbf{T}$ & $\mathbf{t} \leq \mathbf{t}_0 \Rightarrow \mathbf{t} \in \mathbf{T}$ and (ii) $\mathbf{t}', \mathbf{t}'' \in \mathbf{T} \Rightarrow \mathbf{t}' + \mathbf{t}'' \in \mathbf{T}$, $\mathbf{T}(\mathbf{AB})$ being the class of abelian groups G which admit a monomorphism $G \rightarrow \bigoplus\limits_1^n G_i$, where the G_i are rational (n depending on G) and the $\mathbf{t}(G_i) \in \mathbf{T}$.

FACT A torsion free abelian group G of finite rank is in $\mathbf{T}(\mathbf{AB})$ iff for each nonzero homomorphism $\phi : G \rightarrow \mathbf{Q}$, $\mathbf{t}(\phi(G)) \in \mathbf{T}$.

PROPOSITON 2 Let $\mathcal{C}$ be a Serre class. Assume: $\operatorname{tf}(\mathcal{C})$ contains only groups of finite rank and at least one group of positive rank—then $\operatorname{tf}(\mathcal{C}) = \mathbf{T}(\mathbf{AB})$ for some $\mathbf{T}$.

[Let $\mathbf{T}$ be the set of types $\mathbf{t}$ such that a rational group of type $\mathbf{t}$ is in $\operatorname{tf}(\mathcal{C})$. If $G_1, \ldots, G_n$ are rational and if $\mathbf{t}(G_1), \ldots, \mathbf{t}(G_n)$ belong to $\mathbf{T}$, then $\bigoplus\limits_1^n G_i \in \operatorname{tf}(\mathcal{C})$ and every subgroup of $\bigoplus\limits_1^n G_i$ is in $\operatorname{tf}(\mathcal{C})$. On the other hand, for any $G \neq 0$ in $\operatorname{tf}(\mathcal{C})$, there are rational $G_1, \ldots, G_n$ and a monomorphism $G \rightarrow \bigoplus\limits_1^n G_i$. Upon restricting to homomorphic images, one can arrange that the $G_i \in \operatorname{tf}(\mathcal{C})$, so the $\mathbf{t}(G_i) \in \mathbf{T}$. Since $\mathcal{C}$ is closed under subgroups, $\mathbf{T}$ satisfies condition (i) above. As for condition (ii), let $\left\{\begin{matrix} \mathbf{t}' \\ \mathbf{t}'' \end{matrix}\right. \in \mathbf{T}$. Choose $\left\{\begin{matrix} G' \\ G'' \end{matrix}\right. : \mathbf{Z} \subset \left\{\begin{matrix} G' \\ G'' \end{matrix}\right. \subset \mathbf{Q}$ & $\left\{\begin{matrix} \mathbf{t}' = \mathbf{t}(G') \\ \mathbf{t}'' = \mathbf{t}(G'') \end{matrix}\right.$ is represented by the characteristic $\left\{\begin{matrix} \chi' \\ \chi'' \end{matrix}\right.$ corresponding to 1. Suppose first that $\forall\ p$, $\left\{\begin{matrix} \chi_p' \\ \chi_p'' \end{matrix}\right.$ is finite. Let $\mathbf{Z} \subset G \subset \mathbf{Q} : \chi(1) = \chi' + \chi''$. Fix an isomorphism $\phi : G'/\mathbf{Z} \rightarrow G/G''$ and let K be the subgroup of $G' \oplus G$ composed of the $(g', g) : \phi(g' + \mathbf{Z}) = g + G''$—then there is a short exact sequence $0 \rightarrow G'' \rightarrow K \rightarrow G' \rightarrow 0$, hence $K \in \mathcal{C}$. But there is also an epimorphism $K \rightarrow G$, thus $G \in \mathcal{C}$ and $\mathbf{t}' + \mathbf{t}'' \in \mathbf{T}$. Passing to the general case, write $\left\{\begin{matrix} \chi' = \chi_f' + \chi_{0,\infty}' \\ \chi'' = \chi_f'' + \chi_{0,\infty}'' \end{matrix}\right.$, where $\left\{\begin{matrix} \chi_f' \\ \chi_f'' \end{matrix}\right.$ take finite values and $\left\{\begin{matrix} \chi_{0,\infty}' \\ \chi_{0,\infty}'' \end{matrix}\right.$ have values 0 or ∞. Let $\mathbf{Z} \subset G_f \subset \mathbf{Q} : \chi_f(1) = \chi_f' + \chi_f''$; let $\mathbf{Z} \subset \left\{\begin{matrix} G_{0,\infty}' \\ G_{0,\infty}'' \end{matrix}\right. \subset \mathbf{Q}$: $\left\{\begin{matrix} \chi_{0,\infty}'(1) = \chi_{0,\infty}' \\ \chi_{0,\infty}''(1) = \chi_{0,\infty}'' \end{matrix}\right.$. From the foregoing, $G_f \in \mathcal{C}$; in addition, $\left\{\begin{matrix} G_{0,\infty}' \\ G_{0,\infty}'' \end{matrix}\right.$ is isomorphic to a subgroup of $\left\{\begin{matrix} G' \\ G'' \end{matrix}\right. \in \mathcal{C}$. Therefore $G_f \oplus G_{0,\infty}' \oplus G_{0,\infty}'' \in \mathcal{C}$ and $G_f + G_{0,\infty}' + G_{0,\infty}'' \subset \mathbf{Q}$ has type $\mathbf{t}' + \mathbf{t}''$.]

EXAMPLE Given $\mathbf{T}$, let $\mathcal{T}$ be a Serre class of torsion abelian groups with the property that the type determined by a characteristic χ belongs to $\mathbf{T}$ iff $\bigoplus_p \mathbf{Z}/p^{\chi_p}\mathbf{Z} \in \mathcal{T}$—then the class $\mathcal{C}$ consisting of all abelian groups G which are extensions of a group in $\mathcal{T}$ by a group in $\mathbf{T}(\mathbf{AB})$ is a Serre class such that $\mathcal{C}_{\text{tor}} = \mathcal{T}$ and $\text{tf}(\mathcal{C}) = \mathbf{T}(\mathbf{AB})$.

Every torsion abelian group G contains a basic subgroup B, i.e., B is a direct sum of cyclic groups, B is pure in G, and G/B is divisible. If $\begin{cases} G' \\ G'' \end{cases}$ are torsion and if $\begin{cases} B' \subset G' \\ B'' \subset G'' \end{cases}$ are basic, then $G' \otimes G'' \approx B' \otimes B''$. Corollary: The tensor product of two torsion abelian groups is a direct sum of cyclic groups.

LEMMA Let $0 \rightarrow G' \rightarrow G \rightarrow G'' \rightarrow 0$ be a short exact sequence of abelian groups. Suppose that the image of G' in G is pure—then for any K, the sequence $0 \rightarrow G' \otimes K \rightarrow G \otimes K \rightarrow G'' \otimes K \rightarrow 0$ is exact and the image of $G' \otimes K$ in $G \otimes K$ is pure.

[Note: Under the same assumptions, the sequence $0 \rightarrow \text{Tor}(G', K) \rightarrow \text{Tor}(G, K) \rightarrow \text{Tor}(G'', K) \rightarrow 0$ is exact and the image of $\text{Tor}(G', K)$ in $\text{Tor}(G, K)$ is pure.]

A Serre class $\mathcal{C}$ is said to be a ring if $G, K \in \mathcal{C} \Rightarrow G \otimes K \in \mathcal{C}$, $\text{Tor}(G, K) \in \mathcal{C}$.

[Note: $\mathcal{C}$ is a ring provided that $\forall\, G \in \mathcal{C} : G \otimes G \in \mathcal{C}$, $\text{Tor}(G, G) \in \mathcal{C}$. This is because $G, K \in \mathcal{C} \Rightarrow G \otimes K \subset (G \oplus K) \otimes (G \oplus K)$, $\text{Tor}(G, K) \subset \text{Tor}(G \oplus K, G \oplus K)$.]

EXAMPLE Let $\mathcal{C}$ be a ring. Fix a group G—then $G/[G, G] \in \mathcal{C}$ iff $\forall\, i$, $\Gamma^i(G)/\Gamma^{i+1}(G) \in \mathcal{C}$.

[The iterated commutator map $\otimes^{i+1}(G/[G, G]) \rightarrow \Gamma^i(G)/\Gamma^{i+1}(G)$ is surjective.]

EXAMPLE Let $\mathcal{C}$ be a ring. Fix a group G such that $\forall\, n > 0$, $H_n(G) \in \mathcal{C}$. Let $M \in \mathcal{C}$ be a nilpotent G-module—then $\forall\, n \geq 0$, $H_n(G; M) \in \mathcal{C}$.

[Since the $(I[G])^i \cdot M/(I[G])^{i+1} \cdot M \in \mathcal{C}$, it suffices to look at the case when the action of G on M is trivial.]

FACT Let $\mathcal{C}$ be a Serre class. Suppose that $G \in \mathcal{C}$—then for any finitely generated K, $G \otimes K$ and $\text{Tor}(G, K)$ belong to $\mathcal{C}$.

PROPOSITION 3 Let $\mathcal{C}$ be a Serre class—then $\mathcal{C}$ is a ring iff $\mathcal{C}_{\text{tor}}$ is a ring.

[Setting aside the trivial case when $\mathcal{C}$ is the class of all abelian groups, let us assume that $\mathcal{C}_{\text{tor}} \neq \mathcal{C}$ is a ring. Fix $G \in \mathcal{C} - \mathcal{C}_{\text{tor}}$: $\text{Tor}(G, G) \approx \text{Tor}(G_{\text{tor}}, G_{\text{tor}}) \in \mathcal{C}_{\text{tor}}$, G_{tor} the torsion subgroup of G. To deal with $G \otimes G$, put $\text{tf}(G) = G/G_{\text{tor}}$ and consider the exact sequences

$$\begin{cases} 0 \rightarrow G_{\text{tor}} \otimes G \rightarrow G \otimes G \rightarrow \text{tf}(G) \otimes G \rightarrow 0 \\ 0 \rightarrow G_{\text{tor}} \otimes G_{\text{tor}} \rightarrow G_{\text{tor}} \otimes G \rightarrow G_{\text{tor}} \otimes \text{tf}(G) \rightarrow 0 \\ 0 \rightarrow \text{tf}(G) \otimes G_{\text{tor}} \rightarrow \text{tf}(G) \otimes G \rightarrow \text{tf}(G) \otimes \text{tf}(G) \rightarrow 0 \end{cases} .$$

Because $G_{\text{tor}} \otimes G_{\text{tor}} \in \mathcal{C}_{\text{tor}}$, it will be enough to prove that $G_{\text{tor}} \otimes \text{tf}(G)$ and $\text{tf}(G) \otimes \text{tf}(G)$ are in $\mathcal{C}$.

(I) Suppose that $\text{tf}(\mathcal{C})$ contains a group of infinite rank. Choose $\kappa > \omega$ as in Proposition 1 (so $\mathcal{C}$ contains all abelian groups of cardinality $< \kappa$) : $\#(\text{tf}(G)) < \kappa \Rightarrow \#(\text{tf}(G) \otimes \text{tf}(G)) < \kappa \Rightarrow \text{tf}(G) \otimes \text{tf}(G) \in \mathcal{C}$. There is a free group F in $\mathcal{C}$ and an epimorphism $F \rightarrow \text{tf}(G) \rightarrow 0$, where $\text{rank}\, F < \kappa$. Let B be a basic subgroup of G_{tor} and form the exact sequence $0 \rightarrow B \otimes F \rightarrow G_{\text{tor}} \otimes F \rightarrow G_{\text{tor}}/B \otimes F \rightarrow 0$. Using the fact that B is a direct sum of cyclic groups, $B \otimes F \approx B \oplus B_\kappa : \#(B_\kappa) < \kappa \Rightarrow B \otimes F \in \mathcal{C}$. Analogously, by an application of the structure theorem for divisible abelian groups, $G_{\text{tor}}/B \otimes F \in \mathcal{C}$. Conclusion: $G_{\text{tor}} \otimes F \in \mathcal{C} \Rightarrow G_{\text{tor}} \otimes \text{tf}(G) \in \mathcal{C}$.

(II) Suppose that $\text{tf}(\mathcal{C}) = \mathbf{T}(\mathbf{AB})$ (cf. Proposition 2). Let F be the free abelian group generated by a maximal independent system in $\text{tf}(G)$—then there is an exact sequence $0 \rightarrow F \rightarrow \text{tf}(G) \rightarrow \text{tf}(G)/F \rightarrow 0$ and $\text{tf}(G)/F \in \mathcal{C}_{\text{tor}}$. Tensor this sequence with G_{tor} to get another exact sequence $F \otimes G_{\text{tor}} \rightarrow \text{tf}(G) \otimes G_{\text{tor}} \rightarrow \text{tf}(G)/F \otimes G_{\text{tor}}$. Of course, $\text{tf}(G)/F \otimes G_{\text{tor}} \in \mathcal{C}_{\text{tor}}$; moreover, $F \otimes G_{\text{tor}} \in \mathcal{C}$, which implies that $\text{tf}(G) \otimes G_{\text{tor}}$ itself is in $\mathcal{C}$. Finally, the sequence $0 \rightarrow F \otimes \text{tf}(G) \rightarrow \text{tf}(G) \otimes \text{tf}(G) \rightarrow \text{tf}(G)/F \otimes \text{tf}(G) \rightarrow 0$ is exact. Obviously, $F \otimes \text{tf}(G) \in \mathcal{C}$ and, repeating the preceding argument, $\text{tf}(G)/F \otimes \text{tf}(G) \in \mathcal{C}$, hence $\text{tf}(G) \otimes \text{tf}(G) \in \mathcal{C}$.]

In what follows, α and γ are functions having cardinal numbers as values, the domain of α being $\mathbf{\Pi} \times \mathbf{N}$ and the domain of γ being $\mathbf{\Pi}$.

Examples: (1) Let G be a torsion abelian group. Assume: G is a direct sum of cyclic groups—then $G \approx \bigoplus\limits_p \bigoplus\limits_n \alpha(p,n) \cdot (\mathbf{Z}/p^n\mathbf{Z})$; (2) Let G be a torsion abelian group. Assume: G is divisible—then $G \approx \bigoplus\limits_p \gamma(p) \cdot (\mathbf{Z}/p^\infty\mathbf{Z})$; (3) Let G be a torsion abelian group. Assume: G is p-primary and satisfies the descending chain condition on subgroups—then $G \approx \bigoplus\limits_n \alpha(p,n) \cdot (\mathbf{Z}/p^n\mathbf{Z}) \oplus \gamma(p) \cdot (\mathbf{Z}/p^\infty\mathbf{Z})$, where $\sum\limits_n \alpha(p,n) < \omega$ and $\gamma(p)$ is finite.

[Note: For use below, recall that $\mathbf{Z}/p^\infty\mathbf{Z}$ is a homomorphic image of $\bigoplus\limits_n \mathbf{Z}/p^n\mathbf{Z}$ (in fact, every countable p-primary G is a homomorphic image of $\bigoplus\limits_n \mathbf{Z}/p^n\mathbf{Z}$).]

Notation: Given a torsion Serre class $\mathcal{C}$, $\boldsymbol{\alpha}(\mathcal{C}) = \{\alpha : \bigoplus\limits_p \bigoplus\limits_n \alpha(p,n) \cdot (\mathbf{Z}/p^n\mathbf{Z}) \in \mathcal{C}\}$ and $\boldsymbol{\gamma}(\mathcal{C}) = \{\gamma : \bigoplus\limits_p \gamma(p) \cdot (\mathbf{Z}/p^\infty\mathbf{Z}) \in \mathcal{C}\}$.

Observations: (i) $\gamma_0 \in \boldsymbol{\gamma}(\mathcal{C})$ & $\gamma \leq \gamma_0 \Rightarrow \gamma \in \boldsymbol{\gamma}(\mathcal{C})$ and (ii) $\gamma', \gamma'' \in \boldsymbol{\gamma}(\mathcal{C}) \Rightarrow \gamma' + \gamma'' \in \boldsymbol{\gamma}(\mathcal{C})$.

Suppose that $\mathcal{C}$ is a torsion Serre class. Let $G \in \mathcal{C}$—then $G \approx \bigoplus\limits_p G(p)$, $G(p)$ the p-primary component of G. Denote by $\mathcal{C}_0$ the subclass of $\mathcal{C}$ comprised of those G such that each $G(p)$ is bounded, so $\forall\, p$, $\exists\, M(p) : p^{M(p)}G(p) = 0$, and put $\boldsymbol{\alpha}_0(\mathcal{C}) = \boldsymbol{\alpha}(\mathcal{C}_0)$ (meaningful, $\mathcal{C}_0$ being Serre).

CARDINAL LEMMA Let $\mathcal{C}$ be a torsion Serre class—then $\forall\, \alpha \in \boldsymbol{\alpha}(\mathcal{C})$, $\exists\, \alpha_0 \in \boldsymbol{\alpha}_0(\mathcal{C})$ & $\gamma \in \boldsymbol{\gamma}(\mathcal{C})$ such that $\alpha(p,n) \leq \alpha_0(p,n) + \gamma(p)$, where $\gamma(p) \geq \omega$ or $\gamma(p) = 0$.

[Set $\sigma(p,n) = \sum\limits_{m=n}^{\infty} \alpha(p,m)$ and choose $M(p)$ such that $\sigma(p,n) = \sigma(p,n+1) = \cdots$ $(n \geq M(p))$. Define α_0 by $\alpha_0(p,n) = \begin{cases} \alpha(p,n) & (n < M(p)) \\ 0 & (n \geq M(p)) \end{cases}$ and define γ by $\gamma(p) = \sigma(p, M(p)) : \alpha \leq \alpha_0 + \gamma$ and $\alpha_0 \in \boldsymbol{\alpha}_0(\mathcal{C})$, thus the issue is whether $\gamma \in \boldsymbol{\gamma}(\mathcal{C})$. To see this, it need only be shown that $\forall\, p$, $\gamma(p)\cdot(\mathbf{Z}/p^\infty\mathbf{Z})$ is a homomorphic image of $\bigoplus\limits_n \alpha(p,n)\cdot(\mathbf{Z}/p^n\mathbf{Z})$. Case 1: $\gamma(p) = \omega$. Here, $\#\{n : \alpha(p,n) \neq 0\} = \omega$ and there are epimorphisms $\bigoplus\limits_n \alpha(p,n) \cdot (\mathbf{Z}/p^n\mathbf{Z}) \rightarrow \bigoplus\limits_n \mathbf{Z}/p^n\mathbf{Z} \rightarrow \gamma(p)\cdot(\mathbf{Z}/p^\infty\mathbf{Z})$. Case 2: $\gamma(p) > \omega$. Put $N_\infty = \{n : \alpha(p,n) > \omega\}$: $\#(N_\infty) = \omega$ and there are epimorphisms $\bigoplus\limits_n \alpha(p,n)\cdot(\mathbf{Z}/p^n\mathbf{Z}) \rightarrow \bigoplus\limits_{n \in N_\infty} n\alpha(p,n)\cdot(\mathbf{Z}/p^n\mathbf{Z}) \rightarrow \bigoplus\limits_{n \in N_\infty} \alpha(p,n) \cdot (\mathbf{Z}/p\mathbf{Z} \oplus \cdots \oplus \mathbf{Z}/p^n\mathbf{Z}) \rightarrow \gamma(p) \cdot (\bigoplus\limits_n \mathbf{Z}/p^n\mathbf{Z}) \rightarrow \gamma(p) \cdot (\mathbf{Z}/p^\infty\mathbf{Z})$.]

Given a torsion Serre class $\mathcal{C}$, let $\mathcal{C}^*$ be the subclass of those G such that each $G(p)$ satisfies the descending chain condition on subgroups. Note that $\mathcal{C}^*$ is Serre.

PROPOSITION 4 Let $\mathcal{C}$ be a torsion Serre class—then $\mathcal{C}$ is a ring iff $\mathcal{C}^*$ is a ring.

[Straightforward computations establish the necessity. As for the sufficiency, fix $G \in \mathcal{C}$ and let B be a basic subgroup of G. Applying the cardinal lemma, one finds that $B \otimes B \in \mathcal{C}$. But $G \otimes G \approx B \otimes B$, thus $G \otimes G \in \mathcal{C}$. The verification that $\operatorname{Tor}(G,G) \in \mathcal{C}$ hinges on a preliminary remark.

Claim: Suppose that $\mathcal{C}^*$ is a ring—then $\forall\, \gamma \in \boldsymbol{\gamma}(\mathcal{C})$, $\gamma^2 \in \boldsymbol{\gamma}(\mathcal{C})$.

[Write $\gamma = \gamma' + \gamma''$, where $\forall\, p$, $\gamma'(p)$ is finite and $\gamma''(p) \geq \omega$ or $\gamma''(p) = 0$, so $\gamma^2 = (\gamma')^2 + \gamma''$. Since $\mathcal{C}^*$ is a ring, $(\gamma')^2 \in \boldsymbol{\gamma}(\mathcal{C})$, hence $\gamma^2 \in \boldsymbol{\gamma}(\mathcal{C})$.]

Consider the exact sequences

$$\begin{cases} 0 \rightarrow \operatorname{Tor}(B,G) \rightarrow \operatorname{Tor}(G,G) \rightarrow \operatorname{Tor}(G/B,G) \rightarrow 0 \\ 0 \rightarrow \operatorname{Tor}(B,B) \rightarrow \operatorname{Tor}(G,B) \rightarrow \operatorname{Tor}(G/B,B) \rightarrow 0 \\ 0 \rightarrow \operatorname{Tor}(B,G/B) \rightarrow \operatorname{Tor}(G,G/B) \rightarrow \operatorname{Tor}(G/B,G/B) \rightarrow 0 \end{cases}.$$

Owing to the claim, $\operatorname{Tor}(G/B,G/B) \in \mathcal{C}$. Proof: $G/B \approx \bigoplus\limits_p \gamma(p) \cdot (\mathbf{Z}/p^\infty\mathbf{Z}) \Rightarrow \operatorname{Tor}(G/B, G/B) \approx \bigoplus\limits_p \gamma^2(p) \cdot (\mathbf{Z}/p^\infty\mathbf{Z})$. In addition, $\operatorname{Tor}(B,B) \approx B \otimes B \in \mathcal{C}$. Therefore everything comes down to showing that $\operatorname{Tor}(B,G/B) \in \mathcal{C}$ or still, that $\bigoplus\limits_p \gamma(p) \cdot B(p) \in \mathcal{C}$. Using the cardinal lemma, represent B by $B_0 \oplus B_\infty$ with $B_0(p) = \bigoplus\limits_n \alpha_0(p,n) \cdot (\mathbf{Z}/p^n\mathbf{Z})$ and $B_\infty(p) = \bigoplus\limits_n \alpha_\infty(p,n) \cdot (\mathbf{Z}/p^n\mathbf{Z})$, subject to (α_0) $\forall\ p$, $\exists\ M(p) : n \geq M(p) \Rightarrow \alpha_0(p,n) = 0$ and (α_∞) $\exists\ \gamma_\infty \in \boldsymbol{\gamma}(\mathcal{C}) : \forall\ p, \forall\ n, \alpha_\infty(p,n) \leq \gamma_\infty(p)$, where $\gamma_\infty(p) \geq \omega$ or $\gamma_\infty(p) = 0$. From the definitions, $\bigoplus\limits_p \gamma(p) \cdot B_0(p) \approx B_0 \otimes (\bigoplus\limits_p \gamma(p) \cdot (\mathbf{Z}/p^{M(p)}\mathbf{Z})) \in \mathcal{C}$. Turning to B_∞, for each p, there is a monomorphism $\gamma(p) \cdot B_\infty(p) \rightarrow (\gamma(p) + \gamma_\infty(p)) \cdot (\mathbf{Z}/p^\infty\mathbf{Z})$. Because $\gamma + \gamma_\infty \in \boldsymbol{\gamma}(\mathcal{C})$, it follows that $\bigoplus\limits_p \gamma(p) \cdot B_\infty(p) \in \mathcal{C}$.]

Application: Let $\mathcal{C}$ be a Serre class. Assume: $\text{tf}(\mathcal{C})$ contains a free group of infinite rank—then $\mathcal{C}$ is a ring.

EXAMPLE Not every Serre class is a ring. For instance, let $\mathcal{C}$ be the class of all torsion abelian groups G such that $\forall\ p$, $G(p)$ is finite, so $G(p) \approx \bigoplus\limits_n \alpha(p,n) \cdot (\mathbf{Z}/p^n\mathbf{Z})$, where $r(G(p)) = \sum\limits_n \alpha(p,n) < \omega$ (cf. p. 7–1). Enumerate $\mathbf{\Pi} : p_1 < p_2 < \cdots$—then the subclass of $\mathcal{C}$ consisting of those G for which the sequence $\{r(G(p_k))/k\}$ is bounded is a Serre class but it is not a ring (consider $G = \bigoplus\limits_k k \cdot (\mathbf{Z}/p_k\mathbf{Z})$).

[Note: $\mathcal{C}$ is a Serre class and it is a ring.]

A Serre class $\mathcal{C}$ is said to be acyclic if $\forall\ G \in \mathcal{C}$, $H_n(G) \in \mathcal{C}$ $(n > 0)$.

FACT Let $\mathcal{C}$ be a Serre class. Suppose that $G \in \mathcal{C}$ is finitely generated—then $H_n(G) \in \mathcal{C}$ $(n > 0)$.

If G is a torsion abelian group and if $G \approx \bigoplus\limits_p G(p)$ is its primary decomposition, then $\forall\, n > 0$, the $H_n(G)$ are torsion and $\forall\, p$, $H_n(G)(p) \approx H_n(G(p))$ $(\Rightarrow H_n(G) \approx \bigoplus\limits_p H_n(G(p)))$.

[Note: $\forall\ n > 0$, $G(p)$ bounded $\Rightarrow H_n(G(p))$ bounded (in fact, $p^{M(p)}G(p) = 0 \Rightarrow p^{M(p)}H_n(G(p)) = 0$).]

Example: $\mathbf{Q}/\mathbf{Z} \approx \bigoplus\limits_p \mathbf{Z}/p^\infty\mathbf{Z} \Rightarrow H_n(\mathbf{Q}/\mathbf{Z}) \approx \bigoplus\limits_p H_n(\mathbf{Z}/p^\infty\mathbf{Z})$, where for $n > 0$,

$$H_n(\mathbf{Z}/p^\infty\mathbf{Z}) = \operatorname{colim} H_n(\mathbf{Z}/p^k\mathbf{Z}) = \begin{cases} \mathbf{Z}/p^\infty\mathbf{Z} & (n \text{ odd}) \\ 0 & (n \text{ even}) \end{cases}.$$

FACT Fix a prime p. For $k = 1, 2, \ldots$, let G_k be a direct sum of k copies of $\mathbf{Z}/p\mathbf{Z}$—then by the Künneth formula, $\forall\ n > 0$, $H_n(G_k) = G_{d(n,k)}$, where $d(1,k) = k$ and $d(n,k+1) = \sum\limits_{i=1}^{n} d(i,k) + (1-(-1)^n)/2$ (hence $d(n,k) \leq k^n n$).

FACT Fix a prime p. For $k = 1, 2, \ldots$, let G_k be a direct sum of k copies of $\mathbf{Z}/p^\infty\mathbf{Z}$—then by the Künneth formula, $\forall\ n > 0$, $H_n(G_k) = G_{d(n,k)}$, where $d(n,k) = 0$ (n even) and $d(n,k) = \begin{pmatrix} k + \frac{n-1}{2} \\ \frac{n+1}{2} \end{pmatrix}$ (n odd) (hence $d(n,k) \leq k^n$).

LEMMA Suppose that $\mathcal{C}$ is a Serre class. Let $0 \rightarrow K \rightarrow G \rightarrow G/K \rightarrow 0$ be a short exact sequence in $\mathcal{C}$—then for $n > 0$, $H_n(G) \in \mathcal{C}$ provided that the $H_p(G/K; H_q(K)) \in \mathcal{C}$ $(p + q > 0)$.

[Apply the LHS spectral sequence.]

[Note: By the universal coefficient theorem, $H_p(G/K; H_q(K)) \approx H_p(G/K) \otimes H_q(K) \oplus \operatorname{Tor}(H_{p-1}(G/K), H_q(K))$.]

THEOREM OF BALCERZYK Let $\mathcal{C}$ be a Serre class—then $\mathcal{C}$ is acyclic iff $\mathcal{C}$ is a ring.

[Suppose that $\mathcal{C}$ is acyclic. Since G torsion $\Rightarrow H_n(G)$ torsion $(n > 0)$, $\mathcal{C}_{\text{tor}}$ is acyclic, thus one can assume that $\mathcal{C}$ is torsion (cf. Proposition 3) and then, taking into account Proposition 4, work with $\mathcal{C}^*$ (which is acyclic). So let $G \in \mathcal{C}^* : G \approx \bigoplus\limits_p G(p) \Rightarrow G \otimes G \approx \bigoplus\limits_p G(p) \otimes G(p)$ and $\#(G(p) \otimes G(p)) < \omega \Rightarrow G(p) \otimes G(p) \approx H_2(G(p)) \oplus H_2(G(p)) \oplus G(p) \Rightarrow G \otimes G \approx \bigoplus\limits_p (H_2(G(p)) \oplus H_2(G(p) \oplus G(p)) \approx H_2(G) \oplus H_2(G) \oplus G \in \mathcal{C}^*$. To check that $\operatorname{Tor}(G, G) \in \mathcal{C}^*$, it is obviously enough to look at the case when $G \approx \bigoplus\limits_p \gamma(p) \cdot (\mathbf{Z}/p^\infty\mathbf{Z})$, where $\forall\, p$, $\gamma(p) < \omega$. Thus: $H_3(G) \approx \bigoplus\limits_p H_3(\gamma(p) \cdot (\mathbf{Z}/p^\infty\mathbf{Z})) \approx \bigoplus\limits_p \binom{\gamma(p) + 1}{2} \cdot (\mathbf{Z}/p^\infty\mathbf{Z})$ (cf. supra) and $2\binom{\gamma(p) + 1}{2} \geq \gamma(p)^2 \Rightarrow \gamma^2 \in \boldsymbol{\gamma}(\mathcal{C}) \Rightarrow \operatorname{Tor}(G, G) \in \mathcal{C}^*$.

Suppose that $\mathcal{C}$ is a ring. Let $G \in \mathcal{C}$—then there is a short exact sequence $0 \rightarrow G_{\text{tor}} \rightarrow G \rightarrow G/G_{\text{tor}} \rightarrow 0$. Accordingly, in view of the lemma, to prove that $H_n(G) \in \mathcal{C}$ $(n > 0)$, it suffices to prove that $H_p(G/G_{\text{tor}}; H_q(G_{\text{tor}})) \in \mathcal{C}$ $(p + q > 0)$. But $H_p(G/G_{\text{tor}}; H_q(G_{\text{tor}})) \approx H_p(G/G_{\text{tor}}) \otimes H_q(G_{\text{tor}}) \oplus \operatorname{Tor}(H_{p-1}(G/G_{\text{tor}}), H_q(G_{\text{tor}}))$ and the verification that $H_n(G) \in \mathcal{C}$ $(n > 0)$ reduces to when (i) G is torsion free or (ii) G is torsion.

(Torsion Free) If $\operatorname{tf}(\mathcal{C})$ is the class of all torsion free abelian groups of cardinality $< \kappa$ $(\kappa > \omega)$ (cf. Proposition 1), then $G \in \operatorname{tf}(\mathcal{C}) \Rightarrow \#(H_n(G)) < \kappa \Rightarrow H_n(G) \in \mathcal{C}$ $(n > 0)$. The other possibility is that $\operatorname{tf}(\mathcal{C}) = \mathbf{T}(\mathbf{AB})$ for some $\mathbf{T}$ (cf. Proposition 2). Under these circumstances, a given $G \in \operatorname{tf}(\mathcal{C})$ contains a free subgroup $F \approx r \cdot \mathbf{Z}$ of finite rank such that the sequence $0 \rightarrow F \rightarrow G \rightarrow G/F \rightarrow 0$ is exact. Here, $G/F \approx \bigoplus\limits_1^r T_i$ is torsion and the p-primary components of each T_i are isomorphic to $\mathbf{Z}/p^{n_i}\mathbf{Z}$ or $\mathbf{Z}/p^\infty\mathbf{Z}$. Therefore $H_n(T_i) \approx \begin{cases} T_i & (n \text{ odd}) \\ 0 & (n \text{ even}) \end{cases} \in \mathcal{C}$ $(n > 0) \Rightarrow H_n(T) \in \mathcal{C}$ $(n > 0)$ (Künneth). On the other hand, $H_n(F) \approx \begin{cases} \binom{r}{n} \cdot \mathbf{Z} & (n \leq r) \\ 0 & (n > r) \end{cases} \in \mathcal{C}$ $(n > 0)$. The lemma now implies that $H_n(G) \in \mathcal{C}$ $(n > 0)$.

(Torsion) Let $G \in \mathcal{C}_{\text{tor}}$. Choose a basic subgroup B of $G : 0 \to B \to G \to G/B \to 0$—then, thanks to the lemma, one need only consider $H_n(B)$ and $H_n(G/B)$ $(n > 0)$. Using the cardinal lemma, represent B by $B_0 \oplus B_\omega \oplus B_\infty$ with $B_0(p) = \bigoplus_n \alpha_0(p,n) \cdot (\mathbf{Z}/p^n\mathbf{Z})$, $B_\omega(p) = \bigoplus_n \alpha_\omega(p,n) \cdot (\mathbf{Z}/p^n\mathbf{Z})$, and $B_\infty(p) = \bigoplus_n \alpha_\infty(p,n) \cdot (\mathbf{Z}/p^n\mathbf{Z})$, subject to (α_0) $\forall\ p$, $\sum_n \alpha_0(p,n) < \omega$, (α_ω) $\forall\ p$, $\exists\ M(p) : n \geq M(p) \Rightarrow \alpha_\omega(p,n) = 0$ & $\forall\ n : \alpha_\omega(p,n) \geq \omega$ or $\alpha_\omega(p,n) = 0$, and (α_∞) $\exists\ \gamma_\infty \in \boldsymbol{\gamma}(\mathcal{C}) : \forall\ p, \forall\ n,\ \alpha_\infty(p,n) \leq \gamma_\infty(p)$, where $\gamma_\infty(p) \geq \omega$ or $\gamma_\infty(p) = 0$. That $H_n(B) \in \mathcal{C}$ $(n > 0)$ results from the following observations (modulo Künneth): (O_0) $\forall\ p$, $\#(B_0(p)) < \omega$, hence there is a monomorphism $H_n(B_0(p)) \to \otimes^n B_0(p)$; (O_ω) $\forall\ p$, $\forall\ \alpha \geq \omega$, $H_n(\alpha \cdot (\mathbf{Z}/p^k\mathbf{Z})) \approx \alpha \cdot (\mathbf{Z}/p^k\mathbf{Z})$; (O_∞) $\forall\ p$, $\#(B_\infty(p)) \leq \gamma_\infty(p)$, hence there is a monomorphism $H_n(B_\infty(p)) \to \gamma_\infty(p) \cdot (\mathbf{Z}/p^\infty\mathbf{Z})$. Finally, write $G/B \approx \bigoplus_p \gamma(p) \cdot (\mathbf{Z}/p^\infty\mathbf{Z})$ and fix $n > 0$. Case 1: n even $\Rightarrow H_n(G/B) = 0$. Case 2: n odd. If $\gamma(p) \geq \omega$, then $H_n(\gamma(p) \cdot (\mathbf{Z}/p^\infty\mathbf{Z})) \approx \gamma(p) \cdot (\mathbf{Z}/p^\infty\mathbf{Z})$, while if $\gamma(p) < \omega$, then $H_n(\gamma(p) \cdot (\mathbf{Z}/p^\infty\mathbf{Z})) \approx \binom{\gamma(p) + \frac{n-1}{2}}{\frac{n+1}{2}} \cdot (\mathbf{Z}/p^\infty\mathbf{Z})$ (cf. supra). However, $\binom{\gamma(p) + \frac{n-1}{2}}{\frac{n+1}{2}} \leq (\gamma(p))^n$ and $\gamma^n \in \boldsymbol{\gamma}(\mathcal{C})$.]

EXAMPLE Let $\mathcal{C}$ be a ring. Fix a nilpotent group G such that $G/[G,G] \in \mathcal{C}$—then $\forall\ n > 0$, $H_n(G) \in \mathcal{C}$.

FACT Let $\mathcal{C}$ be a ring. Suppose that X is simply connected—then $H_q(X) \in \mathcal{C}\ \forall\ q > 0$ iff $H_q(\Omega X) \in \mathcal{C}\ \forall\ q > 0$.

Application: Let $\mathcal{C}$ be a ring. Fix $\pi \in \mathcal{C}$—then the $H_q(\pi, n) \in \mathcal{C}$ $(q > 0)$.

If $\mathcal{C}$ is a Serre class, then a homomorphism $f : G \to K$ of abelian groups is said to be $\mathcal{C}$-injective ($\mathcal{C}$-surjective) if the kernel (cokernel) of f is in $\mathcal{C}$, f being $\mathcal{C}$-bijective provided that it is both $\mathcal{C}$-injective and $\mathcal{C}$-surjective.

MOD $\mathcal{C}$ HUREWICZ THEOREM Let $\mathcal{C}$ be a Serre class. Assume: $\mathcal{C}$ is a ring. Suppose that X is abelian—then if $n \geq 2$, the condition $\pi_q(X) \in \mathcal{C}$ $(1 \leq q < n)$ is equivalent to the condition $H_q(X) \in \mathcal{C}$ $(1 \leq q < n)$ and either implies that the Hurewicz homomorphism $\pi_n(X) \to H_n(X)$ is $\mathcal{C}$-bijective.

EXAMPLE Let $\mathcal{C}$ be a ring. Suppose that X is a pointed connected CW space which is nilpotent. Agreeing to write $\pi_1(X) \in \mathcal{C}$ if $\pi_1(X)/[\pi_1(X), \pi_1(X)] \in \mathcal{C}$, fix $n \geq 2$—then the following conditions are equivalent: (i) $\pi_q(X) \in \mathcal{C}$ $(1 \leq q < n)$; (ii) $H_q(X) \in \mathcal{C}$ $(1 \leq q < n)$; (iii) $\pi_1(X) \in \mathcal{C}$ & $H_q(\widetilde{X}) \in \mathcal{C}$ $(1 \leq q < n)$. Furthermore, under (i), (ii), or (iii), the Hurewicz homomorphism $\pi_n(X) \to H_n(X)$ induces a $\mathcal{C}$-bijection $\pi_n(X)_{\pi_1(X)} \to H_n(X)$.

[To illustrate the line of argument, assume (iii) and consider the spectral sequence $E^2_{p,q} \approx H_p(\pi_1(X); H_q(\widetilde{X})) \Rightarrow H_{p+q}(X)$ of the covering projection $\widetilde{X} \to X$ (cf. p. 5–62). Since $\pi_1(X) \in \mathcal{C}$ is nilpotent, $E^2_{p,0} \in \mathcal{C}$ $(p > 0)$ (cf. p. 7–9). In addition, the $H_q(\widetilde{X})$ $(q > 0)$ are nilpotent $\pi_1(X)$-modules (cf. §5, Proposition 17), thus $E^2_{p,q} \in \mathcal{C}$ $(p \geq 0, 1 \leq q < n)$ (cf. p. 7–4) $\Rightarrow H_q(X) \in \mathcal{C}$ $(1 \leq q < n)$ and there is a $\mathcal{C}$-bijection $H_n(\widetilde{X})_{\pi_1(X)} \to H_n(X)$. Owing to the mod $\mathcal{C}$ Hurewicz theorem, $\pi_q(X) \approx \pi_q(\widetilde{X}) \in \mathcal{C}$ $(2 \leq q < n)$ and the Hurewicz homomorphism $\pi_n(\widetilde{X}) \to H_n(\widetilde{X})$ is $\mathcal{C}$-bijective. But then the arrow $\pi_n(\widetilde{X})_{\pi_1(X)} \to H_n(\widetilde{X})_{\pi_1(X)}$ is also $\mathcal{C}$-bijective, $\pi_n(\widetilde{X})$ and $H_n(\widetilde{X})$ being nilpotent $\pi_1(X)$-modules.]

A Serre class $\mathcal{C}$ is said to be an ideal if $G \in \mathcal{C} \Rightarrow G \otimes K \in \mathcal{C}$, $\operatorname{Tor}(G, K) \in \mathcal{C}$ for all K in **AB**.

LEMMA Let $\mathcal{C}$ be a Serre class—then $\mathcal{C}$ is an ideal iff $\forall\, G \in \mathcal{C}$, $\bigoplus\limits_i G_i \in \mathcal{C}$, where $\bigoplus\limits_i$ is taken over any index set and $\forall\, i, G_i \approx G$.

Example: Let $\mathcal{C}$ be an ideal. Suppose that $\mathcal{G} \in [(\sin X)^{\mathrm{OP}}, \mathbf{AB}]$ is a coefficient system on X such that $\forall\, \sigma$, $\mathcal{G}\sigma \in \mathcal{C}$—then $\forall\, n \geq 0$, $H_n(X; \mathcal{G}) \in \mathcal{C}$.

EXAMPLE The conglomerate of torsion Serre classes which are ideals is codable by a set. For in the set of subsets of $F(\mathbf{N}, \mathbf{Z}_{\geq 0} \cup \{\infty\})$, write $S \sim T$ iff each sequence in S is $\leq$ a finite sum of sequences in T and each sequence in T is $\leq$ a finite sum of sequences in S. Let E be the resulting set of equivalence classes. Claim: The conglomerate of torsion ideals is in a one-to-one correspondence with E. Thus given a torsion ideal $\mathcal{C}$, assign to $G \in \mathcal{C}$ the sequence $\{x_n(G)\} \in F(\mathbf{N}, \mathbf{Z}_{\geq 0} \cup \{\infty\})$ by letting $x_n(G)$ be the least upper bound of the exponents of the elements in $G(p_n)$, where $\forall\, n$, $p_n < p_{n+1}$. Put $S_{\mathcal{C}} = \{\{x_n(G)\} : G \in \mathcal{C}\}$ and call $[S_{\mathcal{C}}] \in E$ the equivalence class corresponding to $S_{\mathcal{C}}$. To go the other way, take an S and let $\mathcal{C}_S$ be the class of torsion abelian groups G with the property that there exists a finite number of sequences in S such that the nth term of their sum is an upper bound on the exponents of the elements in $G(p_n)$—then $\mathcal{C}_S$ is an ideal and $S \sim T \Rightarrow \mathcal{C}_S = \mathcal{C}_T$, so $\mathcal{C}_{[S]}$ makes sense. One has $\mathcal{C} \to [S_{\mathcal{C}}] \to \mathcal{C}_{[S_{\mathcal{C}}]} = \mathcal{C}$ and $[S] \to \mathcal{C}_{[S]} \to [S_{\mathcal{C}_{[S]}}] = [S]$.

[Note: It is sufficient to consider torsion ideals since any ideal containing a nonzero torsion free group is necessarily the class of all abelian groups.]

MOD $\mathcal{C}$ WHITEHEAD THEOREM Let $\mathcal{C}$ be a Serre class. Assume: $\mathcal{C}$ is an ideal. Suppose that X and Y are abelian and $f : X \to Y$ is a continuous function—then if $n \geq 2$, the condition $f_* : \pi_q(X) \to \pi_q(Y)$ is $\mathcal{C}$-bijective for $1 \leq q < n$ and $\mathcal{C}$-surjective for

$q = n$ is equivalent to the condition $f_* : H_q(X) \to H_q(Y)$ is $\mathcal{C}$-bijective for $1 \leq q < n$ and $\mathcal{C}$-surjective for $q = n$.

EXAMPLE Let $\begin{cases} X \\ Y \end{cases}$ be abelian. Assume: $\forall\, q$, $\begin{cases} H_q(X) \\ H_q(Y) \end{cases}$ is finitely generated ($\Rightarrow$ $\forall\, q$, $\begin{cases} \pi_q(X) \\ \pi_q(Y) \end{cases}$ is finitely generated).

(char $\mathbf{k} = 0$) Let $f : X \to Y$ be a continuous function. Fix a field $\mathbf{k}$ of characteristic 0 and denote by $\mathcal{F}$ the class of finite abelian groups, $\mathcal{T}$ the class of torsion abelian groups—then if $n \geq 2$, the following conditions are equivalent: (1) $f_* : H_q(X) \to H_q(Y)$ is $\mathcal{F}$-bijective for $1 \leq q < n$ and $\mathcal{F}$-surjective for $q = n$; (2) $f_* : H_q(X) \to H_q(Y)$ is $\mathcal{T}$-bijective for $1 \leq q < n$ and $\mathcal{T}$-surjective for $q = n$; (3) $f_* : H_q(X;\mathbf{k}) \to H_q(Y;\mathbf{k})$ is bijective for $1 \leq q < n$ and surjective for $q = n$; (4) $f^* : H^q(Y;\mathbf{k}) \to H^q(X;\mathbf{k})$ is bijective for $1 \leq q < n$ and injective for $q = n$.

(char $\mathbf{k} = p$) Let $f : X \to Y$ be a continuous function. Fix a field $\mathbf{k}$ of characteristic p and denote by $\mathcal{F}_p$ the class of finite abelian groups with order prime to p, $\mathcal{T}_p$ the class of torsion abelian groups with trivial p-primary component—then if $n \geq 2$, the following conditions are equivalent: (1) $f_* : H_q(X) \to H_q(Y)$ is $\mathcal{F}_p$-bijective for $1 \leq q < n$ and $\mathcal{F}_p$-surjective for $q = n$; (2) $f_* : H_q(X) \to H_q(Y)$ is $\mathcal{T}_p$-bijective for $1 \leq q < n$ and $\mathcal{T}_p$-surjective for $q = n$; (3) $f_* : H_q(X;\mathbf{k}) \to H_q(Y;\mathbf{k})$ is bijective for $1 \leq q < n$ and surjective for $q = n$; (4) $f^* : H^q(Y;\mathbf{k}) \to H^q(X;\mathbf{k})$ is bijective for $1 \leq q < n$ and injective for $q = n$.

Example: If $\forall\, n$, f_* induces an isomorphism $H_n(X;\mathbf{F}_p) \to H_n(Y;\mathbf{F}_p)$, then $\forall\, n$, f_* induces an isomorphism $\pi_n(X)(p) \to \pi_n(Y)(p)$ of p-primary components.

FACT Let X be a CW complex. Assume: X is finite and n-connected—then the Hurewicz homomorphism $\pi_q(X) \to H_q(X)$ is $\mathcal{C}$-bijective for $q < 2n + 1$, where $\mathcal{C}$ is the class of finite abelian groups.

§ 7

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§8. LOCALIZATION OF GROUPS

The algebra of this section is the point of departure for the developments in the next §. While the primary focus is on the "abelian-nilpotent" theory, part of the material is presented in a more general setting. I have also included some topological applications that will be of use in the sequel.

The Serre classes in **AB** that are closed under the formation of coproducts (and hence colimits) are in a one-to-one correspondence with the Giraud subcategories of **AB**. Under this correspondence, the class of all abelian groups corresponds to the class of trivial groups. The remaining classes are necessarily torsion ideals and their determination is embedded in abelian localization theory.

[Note: Not every torsion ideal is closed under the formation of coproducts (consider, e.g., the class of bounded abelian groups).]

Notation: P is a set of primes, $\overline{P}$ its complement in the set of all primes.

Given P, put $S_P = \{1\} \cup \{n > 1 : p \in P \Rightarrow p \nmid n\}$—then $\mathbf{Z}_P = S_P^{-1}\mathbf{Z}$ is the localization of $\mathbf{Z}$ at P and the inclusion $\mathbf{Z} \rightarrow \mathbf{Z}_P$ is an epimorphism in **RG**. $\mathbf{Z}_P$ is a principal ideal domain. Moreover, $\mathbf{Z}_P$ is a subring of $\mathbf{Q}$ and every subring of $\mathbf{Q}$ is a $\mathbf{Z}_P$ for a suitable P. The characteristic of 1 in $\mathbf{Z}_P$ is $\begin{cases} 0 & (p \in P) \\ \infty & (p \in \overline{P}) \end{cases} \Rightarrow \mathbf{Z}_P/\mathbf{Z} \approx \bigoplus\limits_{p \in \overline{P}} \mathbf{Z}/p^\infty\mathbf{Z}$. Examples: (1) Take $P = \emptyset : \mathbf{Z}_P = \mathbf{Q}$; (2) Take $P = \mathbf{\Pi} : \mathbf{Z}_P = \mathbf{Z}$; (3) Take $P = \mathbf{\Pi} - \{p\} : \mathbf{Z}_P = \mathbf{Z}\left[\frac{1}{p}\right]$; (4) Take $P = \mathbf{\Pi} - \{2, 5\} : \mathbf{Z}_P =$ all rationals whose decimal expansion is finite.

[Note: Write $\mathbf{Z}_p$ in place of $\mathbf{Z}_{\{p\}} : \mathbf{Z}_p$ is a local ring and its residue field is isomorphic to $\mathbf{F}_p$.]

EXAMPLE Suppose that $P \neq \emptyset$—then as vector spaces over $\mathbf{Q}$, $\operatorname{Ext}(\mathbf{Q}, \mathbf{Z}_P) \approx \mathbf{R}$.

Equip S_P with the structure of a directed set by stipulating that $n' \leq n''$ iff $n' | n''$. View $(S_P, \leq)$ as a filtered category $\mathbf{S}_P$ and let $\Delta_P : \mathbf{S}_P \rightarrow \mathbf{AB}$ be the diagram that sends an object n to $\mathbf{Z}$ and a morphism $n' \rightarrow n''$ to the multiplication $\frac{n''}{n'} : \mathbf{Z} \rightarrow \mathbf{Z}$—then the homomorphism $\operatorname{colim} \Delta_P \rightarrow \mathbf{Z}_P$ is an isomorphism. Example: $\mathbf{Z}_P \otimes \mathbf{Z}/p^n\mathbf{Z} = \begin{cases} 0 & (p \in \overline{P}) \\ \mathbf{Z}/p^n\mathbf{Z} & (p \in P) \end{cases}$.

EXAMPLE Fix $P \neq \mathbf{\Pi}$—then there is a short exact sequence $0 \rightarrow \lim^1 H^1(\mathbf{Z}; \mathbf{Q}[\mathbf{Z}_P]) \rightarrow H^2(\mathbf{Z}_P; \mathbf{Q}[\mathbf{Z}_P]) \rightarrow \lim H^2(\mathbf{Z}; \mathbf{Q}[\mathbf{Z}_P]) \rightarrow 0$. Here, $H^2(\mathbf{Z}_P; \mathbf{Q}[\mathbf{Z}_P]) \neq 0$ (in fact, is uncountable (cf. p. 5–47)).

LEMMA Let P' and P'' be two sets of primes—then (i) $\mathbf{Z}_{P'} + \mathbf{Z}_{P''} = \mathbf{Z}_{P' \cap P''}$ and (ii) $\mathbf{Z}_{P'} \cap \mathbf{Z}_{P''} = \mathbf{Z}_{P' \cup P''}$, the sum and intersection being as subgroups of $\mathbf{Q}$. Furthermore, the biadditive function $\begin{cases} \mathbf{Z}_{P'} \times \mathbf{Z}_{P''} \to \mathbf{Z}_{P' \cap P''} \\ (z', z'') \to z'z'' \end{cases}$ defines an isomorphism of rings: $\mathbf{Z}_{P'} \otimes \mathbf{Z}_{P''} \approx \mathbf{Z}_{P' \cap P''}$ ($\Rightarrow \mathbf{Z}_P \otimes \mathbf{Z}_P \approx \mathbf{Z}_P$).

FACT There is a commutative diagram

$$\begin{tikzcd}
\mathbf{Z}_{P' \cup P''} \arrow[r, "i''"] \arrow[d, "i'"'] & \mathbf{Z}_{P''} \arrow[d, "j''"] \\
\mathbf{Z}_{P'} \arrow[r, "j'"'] & \mathbf{Z}_{P' \cap P''}
\end{tikzcd}$$

and a short exact sequence

$$0 \to \mathbf{Z}_{P' \cup P''} \xrightarrow{\mu} \mathbf{Z}_{P'} \oplus \mathbf{Z}_{P''} \xrightarrow{\nu} \mathbf{Z}_{P' \cap P''} \to 0 \quad (\mu(z) = (i'(z), i''(z)) \ \& \ \nu(z', z'') = j'(z') - j''(z'')),$$

thus the square is simultaneously a pullback and a pushout in **AB**.

An abelian group G is said to be S_P-torsion if $\forall\, g \in G$, $\exists\, n \in S_P : ng = 0$. Denote by $\mathcal{C}_P$ the class of S_P-torsion abelian groups—then $\mathcal{C}_P$ is a Serre class which is closed under the formation of coproducts and every torsion Serre class with this property is a $\mathcal{C}_P$ for some P. Examples: (1) Take $P = \emptyset : \mathcal{C}_P$ is the class of torsion abelian groups; (2) Take $P = \mathbf{\Pi} : \mathcal{C}_P$ is the class of trivial groups; (3) Take $P = \{p\} : \mathcal{C}_P$ is the class of torsion abelian groups with trivial p-primary component; (4) Take $P = \mathbf{\Pi} - \{p\} : \mathcal{C}_P$ is the class of abelian p-groups.

[Note: G is S_P-torsion iff G is $\overline{P}$-primary or still, iff $\mathbf{Z}_P \otimes G = 0$.]

Let $f : G \to K$ be a homomorphism of abelian groups—then f is said to be P-injective (P-surjective) if the kernel (cokernel) of f is S_P-torsion, f being P-bijective provided that it is both P-injective and P-surjective.

[Note: This is the terminology on p. 7–9, specialized to the case $\mathcal{C} = \mathcal{C}_P$.]

FIVE LEMMA Let

$$\begin{tikzcd}
G_1 \arrow[r] \arrow[d, "f_1"] & G_2 \arrow[r] \arrow[d, "f_2"] & G_3 \arrow[r] \arrow[d, "f_3"] & G_4 \arrow[r] \arrow[d, "f_4"] & G_5 \arrow[d, "f_5"] \\
K_1 \arrow[r] & K_2 \arrow[r] & K_3 \arrow[r] & K_4 \arrow[r] & K_5
\end{tikzcd}$$

be a commutative diagram of abelian groups with exact rows.

(1) If f_2 and f_4 are P-surjective and f_5 is P-injective, then f_3 is P-surjective.

(2) If f_2 and f_4 are P-injective and f_1 is P-surjective, then f_3 is P-injective.

The definition of "S_P-torsion" carries over without change to **GR**, as does the definition of "P-injective" but it is best to modify the definition of "P-surjective". Thus let $f : G \to K$ be a homomorphism of groups—then f is said to be P-surjective if $\forall\, k \in K$, $\exists\, n \in S_P$: $k^n \in \operatorname{im} f$ (when G and K are nilpotent, this is equivalent to requiring that $\operatorname{coker} f$ be S_P-torsion). Assigning to the term "P-bijective" the obvious interpretation, the five lemma retains its validity under the following additional assumptions: $(1)_+$ $\operatorname{im}(K_2 \to K_3) \subset \operatorname{Cen} K_3$ or $(2)_+$ $\operatorname{im}(G_1 \to G_2) \subset \operatorname{Cen} G_2$ (no extra conditions are needed in the nilpotent case).

Given an abelian group G, the localization of G at P is the tensor product $G_P = \mathbf{Z}_P \otimes G$. The functor $\mathbf{Z}_P \otimes \text{—} : \mathbf{AB} \to \mathbf{Z}_P\text{-}\mathbf{MOD}$ preserves colimits and is exact. Examples: (1) Suppose that G is finitely generated, say $G \approx \bigoplus\limits_1^r \mathbf{Z} \oplus \bigoplus\limits_p \bigoplus\limits_n \alpha(p,n) \cdot (\mathbf{Z}/p^n\mathbf{Z})$—then $G_P \approx \bigoplus\limits_1^r \mathbf{Z}_P \oplus \bigoplus\limits_{p \in P} \bigoplus\limits_n \alpha(p,n) \cdot (\mathbf{Z}/p^n\mathbf{Z})$; (2) Suppose that G is torsion, say $G \approx \bigoplus\limits_p G(p)$—then $G_P \approx \bigoplus\limits_{p \in P} G(p)$.

[Note: $G_{\mathbf{Q}} = \mathbf{Q} \otimes G$ is the rationalization of G. Example: $\mathbf{Q} \otimes \mathbf{Z}^\omega \neq \mathbf{Q}^\omega$. $G_p = \mathbf{Z}_p \otimes G$ is the p-localization of G. Example: $(\mathbf{Q}/\mathbf{Z})_p = \mathbf{Z}/p^\infty\mathbf{Z}$.]

FACT Let G be an abelian group—then the commutative diagram

$$\begin{tikzcd} G \arrow[r] \arrow[d] & G_{\overline{P}} \arrow[d] \\ G_P \arrow[r] & G_{\mathbf{Q}} \end{tikzcd}$$

is simultaneously a pullback square and a pushout square in $\mathbf{AB}$ and the arrow $\begin{cases} G_P \to G_{\mathbf{Q}} \\ G_{\overline{P}} \to G_{\mathbf{Q}} \end{cases}$ is a $\begin{cases} \overline{P}\text{-bijection} \\ P\text{-bijection} \end{cases}$.

FACT Let G be an abelian group—then G is finitely generated iff $\begin{cases} G_P \\ G_{\overline{P}} \end{cases}$ are finitely generated $\begin{cases} \mathbf{Z}_P \\ \mathbf{Z}_{\overline{P}} \end{cases}$-modules.

[Note: $(\bigoplus\limits_p \mathbf{Z}/p\mathbf{Z})_q$ is a finitely generated $\mathbf{Z}_q$-module for every prime q but $\bigoplus\limits_p \mathbf{Z}/p\mathbf{Z}$ is not a finitely generated abelian group.]

FACT Let G be an abelian group—then $G = 0$ iff $\forall\, p$, $G_p = 0$.

FACT Let $\begin{cases} G \\ K \end{cases}$ be finitely generated abelian groups. Assume: $\forall\, p, G_p \approx K_p$—then $G \approx K$.

[Note: To see the failure of this conclusion when one of G and K is not finitely generated, take $G = \mathbf{Z}$ and let K be the additive subgroup of $\mathbf{Q}$ consisting of those rationals of the form m/n, where n is square free—then $\forall\, p, G_p \approx K_p$, yet $G \not\approx K$. Replacing "square free" by "k^{th}-power free", it follows that there exist infinitely many mutually nonisomorphic abelian groups whose p-localization is isomorphic to $\mathbf{Z}_p$ at every prime p.]

FACT Let $f : G \to K$ be a homomorphism of abelian groups—then f is injective (surjective) iff $\forall\, p$, $f_p : G_p \to K_p$ is injective (surjective).

[Localization is an exact functor, hence preserves kernels and cokernels.]

FACT Let $f, g : G \rightarrow K$ be homomorphisms of abelian groups. Assume: $\forall\, p$, $f_p = g_p$—then $f = g$.

[The vertical arrows in the commutative diagram

$$\begin{tikzcd}
G \arrow[r, "f", "g"'] \arrow[d] & K \arrow[d] \\
\prod G_p \arrow[r, "\Pi f_p", "\Pi g_p"'] & \prod K_p
\end{tikzcd}$$

are one-to-one.]

LEMMA Let G_{tor} be the torsion subgroup of G—then $(G_{\text{tor}})_P$ is the torsion subgroup of G_P.

EXAMPLE Take $G = \prod\limits_p \mathbf{Z}/p\mathbf{Z}$—then $G_{\text{tor}} \approx \bigoplus\limits_p \mathbf{Z}/p\mathbf{Z} \Rightarrow (G_p)_{\text{tor}} \approx \mathbf{Z}/p\mathbf{Z}$, so $\forall\, p$, $(G_p)_{\text{tor}}$ is a direct summand of G_p, yet G_{tor} is not a direct summand of G.

Let G be an abelian group—then one may attach to G a sink $\{r_p : G_p \rightarrow G_{\mathbf{Q}}\}$ and a source $\{l_p : G \rightarrow G_p\}$, where $\forall \begin{cases} p \\ q \end{cases}$, $r_p \circ l_p = r_q \circ l_q$.

FRACTURE LEMMA Suppose that G is a finitely generated abelian group—then the source $\{l_p : G \rightarrow G_p\}$ is the multiple pullback of the sink $\{r_p : G_p \rightarrow G_{\mathbf{Q}}\}$.

[It suffices to look at two cases: (i) $G = \mathbf{Z}/p^n\mathbf{Z}$ and (ii) $G = \mathbf{Z}$.]

EXAMPLE Take $G = \bigoplus\limits_p \mathbf{Z}/p\mathbf{Z}$—then $G_p = \mathbf{Z}/p\mathbf{Z}$ and $G_{\mathbf{Q}} = 0$, the final object in **AB**. Accordingly, the multiple pullback of the sink $\{\mathbf{Z}/p\mathbf{Z} \rightarrow 0\}$ is the source $\{\prod\limits_p \mathbf{Z}/p\mathbf{Z} \rightarrow \mathbf{Z}/p\mathbf{Z}\}$.

An abelian group G is said to be P-local if the map $\begin{cases} G \rightarrow G \\ g \rightarrow ng \end{cases}$ is bijective $\forall\, n \in S_P$. $\mathbf{AB}_P$ is the full subcategory of **AB** whose objects are the P-local abelian groups. $\mathbf{AB}_P$ is a Giraud subcategory of **AB** with exact reflector $L_P : \begin{cases} \mathbf{AB} \rightarrow \mathbf{AB}_P \\ G \rightarrow G_P \end{cases}$ and arrow of localization $l_P : G \rightarrow G_P$. Therefore G is P-local iff l_P is an isomorphism. In general, the kernel and cokernel of $l_P : G \rightarrow G_P$ are S_P-torsion, i.e., l_P is P-bijective.

[Note: The objects of $\mathbf{AB}_P$ are the uniquely $\overline{P}$-divisible abelian groups. Changing the notation momentarily, let $S_P \subset \operatorname{Mor}\mathbf{AB}$ be the class consisting of those s such that $\ker s \in \mathcal{C}_P$ and $\operatorname{coker} s \in \mathcal{C}_P$—then the localization $S_P^{-1}\mathbf{AB}$ is equivalent to $\mathbf{AB}_P$ and the endomorphism ring of $\mathbf{Z}$, considered as an object in $S_P^{-1}\mathbf{AB}$, is isomorphic to $\mathbf{Z}_P$. Moreover, a homomorphism $f : G \rightarrow K$ of abelian groups is P-bijective iff $f_P : G_P \rightarrow K_P$ is bijective.]

RECOGNITION PRINCIPLE Let G be an abelian group—then G is P-local iff it carries the structure of a $\mathbf{Z}_P$-module or satisfies one of the following equivalent conditions.

(REC$_1$) $\mathbf{Z}_P/\mathbf{Z}\otimes G=0$ & $\text{Tor}(\mathbf{Z}_P/\mathbf{Z},G)=0$.
(REC$_2$) $\forall\ n\in S_P, \mathbf{Z}/n\mathbf{Z}\otimes G=0$ & $\text{Tor}(\mathbf{Z}/n\mathbf{Z},G)=0$.
(REC$_3$) $\text{Hom}(\mathbf{Z}_P/\mathbf{Z},G)=0$ & $\text{Ext}(\mathbf{Z}_P/\mathbf{Z},G)=0$.
(REC$_4$) $\forall\ n\in S_P$, $\text{Hom}(\mathbf{Z}/n\mathbf{Z},G)=0$ & $\text{Ext}(\mathbf{Z}/n\mathbf{Z},G)=0$.
[Note: In REC$_2$ or REC$_4$, one can just as well work with the $p\in\overline{P}$.]

FACT Let G be an abelian group. Suppose that G is isomorphic to a subgroup of a P-local abelian group and a quotient group of a P-local abelian group—then G is P-local.

FACT Let $0\rightarrow G'\rightarrow G\rightarrow G''\rightarrow 0$ be a short exact sequence of abelian groups. Assume: Two of the groups are P-local—then so is the third.

[Note: $\mathbf{AB}_P$ is closed with respect to the formation of five term exact sequences but this need not be true of three term exact sequences unless P is the set of all primes, this being the only case when $\mathbf{AB}_P$ is a Serre class.]

EXAMPLE The homology groups attached to a chain complex of P-local abelian groups are P-local.

EXAMPLE Let $f: X\rightarrow Y$ be either a Dold fibration or a Serre fibration. Assume: $\begin{cases} X \\ Y \end{cases}$ and the X_y are path connected and $\begin{cases} \pi_1(X) \\ \pi_1(Y) \end{cases}$ and the $\pi_1(X_y)$ are abelian. Fix $y_0\in Y$ & $x_0\in X_{y_0}$—then there is an exact sequence $\cdots\rightarrow\pi_{n+1}(Y,y_0)\rightarrow\pi_n(X_{y_0},x_0)\rightarrow\pi_n(X,x_0)\rightarrow\pi_n(Y,y_0)\rightarrow\cdots$ and if any two of $\{\pi_n(X_{y_0},x_0)\}$, $\{\pi_n(X,x_0)\}$, $\{\pi_n(Y,y_0)\}$ are P-local, so is the third.

LEMMA $L_P:\mathbf{AB}\rightarrow\mathbf{AB}_P$ preserves finite limits.

EXAMPLE L_P need not preserve arbitrary limits. For instance, take $P=\mathbf{\Pi}-\{2\}$ and define $\mathbf{G}$ in $\mathbf{TOW(AB)}$ by $G_n=\mathbf{Z}\ \forall\ n$ and $\begin{cases} G_{n+1}\rightarrow G_n \\ 1\rightarrow 2 \end{cases}$ —then $\lim\mathbf{G}=0$ but $\lim\mathbf{G}_P=\mathbf{Z}\left[\frac{1}{2}\right]$.

Let $f: G\rightarrow K$ be a homomorphism of abelian groups—then f is said to be P-localizing if $\exists$ an isomorphism $\phi: G_P\rightarrow K$ such that $f=\phi\circ l_P$ (cf. p. 0–30).

LEMMA Let $f: G\rightarrow K$ be a homomorphism of abelian groups—then f is P-localizing iff f is P-bijective and K is P-local.

Example: Let $\begin{cases} X \\ Y \end{cases}$ be path connected topological spaces, $f: X\rightarrow Y$ a continuous function—then by the universal coefficient theorem, $f_*: H_n(X)\rightarrow H_n(Y)$ is P-localizing

$\forall\, n \geq 1$ iff $f_* : H_n(X; \mathbf{Z}_P) \to H_n(Y; \mathbf{Z}_P)$ is an isomorphism $\forall\, n \geq 1$ and $H_n(Y)$ is P-local $\forall\, n \geq 1$.

Example: Let X be a path connected topological space—then by the universal coefficient theorem, $H_n(X)$ is P-local $\forall\, n \geq 1$ iff $\forall\, p \in \overline{P}, H_n(X; \mathbf{Z}/p\mathbf{Z}) = 0 \;\forall\, n \geq 1$.

FACT Let

$$\begin{tikzcd}
G_1 \arrow[r] \arrow[d, "f_1"] & G_2 \arrow[r] \arrow[d, "f_2"] & G_3 \arrow[r] \arrow[d, "f_3"] & G_4 \arrow[r] \arrow[d, "f_4"] & G_5 \arrow[d, "f_5"] \\
K_1 \arrow[r] & K_2 \arrow[r] & K_3 \arrow[r] & K_4 \arrow[r] & K_5
\end{tikzcd}$$

be a commutative diagram of abelian groups with exact rows. Assume: f_1, f_2, f_4, f_5 are P-localizing—then f_3 is P-localizing.

EXAMPLE Let $\begin{cases} G \\ K \end{cases}$ be abelian groups—then $(G \otimes K)_P \approx G_P \otimes K \approx G \otimes K_P \approx G_P \otimes K_P$ and $\mathrm{Tor}(G, K)_P \approx \mathrm{Tor}(G_P, K) \approx \mathrm{Tor}(G, K_P) \approx \mathrm{Tor}(G_P, K_P)$.

EXAMPLE Let $\begin{cases} G \\ K \end{cases}$ be abelian groups.

(R) Assume: G is finitely generated—then $\mathrm{Hom}(G, K)_P \approx \mathrm{Hom}(G, K_P)$ and $\mathrm{Ext}(G, K)_P \approx \mathrm{Ext}(G, K_P)$.

(L) Assume: K is P-local—then $\mathrm{Hom}(G_P, K) \approx \mathrm{Hom}(G, K)$ and $\mathrm{Ext}(G_P, K) \approx \mathrm{Ext}(G, K)$.

[An injective $\mathbf{Z}_P$-module is also injective as an abelian group.]

FACT Let G be an abelian group—then $\forall\, n \geq 1$, $H_n(l_P) : H_n(G) \to H_n(G_P)$ is P-localizing. In particular: G P-local $\Rightarrow H_n(G)$ P-local ($\forall\, n \geq 1$) and conversely.

[There are three steps: (1) $G = \mathbf{Z}/p^n\mathbf{Z}$ or $G = \mathbf{Z}$ (direct verification); (2) G finitely generated (Künneth); (3) G arbitrary (take colimits).]

[Note: It is a corollary that for any abelian group G, $H_n(G; \mathbf{Z}_P) \approx H_n(G_P; \mathbf{Z}_P)$ ($n \geq 1$). This is also true if G is nilpotent (cf. Proposition 8) but is false in general. Example: Take $G = S_3$, $P = \{3\}$—then $H_3(G; \mathbf{Z}_P) \neq 0$ & $H_3(G_P; \mathbf{Z}_P) = 0$.]

PROPOSITION 1 Let $f : X \to Y$ be either a Dold fibration or a Serre fibration such that $\forall\, p \in \overline{P}$, f is $\mathbf{Z}/p\mathbf{Z}$-orientable—then any two of the following conditions imply the third: (1) $\forall\, k \geq 1$, $H_k(Y)$ is P-local; (2) $\forall\, l \geq 1$, $H_l(X_{y_0})$ is P-local; (3) $\forall\, n \geq 1$, $H_n(X)$ is P-local.

[In the notation of p. 4–44, take $\Lambda = \mathbf{Z}/p\mathbf{Z}$. By what has been said there, $\widetilde{H}_*(\text{—}; \Lambda) = 0$ for any two of Y, X_{y_0}, and X entails $\widetilde{H}_*(\text{—}; \Lambda) = 0$ for the third.]

Application: Let π be a P-local abelian group—then $\forall\, q \geq 1$, $H_q(\pi, n)$ is P-local.

[As recorded above, this is true when $n = 1$. To treat the general case, proceed by induction, bearing in mind that the mapping fiber of the projection $\Theta K(\pi, n+1) \to K(\pi, n+1)$ is a $K(\pi, n)$.]

[Note: If π is any abelian group, then the arrow of localization $l_P : \pi \to \pi_P$ induces a map $l_P : K(\pi, n) \to K(\pi_P, n)$ and $\forall\ q \geq 1$, $H_q(l_P) : H_q(\pi, n) \to H_q(\pi_P, n)$ is P-localizing. In fact, $H_q(l_P)$ is P-bijective (mod $\mathcal{C}_P$ Whitehead theorem) and $H_q(\pi_P, n)$ is P-local.]

FACT Let X be a pointed connected CW space. Assume: X is simply connected—then $\forall\ n \geq 1$, $\pi_n(X)$ is P-local iff $\forall\ n \geq 1$, $H_n(X)$ is P-local.

[Pass from homotopy to homology via the Postnikov tower of X and pass from homology to homotopy via the Whitehead tower of X.]

FACT Let $\begin{cases} X \\ Y \end{cases}$ be pointed connected CW spaces, $f : X \to Y$ a pointed continuous function. Assume: X & Y are simply connected—then $\forall\ n \geq 1$, $f_* : \pi_n(X) \to \pi_n(Y)$ is P-localizing iff $\forall\ n \geq 1$, $f_* : H_n(X) \to H_n(Y)$ is P-localizing.

[Taking into account the preceding fact, this follows from the mod $\mathcal{C}_P$ Whitehead theorem.]

If G and K are P-local abelian groups, then $\mathrm{Hom}(G, K)$, $\mathrm{Ext}(G, K)$, $G \otimes K$, $\mathrm{Tor}(G, K)$ are P-local and $\mathbf{Z}_P$-isomorphic to their $\mathbf{Z}_P$-counterparts, hence can be identified with them.

LEMMA Suppose that $P \neq \emptyset$ and let G be P-local. Assume: $\mathrm{Hom}(G, \mathbf{Z}_P) = 0$ & $\mathrm{Ext}(G, \mathbf{Z}_P) = 0$—then $G = 0$.

[To begin with, $\mathrm{Hom}(\mathrm{Tor}(\mathbf{Q}, G), \mathbf{Z}_P) \oplus \mathrm{Ext}(\mathbf{Q} \otimes G, \mathbf{Z}_P) \approx \mathrm{Hom}(\mathbf{Q}, \mathrm{Ext}(G, \mathbf{Z}_P)) \oplus \mathrm{Ext}(\mathbf{Q}, \mathrm{Hom}(G, \mathbf{Z}_P)) \Rightarrow \mathrm{Ext}(\mathbf{Q} \otimes G, \mathbf{Z}_P) = 0$. On the other hand, the condition $\mathrm{Ext}(G, \mathbf{Z}_P) = 0$ implies that G is torsion free, so if $G \neq 0$, then $\mathbf{Q} \otimes G$ is a nontrivial vector space over $\mathbf{Q} : \mathbf{Q} \otimes G \approx I \cdot \mathbf{Q}$ $(\#(I) \geq 1) \Rightarrow \mathrm{Ext}(\mathbf{Q} \otimes G, \mathbf{Z}_P) \approx \mathrm{Ext}(\mathbf{Q}, \mathbf{Z}_P)^I \approx \mathbf{R}^I$ (cf. p. 8–1). Contradiction.]

PROPOSITION 2 Let $\begin{cases} X \\ Y \end{cases}$ be path connected topological spaces, $f : X \to Y$ a continuous function—then $f_* : H_*(X; \mathbf{Z}_P) \to H_*(Y; \mathbf{Z}_P)$ is an isomorphism iff $f^* : H^*(Y; \mathbf{Z}_P) \to H^*(X; \mathbf{Z}_P)$ is an isomorphism.

[There is an exact sequence

$$\cdots \to \widetilde{H}_n(X; \mathbf{Z}_P) \to \widetilde{H}_n(Y; \mathbf{Z}_P) \to \widetilde{H}_n(C_f; \mathbf{Z}_P) \to \widetilde{H}_{n-1}(X; \mathbf{Z}_P) \to \widetilde{H}_{n-1}(Y; \mathbf{Z}_P) \to \cdots$$

in homology and there is an exact sequence

$$\cdots \to \widetilde{H}^{n-1}(Y; \mathbf{Z}_P) \to \widetilde{H}^{n-1}(X; \mathbf{Z}_P) \to \widetilde{H}^n(C_f; \mathbf{Z}_P) \to \widetilde{H}^n(Y; \mathbf{Z}_P) \to \widetilde{H}^n(X; \mathbf{Z}_P) \to \cdots$$

in cohomology. Accordingly, it need only be shown that $\widetilde{H}_*(C_f;\mathbf{Z}_P) = 0$ iff $\widetilde{H}^*(C_f;\mathbf{Z}_P) = 0$. Case 1: $P = \emptyset$. Here, $\widetilde{H}^n(C_f;\mathbf{Q}) \approx \text{Hom}(\widetilde{H}_n(C_f;\mathbf{Q}),\mathbf{Q})$ and the assertion is obvious. Case 2: $P \neq \emptyset$. Since $\widetilde{H}^n(C_f;\mathbf{Z}_P) \approx \text{Hom}(\widetilde{H}_n(C_f;\mathbf{Z}_P),\mathbf{Z}_P) \oplus \text{Ext}(\widetilde{H}_{n-1}(C_f;\mathbf{Z}_P),\mathbf{Z}_P)$, it is clear that $\widetilde{H}_*(C_f;\mathbf{Z}_P) = 0 \Rightarrow \widetilde{H}^*(C_f;\mathbf{Z}_P) = 0$, while if $\widetilde{H}^*(C_f;\mathbf{Z}_P) = 0$, then $\forall\ n$, $\text{Hom}(\widetilde{H}_n(C_f;\mathbf{Z}_P),\mathbf{Z}_P) = 0$ & $\text{Ext}(\widetilde{H}_n(C_f;\mathbf{Z}_P),\mathbf{Z}_P) = 0$, thus from the lemma, $\widetilde{H}_n(C_f;\mathbf{Z}_P) = 0$.]

PROPOSITION 3 Let $\left\{\begin{matrix} X \\ Y \end{matrix}\right.$ be path connected topological spaces, $f : X \rightarrow Y$ a continuous function—then $f_* : H_*(X;\mathbf{Z}_P) \rightarrow H_*(Y;\mathbf{Z}_P)$ is an isomorphism iff $f_* : H_*(X;\mathbf{Q}) \rightarrow H_*(Y;\mathbf{Q})$ is an isomorphism and $\forall\ p \in P$, $f_* : H_*(X;\mathbf{Z}/p\mathbf{Z}) \rightarrow H_*(Y;\mathbf{Z}/p\mathbf{Z})$ is an isomorphism.

[Introducing again the mapping cone, it suffices to prove that $\widetilde{H}_*(C_f;\mathbf{Z}_P) = 0$ iff $\widetilde{H}_*(C_f;\mathbf{Q}) = 0$ and $\forall\ p \in P$, $\widetilde{H}_*(C_f;\mathbf{Z}/p\mathbf{Z}) = 0$. If first $\widetilde{H}_*(C_f;\mathbf{Z}_P) = 0$, then $\widetilde{H}_*(C_f;\mathbf{Q}) \approx \mathbf{Q} \otimes \widetilde{H}_*(C_f) \approx \mathbf{Q} \otimes (\mathbf{Z}_P \otimes \widetilde{H}_*(C_f)) \approx \mathbf{Q} \otimes \widetilde{H}(C_f;\mathbf{Z}_P) = 0$ and because $p \in P \Rightarrow \mathbf{Z}_P \otimes \mathbf{Z}/p\mathbf{Z} = \mathbf{Z}/p\mathbf{Z}$, $\forall\ n$, $\widetilde{H}_n(C_f;\mathbf{Z}/p\mathbf{Z}) \approx \widetilde{H}_n(C_f) \otimes \mathbf{Z}/p\mathbf{Z} \oplus \text{Tor}(\widetilde{H}_{n-1}(C_f),\mathbf{Z}/p\mathbf{Z}) \approx \widetilde{H}_n(C_f) \otimes (\mathbf{Z}_P \otimes \mathbf{Z}/p\mathbf{Z}) \oplus \text{Tor}(\widetilde{H}_{n-1}(C_f),\mathbf{Z}_P \otimes \mathbf{Z}/p\mathbf{Z}) \approx (\widetilde{H}_n(C_f) \otimes \mathbf{Z}_P) \otimes \mathbf{Z}/p\mathbf{Z} \oplus \text{Tor}(\mathbf{Z}_P \otimes \widetilde{H}_{n-1}(C_f),\mathbf{Z}/p\mathbf{Z}) \approx \widetilde{H}_n(C_f;\mathbf{Z}_P) \otimes \mathbf{Z}/p\mathbf{Z} \oplus \text{Tor}(\widetilde{H}_{n-1}(C_f;\mathbf{Z}_P),\mathbf{Z}/p\mathbf{Z}) = 0$. As for the implication in the opposite direction, $\widetilde{H}_*(C_f;\mathbf{Z}_P) = 0$ iff $\widetilde{H}_*(C_f)$ is S_P-torsion, so $\widetilde{H}_*(C_f;\mathbf{Q}) = 0 \Rightarrow \widetilde{H}_*(C_f)$ is torsion and $\forall\ n$, $\widetilde{H}_n(C_f;\mathbf{Z}/p\mathbf{Z}) = 0 \Rightarrow \text{Tor}(\widetilde{H}_n(C_f),\mathbf{Z}/p\mathbf{Z}) = 0 \Rightarrow \widetilde{H}_n(C_f)(p) = 0$ $(p \in P)$, i.e., $\widetilde{H}_*(C_f)$ is S_P-torsion.]

Application: Let $\left\{\begin{matrix} X \\ Y \end{matrix}\right.$ be path connected topological spaces, $f : X \rightarrow Y$ a continuous function—then $f_* : H_n(X) \rightarrow H_n(Y)$ is P-localizing $\forall\ n \geq 1$ iff $f_* : H_n(X;\mathbf{Q}) \rightarrow H_n(Y;\mathbf{Q})$ is an isomorphism $\forall\ n \geq 1$ and $\forall\ p \in P$, $f_* : H_n(X;\mathbf{Z}/p\mathbf{Z}) \rightarrow H_n(Y;\mathbf{Z}/p\mathbf{Z})$ is an isomorphism $\forall\ n \geq 1$ and $\forall\ p \in \overline{P}$, $H_n(Y;\mathbf{Z}/p\mathbf{Z}) = 0\ \forall\ n \geq 1$.

[Note: When $P = \mathbf{\Pi}$, "P-localizing" = "homology equivalence" and the last condition is vacuous.]

FACT Let $\left\{\begin{matrix} X \\ Y \end{matrix}\right.$ be path connected topological spaces, $f : X \rightarrow Y$ a continuous function. Assume: $\forall\ n, \left\{\begin{matrix} H_n(X) \\ H_n(Y) \end{matrix}\right.$ is finitely generated—then for $P \neq \emptyset$, $f_* : H_*(X;\mathbf{Z}_P) \rightarrow H_*(Y;\mathbf{Z}_P)$ is an isomorphism iff $\forall\ p \in P$, $f_* : H_*(X;\mathbf{Z}/p\mathbf{Z}) \rightarrow H_*(Y;\mathbf{Z}/p\mathbf{Z})$ is an isomorphism.

The theory set forth below has been developed by a number of mathematicians and can be approached in a variety of ways. What follows is an account of the bare essentials.

A group G is said to be P-local if the map $\begin{cases} G \to G \\ g \to g^n \end{cases}$ is bijective $\forall\, n \in S_P$. $\mathbf{GR}_P$ is the full subcategory of $\mathbf{GR}$ whose objects are the P-local groups. On general grounds (cf. p. 0–23), $\mathbf{GR}_P$ is a reflective subcategory of $\mathbf{GR}$ with reflector $L_P : \begin{cases} \mathbf{GR} \to \mathbf{GR}_P \\ G \to G_P \end{cases}$ and arrow of localization $l_P : G \to G_P$.

[Note: If G is abelian, then the restriction of L_P to $\mathbf{AB}$ "is" the L_P introduced earlier.]

Example: Fix $P \neq \mathbf{\Pi}$—then no nontrivial free group is P-local.

EXAMPLE Let X be a pointed connected CW space—then $\pi_1(X)$ and the $\pi_q(X) \rtimes \pi_1(X)$ ($q \geq 2$) are P-local iff $\forall\, n \in S_P$, the arrow $\begin{cases} \Omega X \to \Omega X \\ \sigma \to \sigma^n \end{cases}$ is a pointed homotopy equivalence.

[For $[\mathbf{S}^{q-1}, \Omega X]$ (no base points) is isomorphic to $\pi_q(X) \rtimes \pi_1(X)$ ($q \geq 2$).]

The kernel of $l_P : G \to G_P$ contains the set of S_P-torsion elements of G but is ordinarily much larger. Definition: An element $g \in G$ is said to be of type S_P if $\exists\, a, b \in G$ and $n \in S_P : g = ab^{-1}$ & $a^n = b^n$. The subset of G consisting of the elements of type S_P is closed under inversion and conjugation and is annihilated by l_P. Proceeding recursively, construct a sequence $\{1\} = \Lambda_0 \subset \Lambda_1 \subset \cdots$ of normal subgroups of G by letting Λ_{k+1}/Λ_k be the subgroup of G/Λ_k generated by the elements of type S_P. Put $\Lambda_P(G) = \bigcup_k \Lambda_k : \Lambda_P(G)$ is a normal subgroup of G and it is clear that if $f : G \to K$ is a homomorphism of groups, then $f(\Lambda_P(G)) \subset \Lambda_P(K)$. On the other hand, G P-local $\Rightarrow \Lambda_P(G) = \{1\}$, so $\ker l_P \supset \Lambda_P(G)$. The containment can be proper since there are examples where $\Lambda_P(G)$ is trivial but $\ker l_P$ is not trivial (Berrick-Casacuberta[†]). However, for certain G, $\ker l_P$ is always trivial, e.g., when G is locally free (cf. p. 10–6).

Observation: $\Lambda_P(G) = \{1\}$ iff $\forall\, n \in S_P$, the map $\begin{cases} G \to G \\ g \to g^n \end{cases}$ is injective.

EXAMPLE (Generically Trivial Groups) A group G is said to be generically trivial provided that $\forall\, p$, $G_p = 1$. Example: The infinite alternating group is generically trivial. The homomorphic image of a generically trivial group is generically trivial, so generically trivial groups are perfect (but not conversely as there exist perfect groups which are locally free (cf. p. 5–64)). Since a perfect nilpotent group is trivial, the only generically trivial nilpotent group is the trivial group and since a finite p-group is nilpotent, a perfect finite group is generically trivial. Example: Let A be a ring with unit—then $\mathbf{ST}(A)$ is generically trivial (Berrick-Miller[‡]), hence $\mathbf{E}(A)$ is too ($\Rightarrow \mathbf{GL}(\Gamma A) = \mathbf{E}(\Gamma A)$ is acyclic and generically trivial (cf. p. 5–75 ff.)).

[†] *SLN* **1509** (1992), 20–29.

[‡] *Math. Proc. Cambridge Philos. Soc.* **111** (1992), 219–229.

[Note: In the same paper it is shown that if $\{G_n : n \geq 2\}$ is a sequence of abelian groups, then there exists a generically trivial group G such that $H_n(G) \approx G_n$ $(n \geq 2)$.]

EXAMPLE (Separable Groups) A group G is said to be separable provided that the arrow $G \to \prod\limits_p G_p$ is injective. The class of separable groups is closed under the formation of products and subgroups, thus is the object class of an epireflective subcategory of **GR** (cf. p. 0–21). Every nilpotent group is separable as is every locally free group.

FACT A group G is generically trivial iff every homomorphism $f : G \to K$, where K is separable, is trivial.

EXAMPLE Let X be a pointed connected CW space. Assume: X is acyclic and $\pi_1(X)$ is generically trivial—then for every pointed connected CW space Y such that $\pi_1(Y)$ is separable, $C(X, x_0; Y, y_0)$ is homotopically trivial (cf. p. 5–68).

LEMMA Suppose that G is torsion—then G is P-local iff G is $S_{\overline{P}}$-torsion.

[Necessity: Given $g \in G$, $\exists\, t : g^t = e$. Write $t = n\overline{n}$ $(n \in S_P, \overline{n} \in S_{\overline{P}}) : (g^{\overline{n}})^n = e \Rightarrow g^{\overline{n}} = e$. Therefore G is $S_{\overline{P}}$-torsion.

Sufficiency: Fix $n \in S_P$. For each $\overline{n} \in S_{\overline{P}}$, choose $k, l : kn + l\overline{n} = 1$, hence (i) Given $g \in G$, $\exists\, \overline{n} \in S_{\overline{P}} : g^{\overline{n}} = e \Rightarrow g = g^{kn + l\overline{n}} = (g^k)^n$ and (ii) Given $g_1, g_2 \in G$, $\exists\, \overline{n} \in S_{\overline{P}} : g_1^{\overline{n}} = e = g_2^{\overline{n}}$, so $g_1^n = g_2^n \Rightarrow g_1 = (g_1^n)^k (g_1^{\overline{n}})^l = (g_2^n)^k (g_2^{\overline{n}})^l = g_2$.]

LEMMA Suppose that G is torsion—then $l_P : G \to G_P$ is surjective and $\ker l_P$ is generated by the S_P-torsion elements of G.

[Let Λ be the subgroup of G generated by the S_P-torsion elements of G. Since G is torsion, G/Λ is $S_{\overline{P}}$-torsion, thus P-local. In addition, for every homomorphism $f : G \to K$, where K is P-local, $f(\Lambda) = \{1\}$.]

FACT Let $1 \to G' \to G \to G'' \to 1$ be a short exact sequence of groups. Assume: G' is P-local and G'' is $S_{\overline{P}}$-torsion—then G is P-local.

EXAMPLE Let X be a pointed connected CW space. Assume: $\pi_1(X)$ is $S_{\overline{P}}$-torsion and $\forall\, q \geq 2$, $\pi_q(X)$ is P-local—then $\forall\, n \in S_P$, the arrow $\left\{\begin{array}{l} \Omega X \to \Omega X \\ \sigma \to \sigma^n \end{array}\right.$ is a pointed homotopy equivalence.

FACT Let $1 \to G' \to G \to G'' \to 1$ be a short exact sequence of groups. Assume: G'' is $S_{\overline{P}}$-torsion—then the sequence $1 \to G'_P \to G_P \to G''_P \to 1$ is exact.

EXAMPLE Let π be the fundamental group of the Klein bottle—then there is a short exact sequence $1 \to \mathbf{Z} \oplus \mathbf{Z} \to \pi \to \mathbf{Z}/2\mathbf{Z} \to 1$ so if $2 \in P$, there is a short exact sequence $1 \to \mathbf{Z}_P \oplus \mathbf{Z}_P \to \pi_P \to \mathbf{Z}/2\mathbf{Z} \to 1$ and $l_P : \pi \to \pi_P$ is injective (but this is false if $2 \notin P$).

EXAMPLE (Finite Groups) Let G be a finite group—then $l_P : G \to G_P$ is surjective and $\ker l_P$ is the subgroup of G generated by the Sylow p-subgroups ($p \in \overline{P}$), so, e.g., if G is a p-group, $G_P = \begin{cases} G & (p \in P) \\ 1 & (p \in \overline{P}) \end{cases}$. Therefore G is P-local iff $\#(G) \in S_{\overline{P}}$.

FACT Let G be a finite group—then G is P-local iff $\forall\, n \geq 1$, $H_n(G)$ is P-local.

[Given a nontrivial subgroup $K \subset G$, the homomorphism $H_n(K) \to H_n(G)$ is nonzero for infinitely many n (Swan[†]). Since $H_n(G) \approx \bigoplus\limits_{p|\#(G)} H_n(G)(p)$, it follows that $\forall\, p|\#(G)$, $H_n(G)(p) \neq 0$ for infinitely many n.]

FACT Let G be a finite group—then $H_1(l_P) : H_1(G; \mathbf{Z}_P) \to H_1(G_P; \mathbf{Z}_P)$ is bijective and $H_2(l_P) : H_2(G; \mathbf{Z}_P) \to H_2(G_P; \mathbf{Z}_P)$ is surjective.

[The short exact sequence $1 \to \ker l_P \to G \to G_P \to 1$ leads to an exact sequence $H_2(G; \mathbf{Z}_P) \to H_2(G_P; \mathbf{Z}_P) \to \mathbf{Z}_P \otimes \ker l_P/[G, \ker l_P] \to H_1(G; \mathbf{Z}_P) \to H_1(G_P; \mathbf{Z}_P) \to 0$ in which the middle term is zero.]

FACT Let G be a finite group—then $\forall\, n \geq 1$, $H_n(l_P) : H_n(G) \to H_n(G_P)$ is P-localizing iff $\ker l_P$ is S_P-torsion.

Application: Let G be a finite group. Suppose that $\forall\, p$ & $\forall\, n \geq 1$, $H_n(l_p) : H_n(G) \to H_n(G_p)$ is p-localizing—then G is nilpotent.

[The Sylow subgroups of G are normal.]

A subgroup K of a group G is said to be P-isolated if $\forall\, g \in G$, $\forall\, n \in S_P : g^n \in K \Rightarrow g \in K$. The intersection of a collection of P-isolated subgroups of G is P-isolated. Therefore every nonempty subset $X \subset G$ is contained in a unique minimal P-isolated subgroup of G, the P-isolator of X, written $I_P(G, X)$. To describe $I_P(G, X)$, let $X_1 = X$, $I_1 = \langle X_1 \rangle$, and define X_{i+1}, I_{i+1} inductively by setting $X_{i+1} = \{g : g^n \in I_i \ (\exists\, n \in S_P)\}$, $I_{i+1} = \langle X_{i+1} \rangle$—then $I_P(G, X) = \bigcup\limits_i I_i$. Corollary: X conjugation invariant $\Rightarrow I_P(G, X)$ normal.

[Note: A P-isolated subgroup of a P-local group is P-local.]

[†] *Proc. Amer. Math. Soc.* **11** (1960), 885–887.

Example: For any G, $G_P = I_P(G_P, l_P(G))$.

[Note: More generally, if $f : G \rightarrow K$ is a homomorphism of groups, then $f_P(G_P) = I_P(K_P, l_P(f(G)))$. Corollary: f surjective $\Rightarrow$ f_P surjective.]

EXAMPLE Fix a prime p—then $\mathbf{Z}/p^\infty\mathbf{Z}$ is isomorphic to $I_{\overline{p}}(\mathbf{Q}, \mathbf{Z})/\mathbf{Z}$.

EXAMPLE Let F be a free group on $n > 1$ generators—then $F/[F,F] \approx n \cdot \mathbf{Z}$. By contrast, Baumslag[†] has shown that $F_P/I_P(F_P, [F_P, F_P]) \approx n \cdot \mathbf{Z}_P$, while $F_P/[F_P, F_P] \approx n \cdot \mathbf{Z}_P \oplus \bigoplus\limits_{p \in \overline{P}} \omega \cdot (\mathbf{Z}/p^\infty\mathbf{Z})$.]

[Note: Since $\bigoplus\limits_{p \in \overline{P}} \omega \cdot (\mathbf{Z}/p^\infty\mathbf{Z})$ is S_P-torsion, $H_1(F_P)$ is not P-local if $P \neq \mathbf{\Pi}$. This example also shows that in **GR**, the operations $G \rightarrow G/[G,G] \rightarrow (G/[G,G])_P$, $G \rightarrow G_P \rightarrow G_P/[G_P, G_P]$ need not coincide.]

FACT If G is a nilpotent group and if K is a subgroup of G, then $\{g : g^n \in K \ (\exists\, n \in S_P)\}$ is a subgroup of G, hence equals $I_P(G, K)$.

[Assuming that G is generated by the g such that for some $n \in S_P$, $g^n \in K$, one can argue inductively on $d = \operatorname{nil} G > 1$ and suppose that $\forall\, g \in G$, $\exists\, n \in S_P$ & $h \in \Gamma^{d-1}(G)$, $k \in K : g^n = hk$. On the other hand, $\otimes^d([G,G] \cdot K/[G,G]) \rightarrow \otimes^d(G/[G,G]) \rightarrow \Gamma^{d-1}(G)$, so $\exists\, m \in S_P : h^m \in K$. But h is central, thus $g^{nm} = h^m k^m \in K$.]

[Note: In particular, the set of S_P-torsion elements in a nilpotent group is a subgroup (cf. p. 5-54).]

COMMUTATOR FORMULA Suppose that $\Lambda_P(G) = \{1\}$. Let $\begin{cases} K \\ L \end{cases}$ be subgroups of G—then $[K, L] = \{1\} \Rightarrow [I_P(G,K), I_P(G,L)] = \{1\}$.

[Given $\begin{cases} x \in I_P(G,K) \\ y \in I_P(G,L) \end{cases}$, the claim is that $xyx^{-1} = y$. This is trivial if $\begin{cases} x \in I_1(G,K) \\ y \in I_1(G,L) \end{cases}$, so argue by induction on i, assuming that $\begin{cases} x \in I_{i+1}(G,K) \\ y \in I_{i+1}(G,L) \end{cases}$ with $\begin{cases} x^n \in I_i(G,K) \\ y^n \in I_i(G,K) \end{cases}$ $(\exists\, n \in S_P)$—then $(y^{-n}xy^n)^n = y^{-n}x^ny^n = x^n \Rightarrow y^{-n}xy^n = x \Rightarrow xy^nx^{-1} = y^n \Rightarrow (xyx^{-1})^n = y^n \Rightarrow xyx^{-1} = y$.]

Application: Suppose that $\Lambda_P(G) = \{1\}$. Let g_1, g_2 be elements of G such that $[g_1^{n_1}, g_2^{n_2}] = 1$, where $n_1, n_2 \in S_P$—then $[g_1, g_2] = 1$.

LEMMA Suppose that $\Lambda_P(G) = \{1\}$. Let K be a P-isolated central subgroup of G—then $\Lambda_P(G/K) = \{1\}$.

[†] *Acta Math.* **104** (1960), 217–303 (cf. 253–254 & 291–293).

[Consider an element of type S_P in G/K, say $gK = (aK)(b^{-1}K)$ & $a^nK = b^nK$ $(\exists\, n \in S_P)$. So: $a^n = b^nk$ $(\exists\, k \in K) \Rightarrow [a^n, b^n] = 1 \Rightarrow [a,b] = 1 \Rightarrow (ab^{-1})^n \in K \Rightarrow ab^{-1} \in I_P(G,K) = K \Rightarrow aK = bK$.]

TRANSMISSION OF NILPOTENCY Suppose that $\Lambda_P(G) = \{1\}$. Let K be a nilpotent subgroup of G—then $I_P(G,K)$ is nilpotent with $\operatorname{nil} I_P(G,K) = \operatorname{nil} K$.

[The assertion is obvious if K consists of the identity alone. Assume next that K is abelian and nontrivial: $[K,K] = \{1\} \Rightarrow [I_P(G,K), I_P(G,K)] = \{1\} \Rightarrow I_P(G,K)$ is abelian and nontrivial. Induction hypothesis: The assertion is true whenever L is a nilpotent subgroup of H provided that $\Lambda_P(H) = \{1\}$ and $\operatorname{nil} L \leq d-1$, where $d = \operatorname{nil} K > 1$. Let Z be the center of K—then $[K,Z] = \{1\} \Rightarrow [I_P(G,K), I_P(G,Z)] = \{1\}$, thus $I_P(G,Z)$ is a P-isolated central subgroup of $I_P(G,K)$, so by the lemma, $\Lambda_P(I_P(G,K)/I_P(G,Z)) = \{1\}$. Now put $X = K \cdot I_P(G,Z) : I_1 = X_1$ is a group $(X_1 = X)$ and $I_1/I_P(G,Z) \approx K/K \cap I_P(G,Z) \approx K/Z$. Since $\operatorname{nil} K/Z = \operatorname{nil} K - 1$, it follows that I_1 is nilpotent with $\operatorname{nil} I_1 = d$. Write, as above, $I_P(G,X) = \bigcup\limits_i I_i$. Assume that I_i is nilpotent with $\operatorname{nil} I_i = d$ $\forall\, i \leq i_0$. Fix a well ordering of the elements of $X_{i_0+1} : \{x_\beta : 0 \leq \beta < \alpha\}$. Let W_γ be the subgroup of G generated by I_{i_0} and $\{x_\beta : 0 \leq \beta < \gamma\}$—then $I_{i_0+1} = \bigcup\limits_\gamma W_\gamma$ and the claim is that $\forall\, \gamma$, W_γ is nilpotent with $\operatorname{nil} W_\gamma = d$, hence that I_{i_0+1} is nilpotent with $\operatorname{nil} I_{i_0+1} = d$. Consider $W_1 : I_{i_0}/I_P(G,Z)$ is a nilpotent subgroup of $W_1/I_P(G,Z)$ with $\operatorname{nil} I_{i_0}/I_P(G,Z) = d-1$. Therefore the induction hypothesis implies that $W_1/I_P(G,Z) = I_P(W_1/I_P(G,Z), I_{i_0}/I_P(G,Z))$ is nilpotent with $\operatorname{nil} W_1/I_P(G,Z) = d-1$. This means that W_1 is nilpotent with $\operatorname{nil} W_1 = d$, which sets the stage for an evident transfinite recursion. Conclusion: $\forall\, i, I_i$ is nilpotent with $\operatorname{nil} I_i = d$, i.e., $I_P(G,X)$ is nilpotent with $\operatorname{nil} I_P(G,X) = d$ or still, $I_P(G,K)$ is nilpotent with $\operatorname{nil} I_P(G,K) = d$.]

PROPOSITION 4 Let G be a nilpotent group—then G_P is nilpotent and $\operatorname{nil} G_P \leq \operatorname{nil} G$.

[In fact, $G_P = I_P(G_P, l_P(G))$ and transmission of nilpotency ensures that G_P is nilpotent with $\operatorname{nil} G_P = \operatorname{nil} l_P(G) \leq \operatorname{nil} G$.]

Notation: **NIL** is the category of nilpotent groups and $\mathbf{NIL}^d$ is the category of nilpotent groups with degree of nilpotency $\leq d$.

Thanks to Proposition 4, L_P respects **NIL** : G nilpotent $\Rightarrow G_P$ nilpotent, thus $\mathbf{NIL}_P$, the full subcategory of **NIL** whose objects are the P-local nilpotent groups, is a reflective subcategory of **NIL**. More is true: L_P respects $\mathbf{NIL}^d$ and there is a commutative diagram

$$\begin{tikzcd}
\mathbf{NIL}^{d+1} \arrow[r, "L_P"] & \mathbf{NIL}_P^{d+1} \\
\mathbf{NIL}^{d} \arrow[u] \arrow[r, "L_P"'] & \mathbf{NIL}_P^{d} \arrow[u]
\end{tikzcd}$$ (obvious notation).

FACT Let G be a group. Assume: G is locally nilpotent—then G_P is locally nilpotent.

[Note: A group if said to be locally nilpotent if its finitely generated subgroups are nilpotent.]

FACT Let G be a group. Assume: G is virtually nilpotent—then G_P is virtually nilpotent.

[Note: A group is said to be virtually nilpotent if it contains a nilpotent subgroup of finite index.]

Given a set of primes P, a group G is said to be residually finite P if $\forall\, g \neq e$ in G there is a finite $S_{\overline{P}}$-torsion group X_g and an epimorphism $\phi_g : G \rightarrow X_g$ such that $\phi_g(g) \neq e$.

[Note: When $P = \mathbf{\Pi}$, the term is residually finite. Example: $\mathbf{Q}$ is not residually finite but $\mathbf{Z}_P$ $(P \neq \emptyset)$ is residually finite p $\forall\, p \in P$.]

Examples: (1) (Iwasawa) Every free group is residually finite p for all primes p; (2) (Hirsch) Every polycyclic group is residually finite ($\Rightarrow$ every finitely generated nilpotent group is residually finite); (3) (Gruenberg) Every finitely generated torsion free nilpotent group is residually finite p for all primes p; (4) (Hall) Every finitely generated abelian-by-nilpotent group is residually finite.

[Note: Proofs of these results can be found in Robinson[†].]

LEMMA Let G be a finitely generated nilpotent group. Assume: All the torsion in G is $S_{\overline{P}}$-torsion, where $P \neq \emptyset$—then G is residually finite P.

[Fix $g \neq e$ in G. Case 1: $g \notin G_{\text{tor}}$. According to Gruenberg, $\forall\, p$, G/G_{tor} is residually finite p, so a fortiori G/G_{tor} is residually finite P. Case 2: $g \in G_{\text{tor}}$. According to Hirsch, there is a finite nilpotent group X_g and an epimorphism $\phi_g : G \rightarrow X_g$ such that $\phi_g(g) \neq e$. Write $X_g = \prod\limits_p X_g(p), X_g(p)$ the Sylow p-subgroup of X_g. Let π_P be the projection $X_g \rightarrow \prod\limits_{p \in P} X_g(p)$ and consider the composite $\pi_P \circ \phi_g$.]

PROPOSITION 5 Let G be a nilpotent group—then $l_P : G \rightarrow G_P$ is P-bijective.

[Since G_P is nilpotent, $\{g_P : g_P^n \in l_P(G)\ (\exists\, n \in S_P)\}$ equals $I_P(G_P, l_P(G))$ (cf. p. 8–12) or still, G_P, thus l_P is P-surjective. To verify that l_P is P-injective, suppose first that P is nonempty. Because the kernel of l_P contains the S_P-torsion, one can assume that all the torsion in G is $S_{\overline{P}}$-torsion. The claim in this situation is that l_P is injective. If to begin with G is finitely generated, then on the basis of the lemma, there is an embedding $G \to \prod\limits_{g \neq e} X_g$, where each X_g is a finite $S_{\overline{P}}$-torsion group, hence P-local (cf. p. 8–10). Therefore $\prod\limits_{g \neq e} X_g$ is P-local, so l_P is necessarily injective. To see that l_P is injective in general, express G as the colimit of its finitely generated subgroups G_i and compute the kernel of $G \to G_P$ as the colimit of the kernels of the $G_i \to G_{i,P}$. There remains the possibility that P is empty. To finesse this, choose $P : P \neq \emptyset$ & $\overline{P} \neq \emptyset$ and note that the arrow $(G_P)_{\overline{P}} \to G_\emptyset$ $(= G_{\mathbf{Q}})$ is an isomorphism which implies that $G_{\text{tor}} = \ker l_{\mathbf{Q}}$.]

[†] *Finiteness Conditions and Generalized Soluble Groups*, vol. II, Springer Verlag (1972); see also Magnus, *Bull. Amer. Math. Soc.* **75** (1969), 305–316.

Application: Every torsion free nilpotent group embeds in its rationalization.

LEMMA Let $f : G \to K$ be a homomorphism of nilpotent groups. Assume: f is injective (surjective)—then $f_P : G_P \to K_P$ is injective (surjective).

[It will be enough to establish injectivity (see p. 8–12 for surjectivity). Suppose that $f_P(g_P) = e$ $(g_P \in G_P)$. Since l_P is P-surjective, $\exists\ g \in G$ & $n \in S_P : l_P(g) = g_P^n \Rightarrow l_P(f(g)) = e \Rightarrow\ \exists\ m \in S_P : f(g)^m = e \Rightarrow g^m = e \Rightarrow g \in \ker l_P \Rightarrow g_P^n = e \Rightarrow g_P = e$, G_P being P-local.]

PROPOSITION 6 $L_P : \mathbf{NIL} \to \mathbf{NIL}_P$ is exact, i.e., if $1 \to G' \to G \to G'' \to 1$ is a short exact sequence in $\mathbf{NIL}$, then $1 \to G'_P \to G_P \to G''_P \to 1$ is a short exact sequence in $\mathbf{NIL}_P$.

[It is straightforward to check that $\operatorname{im}(G'_P \to G_P) = \ker(G_P \to G''_P)$.]

LEMMA Let G be a nilpotent group. Suppose that K is a central subgroup of G—then K_P is a central subgroup of G_P.

[In fact, $[G, K] = \{1\} \Rightarrow [l_P(G), l_P(K)] = \{1\}$, so by the commutator formula, $[I_P(G_P, l_P(G)),\ I_P(G_P, l_P(K))] = \{1\} \Rightarrow [G_P, K_P] = \{1\}$.]

PROPOSITION 7 $L_P : \mathbf{NIL} \to \mathbf{NIL}_P$ preserves central extensions.

LEMMA Let $G_1 \to G_2 \to G_3 \to G_4 \to G_5$ be an exact sequence of nilpotent groups. Assume: $\left\{\begin{array}{l} G_1, G_2 \\ G_4, G_5 \end{array}\right.$ are P-local—then G_3 is P-local.

Application: Let $1 \to G' \to G \to G'' \to 1$ be a short exact sequence of nilpotent groups. Assume: Two of the groups are P-local—then so is the third.

EXAMPLE Let X be a pointed connected CW space. Assume: X is nilpotent and $\forall\, q \geq 1$, $\pi_q(X)$ is P-local—then $\forall\, n \in S_P$, the arrow $\begin{cases} \Omega X \rightarrow \Omega X \\ \sigma \rightarrow \sigma^n \end{cases}$ is a pointed homotopy equivalence.

[There is a split short exact sequence $1 \rightarrow \pi_q(X) \rightarrow \pi_q(X) \rtimes \pi_1(X) \rightarrow \pi_1(X) \rightarrow 1$, where $\pi_q(X) \rtimes \pi_1(X)$ $(q \geq 2)$ is nilpotent (cf. p. 5–56), hence P-local.]

If $f, g : G \rightarrow K$ are homomorphisms of nilpotent groups such that $\forall\, p$, $f_p = g_p$, then $f = g$. In other words, morphisms in **NIL** (as in **AB**) are determined by their localizations. For finitely generated objects, the situation is different. Definition: Given a finitely generated nilpotent group G, the genus gen G of G is the conglomerate of isomorphism classes of finitely generated nilpotent groups K such that $\forall\, p$, $G_p \approx K_p$. By contrast to what obtains in **AB**, it can happen that $\#(\text{gen } G) > 1$ although always $\#(\text{gen } G) < \omega$ (Pickel[†]).

[Note: If G is a finitely generated abelian group and if K is a finitely generated nilpotent group such that $\forall\, p$, $G_p \approx K_p$, then $G \approx K (\Rightarrow \text{ gen } G = \{[G]\})$.

FACT Let G be a nilpotent group—then two elements of G are conjugate iff their images in every G_p are conjugate.

Let G be a nilpotent group—then one may attach to G a sink $\{r_p : G_p \rightarrow G_{\mathbf{Q}}\}$ and a source $\{l_p : G \rightarrow G_p\}$, where $\forall \begin{cases} p \\ q \end{cases}$, $r_p \circ l_p = r_q \circ l_q$.

LEMMA Let $1 \rightarrow G' \rightarrow G \rightarrow G'' \rightarrow 1$ be a short exact sequence of nilpotent groups. Assume: The source $\begin{cases} \{l_p : G' \rightarrow G'_p\} \\ \{l_p : G'' \rightarrow G''_p\} \end{cases}$ is the multiple pullback of the sink $\begin{cases} \{r_p : G'_p \rightarrow G'_{\mathbf{Q}}\} \\ \{r_p : G''_p \rightarrow G''_{\mathbf{Q}}\} \end{cases}$—then the source $\{l_p : G \rightarrow G_p\}$ is the multiple pullback of the sink $\{r_p : G_p \rightarrow G_{\mathbf{Q}}\}$.

[The verification is a diagram chase, using the exactness of $1 \rightarrow G'_p \rightarrow G_p \rightarrow G''_p \rightarrow 1$. Precisely: Given elements $g_p \in G_p$ & $g_{\mathbf{Q}} \in G_{\mathbf{Q}} : \forall\, p$, $r_p(g_p) = g_{\mathbf{Q}}$, $\exists!\, g \in G : \forall\, p$, $l_p(g) = g_p$.]

FRACTURE LEMMA Suppose that G is a finitely generated nilpotent group—then the source $\{l_p : G \rightarrow G_p\}$ is the multiple pullback of the sink $\{r_p : G_p \rightarrow G_{\mathbf{Q}}\}$.

[Proceed by induction on $\text{nil}\, G$. The assertion is true if $\text{nil}\, G \leq 1$ (cf. p. 8–4). Assume therefore that $\text{nil}\, G > 1$ and consider the short exact sequence $1 \rightarrow \Gamma^1(G) \rightarrow G \rightarrow$

[†] *Trans. Amer. Math. Soc.* **160** (1971), 327–341; see also Mislin, *SLN* **418** (1974), 103–120 and Warfield, *J. Pure Appl. Algebra* **6** (1975), 125–132.

$G/\Gamma^1(G) \to 1$ of nilpotent groups. Since G is finitely generated, $\Gamma^1(G)$ is finitely generated (cf. p. 5–54), as is $G/\Gamma^1(G)$. Furthermore, $\operatorname{nil}\Gamma^1(G) < \operatorname{nil} G$ and $\operatorname{nil} G/\Gamma^1(G) = 1$, thus the lemma is applicable.]

Let $f : G \to K$ be a homomorphism of nilpotent groups—then f is said to be P-localizing if $\exists$ an isomorphism $\phi : G_P \to K$ such that $f = \phi \circ l_P$ (cf. p. 0–30).

LEMMA Let $f : G \to K$ be a homomorphism of nilpotent groups—then f is P-localizing iff f is P-bijective and K is P-local.

[Note: A homomorphism $f : G \to K$ of nilpotent groups is P-bijective iff $f_P : G_P \to K_P$ is bijective (cf. Proposition 5).]

FACT Let

$$\begin{tikzcd}
G_1 \arrow[r] \arrow[d, "f_1"] & G_2 \arrow[r] \arrow[d, "f_2"] & G_3 \arrow[r] \arrow[d, "f_3"] & G_4 \arrow[r] \arrow[d, "f_4"] & G_5 \arrow[d, "f_5"] \\
K_1 \arrow[r] & K_2 \arrow[r] & K_3 \arrow[r] & K_4 \arrow[r] & K_5
\end{tikzcd}$$

be a commutative diagram of nilpotent groups with exact rows. Assume: f_1, f_2, f_4, f_5 are P-localizing—then f_3 is P-localizing.

PROPOSITION 8 Let G be a nilpotent group—then $\forall\, n \geq 1$, $H_n(l_P) : H_n(G) \to H_n(G_P)$ is P-localizing.

[This is true if $\operatorname{nil} G \leq 1$, so argue by induction on $\operatorname{nil} G > 1$. There is a commutative diagram

$$\begin{tikzcd}
1 \arrow[r] & \operatorname{Cen} G \arrow[r] \arrow[d] & G \arrow[r] \arrow[d] & G/\operatorname{Cen} G \arrow[r] \arrow[d] & 1 \\
1 \arrow[r] & (\operatorname{Cen} G)_P \arrow[r] & G_P \arrow[r] & (G/\operatorname{Cen} G)_P \arrow[r] & 1
\end{tikzcd}$$

of central extensions (cf. Proposition 7) and a morphism $\{E^2_{p,q} \approx H_p(G/\operatorname{Cen} G; H_q(\operatorname{Cen} G))\} \to \{\overline{E}^2_{p,q} \approx H_p((G/\operatorname{Cen} G)_P; H_q((\operatorname{Cen} G)_P))\}$ of LHS spectral sequences. Since $\operatorname{nil}\operatorname{Cen} G \leq 1$ and $\operatorname{nil} G/\operatorname{Cen} G \leq \operatorname{nil} G - 1$, it follows from the induction hypothesis and the universal coefficient theorem that the arrow $E^2_{p,q} \to \overline{E}^2_{p,q}$ is P-localizing $(p + q > 0)$. However, the homology groups attached to a chain complex of P-local abelian groups are P-local (cf. p. 8–5), thus this conclusion persists through the spectral sequence and in the end, it is seen that the arrow $E^\infty_{p,q} \to \overline{E}^\infty_{p,q}$ is P-localizing $(p + q > 0)$. Fix now an $n \geq 1$. Consider the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & H_{p-1,q+1} \arrow[r] \arrow[d] & H_{p,q} \arrow[r] \arrow[d] & E^\infty_{p,q} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \overline{H}_{p-1,q+1} \arrow[r] & \overline{H}_{p,q} \arrow[r] & \overline{E}^\infty_{p,q} \arrow[r] & 0
\end{tikzcd},$$

where $p+q=n$—then the obvious recursion argument allows one to say that the arrow $H_{p,q} \rightarrow \overline{H}_{p,q}$ is P-localizing, therefore $H_n(l_P) : H_n(G) \rightarrow H_n(G_P)$ is P-localizing.]

Application: Let G be a nilpotent group—then $\forall\, n \geq 1$, $H_n(G)_P \approx H_n(G_P)$.

FACT Suppose that G and K are finitely generated nilpotent groups. Assume: gen $G =$ gen K—then $\forall\, n \geq 1$, $H_n(G) \approx H_n(K)$.

[The point here is that $H_n(G)$ and $H_n(K)$ are finitely generated (cf. p. 5–54).]

PROPOSITION 9 Let G be a nilpotent group. Assume: $\forall\, n \geq 1$, $H_n(G)$ is P-local—then G is P-local.

[According to Proposition 8, $H_n(l_P) : H_n(G) \rightarrow H_n(G_P)$ is P-localizing or still, is an isomorphism, $H_n(G)$ being P-local. But this means that $l_P : G \rightarrow G_P$ is an isomorphism (cf. p. 5–55).]

PROPOSITION 10 Let $f : G \rightarrow K$ be a homomorphism of nilpotent groups—then f is P-localizing iff $\forall\, n \geq 1$, $H_n(f) : H_n(G) \rightarrow H_n(K)$ is P-localizing.

[Necessity: By definition, $\exists$ an isomorphism $\phi : G_P \rightarrow K$ such that $f = \phi \circ l_P$, so $H_n(f) = H_n(\phi) \circ H_n(l_P)$, where $H_n(\phi)$ is an isomorphism and $H_n(l_P)$ is P-localizing (cf. Proposition 8).

Sufficiency: Since $\forall\, n \geq 1$, $H_n(K)$ is P-local, Proposition 9 implies that K is P-local, hence by universality, $\exists$ a homomorphism $\phi : G_P \rightarrow K$ such that $f = \phi \circ l_P$. Claim: ϕ is an isomorphism. In fact, $H_n(f) = H_n(\phi) \circ H_n(l_P)$, where $H_n(f)$ and $H_n(l_P)$ are P-localizing, thus $\forall\, n \geq 1$, $H_n(\phi)$ is an isomorphism, from which the claim (cf. p. 5–55).]

[Note: Similar considerations show that if $f : G \rightarrow K$ is a homomorphism of nilpotent groups, then f is P-bijective iff $\forall\, n \geq 1$, $H_n(f) : H_n(G; \mathbf{Z}_P) \rightarrow H_n(K; \mathbf{Z}_P)$ is bijective.]

PROPOSITION 11 Let $f : G \rightarrow K$ be a homomorphism of nilpotent groups. Assume: f is P-localizing—then $\forall\, i \geq 0$, $\Gamma^i(f) : \Gamma^i(G) \rightarrow \Gamma^i(K)$ is P-localizing.

[On the basis of the commutative diagram

$$\begin{tikzcd}
1 \arrow[r] & \Gamma^i(G) \arrow[r] \arrow[d] & G \arrow[r] \arrow[d] & G/\Gamma^i(G) \arrow[r] \arrow[d, "f_i"] & 1 \\
1 \arrow[r] & \Gamma^i(K) \arrow[r] & K \arrow[r] & K/\Gamma^i(K) \arrow[r] & 1
\end{tikzcd},$$

it need only be shown that $\forall\, i$, the induced map f_i is P-localizing. This can be done by induction on i. Indeed, the assertion is trivial if $i = 0$ and a consequence of Proposition 10 if $i = 1$, so to pass from i to $i+1$, it suffices to remark that the arrow $\Gamma^i(G)/\Gamma^{i+1}(G) \rightarrow \Gamma^i(K)/\Gamma^{i+1}(K)$ is P-localizing (inspect the proof of Proposition 14 in §5).]

Application: Let G be a nilpotent group—then $\forall\, i \geq 0$, $\Gamma^i(G)_P \approx \Gamma^i(G_P)$.

LEMMA Let $\begin{cases} \phi : G \rightarrow K \\ \psi : H \rightarrow K \end{cases}$ be homomorphisms of nilpotent groups—then $f : G \times_K H \rightarrow G_P \times_{K_P} H_P$ is P-localizing.

[For f is clearly P-injective, being the restriction to $G \times_K H$ of the P-bijection $l_P \times l_P : G \times H \rightarrow G_P \times H_P$. To show that f is P-surjective, take $(g_P, h_P) \in G_P \times_{K_P} H_P$, so $\phi_P(g_P) = \psi_P(h_P)$. Choose $\begin{cases} g \in G \\ h \in H \end{cases}$ & $\begin{cases} m \\ n \end{cases} \in S_P : \begin{cases} g_P^m = l_P(g) \\ h_P^n = l_P(h) \end{cases} \Rightarrow l_P \circ \phi(g^n) = \phi_P \circ l_P(g^n) = \phi_P(g_P^{mn}) = \psi_P(h_P^{mn}) = \psi_P \circ l_P(h^m) = l_P \circ \psi(h^m) \Rightarrow \phi(g^n) = \psi(h^m)k$ $(k \in \ker l_P)$. Choose $t \in S_P : k^t = e$. Fix $d : \operatorname{nil} K \leq d$—then $\phi(g^n)^{t^d} = (\psi(h^m)k)^{t^d} = \psi(h^m)^{t^d}$ (cf. p. 5–54) $\Rightarrow (g^{nt^d}, h^{mt^d}) \in G \times_K H \Rightarrow (g_P, h_P)^{mnt^d} = f(g^{nt^d}, h^{mt^d}) \Rightarrow (g_P, h_P)^{mnt^d} \in \operatorname{im} f$, i.e., f is P-surjective. Since $G_P \times_{K_P} H_P$ is necessarily P-local, it follows that f is P-localizing.]

LEMMA Let $\begin{cases} \phi : G \rightarrow K \\ \psi : G \rightarrow K \end{cases}$ be homomorphisms of nilpotent groups—then $f : \operatorname{eq}(\phi, \psi) \rightarrow \operatorname{eq}(\phi_P, \psi_P)$ is P-localizing.

[Imitate the argument used in the preceding proof.]

PROPOSITION 12 $L_P : \mathbf{NIL} \rightarrow \mathbf{NIL}_P$ preserves finite limits.

[Combine the foregoing lemmas.]

EXAMPLE Let G be a nilpotent group; let $\begin{cases} G' \\ G'' \end{cases}$ be subgroups of G—then $(G' \cap G'')_P \approx G'_P \cap G''_P$.

FACT Let G be a nilpotent group, $\{g_\mu\}$ a subset of G. Fix $n \in \mathbf{N}$. Assume: (1) The set $\{g_\mu[G,G]\}$ generates $G/[G,G]$; (2) Each g_μ is a product of nth powers—then the map $\begin{cases} G \rightarrow G \\ g \rightarrow g^n \end{cases}$ is surjective.

[$\Gamma^i(G)/\Gamma^{i+1}(G)$ has n^{th} roots (consider the arrow $\otimes^{i+1}(G/[G,G]) \rightarrow \Gamma^i(G)/\Gamma^{i+1}(G)$), thus $G/\Gamma^{i+1}(G)$ has n^{th} roots (consider the central extension $1 \rightarrow \Gamma^i(G)/\Gamma^{i+1}(G) \rightarrow G/\Gamma^{i+1}(G) \rightarrow G/\Gamma^i(G) \rightarrow 1$).]

EXAMPLE Let G be a nilpotent group; let K be a subgroup of G. Write $\operatorname{nor}_G K$ for the normal closure of K in G, $\operatorname{nor}_{G_P} K_P$ for the normal closure of K_P in G_P—then $(\operatorname{nor}_G K)_P \approx \operatorname{nor}_{G_P} K_P$.

EXAMPLE Let G be a nilpotent group; let $\begin{cases} G' \\ G'' \end{cases}$ be subgroups of G. Write $\langle G', G'' \rangle$ for the subgroup of G generated by $G' \cup G''$, $\langle G'_P, G''_P \rangle$ for the subgroup of G_P generated by $G'_P \cup G''_P$—then $\langle G', G'' \rangle_P \approx \langle G'_P, G''_P \rangle$.

Notation: Given groups $\begin{cases} G' \\ G'' \end{cases}$, the kernel $\text{car}(G', G'')$ of the epimorphism $G' * G'' \rightarrow G' \times G''$ is the cartesian subgroup of $G' * G''$. It is freely generated by $\{[g', g''] : g' \neq e \ \& \ g'' \neq e\}$. If $\begin{cases} G' \\ G'' \end{cases}$ are subgroups of G, then $\nabla_G(\text{car}(G', G'')) = [G', G'']$, where $\nabla_G : G * G \rightarrow G$ is the folding map.

Suppose that $\begin{cases} G' \\ G'' \end{cases}$ are in $\mathbf{NIL}^d$. Put $G' *_d G'' = G' * G'' / \Gamma^d(G' * G'')$—then $G' *_d G''$ is the coproduct in $\mathbf{NIL}^d$. Call $\text{car}_d(G', G'')$ the kernel of the epimorphism $G' *_d G'' \rightarrow G' \times G''$, so $\text{car}_d(G', G'') \approx \text{car}(G', G'')/\Gamma^d(G' * G'')$.

FACT $\mathbf{NIL}^d$ is a reflective subcategory of $\mathbf{GR}$, hence is complete and cocomplete.

[Note: $\mathbf{NIL}$ is finitely complete but not finitely cocomplete.]

FACT Let G be a nilpotent group—then the commutative diagram

$$\begin{tikzcd} G \arrow[r] \arrow[d] & G_{\overline{P}} \arrow[d] \\ G_P \arrow[r] & G_{\mathbf{Q}} \end{tikzcd}$$

is simultaneously a pullback square and a pushout square in $\mathbf{NIL}$ and the arrow $\begin{cases} G_P \rightarrow G_{\mathbf{Q}} \\ G_{\overline{P}} \rightarrow G_{\mathbf{Q}} \end{cases}$ is a $\begin{cases} \overline{P}\text{-bijection} \\ P\text{-bijection} \end{cases}$.

PROPOSITION 13 Let G be a nilpotent group; let $\begin{cases} G' \\ G'' \end{cases}$ be subgroups of G—then $l_P : G \rightarrow G_P$ restricts to an arrow $f : [G', G''] \rightarrow [G'_P, G''_P]$ which is P-localizing.

[Trivially, f is P-injective. To check that f is P-surjective, look first at the commutative diagram

$$\begin{tikzcd} 1 \arrow[r] & \text{car}_d(G', G'') \arrow[r] \arrow[d] & G' *_d G'' \arrow[r] \arrow[d] & G' \times G'' \arrow[r] \arrow[d] & 1 \\ 1 \arrow[r] & \text{car}_d(G'_P, G''_P) \arrow[r] & G'_P *_d G''_P \arrow[r] & G'_P \times G''_P \arrow[r] & 1 \end{tikzcd},$$

it being assumed that $\text{nil}\, G \leq d$. Since L_P preserves colimits, $(G' *_d G'')_P \approx (G'_P *_d G''_P)_P$ (cf. p. 0–20). Therefore the arrow $G' *_d G'' \rightarrow G'_P *_d G''_P$ is P-bijective, thus the same is true of the arrow $\text{car}_d(G', G'') \rightarrow \text{car}_d(G'_P, G''_P)$. Consequently, upon forming the commutative square

$$\begin{tikzcd} \text{car}_d(G', G'') \arrow[r] \arrow[d] & {[G', G'']} \arrow[d, "f"] \\ \text{car}_d(G'_P, G''_P) \arrow[r] & {[G'_P, G''_P]} \end{tikzcd}$$

in which the horizontal arrows are the epimorphisms induced by the folding maps, it is seen that f is P-surjective. Turning to the verification that $[G'_P, G''_P]$ is P-local, there is no S_P-torsion and $\forall\, n \in S_P$, $G'_P *_d G''_P$ has n^{th} roots (consider generators (cf. p. 8–19)), so $\forall\, n \in S_P$, $\text{car}_d(G'_P, G''_P)$ has n^{th} roots and this suffices.]

Application: Let G be a nilpotent group; let $\begin{cases} G' \\ G'' \end{cases}$ be subgroups of G—then $[G', G'']_P \approx [G'_P, G''_P]$.

Let G and π be groups. Suppose that G operates on π, i.e., suppose given a homomorphism $\chi : G \to \operatorname{Aut} \pi$—then χ determines a homomorphism $\chi_P : G \to \operatorname{Aut} \pi_P$, thus G operates on π_P.

FACT If G operates on π and if π is nilpotent, then $\Gamma^i_\chi(\pi)_P \approx \Gamma^i_{\chi_P}(\pi_P)$ (here the notation is that of p. 5–55). In particular: π χ-nilpotent $\Rightarrow$ π_P χ_P-nilpotent.

[Use induction and Proposition 13, so that $[\pi, \Gamma^i_\chi(\pi)]_P \approx [\pi_P, \Gamma^i_\chi(\pi)_P] \approx [\pi_P, \Gamma^i_{\chi_P}(\pi_P)]$.]

Given groups G and π, let $\operatorname{Hom}_{\text{nil}}(G, \operatorname{Aut} \pi)$ be the subset of $\operatorname{Hom}(G, \operatorname{Aut} \pi)$ consisting of those χ such that π is χ-nilpotent.

[Note: In order that $\operatorname{Hom}_{\text{nil}}(G, \operatorname{Aut} \pi)$ be nonempty, it is necessary that π be nilpotent (cf. p. 5–55).]

Suppose that G and π are nilpotent.

(nil_1) The arrow $\operatorname{Hom}(G, \operatorname{Aut} \pi) \to \operatorname{Hom}(G, \operatorname{Aut} \pi_P)$ restricts to an arrow $\operatorname{Hom}_{\text{nil}}(G, \operatorname{Aut} \pi) \to \operatorname{Hom}_{\text{nil}}(G, \operatorname{Aut} \pi_P)$.

[For, as noted above, π χ-nilpotent $\Rightarrow$ π_P χ_P-nilpotent.]

(nil_2) There is an arrow $\operatorname{Hom}_{\text{nil}}(G, \operatorname{Aut} \pi) \to \operatorname{Hom}_{\text{nil}}(G_P, \operatorname{Aut} \pi_P)$ that sends χ to $\overline{\chi}_P$, where $\overline{\chi}_P \circ l_P = \chi_P$.

[The semidirect product $\Pi = \pi \rtimes_\chi G$ is nilpotent (cf. p. 5–56). Localize the split short exact sequence $1 \to \pi \to \Pi \to G \to 1$ and consider the associated action of G_P on $\pi_P : \Pi_P = \pi_P \rtimes_{\overline{\chi}_P} G_P$.]

(nil_3) The arrow $\operatorname{Hom}(G_P, \operatorname{Aut} \pi_P) \to \operatorname{Hom}(G, \operatorname{Aut} \pi_P)$ restricts to an arrow $\operatorname{Hom}_{\text{nil}}(G_P, \operatorname{Aut} \pi_P) \to \operatorname{Hom}_{\text{nil}}(G, \operatorname{Aut} \pi_P)$ which is bijective. If $\hbar$ is its inverse, then $\forall$ χ, $\hbar(\chi_P) = \overline{\chi}_P$.

[Implicit in the construction of $\hbar$ is the relation $\Gamma^i_{\chi_P}(\pi_P) \approx \Gamma^i_{\overline{\chi}_P}(\pi_P)$.]

FACT Suppose that G operates nilpotently on π and π is abelian—then for any half exact functor $F : \mathbf{AB} \to \mathbf{AB}$, G operates nilpotently on $F\pi$.

EXAMPLE Fix a path connected topological space X and let π be a nilpotent G-module—then $\forall$ $n \geq 0$, $H_n(X; \pi)$ is a nilpotent G-module.

PROPOSITION 14 Let G be a nilpotent group, M a nilpotent G-module—then $\forall$ $n \geq 0$, the arrow $H_n(G; M) \to H_n(G_P; M_P)$ is P-localizing.

[From the definitions, $H_0(G;M) \approx M/\Gamma^1_\chi(M)$ and $H_0(G_P;M_P) \approx M_P/\Gamma^1_{\overline{\chi}_P}(M_P)$. Accordingly, since L_P is exact, $(M/\Gamma^1_\chi(M))_P \approx M_P/\Gamma^1_\chi(M)_P \approx M_P/\Gamma^1_{\chi_P}(M_P) \approx M_P/\Gamma^1_{\overline{\chi}_P}(M_P)$, thereby dispensing with the case $n = 0$. Assume henceforth that $n \geq 1$. Matters are plain when $\text{nil}_\chi M = 0$. If $\text{nil}_\chi M = 1$, i.e., if G operates trivially on M, then G_P operates trivially on M_P and one can apply the universal coefficient theorem, in conjunction with Proposition 10, to derive the desired conclusion. Arguing inductively, suppose that $\text{nil}_\chi M \leq d$ $(d > 1)$ and that the assertion holds for operations having degree of nilpotency $\leq d - 1$. Consider the short exact sequence $0 \to \Gamma^1_\chi(M) \to M \to M/\Gamma^1_\chi(M) \to 0$. The degree of nilpotency of the induced action of G on $\Gamma^1_\chi(M)$ is $\leq d-1$, while that of G on $M/\Gamma^1_\chi(M)$ is ≤ 1. Comparison of the long exact sequence $\cdots \to H_{n+1}(G;M/\Gamma^1_\chi(M)) \to H_n(G;\Gamma^1_\chi(M)) \to H_n(G;M) \to H_n(G;M/\Gamma^1_\chi(M)) \to H_{n-1}(G;\Gamma^1_\chi(M)) \to \cdots$ with its local companion terminates the proof.]

Application: Let G be a nilpotent group, M a nilpotent G-module—then $\forall\ n \geq 0$, $H_n(G;M)_P \approx H_n(G_P;M_P)$.

Given a group G, G-**ACT** is the category whose objects are the groups on which G operates to the left and whose morphisms are the equivariant homomorphisms. An object π in G-**ACT** is really a pair (χ, π), where $\chi : G \to \text{Aut}\,\pi$. One says that π is P-local or that G operates P-locally on π if $\forall\ n \in S_P$ & $\forall\ g \in G$, the map $\pi \to \pi$ that sends α to $\alpha(\chi(g)\alpha)\cdots(\chi(g^{n-1})\alpha)$ is bijective, so π is necessarily a P-local group. Denote by G-**ACT**$_P$ the full subcategory of G-**ACT** whose objects are the P-local π—then G-**ACT**$_P$ is a reflective subcategory of G-**ACT** with reflector $L_{G,P}$. This can be seen by applying the reflective subcategory theorem. Thus let F_G be the free G-group on one generator $*$, i.e., the free group on the symbols $g \cdot *$ $(g \in G)$ with the obvious left action. Write $S_{G,P}$ for the set of G-maps $\begin{cases} F_G \to F_G \\ * \to \rho^n_g(*) \end{cases}$ $(n \in S_P)$, where $\rho^n_g(*) = *(g \cdot *) \cdots (g^{n-1} \cdot *)$. Working through the definitions, one finds that Ob G-**ACT**$_P = S^\perp_{G,P}$.

Example: $\pi \rtimes_\chi G$ is a P-local group iff G operates P-locally on π and G is a P-local group.

[Note: It is a corollary that if G is $S_{\overline{P}}$-torsion, then every P-local group in G-**ACT** is actually in G-**ACT**$_P$. Proof: Consider the short exact sequence $1 \to \pi \to \pi \rtimes_\chi G \to G \to 1$ and quote the generality on p. 8–10.]

FACT Let $f : G \to K$ be a homomorphism of groups—then the functor $f^* : K$-**ACT** $\to G$-**ACT** has a left adjoint $f_* : G$-**ACT** $\to K$-**ACT**.

[Let $\begin{cases}\overline{\pi}_G \\ \overline{\pi}_K\end{cases}$ be the normal closure of π in $\begin{cases}\pi * G \\ \pi * K\end{cases}$. There are pushout squares

$$\begin{tikzcd}
\pi \arrow[r] \arrow[d] & \pi * G \arrow[d] \\
* \arrow[r] & G
\end{tikzcd},
\quad
\begin{tikzcd}
\pi \arrow[r] \arrow[d] & \pi * K \arrow[d] \\
* \arrow[r] & K
\end{tikzcd}$$

, short exact sequences $1 \rightarrow \overline{\pi}_G \rightarrow \pi * G \rightarrow G \rightarrow 1$, $1 \rightarrow \overline{\pi}_K \rightarrow \pi * K \rightarrow K \rightarrow 1$, and a commutative diagram

$$\begin{tikzcd}
\pi * G \arrow[r] \arrow[d, "\mathrm{id} * f"'] & G \arrow[d, "f"] \\
\pi * K \arrow[r] & K
\end{tikzcd}$$

. Let $\pi_{\chi,G}$ be the normal closure in $\pi * G$ of the words $g\alpha g^{-1}(\chi(g)\alpha)^{-1}$, $f(\pi_{\chi,G})$ the normal closure in $\pi * K$ of the words $\mathrm{id} * f(g\alpha g^{-1}(\chi(g)\alpha)^{-1})$—then $\pi_{\chi,G}$ is a normal subgroup of $\overline{\pi}_G$, the quotient $\overline{\pi}_G/\pi_{\chi,G}$ is equivariantly isomorphic to π, and $f(\pi_{\chi,G}) \subset \overline{\pi}_K$. Definition: $f_*(\pi) = \overline{\pi}_K/f(\pi_{\chi,G})$, the action of K being conjugation. Note that the arrow $\pi \rightarrow f^* f_*(\pi)$ is equivariant.]

EXAMPLE For any homomorphism $f : G \rightarrow K$ of groups, the composite $L_{K,P} \circ f_*$ is a functor G-**ACT** $\rightarrow$ K-**ACT** $\rightarrow$ K-**ACT**$_P$. Specialize and take $K = G_P$, $f = l_P$. Given π in G-**ACT**, form $\pi \rtimes_\chi G$—then its localization $(\pi \rtimes_\chi G)_P$ is isomorphic to a semidirect product $? \rtimes G_P$, and $?$ can be identified with $L_{G_P,P} \circ l_{P,*}(\pi)$.

Given a group G, a P-local G-module is a G-module on which G operates P-locally. Every P-local G-module is a P-local abelian group.

[Note: If $(\mathbf{Z}[G])_{S_P}$ is the localization of $\mathbf{Z}[G]$ at the multiplicative closure of the $1 + g + \cdots + g^{n-1}$ $(n \in S_P)$, then the P-local G-modules are the $(\mathbf{Z}[G])_{S_P}$-modules. When G is trivial, $(\mathbf{Z}[G])_{S_P}$ reduces to $\mathbf{Z}_P$.]

PROPOSITION 15 Suppose that G is S_P-torsion—then every P-local G-module is trivial.

[In $\mathbf{Z}[G]$, consider the identity $g^n - 1 = (g-1)\,(1 + g + \cdots + g^{n-1})$.]

FACT Let $0 \rightarrow M' \rightarrow M \rightarrow M'' \rightarrow 0$ be a short exact sequence of G-modules. Assume: Two of the modules are P-local—then so is the third.

EXAMPLE Suppose that M is a P-local G-module and N is a nilpotent G-module—then $M \otimes N$, $\mathrm{Tor}(M, N)$, $\mathrm{Hom}(N, M)$, $\mathrm{Ext}(N, M)$ are P-local G-modules.

FACT Let $G \rightarrow \pi$ be a homomorphism of groups—then every P-local π-module is a P-local G-module.

EXAMPLE Suppose that $1 \to G' \to G \to G'' \to 1$ is a central extension of groups. Let M be a P-local G-module—then $\forall\, n \geq 0$, the action of G'' on $H_n(G';M)$ and $H^n(G';M)$ is P-local.

Given a group G, a $P[G]$-module is a P-local G_P-module. Every $P[G]$-module is a P-local G-module via $l_P : G \to G_P$.

[Note: A G_P-module M is a $P[G]$-module iff the corresponding semidirect product $M \rtimes G_P$ is a P-local group (cf. p. 8–22).]

Example: Suppose that G is nilpotent. Let M be a nilpotent G_P-module which is P-local as an abelian group—then M is a $P[G]$-module.

FACT Let M be a $P[G]$-module—then $H^0(G_P;M) = H^0(G;M)$, i.e., the G_P-invariants in M are equal to the G-invariants in M.

[Let $m \in M^G$. Define homomorphisms $\phi, \psi : G_P \to M \rtimes G_P$ by the rules $\phi(g) = (g \cdot m - m, g)$, $\psi(g) = (0, g) : \phi \circ l_P = \psi \circ l_P \Rightarrow \phi = \psi, M \rtimes G_P$ being P-local, i.e., $m \in M^{G_P}$.]

PROPOSITION 16 Let G be a nilpotent group, M a $P[G]$-module—then $\forall\, n \geq 0$, $H_n(G;M) \approx H_n(G_P;M)$.

[It suffices to treat the case of an abelian G. There are short exact sequences $0 \to \ker l_P \to G \to \operatorname{im} l_P \to 0$, $0 \to \operatorname{im} l_P \to G_P \to \operatorname{coker} l_P \to 0$ and associated LHS spectral sequences. Since $\ker l_P$ is S_P-torsion, $H_q(\ker l_P) \in \mathcal{C}_P$ $(q > 0)$. But the action of $\ker l_P$ on M is by definition trivial, and as an abelian group, M is P-local, thus the universal coefficient theorem implies that $H_q(\ker l_P; M) = 0$ $(q > 0)$. So, $\forall\, n \geq 0$, $H_n(G;M) \approx H_n(\operatorname{im} l_P; M)$. On the other hand, from the above, the action of $\operatorname{coker} l_P$ on the $H_q(\operatorname{im} l_P; M)$ is P-local, hence trivial (cf. Proposition 15). Appealing once again to the universal coefficient theorem, it follows that $H_p(\operatorname{coker} l_P; H_q(\operatorname{im} l_P; M)) = 0$ $(p > 0)$. So, $\forall\, n \geq 0$, $H_n(\operatorname{im} l_P; M) \approx H_n(G_P; M)$.]

FACT Let G be a nilpotent group, M a $P[G]$-module—then $\forall\, n \geq 0$, $H^n(G_P;M) \approx H^n(G;M)$.

EXAMPLE The preceding result can fail if M is not a $P[G]$-module. Thus fix $P \neq \mathbf{\Pi}$ and take $G = \mathbf{Z} : H^2(\mathbf{Z}; \mathbf{Q}[\mathbf{Z}_P]) = 0$ (since $\mathbf{Z}$ has cohomological dimension one) but $H^2(\mathbf{Z}_P; \mathbf{Q}[\mathbf{Z}_P]) \neq 0$ (cf. p. 8–1).

FACT Let G be a finite group—then $\ker l_P$ is S_P-torsion iff $\forall\, n \geq 0$, $H_n(G;M) \approx H_n(G_P;M)$, where M is any $P[G]$-module.

There is another reflective subcategory of **GR** that one can attach to a given $P \subset \mathbf{\Pi}$ whose definition is homological in character. The associated reflector agrees with L_P on **NIL** but differs from L_P on **GR**.

COLIMIT LEMMA Let $\mathbf{C}$ be a cocomplete category with the property that there exists a set $S_0 \subset \operatorname{Ob}\mathbf{C}$ such that each object in $\mathbf{C}$ is a filtered colimit of objects in S_0. Let $F : \mathbf{C} \rightarrow \mathbf{SET}_*$ be a functor which preserves filtered colimits—then there exists a set $K_0 \subset \ker F$ such that each $X \in \ker F$ is a filtered colimit of objects in K_0.

[Note: As the notation suggests, $\ker F = \{X : FX = *\}$.]

Let A be an abelian group—then a homomorphism $f : G \rightarrow K$ of groups is said to be an HA-homomorphism if $f_* : H_1(G;A) \rightarrow H_1(K;A)$ is bijective and $f_* : H_2(G;A) \rightarrow H_2(K;A)$ is surjective. Example: An $H\mathbf{Z}$-homomorphism of nilpotent groups is an isomorphism (cf. p. 5–55).

(HA-Localization) Let $S_{HA} \subset \operatorname{Mor}\mathbf{GR}$ be the class of HA-homomorphisms—then $S_{HA}^{\perp}$ is the object class of a reflective subcategory $\mathbf{GR}_{HA}$ of $\mathbf{GR}$. The reflector $L_{HA} : \begin{cases} \mathbf{GR} \rightarrow \mathbf{GR}_{HA} \\ G \rightarrow G_{HA} \end{cases}$ is called HA-localization and the objects in $\mathbf{GR}_{HA}$ are called the HA-local groups.

[In order to apply the reflective subcategory theorem, it suffices to exhibit a set $S_0 \subset S_{HA} : S_0^{\perp} = S_{HA}^{\perp}$. For this purpose, put $\mathbf{C} = \mathbf{GR}(\rightarrow)$ ($\approx [\mathbf{2}, \mathbf{GR}]$) and let $F : \mathbf{C} \rightarrow \mathbf{SET}_*$ be the functor that sends $f : G \rightarrow K$ to $\ker_1 \oplus \operatorname{coker}_1 \oplus \operatorname{coker}_2$, where $\ker_1$ is the kernel of $f_* : H_1(G;A) \rightarrow H_1(K;A)$ and coker_i is the cokernel of $f_* : H_i(G;A) \rightarrow H_i(K;A)$ $(i = 1,2)$. Owing to the colimit lemma, there exists a set $S_0 \subset S_{HA}$ such that each element of S_{HA} is a filtered colimit of elements in S_0, so $S_0^{\perp} = S_{HA}^{\perp}$.]

[Note: In general, the containment $S_{HA} \subset S_{HA}^{\perp\perp}$ is strict (see below).]

When $A = \mathbf{Z}_P$, the "$\mathbf{Z}$" is dropped from the notation, thus one writes S_{HP} for the class of HP-homomorphisms and $L_{HP} : \begin{cases} \mathbf{GR} \rightarrow \mathbf{GR}_{HP} \\ G \rightarrow G_{HP} \end{cases}$ for the associated reflector, the objects in $\mathbf{GR}_{HP}$ then being referred to as the HP-local groups. Example: Every abelian P-local group is HP-local.

[Note: In the two extreme cases, viz. $P = \emptyset$ or $P = \Pi$, HP is replaced by $H\mathbf{Q}$ or $H\mathbf{Z}$.]

PROPOSITION 17 Every HP-local group is P-local.

[The homomorphisms $\begin{cases} \mathbf{Z} \rightarrow \mathbf{Z} \\ 1 \rightarrow n \end{cases}$ $(n \in S_P)$ are HP-homomorphisms, thus $S_P \subset S_{HP} \Rightarrow S_{HP}^{\perp} = \operatorname{Ob}\mathbf{GR}_{HP} \subset \operatorname{Ob}\mathbf{GR}_P = S_P^{\perp}$.]

Consequently, there is a natural transformation $L_P \rightarrow L_{HP}$.

[Note: For any G, the arrow of localization $l_P : G \rightarrow G_P$ is an HP-homomorphism (cf. p. 9–22). As regards $l_{HP} : G \rightarrow G_{HP}$, it too is an HP-homomorphism (cf. p. 9–23 ff.), although a priori it can only be said that $l_{HP} \in S_{HP}^{\perp\perp}$.]

PROPOSITION 18 Let $f : G \to K$ be an HP-homomorphism—then $\forall\ i \geq 0$, the induced map $(G/\Gamma^i(G))_P \to (K/\Gamma^i(K))_P$ is an isomorphism.

[Taking into account Propositions 6 and 8, one has only to repeat the proof of Proposition 14 in §5.]

LEMMA Let $1 \to G' \to G \to G'' \to 1$ be a central extension of groups. Assume: G' is P-local—then in any commutative diagram

$$\begin{tikzcd} K \arrow[r] \arrow[d, "f"'] & G \arrow[d] \\ L \arrow[r] & G'' \end{tikzcd}$$

of groups, where $f : K \to L$ is an HP-homomorphism, there is a unique lifting

$$\begin{tikzcd} K \arrow[r] \arrow[d, "f"'] & G \arrow[d] \\ L \arrow[r] \arrow[ur, dashed] & G'' \end{tikzcd}$$

rendering the triangles commutative.

[Suppose that $\begin{cases} \phi \\ \psi \end{cases}$ are liftings and $\lambda : L \to G'$ is a homomorphism such that $\phi(l) = \psi(l)\lambda(l)$ $(l \in L)$. Since $\lambda \circ f$ is trivial and $\mathbf{Z}_P \otimes (K/[K,K]) \approx \mathbf{Z}_P \otimes (L/[L,L])$, it follows that λ is trivial, hence $\phi = \psi$, which settles uniqueness. Existence can be established by passing to Eilenberg-MacLane spaces and using obstruction theory (cf. p. 8–38).]

PROPOSITION 19 Let $1 \to G' \to G \to G'' \to 1$ be a central extension of groups. Assume: G' is P-local and G'' is HP-local—then G is HP-local.

[The claim is that $f \perp G$ for every HP-homomorphism $f : K \to L$. This, however, is obviously implied by the lemma.]

[Note: Changing the assumption to G'' is P-local changes the conclusion to G is P-local (but, of course, the proof is different).]

Application: If G is nilpotent, then $G_P \approx G_{HP}$ and $L_P|\mathbf{NIL} \approx L_{HP}|\mathbf{NIL}$.

[Note: It is not necessary to use Proposition 19 to make this deduction. Thus let G be a nilpotent P-local group with $\operatorname{nil} G \leq d$—then for any HP-homomorphism $K \to L$, $\operatorname{Hom}(L, G) \approx \operatorname{Hom}(K, G)$. Proof: $\mathbf{NIL}^d$ is a reflective subcategory of $\mathbf{GR}$, hence $\operatorname{Hom}(L, G) \approx \operatorname{Hom}(L/\Gamma^d(L), G)$, $\operatorname{Hom}(K, G) \approx \operatorname{Hom}(K/\Gamma^d(K), G)$ and $\mathbf{NIL}_P^d$ is a reflective subcategory of $\mathbf{NIL}^d$, hence $\operatorname{Hom}(L/\Gamma^d(L), G) \approx \operatorname{Hom}((L/\Gamma^d(L))_P, G)$, $\operatorname{Hom}(K/\Gamma^d(K), G) \approx \operatorname{Hom}((K/\Gamma^d(K))_P, G)$. And: $(K/\Gamma^d(K))_P \approx (L/\Gamma^d(L))_P$ (cf. Proposition 18).]

FACT Suppose that G is a group such that for some i, $\Gamma^i(G)/\Gamma^{i+1}(G)$ is S_P-torsion—then $G_{HP} \approx (G/\Gamma^i(G))_P$.

[The short exact sequence $1 \to \Gamma^i(G) \to G \to G/\Gamma^i(G) \to 1$ leads to an exact sequence $H_2(G; \mathbf{Z}_P) \to H_2(G/\Gamma^i(G); \mathbf{Z}_P) \to \mathbf{Z}_P \otimes (\Gamma^i(G)/\Gamma^{i+1}(G)) \to H_1(G; \mathbf{Z}_P) \to H_1(G/\Gamma^i(G); \mathbf{Z}_P) \to 0$. Therefore the arrow

$G \to G/\Gamma^i(G)$ is an HP-homomorphism $\Rightarrow G_{HP} \approx (G/\Gamma^i(G))_{HP}$ or still, $G_{HP} \approx (G/\Gamma^i(G))_P$, $G/\Gamma^i(G)$ being nilpotent.]

EXAMPLE The HP-localization of every finite group is nilpotent.

EXAMPLE The HP-localization of every perfect group is trivial. So, if G is perfect and if $H_2(G;\mathbf{Z}_P) \neq 0$, then the arrow $* \to G$ is in $S_{HP}^{\perp\perp}$ but is not in S_{HP}.

FACT The class of HP-homomorphisms admits a calculus of left fractions.

KAN[†] **FACTORIZATION THEOREM** Let $\begin{cases} X \\ Y \end{cases}$ be pointed connected CW spaces, $f : X \to Y$ a pointed continuous function. Assume: $f_* : H_q(X;\mathbf{Z}_P) \to H_q(Y;\mathbf{Z}_P)$ is bijective for $1 \leq q < n$ and surjective for $q = n$—then there exists a pointed connected CW space X_f and pointed continuous functions $\phi_f : X \to X_f$, $\psi_f : X_f \to Y$ with $f = \psi_f \circ \phi_f$ such that $H_*(\phi_f) : H_*(X;\mathbf{Z}_P) \to H_*(X_f;\mathbf{Z}_P)$ is an isomorphism and $\psi_f : X_f \to Y$ is an n-equivalence.

[The case when $n = 1$ is handled by appropriately attaching 1-cells and 2-cells. In general, one iterates the following statement (which can be established by appropriately attaching $(n+1)$-cells and $(n+2)$-cells).

(ST$_n$) Let $\begin{cases} X \\ Y \end{cases}$ be pointed connected CW spaces, $f : X \to Y$ a pointed continuous function. Assume: f is an n-equivalence and $f_* : H_q(X;\mathbf{Z}_P) \to H_q(Y;\mathbf{Z}_P)$ is bijective for $1 \leq q \leq n$ and surjective for $q = n+1$—then there exists a pointed connected CW space X_f and pointed continuous functions $\phi_f : X \to X_f$, $\psi_f : X_f \to Y$ with $f = \psi_f \circ \phi_f$ such that $H_*(\phi_f) : H_*(X;\mathbf{Z}_P) \to H_*(X_f;\mathbf{Z}_P)$ is an isomorphism and $\psi_f : X_f \to Y$ is an $(n+1)$-equivalence.]

Application: Let $f : G \to K$ be a homomorphism of groups. Assume: $f_* : H_1(G;\mathbf{Z}_P) \to H_1(K;\mathbf{Z}_P)$ is surjective—then there exists a factorization $G \xrightarrow{\phi_f} G_f \xrightarrow{\psi_f} K$ of f with ϕ_f an HP-homomorphism and ψ_f surjective.

[Recall that for any pointed path connected space X, there is a surjection $H_2(X;\mathbf{Z}_P) \to H_2(\pi_1(X);\mathbf{Z}_P)$ (cf. p. 5–35).]

EXAMPLE Let $f : G \to K$ be a homomorphism of HP-local groups—then f is surjective iff $f_* : H_1(G;\mathbf{Z}_P) \to H_1(K;\mathbf{Z}_P)$ is surjective.

[To check sufficiency, note that the commutative diagram
$$\begin{tikzcd} G \arrow[r, equal] \arrow[d, "\phi_f"'] & G \arrow[d, "f"] \\ G_f \arrow[r, "\psi_f"'] & K \end{tikzcd}$$
has a filler $G_f \to G$ rendering the triangles commutative.]

[†] In: *Algebra, Topology, and Category Theory*, A. Heller and M. Tierney (ed.), Academic Press (1976), 95–99.

FACT Let $f : G \rightarrow K$ be a homomorphism of HP-local groups—then $\operatorname{im} f$ is HP-local.

Let A be a ring with unit. Fix a right A-module R—then a homomorphism $f : M \rightarrow N$ of left A-modules is said to be an HR-homomorphism provided that $R \otimes_A M \rightarrow R \otimes_A N$ is an isomorphism and $\operatorname{Tor}_1^A(R, M) \rightarrow \operatorname{Tor}_1^A(R, N)$ is an epimorphism.

(HR-Localization) Let $S_{HR} \subset \operatorname{Mor} A\text{-}\mathbf{MOD}$ be the class of HR-homomorphisms—then $S_{HR}^{\perp}$ is the object class of a reflective subcategory $A\text{-}\mathbf{MOD}_{HR}$ of $A\text{-}\mathbf{MOD}$. The reflector $L_{HR} : \begin{cases} A\text{-}\mathbf{MOD} \rightarrow A\text{-}\mathbf{MOD}_{HR} \\ M \rightarrow M_{HR} \end{cases}$ is called HR-localization and the objects in $A\text{-}\mathbf{MOD}_{HR}$ are called the HR-local (left) A-modules.

[Each object in $A\text{-}\mathbf{MOD}$ is κ-definite for some κ. Accordingly, due to the reflective subcategory theorem, one has only to find a set $S_0 \subset S_{HR} : S_0^{\perp} = S_{HR}^{\perp}$, which can be done by using the colimit lemma.]

PROPOSITION 20 $L_{HR} : A\text{-}\mathbf{MOD} \rightarrow A\text{-}\mathbf{MOD}_{HR}$ is an additive functor.

Let G be a group, $A = \mathbf{Z}[G]$ and write $G\text{-}\mathbf{MOD}$ in place of $\mathbf{Z}[G]\text{-}\mathbf{MOD}$. Take $R = \mathbf{Z}$ (trivial G-action)—then a homomorphism $f : M \rightarrow N$ of G-modules is an $H\mathbf{Z}$-homomorphism iff $f_* : H_0(G; M) \rightarrow H_0(G; N)$ is bijective and $f_* : H_1(G; M) \rightarrow H_1(G; N)$ is surjective. The reflector $L_{H\mathbf{Z}} : \begin{cases} G\text{-}\mathbf{MOD} \rightarrow G\text{-}\mathbf{MOD}_{H\mathbf{Z}} \\ M \rightarrow M_{H\mathbf{Z}} \end{cases}$ is called $H\mathbf{Z}$-localization and the objects in $G\text{-}\mathbf{MOD}_{H\mathbf{Z}}$ are called the $H\mathbf{Z}$-local (left) G-modules. Example: Every trivial G-module is $H\mathbf{Z}$-local.

[Note: The arrow of localization $l_{H\mathbf{Z}} : M \rightarrow M_{H\mathbf{Z}}$ is an $H\mathbf{Z}$-homomorphism (cf. p. 9–23 ff.), i.e., $l_{H\mathbf{Z}} \in S_{H\mathbf{Z}} \subset S_{H\mathbf{Z}}^{\perp\perp}$.]

PROPOSITION 21 The $H\mathbf{Z}$ localization of any M in $G\text{-}\mathbf{MOD}$ which is P-local as an abelian group is again P-local: $M = \mathbf{Z}_P \otimes M \Rightarrow M_{H\mathbf{Z}} = \mathbf{Z}_P \otimes M_{H\mathbf{Z}}$.

[This is because $L_{H\mathbf{Z}}$ is an additive functor (cf. Proposition 20).]

SUBLEMMA Suppose that

$$\begin{tikzcd} M \arrow[r] \arrow[d, "f"'] & N \arrow[d, "g"] \\ P \arrow[r] & Q \end{tikzcd}$$

is a pushout square in $G\text{-}\mathbf{MOD}$. Assume: f is an $H\mathbf{Z}$-homomorphism—then g is an $H\mathbf{Z}$-homomorphism.

[There is a commutative diagram

$$\begin{tikzcd} M \arrow[r, "\pi"] \arrow[dr, "f"'] & \overline{M} \arrow[r] \arrow[d] & N \arrow[d, "g"] \\ & P \arrow[r] & Q \end{tikzcd}$$

, where π is surjective and the square is simultaneously a pullback and a pushout in $G\text{-}\mathbf{MOD}$. Observing that the arrow $\overline{M} \rightarrow P$ is an $H\mathbf{Z}$-homomorphism, consider the long exact sequence $H_1(G; \overline{M}) \rightarrow$

$H_1(G;N)\oplus H_1(G;P)\to H_1(G;Q)\to H_0(G;\overline{M})\to H_0(G;N)\oplus H_0(G;P)\to H_0(G;Q)\to 0$.]

LEMMA Let $0\to M'\to M\to M''\to 0$ be a short exact sequence of G-modules. Assume: M' is $H\mathbf{Z}$-local—then in any commutative diagram

$$\begin{tikzcd} P \arrow[r] \arrow[d, "f"'] & M \arrow[d] \\ Q \arrow[r] & M'' \end{tikzcd}$$

of G-modules, where $f:P\to Q$ is an $H\mathbf{Z}$-homomorphism, there is a unique lifting

$$\begin{tikzcd} P \arrow[r] \arrow[d, "f"'] & M \arrow[d] \\ Q \arrow[r] \arrow[ur, dashed] & M'' \end{tikzcd}$$

rendering the triangles commutative.

[Uniqueness is elementary, so we shall deal only with the existence. Define N by the pushout square

$$\begin{tikzcd} P \arrow[r] \arrow[d, "f"'] & M \arrow[d] \\ Q \arrow[r] & N \end{tikzcd}$$

and display the data in a commutative diagram

$$\begin{tikzcd} P \arrow[r] \arrow[d, "f"'] & M \arrow[r, equal] \arrow[d] & M \arrow[d] \\ Q \arrow[r] & N \arrow[r, "\pi"'] & M'' \end{tikzcd}$$

. Put $N'=\ker\pi$, define $\overline{N}$ by the pushout square

$$\begin{tikzcd} N' \arrow[r] \arrow[d] & N \arrow[d] \\ N'_{H\mathbf{Z}} \arrow[r] & \overline{N} \end{tikzcd}$$

, and pass to

$$\begin{tikzcd} 0 \arrow[r] & M' \arrow[r] \arrow[d] & M \arrow[r] \arrow[d] & M'' \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & N' \arrow[r] \arrow[d] & N \arrow[r] \arrow[d] & M'' \arrow[r] \arrow[d, equal] & 0. \\ 0 \arrow[r] & N'_{H\mathbf{Z}} \arrow[r] & \overline{N} \arrow[r] & M'' \arrow[r] & 0 \end{tikzcd}$$

According to the sublemma, the arrows $M\to N$, $N\to\overline{N}$ are $H\mathbf{Z}$-homomorphisms, thus the composite $M'\to N'\to N'_{H\mathbf{Z}}$ is an $H\mathbf{Z}$-homomorphism, hence is an isomorphism (since M' and $N'_{H\mathbf{Z}}$ are $H\mathbf{Z}$-local). Therefore the composite $M\to N\to\overline{N}$ is an isomorphism. Precompose its inverse with the arrow $N\to\overline{N}$ to get a lifting

$$\begin{tikzcd} M \arrow[r, equal] \arrow[d] & M \arrow[d] \\ N \arrow[r, "\pi"'] \arrow[ur, dashed] & M'' \end{tikzcd}$$

, which may then be precomposed with the arrow $Q\to N$ to get a lifting

$$\begin{tikzcd} P \arrow[r] \arrow[d, "f"'] & M \arrow[d] \\ Q \arrow[r] \arrow[ur, dashed] & M'' \end{tikzcd}$$

, as desired.]

PROPOSITION 22 Let $0\to M'\to M\to M''\to 0$ be a short exact sequence of G-modules. Assume: M' and M'' are $H\mathbf{Z}$-local—then M is $H\mathbf{Z}$-local.

Application: Every nilpotent G-module is $H\mathbf{Z}$-local.

[Note: More generally, if M is a G-module such that for some i, $(I[G])^i \cdot M = (I[G])^{i+1} \cdot M$, then $M_{H\mathbf{Z}} \approx M/(I[G])^i \cdot M$. Proof: It follows from the exact sequence $H_1(G;M) \rightarrow H_1(G;M/(I[G])^i \cdot M) \rightarrow (I[G])^i \cdot M/(I[G])^{i+1} \cdot M \rightarrow H_0(G;M) \rightarrow H_0(G;M/(I[G])^i \cdot M) \rightarrow 0$ that the arrow $M \rightarrow M/(I[G])^i \cdot M$ is an $H\mathbf{Z}$-homomorphism. On the other hand, $M/(I[G])^i \cdot M$ is a nilpotent G-module. As for the realizability of the condition, recall that $G/[G,G] \approx I[G]/I[G]^2$, hence G perfect $\Rightarrow I[G] = I[G]^2$ and $G/[G,G]$ divisible + torsion $\Rightarrow I[G]^2 = I[G]^3 = \cdots$.]

FACT The class of $H\mathbf{Z}$-homomorphisms admits a calculus of left fractions.

LEMMA Let $f : M \rightarrow N$ be a homomorphism of G-modules. Assume: $f_* : H_0(G;M) \rightarrow H_0(G;N)$ is surjective—then there exists a factorization $M \xrightarrow{\phi_f} M_f \xrightarrow{\psi_f} N$ of f with ϕ_f an $H\mathbf{Z}$-homomorphism and ψ_f surjective.

[Choose a free G-module P and a surjection $\mu : M \oplus P \rightarrow N$ such that $\mu | M = f$. Since the composite $H_0(G;\ker \mu) \rightarrow H_0(G;M \oplus P) \rightarrow H_0(G;P)$ is surjective and $H_0(G;P)$ is free abelian, one can find a free G-module Q and a homomorphism $\nu : Q \rightarrow \ker \mu$ such that $H_0(G;Q) \approx H_0(G;P)$ through $Q \xrightarrow{\nu} \ker \mu \rightarrow M \oplus P \rightarrow P$. Factor f as $M \xrightarrow{\phi_f} (M \oplus P)/\nu(Q) \xrightarrow{\psi_f} N$, where ϕ_f is induced by the inclusion $M \rightarrow M \oplus P$ and ψ_f is induced by μ.]

PROPOSITION 23 Let $f : M \rightarrow N$ be a homomorphism of $H\mathbf{Z}$-local G-modules—then f is surjective iff $f_* : H_0(G;M) \rightarrow H_0(G;N)$ is surjective.

[To check sufficiency, note that the commutative diagram

$$\begin{tikzcd} M \arrow[r, equal] \arrow[d, "\phi_f"'] & M \arrow[d, "f"] \\ M_f \arrow[r, "\psi_f"'] & N \end{tikzcd}$$

has a filler $M_f \rightarrow M$ rendering the triangles commutative.]

PROPOSITION 24 Let $f : M \rightarrow N$ be a homomorphism of $H\mathbf{Z}$-local G-modules—then $\operatorname{im} f$ is $H\mathbf{Z}$-local.

[Let $\overline{N} \supset f(M)$ be the largest G-submodule of N for which the induced map $H_0(G;f(M)) \rightarrow H_0(G;\overline{N})$ is surjective. There is a commutative triangle

$$\begin{tikzcd} M \arrow[d, "\overline{f}"'] \arrow[dr, "f"] & \\ \overline{N} \arrow[r, "j"'] & N \end{tikzcd}$$

and a factorization $M \xrightarrow{\phi_{\overline{f}}} M_{\overline{f}} \xrightarrow{\psi_{\overline{f}}} \overline{N}$ of $\overline{f}$ with $\phi_{\overline{f}}$ an $H\mathbf{Z}$-homomorphism and $\psi_{\overline{f}}$ surjective. Consider any lifting $M_{\overline{f}} \rightarrow M$ of $j \circ \psi_{\overline{f}}$ to see that $\overline{N} = f(M)$. But $\overline{N}$ is $H\mathbf{Z}$-local.]

PROPOSITION 25 Let $f : M \rightarrow N$ be a homomorphism of $H\mathbf{Z}$-local G-modules—then $\operatorname{coker} f$ is $H\mathbf{Z}$-local.

[Since $\operatorname{im} f$ is $H\mathbf{Z}$-local (cf. Proposition 24), one can assume that f is injective, the claim thus being that N/M is $H\mathbf{Z}$-local. There is a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & M \arrow[r] \arrow[d] & N \arrow[r] \arrow[d, equal] & N/M \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & K \arrow[r] & N \arrow[r] & (N/M)_{H\mathbf{Z}} \arrow[r] & 0
\end{tikzcd}$$

of short exact sequences, where the kernel K is $H\mathbf{Z}$-local. The arrow $M \rightarrow K$ is obviously injective. That it is also surjective can be seen by comparing the exact sequence $H_1(G;N) \rightarrow H_1(G;N/M) \rightarrow H_0(G;M) \rightarrow H_0(G;N) \rightarrow H_0(G;N/M)$ from the first row with its analog from the second row and applying the five lemma: $H_0(G;M) \rightarrow H_0(G;K)$ surjective $\Rightarrow M \rightarrow K$ surjective (cf. Proposition 23). Conclusion: $N/M \approx (N/M)_{H\mathbf{Z}}$.]

[Note: A priori, cokernels in G-$\mathbf{MOD}_{H\mathbf{Z}}$ are calculated first in G-$\mathbf{MOD}$ and then reflected back into G-$\mathbf{MOD}_{H\mathbf{Z}}$. The point of the proposition is that the second step is not needed.]

Application: G-$\mathbf{MOD}_{H\mathbf{Z}}$ is an abelian category and the reflector $L_{H\mathbf{Z}} : G\text{-}\mathbf{MOD} \rightarrow G\text{-}\mathbf{MOD}_{H\mathbf{Z}}$ is right exact.

EXAMPLE Let M be an $H\mathbf{Z}$-local G-module—then $\forall\ n$, $\mathbf{Z}/n\mathbf{Z} \otimes M$ is $H\mathbf{Z}$-local.

EXAMPLE Let $\mathbf{M}$ be a tower in G-$\mathbf{MOD}_{H\mathbf{Z}}$—then $\lim \mathbf{M}$ and $\lim^1 \mathbf{M}$ are $H\mathbf{Z}$-local (cf. p. 5–45).

FACT Let $\mathbf{M}$ be a tower in G-$\mathbf{MOD}_{H\mathbf{Z}}$. Assume: G is finitely generated—then $\lim^1 \mathbf{M} = 0$ iff $\lim^1 H_0(G;\mathbf{M}) = 0$.

[Here, $H_0(G;\mathbf{M})$ stands for the tower determined by the arrows $H_0(G;M_{n+1}) \rightarrow H_0(G;M_n)$. Use Proposition 23 and the fact that G finitely generated $\Rightarrow H_0(G;\prod\limits_n M_n) \approx \prod\limits_n H_0(G;M_n)$ (Brown[†]).]

PROPOSITION 26 Let $G \rightarrow \pi$ be a homomorphism of groups—then every $H\mathbf{Z}$-local π-module is an $H\mathbf{Z}$-local G-module.

[The forgetful functor π-$\mathbf{MOD} \rightarrow G$-$\mathbf{MOD}$ has a left adjoint G-$\mathbf{MOD} \rightarrow \pi$-$\mathbf{MOD}$ that sends M to $\mathbf{Z}[\pi] \otimes_{\mathbf{Z}[G]} M$. Thanks to the change of rings spectral sequence, the homomorphism $H_i(G;M) \rightarrow H_i(\pi;\mathbf{Z}[\pi] \otimes_{\mathbf{Z}[G]} M)$ is bijective for $i = 0$ and surjective for $i = 1$. Therefore an $H\mathbf{Z}$-homomorphism of G-modules goes over to an $H\mathbf{Z}$-homomorphism of π-modules. Suppose now that P is an $H\mathbf{Z}$-local π-module. Let $M \rightarrow N$ be an $H\mathbf{Z}$-homomorphism of G-modules—then the bijectivity of the arrow $\text{Hom}(N, P) \rightarrow \text{Hom}(M, P)$ follows from the bijectivity of the arrow $\text{Hom}(\mathbf{Z}[\pi] \otimes_{\mathbf{Z}[G]} N, P) \rightarrow \text{Hom}(\mathbf{Z}[\pi] \otimes_{\mathbf{Z}[G]} M, P)$.]

[†] *Comment. Math. Helv.* 50 (1975), 129–135; see also Strebel, *Math. Zeit.* 151 (1976), 263–275.

EXAMPLE Let M be an $H\mathbf{Z}$-local G_{HP}-module—then M is an $H\mathbf{Z}$-local G-module.

Although one can consider HA-localization for an arbitrary abelian group A, apart from $A = \mathbf{Z}_P$ the other case of topological significance is when $A = \mathbf{F}_p$. The general aspects of the $H\mathbf{F}_p$-theory are similar to those of the HP-theory. For instance, the analog of Proposition 19 says that if $1 \rightarrow G' \rightarrow G \rightarrow G'' \rightarrow 1$ is a central extension of groups with G' an $\mathbf{F}_p$-module and G'' $H\mathbf{F}_p$-local, then G is $H\mathbf{F}_p$-local.

[Note: An abelian group is a $\mathbf{Z}_P$-module iff it is P-local iff it is HP-local. To perfect the analogy, one can relax the assumption on G' and suppose only that G' is $H\mathbf{F}_p$-local (cf. Proposition 33).]

PROPOSITION 27 Every $H\mathbf{F}_p$-local group is p-local.

EXAMPLE Let G be a finite group—then $G_{H\mathbf{F}_p} \approx G_p$.

The Kan factorization theorem remains valid if $\mathbf{Z}_P$ is replaced by $\mathbf{F}_p$. Therefore a homomorphism $f : G \rightarrow K$ of $H\mathbf{F}_p$-local groups is surjective iff $f_* : H_1(G; \mathbf{F}_p) \rightarrow H_1(K; \mathbf{F}_p)$ is surjective.

The class of $H\mathbf{F}_p$-local abelian groups turns out to be the same as the class of p-cotorsion abelian groups (cf. Proposition 30). It will therefore be convenient to review the theory of the latter starting with the global situation.

An abelian group G is said to be cotorsion if $\text{Hom}(\mathbf{Q}, G) = 0$ & $\text{Ext}(\mathbf{Q}, G) = 0$. Taking into account the exact sequence $\text{Hom}(\mathbf{Q}, G) \rightarrow \text{Hom}(\mathbf{Z}, G) \rightarrow \text{Ext}(\mathbf{Q}/\mathbf{Z}, G) \rightarrow \text{Ext}(\mathbf{Q}, G)$ and making the identification $G \approx \text{Hom}(\mathbf{Z}, G)$, it follows that G is cotorsion iff the arrow $G \rightarrow \text{Ext}(\mathbf{Q}/\mathbf{Z}, G)$ is an isomorphism.

[Note: One motivation for the terminology is that if $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$ is a short exact sequence of abelian groups, then the sequence $0 \rightarrow \text{Hom}(K, A) \rightarrow \text{Hom}(K, B) \rightarrow \text{Hom}(K, C) \rightarrow 0$ is exact for all torsion groups K iff the sequence $0 \rightarrow \text{Hom}(C, L) \rightarrow \text{Hom}(B, L) \rightarrow \text{Hom}(A, L) \rightarrow 0$ is exact for all cotorsion groups L.]

Let $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$ be a short exact sequence of abelian groups—then $0 \rightarrow \text{Hom}(K, A) \rightarrow \text{Hom}(K, B) \rightarrow \text{Hom}(K, C) \rightarrow 0$ is exact $\forall$ torsion K iff $0 \rightarrow \text{Hom}(K, A) \rightarrow \text{Hom}(K, B) \rightarrow \text{Hom}(K, C) \rightarrow 0$ is exact $\forall$ finite cyclic K iff $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$ is pure short exact iff $0 \rightarrow \text{Hom}(C, L) \rightarrow \text{Hom}(B, L) \rightarrow$

Hom$(A, L) \to 0$ is exact $\forall$ finite cyclic L iff $0 \to \text{Hom}(C, L) \to \text{Hom}(B, L) \to \text{Hom}(A, L) \to 0$ is exact $\forall$ cotorsion L.

LEMMA For any abelian group G, $\text{Ext}(\mathbf{Q}/\mathbf{Z}, G)$ is cotorsion.

[Given A, B, C in **AB**, there are isomorphisms

$$\text{Ext}(A, \text{Ext}(B, C)) \approx \text{Ext}(\text{Tor}(A, B), C),$$

$$\text{Ext}(A, \text{Hom}(B, C)) \oplus \text{Hom}(A, \text{Ext}(B, C)) \approx \text{Ext}(A \otimes B, C) \oplus \text{Hom}(\text{Tor}(A, B), C).]$$

LEMMA For any abelian group G, $\text{Ext}(\mathbf{Q}/\mathbf{Z}, \text{Ext}(\mathbf{Q}/\mathbf{Z}, G)) \approx \text{Ext}(\mathbf{Q}/\mathbf{Z}, G)$.

Consequently, the full subcategory of **AB** whose objects are the cotorsion groups is a reflective subcategory of **AB**, the arrow of reflection being $G \to \text{Ext}(\mathbf{Q}/\mathbf{Z}, G)$.

[Note: By comparison, the full subcategory of **AB** whose objects are the torsion groups is a coreflective subcategory of **AB**, the arrow of coreflection being $\text{Tor}(\mathbf{Q}/\mathbf{Z}, G) \to G$.]

EXAMPLE $\mathbf{Z}/n\mathbf{Z}$ is cotorsion but $\mathbf{Z}$ is not cotorsion.

A cotorsion group G is said to be adjusted if G has no torsion free direct summand or, equivalently, if G/G_{tor} is divisible.

COTORSION STRUCTURE LEMMA Suppose that G is cotorsion—then there is a split short exact sequence $0 \to K \to G \to L \to 0$, where $K \approx \text{Ext}(\mathbf{Q}/\mathbf{Z}, G_{\text{tor}})$ is adjusted cotorsion and $L \approx \text{Ext}(\mathbf{Q}/\mathbf{Z}, G/G_{\text{tor}})$ is torsion free cotorsion.

[Note: In the opposite direction, recall that every abelian group is split by its maximal divisible subgroup and the associated quotient is reduced.]

HARRISON'S[†] FIRST THEOREM Let **C** be the full subcategory of **AB** whose objects are the torsion free cotorsion groups; let **D** be the full subcategory of **AB** whose objects are the divisible torsion groups. Define $\Phi : \mathbf{C} \to \mathbf{D}$ by $\Phi G = \mathbf{Q}/\mathbf{Z} \otimes G$; define $\Psi : \mathbf{D} \to \mathbf{C}$ by $\Psi G = \text{Hom}(\mathbf{Q}/\mathbf{Z}, G)$—then the pair (Φ, Ψ) is an adjoint equivalence of categories.

HARRISON'S[†] SECOND THEOREM Let **C** be the full subcategory of **AB** whose objects are the adjusted cotorsion groups; let **D** be the full subcategory of **AB** whose objects are the reduced torsion groups. Define $\Phi : \mathbf{C} \rightarrow \mathbf{D}$ by $\Phi G = \mathrm{Tor}(\mathbf{Q}/\mathbf{Z}, G)$; define $\Psi : \mathbf{D} \rightarrow \mathbf{C}$ by $\Psi G = \mathrm{Ext}(\mathbf{Q}/\mathbf{Z}, G)$—then the pair (Φ, Ψ) is an adjoint equivalence of categories.

[†] *Ann. of Math.* **69** (1959), 366–391.

An abelian group G is said to be p-cotorsion if $\mathrm{Hom}(\mathbf{Z}\left[\frac{1}{p}\right], G) = 0$ & $\mathrm{Ext}(\mathbf{Z}\left[\frac{1}{p}\right], G) = 0$. Taking into account the exact sequence $\mathrm{Hom}(\mathbf{Z}\left[\frac{1}{p}\right], G) \rightarrow \mathrm{Hom}(\mathbf{Z}, G) \rightarrow \mathrm{Ext}(\mathbf{Z}/p^\infty\mathbf{Z}, G)$ $\rightarrow \mathrm{Ext}(\mathbf{Z}\left[\frac{1}{p}\right], G)$ and making the identification $G \approx \mathrm{Hom}(\mathbf{Z}, G)$, it follows that G is p-cotorsion iff the arrow $G \rightarrow \mathrm{Ext}(\mathbf{Z}/p^\infty\mathbf{Z}, G)$ is an isomorphism. Example: $\forall\, n$, $\mathbf{Z}/p^n\mathbf{Z}$ is p-cotorsion.

[Note: The full subcategory of **AB** whose objects are the p-cotorsion groups is a reflective subcategory of **AB** with arrow of reflection $G \rightarrow \mathrm{Ext}(\mathbf{Z}/p^\infty\mathbf{Z}, G)$ and there are evident variants of Harrison's first and second theorems.]

EXAMPLE If $G = \widehat{\mathbf{Z}}_p$, the p-adic integers, then $\widehat{\mathbf{Z}}_p \approx \mathrm{Ext}(\mathbf{Z}/p^\infty\mathbf{Z}, \widehat{\mathbf{Z}}_p)$, hence $\widehat{\mathbf{Z}}_p$ is p-cotorsion. [Note: A subgroup of $\widehat{\mathbf{Z}}_p$ is p-cotorsion iff it is an ideal.]

EXAMPLE The following abelian groups are not p-cotorsion: $\mathbf{Z}/p^\infty\mathbf{Z}$, $\bigoplus_n \mathbf{Z}/p^n\mathbf{Z}$, $\widehat{\mathbf{Z}}_p \otimes \widehat{\mathbf{Z}}_p$.

EXAMPLE For any abelian group G, $\mathrm{Hom}(\mathbf{Z}/p^\infty\mathbf{Z}, G)$ is p-cotorsion. In fact, $\mathrm{Hom}(\mathbf{Z}\left[\frac{1}{p}\right], \mathrm{Hom}(\mathbf{Z}/p^\infty\mathbf{Z}, G)) \approx \mathrm{Hom}(\mathbf{Z}\left[\frac{1}{p}\right] \otimes \mathbf{Z}/p^\infty\mathbf{Z}, G) \approx \mathrm{Hom}(0, G) = 0$ and $\mathrm{Ext}(\mathbf{Z}\left[\frac{1}{p}\right], \mathrm{Hom}(\mathbf{Z}/p^\infty\mathbf{Z}, G)) \approx \mathrm{Ext}(\mathrm{Tor}(\mathbf{Z}\left[\frac{1}{p}\right], \mathbf{Z}/p^\infty\mathbf{Z}), G) \approx \mathrm{Ext}(0, G) = 0$.

FACT Let G be a group and let M be a G-module. Assume: M is $H\mathbf{Z}$-local—then $\mathrm{Ext}(\mathbf{Z}/p^\infty\mathbf{Z}, M)$ is $H\mathbf{Z}$-local.

[The arrow $\mathrm{Ext}(\mathbf{Z}/p^\infty\mathbf{Z}, M) \rightarrow \lim \mathrm{Ext}(\mathbf{Z}/p^n\mathbf{Z}, M)$ is surjective and its kernel can be identified with $\lim^1 \mathrm{Hom}(\mathbf{Z}/p^n\mathbf{Z}, M)$ (Weibel[†]), i.e., there is a short exact sequence $0 \rightarrow \lim^1 \mathrm{Hom}(\mathbf{Z}/p^n\mathbf{Z}, M) \rightarrow \mathrm{Ext}(\mathbf{Z}/p^\infty\mathbf{Z}, M) \rightarrow \lim \mathrm{Ext}(\mathbf{Z}/p^n\mathbf{Z}, M)$. Since $\mathrm{Ext}(\mathbf{Z}/p^n\mathbf{Z}, M) \approx M/p^nM$ and M/p^nM is $H\mathbf{Z}$-local (cf. Proposition 25), $\lim \mathrm{Ext}(\mathbf{Z}/p^n\mathbf{Z}, M)$ must be $H\mathbf{Z}$-local too (G-$\mathbf{MOD}_{H\mathbf{Z}}$ is limit closed). Similar remarks imply that $\lim^1 \mathrm{Hom}(\mathbf{Z}/p^n\mathbf{Z}, M)$ is $H\mathbf{Z}$-local (it is a cokernel (cf. p. 5–45)). Now quote Proposition 22.]

FACT For any abelian group G, the arrow of reflection $G \rightarrow \mathrm{Ext}(\mathbf{Z}/p^\infty\mathbf{Z}, G)$ induces an isomorphism $\mathbf{F}_p \otimes G \rightarrow \mathbf{F}_p \otimes \mathrm{Ext}(\mathbf{Z}/p^\infty\mathbf{Z}, G)$ and an epimorphism $\mathrm{Tor}(\mathbf{F}_p, G) \rightarrow \mathrm{Tor}(\mathbf{F}_p, \mathrm{Ext}(\mathbf{Z}/p^\infty\mathbf{Z}, G))$.

[†] *An Introduction to Homological Algebra*, Cambridge University Press (1994), 85; see also Jensen, *SLN* **254** (1972), 35–37.

[To check the first assertion, observe that $\mathbf{F}_p \otimes G \approx \text{Ext}(\mathbf{F}_p, G) \approx \text{Ext}(\text{Tor}(\mathbf{F}_p, \mathbf{Z}/p^\infty\mathbf{Z}), G) \approx \text{Ext}(\mathbf{F}_p, \text{Ext}(\mathbf{Z}/p^\infty\mathbf{Z}, G)) \approx \mathbf{F}_p \otimes \text{Ext}(\mathbf{Z}/p^\infty\mathbf{Z}, G)$.]

Notation: Given an abelian group G, $\text{div}\, G$ is the maximal divisible subgroup of G and $\text{div}_p\, G$ is the maximal p-divisible subgroup of G.

[Note: The kernel of the arrow of reflection $G \rightarrow \text{Ext}(\mathbf{Z}/p^\infty\mathbf{Z}, G)$ is $\text{div}_p\, G$.]

PROPOSITION 28 Suppose that G is cotorsion—then $G \approx \prod\limits_p G_p$, where $G_p = \bigcap\limits_{q \neq p} \text{div}_q\, G$ is the maximal p-cotorsion subgroup of G.

[The point here is that $\text{Ext}(\mathbf{Q}/\mathbf{Z}, G) \approx \prod\limits_p \text{Ext}(\mathbf{Z}/p^\infty\mathbf{Z}, G)$.]

[Note: This result is the analog for a cotorsion group of the primary decomposition of a torsion group.]

LEMMA If A and G are abelian groups with G p-cotorsion, then (i) $A \otimes \mathbf{F}_p = 0 \Rightarrow \text{Hom}(A, G) = 0$ and (ii) $\text{Tor}(A, \mathbf{F}_p) = 0 \Rightarrow \text{Ext}(A, G) = 0$.

[To check the second assertion, observe that $\text{Ext}(A, G) \approx \text{Ext}(A, \text{Ext}(\mathbf{Z}/p^\infty\mathbf{Z}, G)) \approx \text{Ext}(\text{Tor}(A, \mathbf{Z}/p^\infty\mathbf{Z}), G) \approx \text{Ext}(0, G) = 0$.]

PROPOSITION 29 Let $\begin{cases} X \\ Y \end{cases}$ be path connected topological spaces, $f : X \rightarrow Y$ a continuous function—then $f_* : H_*(X; \mathbf{F}_p) \rightarrow H_*(Y; \mathbf{F}_p)$ is an isomorphism iff $f^* : H^*(Y; G) \rightarrow H^*(X; G)$ is an isomorphism for all p-cotorsion abelian groups G.

[By passing to the mapping cylinder, one can assume that f is an inclusion. If $\forall\, n \geq 1$, $H_n(Y, X; \mathbf{F}_p) = 0$, then $\forall\, n \geq 1$, $H_n(Y, X) \otimes \mathbf{F}_p = 0$ and $\text{Tor}(H_n(Y, X), \mathbf{F}_p) = 0$. So, from the lemma, for any p-cotorsion G, $\text{Hom}(H_n(Y, X), G) = 0$ and $\text{Ext}(H_n(Y, X), G) = 0 \;\forall\, n \geq 1$, thus $H^n(Y, X; G) = 0 \;\forall\, n \geq 1$. To reverse the argument, specialize and take $G = \mathbf{F}_p$.]

In the context of HR-localization, take $A = \mathbf{Z}$ and $R = \mathbf{F}_p$—then the object class of the corresponding reflective subcategory of $\mathbf{Z}$-$\mathbf{MOD} \approx \mathbf{AB}$ is the class of p-cotorsion groups.

PROPOSITION 30 Let G be an abelian group—then G is $H\mathbf{F}_p$-local iff G is p-cotorsion.

[Let $S_1 \subset \text{Mor}\, \mathbf{AB}$ be the class of homomorphisms $f : A \rightarrow B$ such that $A \otimes \mathbf{F}_p \rightarrow B \otimes \mathbf{F}_p$ is an isomorphism and $\text{Tor}(A, \mathbf{F}_p) \rightarrow \text{Tor}(B, \mathbf{F}_p)$ is an epimorphism (thus $S_1^\perp$ is the class of p-cotorsion groups) and let $S_2 \subset \text{Mor}\, \mathbf{AB}$ be the class of homomorphisms $f : A \rightarrow B$ such that $f_* : H_1(A; \mathbf{F}_p) \rightarrow H_1(B; \mathbf{F}_p)$ is bijective and $f_* : H_2(A; \mathbf{F}_p) \rightarrow H_2(B; \mathbf{F}_p)$ is surjective (thus $S_2^\perp$ is the class of abelian $H\mathbf{F}_p$-local groups) (cf. infra). Claim: $S_1 = S_2$. For, in either case, $A/pA \approx B/pB$. This said, consider the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & H_2(A)\otimes \mathbf{F}_p \arrow[r] \arrow[d] & H_2(A;\mathbf{F}_p) \arrow[r] \arrow[d] & \mathrm{Tor}(A,\mathbf{F}_p) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & H_2(B)\otimes \mathbf{F}_p \arrow[r] & H_2(B;\mathbf{F}_p) \arrow[r] & \mathrm{Tor}(B,\mathbf{F}_p) \arrow[r] & 0
\end{tikzcd}$$

of short exact sequences. Since $\begin{cases} H_2(A)\otimes \mathbf{F}_p \approx \wedge^2(A/pA) \\ H_2(B)\otimes \mathbf{F}_p \approx \wedge^2(B/pB) \end{cases}$ (Brown[†]), the five lemma implies that if $\mathrm{Tor}(A,\mathbf{F}_p) \to \mathrm{Tor}(B,\mathbf{F}_p)$ is an epimorphism, then $f_* : H_2(A;\mathbf{F}_p) \to H_2(B;\mathbf{F}_p)$ is surjective. The converse is trivial.]

The reflective subcategory theorem is applicable to **AB**, so one can define the notion "abelian $H\mathbf{F}_p$-local group" internally. That this is the same as "abelian $+H\mathbf{F}_p$-local" is a consequence of the following lemma.

LEMMA An $H\mathbf{F}_p$-homomorphism $G \to K$ of groups induces an $H\mathbf{F}_p$-homomorphism $G/[G,G] \to K/[K,K]$ of abelian groups.

Given a group G, let $\rho_p : G^\omega \to G^\omega$ be the function defined by $\rho_p(g_0, g_1, \ldots) = (g_0 g_1^{-p}, g_1 g_2^{-p}, \ldots)$.

PROPOSITION 31 Suppose that G is abelian—then ρ_p is a homomorphism and $\ker \rho_p \approx \lim \mathbf{G}_p \approx \mathrm{Hom}(\mathbf{Z}\left[\frac{1}{p}\right], G)$, $\mathrm{coker}\, \rho_p \approx \lim^1 \mathbf{G}_p \approx \mathrm{Ext}(\mathbf{Z}\left[\frac{1}{p}\right], G)$, where $\mathbf{G}_p$ is the tower $\cdots \leftarrow G \xleftarrow{p} G \leftarrow \cdots$.

[Representing $\mathbf{Z}\left[\frac{1}{p}\right]$ as a colimit $\cdots \to \mathbf{Z} \xrightarrow{p} \mathbf{Z} \to \cdots$ gives $\lim \mathbf{G}_p \approx \mathrm{Hom}(\mathbf{Z}\left[\frac{1}{p}\right], G)$ and, from the short exact sequence $0 \to \lim^1 \mathrm{Hom}(\mathbf{Z}, G) \to \mathrm{Ext}(\mathbf{Z}\left[\frac{1}{p}\right], G) \to \lim \mathrm{Ext}(\mathbf{Z}, G) \to 0$ (Weibel[‡]), one has $\lim^1 \mathbf{G}_p \approx \mathrm{Ext}(\mathbf{Z}\left[\frac{1}{p}\right], G)$.]

Application: An abelian group G is p-cotorsion ($= H\mathbf{F}_p$-local) iff $\lim \mathbf{G}_p = 0$ & $\lim^1 \mathbf{G}_p = 0$, i.e., iff ρ_p is bijective.

[†] *Cohomology of Groups*, Springer Verlag (1982), 126.

[‡] *An Introduction to Homological Algebra*, Cambridge University Press (1994), 85; see also Jensen, *SLN* **254** (1972), 35–37.

Let G be a group—then G is said to be p-cotorsion provided that ρ_p is bijective. Claim: The full subcategory of **GR** whose objects are the p-cotorsion groups is a reflective subcategory of **GR**. To see this, let F_ω be the free group on generators $x_0, x_1, \ldots,$ define a homomorphism $f : F_\omega \to F_\omega$ by $f(x_i) = x_i x_{i+1}^{-p}$ and consider $f^\perp$ (reflective subcategory theorem).

FACT Suppose that G is p-cotorsion—then $\operatorname{Cen} G$ is p-cotorsion.

PROPOSITION 32 Every $H\mathbf{F}_p$-local group is p-cotorsion.

[It is enough to prove that $f : F_\omega \to F_\omega$ is an $H\mathbf{F}_p$-homomorphism. But $f_* : H_1(F_\omega; \mathbf{F}_p) \to H_1(F_\omega; \mathbf{F}_p)$ is the identity $\omega \cdot \mathbf{F}_p \to \omega \cdot \mathbf{F}_p$ and $H_2(F_\omega; \mathbf{F}_p) \approx H_2(F_\omega) \otimes \mathbf{F}_p \oplus \operatorname{Tor}(H_1(F_\omega), \mathbf{F}_p)$ vanishes.]

The abelian p-cotorsion theory has been extended to **NIL** by Huber-Warfield[†]. Thus the full subcategory of **NIL** whose objects are the p-cotorsion groups is a reflective subcategory of **NIL**. It is traditional to denote the arrow of reflection by $G \to \operatorname{Ext}(\mathbf{Z}/p^\infty\mathbf{Z}, G)$ even though the "Ext" has no a priori connection with extensions of G by $\mathbf{Z}/p^\infty\mathbf{Z}$. One reason for this is that each short exact sequence $1 \to G' \to G \to G'' \to 1$ of nilpotent groups gives rise to an exact sequence $0 \to \operatorname{Hom}(\mathbf{Z}/p^\infty\mathbf{Z}, G') \to \operatorname{Hom}(\mathbf{Z}/p^\infty\mathbf{Z}, G) \to \operatorname{Hom}(\mathbf{Z}/p^\infty\mathbf{Z}, G'') \to \operatorname{Ext}(\mathbf{Z}/p^\infty\mathbf{Z}, G') \to \operatorname{Ext}(\mathbf{Z}/p^\infty\mathbf{Z}, G) \to \operatorname{Ext}(\mathbf{Z}/p^\infty\mathbf{Z}, G'') \to 0$.

[Note: It is reasonable to conjecture that the p-cotorsion reflector in **GR** extends the p-cotorsion reflector in **NIL** but I know of no proof.]

The p-cotorsion reflector in **NIL** respects $\mathbf{NIL}^d$: $\operatorname{nil} \operatorname{Ext}(\mathbf{Z}/p^\infty\mathbf{Z}, G) \leq \operatorname{nil} G$, hence its restriction to **AB** "is" the p-cotorsion reflector in **AB**.

Notation: Given a nilpotent group G, $\operatorname{div} G$ is the maximal divisible subgroup of G and $\operatorname{div}_p G$ is the maximal p-divisible subgroup of G.

[Note: The kernel of the arrow of reflection $G \to \operatorname{Ext}(\mathbf{Z}/p^\infty\mathbf{Z}, G)$ is $\operatorname{div}_p G$.]

LEMMA For any nilpotent group G, $\operatorname{Hom}(\mathbf{Z}/p^\infty\mathbf{Z}, G)$ is a torsion free p-cotorsion abelian group.

[Let $G_{\text{tor}}(p)$ be the maximal p-torsion subgroup of G—then $\operatorname{div} G_{\text{tor}}(p)$ is abelian and the range of every homomorphism $f : \mathbf{Z}/p^\infty\mathbf{Z} \to G$ is contained in $\operatorname{div} G_{\text{tor}}(p)$.]

[Note: Therefore G p-cotorsion $\Rightarrow \operatorname{Hom}(\mathbf{Z}/p^\infty\mathbf{Z}, G) = 0$.]

FACT Let G be a nilpotent group—then the arrow $g \to g^p$ is bijective iff $\operatorname{Hom}(\mathbf{Z}/p^\infty\mathbf{Z}, G) = 0$ & $\operatorname{Ext}(\mathbf{Z}/p^\infty\mathbf{Z}, G) = 0$ or still, iff $\forall\, n > 0$, $H_n(G; \mathbf{F}_p) = 0$.

[†] *J. Algebra* **74** (1982), 402–442.

EXAMPLE There is a short exact sequence $0 \to \mathbf{Z} \to \widehat{\mathbf{Z}}_p \to \widehat{\mathbf{Z}}_p/\mathbf{Z} \to 0$ and $\widehat{\mathbf{Z}}_p/\mathbf{Z}$ is uniquely p-divisible, hence $H_*(\mathbf{Z};\mathbf{F}_p) \approx H_*(\widehat{\mathbf{Z}}_p;\mathbf{F}_p)$ (cf. p. 4–44).

FACT Let $1 \to G' \to G \to G'' \to 1$ be a short exact sequence of nilpotent groups. Assume: Two of the groups are p-cotorsion—then so is the third.

EXAMPLE Suppose that G is nilpotent and p-cotorsion—then $G/\text{Cen}\, G$ is p-cotorsion.

[$\text{Cen}\, G$ is necessarily p-cotorsion (cf. p. 8–37).]

LEMMA Let $1 \to G' \to G \to G'' \to 1$ be a central extension of groups. Assume: G' is $H\mathbf{F}_p$-local—then in any commutative diagram

$$\begin{tikzcd} K \arrow[r] \arrow[d, "f"'] & G \arrow[d] \\ L \arrow[r] & G'' \end{tikzcd}$$

of groups, where $f : K \to L$ is an $H\mathbf{F}_p$-homomorphism, there is a unique lifting

$$\begin{tikzcd} K \arrow[r] \arrow[d, "f"'] & G \arrow[d] \\ L \arrow[r] \arrow[ur, dashed] & G'' \end{tikzcd}$$

rendering the triangles commutative.

[Put $\begin{cases} X = K(K,1) \\ Y = K(L,1) \end{cases}$ and consider the diagram

$$\begin{tikzcd} X \arrow[r] \arrow[d, "f"'] & K(G,1) \arrow[d] \\ Y \arrow[r] \arrow[ur, dashed] & K(G'',1) \end{tikzcd}$$

. Supposing, as we may, that f is an inclusion, the obstruction to lifting lies in $H^2(Y,X;G')$. Claim: $H^2(Y,X;G') = 0$. To verify this, look at the short exact sequence $0 \to \text{Ext}(H_1(Y,X),G') \to H^2(Y,X;G') \to \text{Hom}(H_2(Y,X),G') \to 0$. Since $f_* : H_1(X;\mathbf{F}_p) \to H_1(Y;\mathbf{F}_p)$ is bijective and $f_* : H_2(X;\mathbf{F}_p) \to H_2(Y;\mathbf{F}_p)$ is surjective, $H_2(Y,X) \otimes \mathbf{F}_p = 0$ and $\text{Tor}(H_1(Y,X),\mathbf{F}_p) = 0$. But G' is $H\mathbf{F}_p$-local or still, p-cotorsion (cf. Proposition 30), thus $\text{Hom}(H_2(Y,X),G') = 0$ and $\text{Ext}(H_1(Y,X),G') = 0$ (see the lemma preceding Proposition 29). Therefore $H^2(Y,X;G') = 0$ and the lifting exists. As for its uniqueness, of necessity $H_1(Y,X;\mathbf{F}_p) = 0$, i.e., $H_1(Y,X) \otimes \mathbf{F}_p = 0$, thus $H^1(Y,X;G') \approx \text{Hom}(H_1(Y,X),G') = 0$.]

PROPOSITION 33 Let $1 \to G' \to G \to G'' \to 1$ be a central extension of groups. Assume: G' is $H\mathbf{F}_p$-local and G'' is $H\mathbf{F}_p$-local—then G is $H\mathbf{F}_p$-local.

[The proof is the same as that of Proposition 19.]

Application: If G is nilpotent and p-cotorsion, then G is $H\mathbf{F}_p$-local.

[In fact, $\text{Cen}\, G$ and $G/\text{Cen}\, G$ are p-cotorsion, so one can proceed by induction.]

PROPOSITION 34 Let G be a p-cotorsion nilpotent group—then there exists a central series $G = C^0(G) \supset C^1(G) \supset \cdots$ having the same length as the descending central series of G such that $\forall\, i$, $C^i(G)/C^{i+1}(G)$ is a p-cotorsion abelian group.

[Define $C^i(G)$ to be the kernel of the composite $G \rightarrow G/\Gamma^i(G) \rightarrow \text{Ext}(\mathbf{Z}/p^\infty\mathbf{Z}, G/\Gamma^i(G))$.]

[Note: Here is a variant. Let G be a group. Let M be a nilpotent G-module, $\chi : G \rightarrow \text{Aut}\, M$ the associated homomorphism. Assume: M is p-cotorsion—then there exists a finite filtration $M = C_\chi^0(M) \supset C_\chi^1(M) \supset \cdots \supset C_\chi^d(M) = \{0\}$ of M by G-submodules $C_\chi^i(M)$ such that $\forall\ i, G$ operates trivially on $C_\chi^i(M)/C_\chi^{i+1}(M)$ and $C_\chi^i(M)/C_\chi^{i+1}(M)$ is p-cotorsion.]

§ 8

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§9. HOMOTOPICAL LOCALIZATION

Localization at a set of primes is a powerful tool in commutative algebra and group theory, thus it should come as no surprise that the transcription of this process to algebraic topology is of fundamental importance. More generally, one can interpret "localization" as the search for and construction of reflective subcategories in a homotopy category.

EXAMPLE **HCW** is not a reflective subcategory of **HTOP**. Reason: **HCW** is not isomorphism closed. **HCWSP** is not a reflective subcategory of **HTOP**. Reason: **HCWSP** is not limit closed (e.g., the product $\prod_1^\infty \mathbf{S}^n$ is not a CW space). On the other hand, **HCWSP** is a coreflective subcategory of **HTOP**, the coreflector being the functor that assigns to each topological space X the geometric realization of its singular set (the arrow of adjunction $|\sin X| \rightarrow X$ is a weak homotopy equivalence (Giever-Milnor theorem)). In particular: **HCWSP** has products, viz. the product of $\{X_i\}$ in **HCWSP** is $|\sin \prod_i X_i|$, where $\prod_i X_i$ is the product in **HTOP** (or still, the product in **TOP**).

[Note: Analogous remarks apply in the pointed setting. So, e.g., the nth homotopy group of $\prod_i X_i$ (taken in $\mathbf{HCWSP}_*$) is isomorphic to $\prod_i \pi_n(X_i)$.]

Notation: $\mathbf{CONCWSP}_*$ is the full subcategory of $\mathbf{CWSP}_*$ whose objects are the pointed connected CW spaces and $\mathbf{HCONCWSP}_*$ is the associated homotopy category.

EXAMPLE Write $\mathbf{HCONCWSP}_*[n]$ for the full subcategory of $\mathbf{HCONCWSP}_*$ whose objects have trivial homotopy groups in dimension $> n$ ($n \geq 0$)—then $\mathbf{HCONCWSP}_*[n]$ is a reflective subcategory of $\mathbf{HCONCWSP}_*$, the reflector being the functor that assigns to each X its nth Postnikov approximate $X[n]$. Example: The fundamental group functor $X \rightarrow \pi_1(X)$ sets up an equivalence between $\mathbf{HCONCWSP}_*[1]$ and **GR**.

[Note: The data generates an orthogonal pair (S, D). Here, $[f] : X \rightarrow Y$ is in S iff $f_* : \pi_q(X) \rightarrow \pi_q(Y)$ is bijective for $q \leq n$.]

EXAMPLE Write $\mathbf{HSCONCWSP}_*$ for the full subcategory of $\mathbf{HCONCWSP}_*$ whose objects are simply connected—then $\mathbf{HSCONCWSP}_*$ is not a reflective subcategory of $\mathbf{HCONCWSP}_*$. For suppose it were and, to get a contradiction, take $X = \mathbf{P}^2(\mathbf{R})$. Consider, in the notation of p. 0–22, $\epsilon_X : X \rightarrow TX$. By definition, $\epsilon_X \perp K(\mathbf{Z}, 2) \Rightarrow H^2(TX) \approx H^2(X) \approx \mathbf{Z}/2\mathbf{Z}$. But $H_1(TX) = 0 \Rightarrow H^2(TX) \approx \operatorname{Hom}(H_2(TX), \mathbf{Z})$, which is torsion free.

[Note: Let $f : \mathbf{S}^1 \rightarrow *$—then $f^\perp$ is the object class of $\mathbf{HSCONCWSP}_*$.]

Given a set of primes P, a pointed connected CW space X is said to be P-local in homotopy if $\forall\, n \geq 1$, $\pi_n(X)$ is P-local.

EXAMPLE Fix $P \neq \mathbf{\Pi}$—then the full subcategory of $\mathbf{HCONCWSP}_*$ whose objects are P-local in homotopy is not the object class of a reflective subcategory of $\mathbf{HCONCWSP}_*$. To see this, suppose the opposite and consider $\mathbf{S}^1$. Calling its localization $\mathbf{S}^1_P$, for any P-local group G, the universal arrow $l_P : \mathbf{S}^1 \rightarrow \mathbf{S}^1_P$ necessarily induces a bijection $[\mathbf{S}^1_P, K(G,1)] \approx [\mathbf{S}^1, K(G,1)] \Rightarrow \mathrm{Hom}(\pi_1(\mathbf{S}^1_P), G) \approx \mathrm{Hom}(\pi_1(\mathbf{S}^1), G)$. Since $\pi_1(\mathbf{S}^1_P)$ is by definition P-local, it follows that $\pi_1(\mathbf{S}^1_P) \approx \mathbf{Z}_P$. Form now $K(\mathbf{Q}[\mathbf{Z}_P], 2; \chi)$, where $\chi : \mathbf{Z}_P \rightarrow \mathrm{Aut}\, \mathbf{Q}[\mathbf{Z}_P]$ is the homomorphism corresponding to the action of $\mathbf{Z}_P$ on $\mathbf{Q}[\mathbf{Z}_P]$. Since $K(\mathbf{Q}[\mathbf{Z}_P], 2; \chi)$ is P-local, the bijection $[\mathbf{S}^1_P, K(\mathbf{Q}[\mathbf{Z}_P], 2; \chi)] \approx [\mathbf{S}^1, K(\mathbf{Q}[\mathbf{Z}_P], 2; \chi)]$ restricts to an isomorphism $H^2(\mathbf{S}^1_P; \mathbf{Q}[\mathbf{Z}_P]) \approx H^2(\mathbf{S}^1; \mathbf{Q}[\mathbf{Z}_P])$ (cf. p. 5–34) (locally constant coefficients), thus $H^2(\mathbf{S}^1_P; \mathbf{Q}[\mathbf{Z}_P]) = 0$. But $H^2(\pi_1(\mathbf{S}^1_P); \mathbf{Q}[\mathbf{Z}_P])$ embeds in $H^2(\mathbf{S}^1_P; \mathbf{Q}[\mathbf{Z}_P])$ (consider the spectral sequence $E_2^{p,q} \approx H^p(\pi_1(\mathbf{S}^1_P); H^q(\widetilde{\mathbf{S}}^1_P; \mathbf{Q}[\mathbf{Z}_P])) \Rightarrow H^{p+q}(\mathbf{S}^1_P; \mathbf{Q}[\mathbf{Z}_P])$), which contradicts the fact that $H^2(\mathbf{Z}_P; \mathbf{Q}[\mathbf{Z}_P]) \neq 0$ (cf. p. 8–1).

[Note: Let $\rho^q_n : \mathbf{S}^q \rightarrow \mathbf{S}^q$ $(q \geq 1)$ be a map of degree n $(n \in S_P)$. Working in $\mathbf{HCONCWSP}_*$, put $S_0 = \{[\rho^q_n]\}$—then $S_0^\perp$ is the class of objects in $\mathbf{HCONCWSP}_*$ which are P-local in homotopy.]

Given integers $k, n > 1$, let $k : \mathbf{S}^{n-1} \rightarrow \mathbf{S}^{n-1}$ be a map of degree k—then the adjunction space $\mathbf{P}^n(k) = \mathbf{D}^n \sqcup_k \mathbf{S}^{n-1}$ is a Moore space of type $(\mathbf{Z}/k\mathbf{Z}, n-1)$ and $\Sigma \mathbf{P}^n(k) = \mathbf{P}^{n+1}(k)$.

Given a pointed connected CW space X, the $\underline{n^{\text{th}} \bmod k \text{ homotopy group}}$ of X is $[\mathbf{P}^n(k), X]$, the set of pointed homotopy classes of pointed continuous functions $\mathbf{P}^n(k) \rightarrow X$. Notation: $\pi_n(X; \mathbf{Z}/k\mathbf{Z})$. Here, the language is slightly deceptive. While it is true that $\pi_n(X; \mathbf{Z}/k\mathbf{Z})$ is a group if $n > 2$ (which is abelian if $n > 3$), $\pi_2(X; \mathbf{Z}/k\mathbf{Z})$ is merely a pointed set (but there is a left action $\pi_2(X) \times \pi_2(X; \mathbf{Z}/k\mathbf{Z}) \rightarrow \pi_2(X; \mathbf{Z}/k\mathbf{Z})$). In the event that $\pi_1(X)$ is abelian, put $\pi_1(X; \mathbf{Z}/k\mathbf{Z}) = \pi_1(X) \otimes \mathbf{Z}/k\mathbf{Z}$.

[Note: When X is an H space, $\pi_2(X; \mathbf{Z}/k\mathbf{Z})$ is a group (and $\pi_n(X; \mathbf{Z}/k\mathbf{Z})$ is abelian if $n > 2$).]

A pointed continuous function $f : X \rightarrow Y$ between pointed connected CW spaces induces a map $f_* : \pi_n(X; \mathbf{Z}/k\mathbf{Z}) \rightarrow \pi_n(Y; \mathbf{Z}/k\mathbf{Z})$. It is a homomorphism if $n > 2$ and respects the action of π_2 if $n = 2$.

UNIVERSAL COEFFICIENT THEOREM For each $n > 1$, there is a functorial exact sequence $0 \rightarrow \pi_n(X) \otimes \mathbf{Z}/k\mathbf{Z} \rightarrow \pi_n(X; \mathbf{Z}/k\mathbf{Z}) \rightarrow \mathrm{Tor}(\pi_{n-1}(X), \mathbf{Z}/k\mathbf{Z}) \rightarrow 0$.

[The arrows $\mathbf{S}^{n-1} \xrightarrow{k} \mathbf{S}^{n-1}$, $\mathbf{S}^{n-1} \rightarrow \mathbf{P}^n(k) \rightarrow \mathbf{S}^n$, $\mathbf{S}^n \xrightarrow{k} \mathbf{S}^n$ generate a functorial exact sequence $\pi_n(X) \xrightarrow{k} \pi_n(X) \rightarrow \pi_n(X; \mathbf{Z}/k\mathbf{Z}) \rightarrow \pi_{n-1}(X) \xrightarrow{k} \pi_{n-1}(X)$.]

[Note: If $n = 2$, interpret exactness in $\mathbf{SET}_*$ and if $\pi_1(X)$ is not abelian, interpret $\mathrm{Tor}(\pi_1(X), \mathbf{Z}/k\mathbf{Z})$ as the kernel of $\pi_1(X) \xrightarrow{k} \pi_1(X)$.]

Example: Let X be a pointed connected CW space—then X is P-local in homotopy iff $\pi_1(X)$ is P-local and $\forall\, p \in \overline{P}$, $\pi_n(X;\mathbf{Z}/p\mathbf{Z}) = 0\ \forall\, n > 1$.

[Apply REC_2 of the recognition principle (cf. p. 8–4 ff.).]

Neisendorfer[†] has established a mod k analog of the Hurewicz theorem.

MOD k HUREWICZ THEOREM Suppose that X is a pointed abelian CW space—then if $n \geq 2$, the condition $\pi_q(X;\mathbf{Z}/k\mathbf{Z}) = 0$ $(1 \leq q < n)$ is equivalent to the condition $H_q(X;\mathbf{Z}/k\mathbf{Z}) = 0$ $(1 \leq q < n)$ and either implies that the Hurewicz map $\pi_n(X;\mathbf{Z}/k\mathbf{Z}) \rightarrow H_n(X;\mathbf{Z}/k\mathbf{Z})$ is bijective.

[Note: The arrow $\mathbf{P}^n(k) \rightarrow \mathbf{S}^n$ induces an isomorphism $H_n(\mathbf{P}^n(k);\mathbf{Z}/k\mathbf{Z}) \rightarrow H_n(\mathbf{S}^n;\mathbf{Z}/k\mathbf{Z})$, so there is a generator of $H_n(\mathbf{P}^n(k);\mathbf{Z}/k\mathbf{Z})$ that is sent to the canonical generator of $H_n(\mathbf{S}^n;\mathbf{Z}/k\mathbf{Z})$, from which the Hurewicz map $\pi_n(X;\mathbf{Z}/k\mathbf{Z}) \rightarrow H_n(X;\mathbf{Z}/k\mathbf{Z})$ (it is a homomorphism if $n > 2$).]

The mod k analog of the Whitehead theorem is also true (consult Suslin[‡] for a variant with applications to algebraic K-theory).

Given a set of primes P, a pointed connected CW space X is said to be P-local in homology if $\forall\, n \geq 1$, $H_n(X)$ is P-local.

[Note: X is P-local in homology iff $\forall\, p \in \overline{P}$, $H_n(X;\mathbf{Z}/p\mathbf{Z}) = 0\ \forall\, n \geq 1$ (cf. p. 8–6).]

EXAMPLE Fix $P \neq \Pi$—then there exists a pointed connected CW space X such that $\forall\, n \geq 2$, $\pi_n(X) \approx \mathbf{Z}$ and $\forall\, n \geq 1$, $H_n(X) \approx \mathbf{Z}_P$ (cf. p. 5–77), so P-local in homology need not imply P-local in homotopy.

[Note: In the other direction, P-local in homotopy need not imply P-local in homology. Reason: There exists a P-local group G such that $G/[G,G]$ $(\approx H_1(G))$ has an S_P-torsion direct summand (cf. p. 8–12), e.g., $G = (\mathbf{Z} * \mathbf{Z})_P$.]

PROPOSITION 1 Let $\left\{\begin{matrix} X \\ Y \end{matrix}\right.$ be pointed nilpotent CW spaces, $f : X \rightarrow Y$ a pointed continuous function. Assume: $\forall\, n \geq 1$, $f_* : \pi_n(X) \rightarrow \pi_n(Y)$ is P-localizing—then $\forall\, n \geq 1$, $f_* : H_n(X) \rightarrow H_n(Y)$ is P-localizing.

[There is a commutative diagram

$$\begin{tikzcd} \widetilde{X} \arrow[r] \arrow[d, "\widetilde{f}"'] & X \arrow[d, "f"] \\ \widetilde{Y} \arrow[r] & Y \end{tikzcd}$$

and a morphism $\{E^2_{p,q} \approx H_p(\pi_1(X); H_q(\widetilde{X}))\} \rightarrow \{\overline{E}^2_{p,q} \approx H_p(\pi_1(Y); H_q(\widetilde{Y}))\}$ of fibration spectral sequences. Since $\left\{\begin{matrix} \widetilde{X} \\ \widetilde{Y} \end{matrix}\right.$ are simply connected, $\forall\, q \geq 1$, $\widetilde{f}_* : H_q(\widetilde{X}) \rightarrow H_q(\widetilde{Y})$ is P-localizing (cf. p. 8–7). In addition, $\forall\, q \geq 1$, $\begin{cases} \pi_1(X) \\ \pi_1(Y) \end{cases}$ operates nilpotently on $\begin{cases} H_q(\widetilde{X}) \\ H_q(\widetilde{Y}) \end{cases}$ (cf. §5, Proposition 17), thus $\forall\, q \geq 1$, the arrow $E^2_{p,q} \rightarrow \overline{E}^2_{p,q}$ is P-localizing (cf. §8, Proposition 14). Recalling that $\forall\, p \geq 1$, the arrow $H_p(\pi_1(X)) \rightarrow H_p(\pi_1(Y))$ is P-localizing (cf. §8, Proposition 10), one can pass through the spectral sequence to see that $\forall\, q \geq 1$, $f_* : H_q(X) \rightarrow H_q(Y)$ is P-localizing.]

[†] *Memoirs Amer. Math. Soc.* **232** (1980), 1–67.

[‡] *J. Pure Appl. Algebra* **34** (1984), 301–318.

Application: Let X be a pointed nilpotent CW space. Assume: X is P-local in homotopy—then X is P-local in homology.

[Note: The converse is also true (cf. p. 9–6).]

PROPOSITION 2 Let $\begin{cases} X \\ Y \end{cases}$ be pointed nilpotent CW spaces, $f : X \rightarrow Y$ a pointed continuous function. Assume: $\forall\, n \geq 1$, $f_* : H_n(X) \rightarrow H_n(Y)$ is P-localizing—then for any pointed nilpotent CW space Z which is P-local in homotopy, the precomposition arrow $f^* : [Y, Z] \rightarrow [X, Z]$ is bijective.

[There is no loss of generality in supposing that $\begin{cases} X \\ Y \end{cases}$ are pointed nilpotent CW complexes with X a pointed subcomplex of Y (take f skeletal and replace Y by the pointed mapping cylinder of f). Because the inclusion $X \rightarrow Y$ is a cofibration, this reduction converts the problem into one that can be treated by obstruction theory. Thus given a pointed continuous function $\phi : X \rightarrow Z$, the obstructions to extending ϕ to a pointed continuous function $\Phi : Y \rightarrow Z$ and the obstructions to any two such being homotopic $\operatorname{rel} X$ (hence pointed homotopic) lie in the $H^p(Y, X; \Gamma^i_{X_q}(\pi_q(Z))/\Gamma^{i+1}_{X_q}(\pi_q(Z)))$ for certain p and q (nilpotent obstruction theorem). The claim is that these groups are trivial. But, by hypothesis, $\forall\, n \geq 1$, $f_* : H_n(X; \mathbf{Z}_P) \rightarrow H_n(Y; \mathbf{Z}_P)$ is an isomorphism, hence $\forall\, n \geq 1$, $H_n(Y, X; \mathbf{Z}_P) = 0$. Since $\mathbf{Z}_P$ is a principal ideal domain and since the $\Gamma^i_{X_q}(\pi_q(Z))/\Gamma^{i+1}_{X_q}(\pi_q(Z))$ are $\mathbf{Z}_P$-modules (cf. p. 8–21), the universal coefficient theorem implies that the obstructions to existence and uniqueness do indeed vanish.]

[Note: Otherwise said, under the stated conditions, $[f] \perp Z$ for any pointed nilpotent CW space Z which is P-local in homotopy.]

Notation: $\mathbf{NILCWSP}_*$ is the full subcategory of $\mathbf{CWSP}_*$ whose objects are the pointed nilpotent CW spaces and $\mathbf{HNILCWSP}_*$ is the associated homotopy category, while $\mathbf{NILCWSP}_{*,P}$ is the full subcategory of $\mathbf{NILCWSP}_*$ whose objects are the pointed nilpotent CW spaces which are P-local in homotopy and $\mathbf{HNILCWSP}_{*,P}$ is the associated homotopy category.

NILPOTENT P-LOCALIZATION THEOREM $\mathbf{HNILCWSP}_{*,P}$ is a reflective subcategory of $\mathbf{HNILCWSP}_*$.

[On general grounds, it is a question of assigning to each X in $\mathbf{HNILCWSP}_*$ an object X_P in $\mathbf{HNILCWSP}_{*,P}$ and a pointed homotopy class $[l_P] : X \to X_P$ with the property that for any pointed homotopy class $[f] : X \to Y$, where Y is in $\mathbf{HNILCWSP}_{*,P}$, there exists a unique pointed homotopy class $[\phi] : X_P \to Y$ such that $[f] = [\phi] \circ [l_P]$. In view of Propositions 1 and 2, it will be enough to construct a pair $(X_P, l_P) : \forall\, q \geq 1, \pi_q(l_P) : \pi_q(X) \to \pi_q(X_P)$ is P-localizing. For this, we shall work first with the nth Postnikov approximate $X[n]$ of X and produce $(X[n]_P, l_P)$ inductively. Matters being plain if $n = 0$ ($X[0]$ is contractible), take $n > 0$. Consider a principal refinement of order n of the arrow $X[n] \to X[n-1]$, i.e., a factorization $X[n] \xrightarrow{\Lambda} W_N \xrightarrow{q_N} W_{N-1} \to \cdots \to W_1 \xrightarrow{q_1} W_0 = X[n-1]$, where Λ is a pointed homotopy equivalence and each $q_i : W_i \to W_{i-1}$ is a pointed Hurewicz fibration for which there is an abelian group π_i and a pointed continuous function $\Phi_{i-1} : W_{i-1} \to K(\pi_i, n+1)$ such that the diagram
$$\begin{tikzcd} W_i \arrow[r] \arrow[d, "q_i"'] & \Theta K(\pi_i, n+1) \arrow[d] \\ W_{i-1} \arrow[r, "\Phi_{i-1}"'] & K(\pi_i, n+1) \end{tikzcd}$$
is a pullback square. To exhibit pairs $(W_{i,P}, l_P)$ (and hence produce $(X[n]_P, l_P)$), one can proceed via recursion on $i > 0$, the existence of $(W_{0,P}, l_P)$ being secured by the induction hypothesis. Choose a filler $\Phi_{i-1,P} : W_{i-1,P} \to K(\pi_{i,P}, n+1)$ for
$$\begin{tikzcd} W_{i-1} \arrow[r] \arrow[d] & K(\pi_i, n+1) \arrow[d] \\ W_{i-1,P} \arrow[r, dashed] & K(\pi_{i,P}, n+1) \end{tikzcd}$$
and define $W_{i,P}$ by the pullback square
$$\begin{tikzcd} W_{i,P} \arrow[r] \arrow[d] & \Theta K(\pi_{i,P}, n+1) \arrow[d] \\ W_{i-1,P} \arrow[r, "\Phi_{i-1,P}"'] & K(\pi_{i,P}, n+1) \end{tikzcd}$$
. Since the composite $W_i \to W_{i-1} \to W_{i-1,P} \to K(\pi_{i,P}, n+1)$ is nullhomotopic, there is a filler $l_P : W_i \to W_{i,P}$ for
$$\begin{tikzcd} W_i \arrow[r] \arrow[d, dashed] & W_{i-1} \arrow[d] \\ W_{i,P} \arrow[r] & W_{i-1,P} \end{tikzcd}$$
. From the definitions, $\left\{\begin{array}{l} W_i \\ W_{i,P} \end{array}\right.$ is a pointed connected CW space homeomorphic to $\left\{\begin{array}{l} E_{\Phi_{i-1}} \\ E_{\Phi_{i-1,P}} \end{array}\right.$ (parameter reversal). Moreover, $\left\{\begin{array}{l} W_i \\ W_{i,P} \end{array}\right.$ is nilpotent (cf. §5, Proposition 15) and by comparing the homotopy sequences of $\left\{\begin{array}{l} \Phi_i \\ \Phi_{i,P} \end{array}\right.$ one finds that $\forall\, q \geq 1$, $\pi_q(l_P) : \pi_q(W_i) \to \pi_q(W_{i,P})$ is P-localizing. Recall now that $\forall\, n$, there is a pointed homotopy equivalence $X[n] \to P_n X$ and a pointed Hurewicz fibration $P_n X \to P_{n-1} X$ (cf. p. 5–41). Passing to mapping tracks and changing l_P within its pointed homotopy class, one can always arrange that $\forall\, n$, the arrow $(P_n X)_P \to (P_{n-1} X)_P$ is a pointed Hurewicz

fibration and the diagram
$$\begin{tikzcd} P_nX \arrow[r] \arrow[d] & P_{n-1}X \arrow[d] \\ (P_nX)_P \arrow[r] & (P_{n-1}X)_P \end{tikzcd}$$
commutes. So, $\lim l_P : \lim P_nX \rightarrow \lim(P_nX)_P$ exists and $\forall\ q \geq 1$, $\pi_q(\lim l_P) : \pi_q(\lim P_nX) \rightarrow \pi_q(\lim(P_nX)_P)$ is P-localizing (cf. p. 5–50). Fix a CW resolution $X_P \rightarrow \lim(P_nX)_P$ and let $l_P : X \rightarrow X_P$ be a filler for
$$\begin{tikzcd} X \arrow[r] \arrow[d, dashed] & \lim P_nX \arrow[d] \\ X_P \arrow[r] & \lim(P_nX)_P \end{tikzcd}$$
(cf. §5, Proposition 4). Because the arrow $X \rightarrow \lim P_nX$ is a weak homotopy equivalence (cf. §5, Proposition 13), it follows that $\forall\ q \geq 1$, $\pi_q(l_P) : \pi_q(X) \rightarrow \pi_q(X_P)$ is P-localizing.]

The reflector L_P figuring in the nilpotent P-localization theorem sends X to X_P (special cases: $X_{\mathbf{Q}}, X_p$ $(p \in \Pi)$) with arrow of localization $[l_P] : X \rightarrow X_P$. Brackets are often omitted, e.g., given $f : X \rightarrow Y$, there is a diagram
$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d] & Y \arrow[d] \\ X_P \arrow[r, "f_P"'] & Y_P \end{tikzcd}$$
, commutative up to pointed homotopy.

[Note: L_P respects the "abelian subcategory" and the "simply connected subcategory".]

Let $[f] : X \rightarrow Y$ be a morphism in $\mathbf{HNILCWSP}_*$—then $[f]$ (or f) is said to be P-localizing if $\exists$ an isomorphism $[\phi] : X_P \rightarrow Y$ such that $[f] = [\phi] \circ [l_P]$ (cf. p. 0–30).

PROPOSITION 3 Let $\begin{cases} X \\ Y \end{cases}$ be pointed nilpotent CW spaces, $f : X \rightarrow Y$ a pointed continuous function—then f is P-localizing iff $\forall\ n \geq 1$, $f_* : \pi_n(X) \rightarrow \pi_n(Y)$ is P-localizing.

[This is implicit in the proof of the nilpotent P-localization theorem.]

Example: For any nilpotent group G, $K(G,1)_P \approx K(G_P, 1)$.

PROPOSITION 4 Let $\begin{cases} X \\ Y \end{cases}$ be pointed nilpotent CW spaces, $f : X \rightarrow Y$ a pointed continuous function—then f is P-localizing iff $\forall\ n \geq 1$, $f_* : H_n(X) \rightarrow H_n(Y)$ is P-localizing.

[The point behind the sufficiency is that $\forall\ n \geq 1$, $H_n(Y)$ is P-local, therefore Dror's Whitehead theorem implies that $l_P : Y \rightarrow Y_P$ is a pointed homotopy equivalence, thus Y is P-local in homotopy.]

Application: Let X be a pointed nilpotent CW space. Assume: X is P-local in homology—then X is P-local in homotopy.

[Note: The converse is also true (cf. p. 9–4).]

FACT Let P' and P'' be two sets of primes—then for any pointed nilpotent CW space X, $(X_{P'})_{P''} \approx (X_{P''})_{P'}$.

[The left hand side computes $X_{P' \cap P''}$ and the right hand side computes $X_{P'' \cap P'}$.]

The nilpotent P-localization theorem has been relativized by Llerena[†]. In fact, suppose that $\left\{\begin{matrix} X \\ Y \end{matrix}\right.$ & Z are pointed connected CW spaces. Let $f : X \to Y$ be a pointed Hurewicz fibration with E_f nilpotent—then there exists a pointed connected CW space $X(P)$, a pointed Hurewicz fibration $f(P) : X(P) \to Y$ with $E_{f(P)}$ nilpotent and P-local in homotopy, and a pointed continuous function $l(P) : X \to X(P)$ over Y such that the induced map $E_f \to E_{f(P)}$ is P-localizing: $(E_f)_P \approx E_{f(P)}$. In addition, for any pointed Hurewicz fibration $g : Z \to Y$ with E_g nilpotent and P-local in homotopy, $[f(P), g] \approx [f, g]$ in the sense of pointed fiber homotopy, i.e., given a commutative triangle

$$\begin{tikzcd} X \arrow[rr, "\phi"] \arrow[dr, "f"'] & & Z \arrow[dl, "g"] \\ & Y & \end{tikzcd},$$

there is a commutative triangle

$$\begin{tikzcd} X(P) \arrow[rr, "\phi(P)"] \arrow[dr, "f(P)"'] & & Z \arrow[dl, "g"] \\ & Y & \end{tikzcd}$$

: $[\phi] = [\phi(P)] \circ [l(P)]$, $\phi(P)$ being unique up to pointed fiber homotopy.

EXAMPLE Let X be a pointed connected CW space—then the diagram

$$\begin{tikzcd} \widetilde{X} \arrow[r] \arrow[d, "l_P"'] & X \arrow[r] \arrow[d] & K(\pi_1(X), 1) \arrow[d, equal] \\ \widetilde{X}_P \arrow[r] & X(P) \arrow[r] & K(\pi_1(X), 1) \end{tikzcd}$$

commutes in **HCONCWSP$_*$** (cf. p. 5–62). Here, $\pi_1(X) \approx \pi_1(X(P))$ and $\forall\ n \geq 2$, the arrow $\pi_n(X) \to \pi_n(X(P))$ is P-localizing.

Nilpotent P-localization is compatible with homotopy and homology in that $\forall\ n \geq 1$, $\pi_n(X)_P \approx \pi_n(X_P)$ and $H_n(X)_P \approx H_n(X_P)$ but this is false for cohomology. Example: Take $X = \mathbf{S}^n : \mathbf{S}^n_P = M(\mathbf{Z}_P, n) \Rightarrow H^{n+1}(\mathbf{S}^n_P) \approx \text{Ext}(\mathbf{Z}_P, \mathbf{Z}) \neq 0\ (P \neq \mathbf{\Pi})$.

[Note: By contrast, taking coefficients in $\mathbf{Z}_P$, $\forall\ n \geq 1$, $H^n(X_P; \mathbf{Z}_P) \approx H^n(X; \mathbf{Z}_P)$ (cf. §8, Proposition 2).]

Let $[f] : X \to Y$ be a morphism in **HNILCWSP$_*$**—then $[f]$ (or f) is said to be a P-equivalence if $f_P : X_P \to Y_P$ is a pointed homotopy equivalence. With regard to the underlying orthogonal pair (S, D), $[f]$ is a P-equivalence iff $[f] \in S$, so $[f]$ is P-localizing iff $[f] \in S$ & $Y \in D$ (cf. p. 0–30).

[†] *Math. Zeit.* **188** (1985), 397–410.

[Note: When $P = \emptyset$, the term is rational equivalence. Examples: (1) There is a rational equivalence $\mathbf{S}^3 \to K(\mathbf{Z},3)$ but there is no rational equivalence $K(\mathbf{Z},3) \to \mathbf{S}^3$; (2) There are rational equivalences $\mathbf{S}^3 \vee \mathbf{S}^5 \to \mathbf{S}^3 \vee K(\mathbf{Z},5), \mathbf{S}^3 \vee K(\mathbf{Z},5) \to K(\mathbf{Z},3) \vee K(\mathbf{Z},5), \mathbf{S}^3 \vee \mathbf{S}^5 \to K(\mathbf{Z},3) \vee \mathbf{S}^5$, $K(\mathbf{Z},3) \vee \mathbf{S}^5 \to K(\mathbf{Z},3) \vee K(\mathbf{Z},5)$ but there are no rational equivalences $\mathbf{S}^3 \vee K(\mathbf{Z},5) \to K(\mathbf{Z},3) \vee \mathbf{S}^5$, $K(\mathbf{Z},3) \vee \mathbf{S}^5 \to \mathbf{S}^3 \vee K(\mathbf{Z},5)$.]

PROPOSITION 5 Let $\begin{cases} X \\ Y \end{cases}$ be pointed nilpotent CW spaces, $f : X \to Y$ a pointed continuous function—then f is a P-equivalence iff $f_* : H_*(X;\mathbf{Z}_P) \to H_*(Y;\mathbf{Z}_P)$ is an isomorphism.

[Note: This holds iff $f_* : H_*(X;\mathbf{Q}) \to H_*(Y;\mathbf{Q})$ is an isomorphism and $\forall\ p \in P$, $f_* : H_*(X;\mathbf{Z}/p\mathbf{Z}) \to H_*(Y;\mathbf{Z}/p\mathbf{Z})$ is an isomorphism (cf. §8, Proposition 3).]

Example: Fix a positive integer d. Let P_d be the set of primes that do not divide d—then $\mathbf{S}^n \xrightarrow{d} \mathbf{S}^n$ is a P_d-equivalence.

EXAMPLE (Local Spheres) Given P, let $p_1 < p_2 < \cdots$ be an enumeration of the elements of $\overline{P}$ and put $d_k = p_1^k \cdots p_k^k$ $(k = 1,2,\ldots)$—then a model for $\mathbf{S}_P^n$ is the pointed mapping telescope of the sequence $\mathbf{S}^n \to \mathbf{S}^n \to \cdots$, the kth map having degree d_k. Since $\mathbf{Q}$ is P-local, $H^*(\mathbf{S}_P^n;\mathbf{Q}) \approx H^*(\mathbf{S}^n;\mathbf{Q})$. Accordingly, $\mathbf{S}_P^n$ cannot be an H space if n is even (Hopf). As for what happens when n is odd, Adams[†] has shown that if $2 \notin P$, then $\mathbf{S}_P^n$ is an H space while if $2 \in P$, then $\mathbf{S}_P^n$ is an H space iff $n = 1, 3$, or 7.

EXAMPLE (Rational Spheres) If n is odd, then $\mathbf{S}_{\mathbf{Q}}^n = K(\mathbf{Q},n)$ but if n is even, then $\mathbf{S}_{\mathbf{Q}}^n = E_f$, where $f : K(\mathbf{Q},n) \to K(\mathbf{Q},2n)$ corresponds to $t^2 \in H^{2n}(\mathbf{Q},n;\mathbf{Q})$ $(H^*(\mathbf{Q},n;\mathbf{Q}) = \mathbf{Q}[t], |t| = n)$. Consequently, if n is odd, then $\mathbf{Q} \otimes \pi_q(\mathbf{S}^n) = \begin{cases} \mathbf{Q} & (q = n) \\ 0 & (q \neq n) \end{cases}$ but if n is even, then $\mathbf{Q} \otimes \pi_q(\mathbf{S}^n) = \begin{cases} \mathbf{Q} & (q = n, 2n-1) \\ 0 & (q \neq n, 2n-1) \end{cases}$ (cf. p. 5–44).

PROPOSITION 6 Let $\begin{cases} X \\ Y \end{cases}$ be pointed nilpotent CW spaces, $f : X \to Y$ a pointed continuous function. Suppose that f is a P-equivalence—then for any $P' \subset P$, f is a P'-equivalence.

PROPOSITION 7 Let $\begin{cases} X \\ Y \end{cases}$ be pointed nilpotent CW spaces, $f : X \to Y$ a pointed continuous function. Suppose that f is a P'-equivalence and a P''-equivalence—then f is a $(P' \cup P'')$-equivalence.

[†] *Quart. J. Math.* **12** (1961), 52–60.

FACT Let X be a pointed nilpotent CW space. Fix P—then for any $P' \subset P$, the canonical arrow $X_P \rightarrow X_{P'}$ is a $(P' \cup \overline{P})$-equivalence.

EXAMPLE Let $\begin{cases} X \\ Y \end{cases}$ be pointed nilpotent CW spaces. Assume: $\exists$ a pointed homotopy equivalence $\phi : X_{\mathbf{Q}} \rightarrow Y_{\mathbf{Q}}$—then there is a pointed nilpotent CW space Z such that $\begin{cases} Z_P \approx X_P \\ Z_{\overline{P}} \approx Y_{\overline{P}} \end{cases}$.

[Choose $r_P : X_P \rightarrow X_{\mathbf{Q}}$ & $r_{\overline{P}} : Y_{\overline{P}} \rightarrow Y_{\mathbf{Q}} : l_{\mathbf{Q}} \simeq r_P \circ l_P$ $(l_{\mathbf{Q}} : X \rightarrow X_{\mathbf{Q}})$ & $l_{\mathbf{Q}} \simeq r_{\overline{P}} \circ l_{\overline{P}}$ $(l_{\mathbf{Q}} : Y \rightarrow Y_{\mathbf{Q}})$. The double mapping track Z of the pointed 2-sink $X_P \xrightarrow{\phi \circ r_P} Y_{\mathbf{Q}} \xleftarrow{r_{\overline{P}}} Y_{\overline{P}}$ is a pointed CW space (cf. §6, Proposition 8). To check that Z is path connected (hence nilpotent (cf. p. 5–59)), fix $\gamma \in \pi_1(Y_{\mathbf{Q}})$. Since $\phi \circ r_P$ is a $\overline{P}$-equivalence and $r_{\overline{P}}$ is a P-equivalence, $\exists\ m \in S_{\overline{P}} : \gamma^m = (\phi \circ r_P)_* (\alpha)$ $(\alpha \in \pi_1(X_P))$ & $\exists\ n \in S_P : \gamma^n = (r_{\overline{P}})_* (\beta)$ $(\beta \in \pi_1(Y_{\overline{P}}))$. But m and n are relatively prime, so $\exists$ k and $l : km + ln = 1 \Rightarrow \gamma = (\phi \circ r_P)_* (\alpha^k) \cdot (r_{\overline{P}})_* (\beta^l)$, which means that Z is path connected (cf. p. 4–37). And: $\begin{cases} Z \rightarrow X_P \\ Z \rightarrow Y_{\overline{P}} \end{cases}$ is a $\begin{cases} P\text{-equivalence} \\ \overline{P}\text{-equivalence} \end{cases}$.]

PROPOSITION 8 Let $\begin{cases} X \\ Y \end{cases}$ be pointed nilpotent CW spaces, $f : X \rightarrow Y$ a pointed continuous function—then f is a pointed homotopy equivalence provided that $\forall\ p$, $f_p : X_p \rightarrow Y_p$ is a pointed homotopy equivalence.

[In fact, $\forall\ p$, $H_*(f)_p : H_*(X)_p \rightarrow H_*(Y)_p$ is an isomorphism. Therefore f is a homology equivalence (cf. p. 8–3) and Dror's Whitehead theorem is applicable.]

In the simply connected situation, there is another approach to P-localization which depends on Proposition 2 but not on Proposition 1. Thus let X be a pointed simply connected CW space—then it will be enough to construct a pair $(X_P, l_P) : \forall\ q \geq 1$, $H_q(l_P) : H_q(X) \rightarrow H_q(X_P)$ is P-localizing and for this one can assume that X is a pointed simply connected CW complex.

Observation: A model for X_P, where $X = \bigvee_I \mathbf{S}^n$ $(n > 1)$, is a Moore space of type $(I \cdot \mathbf{Z}_P, n) : X_P = \bigvee_I M(\mathbf{Z}_P, n)$.

$(\dim X < \infty)$ If $\dim X = 2$, then X has the pointed homotopy type of a wedge $\bigvee_I \mathbf{S}^2$, hence (X_P, l_P) exists in this case. Proceeding by induction on the dimension, suppose that (X_P, l_P) has been constructed for all X with $\dim X \leq n$ $(n \geq 2)$ and consider an X with $\dim X = n + 1$. Up to pointed homotopy type, X is the pointed mapping cone C_f of a pointed continuous function $f : \bigvee_I \mathbf{S}^n \rightarrow X^{(n)}$ $(\#(I) = \#(\mathcal{E}_{n+1}))$ and the pointed cofibration $j : X^{(n)} \rightarrow C_f$ is a cofibration (cf. §3, Proposition 19). Choose a filler f_P :

$\bigvee_I \mathbf{S}_P^n \to X_P^{(n)}$ for

$$\begin{tikzcd}
\bigvee_I \mathbf{S}^n \arrow[r] \arrow[d] & X^{(n)} \arrow[d] \\
\bigvee_I \mathbf{S}_P^n \arrow[r, dashed] & X_P^{(n)}
\end{tikzcd}$$

. Since the composite $\bigvee_I \mathbf{S}^n \to X^{(n)} \to X_P^{(n)} \to C_{f_P}$ is nullhomotopic, there is a filler $l_P : C_f \to C_{f_P}$ for

$$\begin{tikzcd}
X^{(n)} \arrow[r] \arrow[d] & C_f \arrow[d, dashed] \\
X_P^{(n)} \arrow[r] & C_{f_P}
\end{tikzcd}$$

. Assembling the data leads to a commutative diagram

$$\begin{tikzcd}
\widetilde{H}_q(\bigvee_I \mathbf{S}^n) \arrow[r] \arrow[d] & \widetilde{H}_q(X^{(n)}) \arrow[r] \arrow[d] & \widetilde{H}_q(C_f) \arrow[r] \arrow[d] & \widetilde{H}_{q-1}(\bigvee_I \mathbf{S}^n) \arrow[r] \arrow[d] & \widetilde{H}_{q-1}(X^{(n)}) \arrow[d] \\
\widetilde{H}_q(\bigvee_I \mathbf{S}_P^n) \arrow[r] & \widetilde{H}_q(X_P^{(n)}) \arrow[r] & \widetilde{H}_q(C_{f_P}) \arrow[r] & \widetilde{H}_{q-1}(\bigvee_I \mathbf{S}_P^n) \arrow[r] & \widetilde{H}_{q-1}(X_P^{(n)})
\end{tikzcd}$$

of abelian groups with exact rows, where both vertical arrows on either side of the arrow $\widetilde{H}_q(C_f) \to \widetilde{H}_q(C_{f_P})$ are P-localizing. But this means that $\widetilde{H}_q(C_f) \to \widetilde{H}_q(C_{f_P})$ is P-localizing as well (cf. p. 8–6).

($\dim X = \infty$) One can arrange matters in such a way that $\forall\, n$, the diagram

$$\begin{tikzcd}
X^{(n)} \arrow[r] \arrow[d] & X^{(n+1)} \arrow[d] \\
X_P^{(n)} \arrow[r] & X_P^{(n+1)}
\end{tikzcd}$$

is commutative and the arrow $X_P^{(n)} \to X_P^{(n+1)}$ is a cofibration. Put $X_P = \operatorname{colim} X_P^{(n)}$ (cf. §5, Proposition 8) and define $l_P : X \to X_P$ in the obvious fashion.

FACT Let X & $\begin{cases} Y \\ Z \end{cases}$ be pointed simply connected CW spaces with finitely generated homotopy groups. Suppose that $g : Y \to Z$ is a rational equivalence—then g induces a bijection $[X_{\mathbf{Q}}, Y] \to [X_{\mathbf{Q}}, Z]$.

[Assuming that X is a pointed simply connected CW complex, construct $X_{\mathbf{Q}}$ as above, and show by induction that $\forall\, n$, $[X_{\mathbf{Q}}^{(n)}, Y] \approx [X_{\mathbf{Q}}^{(n)}, Z]$.]

EXAMPLE (Phantom Maps) The notion of phantom map, as defined on p. 5–90 for pointed connected CW complexes, extends to pointed connected CW spaces $\begin{cases} X \\ Y \end{cases}$: $\operatorname{Ph}(X, Y)$. This said, let $\begin{cases} X \\ Y \end{cases}$ be pointed simply connected CW spaces with finitely generated homotopy groups—then $\operatorname{Ph}(X, Y) = l_{\mathbf{Q}}^*[X_{\mathbf{Q}}, Y] \subset [X, Y]$ (cf. p. 11–6). For instance, take $X = \Omega\mathbf{S}^3$, $Y = \mathbf{S}^3$. To compute $[\Omega\mathbf{S}^3, \mathbf{S}^3]$, note first that $\Sigma\Omega\mathbf{S}^3 \approx \Sigma\Omega\Sigma\mathbf{S}^2 \approx \Sigma(\bigvee_{n\geq 1} \mathbf{S}^{2n}) \approx \bigvee_{n\geq 1} \mathbf{S}^{2n+1}$ (cf. §4, Proposition 28 and subsequent discussion) and $\mathbf{S}^3 \approx \Omega B_{\mathbf{S}3}^{\infty}$ (cf. p. 4–65), hence $[\Omega\mathbf{S}^3, \mathbf{S}^3] \approx [\Omega\mathbf{S}^3, \Omega B_{\mathbf{S}3}^{\infty}] \approx [\Sigma\Omega\mathbf{S}^3, B_{\mathbf{S}3}^{\infty}] \approx [\bigvee_{n\geq 1} \mathbf{S}^{2n+1}, B_{\mathbf{S}3}^{\infty}] \approx$

$\prod_{n\geq 1}[\mathbf{S}^{2n+1}, B^{\infty}_{\mathbf{S}^3}] \approx \prod_{n\geq 1}[\mathbf{S}^{2n}, \mathbf{S}^3]$. By the same token, $[(\Omega\mathbf{S}^3)_{\mathbf{Q}}, \mathbf{S}^3] \approx [\Omega(\mathbf{S}^3_{\mathbf{Q}}), \mathbf{S}^3] \approx \prod_{n\geq 1}[\mathbf{S}^{2n}_{\mathbf{Q}}, \mathbf{S}^3]$ or still, $\approx \prod_{n\geq 1}[\mathbf{S}^{2n}_{\mathbf{Q}}, K(\mathbf{Z}, 3)]$, the arrow $\mathbf{S}^3 \rightarrow K(\mathbf{Z}, 3)$ being a rational equivalence. Conclusion: $\mathrm{Ph}(\Omega\mathbf{S}^3, \mathbf{S}^3) = 0$.

LEMMA Let $\begin{cases} X \\ Y \end{cases}$ be pointed connected CW spaces, $f : X \rightarrow Y$ a pointed Hurewicz fibration with $\pi_0(X_{y_0}) = *$—then there is an exact sequence $\cdots \rightarrow \pi_{n+1}(Y; \mathbf{Z}/k\mathbf{Z}) \rightarrow \pi_n(X_{y_0}; \mathbf{Z}/k\mathbf{Z}) \rightarrow \pi_n(X; \mathbf{Z}/k\mathbf{Z}) \rightarrow \pi_n(Y; \mathbf{Z}/k\mathbf{Z}) \rightarrow \cdots \rightarrow \pi_2(Y; \mathbf{Z}/k\mathbf{Z})$.

EXAMPLE Let $\begin{cases} X \\ Y \end{cases}$ be pointed simply connected CW spaces with finitely generated homotopy groups, $f : X \rightarrow Y$ a pointed continuous function—then f is a p-equivalence iff $\forall\, n \geq 2$, $f_* : \pi_n(X; \mathbf{Z}/p\mathbf{Z}) \rightarrow \pi_n(Y; \mathbf{Z}/p\mathbf{Z})$ is bijective.

(Products) Let $\begin{cases} X \\ Y \end{cases}$ be pointed nilpotent CW spaces—then $(X \times Y)_P \approx X_P \times Y_P$.

EXAMPLE (H Spaces) Suppose that X is a path connected H space—then X_P is a path connected H space and the arrow of localization $l_P : X \rightarrow X_P$ is an H map.

(Mapping Fibers) Let $\begin{cases} X \\ Y \end{cases}$ be pointed nilpotent CW spaces, $f : X \rightarrow Y$ a pointed continuous function. Assume: E_f is nilpotent—then $(E_f)_P \approx E_{f_P}$.

[Since $\pi_0(E_f) = *$, the arrow $\pi_1(X) \rightarrow \pi_1(Y)$ is surjective, thus the same is true of the arrow $\pi_1(X)_P \rightarrow \pi_1(Y)_P$ or still, of the arrow $\pi_1(X_P) \rightarrow \pi_1(Y_P)$. Therefore $\pi_0(E_{f_P}) = *$ and E_{f_P} is nilpotent (cf. p. 5–58). Compare the long exact sequences in homotopy.]

Application: Let (K, k_0) be a pointed finite connected CW complex. Suppose that $f : X \rightarrow Y$ is P-localizing—then for any pointed continuous function $\phi : K \rightarrow X$, the arrow $C(K, k_0; X, x_0 : \phi) \rightarrow C(K, k_0; Y, y_0 : f \circ \phi)$ is P-localizing.

[Note: $C(\cdots : \phi), C(\cdots : f \circ \phi)$ stand for the path component to which ϕ, $f \circ \phi$ belong (cf. p. 5–58 ff.).]

Example: Given a pointed nilpotent CW space X, $(\Omega_0 X)_P \approx \Omega_0(X_P)$, where $\Omega_0?$ is the path component of $\Omega?$ containing the constant loop.

EXAMPLE Let X be a pointed nilpotent CW space. Denote by C_{π_P} the mapping cone of the pointed Hurewicz fibration $\pi_P : E_{l_P} \rightarrow X$—then the projection $C_{\pi_P} \rightarrow X_P$ is a pointed homotopy equivalence iff X is P-local or X_P is simply connected (cf. p. 5–67).

FACT Let K be a finite CW complex; let X be a pointed nilpotent CW space. Fix a continuous function $\phi : K \rightarrow X$. Denote by $C(K, X : \phi)$, $C(K, X_P : l_P \circ \phi)$ the path component of $C(K, X)$, $C(K, X_P)$ containing ϕ, $l_P \circ \phi$—then $C(K, X : \phi)$ is nilpotent (cf. p. 5–61) and $C(K, X : \phi)_P \approx C(K, X_P : l_P \circ \phi)$.

[Reduce to when K is connected and work with the Postnikov tower of X.]

EXAMPLE Let $X = \mathbf{S}^{2m} \times \mathbf{S}^{2n+1}$ $(m, n > 0)$—then $C(X, X : \mathrm{id}_X)_{\mathbf{Q}} \approx \prod\limits_{i=1}^{4m-1} K(\mathbf{Q}^{d_i}, i) \times \prod\limits_{j=1}^{2n+1} K(H^{2n+1-j}(X; \mathbf{Q}), j)$, where $d_i = \dim_{\mathbf{Q}} H^{4m-1-i}(X; \mathbf{Q}) - \dim_{\mathbf{Q}} H^{2m-1-i}(X; \mathbf{Q})$ (cf. p. 5–30).

(Mapping Cones) Let $\begin{cases} X \\ Y \end{cases}$ be pointed nilpotent CW spaces, $f : X \rightarrow Y$ a pointed continuous function. Assume: C_f is nilpotent—then $(C_f)_P \approx C_{f_P}$.

[C_{f_P} is path connected and by Van Kampen, $\pi_1(C_{f_P}) \approx (\pi_1(C_f))_P$. But why is C_{f_P} nilpotent? For this, it is necessary to use the result of Rao mentioned on p. 5–59 (and transferred to the pointed setting). Take, e.g., the third possibility: $\exists$ a prime p such that $\pi_1(C_f)$ is a finite p-group and $\forall\ q > 0$, $H_q(X)$ is a p-group of finite exponent. Case 1: $p \notin P$. Here, $(\pi_1(C_f))_P = 1$ (cf. p. 8–11) and C_{f_P} is simply connected. Case 2: $p \in P$. X is then P-local in homology, hence is P-local in homotopy (cf. p. 9–6), i.e., $X \approx X_P$, and $\pi_1(C_f) \approx \pi_1(C_{f_P})$. Therefore C_{f_P} is nilpotent. Comparing the long exact sequences in homology finishes the proof.]

Example: Given a pointed nilpotent CW space X, $(\Sigma X)_P \approx \Sigma X_P$.

EXAMPLE Let $\begin{cases} X \\ Y \end{cases}$ be pointed simply connected CW spaces—then $(X \# Y)_P \approx X_P \# Y_P$.

[Observing that $(X \vee Y)_P \approx X_P \vee Y_P$, identify $X \# Y$ with the pointed mapping cone $X \overline{\#} Y$ of the inclusion $X \vee Y \rightarrow X \times Y$ (cf. §3, Proposition 23).]

Every nilpotent group G is separable, i.e., the arrow $G \rightarrow \prod\limits_p G_p$ is injective. The following result is its homotopy theoretic analog.

PROPOSITION 9 Let X be a pointed nilpotent CW space—then for any pointed finite connected CW complex K, the arrow $[K, X] \rightarrow \prod\limits_p [K, X_p]$ is injective.

[The assertion is certainly true if K is a finite wedge of circles. Arguing inductively, consider the pushout square

$$\begin{tikzcd} \mathbf{S}^{n-1} \arrow[r, "f"] \arrow[d] & L \arrow[d] \\ \mathbf{D}^n \arrow[r] & K \end{tikzcd}$$

$(n \geq 2)$ and suppose that the arrow

$[L,X] \to \prod\limits_p [L,X_p]$ is injective. Taking f skeletal, there is a factorization $\begin{tikzcd} \mathbf{S}^{n-1} \arrow[r, "f"] \arrow[d, "i"'] & L \\ M_f \arrow[ur, "r"'] & \end{tikzcd}$, where $L \approx M_f$ and $K \approx C_f \approx C_i$, so one can assume that f is a closed cofibration. Restoring the base points, the corresponding arrow of restriction $f^* : C(L, l_0; X, x_0) \to C(\mathbf{S}^{n-1}, s_{n-1}; X, x_0)$ is then a Hurewicz fibration (cf. p. 4–9) and the fiber of f^* over 0 is homeomorphic to $C(L/\mathbf{S}^{n-1}, *_{\mathbf{S}^{n-1}}; X, x_0)$, $*_{\mathbf{S}^{n-1}}$ the image of $\mathbf{S}^{n-1}$ in $L/\mathbf{S}^{n-1}$. But the projection $C_f \to L/\mathbf{S}^{n-1}$ is a pointed homotopy equivalence (cf. p. 3–24), thus $C(K, k_0; X, x_0) \approx C(L/\mathbf{S}^{n-1}, *_{\mathbf{S}^{n-1}}; X, x_0)$ (cf. p. 6–22). This said, given $\phi \in C(K, k_0; X, x_0)$, put $\psi = \phi | L$, let $\begin{cases} (C,\phi) = C(K, k_0; X, x_0 : \phi) \\ (C,\psi) = C(L, l_0; X, x_0 : \psi) \end{cases}$ and call $\begin{cases} [K,X]_\phi \\ [L,X]_\psi \end{cases}$ the pointed set $\begin{cases} [K,X] \\ [L,X] \end{cases}$ with $\begin{cases} \phi \\ \psi \end{cases}$ as the base point. Noting that $\pi_1(C(\mathbf{S}^{n-1}, s_{n-1}; X, x_0), 0) \approx \pi_n(X)$, a portion of the homotopy sequence of our fibration reads: $\pi_1(C,\psi) \to \pi_n(X) \to [K,X]_\phi \to [L,X]_\psi$. Here, $\pi_n(X)$ operates on $[K,X]_\phi$ and the orbit of ϕ consists of those maps which are pointed homotopic to ψ when restricted to L, the stabilizer of ϕ being precisely $\operatorname{im} \pi_1(C,\psi)$. Collect the data and display it in a commutative diagram

$$\begin{tikzcd} \pi_1(C,\psi) \arrow[r] \arrow[d] & \pi_n(X) \arrow[r] \arrow[d] & {[K,X]_\phi} \arrow[r] \arrow[d] & {[L,X]_\psi} \arrow[d] \\ \prod\limits_p \pi_1(C_p, l_p \circ \psi) \arrow[r] & \prod\limits_p \pi_n(X_p) \arrow[r] & \prod\limits_p [K,X_p]_{l_p \circ \phi} \arrow[r] & \prod\limits_p [L,X_p]_{l_p \circ \psi} \end{tikzcd}.$$

The components of the first and second vertical arrows are p-localizing and by hypothesis, the fourth vertical arrow is injective. As for the third vertical arrow, its injectivity amounts to showing that if $\phi' : K \to X$ and if $\forall\ p$, $l_p \circ \phi' \simeq l_p \circ \phi$, then $\phi' \simeq \phi$. To begin, $\forall\ p$, $l_p \circ \psi' \simeq l_p \circ \psi \Rightarrow \psi' \simeq \psi$, hence ϕ' lies on the $\pi_n(X)$-orbit of ϕ, i.e., $\exists!\ \alpha \in \pi_n(X)/\operatorname{im}\pi_1(C,\psi) : [\phi'] = \alpha \cdot [\phi]$. Claim: α is trivial. In fact, $\forall\ p$, $l_p(\alpha)$ is trivial in $\pi_n(X_p)/\operatorname{im}\pi_1(C_p, l_p \circ \psi)$ and the arrow $\pi_n(X)/\operatorname{im}\pi_1(C,\psi) \to \prod\limits_p (\pi_n(X_p)/\operatorname{im}\pi_1(C_p, l_p \circ \psi))$ is one-to-one.]

Application: Let K be a pointed finite nilpotent CW complex; let X be a pointed nilpotent CW complex. Suppose that $f, g : K \to X$ are pointed continuous functions. Assume: $\forall\ p$, $f_p \simeq g_p$—then $f \simeq g$.

EXAMPLE Suppose that $P \neq \emptyset$ & $\overline{P} \neq \emptyset$. Define K by the pushout square $\begin{tikzcd} \mathbf{S}^n \arrow[r, "f"] \arrow[d] & \mathbf{S}^n_P \vee \mathbf{S}^n_{\overline{P}} \arrow[d] \\ \mathbf{D}^{n+1} \arrow[r] & K \end{tikzcd}$ $(n \geq 2)$, where $f = (1,1) \in \pi_n(\mathbf{S}^n_P \vee \mathbf{S}^n_{\overline{P}}) \approx \mathbf{Z}_P \oplus \mathbf{Z}_{\overline{P}}$. Let $\phi : K \to \mathbf{S}^{n+1}$ be the collapsing map—then $\forall\ p$, $l_p \circ \phi \simeq 0$ but $[\phi] \neq [0]$. Therefore, even when X is a sphere, Proposition 9 can fail if K is not finite (but Proposition 9 does imply that $\phi \in \operatorname{Ph}(K, \mathbf{S}^{n+1})$).

FACT Let X be a pointed nilpotent CW space—then for any pointed finite connected CW complex K, the commutative diagram

$$\begin{tikzcd}
{[K,X]} \arrow[r] \arrow[d] & {[K,X_{\overline{P}}]} \arrow[d] \\
{[K,X_P]} \arrow[r] & {[K,X_{\mathbf{Q}}]}
\end{tikzcd}$$

is a pullback square in $\mathbf{SET}_*$.

[Note: X "is" the double mapping track of the pointed 2-sink $X_P \rightarrow X_{\mathbf{Q}} \leftarrow X_{\overline{P}}$.]

EXAMPLE The assumption on K plays a role in the preceding result. Thus suppose that $P \neq \emptyset$ & $\overline{P} \neq \emptyset$—then the commutative diagram

$$\begin{tikzcd}
{[\mathbf{P}^\infty(\mathbf{C}),\mathbf{S}^3]} \arrow[r] \arrow[d] & {[\mathbf{P}^\infty(\mathbf{C}),\mathbf{S}^3_{\overline{P}}]} \arrow[d] \\
{[\mathbf{P}^\infty(\mathbf{C}),\mathbf{S}^3_P]} \arrow[r] & {[\mathbf{P}^\infty(\mathbf{C}),\mathbf{S}^3_{\mathbf{Q}}]}
\end{tikzcd}$$

is not a pullback square in $\mathbf{SET}_*$.

[Show that the arrow $\lim^1[\Sigma\mathbf{P}^n(\mathbf{C}),\mathbf{S}^3] \rightarrow \lim^1[\Sigma\mathbf{P}^n(\mathbf{C}),\mathbf{S}^3_P] \oplus \lim^1[\Sigma\mathbf{P}^n(\mathbf{C}),\mathbf{S}^3_{\overline{P}}]$ is not one-to-one (cf. p. 5–49).]

FACT Let X be a pointed nilpotent CW space—then for any finite CW complex K, the arrow $[K,X] \rightarrow \prod\limits_p [K,X_p]$ is injective.

[Note: In this context, the brackets refer to homotopy classes of maps, not to pointed homotopy classes of pointed maps.]

Let X be a pointed nilpotent CW space—then one may attach to X a sink $\{r_p : X_p \rightarrow X_{\mathbf{Q}}\}$ and a source $\{l_p : X \rightarrow X_p\}$, where $\forall \begin{cases} p \\ q \end{cases}, r_p \circ l_p \simeq r_q \circ l_q$.

PROPOSITION 10 Let X be a pointed nilpotent CW space with finitely generated homotopy groups. Suppose given a pointed finite connected CW complex K and pointed continuous functions $\phi(p) : K \rightarrow X_p$ such that $\forall \begin{cases} p \\ q \end{cases}, r_p \circ \phi(p) \simeq r_q \circ \phi(q)$—then there is a pointed continuous function $\phi : K \rightarrow X$ such that $\forall\, p, l_p \circ \phi \simeq \phi(p)$.

[The fracture lemma on p. 8–16 implies that the result holds if K is a finite wedge of circles. Proceeding via induction, consider the pushout square

$$\begin{tikzcd}
\mathbf{S}^{n-1} \arrow[r, "f"] \arrow[d] & L \arrow[d] \\
\mathbf{D}^n \arrow[r] & K
\end{tikzcd}$$

$(n \geq 2)$ and assume that there is a pointed continuous function $\psi : L \rightarrow X$ such that $\forall\, p, l_p \circ \psi \simeq \psi(p)$, where $\psi(p) = \phi(p)|L$. Since $\forall\, p, \psi(p) \circ f \simeq 0$, from Proposition 9, $\psi \circ f \simeq 0$, so $\exists$ a pointed continuous function $\phi' : K \rightarrow X$ which restricts to ψ. Taking f to be a closed cofibration and following the proof of Proposition 9, form the commutative diagram

$$\begin{tikzcd}
\pi_1(C,\psi) \arrow[r] \arrow[d] & \pi_n(X) \arrow[r] \arrow[d] & {[K,X]_{\phi'}} \arrow[r] \arrow[d] & {[L,X]_\psi} \arrow[d] \\
\pi_1(C_p, l_p \circ \psi) \arrow[r] & \pi_n(X_p) \arrow[r] & {[K,X]_{l_p \circ \phi'}} \arrow[r] & {[L,X_p]_{l_p \circ \psi}}
\end{tikzcd}.$$

Because $\phi(p)|L \simeq l_p \circ \phi'|L$, $\phi(p)$ must be on the $\pi_n(X_p)$-orbit of $l_p \circ \phi'$, i.e., $\exists!\ \alpha(p) \in \pi_n(X_p)/\text{im}\,\pi_1(C_p, l_p \circ \psi) : [\phi(p)] = \alpha(p) \cdot [l_p \circ \phi']$. However, the $\alpha(p)$ all rationalize to the same element of $(\pi_n(X)/\text{im}\,\pi_1(C,\psi))_{\mathbf{Q}}$, thus $\exists!\ \alpha \in \pi_n(X)/\text{im}\,\pi_1(C,\psi) : \forall\, p, l_p(\alpha) = \alpha(p)$. Put $\phi = \alpha \cdot \phi' : l_p \circ \phi \simeq l_p \circ (\alpha \cdot \phi') \simeq l_p(\alpha) \cdot (l_p \circ \phi') \simeq \alpha(p) \cdot (l_p \circ \phi') \simeq \phi(p)$.]

FACT Let X be a pointed nilpotent CW space with finitely generated homotopy groups. Suppose given a finite CW complex K and continuous functions $\phi(p) : K \to X_p$ such that $\forall \begin{cases} p \\ q \end{cases}, r_p \circ \phi(p) \simeq r_q \circ \phi(q)$—then there is a continuous function $\phi : K \to X$ such that $\forall\, p$, $l_p \circ \phi \simeq \phi(p)$.

HASSE PRINCIPLE Let X be a pointed nilpotent CW space with finitely generated homotopy groups—then for any pointed finite connected CW complex K, the source $\{[K,X] \to [K,X_p]\}$ is the multiple pullback of the sink $\{[K,X_p] \to [K,X_{\mathbf{Q}}]\}$.

[This is a consequence of Propositions 9 and 10.]

Given a pointed nilpotent CW space X with finitely generated homotopy groups, the genus $\text{gen}\, X$ of X is the conglomerate of pointed homotopy types $[Y]$, where Y is a pointed nilpotent CW space with finitely generated homotopy groups such that $\forall\, p$, $X_p \approx Y_p$. The members of $\text{gen}\, X$ have isomorphic higher homotopy groups (but their fundamental groups are not necessarily isomorphic) and isomorphic integral singular homology groups (but their integral singular cohomology rings are not necessarily isomorphic).

Examples: (1) $\text{gen}\, \mathbf{S}^n = \{[\mathbf{S}^n]\}$; (2) $\text{gen}\, K(\pi,n) = \{[K(\pi,n)]\}$, π a finitely generated abelian group; (3) $\text{gen}\, M(\pi,n) = \{[M(\pi,n)]\}$, π a finitely generated abelian group ($n \geq 2$).

EXAMPLE Fix a generator $\alpha \in \pi_6(\mathbf{S}^3) \approx \mathbf{Z}/12\mathbf{Z}$. Put $X = \mathbf{D}^7 \sqcup_\alpha \mathbf{S}^3$, $Y = \mathbf{D}^7 \sqcup_{5\alpha} \mathbf{S}^3$—then $\forall\, p$, $X_p \approx Y_p$ but X and Y do not have the same pointed homotopy type.

EXAMPLE It has been shown by Wilkerson[†] that if X is a pointed finite simply connected CW complex, then $\#(\text{gen}\, X) < \omega$ but this can fail when X is not finite. For instance, take $X = \mathbf{P}^\infty(\mathbf{H})$—then $\text{gen}\, X$ is in a one-to-one correspondence with the set of all functions $\mathbf{\Pi} \to \{\pm 1\}$ (Rector[‡]), hence has cardinality 2^ω.

[Note: It is unknown whether $\#(\text{gen}\, X) < \omega$ for an arbitrary pointed finite nilpotent CW complex X.]

EXAMPLE Let $\begin{cases} X \\ Y \end{cases}$ be pointed nilpotent CW spaces—then X and Y are said to be clones if (i) $\forall\, n$, $X[n] \approx Y[n]$ and (ii) $\forall\, p$, $X_p \approx Y_p$. While neither (i) nor (ii) alone suffices to imply that $X \approx Y$, one can ask whether this is the case of their conjunction. In other words, if X and Y are clones, does it follow that X and Y have the same pointed homotopy type? The answer is "no". Take $X = \mathbf{S}^3 \times K(\mathbf{Z}, 3)$—then, up to pointed homotopy type, the number of distinct clones of X is uncountable (McGibbon[†]).

[†] *Topology* **15** (1976), 111–130.

[‡] *SLN* **249** (1971), 99–105.

Given a set of primes P, a pointed connected CW space X is said to be P-local if $\forall\, n \in S_P$, the arrow $\begin{cases} \Omega X \rightarrow \Omega X \\ \sigma \rightarrow \sigma^n \end{cases}$ is a pointed homotopy equivalence.

[Note: X is P-local iff $\pi_1(X)$ and the $\pi_q(X) \rtimes \pi_1(X)$ $(q \geq 2)$ are P-local groups (cf. p. 8–9) or still, iff $\pi_1(X)$ is a P-local group and the $\pi_q(X)$ $(q \geq 2)$ are P-local $\pi_1(X)$-modules (cf. p. 8–22). Therefore a P-local space is P-local in homotopy (but not conversely (cf. p. 9–2)).]

Example: For any P-local group G, $K(G, 1)$ is a P-local space.

[Note: Accordingly, a P-local space is not necessarily P-local in homology (cf. p. 9–3).]

Notation: $\mathbf{CONCWSP}_{*,P}$ is the full subcategory of $\mathbf{CONCWSP}_*$ whose objects are the pointed connected CW spaces which are P-local and $\mathbf{HCONCWSP}_{*,P}$ is the associated homotopy category.

[Note: This notation is a consistent extension of that introduced on p. 9–4 for the nilpotent category, i.e., a pointed nilpotent CW space which is P-local in homotopy is P-local (cf. p. 8–16).]

Observation: Set $\mathbf{S}_T^q = \mathbf{S}^1$ $(q = 1)$, $\mathbf{S}_T^q = (\mathbf{S}^{q-1} \amalg *) \# \mathbf{S}^1$ $(q \geq 2)$ and let $\rho_n^q = \rho_n$ $(q = 1)$, $\rho_n^q = \text{id} \# \rho_n$ $(q \geq 2)$, where $\rho_n : \mathbf{S}^1 \rightarrow \mathbf{S}^1$ is a map of degree n $(n \in S_P)$. Working in $\mathbf{HCONCWSP}_*$, put $S_0 = \{[\rho_n^q]\}$—then $S_0^\perp$ is the object class of $\mathbf{HCONCWSP}_{*,P}$.

[In fact, $[\mathbf{S}_T^1, X] \approx \pi_1(X)$, $[\mathbf{S}_T^q, X] \approx \pi_q(X) \rtimes \pi_1(X)$ $(q \geq 2)$ and $(\rho_n^q)^* : [\mathbf{S}_T^q, X] \rightarrow [\mathbf{S}_T^q, X]$ is the nth power map $\forall\, q \geq 1$.]

Let $[f] : X \rightarrow Y$ be a morphism in $\mathbf{HCONCWSP}_*$—then $[f]$ (or f) is said to be a P-equivalence if $[f]$ is orthogonal to every P-local pointed connected CW space.

[Note: This terminology does not conflict with that used earlier in the nilpotent category (cf. Proposition 12).]

Convention: Given a pointed connected CW space X, a P[X]-module is a $P[\pi_1(X)]$-module.

[Note: If $\begin{cases} X \\ Y \end{cases}$ are pointed connected CW spaces and if $f : X \rightarrow Y$ is a pointed continuous function, then every $P[Y]$-module can be construed as a $P[X]$-module (cf. p. 8–23).]

[†] *Comment. Math. Helv.* **68** (1993), 263–277.

PROPOSITION 11 Let $\begin{cases} X \\ Y \end{cases}$ be pointed connected CW spaces, $f : X \to Y$ a pointed continuous function—then f is a P-equivalence iff $\pi_1(f)_P : \pi_1(X)_P \to \pi_1(Y)_P$ is bijective and for every locally constant coefficient system $\mathcal{G}$ on Y arising from a $P[Y]$-module, $H^n(Y;\mathcal{G}) \approx H^n(X;f^*\mathcal{G})$ $\forall\, n$.

[Necessity: Given a P-local group G, $[f] \perp K(G,1) \Rightarrow [Y,K(G,1)] \approx [X,K(G,1)] \Rightarrow \mathrm{Hom}(\pi_1(Y),G) \approx \mathrm{Hom}(\pi_1(X),G) \Rightarrow \pi_1(f) \perp G \Rightarrow \pi_1(f)_P : \pi_1(X)_P \approx \pi_1(Y)_P$. To check the cohomological assertion, fix a right $P[Y]$-module π and let $\chi : \pi_1(Y)_P \to \mathrm{Aut}\,\pi$ be the associated homomorphism. Denote by $\mathcal{G} : \Pi Y \to \mathbf{AB}$ the cofunctor corresponding to the composite $\chi \circ l_P$, where $l_P : \pi_1(Y) \to \pi_1(Y)_P$. Since for positive n, $K(\pi,n;\chi)$ is P-local, $[f] \perp K(\pi,n;\chi) \Rightarrow [Y,K(\pi,n;\chi)] \approx [X,K(\pi,n;\chi)] \Rightarrow H^n(Y;\mathcal{G}) \approx H^n(X;f^*\mathcal{G})$ (cf. p. 5–34), $n > 0$. There remains the claim that $H^0(Y;\mathcal{G}) \approx H^0(X;f^*\mathcal{G})$, i.e., that the $\pi_1(Y)$-invariants in π equal the $\pi_1(X)$-invariants in π. To see this, consider the commutative diagram

$$\begin{tikzcd}
\pi_1(X) \arrow[r] \arrow[d] & \pi_1(Y) \arrow[d] \\
\pi_1(X)_P \arrow[r] & \pi_1(Y)_P
\end{tikzcd}$$

. From what has been said above, the arrow $\pi_1(X)_P \to \pi_1(Y)_P$ is an isomorphism. The claim thus follows from the fact that the $\pi_1(Y)_P$-invariants in π are equal to the $\pi_1(Y)$-invariants in π (cf. p. 8–24).

Sufficiency: In order to apply the machinery of full blown obstruction theory (locally constant coefficients (Olum[†])), take $\begin{cases} X \\ Y \end{cases}$ to be pointed connected CW complexes with X a pointed subcomplex of Y, so f is the inclusion $X \to Y$. Fix a pointed continuous function $\phi : X \to Z$, where Z is P-local—then $\pi_1(f) \perp \pi_1(Z) \Rightarrow \exists!\ \theta \in \mathrm{Hom}(\pi_1(Y),\pi_1(Z)) : \pi_1(\phi) = \theta \circ \pi_1(f)$. By restriction of scalars, i.e., using the filler for

$$\begin{tikzcd}
\pi_1(Y) \arrow[r, "\theta"] \arrow[d] & \pi_1(Z) \\
\pi_1(Y)_P \arrow[ur, dashed] &
\end{tikzcd}$$

, the $\pi_n(Z)$ $(n \geq 2)$ become $P[Y]$-modules and there is a long exact sequence $H^1(Y;\pi_n(Z)) \to H^1(X;\pi_n(Z)) \to H^2(Y,X;\pi_n(Z)) \to H^2(Y;\pi_n(Z)) \to H^2(X;\pi_n(Z)) \to \cdots$.

(Existence) One can find a pointed continuous function $\psi : (Y,X)^{(2)} \to Z$ such that $\psi|X = \phi$ and $\pi_1(\psi) = \theta$ ($(Y,X)^{(2)} = Y^{(2)} \cup X$ and $\pi_1((Y,X)^{(2)}) \approx \pi_1(Y)$). On the other hand, the higher order obstructions to the existence of a pointed continuous function $\Phi : Y \to Z$ such that $\Phi|X = \phi$ ($\Rightarrow \pi_1(\Phi) = \theta$) lie in the $H^{n+1}(Y,X;\pi_n(Z))$ $(n \geq 2)$. As these groups necessarily vanish, the precomposition arrow $f^* : [Y,Z] \to [X,Z]$ is surjective.

(Uniqueness) Suppose that Φ', $\Phi'' : Y \to Z$ are pointed continuous functions with $\begin{cases}\Phi'|X = \phi \\ \Phi''|X = \phi\end{cases}$—then the claim is that Φ' and Φ'' are pointed homotopic. Indeed, $\pi_1(\Phi') = \theta = \pi_1(\Phi'') \Rightarrow \Phi'|(Y,X)^{(1)} \simeq \Phi''|(Y,X)^{(1)} \operatorname{rel} X$ and since the $H^n(Y,X;\pi_n(Z))$ $(n \geq 2)$ are trivial, Φ' and Φ'' are homotopic $\operatorname{rel} X$.]

[†] *Ann. of Math.* **52** (1950), 1–50; see also Baues, *Obstruction Theory*, Springer Verlag (1977).

LEMMA Let $\begin{cases}X \\ Y\end{cases}$ be pointed connected CW spaces, $f : X \to Y$ a pointed continuous function. Fix a group G and a ring A with unit. Suppose given a homomorphism $\pi_1(Y) \to G$ and a homomorphism $\mathbf{Z}[G] \to A$. Let $\mathcal{A}$ be the locally constant coefficient system on Y corresponding to A. Assume: $\forall\, n \geq 0$, $H_n(X; f^*\mathcal{A}) \approx H_n(Y;\mathcal{A})$—then for every locally constant coefficient system $\mathcal{M}$ on Y corresponding to a $\begin{cases}\text{right} \\ \text{left}\end{cases}$ A-module M, $\begin{cases}H_n(X; f^*\mathcal{M}) \approx H_n(Y;\mathcal{M}) \\ H^n(Y;\mathcal{M}) \approx H^n(X; f^*\mathcal{M})\end{cases} \forall\, n \geq 0$.

[It suffices to work with pointed connected CW complexes X and Y, where X is a pointed subcomplex of Y (f becoming the inclusion). Put $\pi = \pi_1(Y)$ and let $C_*(\widetilde{Y},\widetilde{X})$ be the associated relative skeletal chain complex (Whitehead[†]), so each $C_n(\widetilde{Y},\widetilde{X})$ is a free left $\mathbf{Z}[\pi]$-module and $\forall\, n \geq 0$, $H_n(Y,X;\mathcal{A}) = H_n(A \otimes_{\mathbf{Z}[\pi]} C_*(\widetilde{Y},\widetilde{X}))$. Here, however, $\forall\, n \geq 0$, $H_n(Y,X;\mathcal{A}) = 0$, and this means that $A \otimes_{\mathbf{Z}[\pi]} C_*(\widetilde{Y},\widetilde{X})$ is a free resolution of 0 as an A-module. Therefore, for any right A-module M, $H_n(Y,X;\mathcal{M}) \approx H_n(M \otimes_{\mathbf{Z}[\pi]} C_*(\widetilde{Y},\widetilde{X})) \approx H_n(M \otimes_A A \otimes_{\mathbf{Z}[\pi]} C_*(\widetilde{Y},\widetilde{X})) \approx \operatorname{Tor}_n^A(M,0) = 0 \ \forall\, n \geq 0$ and for any left A-module M, $H^n(Y,X;\mathcal{M}) \approx H^n(\operatorname{Hom}_{\mathbf{Z}[\pi]}(C_*(\widetilde{Y},\widetilde{X}),M)) \approx H^n(\operatorname{Hom}_{\mathbf{Z}[\pi]}(C_*(\widetilde{Y},\widetilde{X}), \operatorname{Hom}_A(A,M))) \approx H^n(\operatorname{Hom}_A(A \otimes_{\mathbf{Z}[\pi]} C_*(\widetilde{Y},\widetilde{X}),M)) \approx \operatorname{Ext}_A^n(0,M) = 0 \ \forall\, n \geq 0$.]

[Note: Recall that when dealing with modules over a group ring, there is no essential distinction between "left" and "right". In particular: The $C_n(\widetilde{Y},\widetilde{X})$ are both left and right free $\mathbf{Z}[\pi]$-modules.]

It is a corollary that f is a P-equivalence provided that $\pi_1(f)_P : \pi_1(X)_P \to \pi_1(Y)_P$ is bijective and for every locally constant coefficient system $\mathcal{G}$ on Y arising from a $P[Y]$-module, $H_n(X; f^*\mathcal{G}) \approx H_n(Y;\mathcal{G}) \ \forall\, n \geq 0$. In fact, to pass from homology to cohomology, one may apply the lemma, taking $G = \pi_1(Y)_P$ and $A = (\mathbf{Z}[G])_{S_P}$ (cf. p. 8–23).

EXAMPLE Let X be a pointed connected CW space. Suppose that N is a perfect normal subgroup of $\pi_1(X)$ which is contained in the kernel of the arrow of localization $\pi_1(X) \to \pi_1(X)_P$—then $f_N^+ : X \to X_N^+$ is a P-equivalence.

[†] *Elements of Homotopy Theory*, Springer Verlag (1978), 287–288.

[The assumption on N guarantees that $\pi_1(X)_P \approx \pi_1(X_N^+)_P$. But f_N^+ is acyclic, so for every locally constant coefficient system $\mathcal{G}$ on X_N^+, $H_*(X;(f_N^+)^*\mathcal{G}) \approx H_*(X_N^+;\mathcal{G})$ (cf. §5, Proposition 22) and the lemma can be quoted.]

[Note: It is not really necessary to use the lemma. This is because acyclic maps can equally well be characterized in terms of cohomology with locally constant coefficients.]

PROPOSITION 12 Let $\left\{\begin{matrix} X \\ Y \end{matrix}\right.$ be pointed nilpotent CW spaces, $f : X \to Y$ a pointed continuous function. Assume: $f_* : H_*(X;\mathbf{Z}_P) \to H_*(Y;\mathbf{Z}_P)$ is an isomorphism—then for every locally constant coefficient system $\mathcal{G}$ on Y arising from a $P[Y]$-module, $H_n(X;f^*\mathcal{G}) \approx H_n(Y;\mathcal{G})$ $\forall\ n \geq 0$.

[According to Proposition 5, $f_P : X_P \to Y_P$ is a pointed homotopy equivalence, so there is no loss of generality in supposing that $Y = X_P$, $f = l_P$. Consider the diagram

$$\begin{tikzcd}
\widetilde{X} \arrow[r, "p"] \arrow[d, "\widetilde{f}"'] & X \arrow[r] \arrow[d] & K(\pi_1(X),1) \arrow[d] \\
\widetilde{Y} \arrow[r, "q"'] & Y \arrow[r] & K(\pi_1(Y),1)
\end{tikzcd}$$

. It commutes up to pointed homotopy and because $\left\{\begin{matrix} \widetilde{X} \\ \widetilde{Y} \end{matrix}\right.$ are simply connected, $H_n(\widetilde{X};p^*f^*\mathcal{G}) \approx H_n(\widetilde{X};\widetilde{f}^*q^*\mathcal{G}) \approx H_n(\widetilde{X};G), H_n(\widetilde{Y};q^*\mathcal{G}) \approx H_n(\widetilde{Y};G)$, G the underlying P-local $\pi_1(Y)$-module. Bearing in mind that G is, in particular, a $\mathbf{Z}_P$-module, the fact that $H_*(\widetilde{X};\mathbf{Z}_P) \approx H_*(\widetilde{Y};\mathbf{Z}_P)$, in conjunction with the universal coefficient theorem, then gives $H_*(\widetilde{X};G) \approx H_*(\widetilde{Y};G)$. Pass now to the morphism $\{E_{p,q}^2 \approx H_p(\pi_1(X);H_q(\widetilde{X};G))\} \to \{\overline{E}_{p,q}^2 \approx H_p(\pi_1(Y);H_q(\widetilde{Y};G))\}$ of fibration spectral sequences. Since the action of $\pi_1(Y)$ on the $H_q(\widetilde{Y})$ is nilpotent (cf. §5, Proposition 17), each $H_q(\widetilde{Y};G)$ is a P-local $\pi_1(Y)$-module (cf. p. 8–23), i.e., is a $P[X]$-module ($\pi_1(X)_P = \pi_1(Y)$). Therefore, $\forall\ p$ & $\forall\ q$, $H_p(\pi_1(X);H_q(\widetilde{X};G)) \approx H_p(\pi_1(Y);H_q(\widetilde{Y};G))$ (cf. §8, Proposition 16), which serves to complete the proof (cf. p. 5–69).]

[Note: In the nilpotent category, the term "P-equivalence" has two possible interpretations. The point of the proposition is that they coincide (cf. §8, Proposition 2).]

If S is the class of P-equivalences and if D is the class of P-local spaces, then (S,D) is an orthogonal pair. Proof: $S = D^\perp$ (by definition) and $S_0^\perp = D \Rightarrow S_0^{\perp\perp} = S \Rightarrow D = S_0^{\perp\perp\perp} = S^\perp$. Consequently, S has the closure properties (1)–(3) formulated on p. 0–22. It will also be necessary to know the interplay between P-equivalences, wedges, and certain weak colimits.

(Wedges) Let $\left\{\begin{matrix} X_i \\ Y_i \end{matrix}\right.$ $(i \in I)$ be pointed connected CW spaces. Suppose that $\forall\ i$, $f_i : X_i \to Y_i$ is a P-equivalence—then $\bigvee_i f_i : \bigvee_i X_i \to \bigvee_i Y_i$ is a P-equivalence.

[By assumption, $\forall\, i$, $(\pi_1(X_i) \to \pi_1(Y_i)) \perp \operatorname{Ob}\mathbf{GR}_P$, hence $(\underset{i}{*}\,\pi_1(X_i) \to \underset{i}{*}\,\pi_1(Y_i)) \perp \operatorname{Ob}\mathbf{GR}_P$, i.e., $(\pi_1(\bigvee_i X_i) \to \pi_1(\bigvee_i Y_i)) \perp \operatorname{Ob}\mathbf{GR}_P$. Let $\mathcal{G}$ be a locally constant coefficient system on $\bigvee_i Y_i$ arising from a $P[\bigvee_i Y_i]$-module. Employing the notation used in the proof of Proposition 11, $[\bigvee_i Y_i, K(\pi,n;\chi)] \approx \prod_i [Y_i, K(\pi,n;\chi)] \approx \prod_i [X_i, K(\pi,n;\chi)] \approx [\bigvee_i X_i, K(\pi,n;\chi)] \Rightarrow H^n(\bigvee_i Y_i;\mathcal{G}) \approx H^n(\bigvee_i X_i;(\bigvee_i f_i)^*\mathcal{G})$ (cf. p. 5–34), $n > 0$. Finally, the $\pi_1(\bigvee_i Y_i)$-invariants in π equal $\bigcap_i \pi^{\pi_1(Y_i)}$ and the $\pi_1(\bigvee_i X_i)$-invariants in π equal $\bigcap_i \pi^{\pi_1(X_i)}$. And: $\forall\, i$, $\pi^{\pi_1(Y_i)} = \pi^{\pi_1(X_i)}$.]

(Double Mapping Cylinders) Let $X \xleftarrow{f} Z \xrightarrow{g} Y$ be a pointed 2-source, where $\begin{cases} X \\ Y \end{cases}$ & Z are pointed connected CW spaces and f is a P-equivalence. Form the pointed double mapping cylinder $M_{f,g}$ of f, g—then the arrow $Y \to M_{f,g}$ is a P-equivalence.

[Assuming that $\begin{cases} X \\ Y \end{cases}$ & Z are pointed connected CW complexes and $\begin{cases} f \\ g \end{cases}$ are skeletal, pass from

$$\begin{tikzcd} Z \arrow[r, "g"] \arrow[d, "f"'] & Y \arrow[d] \\ X \arrow[r] & M_{f,g} \end{tikzcd}$$

to

$$\begin{tikzcd} Z \arrow[r] \arrow[d] & M_g \arrow[d, "j"] \\ M_f \arrow[r, "i"'] & M_{f,g} \end{tikzcd}$$

(cf. p. 3–23), noting that the arrow $Z \to M_f$ is a P-equivalence. Thanks to Van Kampen, the commutative diagram

$$\begin{tikzcd} \pi_1(Z) \arrow[r] \arrow[d] & \pi_1(M_g) \arrow[d] \\ \pi_1(M_f) \arrow[r] & \pi_1(M_{f,g}) \end{tikzcd}$$

is a pushout square in $\mathbf{GR}$, so $(\pi_1(Z) \to \pi_1(M_f)) \perp \operatorname{Ob}\mathbf{GR}_P \Rightarrow (\pi_1(M_g) \to \pi_1(M_{f,g})) \perp \operatorname{Ob}\mathbf{GR}_P$. Let $\mathcal{G}$ be a locally constant coefficient system on $M_{f,g}$ arising from a $P[M_{f,g}]$-module. On general grounds (excision), $H^n(M_{f,g}, M_g;\mathcal{G}) \approx H^n(M_f, Z;\mathcal{G} \mid M_f)\ \forall\, n \geq 0$, thus $H^n(M_{f,g}, M_g;\mathcal{G}) = 0\ \forall\, n \geq 0 \Rightarrow H^n(M_{f,g};\mathcal{G}) \approx H^n(M_g;\mathcal{G} \mid M_g)\ \forall\, n \geq 0$. That the arrow $M_g \to M_{f,g}$ is a P-equivalence is therefore implied by Proposition 11.]

(Mapping Telescopes) Let $\{X_k, f_k\}$ be a sequence, where X_k is a pointed connected CW space and $f_k : X_k \to X_{k+1}$ is a P-equivalence. Form the pointed mapping telescope $\operatorname{tel}(\mathbf{X},\mathbf{f})$ of $(\mathbf{X},\mathbf{f})$—then the arrow $X_0 \to \operatorname{tel}(\mathbf{X},\mathbf{f})$ is a P-equivalence.

[Assuming that the X_k are pointed connected CW complexes and the f_k are skeletal, there is a commutative diagram

$$\begin{tikzcd} \operatorname{tel}_k(\mathbf{X},\mathbf{f}) \arrow[r] \arrow[d] & \operatorname{tel}_{k+1}(\mathbf{X},\mathbf{f}) \arrow[d] \\ X_k \arrow[r] & X_{k+1} \end{tikzcd}$$

in which the vertical arrows are pointed homotopy equivalences (cf. p. 3–21). By hypothesis, $\forall\, k$, $(\pi_1(\operatorname{tel}_0(\mathbf{X},\mathbf{f})) \to \pi_1(\operatorname{tel}_k(\mathbf{X},\mathbf{f}))) \perp \operatorname{Ob}\mathbf{GR}_P$, so $(\pi_1(\operatorname{tel}_0(\mathbf{X},\mathbf{f})) \to \operatorname{colim}\pi_1(\operatorname{tel}_k(\mathbf{X},\mathbf{f}))) \perp \operatorname{Ob}\mathbf{GR}_P \Rightarrow (\pi_1(\operatorname{tel}_0(\mathbf{X},\mathbf{f})) \to \pi_1(\operatorname{tel}(\mathbf{X},\mathbf{f}))) \perp \operatorname{Ob}\mathbf{GR}_P$. Let $\mathcal{G}$ be a locally constant coefficient system on $\text{tel}(\mathbf{X},\mathbf{f})$ arising from a $P[\text{tel}(\mathbf{X},\mathbf{f})]$-module and put $\mathcal{G}_k = \mathcal{G} \mid \text{tel}_k(\mathbf{X},\mathbf{f})$—then $\forall\, n \geq 0$, $H^n(\text{tel}_k(\mathbf{X},\mathbf{f});\mathcal{G}_k) \approx H^n(\text{tel}_0(\mathbf{X},\mathbf{f});\mathcal{G}_0) \Rightarrow \lim H^n(\text{tel}_k(\mathbf{X},\mathbf{f});\mathcal{G}_k) \approx H^n(\text{tel}_0(\mathbf{X},\mathbf{f});\mathcal{G}_0)$. Since $\pi_1(\text{tel}(\mathbf{X},\mathbf{f})) \approx \text{colim}\, \pi_1(\text{tel}_k(\mathbf{X},\mathbf{f}))$, $H^0(\text{tel}(\mathbf{X},\mathbf{f});\mathcal{G}) \approx \lim H^0(\text{tel}_k(\mathbf{X},\mathbf{f}))$. Moreover, $\forall\, n \geq 1$, there is an exact sequence $0 \rightarrow \lim^1 H^{n-1}(\text{tel}_k(\mathbf{X},\mathbf{f});\mathcal{G}_k) \rightarrow H^n(\text{tel}(\mathbf{X},\mathbf{f});\mathcal{G}) \rightarrow \lim H^n(\text{tel}_k(\mathbf{X},\mathbf{f});\mathcal{G}_k) \rightarrow 0$ of abelian groups (Whitehead[†]). But here the $\lim^1$ terms vanish, so $\forall\, n \geq 1$, $H^n(\text{tel}(\mathbf{X},\mathbf{f});\mathcal{G}) \approx \lim H^n(\text{tel}_k(\mathbf{X},\mathbf{f});\mathcal{G}_k)$.]

HOMOTOPICAL P-LOCALIZATION THEOREM **HCONCWSP**$_{*,P}$ is a reflective subcategory of **HCONCWSP**$_*$.

[The theorem will follow provided that one can show that it is possible to assign to each pointed connected CW space X a P-local pointed connected CW space X_P and a P-equivalence $l_P : X \rightarrow X_P$. Let M_n^q be the pointed double mapping cylinder of the pointed 2-source $\mathbf{S}_T^q \xleftarrow{\rho_n^q} \mathbf{S}_T^q \xrightarrow{\rho_n^q} \mathbf{S}_T^q$—then the diagram

$$\begin{tikzcd} \mathbf{S}_T^q \arrow[r, "\rho_n^q"] \arrow[d, "\rho_n^q"'] & \mathbf{S}_T^q \arrow[d, "j_n^q"] \\ \mathbf{S}_T^q \arrow[r, "i_n^q"'] & M_n^q \end{tikzcd}$$

is pointed homotopy commutative and $\begin{cases} i_n^q \\ j_n^q \end{cases}$ are P-equivalences. Choose pointed continuous functions $\phi_n^q : M_n^q \rightarrow \mathbf{S}_T^q$ such that $\begin{cases} \phi_n^q \circ i_n^q \\ \phi_n^q \circ j_n^q \end{cases} = \text{id}$. We shall now construct a sequence $\{X_k, f_k\}$ such that $X_0 = X$ and $f_k : X_k \rightarrow X_{k+1}$ is a P-equivalence. Thus, arguing by recursion, assume that X_k has been constructed. Consider the set of morphisms $[f] \in [\mathbf{S}_T^q, X_k]$ which cannot be factored through ρ_n^q (failure of surjectivity of $(\rho_n^q)^*$) and the set of morphisms $[g] \in [M_n^q, X_k]$ which cannot be factored through ϕ_n^q (failure of injectivity of $(\rho_n^q)^*$). If $\forall\, q$ & $\forall\, n$, these two sets are empty, then X_k is P-local, so one can let $X_P = X_k$ and take for $l_P : X \rightarrow X_P$ the composite $X_0 \rightarrow X_1 \rightarrow \cdots \rightarrow X_k$. Otherwise, form the pointed 2-source

$$X_k \xleftarrow{\vee} \bigvee_{q,n}((\bigvee_f \mathbf{S}_T^q) \vee (\bigvee_g M_n^q)) \xrightarrow{h} \bigvee_{q,n}((\bigvee_f \mathbf{S}_T^q) \vee (\bigvee_g \mathbf{S}_T^q))$$

and call X_{k+1} the pointed double mapping cylinder of $\vee, h$. Since h is a P-equivalence (being a wedge of P-equivalences), the same is true of the arrow $X_k \rightarrow X_{k+1}$, thereby completing the transition from k to $k+1$. Definition: $X_P = \text{tel}(\mathbf{X},\mathbf{f})$. Accordingly, $l_P : X \rightarrow X_P$ is a P-equivalence. To prove that X_P is P-local, it suffices to show that X_P is orthogonal to the ρ_n^q. Due to the compactness of $\begin{cases} \mathbf{S}_T^q \\ M_n^q \end{cases}$, matters may be arranged in such a way that any continuous function $\begin{cases} \mathbf{S}^q_T \to X_P \\ M^q_n \to X_P \end{cases}$ factors through some X_k (cf. p. 1–29), hence the very construction of X_P guarantees that every triangle

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[†] *Elements of Homotopy Theory*, Springer Verlag (1978), 273–274.

$$\begin{tikzcd} \mathbf{S}^q_T \arrow[r, "\rho^q_n"] \arrow[d, "s"'] & \mathbf{S}^q_T \\ X_P & \end{tikzcd}$$

has a unique filler $\mathbf{S}^q_T \xrightarrow{t} X_P$.]

The reflector L_P produced by the homotopical P-localization theorem, when restricted to $\mathbf{HNILCWSP}_*$, "is" the L_P produced by the nilpotent P-localization theorem. Therefore the idempotent triple corresponding to P-localization in $\mathbf{HCONCWSP}_*$ is an extension of the idempotent triple corresponding to P-localization in $\mathbf{HNILCWSP}_*$ (cf. p. 0–30) (however it is not the only such extension (cf. p. 9–27)).

Remarks: (1) $\forall\ X$, $\pi_1(X)_P \approx \pi_1(X_P)$; (2) $\forall\ X$ & $\forall\ n \geq 1$, the arrow $H_n(X) \to H_n(X_P)$ is P-bijective but $H_n(X_P)$ need not be P-local (unless X is nilpotent); (3) $\forall\ X$ & $\forall\ n > 1$, $\pi_n(X_P)$ is P-local but the arrow $\pi_n(X) \to \pi_n(X_P)$ need not be P-bijective (unless X is nilpotent).

EXAMPLE Let G be a group—then $K(G,1)_P \approx K(G_P,1)$ if G is nilpotent (cf. p. 9–6) but this is false in general ($K(G,1)_P$ will ordinarily have nontrivial higher homotopy groups). To illustrate, suppose that G is finite. Claim: $K(G,1)_P \approx K(G_P,1)$ iff $\ker l_P$ is S_P-torsion, $l_P : G \to G_P$ the arrow of localization. In fact, $K(G_P,1)$ is P-local, so the question is whether the arrow $K(G,1) \to K(G_P,1)$ is a P-equivalence, which is the case iff $\ker l_P$ is S_P-torsion (cf. p. 8–24).

[Note: $K(S_3,1)_3$ is simply connected but $\pi_3(K(S_3,1)_3) \approx \mathbf{Z}/3\mathbf{Z}$).]

FACT For any G, the arrow of localization $l_P : G \to G_P$ is an HP-homomorphism.

[The triangle

$$\begin{tikzcd} K(G,1) \arrow[r] \arrow[d] & K(G,1)_P \arrow[dl] \\ K(G_P,1) & \end{tikzcd}$$

commutes in $\mathbf{HCONCWSP}_*$. In addition, $H_*(K(G,1);\mathbf{Z}_P) \approx H_*(K(G,1)_P;\mathbf{Z}_P)$ and $\pi_1(K(G,1)_P) \approx G_P$.]

The methods used in the proof of the homotopical P-localization theorem are of a general character and can easily be abstracted. What follows isolates the essentials.

Fix a category $\mathbf{C}$ with coproducts. Let $S \subset \operatorname{Mor} \mathbf{C}$ be a class of morphisms containing the isomorphisms of $\mathbf{C}$ which is closed under composition and cancellable. Problem: Find additional conditions on S that will ensure that $S^\perp$ is the object class of a reflective subcategory of $\mathbf{C}$. For this, assume that S is closed under coproducts and that for every 2-source $B \xleftarrow{f} A \to A'$, where $f \in S$, there is a weak pushout square

$$\begin{tikzcd} A \arrow[r] \arrow[d, "f"'] & A' \arrow[d, "f'"] \\ B \arrow[r] & B' \end{tikzcd}$$

where $f' \in S$. Suppose further that there is a set $S_0 \subset S : S_0^\perp = S^\perp$

and a regular cardinal κ such that $\forall$ limit ordinal $\lambda \leq \kappa$, every diagram $\Delta : [0, \lambda[\rightarrow \mathbf{C}$ in which the $\Delta_0 \rightarrow \Delta_\alpha$ $(\alpha < \lambda)$ are in S admits a weak colimit Δ_λ such that $\Delta_0 \rightarrow \Delta_\lambda$ is in S and when $\lambda = \kappa$, for each $f : A \rightarrow B$ in S_0 (i) $\forall\ \phi \in \operatorname{Mor}(A, \Delta_\kappa)$, $\exists\ \alpha < \kappa\ \&\ \phi_\alpha \in \operatorname{Mor}(A, \Delta_\alpha)$:
$$\begin{tikzcd} A \arrow[r, "\phi"] \arrow[d, "\phi_\alpha"'] & \Delta_\kappa \\ \Delta_\alpha \arrow[ur] & \end{tikzcd}$$
commutes and (ii) $\forall\ \psi', \psi'' \in \operatorname{Mor}(B, \Delta_\kappa) : \psi' \circ f = \psi'' \circ f$, $\exists\ \alpha < \kappa\ \&\ \psi'_\alpha, \psi''_\alpha \in \operatorname{Mor}(B, \Delta_\alpha)$:
$$\begin{tikzcd} B \arrow[r, "\psi'"] \arrow[d, "\psi'_\alpha"'] & \Delta_\kappa \\ \Delta_\alpha \arrow[ur] & \end{tikzcd}$$
,
$$\begin{tikzcd} B \arrow[r, "\psi''"] \arrow[d, "\psi''_\alpha"'] & \Delta_\kappa \\ \Delta_\alpha \arrow[ur] & \end{tikzcd}$$
commute & $\psi'_\alpha \circ f = \psi''_\alpha \circ f$.

Conclusion: $S = S^{\perp\perp}$ and $S^\perp$ is the object class of a reflective subcategory of $\mathbf{C}$.

[The verification proceeds by transfinite recursion, the only new wrinkle being that at a limit ordinal $\lambda \leq \kappa$, X_λ is taken to be the weak colimit of the $\{X_\alpha : \alpha < \lambda\}$ (as predicated per the hypotheses). Therefore, in the usual notation, $TX \equiv X_\kappa$. It is automatic that the arrow $\epsilon_X : X \rightarrow TX$ is in S. Since $TX \in S_0^\perp = S^\perp$, what remains to be shown is that $S = S^{\perp\perp}$. Thus let $f : A \rightarrow B$ be orthogonal to $S^\perp$. Since $\epsilon_A : A \rightarrow TA$ is in S and $TB \in S^\perp$, there is a unique filler $Tf : TA \rightarrow TB$ for the diagram
$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "\epsilon_A"'] & B \arrow[d, "\epsilon_B"] \\ TA \arrow[r, dashed] & TB \end{tikzcd}$$
. On the other hand, $\epsilon_B \circ f$ is orthogonal to TA, so one can find an arrow $TB \rightarrow TA$ inverting Tf. It follows that Tf is an isomorphism, hence $Tf \in S \Rightarrow \epsilon_B \circ f \in S \Rightarrow f \in S$, S being cancellable.]

[Note: If $\mathbf{C}$ is cocomplete, then the statement simplifies. Example: The reflective subcategory theorem is a special case of these considerations (Adámek-Rosicky[†]). Applied to $\mathbf{GR}$, one sees, e.g., that the P-localization of a countable group is countable.]

There are situations where the preceding remarks are not applicable since the assumption of cancellability on S may not be satisfied. The point is that cancellable means right cancellable and left cancellable, i.e., $g \circ f \in S\ \&\ f \in S \Rightarrow g \in S$ and $g \circ f \in S\ \&\ g \in S \Rightarrow f \in S$. Let us drop the supposition that S is left cancellable (but retain everything else, including right cancellable)—then the argument above still implies that it is possible to assign to each object $X \in \operatorname{Ob}\mathbf{C}$ another object $TX \in \operatorname{Ob}\mathbf{C}$ and a morphism $\epsilon_X : X \rightarrow TX$ in S. Again, $TX \in S_0^\perp = S^\perp$, thus $S^\perp$ is the object class of a reflective subcategory of $\mathbf{C}$ but now the containment $S \subset S^{\perp\perp}$ can be strict (left cancellable is used to get $S = S^{\perp\perp}$).

EXAMPLE Let $\mathbf{C}$ be a cocomplete category, each object of which is κ-definite for some κ. Let

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[†] *Locally Presentable and Accessible Categories*, Cambridge University Press (1994), 30–35; see also Borceux, *Handbook of Categorical Algebra* 1, Cambridge University Press (1994), 193–209.

$S \subset \text{Mor}\,\mathbf{C}$ be a class of morphisms containing the isomorphisms of $\mathbf{C}$ which is closed under composition and right cancellable. Assume that if

$$\begin{tikzcd} A \arrow[r] \arrow[d, "f"'] & A' \arrow[d, "f'"] \\ B \arrow[r] & B' \end{tikzcd}$$

is a pushout square, then $f \in S \Rightarrow f' \in S$ and if $\Xi \in \text{Nat}(\Delta, \Delta')$, where $\Delta, \Delta' : \mathbf{I} \rightarrow \mathbf{C}$, then $\Xi_i \in S$ $(\forall\, i) \Rightarrow \text{colim}\,\Xi \in S$. Finally, suppose that there is a set $S_0 \subset S : S_0^{\perp} = S^{\perp}$. Accordingly, $S^{\perp}$ is the object class of a reflective subcategory of $\mathbf{C}$ and $\forall\, X$, the arrow $\epsilon_X : X \rightarrow TX$ is in S. Examples: (1) Take $\mathbf{C} = \mathbf{GR}$—then the class of HP-homomorphisms satisfies these conditions, hence $\forall\, G$, the arrow of localization $l_{HP} : G \rightarrow G_{HP}$ is in S_{HP} (cf. p. 8–25); (2) Take $\mathbf{C} = G\text{-}\mathbf{MOD}$—then the class of $H\mathbf{Z}$-homomorphisms satisfies these conditions, hence $\forall\, M$, the arrow of localization $l_{H\mathbf{Z}} : M \rightarrow M_{H\mathbf{Z}}$ is in $S_{H\mathbf{Z}}$ (cf. p. 8–28).

The role played in the theory by "closure" properties can be pinned down.

Given a category $\mathbf{C}$, let $S \subset \text{Mor}\,\mathbf{C}$ be a class of morphisms containing the isomorphisms of $\mathbf{C}$ and closed under composition with them. Definition: S is said to be a localization class provided that it is possible to assign to each object $X \in \text{Ob}\,\mathbf{C}$ another object $TX \in \text{Ob}\,\mathbf{C}$ and a morphism $\epsilon_X : X \rightarrow TX$ in S with the following universal property: For every $f : A \rightarrow B$ in S and for every $g : A \rightarrow X$ there is a unique $t : B \rightarrow TX$ such that $\epsilon_X \circ g = t \circ f$. So, for any arrow $X \rightarrow Y$, there is a commutative diagram

$$\begin{tikzcd} X \arrow[r, "\epsilon_X"] \arrow[d] & TX \arrow[d] \\ Y \arrow[r, "\epsilon_Y"'] & TY \end{tikzcd}$$

, thus T defines a functor $\mathbf{C} \rightarrow \mathbf{C}$ and $\epsilon : \text{id}_{\mathbf{C}} \rightarrow T$ is a natural transformation. Here, $\epsilon T = T\epsilon$ is not necessarily a natural isomorphism (it is if S is closed under composition).

THEOREM OF KOROSTENSKI-THOLEN[†] Let S be a localization class in a category $\mathbf{C}$—then $S = S^{\perp\perp}$ iff S is closed under composition and left cancellable. In addition, the assignment $S \rightarrow S^{\perp}$ sets up a one-to-one correspondence between those localization classes S such that $S = S^{\perp\perp}$ and the conglomerate of reflective subcategories of $\mathbf{C}$.

Let $[f] : X \rightarrow Y$ be a morphism in $\mathbf{HCONCWSP}_*$—then $[f]$ (or f) is said to be an HP-equivalence if $\forall\, n \geq 0$, $f_* : H_n(X; \mathbf{Z}_P) \rightarrow H_n(Y; \mathbf{Z}_P)$ is an isomorphism.

[Note: In the two extreme cases, viz. $P = \emptyset$ or $P = \mathbf{\Pi}$, HP is replaced by $H\mathbf{Q}$ or $H\mathbf{Z}$.]

(Wedges) Let $\left\{\begin{matrix} X_i \\ Y_i \end{matrix}\right.$ $(i \in I)$ be pointed connected CW spaces. Suppose that $\forall\, i$, $f_i : X_i \rightarrow Y_i$ is an HP-equivalence—then $\bigvee\limits_i f_i : \bigvee\limits_i X_i \rightarrow \bigvee\limits_i Y_i$ is an HP-equivalence.

[†] *Comm. Algebra* **14** (1986), 741–766.

[This is because $\forall\ n \geq 1$, $H_n(\bigvee_i X_i; \mathbf{Z}_P) \approx \bigoplus_i H_n(X_i; \mathbf{Z}_P)$ and $H_n(\bigvee_i Y_i; \mathbf{Z}_P) \approx \bigoplus_i H_n(Y_i; \mathbf{Z}_P)$.]

(Pushouts) Suppose that $\begin{cases} X \\ Y \end{cases}$ are pointed connected CW spaces, $A \subset X$ a pointed connected CW subspace, and $f : A \to Y$ a pointed continuous function. Assume: The inclusion $A \to X$ is a closed cofibration and an HP-equivalence—then the arrow $Y \to X \sqcup_f Y$ is an HP-equivalence.

[The adjunction space $X \sqcup_f Y$ is a pointed connected CW space (cf. §5, Proposition 7) and it has the same pointed homotopy type as the pointed double mapping cylinder of the pointed 2-source $X \leftarrow A \to Y$ (cf. §3, Proposition 18).]

PROPOSITION 13 Every P-equivalence $f : X \to Y$ is an HP-equivalence.

[Specializing Proposition 11, one can say that $\forall\ n \geq 0$, $f^* : H^n(Y; \mathbf{Z}_P) \to H^n(X; \mathbf{Z}_P)$ is an isomorphism, hence $\forall\ n \geq 0$, $f_* : H_n(X; \mathbf{Z}_P) \to H_n(Y; \mathbf{Z}_P)$ is an isomorphism (cf. §8, Proposition 2).]

[Note: An HP-equivalence need not be a P-equivalence. For instance, take $P = \Pi$—then HP-equivalence = homology equivalence and P-equivalence = pointed homotopy equivalence.]

Given a set of primes P, a pointed connected CW space X is said to be HP-local provided that $[f] \perp X$ for every HP-equivalence f.

SUBLEMMA Let K be a pointed connected CW complex, $L \subset K$ $(L \neq K)$ a pointed connected subcomplex such that $H_*(K, L; \mathbf{Z}_P) = 0$—then there exists a pointed countable connected subcomplex $A \subset K$ such that $A \not\subset L$ and $H_*(A, A \cap L; \mathbf{Z}_P) = 0$.

[We shall construct an expanding sequence of pointed countable connected subcomplexes $A_1, A_2, \ldots$ of K such that $\forall\ n$, $A_n \not\subset L$ and the arrow $H_*(A_n, A_n \cap L; \mathbf{Z}_P) \to H_*(A_{n+1}, A_{n+1} \cap L; \mathbf{Z}_P)$ is the zero map. Thus fix $A_1 : A_1 \not\subset L$. Given A_n, for each element $x \in H_*(A_n, A_n \cap L; \mathbf{Z}_P)$ choose a pointed finite connected subcomplex $K_x \subset K$ such that x goes to zero in $H_*(A_n \cup K_x, (A_n \cup K_x) \cap L; \mathbf{Z}_P)$. Let A_{n+1} be the union of the A_n and the K_x and put $A = \bigcup_n A_n$.]

[Note: $A \cap L$ is necessarily connected.]

LEMMA Let Z be a pointed connected CW space. Suppose that for any CW pair (K, L), where K is a pointed countable connected CW complex and $L \subset K$ $(L \neq K)$ is a pointed connected subcomplex such that $H_*(K, L; \mathbf{Z}_P) = 0$, the arrow $[K, Z] \to [L, Z]$ is surjective—then Z is HP-local.

[The claim is that for every HP-equivalence $f : X \to Y$, the precomposition arrow $f^* : [Y, Z] \to [X, Z]$ is bijective. Since it is clear that the class of HP-equivalences admits a calculus of left fractions (cf. p. 0–31), it need only be shown that $f^* : [Y, Z] \to [X, Z]$ is surjective. For this purpose, one can make the usual adjustments and take $\begin{cases} X \\ Y \end{cases}$ to be pointed connected CW complexes and $f : X \to Y$ the inclusion, with $X \neq Y$. Build now a transfinite sequence of pointed connected subcomplexes X_α of Y via the following procedure. Set $X_0 = X$. Owing to the sublemma, there exists a pointed countable connected subcomplex $A_0 \subset Y$ such that $A_0 \not\subset X_0$ and $H_*(A_0, A_0 \cap X_0; \mathbf{Z}_P) = 0$. Set $X_1 = A_0 \cup X_0$. Case 1: $X_1 = Y$. In this situation, the arrow $[Y, Z] \to [X, Z]$ is surjective. For let $\phi : X \to Z$ be a pointed continuous function. Since the inclusion $A_0 \cap X_0 \to A_0$ is a cofibration, our assumptions imply that the restriction of ϕ to $A_0 \cap X_0$ extends to a pointed continuous function $A_0 \to Z$, thus ϕ extends to a pointed continuous function $\Phi : Y \to Z$. Case 2: $X_1 \neq Y$. Utilizing excision, $H_*(X_1, X_0; \mathbf{Z}_P) = 0$, so from the exact sequence of the triple (Y, X_1, X_0), $H_*(Y, X_1; \mathbf{Z}_P) = 0$. Therefore the sublemma is applicable to the pair (Y, X_1), hence there exists a pointed countable connected subcomplex $A_1 \subset Y$ such that $A_1 \not\subset X_1$ and $H_*(A_1, A_1 \cap X_1; \mathbf{Z}_P) = 0$. Set $X_2 = A_1 \cup X_1$. Continue on out to a sufficiently large regular cardinal κ (if necessary), taking $X_\lambda = \bigcup_{\alpha<\lambda} X_\alpha$ at a limit ordinal $\lambda \leq \kappa$ (observe that $H_*(Y, X_\lambda; \mathbf{Z}_P) = 0$), where $X_\kappa = Y$.]

Notation: $\mathbf{CONCWSP}_{*,HP}$ is the full subcategory of $\mathbf{CONCWSP}_*$ whose objects are the pointed connected CW spaces which are HP-local and $\mathbf{HCONCWSP}_{*,HP}$ is the associated homotopy category.

HOMOTOPICAL *HP*-LOCALIZATION THEOREM $\mathbf{HCONCWSP}_{*,HP}$ is a reflective subcategory of $\mathbf{HCONCWSP}_*$.

[The theorem will follow provided that one can show that it is possible to assign to each pointed connected CW space X an HP-local pointed connected CW space X_{HP} and an HP-equivalence $l_{HP} : X \to X_{HP}$. The full subcategory of $\mathbf{HCW}_*$ whose objects are the pointed countable connected CW complexes has a small skeleton. One can therefore choose a set of CW pairs (K_i, L_i), where K_i is a pointed countable connected CW complex and $L_i \subset K_i$ ($L_i \neq K_i$) is a pointed connected subcomplex such that $H_*(K_i, L_i; \mathbf{Z}_P) = 0$, which contains up to isomorphism all such CW pairs with these properties. Assuming that X is a pointed connected CW complex, construct an expanding transfinite sequence $X = X_0 \subset X_1 \subset \cdots \subset X_\alpha \subset X_{\alpha+1} \subset \cdots \subset X_\Omega$ of pointed connected CW complexes by setting $X_\lambda = \bigcup_{\alpha<\lambda} X_\alpha$ at a limit ordinal $\lambda \leq \Omega$ and defining $X_{\alpha+1}$ by the pushout square

$$\begin{tikzcd}\bigvee_i \bigvee_f L_i \arrow[r] \arrow[d] & X_\alpha \arrow[d] \\ \bigvee_i \bigvee_f K_i \arrow[r] & X_{\alpha+1}\end{tikzcd}$$

. Here, f runs over a set of skeletal representatives in $[L_i, X_\alpha]$ and the arrow $X_\alpha \to X_{\alpha+1}$ is an HP-equivalence. Put $X_{HP} = X_\Omega$—then $\forall\ i$, $[K_i, X_{HP}] \to [L_i, X_{HP}]$ is surjective, thus by the lemma, X_{HP} is HP-local. That the inclusion $X \to X_{HP}$ is an HP-equivalence is automatic.]

The reflector L_{HP} produced by the homotopical HP-localization theorem, when restricted to **HNILCWSP**$_*$, "is" the L_P produced by the nilpotent P-localization theorem. Proof: If X is nilpotent and P-local, then X is HP-local, as can be seen by appealing to the preceding lemma and using the nilpotent obstruction theorem (cf. Proposition 2). Therefore the idempotent triple corresponding to HP-localization in **HCONCWSP**$_*$ is an extension of the idempotent triple corresponding to localization in **HNILCWSP**$_*$ (cf. p. 0–30). On the other hand, Proposition 13 implies that every HP-local space is P-local, so there is a natural transformation $L_P \to L_{HP}$.

PROPOSITION 14 Let $[f] : X \to Y$ be a morphism in **HCONCWSP**$_*$. Assume: $[f]$ is orthogonal to every HP-local pointed connected CW space—then $[f]$ is an HP-equivalence.

[By hypothesis, for every HP-local Z, $[Y, Z] \approx [X, Z]$. Specialize and substitute in $Z = K(\mathbf{Z}_P, n)$ (which is HP-local) to get $H^n(Y; \mathbf{Z}_P) \approx H^n(X; \mathbf{Z}_P)$ $\forall\ n \geq 1$ or still, $H_n(X; \mathbf{Z}_P) \approx H_n(Y; \mathbf{Z}_P)$ $\forall\ n \geq 1$ (cf. §8, Proposition 2).]

[Note: Thus, in the homotopy category, the class of HP-equivalences is "saturated" but the group theoretic analog of this is false (cf. p. 8–27).]

In the P-local situation, one starts with an intrinsic definition of the P-local objects and defines the P-equivalences via orthogonality, while in the HP-local situation, one starts with an intrinsic definition of the HP-equivalences and defines the HP-local objects via orthogonality. The P-equivalences are characterized in Proposition 11, so to complete the picture, it is necessary to characterize the HP-local objects.

A pointed connected CW space X is said to satisfy Bousfield's condition if $\forall\ n \geq 1$, $\pi_n(X)$ is an HP-local group and $\forall\ n \geq 2$, $\pi_n(X)$ is an $H\mathbf{Z}$-local $\pi_1(X)$-module.

[Note: Recall that an abelian group is P-local iff it is HP-local.]

LEMMA B Let X be a pointed connected CW space. Fix $n > 1$ and suppose that $\phi : \pi_n(X) \to M$ is a homomorphism of $\pi_1(X)$-modules—then $\phi_P : \pi_n(X)_P \to M_P$ is an $H\mathbf{Z}$-homomorphism iff there exists a pointed connected CW space Y and a pointed continuous function $f : X \to Y$ such that $H_*(f) : H_*(X;\mathbf{Z}_P) \approx H_*(Y;\mathbf{Z}_P)$, $\pi_q(f) : \pi_q(X) \approx \pi_q(Y)$ $(q < n)$, and $\pi_n(f) \approx \phi$ in $\pi_n(X)\backslash\pi_1(X)$-**MOD**.

[To establish the sufficiency, compare the exact sequence $H_{n+2}(P_{n-1}X;\mathbf{Z}_P) \to H_1(P_{n-1}X;\pi_n(X)_P) \to H_{n+1}(P_nX;\mathbf{Z}_P) \to H_{n+1}(P_{n-1}X;\mathbf{Z}_P) \to H_0(P_{n-1}X;\pi_n(X)_P) \to H_n(P_nX;\mathbf{Z}_P) \to H_n(P_{n-1}X;\mathbf{Z}_P) \to 0$ on p. 5–41 with its analog for Y, noting that $H_1(P_{n-1}X;\pi_n(X)_P) \approx H_1(\pi_1(X);\pi_n(X)_P)$, $H_0(P_{n-1}X;\pi_n(X)_P) \approx H_0(\pi_1(X);\pi_n(X)_P)$. Indeed, there are bijections $H_q(P_nX;\mathbf{Z}_P) \approx H_q(P_nY;\mathbf{Z}_P)$ $(q \leq n)$ and a surjection $H_{n+1}(P_nX;\mathbf{Z}_P) \to H_{n+1}(P_nY;\mathbf{Z}_P)$ (cf. p. 5–51).

To establish the necessity, attach certain n-cells and $(n+1)$-cells to X so as to produce a relative CW complex $(\overline{X},X)$ and an isomorphism $\pi_n(\overline{X}) \to M$ such that $X[n-1] \approx \overline{X}[n-1]$ and the triangle

$$\begin{tikzcd} & \pi_n(X) \arrow[dl] \arrow[dr, "\phi"] & \\ \pi_n(\overline{X}) \arrow[rr] & & M \end{tikzcd}$$

commutes. The composite $X \to X[n] \to \overline{X}[n]$ induces an arrow $H_q(X;\mathbf{Z}_P) \to H_q(\overline{X}[n];\mathbf{Z}_P)$ which is bijective for $q \leq n$ and surjective for $q = n+1$. Apply the Kan factorization theorem.]

PROPOSITION 15 Let $\begin{cases} X \\ Y \end{cases}$ be pointed connected CW spaces, $f : X \to Y$ a pointed continuous function. Assume: $\begin{cases} X \\ Y \end{cases}$ satisfy Bousfield's condition and f is an HP-equivalence—then f is a pointed homotopy equivalence.

[Obviously, $\mathbf{Z}_P \otimes \pi_1(X)/[\pi_1(X),\pi_1(X)] \approx \mathbf{Z}_P \otimes \pi_1(Y)/[\pi_1(Y),\pi_1(Y)]$. Furthermore, the horizontal arrows in the commutative diagram

$$\begin{tikzcd} H_2(X;\mathbf{Z}_P) \arrow[r] \arrow[d] & H_2(\pi_1(X);\mathbf{Z}_P) \arrow[d] \\ H_2(Y;\mathbf{Z}_P) \arrow[r] & H_2(\pi_1(Y);\mathbf{Z}_P) \end{tikzcd}$$

are surjective (cf. p. 5–35) and $H_2(X;\mathbf{Z}_P) \approx H_2(Y;\mathbf{Z}_P)$. Therefore $f_* : \pi_1(X) \to \pi_1(Y)$ is an HP-homomorphism. But this means that f_* is an isomorphism, $\begin{cases} \pi_1(X) \\ \pi_1(Y) \end{cases}$ being HP-local. Next, consider the commutative diagram

$$\begin{tikzcd} \pi_2(X) \arrow[r, "f_*"] \arrow[d] & \pi_2(Y) \arrow[d] \\ \pi_2(X)_P \arrow[r, "(f_*)_P"'] & \pi_2(Y)_P \end{tikzcd}$$

. The vertical arrows are isomorphisms and $(f_*)_P$ is an $H\mathbf{Z}$-homomorphism (cf. Lemma B). Consequently, $f_* : \pi_2(X) \to \pi_2(Y)$ is an $H\mathbf{Z}$-homomorphism between $H\mathbf{Z}$-local $\pi_1(X)$-modules, hence is an isomorphism. That f is a weak homotopy equivalence then follows by iteration.]

LEMMA For any pointed connected CW space X, there exists a pointed connected CW space X_B which satisfies Bousfield's condition and an HP-equivalence $l_B : X \to X_B$, where $\pi_1(X)_{HP} \approx \pi_1(X_B)$.

[Fix a pointed continuous function $\phi : X \to K(\pi_1(X)_{HP}, 1)$ such that $\phi_* = l_{HP}$, where $l_{HP} : \pi_1(X) \to \pi_1(X)_{HP}$ is the arrow of localization. Since l_{HP} is an HP-homomorphism, the Kan factorization theorem implies that there exists a pointed connected CW space X_1 and pointed continuous functions $f_1 : X \to X_1$, $\psi_1 : X_1 \to K(\pi_1(X)_{HP}, 1)$ with $\phi = \psi_1 \circ f_1$ such that f_1 is an HP-equivalence and $\pi_1(\psi_1) : \pi_1(X_1) \to \pi_1(X)_{HP}$ is an isomorphism. Continuing, construct a pointed connected CW space X_2, a pointed continuous function $f_2 : X_1 \to X_2$, and an isomorphism $\pi_2(X_2) \to (\pi_2(X_1)_P)_{H\mathbf{Z}}$ such that f_2 is an HP-equivalence, $\pi_1(f_2) : \pi_1(X_1) \to \pi_1(X_2)$ is an isomorphism, and the composite $\pi_2(X_1) \to \pi_2(X_2) \to (\pi_2(X_1)_P)_{H\mathbf{Z}}$ equals the composite $\pi_2(X_1) \to \pi_2(X_1)_P \to (\pi_2(X_1)_P)_{H\mathbf{Z}}$ (cf. Lemma B and §8, Proposition 21). This gives $X \to X_1 \to X_2$. Proceed from here inductively and let X_B be the pointed mapping telescope of the sequence thereby obtained.]

[Note: It is apparent from the construction of X_B that if $\pi_q(X)$ is an HP-local group for $1 \leq q \leq n$ and if $\pi_q(X)$ is an $H\mathbf{Z}$-local $\pi_1(X)$-module for $2 \leq q \leq n$, then $\forall\, q \leq n$, $\pi_q(X) \approx \pi_q(X_B)$.]

PROPOSITION 16 Let X be a pointed connected CW space—then X is HP-local iff X satisfies Bousfield's condition.

[Suppose that X satisfies Bousfield's condition. Bearing in mind that the class of HP-equivalences admits a calculus of left fractions, to prove that X is HP-local, it suffices to show that every HP-equivalence $f : X \to Y$ has a left inverse $g : Y \to X$ in $\mathbf{HCONCWSP}_*$, i.e., $g \circ f \simeq \mathrm{id}_X$. For this purpose, apply the lemma to get $l_B : Y \to Y_B$—then the composite $l_B \circ f : X \to Y_B$ is a pointed homotopy equivalence (cf. Proposition 15), so $\exists\, h : Y_B \to X$ such that $h \circ l_B \circ f \simeq \mathrm{id}_X$ and we can take $g = h \circ l_B$. Conversely, suppose that X is HP-local. By what has just been said, X_B is HP-local, thus $l_B : X \to X_B$ is a pointed homotopy equivalence.]

Application: $\forall\, X$, $\pi_1(X)_{HP} \approx \pi_1(X_{HP})$.

EXAMPLE Take $X = \mathbf{S}^1 \vee \mathbf{S}^1$: $\pi_1(X)_P$ is countable but $\pi_1(X)_{HP}$ is uncountable if $2 \in P$.

EXAMPLE When P is the set of all primes, every space is P-local. However, not every space is $H\mathbf{Z}$-local and in fact the effect of $H\mathbf{Z}$-localization on the higher homotopy groups can be drastic even if the fundamental group is nilpotent. Thus let X be a pointed connected CW space and for $q > 1$, put $\widehat{\pi}_q(X) = \lim \pi_q(X)/(I[\pi_1(X)])^i \cdot \pi_q(X)$. Note that $\widehat{\pi}_q(X)$ is an $H\mathbf{Z}$-local $\pi_1(X)$-module, being the limit of nilpotent $\pi_1(X)$-modules (cf. p. 8–29). Assume now that $\pi_1(X)$ is a finitely generated nilpotent group. Suppose further that (i) $\pi_q(X)$ is a nilpotent $\pi_1(X)$-module $(1 < q \leq n)$ and (ii) $\pi_q(X)$ is a finitely generated $\pi_1(X)$-module $(n < q \leq 2n)$ $(n \geq 1)$—then Dror-Dwyer[†] have shown that (i) $\pi_q(X_{H\mathbf{Z}}) \approx \pi_q(X)$ $(1 < q \leq n)$ and (ii) $\pi_q(X_{H\mathbf{Z}}) \approx \widehat{\pi}_q(X)$ $(n < q \leq 2n)$ $(n \geq 1)$. In this situation, the first conclusion is actually automatic, so the impact lies in the second. Example: Take $X = \mathbf{P}^2(\mathbf{R})$ and $n = 1$ to see that $\pi_2(X_{H\mathbf{Z}}) \approx \widehat{\mathbf{Z}}_2$, the 2-adic integers.

HP WHITEHEAD THEOREM Suppose that X and Y are HP-local and let $f : X \to Y$ be a pointed continuous function. Assume: $f_* : H_q(X; \mathbf{Z}_P) \to H_q(Y; \mathbf{Z}_P)$ is bijective for $1 \leq q < n$ and surjective for $q = n$—then f is an n-equivalence.

[If $n = 1$, the claim is that $f_* : H_1(\pi_1(X); \mathbf{Z}_P) \to H_1(\pi_1(Y); \mathbf{Z}_P)$ surjective $\Rightarrow f_* : \pi_1(X) \to \pi_1(Y)$ surjective, which is true (cf. p. 8–27). If $n > 1$, use the Kan factorization theorem to write $f = \psi_f \circ \phi_f$, where $\phi_f : X \to X_f$ is an HP-equivalence and $\psi_f : X_f \to Y$ is an n-equivalence. Since X is HP-local, $X \approx (X_f)_{HP}$ and since Y is HP-local, $\pi_q(X_f) \approx \pi_q(Y)$ $(1 \leq q < n) \Rightarrow \pi_q(X_f) \approx \pi_q((X_f)_{HP})$ $(1 \leq q < n)$. Therefore the arrow $\pi_q(X) \to \pi_q(Y)$ is bijective for $1 \leq q < n$ and surjective for $q = n$, i.e., f is an n-equivalence.]

[Note: Taking $\mathbf{Z}_P = \mathbf{Z}$ and $\begin{cases} X \\ Y \end{cases}$ nilpotent leads to a refinement of Dror's Whitehead theorem (which, of course, can also be derived directly).]

EXAMPLE Let X be a pointed connected CW space. Assume: $\widetilde{H}_*(X; \mathbf{Z}_P) = 0$, i.e., X is $\mathbf{Z}_P$-acyclic—then X_{HP} is contractible.

Given an abelian group G, one can introduce the notion of "HG-equivalence" and play the tape again. So, employing obvious notation, the upshot is that $\mathbf{HCONCWSP}_{*,HG}$ is a reflective subcategory of $\mathbf{HCONCWSP}_*$, with reflector L_{HG} which sends X to X_{HG}.

[Note: The CW pairs (K, L) that intervene when testing for "HG-local" have the property that the cardinality of the set of cells in K is $\leq \#(G)$ if $\#(G)$ is infinite and $\leq \omega$ if $\#(G)$ is finite.]

While the number of distinct homological localizations appears to be large, the reality is that all the possibilities can be described in a simple way. Definition: $L_{HG'}$ and $L_{HG''}$ have the same acyclic spaces if $\widetilde{H}_*(X; G') = 0 \Leftrightarrow \widetilde{H}_*(X; G'') = 0$ or still, if the HG'-equivalences are the same as the HG''-equivalences, hence that $L_{HG'}$ and $L_{HG''}$ are naturally isomorphic.

Given an abelian group G, call $\mathcal{S}(G)$ the class of abelian groups A such that $A \otimes G = 0 = \mathrm{Tor}(A, G)$.

[†] *Illinois J. Math.* **21** (1977), 675–684.

PROPOSITION 17 Let Acy_G be the class of G-acyclic spaces—then $\mathcal{S}(G) = \{\widetilde{H}_n(X) : n \geq 0 \ \& \ X \in \text{Acy}_G\}$.

[This follows from the universal coefficient theorem and the existence of Moore spaces.]

Application: $\mathcal{S}(G') = \mathcal{S}(G'')$ iff $\text{Acy}_{G'} = \text{Acy}_{G''}$.

Given an abelian group G, let P_G be the set of primes p such that G is not uniquely divisible by p and put $S_G = \begin{cases} \bigoplus\limits_{p \in P_G} \mathbf{Z}/p\mathbf{Z} & \text{if } \mathbf{Q} \otimes G = 0 \\ \mathbf{Z}_{P_G} & \text{if } \mathbf{Q} \otimes G \neq 0 \end{cases}$—then $\mathcal{S}(G) = \mathcal{S}(S_G)$ (cf. p. 5–66 ff.). Corollary: $L_{HG} \approx L_{HS_G}$. Therefore, besides the L_{HP}, the only other homological localizations that need be considered are those corresponding to $\bigoplus\limits_{p \in P} \mathbf{Z}/p\mathbf{Z}$ for some P.

FACT Let X be a pointed connected CW space—then $\widetilde{H}_*(X; \bigoplus\limits_{p \in P} \mathbf{Z}/p\mathbf{Z}) = 0$ iff $\widetilde{H}^*(X; \prod\limits_{p \in P} \mathbf{Z}/p\mathbf{Z}) = 0$.

The "$\mathbf{Z}/p\mathbf{Z}$-theory" (="$\mathbf{F}_p$-theory"), in its general aspects, runs parallel to the "$\mathbf{Z}_P$-theory" but there are some differences in detail.

A pointed connected CW space X is said to satisfy Bousfield's condition mod p if $\forall\, n \geq 1$, $\pi_n(X)$ is an $H\mathbf{F}_p$-local group and $\forall\, n \geq 2$, $\pi_n(X)$ is an $H\mathbf{Z}$-local $\pi_1(X)$-module.

[Note: Recall that an abelian group is $H\mathbf{F}_p$-local iff it is p-cotorsion.]

LEMMA B mod p Let X be a pointed connected CW space. Fix $n > 1$ and suppose that $\phi : \pi_n(X) \to M$ is a homomorphism of $\pi_1(X)$-modules. Consider the following conditions.

(C_1) $\text{id} \otimes \phi : \mathbf{F}_p \otimes \pi_n(X) \to \mathbf{F}_p \otimes M$ is an $H\mathbf{Z}$-homomorphism.

(C_2) $\phi_* : H_0(\pi_1(X); \text{Tor}(\mathbf{F}_p, \pi_n(X))) \to H_0(\pi_1(X); \text{Tor}(\mathbf{F}_p, M))$ is surjective.

(C_3) $\text{id} \otimes \phi : \mathbf{F}_p \otimes \pi_n(X) \to \mathbf{F}_p \otimes M$ is an isomorphism.

Then $C_1 + C_2 \Rightarrow$

(E) There exists a pointed connected CW space Y and a pointed continuous function $f : X \to Y$ such that $H_*(f) : H_*(X; \mathbf{F}_p) \approx H_*(Y; \mathbf{F}_p)$, $\pi_q(f) : \pi_q(X) \approx \pi_q(Y)$ $(q < n)$, and $\pi_n(f) \approx \phi$ in $\pi_n(X)\backslash\pi_1(X)$-**MOD**.

Conversely, $E \Rightarrow C_1$ and $E + C_3 \Rightarrow C_2$.

PROPOSITION 18 Let $\left\{\begin{matrix} X \\ Y \end{matrix}\right.$ be pointed connected CW spaces, $f : X \to Y$ a pointed continuous function. Assume: $\left\{\begin{matrix} X \\ Y \end{matrix}\right.$ satisfy Bousfield's condition mod p and f is an $H\mathbf{F}_p$-equivalence—then f is a pointed homotopy equivalence.

[Arguing as in the proof of Proposition 15, one finds that $f_* : \pi_1(X) \rightarrow \pi_1(Y)$ is an isomorphism. To discuss $f_* : \pi_2(X) \rightarrow \pi_2(Y)$, define M, N in $\pi_1(X)$-**MOD** by the exact sequence $0 \rightarrow M \rightarrow \pi_2(X) \rightarrow \pi_2(Y) \rightarrow N \rightarrow 0$. The claim is that $M = 0 = N$, hence that $f_* : \pi_2(X) \rightarrow \pi_2(Y)$ is an isomorphism. For this, it need only be shown that $\mathbf{F}_p \otimes M = 0 = \mathbf{F}_p \otimes N$ (both M and N are $H\mathbf{F}_p$-local). Since f is an $H\mathbf{F}_p$-equivalence, $\mathrm{id} \otimes f_* : \mathbf{F}_p \otimes \pi_2(X) \rightarrow \mathbf{F}_p \otimes \pi_2(Y)$ is an $H\mathbf{Z}$-homomorphism ($E \Rightarrow C_1$). But $\begin{cases} \mathbf{F}_p \otimes \pi_2(X) \\ \mathbf{F}_p \otimes \pi_2(Y) \end{cases}$ are $H\mathbf{Z}$-local (cf. p. 8–31), so $\mathbf{F}_p \otimes \pi_2(X) \approx \mathbf{F}_p \otimes \pi_2(Y)$, from which $\mathbf{F}_p \otimes N = 0$. Using now the exact sequence $\mathrm{Tor}(\mathbf{F}_p, \pi_2(X)) \rightarrow \mathrm{Tor}(\mathbf{F}_p, \pi_2(Y)) \rightarrow \mathbf{F}_p \otimes M \rightarrow 0$, $E + C_3 \Rightarrow C_2$ gives $H_0(\pi_1(X); \mathbf{F}_p \otimes M) = 0$. However, M is $H\mathbf{Z}$-local (being a kernel), thus $\mathbf{F}_p \otimes M$ is $H\mathbf{Z}$-local (cf. p. 8–31). And: $\mathbf{F}_p \otimes M = I[\pi_1(X)] \cdot (\mathbf{F}_p \otimes M) \Rightarrow (\mathbf{F}_p \otimes M)_{H\mathbf{Z}} = 0$ (cf. p. 8–30) $\Rightarrow \mathbf{F}_p \otimes M = 0$. That f is a weak homotopy equivalence then follows by iteration.]

LEMMA For any pointed connected CW space X, there exists a pointed connected CW space X_B which satisfies Bousfield's condition mod p and an $H\mathbf{F}_p$-equivalence $l_B : X \rightarrow X_B$, where $\pi_1(X)_{H\mathbf{F}_p} \approx \pi_1(X_B)$.

[Construct $f_1 : X \rightarrow X_1$ as before (the Kan factorization theorem holds mod p (cf. p. 8–32)). Continuing, construct a pointed connected CW space X_1', a pointed continuous function $f_1' : X_1 \rightarrow X_1'$, and an isomorphism $\pi_2(X_1') \rightarrow \pi_2(X_1)_{H\mathbf{Z}}$ such that f_1' is an $H\mathbf{Z}$-equivalence, $\pi_1(f_1') : \pi_1(X_1) \rightarrow \pi_1(X_1')$ is an isomorphism, and the composite $\pi_2(X_1) \rightarrow \pi_2(X_1') \rightarrow \pi_2(X_1)_{H\mathbf{Z}}$ is the arrow $\pi_2(X_1) \rightarrow \pi_2(X_1)_{H\mathbf{Z}}$ (cf. Lemma B ($P = \mathbf{\Pi}$)). This gives $X \rightarrow X_1 \rightarrow X_1'$. Next, construct a pointed connected CW space X_2, a pointed continuous function $f_1'' : X_1' \rightarrow X_2$, and an isomorphism $\pi_2(X_2) \rightarrow \mathrm{Ext}(\mathbf{Z}/p^\infty\mathbf{Z}, \pi_2(X_1'))$ such that f_1'' is an $H\mathbf{F}_p$-equivalence, $\pi_1(f_1'') : \pi_1(X_1') \rightarrow \pi_1(X_2)$ is an isomorphism, and the composite $\pi_2(X_1') \rightarrow \pi_2(X_2) \rightarrow \mathrm{Ext}(\mathbf{Z}/p^\infty\mathbf{Z}, \pi_2(X_1'))$ is the arrow $\pi_2(X_1') \rightarrow \mathrm{Ext}(\mathbf{Z}/p^\infty\mathbf{Z}, \pi_2(X_1'))$ (cf. Lemma B mod p and p. 8–34). To justify the application of $\mathrm{C}_1 + \mathrm{C}_2 \Rightarrow \mathrm{E}$, note that the arrow $\mathbf{F}_p \otimes \pi_2(X_1') \rightarrow \mathbf{F}_p \otimes \mathrm{Ext}(\mathbf{Z}/p^\infty\mathbf{Z}, \pi_2(X_1'))$ is bijective and the arrow $\mathrm{Tor}(\mathbf{F}_p, \pi_2(X_1')) \rightarrow \mathrm{Tor}(\mathbf{F}_p, \mathrm{Ext}(\mathbf{Z}/p^\infty\mathbf{Z}, \pi_2(X_1')))$ is surjective (cf. p. 8–34). This gives $X \rightarrow X_1 \rightarrow X_1' \rightarrow X_2$. Proceed from here inductively and let X_B be the pointed mapping telescope of the sequence thereby obtained.]

[Note: It is apparent from the construction of X_B that if $\pi_q(X)$ is an $H\mathbf{F}_p$-local group for $1 \leq q \leq n$ and if $\pi_q(X)$ is an $H\mathbf{Z}$-local $\pi_1(X)$-module for $2 \leq q \leq n$, then $\forall\ q \leq n$, $\pi_q(X) \approx \pi_q(X_B)$.]

PROPOSITION 19 Let X be a pointed connected CW space—then X is $H\mathbf{F}_p$-local iff X satisfies Bousfield's condition mod p.

[The proof is the same as that of Proposition 16.]

Application: $\forall\, X$, $\pi_1(X)_{H\mathbf{F}_p} \approx \pi_1(X_{H\mathbf{F}_p})$.

EXAMPLE Let X be a pointed connected CW space. Assume: The homotopy groups of X are finite—then $\forall\, n \geq 1$, $\pi_n(X_{H\mathbf{F}_p})$ is a finite p-group, thus $X_{H\mathbf{F}_p}$ is nilpotent.

[For here $\pi_1(X)_{H\mathbf{F}_p} \approx \pi_1(X)_p$ (cf. p. 8–32), which is a finite p-group (cf. p. 8–11).]

EXAMPLE Every $H\mathbf{F}_p$-local space is p-local (cf. Proposition 13 and §8, Proposition 3), so there is a natural transformation $L_p \to L_{H\mathbf{F}_p}$. If G is a finite group, then $K(G,1)_p \approx K(G,1)_{H\mathbf{F}_p}$ but if G is infinite, this is false (consider $G = \mathbf{Z}$).

EXAMPLE Suppose that X is a pointed nilpotent CW space—then $X_{H\mathbf{F}_p}$ is nilpotent and $\forall\, n \geq 1$, there is a split short exact sequence $0 \to \mathrm{Ext}(\mathbf{Z}/p^\infty\mathbf{Z}, \pi_n(X)) \to \pi_n(X_{H\mathbf{F}_p}) \to \mathrm{Hom}(\mathbf{Z}/p^\infty\mathbf{Z}, \pi_{n-1}(X)) \to 0$ (see below). Therefore, even in the nilpotent case, it need not be true that $\pi_n(X)_{H\mathbf{F}_p}$ "is" $\pi_n(X_{H\mathbf{F}_p})$ when $n > 1$.

$H\mathbf{F}_p$ WHITEHEAD THEOREM Suppose that X and Y are $H\mathbf{F}_p$-local and let $f : X \to Y$ be a pointed continuous function. Assume: $f_* : H_q(X;\mathbf{F}_p) \to H_q(Y;\mathbf{F}_p)$ is bijective for $1 \leq q < n$ and surjective for $q = n$—then f is an n-equivalence.

[The proof is the same as that of the HP Whitehead theorem.]

EXAMPLE Let X be a pointed connected CW space. Assume: $\widetilde{H}_*(X;\mathbf{F}_p) = 0$, i.e., X is $\mathbf{F}_p$-acyclic—then $X_{H\mathbf{F}_p}$ is contractible.

[Note: A pointed nilpotent CW space X is $\mathbf{F}_p$-acyclic iff $\forall\, n \geq 1$, $\mathrm{Hom}(\mathbf{Z}/p^\infty\mathbf{Z}, \pi_n(X)) = 0$ & $\mathrm{Ext}(\mathbf{Z}/p^\infty\mathbf{Z}, \pi_n(X)) = 0$ (cf. p. 8–37).]

PROPOSITION 20 Let Z be a pointed nilpotent CW space—then Z is $H\mathbf{F}_p$-local iff $\forall\, n \geq 1$, $\pi_n(Z)$ is p-cotorsion.

[Necessity: Since Z satisfies Bousfield's condition mod p (cf. Proposition 19), the $\pi_n(Z)$ are $H\mathbf{F}_p$-local, hence are p-cotorsion (cf. §8, Proposition 32).

Sufficiency: The claim is that for every $H\mathbf{F}_p$-equivalence $f : X \to Y$, the precomposition arrow $f^* : [Y,Z] \to [X,Z]$ is bijective. For this, one can assume that $\begin{cases} X \\ Y \end{cases}$ are pointed connected CW complexes with X a pointed subcomplex of Y and argue as in the proof of Proposition 2. However, it is no longer possible to work with the $\Gamma^i_{\chi_q}(\pi_q(Z))/\Gamma^{i+1}_{\chi_q}(\pi_q(Z))$ (since they need not be p-cotorsion). Instead, one uses the $C^i_{\chi_q}(\pi_q(Z))/C^{i+1}_{\chi_q}(\pi_q(Z))$ (which are p-cotorsion) (cf. §8, Proposition 34). Thus now, $\forall\, n \geq 1$, $H_n(Y,X;\mathbf{F}_p) = 0 \Rightarrow H^p(Y,X;C^i_{\chi_q}(\pi_q(Z))/C^{i+1}_{\chi_q}(\pi_q(Z))) = 0$ (cf. §8, Proposition 29) and the obvious modification of the nilpotent obstruction theorem is applicable.]

EXAMPLE Fix a prime p—then every $H\mathbf{F}_p$-local pointed nilpotent CW space is $\mathbf{F}_q$-acyclic for all primes $q \neq p$.

[A p-cotorsion nilpotent group is uniquely q-divisible for all primes $q \neq p$ (cf. p. 8–37).]

LEMMA Let F be a free abelian group—then the arrow $K(F,n) \rightarrow K(\widehat{F}_p,n)$ is an $H\mathbf{F}_p$-equivalence.

[Since $\widehat{F}_p/F$ is uniquely p-divisible, $K(\widehat{F}_p/F,n)$ is $\mathbf{F}_p$-acyclic. On the other hand, $K(F,n)$ is the mapping fiber of the arrow $K(\widehat{F}_p,n) \rightarrow K(\widehat{F}_p/F,n)$, so $H_*(F,n;\mathbf{F}_p) \approx H_*(\widehat{F}_p,n;\mathbf{F}_p)$ (cf. p. 4–44).]

[Note: $\widehat{F}_p$ is the p-adic completion of F. Since F is torsion free, $\operatorname{Ext}(\mathbf{Z}/p^\infty\mathbf{Z},F) \approx \widehat{F}_p$ (cf. p. 10–2).]

Let G be an abelian group. Fix a presentation $0 \rightarrow R \rightarrow F \rightarrow G \rightarrow 0$ of G, i.e., a short exact sequence with R and F free abelian—then there is an exact sequence $0 \rightarrow \operatorname{Hom}(\mathbf{Z}/p^\infty\mathbf{Z},G) \rightarrow \operatorname{Ext}(\mathbf{Z}/p^\infty\mathbf{Z},R) \rightarrow \operatorname{Ext}(\mathbf{Z}/p^\infty\mathbf{Z},F) \rightarrow \operatorname{Ext}(\mathbf{Z}/p^\infty\mathbf{Z},G) \rightarrow 0$ or still, an exact sequence $0 \rightarrow \operatorname{Hom}(\mathbf{Z}/p^\infty\mathbf{Z},G) \rightarrow \widehat{R}_p \rightarrow \widehat{F}_p \rightarrow \operatorname{Ext}(\mathbf{Z}/p^\infty\mathbf{Z},G) \rightarrow 0$. Consider the following diagram

$$\begin{tikzcd}
K(F,n) \arrow[r] \arrow[d] & K(G,n) \arrow[r] \arrow[d, dashed] & K(R,n+1) \arrow[r] \arrow[d] & K(F,n+1) \arrow[d] \\
K(\widehat{F}_p,n) \arrow[r] & K(G,n)_{H\mathbf{F}_p} \arrow[r] & K(\widehat{R}_p,n+1) \arrow[r] & K(\widehat{F}_p,n+1)
\end{tikzcd},$$

where by definition $K(G,n)_{H\mathbf{F}_p}$ is the mapping fiber of the arrow $K(\widehat{R}_p,n+1) \rightarrow K(\widehat{F}_p,n+1)$. To justify the notation, first note that $K(G,n)_{H\mathbf{F}_p}$ has two nontrivial homotopy groups, namely $\pi_n(K(G,n)_{H\mathbf{F}_p}) \approx \operatorname{Ext}(\mathbf{Z}/p^\infty\mathbf{Z},G)$ and $\pi_{n+1}(K(G,n)_{H\mathbf{F}_p}) \approx \operatorname{Hom}(\mathbf{Z}/p^\infty\mathbf{Z},G)$. Since both of these groups are p-cotorsion, Proposition 20 implies that $K(G,n)_{H\mathbf{F}_p}$ is $H\mathbf{F}_p$-local. Taking into account the lemma, standard spectral sequence generalities allow one to infer that the filler $K(G,n) \dashrightarrow K(G,n)_{H\mathbf{F}_p}$ is an $H\mathbf{F}_p$-equivalence. Therefore $K(G,n)_{H\mathbf{F}_p}$ is the $H\mathbf{F}_p$-localization of $K(G,n)$. Example: $K(\mathbf{Q},n)_{H\mathbf{F}_p} \approx *$.

EXAMPLE Suppose that $G = \mathbf{Z}/p^\infty\mathbf{Z}$—then $K(\mathbf{Z}/p^\infty\mathbf{Z},n)_{H\mathbf{F}_p} \approx K(n+1,\widehat{\mathbf{Z}}_p)$ (cf. p. 10–3).

Let X be a pointed nilpotent CW space. Thanks to the preceding considerations, one can copy the proof of the nilpotent P-localization theorem to see that $X_{H\mathbf{F}_p}$ is nilpotent. In so doing, one finds that there is a short exact sequence $0 \rightarrow \operatorname{Ext}(\mathbf{Z}/p^\infty\mathbf{Z},\pi_n(X)) \rightarrow$

$\pi_n(X_{H\mathbf{F}_p}) \to \operatorname{Hom}(\mathbf{Z}/p^\infty\mathbf{Z}, \pi_{n-1}(X)) \to 0$ which necessarily splits (Ext(torsion free, p-cotorsion)$= 0$ (cf. p. 8–35)). Moreover, the triangle

$$\begin{tikzcd} \pi_n(X) \arrow[r] \arrow[dr] & \operatorname{Ext}(\mathbf{Z}/p^\infty\mathbf{Z}, \pi_n(X)) \arrow[d] \\ & \pi_n(X_{H\mathbf{F}_p}) \end{tikzcd}$$

commutes. When the homotopy groups of X are finitely generated, it is common to write $\widehat{X}_p$ in place of $X_{H\mathbf{F}_p}$ and to refer to $\widehat{X}_p$ as the p-adic completion of X, the rationale being that in this case, $\forall\, n$, $\pi_n(\widehat{X}_p) \approx \pi_n(X)_p^\wedge$ (cf. p. 10–2).

Observation: Let X be a pointed nilpotent CW space—then $\forall\, p \in P$, $(X_P)_{H\mathbf{F}_p} \approx X_{H\mathbf{F}_p}$ and $\forall\, p \notin P$, $(X_P)_{H\mathbf{F}_p} \approx *$.

EXAMPLE Given $n \geq 1$, $[\widehat{\mathbf{S}}_p^n, \widehat{\mathbf{S}}_p^n] \approx [\mathbf{S}^n, \widehat{\mathbf{S}}_p^n] \approx \pi_n(\widehat{\mathbf{S}}_p^n) \approx \widehat{\mathbf{Z}}_p$, the p-adic integers. This correspondence is an isomorphism of rings, thus a pointed homotopy equivalence $\widehat{\mathbf{S}}_p^n \to \widehat{\mathbf{S}}_p^n$ determines a p-adic unit (i.e., in the notation of p. 10–10, an element of $\widehat{\mathbf{U}}_p$) and vice versa.

[Note: $\mathbf{S}_p^n = M(\mathbf{Z}_p, n)$ but $\widehat{\mathbf{S}}_p^n \neq M(\widehat{\mathbf{Z}}_p, n)$.]

LEMMA Let G be a finite group whose order is prime to p. Suppose that X is a path connected free right G-space—then $H^*(X/G; \mathbf{F}_p) \approx H^*(X; \mathbf{F}_p)^G$.

EXAMPLE (Sullivan's Loop Space) Assume that p is odd and that n divides $p-1$—then $\widehat{\mathbf{S}}_p^{2n-1}$ has the pointed homotopy type of a loop space. This is seen as follows. Since $\widehat{\mathbf{U}}_p \approx \mathbf{Z}/(p-1)\mathbf{Z} \oplus \widehat{\mathbf{Z}}_p$ (cf. p. 10–10), $\mathbf{Z}/n\mathbf{Z}$ ($\subset \mathbf{Z}/(p-1)\mathbf{Z}$) operates on $\widehat{\mathbf{Z}}_p$ (but the action is not nilpotent). Realize $K(\widehat{\mathbf{Z}}_p, 2)$ per p. 5–31 and form $K(\widehat{\mathbf{Z}}_p, 2; \chi) = (\widetilde{K}(\mathbf{Z}/n\mathbf{Z}, 1) \times K(\widehat{\mathbf{Z}}_p, 2))/(\mathbf{Z}/n\mathbf{Z})$, where $\chi : \mathbf{Z}/n\mathbf{Z} \to \operatorname{Aut}\widehat{\mathbf{Z}}_p$ (thus $\pi_1(K(\widehat{\mathbf{Z}}_p, 2; \chi)) \approx \mathbf{Z}/n\mathbf{Z}$ and $\pi_2(K(\widehat{\mathbf{Z}}_p, 2; \chi)) \approx \widehat{\mathbf{Z}}_p$). Since $H^*(\widehat{\mathbf{Z}}_p, 2; \mathbf{F}_p) \approx \mathbf{F}_p[t]$ ($|t| = 2$), the lemma implies that $H^*(\widehat{\mathbf{Z}}_p, 2; \chi; \mathbf{F}_p) \approx \mathbf{F}_p[t]$ ($|t| = 2n$). Fix a pointed continuous function $f : \mathbf{P}^2(n) \to K(\widehat{\mathbf{Z}}_p, 2; \chi)$ which induces an isomorphism of fundamental groups ($\mathbf{P}^2(n) = M(\mathbf{Z}/n\mathbf{Z}, 1)$ (cf. p. 9–2))—then C_f is simply connected (Van Kampen) and the arrow $K(\widehat{\mathbf{Z}}_p, 2; \chi) \to C_f$ is an $H\mathbf{F}_p$-equivalence, hence $K(\widehat{\mathbf{Z}}_p, 2; \chi)_{H\mathbf{F}_p} \approx (C_f)_{H\mathbf{F}_p} \equiv B$.

Claim: B is $(2n-1)$-connected.

[$H^q(B; \mathbf{F}_p) = 0$ $(1 \leq q < 2n) \Rightarrow H_q(B) \otimes \mathbf{F}_p = 0$ $(1 \leq q < 2n)$ & $\pi_1(B) = * \Rightarrow \pi_2(B) \approx H_2(B)$ (Hurewicz) $\Rightarrow \pi_2(B) = 0$ ($\pi_2(B)$ is p-cotorsion and p-divisible), so by iteration, $\pi_q(B) = 0$ $(1 \leq q < 2n)$.]

The cohomology algebra $H^*(\Omega B; \mathbf{F}_p)$ is an exterior algebra on one generator of degree $2n-1$ and there is an $H\mathbf{F}_p$-equivalence $\mathbf{S}^{2n-1} \to \Omega B$. Accordingly, $\widehat{\mathbf{S}}_p^{2n-1} \approx \Omega B$, ΩB being $H\mathbf{F}_p$-local (cf. p. 9–37).

EXAMPLE Let A be a ring with unit—then $B\mathbf{GL}(A)^+$ is nilpotent (in fact, abelian (cf. p. 5–74 ff.)). Supposing that the $K_n(A)$ are finitely generated, $\forall\, n \geq 1$, $\pi_n(B\mathbf{GL}(A)^+_{H\mathbf{F}_p}) \approx \operatorname{Ext}(\mathbf{Z}/p^\infty\mathbf{Z}, K_n(A)) \approx \widehat{\mathbf{Z}}_p \otimes K_n(A)$.

[Note: This assumption is in force whenever A is a finite field (Quillen[†]) or the ring of integers in an algebraic number field (Quillen[‡]).]

FACT Suppose that X is a pointed simply connected CW space which is $H\mathbf{F}_p$-local—then $H^n(X;\widehat{\mathbf{Z}}_p)$ is a finite p-group $\forall\ n \geq 1$ iff $\pi_n(X)$ is a finite p-group $\forall\ n \geq 1$.

[Since $X_{\mathbf{Q}}$ is $\mathbf{F}_p$-acyclic, the projection $E_{l_{\mathbf{Q}}} \rightarrow X$ is an $H\mathbf{F}_p$-equivalence, so $(E_{l_{\mathbf{Q}}})_{H\mathbf{F}_p} \approx X$. In addition, the homotopy groups of X are p-cotorsion, thus are uniquely q-divisible for all primes $q \neq p$. Therefore the $\pi_n(E_{l_{\mathbf{Q}}})$ are p-primary. The mod $\mathcal{C}$ Hurewicz theorem then implies that $\forall\ n \geq 1$, $H_n(E_{l_{\mathbf{Q}}})$ is p-primary ($E_{l_{\mathbf{Q}}}$ is abelian). Finally, if the homotopy groups of either $E_{l_{\mathbf{Q}}}$ or X are finite p-groups, then $E_{l_{\mathbf{Q}}} \approx X$.]

PROPOSITION 21 Let $[f] : X \rightarrow Y$ be a morphism in $\mathbf{HCONCWSP}_*$. Assume: $[f]$ is orthogonal to every $H\mathbf{F}_p$-local pointed connected CW space—then $[f]$ is an $H\mathbf{F}_p$-equivalence.

[This is the $H\mathbf{F}_p$ version of Proposition 14 and is proved in the same way (cf. §8, Proposition 29).]

Given a set of primes P, put $\mathbf{F}_P = \bigoplus\limits_{p \in P} \mathbf{F}_p$.

PROPOSITION 22 Let X be a pointed nilpotent CW space—then $\forall\ P$, $X_{H\mathbf{F}_P}$ is nilpotent and $X_{H\mathbf{F}_P} \approx \prod\limits_{p \in P} X_{H\mathbf{F}_p}$.

[Extending the algebra of p-cotorsion abelian or nilpotent groups to a P-cotorsion theory is a formality. The other point is that the product may be infinite, hence has to be interpreted as on p. 9–1.]

EXAMPLE (Arithmetic Square) Suppose that X is a pointed nilpotent CW space—then for any P, the diagram

$$\begin{tikzcd} X_P \arrow[r] \arrow[d] & X_{H\mathbf{F}_P} \arrow[d] \\ (X_P)_{\mathbf{Q}} \arrow[r] & (X_{H\mathbf{F}_P})_{\mathbf{Q}} \end{tikzcd}$$

is pointed homotopy commutative and X_P "is" the double mapping track of the pointed 2-sink $(X_P)_{\mathbf{Q}} \rightarrow (X_{H\mathbf{F}_P})_{\mathbf{Q}} \leftarrow X_{H\mathbf{F}_P}$ (Dror-Dwyer-Kan[‖]).

[Note: When $P = \mathbf{\Pi}$, the result asserts that X "is" the double mapping track of the pointed 2-sink $X_{\mathbf{Q}} \rightarrow (\prod\limits_p X_{H\mathbf{F}_p})_{\mathbf{Q}} \leftarrow \prod\limits_p X_{H\mathbf{F}_p}$. Replacing the $X_{H\mathbf{F}_p}$ by the X_p, it can also be shown that X "is" the double mapping track of the pointed 2-sink $X_{\mathbf{Q}} \to (\prod_p X_p)_{\mathbf{Q}} \leftarrow \prod_p X_p$ (Hilton-Mislin[†]).]

[†] *Ann. of Math.* **96** (1972), 552–586.

[‡] *SLN* **341** (1973), 179–198.

[‖] *Illinois J. Math.* **21** (1977), 242–254.

PROPOSITION 23 Let G be an abelian group. Suppose that $\begin{cases} f \\ g \end{cases}$ are HG-equivalences—then so is $f \times g$.

Application: Let $\begin{cases} X \\ Y \end{cases}$ be pointed connected CW spaces—then $(X \times Y)_{HG} \approx X_{HG} \times Y_{HG}$.

[Note: The product $X_{HG} \times Y_{HG}$ is, a priori, HG-local.]

PROPOSITION 24 Let G be an abelian group. Suppose that $X \xrightarrow{f} Z \xleftarrow{g} Y$ is a pointed 2-sink, where $\begin{cases} X \\ Y \end{cases}$ & Z are HG-local pointed connected CW spaces—then the path component W_0 of $W_{f,g}$ which contains the base point $(x_0, y_0, j(z_0))$ is HG-local.

[It suffices to prove that if K is a pointed connected CW complex and $L \subset K$ $(L \neq K)$ is a pointed connected subcomplex such that $H_*(K, L; G) = 0$, then any pointed continuous function $\phi : L \to W_0$ admits a pointed continuous extension $\Phi : K \to W_0$. Thus write $\phi = (x_\phi, y_\phi, \tau_\phi)$ and view τ_ϕ as a pointed homotopy $I(L, l_0) \to Z$ between $f \circ x_\phi$ and $g \circ y_\phi$ (note that $\phi(l_0) = (x_0, y_0, j(z_0))$). Fix pointed continuous functions $\begin{cases} x_\Phi : K \to X \\ y_\Phi : K \to Y \end{cases}$ extending $\begin{cases} x_\phi \\ y_\phi \end{cases}$ and define $H : i_0 K \cup I(L, l_0) \cup i_1 K \to Z$ accordingly ($\begin{cases} X \\ Y \end{cases}$ are HG-local). Since the inclusion $i_0 K \cup I(L, l_0) \cup i_1 K \to I(K, k_0)$ is an HG-equivalence and Z is HG-local, H can be extended to $\tau_\Phi : I(K, k_0) \to Z$. Therefore one can take $\Phi = (x_\Phi, y_\Phi, \tau_\Phi)$.]

Application: For any HG-local pointed connected CW space X, the path component $\Omega_0 X$ of ΩX which contains the constant loop is HG-local.

Notation: Given compactly generated Hausdorff spaces $\begin{cases} X \\ Y \end{cases}$, put $\text{map}(X, Y) = kC(X, Y)$, where $C(X, Y)$ carries the compact open topology (cf. p. 1–32).

[Note: If $\begin{cases} (X, x_0) \\ (Y, y_0) \end{cases}$ are pointed compactly generated Hausdorff spaces, then $\text{map}_*(X, Y)$ is the closed subspace of $\text{map}(X, Y)$ consisting of the base point preserving continuous functions.]

[†] *Comment. Math. Helv.* **50** (1975), 477–491.

Let $\begin{cases}(X,x_0)\\(Y,y_0)\end{cases}$ be pointed connected CW spaces. Consider $C(X,x_0;Y,y_0)$ (compact open topology)—then the pointed homotopy type of $C(X,x_0;Y,y_0)$ depends only on the pointed homotopy types of (X,x_0) and (Y,y_0) (cf. p. 6–22). Therefore, when dealing with questions involving the pointed homotopy type of $C(X,x_0;Y,y_0)$, one can always assume that (X,x_0) and (Y,y_0) are pointed connected CW complexes, hence are wellpointed compactly generated Hausdorff spaces. Of course, the homotopy type of $\operatorname{map}_*(X,Y)$ is not necessarily that of $C(X,x_0;Y,y_0)$ but the arrow $\operatorname{map}_*(X,Y)\to C(X,x_0;Y,y_0)$ is at least a weak homotopy equivalence (cf. p. 1–32).

[Note: The evaluation $f\to f(x_0)$ defines a **CG** fibration $\operatorname{map}(X,Y)\to Y$ whose fiber over y_0 is $\operatorname{map}_*(X,Y)$.]

Observation: If $\pi_q(\operatorname{map}_*(X,Y))$ is computed on the path component containing the constant map, then $\pi_q(\operatorname{map}_*(X,Y))\approx[\Sigma^q X,Y]$.

Examples: (1) $\forall$ HP-local X, $\pi_q(\operatorname{map}_*(\mathbf{S}^n_{HP},X))\approx\pi_{n+q}(X)$ $(\Sigma^q\mathbf{S}^n_{HP}\approx\mathbf{S}^{n+q}_{HP})$; (2) $\forall$ $H\mathbf{F}_p$-local X, $\pi_q(\operatorname{map}_*(\mathbf{S}^n_{H\mathbf{F}_p},X))\approx\pi_{n+q}(X)$ $((\Sigma^q\mathbf{S}^n_{H\mathbf{F}_p})_{H\mathbf{F}_p}\approx\mathbf{S}^{n+q}_{H\mathbf{F}_p})$.

Let (X,x_0) be a pointed connected CW space—then (X,x_0) is nondegenerate (cf. p. 5–22), thus satisfies Puppe's condition (cf. §3, Proposition 20). On the other hand, the identity map $kX\to X$ is a homotopy equivalence (cf. p. 5–22). Moreover, (kX,x_0) satisfies Puppe's condition. Therefore (kX,x_0) is nondegenerate (cf. §3, Proposition 20) and the identity map $kX\to X$ is a pointed homotopy equivalence (cf. p. 3–35).

PROPOSITION 25 Fix an abelian group G. Let $\begin{cases}(X,x_0)\\(Y,y_0)\end{cases}$ be pointed connected CW spaces, $f:X\to Y$ a pointed continuous function. Assume: f is an HG-equivalence—then for any HG-local pointed connected CW space (Z,z_0), the precomposition arrow $f^*:C(Y,y_0;Z,z_0)\to C(X,x_0;Z,z_0)$ is a weak homotopy equivalence.

[Make the transition spelled out above and consider instead $f^*:\operatorname{map}_*(Y,Z)\to\operatorname{map}_*(X,Z)$, there being no loss of generality in supposing that f is an inclusion. Since $\begin{cases}\operatorname{map}(Y,Z)\to Z\\\operatorname{map}(X,Z)\to Z\end{cases}$ are **CG** fibrations, thus are Serre, and since the diagram

$$\begin{tikzcd}
\operatorname{map}_*(Y,Z) \arrow[r] \arrow[d] & \operatorname{map}(Y,Z) \arrow[r] \arrow[d] & Z \arrow[d, equal] \\
\operatorname{map}_*(X,Z) \arrow[r] & \operatorname{map}(X,Z) \arrow[r] & Z
\end{tikzcd}$$

commutes, it need only be shown that $f^*:\operatorname{map}(Y,Z)\to\operatorname{map}(X,Z)$ is a weak homotopy equivalence (cf. p. 4–41 ff.). Claim: $\forall$ finite connected CW pair (K,L), the diagram

$$\begin{tikzcd}
L \arrow[r, "\phi"] \arrow[d] & \operatorname{map}(Y,Z) \arrow[d, "f^*"] \\
K \arrow[r, "\psi"'] & \operatorname{map}(X,Z)
\end{tikzcd}$$

admits a

filler $\Phi : K \rightarrow \mathrm{map}(Y, Z)$ such that $\Phi|L = \phi$ and $f^* \circ \Phi = \psi$. For this, convert to

$$\begin{tikzcd}
K \times X \cup L \times Y \arrow[r, "i"] \arrow[dr] & K \times Y \arrow[d, dashed] \\
& Z
\end{tikzcd}$$

. Because i is a cofibration (cf. §3, Proposition 7) and an HG-equivalence (Mayer-Vietoris), there exists an arrow $K \times Y \rightarrow Z$ rendering the triangle strictly commutative. Now quote the WHE criterion.]

[Note: The fact that Z is HG-local gives $[Y, Z] \approx [X, Z]$, i.e., $\pi_0(\mathrm{map}_*(Y, Z)) \approx \pi_0(\mathrm{map}_*(X, Z))$, so f^* automatically induces a bijection of path components.]

Application: Fix an abelian group G. Let $\begin{cases} (X, x_0) \\ (Y, y_0) \end{cases}$ be pointed connected CW spaces. Assume: X is HG-acyclic and Y is HG-local—then $C(X, x_0; Y, y_0)$ is homotopically trivial.

[The constant map $X \rightarrow x_0$ is an HG-equivalence.]

LEMMA Let $\begin{cases} X \\ Y \end{cases}$ be topological spaces, $f : X \rightarrow Y$ a continuous function. Assume: f is a weak homotopy equivalence—then for any CW complex Z, the postcomposition arrow $f_* : C(Z, X) \rightarrow C(Z, Y)$ is a weak homotopy equivalence.

[Given a finite CW pair (K, L), convert

$$\begin{tikzcd}
L \arrow[r] \arrow[d] & C(Z,X) \arrow[d, "f_*"] \\
K \arrow[r] \arrow[ur, dashed] & C(Z,Y)
\end{tikzcd}
\quad \text{to} \quad
\begin{tikzcd}
L \times Z \arrow[r] \arrow[d] & X \arrow[d, "f"] \\
K \times Z \arrow[r] \arrow[ur, dashed] & Y
\end{tikzcd}
\quad .$$

This is permissible: $\begin{cases} L \times Z \\ K \times Z \end{cases}$ are CW complexes, hence are compactly generated Hausdorff spaces. Accordingly, the arrows $\begin{cases} C(L \times Z, X) \rightarrow C(L, C(Z, X)) \\ C(K \times Z, Y) \rightarrow C(K, C(Z, Y)) \end{cases}$ are homeomorphisms (compact open topology) (Engelking[†]).]

[Note: Let $\begin{cases} X \\ Y \end{cases}$ be compactly generated Hausdorff spaces, $f : X \rightarrow Y$ a continuous function. Assume: f is a weak homotopy equivalence—then for any CW complex Z, the postcomposition arrow $f_* : \mathrm{map}(Z, X) \rightarrow \mathrm{map}(Z, Y)$ is a weak homotopy equivalence (same argument). When $\begin{cases} X \\ Y \end{cases}$ & f are pointed, consideration of

$$\begin{tikzcd}
\mathrm{map}_*(Z,X) \arrow[r] \arrow[d] & \mathrm{map}(Z,X) \arrow[r] \arrow[d] & X \arrow[d] \\
\mathrm{map}_*(Z,Y) \arrow[r] & \mathrm{map}(Z,Y) \arrow[r] & Y
\end{tikzcd}$$

implies that $f_* : \mathrm{map}_*(Z, X) \rightarrow \mathrm{map}_*(Z, Y)$ is also a weak homotopy equivalence (cf. p. 4–41 ff.).]

[†] *General Topology*, Heldermann Verlag (1989), 160.

EXAMPLE Fix a prime p. Let K be a pointed connected CW complex; let X be a pointed nilpotent CW complex. Assume: K is $\mathbf{Z}\left[\frac{1}{p}\right]$-acyclic, i.e., $\widetilde{H}_*(K;\mathbf{Z}\left[\frac{1}{p}\right]) = 0$—then the arrow of localization $l_{H\mathbf{F}_p} : X \rightarrow X_{H\mathbf{F}_p}$ induces a weak homotopy equivalence $\text{map}_*(K, X) \rightarrow \text{map}_*(K, X_{H\mathbf{F}_p})$.

[Every pointed nilpotent CW complex Z which is either rational or $H\mathbf{F}_q$-local ($q \neq p$) is necessarily $H\mathbf{Z}\left[\frac{1}{p}\right]$-local. Therefore $\text{map}_*(K, Z)$ is homotopically trivial. This said, work in the compactly generated category and consider the arithmetic square

$$\begin{tikzcd} X \arrow[r] \arrow[d] & L \arrow[d] \\ X_{\mathbf{Q}} \arrow[r] & L_{\mathbf{Q}} \end{tikzcd}$$

, where $L = X_{H\mathbf{F}_\Pi}$ ($P = \Pi$). Since X can be identified with the double mapping track of the pointed 2-sink $X_{\mathbf{Q}} \rightarrow L_{\mathbf{Q}} \leftarrow L$, $\text{map}_*(K, X)$ is the double mapping track of the pointed 2-sink $\text{map}_*(K, X_{\mathbf{Q}}) \rightarrow \text{map}_*(K, L_{\mathbf{Q}}) \leftarrow \text{map}_*(K, L)$. Because $\text{map}_*(K, X_{\mathbf{Q}})$ and $\text{map}_*(K, L_{\mathbf{Q}})$ are both homotopically trivial, the arrow $\text{map}_*(K, X) \rightarrow \text{map}_*(K, L)$ is a weak homotopy equivalence (cf. p. 4–50). However, by definition, there is a weak homotopy equivalence $L \rightarrow X_{H\mathbf{F}_p} \times_k \prod_{q \neq p} X_{H\mathbf{F}_q}$, so from the above, the arrow $\text{map}_*(K, L) \rightarrow \text{map}_*(K, X_{H\mathbf{F}_p}) \times_k \prod_{q \neq p} \text{map}_*(K, X_{H\mathbf{F}_q})$ is a weak homotopy equivalence. But $\prod_{q \neq p} \text{map}_*(K, X_{H\mathbf{F}_q})$ is homotopically trivial, thus the projection $\text{map}_*(K, L) \rightarrow \text{map}_*(K, X_{H\mathbf{F}_p})$ is a weak homotopy equivalence.]

EXAMPLE Let G be a finite p-group—then $BG(= K(G,1)$ (cf. p. 5–72)) is $\mathbf{Z}\left[\frac{1}{p}\right]$-acyclic (Brown[†]). So, for any pointed nilpotent CW space X, $[BG, X] \approx [BG, X_{H\mathbf{F}_p}]$.

[Note: If X is a simply connected CW space and if the homotopy groups of X are finite p-groups, then X is $\mathbf{Z}\left[\frac{1}{p}\right]$-acyclic. Proof: $\forall\ n > 0$, $H_n(X)$ is a finite p-group (mod $\mathcal{C}$ Hurewicz), hence $\forall\ n > 0$, $H_n(X; \mathbf{Z}\left[\frac{1}{p}\right]) = \mathbf{Z}\left[\frac{1}{p}\right] \otimes H_n(X) = 0$.]

EXAMPLE Fix a prime p. Let X be a pointed nilpotent CW complex—then the arrow of localization $l_p : X \rightarrow X_p$ induces a weak homotopy equivalence $\text{map}_*(B\mathbf{Z}/p\mathbf{Z}, X) \rightarrow \text{map}_*(B\mathbf{Z}/p\mathbf{Z}, X_p)$.

[The point is that $X_{H\mathbf{F}_p}$ can be identified with $(X_p)_{H\mathbf{F}_p}$.]

If $\begin{cases} A \\ B \end{cases}$ are pointed connected CW complexes and if $\rho : A \rightarrow B$ is a pointed continuous function, then $\rho^\perp$ need not be the object class of a reflective subcategory of **HCONCWSP**$_*$ (cf. p. 9–1). Of course, $Z \in \rho^\perp$ iff $\rho^* : \pi_0(C(B, b_0; Z, z_0)) \rightarrow \pi_0(C(A, a_0; Z, z_0))$ is bijective and it is a fundamental point of principle that the class of Z for which $\rho^* : C(B, b_0; Z, z_0) \rightarrow C(A, a_0; Z, z_0)$ is a weak homotopy equivalence is the object class of a reflective subcategory of **HCONCWSP**$_*$ (cf. p. 9–46). This means that the "orthogonal subcategory problem" in **HCONCWSP**$_*$ has a positive solution if the notion of

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[†] *Cohomology of Groups*, Springer Verlag (1982), 84.

"orthogonality" is strengthened so as to include not just π_0 but all the π_n $(n > 0)$ as well (cf. Proposition 25 (and its proof)).

The formalities are best handled by working in **CGH**$_*$. In fact, it is actually more convenient to work in **CGH**. Thus let $\begin{cases} A \\ B \end{cases}$ be CW complexes, $\rho : A \to B$ a continuous function—then an object Z in **CGH** is said to be ρ-local if $\rho^* : \text{map}(B, Z) \to \text{map}(A, Z)$ is a weak homotopy equivalence.

[Note: Since the diagram

$$\begin{tikzcd} \text{map}(B,Z) \arrow[r] \arrow[d] & \text{map}(A,Z) \arrow[d] \\ C(B,Z) \arrow[r] & C(A,Z) \end{tikzcd}$$

commutes and the vertical arrows are weak homotopy equivalences, Z is ρ-local iff $\rho^* : C(B, Z) \to C(A, Z)$ is a weak homotopy equivalence.]

Notation: ρ-**loc** is the full subcategory of **CGH** whose objects are ρ-local.

[Note: If $\begin{cases} \rho_1 \\ \rho_2 \end{cases}$ are homotopic, then the same holds for $\begin{cases} \rho_1^* \\ \rho_2^* \end{cases}$ (cf. p. 6–22). Therefore Z is in ρ_1-**loc** iff Z is in ρ_2-**loc**.]

ρ-**loc** is closed under the formation of products in **CGH** and is invariant under homotopy equivalence.

LEMMA Let $\begin{cases} A \\ B \end{cases}$ be pointed CW complexes, $\rho : A \to B$ a pointed continuous function. Suppose that Z is a pointed compactly generated Hausdorff space—then $\rho^* : \text{map}_*(B, Z) \to \text{map}_*(A, Z)$ is a weak homotopy equivalence if Z is ρ-local and conversely if $\pi_0(Z) = *$.

EXAMPLE Take $\begin{cases} A = W \\ B = * \end{cases}$, where W is path connected, and let $\rho : W \to *$—then the ρ-local objects are said to be W-null. So, Z is W-null iff the arrow $Z \to \text{map}(W, Z)$ is a weak homotopy equivalence. On the other hand, relative to some choice of a base point in W, a pointed path connected Z is W-null iff the arrow $* \to \text{map}_*(W, Z)$ is a weak homotopy equivalence or still, iff $\text{map}_*(W, Z)$ is homotopically trivial, i.e., iff $\forall\, q \geq 0$, $[\Sigma^q W, Z] = 0$. Example: When $W = \mathbf{S}^{n+1}$ $(n \geq 0)$, a pointed path connected Z is W-null iff $\pi_q(Z) = 0$ $(q > n)$.

FACT Let $f : X \to Y$ be a CG fibration, where Y is path connected. Fix $y_0 \in Y$ and assume that X_{y_0} & Y are W-null—then X is W-null.

[Observing that the arrow $\text{map}(W, X) \to \text{map}(W, Y)$ is a CG fibration, consider the commutative diagram

$$\begin{tikzcd} X_{y_0} \arrow[r] \arrow[d] & X \arrow[r] \arrow[d] & Y \arrow[d] \\ \text{map}(W,X_{y_0}) \arrow[r] & \text{map}(W,X) \arrow[r] & \text{map}(W,Y) \end{tikzcd}$$

.]

[Note: By the same token, X & Y W-null $\Rightarrow$ X_{y_0} W-null.]

PROPOSITION 26 Let $\left\{\begin{matrix} A \\ B \end{matrix}\right.$ be CW complexes, $\rho : A \rightarrow B$ a continuous function. Suppose that Z is ρ-local—then $\forall\ Y$ in **CW**, $\mathrm{map}(Y, Z)$ is ρ-local.

[The arrow $\mathrm{map}(B, \mathrm{map}(Y, Z)) \rightarrow \mathrm{map}(A, \mathrm{map}(Y, Z))$ is a weak homotopy equivalence iff the arrow $\mathrm{map}(B \times_k Y, Z) \rightarrow \mathrm{map}(A \times_k Y, Z)$ is a weak homotopy equivalence, i.e., iff the arrow $\mathrm{map}(Y, \mathrm{map}(B, Z)) \rightarrow \mathrm{map}(Y, \mathrm{map}(A, Z))$ is a weak homotopy equivalence.]

LEMMA Given X in **CGH**, $\left\{\begin{matrix} Y \\ Z \end{matrix}\right.$ in $\mathbf{CGH}_*$, $\mathrm{map}(X, \mathrm{map}_*(Y, Z))$ is homeomorphic to $\mathrm{map}_*(Y, \mathrm{map}(X, Z))$.

[$\mathrm{map}(X, \mathrm{map}_*(Y, Z)) \approx \mathrm{map}_*(X_+, \mathrm{map}_*(Y, Z)) \approx \mathrm{map}_*(X_+ \#_k Y, Z) \approx \mathrm{map}_*(Y, \mathrm{map}_*(X_+, Z)) \approx \mathrm{map}_*(Y, \mathrm{map}(X, Z))$.]

PROPOSITION 27 Let $\left\{\begin{matrix} A \\ B \end{matrix}\right.$ be pointed CW complexes, $\rho : A \rightarrow B$ a pointed continuous function. Suppose that Z is pointed and ρ-local—then $\forall\ Y$ in $\mathbf{CW}_*$, $\mathrm{map}_*(Y, Z)$ is ρ-local.

[The arrow $\mathrm{map}(B, \mathrm{map}_*(Y, Z)) \rightarrow \mathrm{map}(A, \mathrm{map}_*(Y, Z))$ is a weak homotopy equivalence iff the arrow $\mathrm{map}_*(Y, \mathrm{map}(B, Z)) \rightarrow \mathrm{map}_*(Y, \mathrm{map}(A, Z))$ is a weak homotopy equivalence.]

Given a pointed compactly generated Hausdorff space X, put $\Sigma_k X = X \#_k \mathbf{S}^1$, $\Omega_k X = \mathrm{map}_*(\mathbf{S}^1, X)$ —then the assignments $X \rightarrow \Sigma_k X$, $X \rightarrow \Omega_k X$ define functors $\mathbf{CGH}_* \rightarrow \mathbf{CGH}_*$ and (Σ_k, Ω_k) is an adjoint pair.

EXAMPLE Let $\left\{\begin{matrix} A \\ B \end{matrix}\right.$ be pointed CW complexes, $\rho : A \rightarrow B$ a pointed continuous function. Suppose that Z is pointed and ρ-local—then $\Omega_k Z$ is ρ-local. Therefore the arrow $\mathrm{map}_*(B, \Omega_k Z) \rightarrow \mathrm{map}_*(A, \Omega_k Z)$ is a weak homotopy equivalence, i.e., the arrow $\mathrm{map}_*(\Sigma_k B, Z) \rightarrow \mathrm{map}_*(\Sigma_k A, Z)$ is a weak homotopy equivalence, so Z is $\Sigma_k \rho$-local provided that Z is path connected.

PROPOSITION 28 Let $\left\{\begin{matrix} A \\ B \end{matrix}\right.$ be CW complexes, $\rho : A \rightarrow B$ a continuous function. Suppose that $X \rightarrow Z \leftarrow Y$ is a 2-sink of compactly generated Hausdorff spaces. Assume: $\left\{\begin{matrix} X \\ Y \end{matrix}\right.$ & Z are ρ-local—then the compactly generated double mapping track W is ρ-local.

[The vertical arrows in the commutative diagram

$$\begin{tikzcd}
\mathrm{map}(B, X) \arrow[r] \arrow[d] & \mathrm{map}(B, Z) \arrow[d] & \mathrm{map}(B, Y) \arrow[l] \arrow[d] \\
\mathrm{map}(A, X) \arrow[r] & \mathrm{map}(A, Z) & \mathrm{map}(A, Y) \arrow[l]
\end{tikzcd}$$

are weak homotopy equivalences, thus the arrow $\mathrm{map}(B, W) \rightarrow \mathrm{map}(A, W)$ is a weak homotopy equivalence (cf. p. 4–48).]

PROPOSITION 29 Let $\begin{cases} A \\ B \end{cases}$ be CW complexes, $\rho : A \to B$ a continuous function. Suppose that W is a retract of Z, where Z is ρ-local—then W is ρ-local.

[There is a commutative diagram

$$\begin{tikzcd}
\mathrm{map}(B,W) \arrow[r] \arrow[d] & \mathrm{map}(B,Z) \arrow[r] \arrow[d] & \mathrm{map}(B,W) \arrow[d] \\
\mathrm{map}(A,W) \arrow[r] & \mathrm{map}(A,Z) \arrow[r] & \mathrm{map}(A,W)
\end{tikzcd}$$

in which the composite of the horizontal arrows across the top and the bottom is the respective identity map, i.e., the arrow $\mathrm{map}(B,W) \to \mathrm{map}(A,W)$ is a retract of the arrow $\mathrm{map}(B,Z) \to \mathrm{map}(A,Z)$ (cf. p. 12–1). But the retract of a weak homotopy equivalence is a weak homotopy equivalence.]

EXAMPLE If Z is ρ-local and a CW space, then any nonempty union of its path components is again ρ-local.

[Z is the coproduct of its path components (cf p. 5–19).]

((A,B) Construction) Let $\begin{cases} A \\ B \end{cases}$ be CW complexes, $\rho : A \to B$ a continuous function. Because the objects in ρ-**loc** depend only on $[\rho]$, there is no loss of generality in taking ρ skeletal. The mapping cylinder M_ρ of ρ is then a CW complex and it is clear that the ρ-local spaces are the same as the i-local spaces, $i : A \to M_\rho$ the embedding. One can therefore assume that A is a subcomplex of B and $\rho : A \to B$ the inclusion (which is a closed cofibration). Let (K,L) be $(\mathbf{D}^n, \mathbf{S}^{n-1})$ $(n \geq 0)$. Given an X in **CGH**, put $X_0 = X$ and with f running over $\mathrm{map}(K \times A \cup L \times B, X_0)$, define X_1 by the pushout square

$$\begin{tikzcd}
\coprod\limits_{(K,L)} \coprod\limits_{f} K \times A \cup L \times B \arrow[r] \arrow[d] & X_0 \arrow[d] \\
\coprod\limits_{(K,L)} \coprod\limits_{f} K \times B \arrow[r] & X_1
\end{tikzcd}.$$

Since $K \times A \cup L \times B \to K \times B$ is a closed cofibration (cf. §3, Proposition 7), $X_0 \to X_1$ is a closed cofibration and X_1 is in **CGH** (cf. p. 3–8). Proceeding, construct an expanding transfinite sequence $X = X_0 \subset X_1 \subset \cdots \subset X_\alpha \subset X_{\alpha+1} \subset \cdots \subset X_\kappa$ of compactly generated Hausdorff spaces by setting $X_\lambda = \bigcup\limits_{\alpha<\lambda} X_\alpha$ at a limit ordinal $\lambda \leq \kappa$ and defining $X_{\alpha+1}$ by the pushout square

$$\begin{tikzcd}
\coprod\limits_{(K,L)} \coprod\limits_{f} K \times A \cup L \times B \arrow[r] \arrow[d] & X_\alpha \arrow[d] \\
\coprod\limits_{(K,L)} \coprod\limits_{f} K \times B \arrow[r] & X_{\alpha+1}
\end{tikzcd},$$

where f runs over $\text{map}(K \times A \cup L \times B, X_\alpha)$. Here, it is understood that each X_λ has the final topology per the $X_\alpha \rightarrow X_\lambda$ $(\alpha < \lambda)$. Transfinite induction then implies that all the X_α $(\alpha \leq \kappa)$ are in **CGH** and every embedding $X_\alpha \rightarrow X_\beta$ $(\alpha < \beta \leq \kappa)$ is a closed cofibration. As for κ, choose it to be a regular cardinal $> \sup\limits_{(K,L)} \#(K \times A \cup L \times B)$ (thus κ is independent of X). Now fix a pair (K, L). Claim: The arrow of restriction $\text{map}(K \times B, X_\kappa) \rightarrow \text{map}(K \times A \cup L \times B, X_\kappa)$ is surjective. To see this, let $f : K \times A \cup L \times B \rightarrow X_\kappa$. Given $x \in K \times A \cup L \times B$, $\exists\ \alpha_x < \kappa : f(x) \in X_{\alpha_x} \Rightarrow \alpha = \sup\limits_x \alpha_x < \kappa$, so f factors through X_α, hence the claim. Consequently, $\rho^* : \text{map}(B, X_\kappa) \rightarrow \text{map}(A, X_\kappa)$ is a weak homotopy equivalence (cf. p. 5–16) (the arrow $\text{map}(B, X_\kappa) \rightarrow \text{map}(A, X_\kappa)$ is a **CG** fibration (cf. §4, Proposition 6)), i.e., X_κ is ρ-local.

Definition: Given an X in **CGH**, put $L_\rho X = X_\kappa$—then this assignment defines a functor $L_\rho : \mathbf{CGH} \rightarrow \mathbf{CGH}$ and there is a natural transformation $\text{id} \rightarrow L_\rho$.

[Note: The very construction of L_ρ guarantees that the embedding $l_\rho : X \rightarrow L_\rho X$ is a closed cofibration.]

Remarks: (1) $\begin{cases} A \\ B \end{cases}$ & X path connected $\Rightarrow$ $L_\rho X$ path connected; (2) X in **CWSP** $\Rightarrow$ $L_\rho X$ in **CWSP**.

PROPOSITION 30 Let $\begin{cases} A \\ B \end{cases}$ be CW complexes, $\rho : A \rightarrow B$ a continuous function. Suppose that Z is ρ-local—then $\forall\ X$, the arrow $\text{map}(L_\rho X, Z) \rightarrow \text{map}(X, Z)$ is a weak homotopy equivalence.

[By definition, $L_\rho X = \operatorname*{colim}\limits_{\alpha<\kappa} X_\alpha$, hence $\text{map}(L_\rho X, Z) \approx \lim\limits_{\alpha<\kappa} \text{map}(X_\alpha, Z)$ (homeomorphism of compactly generated Hausdorff spaces) (limit in **CGH**). On the other hand, the arrows in the "long" tower $\text{map}(X_0, Z) \leftarrow \text{map}(X_1, Z) \leftarrow \cdots \leftarrow \text{map}(X_\alpha, Z) \leftarrow \text{map}(X_{\alpha+1}, Z) \leftarrow \cdots$ are **CG** fibrations and at a limit ordinal λ, $\text{map}(X_\lambda, Z) \approx \lim\limits_{\alpha<\lambda} \text{map}(X_\alpha, Z)$, so it will be enough to prove that $\forall\ \alpha$, $\text{map}(X_{\alpha+1}, Z) \rightarrow \text{map}(X_\alpha, Z)$ is a weak homotopy equivalence. But the commutative diagram

$$\begin{tikzcd}
\text{map}(X_{\alpha+1}, Z) \arrow[r] \arrow[d] & \text{map}\Big(\coprod\limits_{(K,L)} \coprod\limits_{f} K \times B, Z\Big) \arrow[d, "p"] \\
\text{map}(X_\alpha, Z) \arrow[r] & \text{map}\Big(\coprod\limits_{(K,L)} \coprod\limits_{f} K \times A \cup L \times B, Z\Big)
\end{tikzcd}$$

is a pullback square in **CGH** and p is a **CG** fibration, thus one has only to show that p is a weak homotopy equivalence (cf. p. 5–16). To this end, fix a pair (K, L) and consider the triangle

$$\begin{tikzcd}
\mathrm{map}(K \times B, Z) \arrow[r] \arrow[dr] & \mathrm{map}(K \times A \cup L \times B, Z) \arrow[d] \\
& \mathrm{map}(K \times A, Z)
\end{tikzcd}.$$

According to Proposition 26, the oblique arrow is a weak homotopy equivalence. In addition, the commutative diagram

$$\begin{tikzcd}
\mathrm{map}(K \times A \cup L \times B, Z) \arrow[r] \arrow[d] & \mathrm{map}(L \times B, Z) \arrow[d] \\
\mathrm{map}(K \times A, Z) \arrow[r] & \mathrm{map}(L \times A, Z)
\end{tikzcd}$$

is a pullback square in **CGH** and another appeal to Proposition 26 says that the CG fibration $\mathrm{map}(L \times B, Z) \to \mathrm{map}(L \times A, Z)$ is a weak homotopy equivalence. Therefore the arrow $\mathrm{map}(K \times A \cup L \times B, Z) \to \mathrm{map}(K \times A, Z)$ is a weak homotopy equivalence (cf. p. 5–16). Finally, then, the arrow $\mathrm{map}(K \times B, Z) \to \mathrm{map}(K \times A \cup L \times B, Z)$ is a weak homotopy equivalence and our assertion follows.]

Application: Suppose that Z is ρ-local—then every diagram $\begin{tikzcd} X \arrow[r, "\phi"] \arrow[d, "l_\rho"'] & Z \\ L_\rho X \end{tikzcd}$ has a filler $\Phi : L_\rho X \to Z$ in the homotopy category: $\phi \simeq \Phi \circ l_\rho$. And: Φ is unique up to homotopy.

Because L_ρ is a functor **CGH** $\to$ **CGH**, given $f, g \in \mathrm{map}(X, Y)$, there are commutative diagrams $\begin{tikzcd} X \arrow[r, "f"] \arrow[d] & Y \arrow[d] \\ L_\rho X \arrow[r, "L_\rho f"'] & L_\rho Y \end{tikzcd}$, $\begin{tikzcd} X \arrow[r, "g"] \arrow[d] & Y \arrow[d] \\ L_\rho X \arrow[r, "L_\rho g"'] & L_\rho Y \end{tikzcd}$. If further $f \simeq g$, then $L_\rho f \simeq L_\rho g$, i.e., L_ρ respects the homotopy congruence.

HOMOTOPICAL ρ-LOCALIZATION THEOREM Let $\begin{cases} A \\ B \end{cases}$ be CW complexes, $\rho : A \to B$ a continuous function. Let **C** be either the homotopy category of compactly generated Hausdorff spaces or the homotopy category of compactly generated CW Hausdorff spaces—then the full subcategory of **C** whose objects are ρ-local is reflective.

[Note: Analogous conclusions can be drawn in the path connected situation provided that $\begin{cases} A \\ B \end{cases}$ themselves are path connected.]

Let $f \in \operatorname{map}(X,Y)$—then f is said to be a ρ-equivalence if $L_\rho f : L_\rho X \to L_\rho Y$ is a homotopy equivalence. On general grounds, f is a ρ-equivalence iff $\forall$ ρ-local Z, $f^* : [Y,Z] \to [X,Z]$ is bijective. More is true: f is a ρ-equivalence iff $\forall$ ρ-local Z, $f^* : \operatorname{map}(Y,Z) \to \operatorname{map}(X,Z)$ is a weak homotopy equivalence. Proof: Consider the commutative diagram

$$\begin{tikzcd}
\operatorname{map}(L_\rho Y, Z) \arrow[r] \arrow[d] & \operatorname{map}(L_\rho X, Z) \arrow[d] \\
\operatorname{map}(Y, Z) \arrow[r] & \operatorname{map}(X, Z)
\end{tikzcd}.$$

In the special case when $\rho : W \to *$, where W is path connected, homotopical ρ-localization is referred to as W-nullification and one writes $l_W : X \to L_W X$ in place of $l_\rho : X \to L_\rho X$, the ρ-equivalences being termed W-equivalences.

PROPOSITION 31 Let $\begin{cases} X \\ Y \end{cases}$ be compactly generated CW Hausdorff spaces—then $L_\rho(X \times_k Y) \approx L_\rho X \times_k L_\rho Y$.

[The product $L_\rho X \times_k L_\rho Y$ is necessarily ρ-local, thus it suffices to prove that the arrow $X \times_k Y \to L_\rho X \times_k L_\rho Y$ is a ρ-equivalence. To see this, let Z be ρ-local. Thanks to Proposition 26, $\operatorname{map}(L_\rho Y, Z)$ and $\operatorname{map}(X,Z)$ are ρ-local. Consider the composite $\operatorname{map}(L_\rho X \times_k L_\rho Y, Z) \to \operatorname{map}(L_\rho X, \operatorname{map}(L_\rho Y, Z)) \to \operatorname{map}(X, \operatorname{map}(L_\rho Y, Z)) \to \operatorname{map}(X \times_k L_\rho Y, Z) \to \operatorname{map}(L_\rho Y, \operatorname{map}(X,Z)) \to \operatorname{map}(Y, \operatorname{map}(X,Z)) \to \operatorname{map}(X \times_k Y, Z)$.]

[Note: L_ρ need not preserve arbitrary products.]

As it stands, base points play no role in the homotopical ρ-localization theorem but they can be incorporated.

Let $\begin{cases} A \\ B \end{cases}$ be pointed CW complexes, $\rho : A \to B$ a pointed continuous function. Since $l_\rho : X \to L_\rho X$ is a closed cofibration, X wellpointed $\Rightarrow$ $L_\rho X$ wellpointed. Accordingly, for any ρ-local, wellpointed Z, the arrow $\operatorname{map}_*(L_\rho X, Z) \to \operatorname{map}_*(X,Z)$ is a weak homotopy equivalence. Therefore if **C** is either the homotopy category of wellpointed compactly generated Hausdorff spaces or the homotopy category of wellpointed compactly generated CW Hausdorff spaces, then the full subcategory of **C** whose objects are ρ-local is reflective.

[Note: While the data is pointed, ρ-local is defined in terms of map, not map_* (but one can use map_* for path connected objects (cf. p. 9–41)).]

Let $\begin{cases} A \\ B \end{cases}$ be pointed connected CW complexes, $\rho : A \to B$ a pointed continuous function—then an object Z in $\mathbf{CONCWSP}_*$ is said to be ρ-local if $\rho^* : C(B, b_0; Z, z_0) \to C(A, a_0; Z, z_0)$ is a weak homotopy equivalence.

LOCALIZATION THEOREM OF DROR FARJOUN The ρ-local Z constitute the object class of a reflective subcategory of $\mathbf{HCONCWSP}_*$.

[It is a question of assigning to each X a ρ-local object $L_\rho X$ and an arrow $l_\rho : X \to L_\rho X$ such that $\forall$ ρ-local Z, l_ρ^* induces a bijection $[L_\rho X, Z] \to [X, Z]$. Fix a pointed CW complex $(\overline{X}, \overline{x}_0)$ and a pointed homotopy equivalence $(X, x_0) \to (\overline{X}, \overline{x}_0)$. Definition: $L_\rho X = L_\rho \overline{X}$, $l_\rho : X \to L_\rho X$ being the composite $X \to \overline{X} \to L_\rho \overline{X}$.

Claim: $L_\rho X$ is ρ-local.

[Setting $Y = L_\rho X$, by construction, the arrow $\text{map}(B, Y) \to \text{map}(A, Y)$ is a weak homotopy equivalence. Therefore the arrow $\text{map}_*(B, Y) \to \text{map}_*(A, Y)$ is a weak homotopy equivalence, so inspection of

$$\begin{tikzcd}
\text{map}_*(B, Y) \arrow[r] \arrow[d] & \text{map}_*(A, Y) \arrow[d] \\
C(B, b_0; Y, y_0) \arrow[r] & C(A, a_0; Y, y_0)
\end{tikzcd}$$

shows that $L_\rho X$ is ρ-local.]

Given a ρ-local Z, choose a pointed CW complex $(\overline{Z}, \overline{z}_0)$ and a pointed homotopy equivalence $(Z, z_0) \to (\overline{Z}, \overline{z}_0)$. Consideration of

$$\begin{tikzcd}
C(B, b_0; \overline{Z}, \overline{z}_0) \arrow[r] \arrow[d] & C(A, a_0; \overline{Z}, \overline{z}_0) \arrow[d] \\
C(B, b_0; Z, z_0) \arrow[r] & C(A, a_0; Z, z_0)
\end{tikzcd}$$

and

$$\begin{tikzcd}
\text{map}_*(B, \overline{Z}) \arrow[r] \arrow[d] & \text{map}_*(A, \overline{Z}) \arrow[d] \\
C(B, b_0; \overline{Z}, \overline{z}_0) \arrow[r] & C(A, a_0; \overline{Z}, \overline{z}_0)
\end{tikzcd}$$

allows one to infer that the arrow $\text{map}_*(B, \overline{Z}) \to \text{map}_*(A, \overline{Z})$ is a weak homotopy equivalence. In turn, this means that the arrow $\text{map}(B, \overline{Z}) \to \text{map}(A, \overline{Z})$ is a weak homotopy equivalence $(\pi_0(\overline{Z}) = *)$. Take now any $\phi : X \to Z$ and chase the diagram

$$\begin{tikzcd}
X \arrow[r, "\phi"] \arrow[d, shift left] & Z \arrow[r, shift left] & \overline{Z} \arrow[l, shift left] \\
\overline{X} \arrow[u, shift left] \arrow[d] & & \\
L_\rho \overline{X} & &
\end{tikzcd}$$

to see that up to pointed homotopy, there exists a unique $\Phi : L_\rho X \to Z$ such that $\phi \simeq \Phi \circ l_\rho$.]

[Note: If $\left\{\begin{array}{l} A \\ B \end{array}\right.$ are n-connected, then $\pi_q(l_\rho) : \pi_q(X) \to \pi_q(L_\rho X)$ is an isomorphism for $q \leq n$ (cf. p. 9–49).]

EXAMPLE Consider $L_{\mathbf{S}^{n+1}}$, the nullification functor corresponding to $\mathbf{S}^{n+1} \to *$ $(n \geq 0)$—then, in this situation, one recovers the fact that **HCONCWSP**$_*[n]$ is a reflective subcategory of **HCONCWSP**$_*$ (cf. p. 9–1), where $\forall X$, $L_{\mathbf{S}^{n+1}} X \approx X[n]$.

EXAMPLE Fix a set of primes P. Given a pointed connected CW space X, its loop space ΩX is a pointed CW space (loop space theorem), thus the arrow $\left\{\begin{array}{l} \Omega X \to \Omega X \\ \sigma \to \sigma^n \end{array}\right.$ $(n \in S_P)$ is a pointed homotopy equivalence iff it is a weak homotopy equivalence. To interpret this, put $\rho = \bigvee_n \rho_n$, where $\rho_n : \mathbf{S}^1 \to \mathbf{S}^1$ is a map of degree n $(n \in S_P)$—then the ρ-local objects in **CONCWSP**$_*$ are precisely the objects of

$\mathbf{CONCWSP}_{*,P}$ and the homotopical P-localization theorem is seen to be a special case of the localization theorem of Dror Farjoun.

[Note: The full subcategory of $\mathbf{HCONCWSP}_*$ whose objects are P-local in homotopy is not the object class of a reflective subcategory of $\mathbf{HCONCWSP}_*$ (cf. p. 9–2). However, the full subcategory of $\mathbf{HCONCWSP}_*$ whose objects are P-local in "higher homotopy" is the object class of a reflective subcategory of $\mathbf{HCONCWSP}_*$. Proof: Consider the pointed suspension of ρ. Therefore $L_{\Sigma\rho}$ induces an isomorphism of fundamental groups and P-localizes the higher homotopy groups.]

EXAMPLE Fix an abelian group G. Choose a set of CW pairs (K_i, L_i), where K_i is a pointed connected CW complex and $L_i \subset K_i$ $(L_i \neq K_i)$ is a pointed connected subcomplex such that $H_*(K_i, L_i; G) = 0$ subject to the restriction that the cardinality of the set of cells in K_i is $\leq \#(G)$ if $\#(G)$ is infinite and $\leq \omega$ if $\#(G)$ is finite, which contains up to isomorphism all such CW pairs with these properties. Let $\rho : \bigvee\limits_i L_i \to \bigvee\limits_i K_i$—then a pointed connected CW space is HG-local iff it is ρ-local, proving once again that $\mathbf{HCONCWSP}_{*,HG}$ is a reflective subcategory of $\mathbf{HCONCWSP}_*$.

[Note: Take $G = \mathbf{Z}$ and let W be the pointed mapping cone of ρ—then the nullification functor L_W assigns to each X its plus construction X^+.]

EXAMPLE Fix a prime p. Let $W = M(\mathbf{Z}/p\mathbf{Z}, 1)$ be the "standard" Moore space of type $(\mathbf{Z}/p\mathbf{Z}, 1)$—then a simply connected Z is W-null iff $\forall\ n \geq 2$, $\pi_n(Z)$ is $\overline{p}$-local.

EXAMPLE Fix a prime p. Let $W = M(\mathbf{Z}\left[\frac{1}{p}\right], 1)$ be the "standard" Moore space of type $(\mathbf{Z}\left[\frac{1}{p}\right], 1)$—then a simply connected Z is W-null iff $\forall\ n \geq 2$, $\pi_n(Z)$ is p-cotorsion.

EXAMPLE Fix a prime p. Put $W = B\mathbf{Z}/p\mathbf{Z}$—then a nilpotent Z is W-null iff Z_p is W-null iff $Z_{H\mathbf{F}_p}$ is W-null (cf. p. 9–40). In general, a W-null Z is W_k-null, where $W_k = B\mathbf{Z}/p^k\mathbf{Z}$ $(1 \leq k < \infty)$ (consider the short exact sequence $0 \to \mathbf{Z}/p\mathbf{Z} \to \mathbf{Z}/p^{k+1}\mathbf{Z} \to \mathbf{Z}/p^k\mathbf{Z} \to 0$, show that the pointed mapping cone of $B\mathbf{Z}/p\mathbf{Z} \to B\mathbf{Z}/p^{k+1}\mathbf{Z}$ is $B\mathbf{Z}/p^k\mathbf{Z}$, and use induction (replication theorem)), hence Z is W_∞-null, where $W_\infty = B\mathbf{Z}/p^\infty\mathbf{Z}$.

[Note: The arrow $W \to *$ is a $\overline{p}$-equivalence, so every $\overline{p}$-local space is W-null. Example: $K(\mathbf{Z}\left[\frac{1}{p}\right], 1)$ is W-null.]

LEMMA Let $\begin{cases} A \\ B \end{cases}$ be pointed connected CW complexes, $\rho : A \to B$ a pointed continuous function. Assume: $\pi_1(\rho) : \pi_1(A) \to \pi_1(B)$ is surjective—then for any ρ-local Z, its universal covering space $\widetilde{Z}$ is ρ-local.

[Note: Therefore $\pi_1(Z) = * \Rightarrow \pi_1(L_\rho Z) = *$.]

EXAMPLE Fix a prime p. Put $W = B\mathbf{Z}/p\mathbf{Z}$—then Z W-null $\Rightarrow \widetilde{Z}$ W-null. Suppose now that X is a simply connected CW space. Assume: The homotopy groups of $X_{H\mathbf{F}_p}$ are finite p-groups. Claim: $L_W X_{H\mathbf{F}_p}$ is contractible if $(L_W X)_{H\mathbf{F}_p}$ is contractible. For let Z be W-null. Since $X_{H\mathbf{F}_p}$ is simply connected and $\widetilde{Z}$ is W-null, one need only show that $[X_{H\mathbf{F}_p}, \widetilde{Z}] = *$. But $X_{H\mathbf{F}_p}$ is $\mathbf{Z}\left[\frac{1}{p}\right]$-acyclic (cf. p. 9–40) and $\widetilde{Z}_{H\mathbf{F}_p}$ is W-null (cf. supra), hence $[X_{H\mathbf{F}_p}, \widetilde{Z}] \approx [X_{H\mathbf{F}_p}, \widetilde{Z}_{H\mathbf{F}_p}] \approx [X, \widetilde{Z}_{H\mathbf{F}_p}] \approx [L_W X, \widetilde{Z}_{H\mathbf{F}_p}] \approx [(L_W X)_{H\mathbf{F}_p}, \widetilde{Z}_{H\mathbf{F}_p}] \approx [*, \widetilde{Z}_{H\mathbf{F}_p}] = *$.

LEMMA Let π be a group—then for any pointed connected CW space X, the path components of $C(X, x_0; K(\pi, 1), k_{\pi,1})$ are homotopically trivial.

EXAMPLE Let $\left\{\begin{matrix} A \\ B \end{matrix}\right.$ be pointed connected CW complexes, $\rho : A \rightarrow B$ a pointed continuous function—then the precomposition arrow $\mathrm{Hom}(\pi_1(B), \pi) \rightarrow \mathrm{Hom}(\pi_1(A), \pi)$ determined by $\pi_1(\rho)$ is bijective iff $K(\pi, 1)$ is ρ-local.

EXAMPLE Fix a prime p. Put $W = B\mathbf{Z}/p\mathbf{Z}$—then $K(\pi, 1)$ is W-null iff π has no p-torsion. Example: Z is W-null provided that $\pi_1(Z)$ has no p-torsion and $\widetilde{Z}$ is W-null.

FACT Fix a pointed connected CW complex W—then W is acyclic iff $\forall$ Z, $l_W : X \rightarrow L_W X$ is a homology equivalence.

[Note: Assuming that W is acyclic, X is W-null iff $[W, X] = 0$.]

LEMMA Given ρ_1 & ρ_2, suppose that ρ_2 is a ρ_1-equivalence—then there exists a natural transformation $L_{\rho_2} \rightarrow L_{\rho_1}$ in **HCONCWSP**$_*$ and the class of ρ_2-equivalences is contained in the class of ρ_1-equivalences.

Let $\left\{\begin{matrix} A \\ B \end{matrix}\right.$ be pointed connected CW complexes, $\rho : A \rightarrow B$ a pointed continuous function.

Application: If $\left\{\begin{matrix} A \\ B \end{matrix}\right.$ are n-connected, then $\pi_q(l_\rho) : \pi_q(X) \rightarrow \pi_q(L_\rho X)$ is an isomorphism for $q \leq n$.

[The class of ρ_{n+1}-equivalences, where $\rho_{n+1} : \mathbf{S}^{n+1} \rightarrow *$, is the class of maps $X \rightarrow Y$ inducing isomorphisms in homotopy up to degree n. But ρ is a ρ_{n+1}-equivalence and $X \rightarrow L_\rho X$ is a ρ-equivalence.]

FACT If W is n-connected, then $\pi_{n+1}(l_W) : \pi_{n+1}(X) \rightarrow \pi_{n+1}(L_W X)$ is surjective.

Localization theory has been developed in extenso by Bousfield[†] and Dror Farjoun[‡]. While I shall not pursue these developments in detail, let us at least set up some of the machinery without proof and see how it is used to make computations.

The simplest situation is that of W-nullification, where W is a pointed connected CW complex.

FIBRATION RULE Let $\left\{\begin{matrix} X \\ Y \end{matrix}\right.$ be pointed connected CW spaces, $f : X \to Y$ a pointed continuous function with $\pi_0(E_f) = *$. Suppose that $L_W E_f$ is contractible—then f is a W-equivalence, i.e., the arrow $L_W f : L_W X \to L_W Y$ is a pointed homotopy equivalence.

EXAMPLE Fix a prime p. Put $W = B\mathbf{Z}/p\mathbf{Z}$—then the arrow $W \to *$ is a W-equivalence, thus $L_W K(\mathbf{Z}/p\mathbf{Z}, 1)$ is contractible. So, $\forall\ k$, $L_W K(\mathbf{Z}/p^k\mathbf{Z}, 1)$ is contractible and this implies that $L_W K(\mathbf{Z}/p^\infty\mathbf{Z}, 1)$ is contractible. Examples: (1) From the short exact sequence $0 \to \mathbf{Z} \to \mathbf{Z}\left[\frac{1}{p}\right] \to \mathbf{Z}/p^\infty\mathbf{Z} \to 0$, $\forall\ n \geq 2$, $L_W K(\mathbf{Z}, n) \approx K(\mathbf{Z}\left[\frac{1}{p}\right], n)$; (2) From the short exact sequence $0 \to \widehat{\mathbf{Z}}_p \to \widehat{\mathbf{Q}}_p \to \mathbf{Z}/p^\infty\mathbf{Z} \to 0$ (cf. p. 10–3), $\forall\ n \geq 2$, $L_W K(\widehat{\mathbf{Z}}_p, n) \approx K(\widehat{\mathbf{Q}}_p, n)$.

[Note: $L_W K(\pi, 1)$ is contractible if π is a finite p-group and, when π is in addition abelian, $L_W K(\pi, n)$ is contractible as can be checked by considering $K(\pi, n-1) \to \Theta K(\pi, n) \to K(\pi, n)$.]

ZABRODSKY LEMMA Let $\left\{\begin{matrix} X \\ Y \end{matrix}\right.$ & Z be wellpointed compactly generated connected CW Hausdorff spaces, $f : X \to Y$ a pointed continuous function with $\pi_0(E_f) = *$. Assume: $\text{map}_*(E_f, Z)$ and $\text{map}_*(X, Z)$ are homotopically trivial—then $\text{map}_*(Y, Z)$ is homotopically trivial.

[Note: In this setting, E_f is the compactly generated mapping track. Its base point is $(x_0, j(y_0))$ and the inclusion $\{(x_0, j(y_0))\} \to E_f$ is a closed cofibration (cf. p. 4–33).]

EXAMPLE Miller[‖] has shown that if G is a locally finite group, then every pointed finite dimensional connected CW complex Z is W-null, where $W = BG$. Using the Zabrodsky lemma, it follows by induction that for any locally finite abelian group π, all such Z are $K(\pi, n)$-null.

[Note: A group is said to be locally finite if its finitely generated subgroups are finite. Example: Let X be a pointed simply connected CW space with finitely generated homotopy groups—then the homotopy groups of $E_{l_\mathbf{Q}}(l_\mathbf{Q} : X \to X_\mathbf{Q})$ are locally finite.]

EXAMPLE Suppose that G is a locally finite group with the property that $\#\{n : H_n(G) \neq 0\} < \omega$—then G is acyclic.

[†] *J. Amer. Math. Soc.* **7** (1994), 831–873.

[‡] *Cellular Spaces, Null Spaces and Homotopy Localization*, Springer Verlag (1996).

[‖] *Ann. of Math.* **120** (1984), 39–87.

[ΣBG has the pointed homotopy type of a pointed finite dimensional connected CW complex, so by Miller, $[\Sigma BG, \Sigma BG] = *$. Therefore ΣBG is contractible, thus G is acyclic.]

EXAMPLE Miller (ibid.) has shown that if Z is a pointed nilpotent CW space such that $H_n(Z; \mathbf{F}_p) = 0$ for $n >> 0$, then Z is W-null, where $W = B\mathbf{Z}/p\mathbf{Z}$. Example: $\forall\ n > 0$, $\mathbf{S}^n$ and $\widehat{\mathbf{S}}_p^n$ are W-null.

PRESERVATION RULE Let $\left\{\begin{array}{l} X \\ Y \end{array}\right.$ be pointed connected CW spaces, $f : X \rightarrow Y$ a pointed continuous function with $\pi_0(E_f) = *$. Suppose that Y is W-null—then the arrow $L_W E_f \rightarrow E_{L_W f}$ is a pointed homotopy equivalence.

[Note: The assumption that Y is W-null can be weakened to $L_{\Sigma W} Y \approx L_W Y$.]

EXAMPLE Let X be a pointed simply connected CW complex. Assume: X is finite and $\pi_2(X)$ is torsion—then $\forall\ n \geq 2$, $(L_W \widetilde{X}_n)_{H\mathbf{F}_p} \approx X_{H\mathbf{F}_p}$ ($\widetilde{X}_n$ as on p. 5–38), where $W = B\mathbf{Z}/p\mathbf{Z}$.

[Let E be the mapping fiber of the pointed Hurewicz fibration $\widetilde{X}_n \rightarrow X$. According to Miller's theorem, X is W-null, so $L_W E$ can be identified with the mapping fiber of the arrow $L_W \widetilde{X}_n \rightarrow X$, hence $(L_W E)_{H\mathbf{F}_p}$ can be identified with the mapping fiber of the arrow $(L_W \widetilde{X}_n)_{H\mathbf{F}_p} \rightarrow X_{H\mathbf{F}_p}$. Let $\overline{E}$ be the mapping fiber of the arrow of localization $l_{\overline{p}} : E \rightarrow E_{\overline{p}}$. Since $\pi_2(X) \approx \pi_1(E)$ and $\pi_2(X)$ is torsion, $\pi_1(E)$ maps onto $\pi_1(E)_{\overline{p}}$ (cf. p. 8–10). Therefore $\overline{E}$ is path connected. On the other hand, the nonzero homotopy groups of $\overline{E}$ are finite in number and each of them is a locally finite p-group. From this it follows that $L_W \overline{E}$ is contractible, thus $L_W E \approx L_W E_{\overline{p}} \approx E_{\overline{p}}$. But the homotopy groups of $E_{\overline{p}}$ are uniquely p-divisible which means that $E_{\overline{p}}$ is $\mathbf{F}_p$-acyclic or still, that $(E_{\overline{p}})_{H\mathbf{F}_p}$ is contractible (cf. p. 9–33). Consequently, $(L_W E)_{H\mathbf{F}_p}$ is contractible and $(L_W \widetilde{X}_n)_{H\mathbf{F}_p} \approx X_{H\mathbf{F}_p}$.]

[Note: Here is a numerical illustration. Take $X = \mathbf{S}^3$—then the fibers of the projection $\widetilde{X}_3 \rightarrow X$ have homotopy type $(\mathbf{Z}, 2)$ and $L_W K(\mathbf{Z}, 2) \approx K(\mathbf{Z}\left[\frac{1}{p}\right], 2)$, the mapping fiber of the arrow $L_W \widetilde{X}_3 \rightarrow X$. The potentially nonzero homotopy groups of $K(\mathbf{Z}\left[\frac{1}{p}\right], 2)_{H\mathbf{F}_p}$ are $\text{Ext}(\mathbf{Z}/p^\infty\mathbf{Z}, \mathbf{Z}\left[\frac{1}{p}\right])$ and $\text{Hom}(\mathbf{Z}/p^\infty\mathbf{Z}, \mathbf{Z}\left[\frac{1}{p}\right])$, which in fact vanish, $\mathbf{Z}\left[\frac{1}{p}\right]$ being uniquely p-divisible. Therefore $(L_W K(\mathbf{Z}, 2))_{H\mathbf{F}_p}$ is contractible. Observe too that the mapping fiber of the arrow $(\widetilde{X}_3)_{H\mathbf{F}_p} \rightarrow X_{H\mathbf{F}_p}$ is a $K(\widehat{\mathbf{Z}}_p, 2)$. Because $X_{H\mathbf{F}_p}$ is W-null, $L_W K(\widehat{\mathbf{Z}}_p, 2) \approx K(\widehat{\mathbf{Q}}_p, 2)$ can be identified with the mapping fiber of the arrow $L_W((\widetilde{X}_3)_{H\mathbf{F}_p}) \rightarrow X_{H\mathbf{F}_p}$.]

Given abelian groups G and A, call A G-null if $\text{Hom}(G, A) = 0$. Every abelian group A has a maximal G-null quotient $A//G$.

EXAMPLE Fix an abelian group G. Put $W = M(G, n)$ $(n \geq 2)$ and let P_G be the set of primes p such that G is uniquely divisible by p (P_G has the opposite meaning on p. 9–31). Let X be a pointed connected CW space—then $\pi_q(L_W X) \approx \pi_q(X)$ $(q < n)$ and $\pi_n(L_W X) \approx \pi_n(X)//G$. Moreover, for

$q > n$, $\pi_q(L_W X) \approx \mathbf{Z}_{P_G} \otimes \pi_q(X)$ if $\mathbf{Q} \otimes G = 0$, while if $\mathbf{Q} \otimes G \neq 0$, there is a split short exact sequence $0 \rightarrow \prod_{p \in P_G} \operatorname{Ext}(\mathbf{Z}/p^\infty\mathbf{Z}, \pi_q(X)) \rightarrow \pi_q(L_W X) \rightarrow \prod_{p \in P_G} \operatorname{Hom}(\mathbf{Z}/p^\infty\mathbf{Z}, \pi_{q-1}(X)) \rightarrow 0$.

[Z is W-null iff $\operatorname{Hom}(G, \pi_q(Z)) = 0 = \operatorname{Ext}(G, \pi_q(Z))$ $\forall\, q > n$ and $\operatorname{Hom}(G, \pi_n(Z)) = 0$. This said, reduce to when X is $(n-1)$-connected and show first that $\pi_n(L_W X) \approx \pi_n(X)//G$. Next, set $S_G = \begin{cases} \mathbf{Z}_{P_G} & \text{if } \mathbf{Q} \otimes G = 0 \\ \bigoplus\limits_{p \in P_G} \mathbf{Z}/p\mathbf{Z} & \text{if } \mathbf{Q} \otimes G \neq 0 \end{cases}$. Since $\widetilde{H}_*(W; S_G) = 0$, each HS_G-local space is W-null, thus there is a natural transformation $L_W \rightarrow L_{HS_G}$. Deduce from this that $\pi_q(L_W X) \approx \pi_q(X_{HS_G})$ for $q > n$.]

§ 9

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§10. COMPLETION OF GROUPS

There are many ways to "complete" a group. While the various procedures are related by a web of interconnections, the theory is less systematic than that of §8, one reason for this being that completion functors are generally not idempotent. Still, the material is more or less standard, so I shall omit the details and settle for a survey of what is relevant.

Let G be a topological group. Assume: The left and right uniform structures on G coincide—then the completion $\widehat{G}$ of G is the uniform completion of $G/\overline{\{e\}}$. Therefore $\widehat{G}$ is a uniformly complete Hausdorff topological group which is universal with respect to continuous homomorphisms $G \rightarrow K$, where K is a uniformly complete Hausdorff topological group:

$$\begin{tikzcd} G \arrow[r] \arrow[d] & K \\ \widehat{G} \arrow[ur, dashed] & \end{tikzcd}$$

.

[Note: The assumption is automatic if G is abelian. In this case, $\widehat{G}$ is also abelian. Example: Each prime p determines a metrizable topology on $\mathbf{Q}$ and a corresponding completion $\widehat{\mathbf{Q}}_p$, the field of p-adic numbers. It is homeomorphic to $\coprod\limits_1^\infty C$, C the Cantor set.]

EXAMPLE Let G be a group and let $\{G_i\}$ be a collection of normal subgroups of G directed by inclusion (i.e., $i \leq j \Leftrightarrow G_j \subset G_i$). Equip G with the structure of a topological group by stipulating that the G_i are to be a fundamental system of neighborhoods of e, thus the underlying topology is Hausdorff iff $\bigcap\limits_i G_i = \{e\}$. Because the G_i are normal, the left and right uniform structures on G coincide. On the other hand, the G/G_i are discrete, therefore $\lim G/G_i$ is a uniformly complete Hausdorff topological group and the canonical arrow $\widehat{G} \rightarrow \lim G/G_i$ is an isomorphism of topological groups.

Let G be a group—then by a filtration on G we understand a sequence $\{G_n\}$ of normal subgroups of G such that $\forall\, n$, $G_n \supset G_{n+1}$. The filtration is said to be exhaustive provided that $\bigcup\limits_n G_n = G$. If K is a subgroup of G, $\{K \cap G_n\}$ is a filtration on K (the induced filtration) and if K is a normal subgroup of G, $\{K \cdot G_n/K\}$ is a filtration on G/K (the quotient filtration).

[Note: The n run over $\mathbf{Z}$ but in practice it often happens that $G_0 = G$.]

Let G be a group with a filtration, i.e., a filtered group. Endow G with the structure of a topological group in which the G_n become a fundamental system of neighborhoods of e—then the canonical arrow $\widehat{G} \rightarrow \lim G/G_n$ is an isomorphism of topological groups (cf. supra). More is true: $\widehat{G}_n$ can be identified with the closure of the image of G_n in $\widehat{G}$ and the $\widehat{G}_n$ form a fundamental system of neighborhoods of e in $\widehat{G}$, hence are normal open subgroups of $\widehat{G}$. The topology on $\widehat{G}$ is defined by the filtration $\{\widehat{G}_n\}$. In addition: $G/G_n \approx \widehat{G}/\widehat{G}_n \Rightarrow \lim G/G_n \approx \lim \widehat{G}/\widehat{G}_n \Rightarrow \widehat{G} \approx (\widehat{G})\widehat{\ }$.

[Note: If K is a subgroup of G, the induced topology on K is the topology defined by the induced filtration and if K is a normal subgroup of G, the quotient topology on G/K is the topology defined by the quotient filtration.]

EXAMPLE Let G be a filtered abelian group—then $\forall$ n, there is a short exact sequence $0 \rightarrow G_n \rightarrow G \rightarrow G/G_n \rightarrow 0$. Since $\lim^1 G = 0$, it follows that there is an exact sequence $0 \rightarrow \lim G_n \rightarrow G \rightarrow \lim G/G_n \rightarrow \lim^1 G_n \rightarrow 0$, hence $\lim^1 G_n \approx \widehat{G}/G$ provided that $\bigcap\limits_n G_n = 0$.

(p-Adic Completions) Fix a prime p. Given a group G, let G^{p^n} $(n \geq 0)$ be the subgroup of G generated by the g^{p^n} $(g \in G)$ (take $G^{p^n} = G$ for $n < 0$) and set $G^{p^\omega} = \bigcap\limits_1^\infty G^{p^n}$—then the G^{p^n} filter G, thus one can form $\widehat{G}_p = \lim G/G^{p^n}$, the p-adic completion of G. The assignment $G \rightarrow \widehat{G}_p$ defines a functor $\mathbf{GR} \rightarrow \mathbf{GR}$ and this data generates a triple in $\mathbf{GR}$. In general, $\widehat{G}_p \not\approx (\widehat{G}_p)\widehat{_p}$ but if G is nilpotent, then $\widehat{G}_p$ is nilpotent with $\text{nil}\,\widehat{G}_p = \text{nil}\,G/G^{p^\omega}$ and $\widehat{G}_p \approx (\widehat{G}_p)\widehat{_p}$ (the kernel of the projection $\widehat{G}_p \rightarrow G/G^{p^n}$ is $(\widehat{G}_p)^{p^n}$) (Warfield[†]). Accordingly, p-adic completion restricts to a functor $\mathbf{NIL} \rightarrow \mathbf{NIL}$ and $\mathbf{NIL}\widehat{_p}$, the full subcategory of $\mathbf{NIL}$ whose objects are Hausdorff and complete in the p-adic topology, is a reflective subcategory of $\mathbf{NIL}$. Every object in $\mathbf{NIL}\widehat{_p}$ is p-cotorsion.

[Note: On a subgroup of G, the induced p-adic topology need not agree with the intrinsic p-adic topology. Moreover, the image of G in $\widehat{G}_p$ need not be normal and $(\widehat{G}_p)\widehat{\ }$ is conceptually distinct from $(\widehat{G}_p)\widehat{_p}$.]

Example: Take $G = \mathbf{Z}$—then $\widehat{G}_p = \lim \mathbf{Z}/p^n\mathbf{Z}$ is $\widehat{\mathbf{Z}}_p$, the (ring of) p-adic integers.

[Note: $\widehat{\mathbf{Z}}_p$ is homeomorphic to the Cantor set, hence is uncountable. A p-adic module is a $\widehat{\mathbf{Z}}_p$-module. Example: Let G be an abelian group—then G is a p-adic module if G is p-primary or p-cotorsion.]

EXAMPLE (Nilpotent Groups) Suppose that G is nilpotent—then there is a short exact sequence $1 \rightarrow \text{Ext}(\mathbf{Z}/p^\infty\mathbf{Z}, G)^{p^\omega} \rightarrow \text{Ext}(\mathbf{Z}/p^\infty\mathbf{Z}, G) \rightarrow \widehat{G}_p \rightarrow 1$, hence $\text{Ext}(\mathbf{Z}/p^\infty\mathbf{Z}, G)\widehat{_p} \approx \widehat{G}_p$. Here, $\text{Ext}(\mathbf{Z}/p^\infty\mathbf{Z}, G)^{p^\omega}$ is a p-cotorsion abelian group. It is trivial if $G_{\text{tor}}(p)$ has finite exponent, in particular, if G is finitely generated or torsion free. When G is abelian, $\text{Ext}(\mathbf{Z}/p^\infty\mathbf{Z}, G)^{p^\omega}$ can be alternatively described as $\text{PurExt}(\mathbf{Z}/p^\infty\mathbf{Z}, G)$ (the subgroup of $\text{Ext}(\mathbf{Z}/p^\infty\mathbf{Z}, G)$ which classifies the pure extensions of G by $\mathbf{Z}/p^\infty\mathbf{Z}$) or as $\lim^1 \text{Hom}(\mathbf{Z}/p^n\mathbf{Z}, G)$.

[†] *SLN* **513** (1976), 59–60.

[Note: Proofs of the above assertions can be found in Huber-Warfield[†]. They also show that if $1 \rightarrow G' \rightarrow G \rightarrow G'' \rightarrow 1$ is a short exact sequence of nilpotent groups and if $G''_{\text{tor}}(p)$ has finite exponent, then the sequence $1 \rightarrow \widehat{G}'_p \rightarrow \widehat{G}_p \rightarrow \widehat{G}''_p \rightarrow 1$ is short exact.]

EXAMPLE (*p*-Adic Integers) $\widehat{\mathbf{Z}}_p$ is a principal ideal domain. It is the closure of $\mathbf{Z}$ in $\widehat{\mathbf{Q}}_p$ and $\mathbf{Q} \otimes \widehat{\mathbf{Z}}_p \approx \widehat{\mathbf{Q}}_p$. $\widehat{\mathbf{Z}}_p$ is a local ring with unique maximal ideal $p\widehat{\mathbf{Z}}_p$ and $\widehat{\mathbf{Z}}_p / p\widehat{\mathbf{Z}}_p \approx \mathbf{F}_p$. Examples: (1) $\text{Hom}(\widehat{\mathbf{Z}}_p, \widehat{\mathbf{Z}}_p) \approx \widehat{\mathbf{Z}}_p$; (2) $\text{Hom}(\mathbf{Z}/p^\infty\mathbf{Z}, \mathbf{Z}/p^\infty\mathbf{Z}) \approx \widehat{\mathbf{Z}}_p$; (3) $\widehat{\mathbf{Q}}_p / \widehat{\mathbf{Z}}_p \approx \mathbf{Z}/p^\infty\mathbf{Z}$; (4) $\widehat{\mathbf{Z}}_p \otimes \widehat{\mathbf{Z}}_p \approx \widehat{\mathbf{Z}}_p \oplus 2^\omega \cdot \mathbf{Q}$; (5) $\widehat{\mathbf{Z}}_p^\omega \approx (2^\omega \cdot \widehat{\mathbf{Z}}_p)\widehat{\ }_p$; (6) $\mathbf{Z}^\omega / \omega \cdot \mathbf{Z} \approx 2^\omega \cdot \mathbf{Q} \oplus \prod\limits_p \widehat{\mathbf{Z}}_p^\omega$; (7) $\text{Ext}(\mathbf{Z}/p^\infty\mathbf{Z}, \omega \cdot \mathbf{Z}) \approx (\omega \cdot \mathbf{Z})\widehat{\ }_p$; (8) $\text{Ext}(\widehat{\mathbf{Z}}_p, \mathbf{Z}) \approx \mathbf{Z}/p^\infty\mathbf{Z} \oplus \mathbf{Q}^{2^\omega}$.

EXAMPLE The commutative diagram

$$\begin{tikzcd}
\mathbf{Z}_p \arrow[r] \arrow[d] & \widehat{\mathbf{Z}}_p \arrow[d] \\
\mathbf{Q} \arrow[r] & \widehat{\mathbf{Q}}_p
\end{tikzcd}$$

is simultaneously a pullback and a pushout in **AB**.

FACT The *p*-adic completion functor on **AB** is not right exact. Its 0th left derived functor is $\text{Ext}(\mathbf{Z}/p^\infty\mathbf{Z}, —)$ and its 1st left derived functor is $\text{Hom}(\mathbf{Z}/p^\infty\mathbf{Z}, —)$.

($\mathbf{F}_p$-Completions) Fix a prime p. Given a group G, let $G = \Gamma_p^0(G) \supset \Gamma_p^1(G) \supset \cdots$ be its descending *p*-central series, so $\Gamma_p^{i+1}(G)$ is the subgroup of G generated by $[G, \Gamma_p^i(G)]$ and the g^p $(g \in \Gamma_p^i(G))$. Note that $\Gamma_p^i(G)/\Gamma_p^{i+1}(G)$ is central in $G/\Gamma_p^{i+1}(G)$ and $\Gamma_p^i(G)/\Gamma_p^{i+1}(G)$ is an $\mathbf{F}_p$-module. Moreover, $H_1(G; \mathbf{F}_p) \approx \mathbf{F}_p \otimes (G/[G,G]) \approx G/\Gamma_p^1(G)$. Definition: $\mathbf{F}_p G = \lim G/\Gamma_p^i(G)$ is the $\mathbf{F}_p$-completion of G. The assignment $G \rightarrow \mathbf{F}_p G$ defines a functor $\mathbf{GR} \rightarrow \mathbf{GR}$ and this data generates a triple in $\mathbf{GR}$. In general, $\mathbf{F}_p G \not\approx \mathbf{F}_p \mathbf{F}_p G$ but Bousfield[‡] has shown that if $H_1(G; \mathbf{F}_p)$ is a finitely generated $\mathbf{F}_p$-module, then $\mathbf{F}_p G \approx \mathbf{F}_p \mathbf{F}_p G$. Therefore $\mathbf{F}_p$-completion is idempotent on the class of finitely generated groups or the class of perfect groups.

LEMMA A group G has a finite central series whose factors are elementary abelian *p*-groups iff $\exists\, i : \Gamma_p^i(G) = \{1\}$ or still, iff G is nilpotent and $\exists\, n : G^{p^n} = \{1\}$.

EXAMPLE (Nilpotent Groups) For any group G, $G^{p^i} \subset \Gamma_p^i(G)$ $\forall\, i$, thus there is an arrow $\widehat{G}_p \rightarrow \mathbf{F}_p G$. If in addition G is nilpotent, then $\forall\, n$, G/G^{p^n} is nilpotent and $(G/G^{p^n})^{p^n} = \{1\}$, hence by the lemma $\exists\, i : \Gamma_p^i(G/G^{p^n}) = \{1\} \Rightarrow \Gamma_p^i(G) \subset G^{p^n} \Rightarrow \widehat{G}_p \approx \mathbf{F}_p G$. Corollary: G nilpotent $\Rightarrow \mathbf{F}_p G \approx \mathbf{F}_p \mathbf{F}_p G$.

[†] *J. Algebra* **74** (1982), 402–442.

[‡] *Memoirs Amer. Math. Soc.* **186** (1977), 1–68.

Recall that if $1 \to G' \to G \to G'' \to 1$ is a central extension of groups with G' an $\mathbf{F}_p$-module and G'' $H\mathbf{F}_p$-local, then G is $H\mathbf{F}_p$-local (cf. p. 8–32). Consequently, given any G, it follows by induction that $\forall\ i$, $G/\Gamma_p^i(G)$ is $H\mathbf{F}_p$-local which means that $\mathbf{F}_pG$ is $H\mathbf{F}_p$-local as well (for, being reflective in $\mathbf{GR}$, $\mathbf{GR}_{H\mathbf{F}_p}$ is limit closed). Accordingly, there is a commutative triangle

$$\begin{tikzcd} G \arrow[r] \arrow[d] & \mathbf{F}_pG \\ G_{H\mathbf{F}_p} \arrow[ur] & \end{tikzcd}$$

and the arrow $G_{H\mathbf{F}_p} \to \mathbf{F}_pG$ is an isomorphism iff $G \to \mathbf{F}_pG$ is an $H\mathbf{F}_p$-homomorphism. Example: Suppose that G is a nilpotent group for which $G_{\text{tor}}(p)$ has finite exponent—then $G_{H\mathbf{F}_p} \approx \mathbf{F}_pG$. Proof: $G_{H\mathbf{F}_p} \approx \text{Ext}(\mathbf{Z}/p^\infty\mathbf{Z}, G) \approx \widehat{G}_p \approx \mathbf{F}_pG$.

EXAMPLE Take $G = \bigoplus_1^\infty \mathbf{Z}/p^n\mathbf{Z}$—then the arrow $G_{H\mathbf{F}_p} \to \mathbf{F}_pG$ is not an isomorphism.

[Show that the induced map $H_2(G;\mathbf{F}_p) \to H_2(\mathbf{F}_pG;\mathbf{F}_p)$ is not surjective, hence that $G \to \mathbf{F}_pG$ is not an $H\mathbf{F}_p$-homomorphism.]

FACT Let $f: G \to K$ be an $H\mathbf{F}_p$-homomorphism—then $\forall\ i \geq 0$, the induced map $G/\Gamma_p^i(G) \to K/\Gamma_p^i(K)$ is an isomorphism.

[Note: Compare this result with Proposition 18 in §8.]

Fix a set of primes P. Given a group G, its P-completion PG is $\lim(G/\Gamma^i(G))_P$. The assignment $G \to PG$ defines a functor $\mathbf{GR} \to \mathbf{GR}$ and this data generates a triple in $\mathbf{GR}$. In general, $PG \not\approx PPG$ but Bousfield[†] has shown that if $H_1(G;\mathbf{Z}_P)$ is a finitely generated $\mathbf{Z}_P$-module, then $PG \approx PPG$. Therefore P-completion is idempotent on the class of finitely generated groups or the class of perfect groups.

[Note: It is clear that $PG \approx PPG$ if G is nilpotent.]

P-completion is related to HP-localization in the same way that $\mathbf{F}_p$-completion is related to $H\mathbf{F}_p$-localization. In fact, since $G/\Gamma^i(G)$ is nilpotent, $(G/\Gamma^i(G))_P \approx (G/\Gamma^i(G))_{HP}$ (cf. p. 8–26) $\Rightarrow$ PG is HP-local. Thus there is a commutative triangle

$$\begin{tikzcd} G \arrow[r] \arrow[d] & PG \\ G_{HP} \arrow[ur] & \end{tikzcd}$$

and the arrow $G_{HP} \to PG$ is an isomorphism iff $G \to PG$ is an HP-homomorphism.

EXAMPLE Let π be the fundamental group of the Klein bottle—then the arrow $\pi_{HP} \to P\pi$ is not an isomorphism if $2 \in P$.

[By definition, $\pi \to \pi_{HP}$ is an HP-homomorphism, so $H_2(\pi_{HP};\mathbf{Q}) = 0$. On the other hand, there is a short exact sequence $1 \to \mathbf{Z}_P \oplus \widehat{\mathbf{Z}}_2 \to P\pi \to \mathbf{Z}/2\mathbf{Z} \to 1$ and, from the LHS spectral sequence, $H_2(P\pi;\mathbf{Q}) \approx H_2(\widehat{\mathbf{Z}}_2;\mathbf{Q}) \approx \bigwedge^2_{\mathbf{Q}}(\widehat{\mathbf{Z}}_2 \otimes \mathbf{Q})$, which is uncountable.]

[†] *Memoirs Amer. Math. Soc.* **186** (1977), 1–68.

Notation: Given a category $\mathbf{C}$, $\mathbf{TRI_C}$ is the metacategory whose objects are the triples in $\mathbf{C}$ and $\mathbf{IDTRI_C}$ is the full submetacategory of $\mathbf{TRI_C}$ whose objects are the idempotent triples in $\mathbf{C}$.

[Note: Recall that a morphism of triples is a morphism in the metacategory $\mathbf{MON}_{[\mathbf{C},\mathbf{C}]}$ (cf. p. 0-27).]

THEOREM OF FAKIR[†] Let $\mathbf{C}$ be a category. Assume: $\mathbf{C}$ is complete and wellpowered—then $\mathbf{IDTRI_C}$ is a monocoreflective submetacategory of $\mathbf{TRI_C}$.

[Note: The coreflector sends $\mathbf{T} = (T, m, \epsilon)$ to its idempotent modification $\mathbf{T}^\infty = (T^\infty, m^\infty, \epsilon^\infty)$. In addition: (1) $\forall$ $\mathbf{T}$, $\mathbf{T}$ and $\mathbf{T}^\infty$ have the same equivalences, i.e., a morphism is rendered invertible by T iff it is rendered invertible by T^∞; (2) $\forall$ $\mathbf{T}$, $\epsilon^\infty T : T \to T^\infty \circ T$ is a natural isomorphism.]

Let us take $\mathbf{C} = \mathbf{GR}$ and apply this result to the triple determined by P-completion. Thus, in obvious notation, $P^\infty G$ is the idempotent modification of PG, so $P^\infty G$ embeds in PG while $PG \approx PP^\infty G$ (by (1)) & $PG \approx P^\infty PG$ (by (2)). Of course, those G for which the arrow $G \to P^\infty G$ is an isomorphism constitute the object class of a reflective subcategory of $\mathbf{GR}$. Moreover, $P^\infty G$ is HP-local, hence there is a commutative diagram

$$\begin{tikzcd}
 & G \arrow[dl] \arrow[d] \arrow[dr] & \\
G_P \arrow[r] & G_{HP} \arrow[r] & P^\infty G
\end{tikzcd}$$

. When restricted to $\mathbf{NIL}$, L_P, L_{HP}, and P^∞ are naturally isomorphic but on $\mathbf{GR}$, these functors are distinct (see below).

FACT The arrow $PG \to PPG$ is surjective iff the induced map $H_1(G; \mathbf{Z}_P) \to H_1(PG; \mathbf{Z}_P)$ is surjective.

Claim: $\forall$ G, PG embeds in PPG.

[For $P^\infty G$ embeds in $PG \Rightarrow P^\infty PG$ embeds in PPG, i.e., PG embeds in PPG.]

Therefore $PG \approx PPG$ iff the induced map $H_1(G; \mathbf{Z}_P) \to H_1(PG; \mathbf{Z}_P)$ is surjective. This can be rephrased: $PG \approx PPG$ iff the arrow $G_{HP} \to PG$ is surjective. Proof: Since G_{HP} and PG are HP-local, the arrow $G_{HP} \to PG$ is surjective iff the induced map $H_1(G_{HP}; \mathbf{Z}_P) \to H_1(PG; \mathbf{Z}_P)$ is surjective (cf. p. 8-27).

EXAMPLE Let π be the fundamental group of the Klein bottle—then π is finitely generated, hence $P\pi \approx PP\pi$ and the arrow $\pi_{HP} \to P\pi$ is surjective but, as seen above, it is not an isomorphism if $2 \in P$.

FACT Let $f : G \to K$ be a homomorphism of groups—then the following conditions are equivalent: (1) $P^\infty f : P^\infty G \to P^\infty K$ is an isomorphism; (2) $Pf : PG \to PK$ is an isomorphism; (3) $f \perp PX$ for every group X; (4) $f_* : (G/\Gamma^i(G))_P \to (K/\Gamma^i(K))_P$ is an isomorphism $\forall$ i.

[†] *C. R. Acad. Sci. Paris* **270** (1970), 99–101.

Application: $\forall\, G$, $H_1(G;\mathbf{Z}_P) \approx H_1(P^\infty G;\mathbf{Z}_P)$.

Thus, as a consequence, $\forall\, G$, the induced map $H_1(G_{HP};\mathbf{Z}_P) \rightarrow H_1(P^\infty G;\mathbf{Z}_P)$ is an isomorphism which means that the arrow $G_{HP} \rightarrow P^\infty G$ is surjective (cf. p. 8–27). Corollary: The range of the arrow $G_{HP} \rightarrow PG$ is $P^\infty G$.

[Note: Accordingly, $P^\infty G \approx PG \Leftrightarrow PG \approx PPG \Leftrightarrow H_1(G;\mathbf{Z}_P) \approx H_1(PG;\mathbf{Z}_P)$.]

EXAMPLE Let π be the fundamental group of the Klein bottle—then for any P, π_P is countable (cf. p. 9–23). If now $2 \in P$, then $P^\infty\pi \approx P\pi$ is uncountable, so $\pi_P \not\approx \pi_{HP}$. On the other hand, $\pi_{HP} \not\approx P^\infty\pi$.

FACT Suppose that G is a free group—then the arrow of localization $l_P : G \rightarrow G_P$ is one-to-one.

[Since G is free, the quotients $G/\Gamma^i(G)$ are torsion free nilpotent groups and the intersection $\bigcap_i \Gamma^i(G)$ is trivial.]

(I-Adic Completions) Let A be a ring with unit, $I \subset A$ a two sided ideal. Put $A_n = I^n$ $(n \geq 0)$, $A_n = A$ $(n < 0)$—then $\{A_n\}$ is an exhaustive filtration on A, the associated topology being the I-adic topology. A is a topological ring in the I-adic topology. Moreover, $\widehat{A}$ is a topological ring but in general, $(\widehat{I})^n \neq \widehat{I^n}$ and the $\widehat{I}$-adic topology on $\widehat{A}$ need not agree with the filtration topology.

[Note: Given a left A-module M, put $M_n = I^n \cdot M$ $(n \geq 0)$, $M_n = M$ $(n < 0)$—then $\{M_n\}$ is an exhaustive filtration on M, the associated topology being the I-adic topology. M is a topological left A-module in the I-adic topology. Moreover, $\widehat{M}$ is a topological left $\widehat{A}$-module and $\widehat{M_n} = \widehat{I^n} \cdot \widehat{M} = \widehat{I^n} \cdot \operatorname{im} M$ $\forall\, n$ provided that M is finitely generated (in which case $\widehat{M}$ is finitely generated). Example: Take A commutative and I finitely generated: $\widehat{I^n} = I^n \cdot \widehat{A} \Rightarrow \widehat{I} = I \cdot \widehat{A} \Rightarrow (\widehat{I})^n = I^n \cdot \widehat{A} = \widehat{I^n}$, so, in this situation, the $\widehat{I}$-adic topology on $\widehat{A}$ agrees with the filtration topology.]

Let A be a left Noetherian ring with unit, $I \subset A$ a two sided ideal—then I is said to have the left Artin-Rees property if for every finitely generated left A-module M and every left submodule $N \subset M$, the I-adic topology on N is the restriction of the I-adic topology on M. Example: I has the left Artin-Rees property if $\forall\, M,\, N,\, \exists\, i : I^i \cdot M \cap N \subset I \cdot N$.

[Note: The theory has been surveyed by Smith[†].]

[†] *SLN* **924** (1982), 197–240.

EXAMPLE Fix a group G. Definition: G is said to have the Artin-Rees property if $\mathbf{Z}[G]$ is noetherian and $I[G]$ has the Artin-Rees property. Here, it is not necessary to distinguish between "left" and "right". Example: Every finitely generated nilpotent group G has the Artin-Rees property.

Let A be a ring with unit, $I \subset A$ a two sided ideal—then there is a homomorphism of rings $A \rightarrow \widehat{A}$, hence $\widehat{A}$ can be viewed as an A-bimodule. Given a left A-module M, its formal completion is the left $\widehat{A}$-module obtained from M by extension of the scalars, i.e., the tensor product $\widehat{A} \otimes_A M$.

[Note: A homomorphism $f : M \rightarrow N$ of left A-modules leads to a commutative diagram

$$\begin{tikzcd} \widehat{A} \otimes_A M \arrow[r] \arrow[d] & \widehat{A} \otimes_A N \arrow[d] \\ \widehat{M} \arrow[r, "\widehat{f}"'] & \widehat{N} \end{tikzcd}$$

of left $\widehat{A}$-modules.]

Assume again that A is left noetherian and I has the left Artin-Rees property—then, like in the commutative case, the functor $M \rightarrow \widehat{M}$ is exact on the category of finitely generated left A-modules and for all such M, the arrow $\widehat{A} \otimes_A M \rightarrow \widehat{M}$ is bijective. Moreover $\widehat{A}$, as a right A-module, is flat.

FACT Suppose that A is left and right noetherian and I has the left and right Artin-Rees property. Let M be a left A-module—then $\operatorname{Tor}_*^A(A/I, M) \approx \operatorname{Tor}_*^A(A/I, \widehat{A} \otimes_A M)$.

EXAMPLE Fix a group G with the Artin-Rees property. Let M be a finitely generated G-module—then $H_*(G; M) \approx H_*(G; \widehat{M})$. Consequently, a homomorphism $f : M \rightarrow N$ of finitely generated G-modules is an $H\mathbf{Z}$-homomorphism iff $\widehat{f} : \widehat{M} \rightarrow \widehat{N}$ is an isomorphism.

FACT Suppose that G is a finitely generated nilpotent group. Let M be a finitely generated G-module—then $\widehat{M}$ is $H\mathbf{Z}$-local and the arrow of completion $M \rightarrow \widehat{M}$ is an $H\mathbf{Z}$-homomorphism, thus $M_{H\mathbf{Z}} \approx \widehat{M}$.

EXAMPLE Take $G = \mathbf{Z}/2\mathbf{Z}$ and for any abelian group M, let G operate on M by "negation". In this situation, $M_{H\mathbf{Z}} \approx \operatorname{Ext}(\mathbf{Z}/2^\infty\mathbf{Z}, M)$ and there is a short exact sequence $0 \rightarrow \lim^1 \operatorname{Hom}(\mathbf{Z}/2^n\mathbf{Z}, M) \rightarrow \operatorname{Ext}(\mathbf{Z}/2^\infty\mathbf{Z}, M) \rightarrow \widehat{M} \rightarrow 0$ (cf. p. 8–34). And: The epimorphism $\operatorname{Ext}(\mathbf{Z}/2^\infty\mathbf{Z}, M) \rightarrow \widehat{M}$ has a nonzero kernel if $M = \bigoplus_1^\infty \mathbf{Z}/2^n\mathbf{Z}$.

A Hausdorff topological group G is said to be profinite if it is compact and totally disconnected or, equivalently, that $G \approx \lim G_i$, where i runs over a directed set and $\forall\ i$, G_i is a finite group (discrete topology).

[Note: If G is profinite, then $G \approx \lim G/U$, U open and normal.]

EXAMPLE Let G be a Hausdorff topological group. Assume: G is compact and torsion—then G is profinite.

EXAMPLE Let G be an abelian group—then G is algebraically isomorphic to a profinite abelian group iff G is algebraically isomorphic to a product $\prod\limits_p [\widehat{\mathbf{Z}}_p^{\kappa_p} \times \prod\limits_{i \in I_p} \mathbf{Z}/p^{n_i}\mathbf{Z}]$. Here, κ_p is a cardinal number (possibly zero), I_p is an index set (possibly empty), and n_i is a positive integer.

EXAMPLE Let k be a field, K a Galois extension of k. Put $G = \mathrm{Gal}(K/k)$—then G is a profinite group. In fact, $G \approx \lim G_i$, where $G_i = \mathrm{Gal}(K_i/k)$, K_i a finite Galois extension of k.

[Note: The quotient $G/\overline{[G,G]}$ can be identified with $\mathrm{Gal}(k^{\mathrm{ab}}/k)$, k^{ab} the maximal abelian extension of k in K.]

Given a group G, the profinite completion $\operatorname{pro} G$ of G is $\lim G/U$, the limit being taken over the normal subgroups of finite index in G. The assignment $G \rightarrow \operatorname{pro} G$ defines a functor $\mathbf{GR} \rightarrow \mathbf{GR}$ and this data generates a triple in $\mathbf{GR}$ which, however, is not idempotent.

Example: Take $G = \mathbf{Z}$—then $\operatorname{pro} \mathbf{Z} = \lim \mathbf{Z}/n\mathbf{Z}$ is $\widehat{\mathbf{Z}}$, the (ring of) $\mathbf{\Pi}$-adic integers.

EXAMPLE Every residually finite group embeds in its profinite completion. This said, Evans[†] has shown that for each prime p, there exists a countable, torsion free, residually finite group G such that $\operatorname{pro} G$ contains an element of order p.

EXAMPLE Let $k = \mathbf{F}_p$—then $\mathrm{Gal}(\overline{k}/k) \approx \widehat{\mathbf{Z}}$. Moreover, the infinite cyclic group generated by the Frobenius is dense in $\mathrm{Gal}(\overline{k}/k)$.

EXAMPLE It follows from the positive solution to the congruence subgroup problem for $\mathbf{SL}(n, \mathbf{Z})$ $(n > 2)$ that $\operatorname{pro} \mathbf{SL}(n, \mathbf{Z}) \approx \prod\limits_p \mathbf{SL}(n, \widehat{\mathbf{Z}}_p)$.

EXAMPLE Define a homomorphism $\chi : \widehat{\mathbf{Z}} \rightarrow \operatorname{Aut} \widehat{\mathbf{Z}}$ by $\chi(\widehat{n}) = \mathrm{id}_{\widehat{\mathbf{Z}}}$ if $\widehat{n} \in 2\widehat{\mathbf{Z}}$ and $\chi(\widehat{n}) = -\mathrm{id}_{\widehat{\mathbf{Z}}}$ if $\widehat{n} \notin 2\widehat{\mathbf{Z}}$—then the semidirect product $\widehat{\mathbf{Z}} \rtimes_\chi \widehat{\mathbf{Z}}$ is isomorphic to $\operatorname{pro} \pi$, π the fundamental group of the Klein bottle.

EXAMPLE Let G be a finitely generated nilpotent group—then $\operatorname{pro} G$ is nilpotent and $\operatorname{nil} G = \operatorname{nil} \operatorname{pro} G$. Proof: G is residually finite (cf. p. 8–14), hence embeds in $\operatorname{pro} G$.

[Note: Blackburn's[‡] theorem says that two elements of G are conjugate iff their images in every finite quotient of G are conjugate, i.e., two elements of G are conjugate iff they are conjugate in $\operatorname{pro} G$.]

[†] *J. Pure Appl. Algebra* **65** (1990), 101–104.

[‡] *Proc. Amer. Math. Soc.* **16** (1965), 143–148.

EXAMPLE If $1 \rightarrow G' \rightarrow G \rightarrow G'' \rightarrow 1$ is short exact, then $1 \rightarrow \text{pro}\, G' \rightarrow \text{pro}\, G \rightarrow \text{pro}\, G'' \rightarrow 1$ need not be short exact even when the data is abelian (e.g., pro turns $0 \rightarrow \mathbf{Z} \rightarrow \mathbf{Q} \rightarrow \mathbf{Q}/\mathbf{Z} \rightarrow 0$ into $0 \rightarrow \widehat{\mathbf{Z}} \rightarrow 0 \rightarrow 0 \rightarrow 0$). However, there are positive results. For instance Schneebeli† has shown that pro preserves short exact sequences in the class of polycyclic groups, thus in the class of finitely generated nilpotent groups.

FACT Suppose that G is a finitely generated nilpotent group—then $\forall\, i \geq 0$, $\text{pro}\, \Gamma^i(G) \approx \Gamma^i(\text{pro}\, G)$.

FACT Suppose that G is a finitely generated nilpotent group—then every normal subgroup of $\text{pro}\, G$ of finite index is open.

[Note: This can fail if G is not finitely generated (consider a discontinuous homomorphism $(\mathbf{Z}/p\mathbf{Z})^\omega \rightarrow \mathbf{Z}/p\mathbf{Z}$).]

A group G is said to have property S if for any $\text{pro}\, G$-module M which is finite as an abelian group, $H^n(\text{pro}\, G; M) \approx H^n(G; M)$ $\forall\, n$. Example: Every cyclic group has property S.

FACT Suppose that G is a finitely generated nilpotent group—then G has property S.

[Consider first the case of a central extension $1 \rightarrow K \rightarrow G \rightarrow G/K \rightarrow 1$, where K is cyclic and assume that the assertion holds for G/K. Claim: The assertion holds for G. Indeed, since G is a finitely generated nilpotent group, the sequence $1 \rightarrow \text{pro}\, K \rightarrow \text{pro}\, G \rightarrow \text{pro}\, G/K \rightarrow 1$ is exact (cf. supra), so there is a morphism of LHS spectral sequences

$$\begin{tikzcd}
H^p(\text{pro}\, G/K; H^q(\text{pro}\, K; M)) \arrow[r, Rightarrow] \arrow[d] & H^{p+q}(\text{pro}\, G; M) \arrow[d] \\
H^p(G/K; H^q(K; M)) \arrow[r, Rightarrow] & H^{p+q}(G; M)
\end{tikzcd}$$

which is an isomorphism on the $E_2^{p,q}$. In general, one can find a central series $G = G^0 \supset \cdots \supset G^n = \{1\}$, where $\forall\, i$, G_i is normal in G and G^i/G^{i+1} is cyclic. Proceed from here inductively to see that the G/G_i have property S.]

Although profinite completion is not an idemopotent functor on **GR**, it is idempotent on **TOPGR**, the category of topological groups. Thus let G be a topological group—then its continuous profinite completion $\text{pro}_c G$ is $\lim G/U$, the limit being taken over the open, normal subgroups of finite index in G. With this understanding, $\text{pro}_c G \approx \text{pro}_c \text{pro}_c G$.

[Note: Given a group G, $\text{pro}\, G \approx \text{pro}\,\text{pro}\, G$ iff every normal subgroup of $\text{pro}\, G$ of finite index is open. Corollary: $\text{pro}\, G \approx \text{pro}\,\text{pro}\, G$ iff every homomorphism $G \rightarrow F$, where F is finite, can be extended uniquely to a homomorphism $\text{pro}\, G \rightarrow F$ (in general, $\text{Hom}_c(\text{pro}\, G, F) \approx \text{Hom}(G, F)$, the subscript standing for "continuous". Example: pro is idempotent on the class of finitely generated nilpotent groups.]

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† *Arch. Math.* **31** (1978), 244–253.

FACT Let $f : G \rightarrow K$ be a homomorphism of groups—then $\text{pro}\, f : \text{pro}\, G \rightarrow \text{pro}\, K$ is an isomorphism of topological groups iff $\forall$ finite group F, $\text{Hom}(K, F) \approx \text{Hom}(G, F)$.

[Note: pro is not a conservative functor (Platonov-Tavgen[†]).]

Let G be a profinite group—then G is said to be p-profinite if G is p-local. In this connection, recall that a finite group is a p-group iff it is p-local (cf. p. 8–11). Upon representing G as $\lim G_i$ (cf. p. 10–7), it follows that G is p-profinite iff $\forall\, i$, G_i is p-local.

[Note: Let G be a finite group—then G is p-local iff $\forall\, q \neq p$, the arrow $g \rightarrow g^q$ is surjective.]

EXAMPLE (p-Adic Units) Put $\widehat{\mathbf{U}}_p = \lim(\mathbf{Z}/p^n\mathbf{Z})^\times$—then $\widehat{\mathbf{U}}_p$ is p-profinite. It is the group of units in $\widehat{\mathbf{Z}}_p$. Using the "exp-log" correspondence, one shows that $\widehat{\mathbf{U}}_p \approx \mathbf{Z}/(p-1)\mathbf{Z} \oplus \widehat{\mathbf{Z}}_p$ if p is odd, while $\widehat{\mathbf{U}}_2 \approx \mathbf{Z}/2\mathbf{Z} \oplus \widehat{\mathbf{Z}}_2$.

EXAMPLE Let $\mathbf{Q}^{\text{cy}}$ be the field generated over $\mathbf{Q}$ by the roots of unity in $\overline{\mathbf{Q}}$. For each prime p, choose ω_n subject to $\omega_n^{p^n} = 1$ & $\omega_{n+1}^p = \omega_n$ $(n \geq 1)$. Let K_p be the field generated over $\mathbf{Q}$ by the roots of unity in $\overline{\mathbf{Q}}$ whose order is a power of p—then $K_p = \bigcup\limits_n \mathbf{Q}(\omega_n) \Rightarrow \text{Gal}(K_p/\mathbf{Q}) \approx \lim \text{Gal}(\mathbf{Q}(\omega_n)/\mathbf{Q})$. But $\text{Gal}(\mathbf{Q}(\omega_n)/\mathbf{Q}) \approx (\mathbf{Z}/p^n\mathbf{Z})^\times \Rightarrow \text{Gal}(K_p/\mathbf{Q}) \approx \widehat{\mathbf{U}}_p \Rightarrow \text{Gal}(\mathbf{Q}^{\text{cy}}/\mathbf{Q}) \approx \prod\limits_p \widehat{\mathbf{U}}_p \approx \widehat{\mathbf{Z}}^\times$.

[Note: It follows from global class field theory that $\mathbf{Q}^{\text{cy}}$ is the maximal abelian extension $\mathbf{Q}^{\text{ab}}$ of $\mathbf{Q}$ in $\overline{\mathbf{Q}}$.]

EXAMPLE Suppose that G is p-profinite. Assume: G is torsion—then Zelmanov[‡] has shown that G is locally finite.

Platonov had conjectured that every Hausdorff topological group which is compact and torsion is locally finite (such a group is necessarily profinite (cf. p. 10–7)). Wilson[‖] reduced this to the p-profinite case which was then disposed of by Zelmanov.

Given a group G, the p-profinite completion $\text{pro}_p\, G$ of G is $\lim G/U$, the limit being taken over the normal subgroups of finite index in G subject to $[G : U] \in \{p^n\}$. The assignment $G \to \mathrm{pro}_p\, G$ defines a functor $\mathbf{GR} \to \mathbf{GR}$ and this data generates a triple in $\mathbf{GR}$ which, however, is not idempotent.

[†] *K-Theory* **4** (1990), 89–101.

[‡] *Israel J. Math.* **77** (1992), 83–95.

[‖] *Monatsh. Math.* **96** (1983), 57–66.

[Note: Since $\mathrm{pro}_p\, G$ is p-local, there is a commutative triangle

$$\begin{tikzcd} G \arrow[r] \arrow[d] & \mathrm{pro}_p\, G \\ G_p \arrow[ur] & \end{tikzcd}$$

and a natural transformation $L_p \to \mathrm{pro}_p$.]

Example: Take $G = \mathbf{Z}$—then $\mathrm{pro}_p\, \mathbf{Z} = \lim \mathbf{Z}/p^n\mathbf{Z}$ is $\widehat{\mathbf{Z}}_p$, the (ring of) p-adic integers.

EXAMPLE Define a homomorphism $\chi : \widehat{\mathbf{Z}}_2 \to \mathrm{Aut}\, \widehat{\mathbf{Z}}_2$ by $\chi(\widehat{n}) = \mathrm{id}_{\widehat{\mathbf{Z}}_2}$ if $\widehat{n} \in 2\widehat{\mathbf{Z}}_2$ and $\chi(\widehat{n}) = -\mathrm{id}_{\widehat{\mathbf{Z}}_2}$ if $\widehat{n} \notin 2\widehat{\mathbf{Z}}_2$—then the semidirect product $\widehat{\mathbf{Z}}_2 \rtimes_\chi \widehat{\mathbf{Z}}_2$ is isomorphic to $\mathrm{pro}_2 \pi$, π the fundamental group of the Klein bottle.

[Note: For p odd, $\mathrm{pro}_p\, \pi \approx \widehat{\mathbf{Z}}_p$. Therefore a nonabelian group can have an abelian p-profinite completion.]

LEMMA Suppose that $G/\Gamma_p^1(G)$ is finite—then $\forall\ i > 1$, $G/\Gamma_p^i(G)$ is a finite p-group.

Application: $\dim H_1(G; \mathbf{F}_p) < \omega \Rightarrow \mathrm{pro}_p\, G \approx \mathbf{F}_p G$.

EXAMPLE Let F be a free group on $n > 1$ generators—then $\mathrm{pro}_p\, F \approx \mathbf{F}_p F$ and Bousfield[†] has shown that $H_1(\mathrm{pro}_p\, F; \mathbf{F}_p) \approx n \cdot \mathbf{F}_p$ but for some $q > 1$, $H_q(\mathrm{pro}_p\, F; \mathbf{F}_p)$ is uncountable.

[Note: If F^k is the subgroup of F generated by the kth powers, then it follows from the negative solution to the Burnside problem that F/F^k is infinite provided that $k >> 0$ (Ivanov[‡]). This circumstance makes it difficult to compare $\widehat{F}_p$ and $\mathrm{pro}_p\, F$.]

EXAMPLE For any G there is an arrow $\widehat{G}_p \to \mathrm{pro}_p\, G$. It is an isomorphism if G is finitely generated and nilpotent but not in general (consider $\omega \cdot (\mathbf{Z}/p\mathbf{Z})$).

FACT Suppose that G is a finitely generated nilpotent group—then the arrow $\mathrm{pro}\, G \to \prod_p \mathrm{pro}_p\, G$ is an isomorphism.

[Note: This can fail if G is not nilpotent (consider S_3).]

FACT Suppose that G is a finitely generated nilpotent group. Let K be a subgroup of G—then the p-profinite topology on K is the restriction of the p-profinite topology on G.

[†] *Trans. Amer. Math. Soc.* **331** (1992), 335–359.

[‡] *Bull. Amer. Math. Soc.* **27** (1992), 257–260.

§ 10

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§11. HOMOTOPICAL COMPLETION

In homotopy theory, completion appeared on the scene before localization and, to a certain extent, has been superseded by it. Because of this, a semiproofless account will suffice.

One approach to completing a space at a prime p is due to Bousfield-Kan[†]. It is the analog of the $\mathbf{F}_p$-completion process for groups. Thus there is a functor $X \to \mathbf{F}_pX$ on $\mathbf{HCONCWSP}_*$ called $\underline{\mathbf{F}_p\text{-completion}}$ which is part of a triple. It is not idempotent but $\mathbf{F}_pX$ is $H\mathbf{F}_p$-local so there is a triangle

$$\begin{tikzcd} X \arrow[r] \arrow[d] & \mathbf{F}_pX \\ X_{H\mathbf{F}_p} \arrow[ur] & \end{tikzcd}$$

, commutative up to pointed homotopy. Definition: X is said to be $\underline{\mathbf{F}_p\text{-good}}$ provided that the arrow $X_{H\mathbf{F}_p} \to \mathbf{F}_pX$ is a pointed homotopy equivalence; otherwise, X is said to be $\underline{\mathbf{F}_p\text{-bad}}$. For X to be $\mathbf{F}_p$-good, it is necessary and sufficient that the arrow $\mathbf{F}_pX \to \mathbf{F}_p\mathbf{F}_pX$ be a pointed homotopy equivalence. Therefore $\mathbf{F}_p$-completion is idempotent on the class of $\mathbf{F}_p$-good spaces.

[Note: X is $\mathbf{F}_p$-good iff the arrow $X \to \mathbf{F}_pX$ is an $H\mathbf{F}_p$-equivalence.]

Examples: (1) Let X be a pointed connected CW space—then X is $\mathbf{F}_p$-good if (i) X is nilpotent or (ii) $\pi_1(X)$ is finite or (iii) $H_1(X;\mathbf{F}_p)$ is trivial; (2) Let F be a free group—then $\mathbf{F}_pK(F,1) \approx K(\mathbf{F}_pF,1)$ but $K(F,1)$ is $\mathbf{F}_p$-bad if F is free on two generators, i.e., $\mathbf{S}^1 \vee \mathbf{S}^1$ is $\mathbf{F}_p$-bad (Bousfield[‡]).

As a heuristic guide, $H\mathbf{F}_p$-localization can be thought of as the "idempotent modification" of $\mathbf{F}_p$-completion. Reason: $f : X \to Y$ is an $H\mathbf{F}_p$-equivalence iff $\mathbf{F}_pf : \mathbf{F}_pX \to \mathbf{F}_pY$ is a pointed homotopy equivalence, thus $H\mathbf{F}_p$-localization and $\mathbf{F}_p$-completion have the same equivalences (cf. §9, Proposition 21).

[Note: In a sense that can be made precise, the $\mathbf{F}_p$-completion of a space is but an initial step along the transfinite road to its $H\mathbf{F}_p$-localization (Dror-Dwyer[‖]).]

FIBER THEOREM Let $\begin{cases} X \\ Y \end{cases}$ be pointed connected CW spaces, $f : X \to Y$ a pointed continuous function with $\pi_0(E_f) = *$. Assume: The action of $\pi_1(Y)$ on the $H_n(E_f;\mathbf{F}_p)$ is nilpotent $\forall\ n$—then $\mathbf{F}_pE_f$ can be identified with the mapping fiber of the arrow $\mathbf{F}_pX \to \mathbf{F}_pY$.

[†] *SLN* **304** (1972); see also Iwase, *Trans. Amer. Math. Soc.* **320** (1990), 77–90.

[‡] *Trans. Amer. Math. Soc.* **331** (1992), 335–359.

[‖] *Comment. Math. Helv.* **52** (1977), 185–210; see also *Israel J. Math.* **29** (1978), 141–154.

[Note: The action of $\pi_1(\mathbf{F}_pY)$ on the $H_n(\mathbf{F}_pE_f;\mathbf{F}_p)$ is nilpotent $\forall$ n if E_f is $\mathbf{F}_p$-good, thus if E_f and Y are both $\mathbf{F}_p$-good, then so is X.]

EXAMPLE Suppose that X is a pointed connected CW space with the property that $\pi_1(X)$ operates nilpotently on the $H_n(\widetilde{X};\mathbf{F}_p)$ $\forall$ n—then X is $\mathbf{F}_p$-good if in addition $\pi_1(X)$ is nilpotent.

$\mathbf{F}_p$ WHITEHEAD THEOREM Let $\begin{cases} X \\ Y \end{cases}$ be pointed connected CW spaces, $f : X \rightarrow Y$ a pointed continuous function. Assume: $f_* : H_q(X;\mathbf{F}_p) \rightarrow H_q(Y;\mathbf{F}_p)$ is bijective for $1 \leq q < n$ and surjective for $q = n$—then $\mathbf{F}_pf$ is an n-equivalence.

[Note: To explain the difference in formulation between the $\mathbf{F}_p$ Whitehead theorem and the $H\mathbf{F}_p$ Whitehead theorem (cf. p. 9–33), one has only to recall that the arrows $\begin{cases} X \rightarrow X_{H\mathbf{F}_p} \\ Y \rightarrow Y_{H\mathbf{F}_p} \end{cases}$ are $H\mathbf{F}_p$-equivalences.]

Application: X n-connected $\Rightarrow$ $\mathbf{F}_pX$ n-connected.

EXAMPLE Define functors $L_n^p : \mathbf{GR} \rightarrow \mathbf{GR}$ by writing $L_n^pG = \pi_{n+1}(\mathbf{F}_pK(G,1))$ $(n \geq 0)$. So, e.g., for any pointed connected CW space X, $\pi_1(\mathbf{F}_pX) \approx L_0^p\pi_1(X)$ ($\mathbf{F}_p$ Whitehead theorem). Since $\mathbf{F}_pK(G,1)$ is $H\mathbf{F}_p$-local, L_n^pG is abelian p-cotorsion $(n \geq 1)$. Examples: (1) If G is free, then $L_0^pG \approx \mathbf{F}_pG$ and $L_n^pG = 0$ $(n \geq 1)$; (2) If G is nilpotent, then $L_0^pG \approx \mathrm{Ext}(\mathbf{Z}/p^\infty\mathbf{Z}, G)$, $L_1^pG \approx \mathrm{Hom}(\mathbf{Z}/p^\infty\mathbf{Z}, G)$, and $L_n^pG = 0$ $(n \geq 2)$; (3) If G is finite, then L_n^pG is a finite p-group which is trivial when p and $\#(G)$ are relatively prime.

[Note: $\forall$ G, there is a surjection $L_0^pG \rightarrow \mathbf{F}_pG$ (Bousfield[†]) which is a bijection whenever $H_1(G;\mathbf{F}_p)$ and $H_2(G;\mathbf{F}_p)$ are finite dimensional, e.g., if G is finitely presented (Brown[‡]).]

EXAMPLE Let A be a ring with unit—then the arrow $B\mathbf{GL}(A) \rightarrow B\mathbf{GL}(A)^+$ is a homology equivalence, hence is an $H\mathbf{F}_p$-equivalence. Therefore $L_n^p\mathbf{GL}(A) \approx \pi_{n+1}(\mathbf{F}_pB\mathbf{GL}(A)^+)$, so if the $K_n(A)$ are finitely generated, $L_n^p\mathbf{GL}(A) \approx \widehat{\mathbf{Z}}_p \otimes K_{n+1}(A)$ (cf. p. 9–35).

Here is a final point. Fix a set of primes P—then Bousfield-Kan (ibid.) have shown that the P-completion process for groups can be imitated in the homotopy category, i.e., there is a functor $X \rightarrow PX$ on $\mathbf{HCONCWSP}_*$ called P-completion which is part of a triple. Its formal properties are identical to those of $\mathbf{F}_p$-completion and its "idempotent modification" is HP-localization. Example: $\mathbf{P}^2(\mathbf{R})$ is P-bad if $2 \in P$ but $\mathbf{P}^2(\mathbf{R})$ is $\mathbf{F}_p$-good $\forall$ p (since $\pi_1(\mathbf{P}^2(\mathbf{R})) \approx \mathbf{Z}/2\mathbf{Z}$ is finite).

Another approach to completing a space at a prime p is due to Sullivan[‖]. In this context, there is also an analog of the profinite completion process for groups and we shall consider it first.

[†] *Memoirs Amer. Math. Soc.* **186** (1977), 1–68 (cf. 66).

[‡] *Cohomology of Groups*, Springer Verlag (1982), 197–198.

[‖] *Ann. of Math.* **100** (1974), 1–79.

Notation: $\mathbf{F}_*$ is the full subcategory of $\mathbf{CONCW}_*$ whose objects are the pointed connected CW complexes with finite homotopy groups and $\mathbf{HF}_*$ is the associated homotopy category.

[Note: Any skeleton $\overline{\mathbf{HF}}_*$ of $\mathbf{HF}_*$ is small.]

LEMMA For every pointed connected CW complex X, the category $X\backslash\overline{\mathbf{HF}}_*$ is cofiltered.

[This is because $\mathbf{HF}_*$ has finite products and weak pullbacks.]

[Note: The objects of $X\backslash\overline{\mathbf{HF}}_*$ are the pointed homotopy classes of maps $X \rightarrow K$ and the morphisms $(X \rightarrow K) \rightarrow (X \rightarrow L)$ are the pointed homotopy commutative triangles

$$\begin{tikzcd} & X \arrow[dl] \arrow[dr] & \\ K \arrow[rr] & & L \end{tikzcd}$$

.]

In what follows, $\lim\limits_{X}$ stands for a limit calculated over $X\backslash\overline{\mathbf{HF}}_*$.

PROPOSITION 1 For every pointed connected CW complex X, the cofunctor F_X : $\mathbf{HCONCW}_* \rightarrow \mathbf{SET}$ defined by $F_X Y = \lim\limits_{X}[Y, K]$ is representable.

[It is a question of applying the Brown representability theorem. That F_X satisfies the wedge condition is automatic. Turning to the Mayer-Vietoris condition, if Y_k is a pointed finite connected subcomplex of Y, then $[Y_k, K]$ is finite (cf. p. 5–49). Give it the discrete topology and form $\lim[Y_k, K]$, a nonempty compact Hausdorff space. Since $[Y, K] \approx \lim[Y_k, K]$ (cf. p. 5–89), it follows that there is a factorization

$$\begin{tikzcd} \mathbf{HCONCW}_* \arrow[r, dashed] \arrow[dr, "F_X"'] & \mathbf{CPTHAUS} \arrow[d, "U"] \\ & \mathbf{SET} \end{tikzcd}$$

, where U is the forgetful functor. The verification that F_X satisfies the Mayer-Vietoris condition is now straightforward.]

The profinite completion of X, denoted $\text{pro}\,X$, is an object that represents F_X. There is a natural transformation $[—, X] \rightarrow [—, \text{pro}\,X]$ and an arrow $\text{pro}_X : X \rightarrow \text{pro}\,X$ (Yoneda).

[Note: Profinite completion generates a triple in $\mathbf{HCONCW}_*$ (or $\mathbf{HCONCWSP}_*$) which, however, is not idempotent.]

EXAMPLE Let G be a topological group. Assume: G is Lie and $\#(\pi_0(G)) < \omega$—then B_G^∞ is metrizable (cf. p. 4–64) (B_G^∞ is even an ANR (cf. p. 6–45)), in particular B_G^∞ is a compactly generated

Hausdorff space. And: For every pointed finite dimensional connected CW complex X, $\mathrm{map}_*(B_G^\infty, \mathrm{pro}\,X)$ is homotopically trivial (Friedlander-Mislin[†]).

[Note: Taking $G = \mathbf{S}^1$, the Zabrodsky lemma and induction imply that $\forall\, n \geq 2$, $\mathrm{map}_*(K(\mathbf{Z}, n), \mathrm{pro}\,X)$ is homotopically trivial.]

FACT Let X be a pointed connected CW complex—then for any CW complex Y, the arrow $[Y, \mathrm{pro}\,X] \rightarrow \lim\limits_{X}[Y, K]$ is bijective.

[Note: In this context, the brackets refer to homotopy classes of maps, not to pointed homotopy classes of pointed maps.]

The homotopy groups of $\mathrm{pro}\,X$ are profinite. Proof: $\pi_n(\mathrm{pro}\,X) \approx [\mathbf{S}^n, \mathrm{pro}\,X] \approx \lim\limits_{X}[\mathbf{S}^n, K]$ and the $[\mathbf{S}^n, K]$ are finite.

[Note: It follows that $\forall\, n$, there is a commutative triangle

$$\begin{tikzcd} & \pi_n(X) \arrow[dl] \arrow[dr] & \\ \mathrm{pro}\,\pi_n(X) \arrow[rr] & & \pi_n(\mathrm{pro}\,X) \end{tikzcd}$$

.]

PROPOSITION 2 Let X be a pointed connected CW complex—then $\pi_1(\mathrm{pro}\,X) \approx \mathrm{pro}\,\pi_1(X)$.

[The full subcategory of $X\backslash\overline{\mathbf{HF}}_*$ consisting of those objects $X \rightarrow K$ such that the induced map $\pi_1(X) \rightarrow \pi_1(K)$ is surjective is an initial subcategory. To see this, let $\widetilde{K} \rightarrow K$ be the covering of K corresponding to $\mathrm{im}\,\pi_1(X)$, and consider

$$\begin{tikzcd} & \widetilde{K} \arrow[d] \\ X \arrow[ur] \arrow[r] & K \end{tikzcd}$$

. On the other hand, for any normal subgroup G of $\pi_1(X)$ of finite index, there is an arrow $X \rightarrow K(\pi_1(X)/G, 1)$.]

EXAMPLE The arrow $\mathrm{pro}\,\pi_n(X) \rightarrow \pi_n(\mathrm{pro}\,X)$ is not necessarily bijective when $n > 1$. Thus take $X = \mathbf{S}^1 \vee \Sigma\mathbf{P}^2(\mathbf{R})$—then $\pi_1(X) \approx \mathbf{Z}$, $\pi_2(X) \approx \omega \cdot (\mathbf{Z}/2\mathbf{Z})$ and $\pi_1(\mathrm{pro}\,X) \approx \widehat{\mathbf{Z}}$, $\pi_2(\mathrm{pro}\,X) \approx (\mathbf{Z}/2\mathbf{Z})^\omega$ but $\mathrm{pro}\,\pi_2(X) \approx \mathrm{Hom}((\mathbf{Z}/2\mathbf{Z})^\omega, \mathbf{Z}/2\mathbf{Z})$.

LEMMA Suppose that G is a finitely generated abelian group—then $\mathrm{pro}\,K(G, n) \approx K(\mathrm{pro}\,G, n)$.

EXAMPLE $\mathrm{pro}\,K(\mathbf{Z}, n) \approx K(\widehat{\mathbf{Z}}, n)$ but $\mathrm{pro}\,K(\mathbf{Q}/\mathbf{Z}, n) \approx K(\widehat{\mathbf{Z}}, n + 1)$.

[†] *Invent. Math.* **83** (1986), 425–436.

EXAMPLE Consider $K(\mathbf{Z},2;\chi)$, where $\chi : \mathbf{Z}/2\mathbf{Z} \to \operatorname{Aut}\mathbf{Z}$ is the nontrivial homomorphism (so $K(\mathbf{Z},2;\chi) \approx B_{\mathbf{O}(2)}$ (cf. p. 5–32))—then χ extends to a homomorphism $\widehat{\chi} : \mathbf{Z}/2\mathbf{Z} \to \operatorname{Aut}\widehat{\mathbf{Z}}$ and $\operatorname{pro} K(\mathbf{Z},2;\chi) \approx K(\widehat{\mathbf{Z}},2;\widehat{\chi})$.

FACT Let X be a pointed connected CW complex—then $\forall\ q$, $H^q(X;\widehat{\mathbf{Z}}) \approx \lim_n H^q(X;\mathbf{Z}/n\mathbf{Z})$.

$[H^q(X;\widehat{\mathbf{Z}}) \approx [X,K(\widehat{\mathbf{Z}},q)] \approx [X,\operatorname{pro} K(\mathbf{Z},q)] \approx \lim_n [X,K(\mathbf{Z}/n\mathbf{Z},q)] \approx \lim_n H^q(X;\mathbf{Z}/n\mathbf{Z}).]$

FACT Let X be a pointed connected CW complex—then $\forall\ q$, $H^q(X;\widehat{\mathbf{Z}}) \approx \lim H^q(X_k;\widehat{\mathbf{Z}})$, where X_k runs over the pointed finite connected subcomplexes of X.

$[H^q(X;\mathbf{Z}/n\mathbf{Z}) \approx [X,K(\mathbf{Z}/n\mathbf{Z},q)] \approx \lim[X_k,K(\mathbf{Z}/n\mathbf{Z},q)] \approx \lim H^q(X_k;\mathbf{Z}/n\mathbf{Z})$ (cf. p. 5–89) $\Rightarrow$ $H^q(X;\widehat{\mathbf{Z}}) \approx \lim_n H^q(X;\mathbf{Z}/n\mathbf{Z}) \approx \lim_n \lim H^q(X_k;\mathbf{Z}/n\mathbf{Z}) \approx \lim \lim_n H^q(X_k;\mathbf{Z}/n\mathbf{Z}) \approx \lim H^q(X_k;\widehat{\mathbf{Z}}).]$

In general, it is difficult to relate the higher homotopy groups of $\operatorname{pro} X$ to those of X itself except under the most favorable circumstances.

PROPOSITION 3 Let X be a pointed nilpotent CW space with finitely generated homotopy groups—then $\forall\ n$, $\pi_n(\operatorname{pro} X) \approx \operatorname{pro} \pi_n(X)$.

[Note: Recall that a particular choice for the abelian groups figuring in a principal refinement of order n of $X[n] \to X[n-1]$ are the $\Gamma^i_{\chi_n}(\pi_n(X))/\Gamma^{i+1}_{\chi_n}(\pi_n(X))$ (cf. p. 5–60). Since the π_n are finitely generated, there is a unique continuous nilpotent action of $\operatorname{pro}\pi_1(X)$ on $\operatorname{pro}\pi_n(X)$ compatible with the action of $\pi_1(X)$ on $\pi_n(X)$. This said, Hilton-Roitberg[†] have shown that, in obvious notation, (i) $\operatorname{nil}_{\chi_n}\pi_n(X) = \operatorname{nil}_{\operatorname{pro}\chi_n}\operatorname{pro}\pi_n(X)$ and (ii) $\operatorname{pro}(\Gamma^i_{\chi_n}(\pi_n(X))/\Gamma^{i+1}_{\chi_n}(\pi_n(X))) \approx \Gamma^i_{\operatorname{pro}\chi_n}(\operatorname{pro}\pi_n(X))/\Gamma^{i+1}_{\operatorname{pro}\chi_n}(\operatorname{pro}\pi_n(X))$. Since profinite completion preserves short exact sequences of finitely generated nilpotent groups (cf. p. 10–9), the conclusion is that the arrow $(\operatorname{pro} X)[n] \to (\operatorname{pro} X)[n-1]$ admits a "canonical" principal refinement of order n, viz. apply pro to the "canonical" principal refinement of order n of $X[n] \to X[n-1]$. Corollary: Under the stated assumptions on X, $\operatorname{pro} X$ is nilpotent (but the unconditional assertion "X nilpotent $\Rightarrow$ $\operatorname{pro} X$ nilpotent" is seemingly in limbo).]

Example: $\mathbf{S}^n = M(\mathbf{Z},n)$ but $\operatorname{pro}\mathbf{S}^n \neq M(\operatorname{pro}\mathbf{Z},n)$.

FACT Let X be a pointed nilpotent CW space with finitely generated homotopy groups—then for every pointed finite connected CW complex K, the arrow $[K,X] \to [K,\operatorname{pro} X]$ is injective.

[Note: As a reality check, take $K = \mathbf{S}^1$ and $X = K(G,1)$, where G is a finitely generated nilpotent group, and observe that the injectivity of the arrow $[\mathbf{S}^1,K(G,1)] \to [\mathbf{S}^1,K(\operatorname{pro} G,1)]$ is equivalent to the assertion that G embeds in $\operatorname{pro} G$ (cf. p. 10–8).]

[†] *J. Algebra* **60** (1979), 289–306.

Application: Let Y be a pointed nilpotent CW space with finitely generated homotopy groups—then for every pointed connected CW space X, $\mathrm{Ph}(X, Y)$ is the kernel of the arrow $[X, Y] \rightarrow [X, \mathrm{pro}\,Y]$.

LEMMA Let $\{G_n, f_n : G_{n+1} \rightarrow G_n\}$ be a tower in **GR**. Assume: $\forall\, n$, G_n is a compact Hausdorff topological group and f_n is a continuous homomorphism—then $\lim^1 G_n = *$.

[Note: The result is false if the "Hausdorff" hypothesis is dropped.]

EXAMPLE Let X be a pointed connected CW complex with a finite number of cells in each dimension; let Y be a pointed nilpotent CW space with finitely generated homotopy groups—then $\forall\, n$, $[\Sigma X^{(n)}, \mathrm{pro}\,Y]$ is a compact Hausdorff topological group and the arrow $[\Sigma X^{(n+1)}, \mathrm{pro}\,Y] \rightarrow [\Sigma X^{(n)}, \mathrm{pro}\,Y]$ is a continuous homomorphism. So, by the lemma, $\lim^1 [\Sigma X^{(n)}, \mathrm{pro}\,Y] = *$, i.e., $\mathrm{Ph}(X, \mathrm{pro}\,Y) = *$ (cf. p. 5–49).

Claim: A pointed continuous function $f : X \rightarrow Y$ is a phantom map iff $\mathrm{pro}_Y \circ f \simeq 0$.

[Necessity: $f \in \mathrm{Ph}(X, Y) \Rightarrow \mathrm{pro}_Y \circ f \in \mathrm{Ph}(X, \mathrm{pro}\,Y) \Rightarrow \mathrm{pro}_Y \circ f \simeq 0$.

Sufficiency: Let $\phi : K \rightarrow X$ be a pointed continuous function, where K is a pointed finite connected CW complex—then $\mathrm{pro}_Y \circ f \circ \phi \simeq 0 \Rightarrow f \circ \phi \simeq 0$, the arrow $[K, Y] \rightarrow [K, \mathrm{pro}\,Y]$ being one-to-one.]

LEMMA Let $\begin{cases} X \\ Y \end{cases}$ be pointed simply connected CW spaces with finitely generated homotopy groups—then the function space of pointed continuous functions $X_{\mathbf{Q}} \rightarrow \mathrm{pro}\,Y$ is homotopically trivial (compact open topology).

[Adopt the conventions on p. 9–38 and work with $\mathrm{map}_*(X_{\mathbf{Q}}, \mathrm{pro}\,Y)$. Since $\Sigma^n X_{\mathbf{Q}} \approx (\Sigma^n X)_{\mathbf{Q}}$ (cf. p. 9–12), $\widetilde{H}_*(\Sigma^n X_{\mathbf{Q}}; \mathbf{F}_p) = 0\ \forall\, p$, thus $\widetilde{H}^*(\Sigma^n X_{\mathbf{Q}}; \pi_q(\mathrm{pro}\,Y)) = 0\ \forall\, q$ (the $\pi_q(\mathrm{pro}\,Y)$ are cotorsion). Accordingly, by obstruction theory (cf. p. 5–43), $\forall\, n \geq 0$, $[\Sigma^n X_{\mathbf{Q}}, \mathrm{pro}\,Y] = *$.]

EXAMPLE Let $\begin{cases} X \\ Y \end{cases}$ be pointed simply connected CW spaces with finitely generated homotopy groups—then $\mathrm{Ph}(X, Y) = l_{\mathbf{Q}}^*[X_{\mathbf{Q}}, Y] \subset [X, Y]$.

[There is no loss of generality in supposing that X is a pointed simply connected CW complex with a finite number of cells in each dimension (cf. p. 5–23).

($\mathrm{Ph}(X, Y) \subset l_{\mathbf{Q}}^*[X_{\mathbf{Q}}, Y]$) Fix an $f \in \mathrm{Ph}(X, Y)$. From the above, $\mathrm{pro}_Y \circ f \simeq 0$, so $\exists$ a $g : X \rightarrow E$ such that $f = \pi \circ g$, E the mapping fiber of pro_Y and $\pi : E \rightarrow Y$ the projection. Since E is rational (each of its homotopy groups is a direct sum of copies of $\widehat{\mathbf{Z}}/\mathbf{Z}$), $\exists$ an $h : X_{\mathbf{Q}} \rightarrow E$ such that $g \simeq h \circ l_{\mathbf{Q}}$, thus $f \simeq f_{\mathbf{Q}} \circ l_{\mathbf{Q}}$, where $f_{\mathbf{Q}} = \pi \circ h : X_{\mathbf{Q}} \rightarrow Y$.

($l_{\mathbf{Q}}^*[X_{\mathbf{Q}}, Y] \subset \mathrm{Ph}(X, Y)$) Assume that $f \simeq f_{\mathbf{Q}} \circ l_{\mathbf{Q}}$, where $f_{\mathbf{Q}} : X_{\mathbf{Q}} \rightarrow Y$. Thanks to the lemma, the composite $\mathrm{pro}_Y \circ f_{\mathbf{Q}}$ is nullhomotopic, hence $\mathrm{pro}_Y \circ f$ is too.]

FACT Let X be a pointed nilpotent CW space with finitely generated homotopy groups—then for every finite CW complex K, the arrow $[K, X] \rightarrow [K, \mathrm{pro}\,X]$ is injective.

[Note: In this context, the brackets refer to homotopy classes of maps, not to pointed homotopy classes of pointed maps.]

EXAMPLE The preceding result has content even when K is connected. Thus, restoring the base points, it follows that the arrow $\pi_1(X)\backslash[K, k_0; X, x_0] \rightarrow \pi_1(\text{pro}\, X)\backslash[K, k_0; \text{pro}\, X, \text{pro}\, x_0]$ is one-to-one. Specializing this to $K = \mathbf{S}^1$, $X = K(G, 1)$, where G is a finitely generated nilpotent group, one recovers Blackburn's theorem (cf. p. 10–8).

PROPOSITION 4 Let X be a pointed nilpotent CW space with finitely generated homotopy groups—then for every locally constant coefficient system $\mathcal{G}$ on $\text{pro}\, X$ arising from a finite $\text{pro}\, \pi_1(X)$-module, $H^*(\text{pro}\, X; \mathcal{G}) \approx H^*(X; \text{pro}_X^* \mathcal{G})$.

[The main idea here is to proceed inductively, playing off $K(\pi_n(X), n) \rightarrow P_n X \rightarrow P_{n-1} X$ against $\text{pro}\, K(\pi_n(X), n) \rightarrow \text{pro}\, P_n X \rightarrow \text{pro}\, P_{n-1} X$ (use the cohomological version of the fibration spectral sequence formulated on p. 5–69). To get the induction off the ground, one has to deal with $K(\pi_1(X), 1)$, the point being that $\pi_1(X)$ has property S (cf. p. 10–9).]

LEMMA Let $\begin{cases} X \\ Y \end{cases}$ & Z be pointed connected CW spaces, $f: X \rightarrow Y$ a pointed continuous function—then the precomposition arrow $f^*: [Y, Z] \rightarrow [X, Z]$ is bijective whenever Z has finite homotopy groups iff

(A₁) $\text{Hom}(\pi_1(Y), F) \approx \text{Hom}(\pi_1(X), F)$ for any finite group F

and

(A₂) $H^n(Y; \mathcal{G}) \approx H^n(X; f^*\mathcal{G})$ $\forall\, n$ for any locally constant coefficient system $\mathcal{G}$ on Y arising from a finite $\pi_1(Y)$-module.

[Tailor the proof of Proposition 11 in §9 to the setup at hand.]

PROPOSITION 5 Let X be a pointed nilpotent CW space with finitely generated homotopy groups—then every pointed continuous function $\phi: X \rightarrow K$, where K is a pointed connected CW complex with finite homotopy groups, admits a continuous extension $\text{pro}\, \phi: \text{pro}\, X \rightarrow K$ which is unique up to pointed homotopy.

[Each homomorphism $\pi_1(X) \rightarrow F$, where F is finite, can be extended uniquely to a homomorphism $\text{pro}\, \pi_1(X) \rightarrow F$ (cf. p. 10–9 ff.), therefore A₁ holds. That A₂ holds is the content of Proposition 4.]

Application: pro is idempotent on the class of pointed nilpotent CW spaces with finitely generated homotopy groups.

Fix a prime p—then upon replacing "finite group" by "finite p-group" in the foregoing, one arrives at the p-profinite completion $\text{pro}_p X$ of X. Modulo minor changes, the theory carries over in the expected way. Consider, e.g., Proposition 4. There it is necessary to look only at those $\mathcal{G}$ whose underlying $\text{pro}_p\pi_1(X)$-module G is a finite abelian p-group such that the associated homomorphism $\text{pro}_p\pi_1(X) \to \text{Aut}\, G$ factors through a p-subgroup of $\text{Aut}\, G$. Another point to bear in mind is that p-adic completion preserves short exact sequences of finitely generated nilpotent groups (cf. p. 10–9) and p-adic completion = p-profinite completion in the class of finitely generated nilpotent groups (cf. p. 10–11).

EXAMPLE Let X be a pointed simply connected CW complex with a finite number of cells in each dimension. Denote by $\text{pro}_{p,T}X$ the pointed mapping telescope of the sequence $\{\text{pro}_p X^{(n)} \to \text{pro}_p X^{(n+1)}\}$—then $\forall\, n$, $\pi_n(\text{pro}_{p,T}X) \approx \widehat{\mathbf{Z}}_p \otimes \pi_n(X) \Rightarrow \text{pro}_{p,T}X \approx \text{pro}_p X$.

It is clear that $\forall\, p$, there is an arrow $\text{pro}\, X \to \text{pro}_p X$, from which an arrow $\text{pro}\, X \to \prod\limits_p \text{pro}_p X$ (product in **HTOP**$_*$). And: $[\mathbf{S}^n, \text{pro}\, X] \to [\mathbf{S}^n, \prod\limits_p \text{pro}_p X] \Rightarrow \pi_n(\text{pro}\, X) \to \prod\limits_p \pi_n(\text{pro}_p X)$.

PROPOSITION 6 Let X be a pointed nilpotent CW space with finitely generated homotopy groups—then the arrow $\text{pro}\, X \to \prod\limits_p \text{pro}_p X$ is a weak homotopy equivalence.

[In this situation, $\forall\, n$, $\pi_n(\text{pro}\, X) \approx \text{pro}\, \pi_n(X)$ & $\pi_n(\text{pro}_p X) \approx \text{pro}_p \pi_n(X)$. Moreover, for any finitely generated nilpotent group G, the arrow $\text{pro}\, G \to \prod\limits_p \text{pro}_p G$ is an isomorphism (cf. p. 10–11).]

[Note: If the product is taken in **HCWSP**$_*$ (cf. p. 9–1), then the arrow $\text{pro}\, X \to \prod\limits_p \text{pro}_p X$ is a pointed homotopy equivalence.]

EXAMPLE Let $X = B_{\mathbf{O}(2)}$ (cf. p. 11–5)—then, in obvious notation, $\text{pro}_2 X \approx K(\widehat{\mathbf{Z}}_2, 2; \widehat{\chi}_2)$ but at an odd prime p, $\text{pro}_p X$ is simply connected and in fact $\Omega \text{pro}_p X \approx \widehat{\mathbf{S}}^3_p$. Thus here, it is false that the arrow $\text{pro}\, X \to \prod\limits_p \text{pro}_p X$ is a weak homotopy equivalence.

Let X be a pointed nilpotent CW space—then $\text{pro}_p X$ and $X_{H\mathbf{F}_p} (= \mathbf{F}_p X)$ are, in general, not the "same." Reason: pro_p fails to be idempotent. However, when the homotopy groups of X are finitely generated, $\pi_n(\text{pro}_p X) \approx \text{pro}_p \pi_n(X) \approx \text{Ext}(\mathbf{Z}/p^\infty\mathbf{Z}, \pi_n(X)) \approx \pi_n(X_{H\mathbf{F}_p})$. Therefore $\text{pro}_p X$ is $H\mathbf{F}_p$-local (cf. §9, Proposition 20) ($\text{pro}_p X$ is nilpotent) and in this case, $\text{pro}_p X \approx X_{H\mathbf{F}_p} (= \widehat{X}_p)$.

[Note: It is a fact that for nilpotent X, $\text{pro}_p X \approx X_{H\mathbf{F}_p}$ under the sole hypothesis that $\forall\, n$, $H^n(X; \mathbf{F}_p)$ is finite dimensional (cf. p. 11–11). In this connection, recall that if the homotopy groups of a nilpotent X are finitely generated, then the $H_n(X)$ are finitely generated (cf. §5, Proposition 18), hence $\forall\, n$, $H^n(X; \mathbf{F}_p)$ is finite dimensional.]

PROPOSITION 7 Let X be a path connected topological space—then the following conditions are equivalent:

(CO$_1$) $\forall\, n$, $H^n(X;\mathbf{F}_p)$ is finite dimensional;
(HO$_1$) $\forall\, n$, $H_n(X;\mathbf{F}_p)$ is finite dimensional;
(CO$_2$) $\forall\, n$, $H^n(X;\widehat{\mathbf{Z}}_p)$ is finitely generated over $\widehat{\mathbf{Z}}_p$;
(HO$_2$) $\forall\, n$, $H_n(X;\widehat{\mathbf{Z}}_p)$ is finitely generated over $\widehat{\mathbf{Z}}_p$;
(CO$_3$) $\forall\, n$, $H^n(X;\mathbf{Z}_p)$ is finitely generated over $\mathbf{Z}_p$;
(HO$_3$) $\forall\, n$, $H_n(X;\mathbf{Z}_p)$ is finitely generated over $\mathbf{Z}_p$;
(CO$_4$) $\forall\, n$, $H^n(X;\mathbf{Q})$ is finite dimensional and $H^n(X;\mathbf{Z})_{\text{tor}}(p)$ is finite;
(HO$_4$) $\forall\, n$, $H_n(X;\mathbf{Q})$ is finite dimensional and $H_n(X;\mathbf{Z})_{\text{tor}}(p)$ is finite.

EXAMPLE Suppose that X is a pointed simply connected CW space which is $H\mathbf{F}_p$-local—then $H^n(X;\mathbf{F}_p)$ is finite dimensional $\forall\, n$ iff $\pi_n(X)$ is a finitely generated $\widehat{\mathbf{Z}}_p$-module $\forall\, n$.

[Note: $\pi_n(X)$ is p-cotorsion, hence is a p-adic module (cf. p. 10–2).]

A group G is said to be $\mathbf{F}_p$-finite provided that $H^1(G;\mathbf{F}_p)$ and $H^2(G;\mathbf{F}_p)$ are finite dimensional. Example: Every finitely generated nilpotent group is $\mathbf{F}_p$-finite (cf. p. 5–56).

[Note: Let G be an abelian group—then G is $\mathbf{F}_p$-finite iff $G\otimes\mathbf{F}_p$ and $\text{Tor}(G,\mathbf{F}_p)$ are finite or still, G is $\mathbf{F}_p$-finite iff $H^n(G,n;\mathbf{F}_p)$ and $H^{n+1}(G,n;\mathbf{F}_p)$ are finite dimensional.]

EXAMPLE Suppose that G is $\mathbf{F}_p$-finite—then $H_1(G;\mathbf{F}_p)$ and $H_2(G;\mathbf{F}_p)$ are finite dimensional. Therefore, $L_0^pG \approx \mathbf{F}_pG$ (cf. p. 11–2). In particular, for any nilpotent $\mathbf{F}_p$-finite group G, $\text{Ext}(\mathbf{Z}/p^\infty\mathbf{Z},G) \approx \mathbf{F}_pG \approx \widehat{G}_p \approx \text{pro}_p G$.

[Note: In the abelian case, one may proceed directly. Thus observe first that if G is abelian and $\mathbf{F}_p$-finite, then $\forall\, n$, $\text{Tor}(G,\mathbf{Z}/p^n\mathbf{Z})$ is finite (argue by induction, using the coefficient sequence associated with the short exact sequence $0\rightarrow\mathbf{Z}/p\mathbf{Z}\rightarrow\mathbf{Z}/p^{n+1}\mathbf{Z}\rightarrow\mathbf{Z}/p^n\mathbf{Z}\rightarrow 0$). Accordingly, $\forall\, n$, $\text{Hom}(\mathbf{Z}/p^n\mathbf{Z},G)$ is finite $\Rightarrow \lim^1 \text{Hom}(\mathbf{Z}/p^n\mathbf{Z},G)=0$ (cf. p. 5–45) $\Rightarrow \text{Ext}(\mathbf{Z}/p^\infty\mathbf{Z},G)\approx\widehat{G}_p$ (cf. p. 10–2).]

EXAMPLE Any abelian group in any of the following four classes is $\mathbf{F}_p$-finite: (C$_1$) The finite abelian p-groups; (C$_2$) The free abelian groups of finite rank; (C$_3$) The uniquely p-divisible abelian groups; (C$_4$) The p-primary divisible abelian groups satisfying the descending chain condition on subgroups. Moreover, every $\mathbf{F}_p$-finite abelian group G admits a composition series $G=G^0\supset G^1\supset\cdots\supset G^n=\{0\}$ such that $\forall\, i$, G^i/G^{i+1} is in one of these four classes.

[Given an $\mathbf{F}_p$-finite abelian G, $\exists$ a short exact sequence $0\rightarrow G'\rightarrow G\rightarrow G''\rightarrow 0$, where $\begin{cases} G' \\ G'' \end{cases}$ are $\mathbf{F}_p$-finite with G' finitely generated and G'' p-divisible. Proof: One may take $G''=G/G'$, where G' is a finitely generated subgroup of G mapping onto G/pG.]

FACT Let G be an abelian group. Assume: G is $\mathbf{F}_p$-finite—then $\forall\ n$, $H^n(G;\mathbf{F}_p)$ is finite dimensional.

PROPOSITION 8 Let G be an $\mathbf{F}_p$-finite nilpotent group—then $\forall\ n$, $H^n(G;\mathbf{F}_p)$ is finite dimensional.

[This is true if G is abelian (cf. supra). Since in general, the iterated commutator map $\otimes^{i+1}(G/[G,G]) \rightarrow \Gamma^i(G)/\Gamma^{i+1}(G)$ is surjective, $H^1(\Gamma^i(G)/\Gamma^{i+1}(G);\mathbf{F}_p)$ is finite dimensional $\forall\ i$. In particular: $H^1(\Gamma^{d-1}(G);\mathbf{F}_p)$ is finite dimensional ($d = \operatorname{nil} G > 1$). Put $K = \Gamma^{d-1}(G)$ and consider the central extension $1 \rightarrow K \rightarrow G \rightarrow G/K \rightarrow 1$. The associated LHS spectral sequence is $H^p(G/K; H^q(K;\mathbf{F}_p)) \Rightarrow H^{p+q}(G;\mathbf{F}_p)$, so it need only be shown that the $E_2^{p,q}$ are finite dimensional. Specialized to the present situation, the fundamental exact sequence in cohomology reads $0 \rightarrow H^1(G/K;\mathbf{F}_p) \rightarrow H^1(G;\mathbf{F}_p) \rightarrow H^1(K;\mathbf{F}_p) \rightarrow H^2(G/K;\mathbf{F}_p) \rightarrow H^2(G;\mathbf{F}_p)$ (cf. p. 5–54). Therefore $H^1(G/K;\mathbf{F}_p)$ and $H^2(G/K;\mathbf{F}_p)$ are finite dimensional, hence by induction, $\forall\ n$, $H^n(G/K;\mathbf{F}_p)$ is finite dimensional. Claim: $H^2(K;\mathbf{F}_p)$ is finite dimensional. To see this, suppose the contrary. Because $\dim E_2^{2,1} < \omega$, $E_3^{0,2}$ (the kernel of the differential $E_2^{0,2} \rightarrow E_2^{2,1}$) would be infinite dimensional. But $\dim E_2^{3,0} < \omega \Rightarrow \dim E_3^{3,0} < \omega$, which means that $E_4^{0,2}$ (the kernel of the differential $E_3^{0,2} \rightarrow E_3^{3,0}$) would be infinite dimensional. This, however, is untenable: $E_4^{0,2} = E_\infty^{0,2}$ and $H^2(G;\mathbf{F}_p)$ is finite dimensional. Thus the conclusion is that K is $\mathbf{F}_p$-finite and, being abelian, $H^n(K;\mathbf{F}_p)$ is finite dimensional $\forall\ n$. It now follows that $\forall\ p$ & $\forall\ q$, $E_2^{p,q}$ is finite dimensional.]

Application: Let G be an $\mathbf{F}_p$-finite nilpotent group—then $\forall\ i$, $\Gamma^i(G)/\Gamma^{i+1}(G)$ is an $\mathbf{F}_p$-finite abelian group.

FACT Let G be a group, M a nilpotent G-module. Assume: $H^1(G;\mathbf{F}_p)$ is finite dimensional and M is $\mathbf{F}_p$-finite—then $\forall\ i$, $\Gamma_\chi^i(M)/\Gamma_\chi^{i+1}(M)$ is $\mathbf{F}_p$-finite.

LEMMA Let G be a group, M a nilpotent G-module which is a vector space over $\mathbf{F}_p$. Assume: $H^1(G;\mathbf{F}_p)$ is finite dimensional and $H^0(G;M)$ is finite dimensional—then M is finite dimensional.

[The assertion is clear if G operates trivially on M. Agreeing to argue inductively on $d = \operatorname{nil}_\chi M > 1$, put $N = \Gamma_\chi^{d-1}(M)$ and consider the exact sequence $0 \rightarrow H^0(G;N) \rightarrow H^0(G;M) \rightarrow H^0(G;M/N) \rightarrow H^1(G;N) \rightarrow \cdots$. Since G operates trivially on N, $H^0(G;N) = N$, thus N is finite dimensional. Consequently, $H^1(G;N)$ is finite dimensional, so $H^0(G;M/N)$ is finite dimensional. Owing to the induction hypothesis, M/N is finite dimensional, hence the same holds for M itself.]

PROPOSITION 9 Let X be a pointed nilpotent CW space—then $\forall\, n$, $H^n(X;\mathbf{F}_p)$ is finite dimensional iff $\forall\, n$, $\pi_n(X)$ is $\mathbf{F}_p$-finite.

[We shall prove that the condition on the homotopy groups is necessary, the verification that it is also sufficient being similar. For this, consider the 5-term exact sequence $0 \to E_2^{1,0} \to H^1(X;\mathbf{F}_p) \to E_2^{0,1} \to E_2^{2,0} \to H^2(X;\mathbf{F}_p)$ associated with the fibration spectral sequence $H^p(\pi_1(X); H^q(\widetilde{X};\mathbf{F}_p)) \Rightarrow H^{p+q}(X;\mathbf{F}_p)$ to see that $H^1(\pi_1(X);\mathbf{F}_p)$ and $H^2(\pi_1(X);\mathbf{F}_p)$ are finite dimensional, i.e., that $\pi_1(X)$ is $\mathbf{F}_p$-finite. Since $\pi_1(X)$ operates nilpotently on the $H_n(\widetilde{X})$ (cf. §5, Proposition 17), $H_n(\widetilde{X};\mathbf{F}_p)$ is a nilpotent $\pi_1(X)$-module, as is its dual $H^n(\widetilde{X};\mathbf{F}_p)$. Taking into account Proposition 8, one finds from the lemma that $H^2(\widetilde{X};\mathbf{F}_p)$ is finite dimensional and then by iteration that $H^n(\widetilde{X};\mathbf{F}_p)$ is finite dimensional $\forall\, n$. This sets the stage for the discussion of $\pi_2(X)$. Thus, in the notation of p. 5–38, consider $\widetilde{X}_2 \to \widetilde{X}_1 \to K(\pi_2(X),2)$ $(\widetilde{X}_1 \approx \widetilde{X})$. Once again, there is a fibration spectral sequence $H^p(K(\pi_2(X),2); H^q(\widetilde{X}_2;\mathbf{F}_p)) \Rightarrow H^{p+q}(\widetilde{X}_1;\mathbf{F}_p)$ and a low degree exact sequence $H^2(\pi_2(X),2;\mathbf{F}_p) \to H^2(\widetilde{X}_1;\mathbf{F}_p) \to H^2(\widetilde{X}_2;\mathbf{F}_p) \to H^3(\pi_2(X),2;\mathbf{F}_p) \to H^3(\widetilde{X}_1;\mathbf{F}_p)$. Because $H^2(\widetilde{X}_1;\mathbf{F}_p)$ and $H^3(\widetilde{X}_1;\mathbf{F}_p)$ are finite dimensional and $H^2(\widetilde{X}_2;\mathbf{F}_p) = 0$, it follows that $H^2(\pi_2(X),2;\mathbf{F}_p)$ and $H^3(\pi_2(X),2;\mathbf{F}_p)$ are finite dimensional. Therefore $\pi_2(X)$ is $\mathbf{F}_p$-finite and the process can be continued.]

FACT Let G be an $\mathbf{F}_p$-finite nilpotent group—then $\text{pro}_p G$ operates nilpotently on the $L_n^p G$.

FACT Let G be an $\mathbf{F}_p$-finite nilpotent group, M an $\mathbf{F}_p$-finite nilpotent G-module—then $\text{pro}_p G$ operates nilpotently on the $L_n^p M$.

COINCIDENCE CRITERION Let X be a pointed nilpotent CW space such that $\forall\, n$, $H^n(X;\mathbf{F}_p)$ is finite dimensional—then $\forall\, n$, there is a split short exact sequence $0 \to \text{pro}_p \pi_n(X) \to \pi_n(\text{pro}_p X) \to \text{Hom}(\mathbf{Z}/p^\infty\mathbf{Z}, \pi_{n-1}(X)) \to 0$, hence $\text{pro}_p X \approx X_{H\mathbf{F}_p}$.

[Note: Recall that here, $\text{Ext}(\mathbf{Z}/p^\infty\mathbf{Z}, \pi_n(X)) \approx \mathbf{F}_p\pi_n(X) \approx \pi_n(X)_p^\wedge \approx \text{pro}_p\pi_n(X)$ (cf. p. 11–9).]

EXAMPLE Let X be a pointed nilpotent CW space such that $\forall\, n$, $H^n(X;\mathbf{F}_p)$ is finite dimensional. Let $\mathcal{A}_p$ be the mod p Steenrod algebra—then $H^*(X;\mathbf{F}_p)$ is an unstable $\mathcal{A}_p$-module and Lannes-Schwartz[†] have shown that X is W-null, where $W = B\mathbf{Z}/p\mathbf{Z}$, iff every cyclic submodule of $H^*(X;\mathbf{F}_p)$ is finite.

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§12. MODEL CATEGORIES

Of the various proposals that have been advanced for the development of abstract homotopy theory, perhaps the most widely used and successful axiomization is Quillen's. The resulting unification is striking and the underlying techniques are applicable not only in topology but also in algebra.

Let $i : A \to Y$, $p : X \to B$ be morphisms in a category **C**—then i is said to have the left lifting property with respect to p (LLP w.r.t. p) and p is said to have the right lifting property with respect to i (RLP w.r.t. i) if for all $u : A \to X$, $v : Y \to B$ such that $p \circ u = v \circ i$, there is a $w : Y \to X$ such that $w \circ i = u$, $p \circ w = v$.

For instance, take $\mathbf{C} = \mathbf{TOP}$—then $i : A \to Y$ is a cofibration iff $\forall$ X, i has the LLP w.r.t. $p_0 : PX \to X$, i.e.,

$$\begin{tikzcd} A \arrow[r] \arrow[d] & PX \arrow[d, "p_0"] \\ Y \arrow[r] \arrow[ur, dashed] & X \end{tikzcd}$$

and $p : X \to B$ is a Hurewicz fibration iff $\forall$ Y, p has the RLP w.r.t. $i_0 : Y \to IY$, i.e.,

$$\begin{tikzcd} Y \arrow[r] \arrow[d, "i_0"'] & X \arrow[d, "p"] \\ IY \arrow[r] \arrow[ur, dashed] & B \end{tikzcd}$$

Consider a category **C** equipped with three composition closed classes of morphisms termed weak equivalences (denoted $\xrightarrow{\sim}$), cofibrations (denoted $\rightarrowtail$), and fibrations (denoted $\twoheadrightarrow$), each containing the isomorphisms of **C**. Agreeing to call a morphism which is both a weak equivalence and a cofibration (fibration) an acyclic cofibration (fibration), **C** is said to be a model category provided that the following axioms are satisfied.

(MC–1) **C** is finitely complete and finitely cocomplete.

(MC–2) Given composable morphisms f, g, if any two of f, g, $g \circ f$ are weak equivalences, so is the third.

(MC–3) Every retract of a weak equivalence, cofibration, or fibration is again a weak equivalence, cofibration, or fibration.

[Note: To say that $f : X \to Y$ is a retract of $g : W \to Z$ means that there exist morphisms $i : X \to W$, $r : W \to X$, $j : Y \to Z$, $s : Z \to Y$ with $g \circ i = j \circ f$, $f \circ r = s \circ g$, $r \circ i = \mathrm{id}_X$, $s \circ j = \mathrm{id}_Y$. A retract of an isomorphism is an isomorphism.]

(MC–4) Every cofibration has the LLP w.r.t. every acyclic fibration and every fibration has the RLP w.r.t. every acyclic cofibration.

(MC–5) Every morphism can be written as the composite of a cofibration and an acyclic fibration and the composite of an acyclic cofibration and a fibration.

[Note: In proofs, the axioms for a model category are often used without citation.]

Remark: A weak equivalence which is a cofibration and a fibration is an isomorphism.

A model category **C** has an initial object (denoted $\emptyset$) and a final object (denoted $*$). An object X in **C** is said to be cofibrant if $\emptyset \rightarrow X$ is a cofibration and fibrant if $X \rightarrow *$ is a fibration.

FACT Suppose that **C** is a model category. Let $X \in \operatorname{Ob} \mathbf{C}$—then X is cofibrant iff every acyclic fibration $Y \rightarrow X$ has a right inverse and X is fibrant iff every acyclic cofibration $X \rightarrow Y$ has a left inverse.

Example: Take $\mathbf{C} = \mathbf{TOP}$—then **TOP** is a model category if weak equivalence= homotopy equivalence, cofibration=closed cofibration, fibration=Hurewicz fibration. All objects are cofibrant and fibrant.

[MC–1 is clear, as is MC–2. That MC–4 obtains is implied by what can be found on p. 4–16 & p. 4–17, p. 4–21 & p. 4–22 and that MC–5 obtains is implied by what can be found on p. 4–12. There remains the verification of MC–3. That MC–3 obtains for closed cofibrations or Hurewicz fibrations is implied by what can be found on p. 4–16 & p. 4–17, p. 4–21 & p. 4–22. Finally, suppose that f is the retract of a homotopy equivalence—then $[f]$ is the retract of an isomorphism in **HTOP**, so $[f]$ is an isomorphism in **HTOP**, i.e., f is a homotopy equivalence.]

[Note: We shall refer to this structure of a model category on **TOP** as the standard structure.]

Addendum: **CG** has a standard model category structure, viz. weak equivalence= homotopy equivalence, cofibration=closed cofibration, fibration=**CG** fibration.

[The verification of MC–4 for **CG** is essentially the same as it is for **TOP**. To check MC–5, note that k preserves homotopy equivalences, sends closed cofibrations to closed cofibrations (cf. p. 3–8), and takes Hurewicz fibrations to **CG** fibrations (cf. p. 4–7). Therefore, if $\begin{cases} X \\ Y \end{cases}$ are in **CG** and if $f : X \rightarrow Y$ is a continuous function, one can first factor f in **TOP** and then apply k to get the desired factorization of f in **CG**.]

EXAMPLE Let **A** be an abelian category. Write **CXA** for the abelian category of chain complexes over **A**. Given a morphism $f : X \rightarrow Y$ in **CXA**, call f a weak equivalence if f is a chain homotopy equivalence, a cofibration if $\forall\, n$, $f_n : X_n \rightarrow Y_n$ has a left inverse, and a fibration if $\forall\, n$, $f_n : X_n \rightarrow Y_n$ has a right inverse—then **CXA** is a model category. Every object is cofibrant and fibrant.

EXAMPLE Let **A** be an abelian category with enough projectives. Write $\mathbf{CXA}_{\geq 0}$ for the full subcategory of **CXA** whose objects X have the property that $X_n = 0$ if $n < 0$. Given a morphism $f : X \rightarrow Y$ in $\mathbf{CXA}_{\geq 0}$, call f a weak equivalence if f is a homology equivalence, a cofibration if $\forall\, n$, $f_n : X_n \rightarrow Y_n$ is a monomorphism with a projective cokernel, and a fibration if $\forall\, n > 0$, $f_n : X_n \rightarrow Y_n$ is an epimorphism—then $\mathbf{CXA}_{\geq 0}$ is a model category. Every object is fibrant and the cofibrant objects are those X such that $\forall\, n$, X_n is projective.

There are lots of other "algebraic" examples of model categories, many of which figure prominently in rational homotopy theory (specifics can be found in the references at the end of the §).

Given a model category $\mathbf{C}$, $\mathbf{C}^{\mathrm{OP}}$ acquires the structure of a model category by stipulating that f^{OP} is a weak equivalence in $\mathbf{C}^{\mathrm{OP}}$ iff f is a weak equivalence in $\mathbf{C}$, that f^{OP} is a cofibration in $\mathbf{C}^{\mathrm{OP}}$ iff f is a fibration in $\mathbf{C}$, and that f^{OP} is a fibration in $\mathbf{C}^{\mathrm{OP}}$ iff f is a cofibration in $\mathbf{C}$.

Given a model category $\mathbf{C}$ and objects A, B in $\mathbf{C}$, the categories $A\backslash\mathbf{C}$, $\mathbf{C}/B$ are again model categories, a morphism in either case being declared a weak equivalence, cofibration, or fibration if it is such when viewed in $\mathbf{C}$ alone.

Example: Take $\mathbf{C} = \mathbf{TOP}$ (standard structure)—then an object (X, x_0) in $\mathbf{TOP}_*$ is cofibrant iff $* \rightarrow (X, x_0)$ is a closed cofibration (in $\mathbf{TOP}$), i.e., iff (X, x_0) is wellpointed with $\{x_0\} \subset X$ closed.

PROPOSITION 1 Let $\mathbf{C}$ be a model category.

(1) The cofibrations in $\mathbf{C}$ are the morphisms that have the LLP w.r.t. acyclic fibrations.

(2) The acyclic cofibrations in $\mathbf{C}$ are the morphisms that have the LLP w.r.t. fibrations.

(3) The fibrations in $\mathbf{C}$ are the morphisms that have the RLP w.r.t. acyclic cofibrations.

(4) The acyclic fibrations in $\mathbf{C}$ are the morphisms that have the RLP w.r.t. cofibrations.

[Statements (3) and (4) follow from statements (1) and (2) by duality. The proofs of (1) and (2) being analogous, consider (1). Thus suppose that $i : A \rightarrow Y$ has the LLP w.r.t. acyclic fibrations. Using MC–5, write $i = p \circ j$, where $j : A \rightarrow X$ is a cofibration and $p : X \rightarrow Y$ is an acyclic fibration. By hypothesis, $\exists$ a w such that $w \circ i = j$, $p \circ w = \mathrm{id}_Y$, and this implies that i is a retract of j, so i is a cofibration.]

Example: Take $\mathbf{C} = \mathbf{CG}$ (standard structure)—then an arrow $A \rightarrow Y$ that has the LLP w.r.t. acyclic $\mathbf{CG}$ fibrations must be a closed cofibration.

EXAMPLE Let $\mathbf{C}$ and $\mathbf{D}$ be model categories. Suppose that $\begin{cases} F : \mathbf{C} \rightarrow \mathbf{D} \\ G : \mathbf{D} \rightarrow \mathbf{C} \end{cases}$ are functors and (F, G) is an adjoint pair—then F preserves cofibrations and acyclic cofibrations iff G preserves fibrations and acyclic fibrations.

[Note: Either condition is equivalent to requiring that F preserve cofibrations and G preserve fibrations.]

In a model category **C**, the classes of cofibrations and fibrations possess a number of "closure" properties (all verifications are simple consequences of Proposition 1).

(Coproducts) If $\forall\ i$, $f_i : X_i \rightarrow Y_i$ is a cofibration (acyclic cofibration), then $\coprod_i f_i : \coprod_i X_i \rightarrow \coprod_i Y_i$ is a cofibration (acyclic cofibration).

(Products) If $\forall\ i$, $f_i : X_i \rightarrow Y_i$ is a fibration (acyclic fibration), then $\prod_i f_i : \prod_i X_i \rightarrow \prod_i Y_i$ is a fibration (acyclic fibration).

(Pushouts) Given a 2-source $X \xleftarrow{f} Z \xrightarrow{g} Y$, define P by the pushout square

$$\begin{tikzcd} Z \arrow[r, "g"] \arrow[d, "f"'] & Y \arrow[d, "\eta"] \\ X \arrow[r, "\xi"'] & P \end{tikzcd}$$

. Assume: f is a cofibration (acyclic cofibration)—then η is a cofibration (acyclic cofibration).

(Pullbacks) Given a 2-sink $X \xrightarrow{f} Z \xleftarrow{g} Y$, define P by the pullback square

$$\begin{tikzcd} P \arrow[r, "\eta"] \arrow[d, "\xi"'] & Y \arrow[d, "g"] \\ X \arrow[r, "f"'] & Z \end{tikzcd}$$

. Assume: g is a fibration (acyclic fibration)—then ξ is a fibration (acyclic fibration).

(Sequential Colimits) If $\forall\ n$, $f_n : X_n \rightarrow X_{n+1}$ is a cofibration (acyclic cofibration), then $\forall\ n$, $i_n : X_n \rightarrow \operatorname{colim} X_n$ is a cofibration (acyclic cofibration).

(Sequential Limits) If $\forall\ n$, $f_n : X_{n+1} \rightarrow X_n$ is a fibration (acyclic fibration), then $\forall\ n$, $p_n : \lim X_n \rightarrow X_n$ is a fibration (acyclic fibration).

[Note: It is assumed that the relevant coproducts, products, sequential colimits, and sequential limits exist.]

EXAMPLE (Pushouts) Fix a model category **C**. Let **I** be the category $1\bullet \xleftarrow{a} \underset{3}{\bullet} \xrightarrow{b} \bullet 2$ (cf. p. 0–9)—then the functor category $[\mathbf{I}, \mathbf{C}]$ is again a model category. Thus an object of $[\mathbf{I}, \mathbf{C}]$ is a 2-source $X \xleftarrow{f} Z \xrightarrow{g} Y$ and a morphism Ξ of 2-sources is a commutative diagram

$$\begin{tikzcd} X \arrow[d] & Z \arrow[l, "f"'] \arrow[r, "g"] \arrow[d] & Y \arrow[d] \\ X' & Z' \arrow[l, "f'"] \arrow[r, "g'"'] & Y' \end{tikzcd}$$

. Stipulate that Ξ is a weak equivalence or a fibration if this is the case of each of its vertical constituents. Define now P_L, P_R by the pushout squares

$$\begin{tikzcd} X \arrow[d] & Z \arrow[l, "f"'] \arrow[d] \\ P_L & Z' \arrow[l] \end{tikzcd} , \quad \begin{tikzcd} Z \arrow[r, "g"] \arrow[d] & Y \arrow[d] \\ Z' \arrow[r] & P_R \end{tikzcd}$$

, let $\rho_L : P_L \rightarrow X'$, $\rho_R : P_R \rightarrow Y'$ be the induced morphisms, and call Ξ a cofibration provided that $Z \rightarrow Z'$, ρ_L, and ρ_R are cofibrations. With these choices, $[\mathbf{I}, \mathbf{C}]$ is a model category. The fibrant objects $X \xleftarrow{f} Z \xrightarrow{g} Y$ in $[\mathbf{I}, \mathbf{C}]$ are those for which X, Y, and Z are fibrant. The cofibrant objects $X \xleftarrow{f} Z \xrightarrow{g} Y$ in $[\mathbf{I}, \mathbf{C}]$ are those for which Z is cofibrant and $\begin{cases} f : Z \rightarrow X \\ g : Z \rightarrow Y \end{cases}$ are cofibrations.

[Note: The story for pullbacks is analogous.]

EXAMPLE Fix a model category $\mathbf{C}$—then $\mathbf{FIL}(\mathbf{C})$ is again a model category. Thus let $\phi : (\mathbf{X}, \mathbf{f}) \rightarrow (\mathbf{Y}, \mathbf{g})$ be a morphism in $\mathbf{FIL}(\mathbf{C})$. Stipulate that ϕ is a weak equivalence or a fibration if this is the case of each ϕ_n. Define now P_{n+1} by the pushout square
$$\begin{tikzcd} X_n \arrow[r, "f_n"] \arrow[d, "\phi_n"'] & X_{n+1} \arrow[d] \\ Y_n \arrow[r, "g_n"'] & P_{n+1} \end{tikzcd}$$
, let $\rho_{n+1} : P_{n+1} \rightarrow Y_{n+1}$ be the induced morphism, and call ϕ a cofibration provided that ϕ_0 and all the ρ_{n+1} are cofibrations (each ϕ_n $(n > 0)$ is then a cofibration as well). With these choices, $\mathbf{FIL}(\mathbf{C})$ is a model category. The fibrant objects $(\mathbf{X}, \mathbf{f})$ in $\mathbf{FIL}(\mathbf{C})$ are those for which X_n is fibrant $\forall\ n$. The cofibrant objects $(\mathbf{X}, \mathbf{f})$ in $\mathbf{FIL}(\mathbf{C})$ are those for which X_0 is cofibrant and $\forall\ n$, $f_n : X_n \rightarrow X_{n+1}$ is a cofibration.

[Note: The story for $\mathbf{TOW}(\mathbf{C})$ is analogous.]

FACT Let $\mathbf{C}$ be a model category. Suppose that
$$\begin{tikzcd} A \arrow[r, "u"] \arrow[d, "i"'] & X \arrow[d, "p"] \\ Y \arrow[r, "v"'] & B \end{tikzcd}$$
is a commutative diagram in $\mathbf{C}$, where i is a cofibration, p is a weak equivalence, and X is fibrant—then $\exists$ a $w : Y \rightarrow X$ such that $w \circ i = u$.

[Note: There is a similar assertion for fibrations and cofibrant objects.]

Given a model category $\mathbf{C}$, objects X' and X'' are said to be weakly equivalent if there exists a path beginning at X' and ending at X'': $X' = X_0 \rightarrow X_1 \leftarrow \cdots \rightarrow X_{2n-1} \leftarrow X_{2n} = X''$, where all the arrows are weak equivalences. Example: Take $\mathbf{C} = \mathbf{TOP}$ (standard structure)—then X' and X'' are weakly equivalent iff they have the same homotopy type.

EXAMPLE The arrow category $\mathbf{C}(\rightarrow)$ of a model category $\mathbf{C}$ is again a model category (cf. p. 12–26). Therefore it makes sense to consider weakly equivalent morphisms. Example: Every morphism in $\mathbf{C}$ is weakly equivalent to a fibration with a fibrant domain and codomain.

COMPOSITION LEMMA Consider the commutative diagram
$$\begin{tikzcd} \bullet \arrow[r] \arrow[d] & \bullet \arrow[r] \arrow[d] & \bullet \arrow[d] \\ \bullet \arrow[r] & \bullet \arrow[r] & \bullet \end{tikzcd}$$
in a category $\mathbf{C}$. Suppose that both the squares are pushouts—then the rectangle is a pushout. Conversely, if the rectangle and the first square are pushouts, then the second square is a pushout.

Application: Consider the commutative cube

$$\begin{tikzcd}
& \bullet \arrow[rr] \arrow[dd] & & \bullet \arrow[dd] \\
\bullet \arrow[ur] \arrow[rr] \arrow[dd] & & \bullet \arrow[ur] \arrow[dd] & \\
& \bullet \arrow[rr] & & \bullet \\
\bullet \arrow[ur] \arrow[rr] & & \bullet \arrow[ur] &
\end{tikzcd}$$

in a category **C**. Suppose that the top and the left and right hand sides are pushouts—then the bottom is a pushout.

PROPOSITION 2 Let **C** be a model category. Given a 2-source $X \xleftarrow{f} Z \xrightarrow{g} Y$, define P by the pushout square

$$\begin{tikzcd}
Z \arrow[r, "g"] \arrow[d, "f"'] & Y \arrow[d, "\eta"] \\
X \arrow[r, "\xi"'] & P
\end{tikzcd}$$

. Assume: f is a cofibration and g is a weak equivalence—then ξ is a weak equivalence provided that Z & Y are cofibrant.

[Introduce the cylinder object IZ for Z (cf. p. 12–16) and define M_g by the pushout square

$$\begin{tikzcd}
Z \amalg Z \arrow[r, "g \amalg \mathrm{id}_Z"] \arrow[d, "\iota"'] & Y \amalg Z \arrow[d] \\
IZ \arrow[r] & M_g
\end{tikzcd}$$

(cf. p. 3–21). Noting that

$$\begin{tikzcd}
& Z \amalg Z \arrow[dl] \arrow[r] & Y \amalg Z \arrow[dd, "{(\mathrm{id}_Y, g)}"'] \\
IZ \arrow[dr] & & \\
& Z \arrow[r, "g"'] & Y
\end{tikzcd}$$

commutes, choose $r : M_g \to Y$ accordingly, so $g = r \circ i$ and $r \circ j = \mathrm{id}_Y$, where $i : Z \to M_g$ is the composite $Z \to Y \amalg Z \to M_g$ and $j : Y \to M_g$ is the composite $Y \to Y \amalg Z \to M_g$. Since ι is a cofibration and

$$\begin{tikzcd}
\emptyset \arrow[r] \arrow[d] & Z \arrow[d] \\
Y \arrow[r] & Y \amalg Z
\end{tikzcd}$$

is a pushout square, i and j are cofibrations. Moreover, j is acyclic. This is because $i_0 : Z \to IZ$ is an acyclic cofibration and j is obtained from i_0 via

$$\begin{tikzcd}
Z \arrow[r, "g"] \arrow[d, "\mathrm{in}_0"'] & Y \arrow[d] \\
Z \amalg Z \arrow[r] \arrow[d, "\iota"'] & Y \amalg Z \arrow[d] \\
IZ \arrow[r] & M_g
\end{tikzcd}$$

($i_0 = \iota \circ \mathrm{in}_0$). Therefore r is a weak equivalence. But, by assumption, g is a weak equivalence. Therefore i is a weak equivalence. Define $\bar{I}$ by the pushout square

$$\begin{tikzcd}
Z \arrow[r, "i"] \arrow[d, "f"'] & M_g \arrow[d, "\bar{f}"] \\
X \arrow[r, "\bar{i}"'] & \bar{I}
\end{tikzcd}$$

. Since f is a cofibration and i is an acyclic cofibration, $\bar{f}$ is a cofibration and $\bar{i}$ is an acyclic cofibration. The commutative diagram

$$\begin{tikzcd}
M_g \arrow[r, "r"] \arrow[d, "\bar{f}"'] & Y \arrow[d, "\eta"] \\
\bar{I} \arrow[r, "r_i"'] & P
\end{tikzcd}$$

is a pushout square and $\xi = r_i \circ \bar{i}$. Define $\bar{J}$ by the pushout square

$$\begin{tikzcd}
Y \arrow[r, "j"] \arrow[d, "\eta"'] & M_g \arrow[d, "\bar{\eta}"] \\
P \arrow[r, "\bar{j}"'] & \bar{J}
\end{tikzcd}$$

.

Since η is a cofibration and j is an acyclic cofibration, $\bar{\eta}$ is a cofibration and $\bar{j}$ is an

acyclic cofibration. The commutative diagram
$$\begin{tikzcd} M_g \arrow[r, "r"] \arrow[d, "\overline{\eta}"'] & Y \arrow[d, "\eta"] \\ \overline{J} \arrow[r, "r_j"'] & P \end{tikzcd}$$
is a pushout square and $\mathrm{id}_P = r_j \circ \overline{j}$. Therefore r_j is a weak equivalence. Define Z_0, Z_1 by the pushout squares
$$\begin{tikzcd} Z \arrow[r, "i_0"] \arrow[d, "f"'] & IZ \arrow[d, "f_0"] \\ X \arrow[r, "\sim"] & Z_0 \end{tikzcd}, \quad \begin{tikzcd} Z \arrow[r, "i_1"] \arrow[d, "f"'] & IZ \arrow[d, "f_1"] \\ X \arrow[r, "\sim"] & Z_1 \end{tikzcd}.$$
The composites $Z \xrightarrow{i_0} IZ \xrightarrow{\sim} Z$, $Z \xrightarrow{i_1} IZ \xrightarrow{\sim} Z$ being id_Z, there are weak equivalences $\zeta_0 : Z_0 \to X$, $\zeta_1 : Z_1 \to X$ and factorizations $X \xrightarrow{\sim} Z_0 \xrightarrow{\zeta_0} X$, $X \xrightarrow{\sim} Z_1 \xrightarrow{\zeta_1} X$ of id_X. Define W by the pushout square
$$\begin{tikzcd} IZ \arrow[r, "f_1"] \arrow[d, "f_0"'] & Z_1 \arrow[d] \\ Z_0 \arrow[r] & W \end{tikzcd}$$
and determine $\zeta : W \to X$ so that ζ_0 is the composite $Z_0 \to W \xrightarrow{\zeta} X$ and ζ_1 is the composite $Z_1 \to W \xrightarrow{\zeta} X$. Decompose ζ per $W \rightarrowtail \overline{W} \overset{\sim}{\twoheadrightarrow} X$—then the composites $Z_0 \to \overline{W}$, $Z_1 \to \overline{W}$ are acyclic cofibrations. To go from Z to $\overline{I}$ through $Z \xrightarrow{i_1} IZ \xrightarrow{\sim} Z \xrightarrow{i} M_g \xrightarrow{\overline{f}} \overline{I}$ is the same as going from Z to $\overline{I}$ through $Z \xrightarrow{f} X \xrightarrow{\overline{i}} \overline{I}$. Consequently, there is an arrow $\overline{i}_1 : Z_1 \to \overline{I}$ such that the composite $X \xrightarrow{\sim} Z_1 \xrightarrow{\overline{i}_1} \overline{I}$ is $\overline{i}$ and the commutative diagram
$$\begin{tikzcd} IZ \arrow[r] \arrow[d, "f_1"'] & M_g \arrow[d] \\ Z_1 \arrow[r, "\overline{i}_1"'] & \overline{I} \end{tikzcd}$$
is a pushout square. But $\overline{i}$ is a weak equivalence. Therefore $\overline{i}_1$ is a weak equivalence. Define $\overline{K}$ by the pushout square
$$\begin{tikzcd} Z_1 \arrow[r, "\overline{i}_1"] \arrow[d] & \overline{I} \arrow[d] \\ \overline{W} \arrow[r] & \overline{K} \end{tikzcd}.$$
Since $\overline{i}_1$ is a weak equivalence, the same holds for $\overline{W} \to \overline{K}$.

To go from Z to $\overline{J}$ through $Z \xrightarrow{i_0} IZ \xrightarrow{\sim} Z \xrightarrow{g} Y \xrightarrow{j} M_g \xrightarrow{\overline{\eta}} \overline{J}$ is the same as going from Z to $\overline{J}$ through $Z \xrightarrow{f} X \xrightarrow{\xi} P \xrightarrow{\overline{j}} \overline{J}$. Consequently, there is an arrow $\overline{j}_0 : Z_0 \to \overline{J}$ such that the composite $X \xrightarrow{\sim} Z_0 \xrightarrow{\overline{j}_0} \overline{J}$ is $\overline{j} \circ \xi$ and the commutative diagram
$$\begin{tikzcd} IZ \arrow[r] \arrow[d, "f_0"'] & M_g \arrow[d] \\ Z_0 \arrow[r, "\overline{j}_0"'] & \overline{J} \end{tikzcd}$$
is a pushout square. To go from IZ to $\overline{K}$ by $IZ \to M_g \to \overline{I} \to \overline{K}$ is the same as going from IZ to $\overline{K}$ by $IZ \to Z_0 \to \overline{W} \xrightarrow{\sim} \overline{K}$, thus there is an arrow $\overline{J} \to \overline{K}$ and a commutative diagram
$$\begin{tikzcd} Z_0 \arrow[r, "\overline{j}_0"] \arrow[d] & \overline{J} \arrow[d] \\ \overline{W} \arrow[r] & \overline{K} \end{tikzcd}$$
which is a pushout square:
$$\begin{tikzcd} & M_g \arrow[rr] \arrow[dd] & & \overline{I} \arrow[dd] \\ IZ \arrow[ur] \arrow[rr] \arrow[dd] & & Z_1 \arrow[ur] \arrow[dd] & \\ & \overline{J} \arrow[rr] & & \overline{K} \\ Z_0 \arrow[ur] \arrow[rr] & & \overline{W} \arrow[ur] & \end{tikzcd}.$$

It follows that $\bar{j}_0$ is a weak equivalence and this implies that $\bar{j} \circ \xi$ is a weak equivalence. Finally, $\xi = \mathrm{id}_P \circ \xi = r_j \circ \bar{j} \circ \xi$ is a weak equivalence.]

[Note: There is a parallel statement for fibrations and pullbacks.]

EXAMPLE Working in $\mathbf{C} = \mathbf{TOP}$ (standard structure), suppose that $A \to X$ is a closed cofibration. Let $f : A \to Y$ be a homotopy equivalence—then the arrow $X \to X \sqcup_f Y$ is a homotopy equivalence (cf. p. 3–24).

PROPOSITION 3 Let $\mathbf{C}$ be a model category. Suppose given a commutative diagram

$$\begin{tikzcd} X \arrow[d] & Z \arrow[l, "f"'] \arrow[r, "g"] \arrow[d] & Y \arrow[d] \\ X' & Z' \arrow[l, "f'"] \arrow[r, "g'"'] & Y' \end{tikzcd}$$

, where $\left\{\begin{array}{l} f \\ f' \end{array}\right.$ are cofibrations and the vertical arrows are weak equivalences—then the induced morphism $P \to P'$ of pushouts is a weak equivalence provided that $\left\{\begin{array}{l} Z \,\&\, Y \\ Z' \,\&\, Y' \end{array}\right.$ are cofibrant.

[We shall first treat the special case when g is a cofibration. In this situation, the arrow $Y \to Z' \underset{Z}{\sqcup} Y$ is a weak equivalence (cf. Proposition 2) and $Z' \underset{Z}{\sqcup} Y$ is cofibrant. Form the pushout square

$$\begin{tikzcd} Y \arrow[r] \arrow[d] & Z' \underset{Z}{\sqcup} Y \arrow[d] \\ X \underset{Z}{\sqcup} Y \arrow[r] & X \underset{Z}{\sqcup} (Z' \underset{Z}{\sqcup} Y) \end{tikzcd}$$

and apply Proposition 2 once again to see that the arrow $X \underset{Z}{\sqcup} Y \to X \underset{Z}{\sqcup} (Z' \underset{Z}{\sqcup} Y)$ is a weak equivalence. Next write $X \underset{Z}{\sqcup} (Z' \underset{Z}{\sqcup} Y) \approx (X \underset{Z}{\sqcup} Z') \underset{Z'}{\sqcup} (Z' \underset{Z}{\sqcup} Y)$ and note that the arrow $X \underset{Z}{\sqcup} Z' \to X'$ is a weak equivalence (cf. Proposition 2). Consider now the commutative diagram

$$\begin{tikzcd} Z' \arrow[r] \arrow[d] & X \underset{Z}{\sqcup} Z' \arrow[r] \arrow[d] & X' \arrow[d] \\ Z' \underset{Z}{\sqcup} Y \arrow[r] & (X \underset{Z}{\sqcup} Z') \underset{Z'}{\sqcup} (Z' \underset{Z}{\sqcup} Y) \arrow[r] & X' \underset{Z'}{\sqcup} (Z' \underset{Z}{\sqcup} Y) \end{tikzcd}$$

in which both the squares and the rectangle are pushouts. Since $Z' \rightarrowtail Z' \underset{Z}{\sqcup} Y \Rightarrow X \underset{Z}{\sqcup} Z' \rightarrowtail (X \underset{Z}{\sqcup} Z') \underset{Z'}{\sqcup} (Z' \underset{Z}{\sqcup} Y)$ and $X \underset{Z}{\sqcup} Z'$ is cofibrant, still another application of Proposition 2 implies that the arrow $(X \underset{Z}{\sqcup} Z') \underset{Z'}{\sqcup} (Z' \underset{Z}{\sqcup} Y) \to X' \underset{Z'}{\sqcup} (Z' \underset{Z}{\sqcup} Y)$ is a weak equivalence. Repeating the reasoning with

$$\begin{tikzcd} Z' \arrow[r] \arrow[d] & Z' \underset{Z}{\sqcup} Y \arrow[r] \arrow[d] & Y' \arrow[d] \\ X' \arrow[r] & X' \underset{Z'}{\sqcup} (Z' \underset{Z}{\sqcup} Y) \arrow[r] & X' \underset{Z'}{\sqcup} Y' \end{tikzcd}$$

leads to the conclusion that the arrow $X' \underset{Z'}{\sqcup} (Z' \underset{Z}{\sqcup} Y) \to X' \underset{Z'}{\sqcup} Y'$ is a weak equivalence. We have therefore built a weak equivalence from P to P'. To proceed in general, factor g as $Z \rightarrowtail \overline{Y} \overset{\sim}{\twoheadrightarrow} Y$. Define $\overline{X}'$, $\overline{Y}'$ by the pushout squares

$$\begin{tikzcd} Z \arrow[r] \arrow[d] & X \arrow[d] \\ Z' \arrow[r] & \overline{X}' \end{tikzcd} , \quad \begin{tikzcd} Z \arrow[r] \arrow[d] & \overline{Y} \arrow[d] \\ Z' \arrow[r] & \overline{Y}' \end{tikzcd} —$$

then there are weak equivalences $\overline{X}' \to X'$, $\overline{Y}' \to Y'$. The 2-sources $X \leftarrow Z \to \overline{Y}$, $\overline{X}' \leftarrow Z' \to \overline{Y}'$ generate pushouts $\overline{P}$, $\overline{P}'$. Since the arrows on the "right" are cofibrations, the induced morphisms $\overline{P} \to P$, $\overline{P} \to \overline{P}'$, $\overline{P}' \to P'$ are weak equivalences. The assertion thus follows from the fact that the diagram

$$\begin{tikzcd} \overline{P} \arrow[r] \arrow[d] & P \arrow[d] \\ \overline{P}' \arrow[r] & P' \end{tikzcd}$$

commutes.]

[Note: There is a parallel statement for fibrations and pullbacks.]

EXAMPLE Working in **C** = **TOP** (standard structure), suppose that $\begin{cases} A \to X \\ A' \to X' \end{cases}$ are closed cofibrations. Let $\begin{cases} f : A \to Y \\ f' : A' \to Y' \end{cases}$ be continuous functions. Assume that the diagram

$$\begin{tikzcd} X \arrow[d] & A \arrow[l] \arrow[r, "f"] \arrow[d] & Y \arrow[d] \\ X' & A' \arrow[l] \arrow[r, "f'"'] & Y' \end{tikzcd}$$

commutes and that the vertical arrows are homotopy equivalences—then the induced map $X \sqcup_f Y \to X' \sqcup_{f'} Y'$ is a homotopy equivalence (cf. p. 3–24 ff.).

PROPOSITION 4 Let **C** be a model category. Suppose given a commutative diagram

$$\begin{tikzcd} X \arrow[d] & Z \arrow[l, "f"'] \arrow[r, "g"] \arrow[d] & Y \arrow[d] \\ X' & Z' \arrow[l, "f'"] \arrow[r, "g'"'] & Y' \end{tikzcd}$$

, where $Y \to Y'$ and $X \underset{Z}{\sqcup} Z' \to X'$ are cofibrations (acyclic cofibrations)—then the induced morphism $P \to P'$ of pushouts is a cofibration (acyclic cofibration).

[Each morphism in the string $P = X \underset{Z}{\sqcup} Y \to X \underset{Z}{\sqcup} Y' \approx (X \underset{Z}{\sqcup} Z') \underset{Z'}{\sqcup} Y' \to X' \underset{Z'}{\sqcup} Y' = P'$ is a cofibration (acyclic cofibration).]

[Note: There is a parallel statement for fibrations and pullbacks.]

In the topological setting, Proposition 4 is related to but does not directly imply the lemma on p. 3–15 ff.

(Small Object Argument) Suppose that **C** is a cocomplete category. Let $S_0 = \{L_i \xrightarrow{\phi_i} K_i \ (i \in I)\}$ be a set of morphisms in **C**. Given a morphism $f : X \to Y$, consider the set of pairs of morphisms (g, h) such that the diagram

$$\begin{tikzcd} L_i \arrow[r, "g"] \arrow[d, "\phi_i"'] & X \arrow[d, "f"] \\ K_i \arrow[r, "h"'] & Y \end{tikzcd}$$

commutes. Put

$X_0 = X$ and define X_1 by the pushout square

$$\begin{tikzcd}\coprod\limits_i \coprod\limits_{(g,h)} L_i \arrow[r] \arrow[d] & X_0 \arrow[d] \\ \coprod\limits_i \coprod\limits_{(g,h)} K_i \arrow[r] & X_1\end{tikzcd}$$

. Observing that the data furnishes a commutative triangle

$$\begin{tikzcd}X_0 \arrow[rr] \arrow[dr] & & X_1 \arrow[dl] \\ & Y &\end{tikzcd}$$

, one may proceed and construct a sequence $X = X_0 \rightarrow X_1 \rightarrow \cdots \rightarrow X_\omega$ of objects in $\mathbf{C}$, taking $X_\omega = \operatorname{colim} X_n$. There is a commutative triangle

$$\begin{tikzcd}X \arrow[rr, "i_\omega"] \arrow[dr, "f"'] & & X_\omega \arrow[dl, "f_\omega"] \\ & Y &\end{tikzcd}$$

and if $\forall\ i$, L_i is ω-definite, then the conclusion is that $f_\omega : X_\omega \rightarrow Y$ has the RLP w.r.t. each ϕ_i.

[Note: All that's really required of the L_i is that the arrow colim $\operatorname{Mor}(L_i, X_n) \rightarrow \operatorname{Mor}(L_i, X_\omega)$ be surjective $\forall\ i$.]

Example: Take $\mathbf{C} = \mathbf{TOP}$—then $\mathbf{TOP}$ is a model category if weak equivalence=weak homotopy equivalence, fibration=Serre fibration, cofibration=all continuous functions which have the LLP w.r.t. Serre fibrations that are weak homotopy equivalences. Every object is fibrant and every CW complex is cofibrant. Every object is weakly equivalent to a CW complex.

[Axioms MC–1, MC–2, and MC–3 are immediate.

Claim: Every continuous function $f : X \rightarrow Y$ can be written as a composite $f_\omega \circ i_\omega$, where $i_\omega : X \rightarrow X_\omega$ is a weak homotopy equivalence and has the LLP w.r.t. Serre fibrations and $f_\omega : X_\omega \rightarrow Y$ is a Serre fibration.

[Serre fibrations can be characterized by the property that they have the RLP w.r.t. the embeddings $i_0 : [0,1]^n \rightarrow I[0,1]^n$ $(n \geq 0)$ (cf. p. 4–8). Accordingly, in the small object argument, take $S_0 = \{[0,1]^n \xrightarrow{i_0} I[0,1]^n\ (n \geq 0)\}$—then $\forall\ k$, the arrow $X_k \rightarrow X_{k+1}$ is a homotopy equivalence and has the LLP w.r.t. Serre fibrations. Consider the factorization of f arising from the small object argument:

$$\begin{tikzcd}X \arrow[rr, "i_\omega"] \arrow[dr, "f"'] & & X_\omega \arrow[dl, "f_\omega"] \\ & Y &\end{tikzcd}$$

. It is clear that i_ω has the LLP w.r.t. Serre fibrations. On the other hand, since the points of $X_\omega - i_\omega(X)$ are closed, every compact subset of X_ω lies in some X_k, thus the arrow colim $C([0,1]^n, X_k) \rightarrow C([0,1]^n, X_\omega)$ is surjective $\forall\ n$. Therefore f_ω has the RLP w.r.t. each $i_0 : [0,1]^n \rightarrow I[0,1]^n$, hence is a Serre fibration. And: i_ω is a homotopy equivalence (cf. §3, Proposition 15), hence is a weak homotopy equivalence.]

Claim: Every continuous function $f : X \to Y$ can be written as a composite $f_\omega \circ i_\omega$, where $i_\omega : X \to X_\omega$ has the LLP w.r.t. Serre fibrations that are weak homotopy equivalences and f_ω is both a weak homotopy equivalence and a Serre fibration.

[Serre fibrations that are weak homotopy equivalences can be characterized by the property that they have the RLP w.r.t. the inclusions $\mathbf{S}^{n-1} \to \mathbf{D}^n$ $(n \geq 0)$ (cf. p. 5–16). Accordingly, in the small object argument, take $S_0 = \{\mathbf{S}^{n-1} \to \mathbf{D}^n \ (n \geq 0)\}$ and reason as above.]

Combining the claims gives MC–5. Turning to the nontrivial half of MC–4, viz. that "every fibration has the RLP w.r.t. every acyclic cofibration", suppose that $f : X \to Y$ is an "acyclic cofibration". Decompose f per the first claim: $f = f_\omega \circ i_\omega$. Since f and i_ω are weak homotopy equivalences, the same is true of f_ω, so $\exists$ a $g : Y \to X_\omega$ such that $g \circ f = i_\omega$, $f_\omega \circ g = \mathrm{id}_Y$. This means that f is a retract of i_ω. But the class of maps which have the LLP w.r.t. Serre fibrations is closed under the formation of retracts.]

[Note: We shall refer to this structure of a model category on **TOP** as the singular structure.]

Remark: If (K, L) is a relative CW complex, then the inclusion $L \to K$ has the LLP w.r.t. Serre fibrations that are weak homotopy equivalences (cf. p. 5–16), hence is a cofibration in the singular structure.

[Note: Every cofibration in the singular structure is a cofibration in the standard structure, thus is a closed cofibration. In fact there is a characterization: A continuous function is a cofibration in the singular structure iff it is a retract of a "countable composition" $X_0 \to X_1 \to \cdots \to X_\omega$, where $\forall$ k the arrow $X_k \to X_{k+1}$ is defined by a pushout square

$$\begin{tikzcd}
\coprod\limits_{n \geq 0} \coprod \mathbf{S}^{n-1} \arrow[r] \arrow[d] & X_k \arrow[d] \\
\coprod\limits_{n \geq 0} \coprod \mathbf{D}^n \arrow[r] & X_{k+1}
\end{tikzcd}$$

.]

Addendum: **CG**, **$\boldsymbol{\Delta}$-CG**, and **CGH** have a singular model category structure, viz. weak equivalence=weak homotopy equivalence, fibration=Serre fibration, cofibration=all continuous functions which have the LLP w.r.t. Serre fibrations that are weak homotopy equivalences.

[In fact, if $f : X \to Y$ is a continuous function, where $\begin{cases} X \\ Y \end{cases}$ are in **CG**, **$\boldsymbol{\Delta}$-CG**, or **CGH**, then the X_ω that figures in either of the small object arguments used above is again in **CG**, **$\boldsymbol{\Delta}$-CG**, or **CGH**.]

EXAMPLE Take $\mathbf{C} = \mathbf{TOP}$ (singular structure)—then any cofibrant X is a CW space. Thus fix a CW resolution $f : K \to X$. Factor f as $K \underset{i}{\rightarrowtail} L \underset{p}{\overset{\sim}{\twoheadrightarrow}} X$, where L is a cofibrant CW space (that this is possible is implicit in the relevant small object argument). Since X is cofibrant, $\exists$ an $s : X \to L$ such that $p \circ s = \mathrm{id}_X$. Fix a $j : L \to K$ for which $\begin{cases} i \circ j \simeq \mathrm{id}_L \\ j \circ i \simeq \mathrm{id}_K \end{cases}$ (i is a weak homotopy equivalence, hence a homotopy equivalence (realization theorem)). So: $f \circ (j \circ s) = (p \circ i) \circ (j \circ s) \simeq p \circ s = \mathrm{id}_X$. Therefore X is dominated in homotopy by K, thus by the domination theorem is a CW space.

[Note: L is a compactly generated Hausdorff space and $s : X \to L$ is a closed embedding. Conclusion: Every cofibrant X is in **CGH**. Example: $[0,1]/[0,1[$ is compactly generated (and contractible) but not Hausdorff, hence not cofibrant.]

A model category **C** is said to be proper provided that the following axiom is satisfied.

(PMC) Given a 2-source $X \xleftarrow{f} Z \xrightarrow{g} Y$, define P by the pushout square

$$\begin{tikzcd} Z \arrow[r, "g"] \arrow[d, "f"'] & Y \arrow[d, "\eta"] \\ X \arrow[r, "\xi"'] & P \end{tikzcd}$$

. Assume: f is a cofibration and g is a weak equivalence—then ξ is a weak equivalence. Given a 2-sink $X \xrightarrow{f} Z \xleftarrow{g} Y$, define P by the pullback square

$$\begin{tikzcd} P \arrow[r, "\eta"] \arrow[d, "\xi"'] & Y \arrow[d, "g"] \\ X \arrow[r, "f"'] & Z \end{tikzcd}$$

. Assume: g is a fibration and f is a weak equivalence—then η is a weak equivalence.

Remark: In a proper model category, Proposition 2 becomes an axiom (no cofibrancy conditions), which suffices to ensure the validity of Proposition 3 (no cofibrancy conditions).

PROPOSITION 5 Let **C** be a model category. Assume: All the objects of **C** are cofibrant and fibrant—then **C** is proper.

[This follows from Proposition 2.]

[Note: Not every model category is proper (cf. p. 13–40).]

Example: **TOP** (or **CG**), in its standard structure, is a proper model category.

EXAMPLE **TOP** (or **CG**, **Δ-CG**, **CGH**), in its singular structure, is a proper model category. In fact, since every object is fibrant, half of Proposition 5 is immediately applicable. However, not every object is cofibrant so for this part an ad hoc argument is necessary. Thus consider the commutative diagram

$$\begin{tikzcd} X \arrow[d, equal] & Z \arrow[l, "f"'] \arrow[r, "\mathrm{id}_Z"] \arrow[d, equal] & Z \arrow[d, "g"] \\ X & Z \arrow[l, "f"] \arrow[r, "g"'] & Y \end{tikzcd}$$

, where f is a cofibration in the singular structure and g is a weak homotopy equivalence—then f is a closed cofibration, therefore $\xi : X \to P$ is a weak homotopy equivalence (cf. p. 4–51).

[Note: Let X be a topological space which is not compactly generated—then ΓX is not compactly generated and the identity map $k\Gamma X \to \Gamma X$ is an acyclic Serre fibration, so ΓX is not cofibrant (but ΓX is a CW space).]

Let $\mathbf{C}$ be a proper model category—then a commutative diagram

$$\begin{tikzcd} W \arrow[r] \arrow[d] & Y \arrow[d, "g"] \\ X \arrow[r, "f"'] & Z \end{tikzcd}$$

in $\mathbf{C}$ is said to be a homotopy pullback if for some factorization $Y \xrightarrow{\sim} \overline{Y} \twoheadrightarrow Z$ of g, the induced morphism $W \to X \times_Z \overline{Y}$ is a weak equivalence. This definition is essentially independent of the choice of the factorization of g since any two such factorizations $\left\{\begin{array}{l} Y \xrightarrow{\sim} \overline{Y}' \twoheadrightarrow Z \\ Y \xrightarrow{\sim} \overline{Y}'' \twoheadrightarrow Z \end{array}\right.$ lead to a commutative diagram

$$\begin{tikzcd} & X \times_Z \overline{Y}' \\ W \arrow[ur] \arrow[r] \arrow[dr] & \bullet \arrow[u, "\sim"'] \arrow[d, "\sim"'] \\ & X \times_Z \overline{Y}'' \end{tikzcd}$$

and it does not matter whether one factors g or f (see below). Example: A pullback square

$$\begin{tikzcd} P \arrow[r, "\eta"] \arrow[d, "\xi"'] & Y \arrow[d, "g"] \\ X \arrow[r, "f"'] & Z \end{tikzcd}$$

is a homotopy pullback provided that g is a fibration.

[Note: The dual notion is homotopy pushout.]

Take two factorizations $\left\{\begin{array}{l} Y \xrightarrow{\sim} \overline{Y}' \twoheadrightarrow Z \\ Y \xrightarrow{\sim} \overline{Y}'' \twoheadrightarrow Z \end{array}\right.$ of g, form the pullback $\overline{Y}' \times_Z \overline{Y}''$, and note that the projections $\overline{Y}' \times_Z \overline{Y}'' \to \overline{Y}'$, $\overline{Y}' \times_Z \overline{Y}'' \to \overline{Y}''$ are fibrations. Factor the arrow $Y \to \overline{Y}' \times_Z \overline{Y}''$ as $Y \xrightarrow{\sim} \overline{W} \twoheadrightarrow \overline{Y}' \times_Z \overline{Y}''$. Since the diagram

$$\begin{tikzcd} Y \arrow[r] \arrow[d] \arrow[dr] & \overline{Y}'' \\ \overline{Y}' & \overline{W} \arrow[l] \arrow[u] \end{tikzcd}$$

commutes, the arrows $\overline{W} \to \overline{Y}'$, $\overline{W} \to \overline{Y}''$ are weak equivalences. Consider the commutative diagrams

$$\begin{tikzcd} X \arrow[r] \arrow[d, equal] & Z \arrow[d, equal] & \overline{W} \arrow[l] \arrow[d] \\ X \arrow[r] & Z & \overline{Y}' \arrow[l] \end{tikzcd}, \quad \begin{tikzcd} X \arrow[r] \arrow[d, equal] & Z \arrow[d, equal] & \overline{W} \arrow[l] \arrow[d] \\ X \arrow[r] & Z & \overline{Y}'' \arrow[l] \end{tikzcd}.$$

Because the arrows $\overline{W} \to Z$, $\overline{Y}' \to Z$, $\overline{Y}'' \to Z$ are fibrations, Proposition 3 implies that the induced morphisms $X \times_Z \overline{W} \to X \times_Z \overline{Y}'$, $X \times_Z \overline{W} \to X \times_Z \overline{Y}''$ are weak equivalences. Therefore one may put $\bullet = X \times_Z \overline{W}$ in the above.

[Note: Take a factorization $Y \xrightarrow{\sim} \overline{Y} \twoheadrightarrow Z$ of g and a factorization $X \xrightarrow{\sim} \overline{X} \twoheadrightarrow Z$ of f. Claim: The induced morphism $W \to X \times_Z \overline{Y}$ is a weak equivalence iff the induced morphism $W \to \overline{X} \times_Z Y$ is a weak equivalence. Proof: The diagram

$$\begin{tikzcd} W \arrow[r] \arrow[d] & X \times_Z \overline{Y} \arrow[d] \\ \overline{X} \times_Z Y \arrow[r] & \overline{X} \times_Z \overline{Y} \end{tikzcd}$$

commutes and the arrows $X \times_Z \overline{Y} \to \overline{X} \times_Z \overline{Y}$,

$\overline{X} \times_Z Y \to \overline{X} \times_Z \overline{Y}$ are weak equivalences (cf. Proposition 3).]

Example: In a proper model category **C**, a commutative diagram $\begin{tikzcd} W \arrow[r] \arrow[d] & Y \arrow[d, "g"] \\ X \arrow[r, "f"'] & Z \end{tikzcd}$, where f is a weak equivalence, is a homotopy pullback iff the arrow $W \to Y$ is a weak equivalence.

COMPOSITION LEMMA Consider the commutative diagram $\begin{tikzcd} \bullet \arrow[r] \arrow[d] & \bullet \arrow[r] \arrow[d] & \bullet \arrow[d] \\ \bullet \arrow[r] & \bullet \arrow[r] & \bullet \end{tikzcd}$ in a proper model category **C**. Suppose that both the squares are homotopy pullbacks—then the rectangle is a homotopy pullback. Conversely, if the rectangle and the second square are homotopy pullbacks, then the first square is a homotopy pullback.

EXAMPLE Take **C** = **TOP** (standard structure)—then the commutative diagram $\begin{tikzcd} W \arrow[r] \arrow[d] & Y \arrow[d, "g"] \\ X \arrow[r, "f"'] & Z \end{tikzcd}$ is a homotopy pullback iff the arrow $W \to W_{f,g}$ is a homotopy equivalence. Proof: The commutative diagram $\begin{tikzcd} W_{f,g} \arrow[r] \arrow[d] & Y \arrow[d, "g"] \\ W_f \arrow[r, "q"'] & Z \end{tikzcd}$ is a pullback square ($f = q \circ s$) (cf. p. 4–23). One may therefore take this condition as the definition of homotopy pullback in **TOP**. Example: A pullback square $\begin{tikzcd} P \arrow[r, "\eta"] \arrow[d, "\xi"'] & Y \arrow[d, "g"] \\ X \arrow[r, "f"'] & Z \end{tikzcd}$ is a homotopy pullback provided that g is a Dold fibration (cf. §4, Proposition 18 (with "Hurewicz" replaced by "Dold")).

[Note: Let W be a topological space; let $\begin{cases} X \\ Y \end{cases}$ be pointed topological spaces, $f : X \to Y$ a pointed continuous function—then a sequence $W \to X \xrightarrow{f} Y$ is said to be a fibration up to homotopy (or a homotopy fiber sequence) if the diagram $\begin{tikzcd} W \arrow[r] \arrow[d] & \{y_0\} \arrow[d] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$ commutes and the induced map $W \to E_f$ is a homotopy equivalence. Because E_f is the double mapping track of the 2-sink $X \xrightarrow{f} Y \leftarrow \{y_0\}$, a sequence $W \to X \xrightarrow{f} Y$ is a fibration up to homotopy if the composite $W \to Y$ is the constant map $W \to y_0$

and the commutative diagram

$$\begin{tikzcd} W \arrow[r] \arrow[d] & \{y_0\} \arrow[d] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

is a homotopy pullback.]

FACT Let

$$\begin{tikzcd} X \arrow[d] & Z \arrow[l, "f"'] \arrow[r, "g"] \arrow[d] & Y \arrow[d] \\ X' & Z' \arrow[l, "f'"] \arrow[r, "g'"'] & Y' \end{tikzcd}$$

be a commutative diagram of topological spaces in which the squares are homotopy pullbacks—then in the commutative diagram

$$\begin{tikzcd} X \arrow[r] \arrow[d] & M_{f,g} \arrow[d] & Y \arrow[l] \arrow[d] \\ X' \arrow[r] & M_{f',g'} & Y' \arrow[l] \end{tikzcd},$$

the squares are homotopy pullbacks.

Application: Suppose that $\begin{cases} A \rightarrow X \\ A' \rightarrow X' \end{cases}$ are closed cofibrations. Let $\begin{cases} f : A \rightarrow Y \\ f' : A' \rightarrow Y' \end{cases}$ be continuous functions. Assume that the diagram

$$\begin{tikzcd} X \arrow[d] & A \arrow[l] \arrow[r, "f"] \arrow[d] & Y \arrow[d] \\ X' & A' \arrow[l] \arrow[r, "f'"'] & Y' \end{tikzcd}$$

commutes and that the squares are homotopy pullbacks—then in the commutative diagram

$$\begin{tikzcd} X \arrow[r] \arrow[d] & X \sqcup_f Y \arrow[d] & Y \arrow[l] \arrow[d] \\ X' \arrow[r] & X' \sqcup_{f'} Y' & Y' \arrow[l] \end{tikzcd},$$

the squares are homotopy pullbacks.

FACT Let $\begin{cases} (\mathbf{X},\mathbf{f}) \\ (\mathbf{Y},\mathbf{g}) \end{cases}$ be objects in **FIL(TOP)**, $\phi : (\mathbf{X},\mathbf{f}) \rightarrow (\mathbf{Y},\mathbf{g})$ a morphism. Assume: $\forall\ n$,

$$\begin{tikzcd} X_n \arrow[r, "f_n"] \arrow[d, "\phi_n"'] & X_{n+1} \arrow[d, "\phi_{n+1}"] \\ Y_n \arrow[r, "g_n"'] & Y_{n+1} \end{tikzcd}$$

is a homotopy pullback—then $\forall\ n$,

$$\begin{tikzcd} X_n \arrow[r] \arrow[d] & \mathrm{tel}(\mathbf{X},\mathbf{f}) \arrow[d] \\ Y_n \arrow[r] & \mathrm{tel}(\mathbf{Y},\mathbf{g}) \end{tikzcd}$$

is a homotopy pullback.

Application: Let

$$\begin{tikzcd} X^0 \arrow[r] \arrow[d] & X^1 \arrow[r] \arrow[d] & \cdots \\ Y^0 \arrow[r] & Y^1 \arrow[r] & \cdots \end{tikzcd}$$

be a commutative ladder connecting two expanding sequences of topological spaces. Assume: $\forall\ n$, the inclusions $\begin{cases} X^n \rightarrow X^{n+1} \\ Y^n \rightarrow Y^{n+1} \end{cases}$ are cofibrations and

$$\begin{tikzcd} X_n \arrow[r] \arrow[d] & X_{n+1} \arrow[d] \\ Y_n \arrow[r] & Y_{n+1} \end{tikzcd}$$

is a homotopy pullback—then $\forall\ n$,

$$\begin{tikzcd} X^n \arrow[r] \arrow[d] & X^\infty \arrow[d] \\ Y^n \arrow[r] & Y^\infty \end{tikzcd}$$

is a homotopy pullback.

Let $\mathbf{C}$ be a model category—then a morphism $g : Y \to Z$ in $\mathbf{C}$ is said to be a homotopy fibration if in any commutative diagram

$$\begin{tikzcd} X' \times_Z Y \arrow[r, "\Phi"] \arrow[d] & X \times_Z Y \arrow[r] \arrow[d] & Y \arrow[d, "g"] \\ X' \arrow[r, "\phi"'] & X \arrow[r, "f"'] & Z \end{tikzcd},$$

Φ is a weak equivalence whenever ϕ is a weak equivalence. Example: Every fibration in a proper model category is a homotopy fibration.

LEMMA Let $\mathbf{C}$ be a proper model category. Suppose that $g : Y \to Z$ is a homotopy fibration—then the pullback square

$$\begin{tikzcd} X \times_Z Y \arrow[r] \arrow[d] & Y \arrow[d, "g"] \\ X \arrow[r, "f"'] & Z \end{tikzcd}$$

is a homotopy pullback.

[Fix factorizations $Y \xrightarrow{\sim} \overline{Y} \twoheadrightarrow Z, X \xrightarrow{\sim} \overline{X} \twoheadrightarrow Z$ of g, f and form the commutative diagram

$$\begin{tikzcd} X \times_Z Y \arrow[r] \arrow[d] & \overline{X} \times_Z Y \arrow[r] \arrow[d] & Y \arrow[d, "\sim"] \\ X \times_Z \overline{Y} \arrow[r] \arrow[d] & \overline{X} \times_Z \overline{Y} \arrow[r] \arrow[d] & \overline{Y} \arrow[d, two heads] \\ X \arrow[r, "\sim"'] & \overline{X} \arrow[r, two heads] & Z \end{tikzcd}.$$

Isolate the upper left hand corner:

$$\begin{tikzcd} X \times_Z Y \arrow[r] \arrow[d, "\Phi"'] & \overline{X} \times_Z Y \arrow[d] \\ X \times_Z \overline{Y} \arrow[r] & \overline{X} \times_Z \overline{Y} \end{tikzcd}.$$

From the assumptions, the three unlabeled arrows are weak equivalences. Therefore Φ is a weak equivalence.]

FACT The class of homotopy fibrations is closed under composition and the formation of retracts and is pullback stable.

In a model category $\mathbf{C}$, one can introduce two notions of "homotopy", which are defined respectively via "cylinder objects" and "path objects". These considerations then lead to the construction of the homotopy category $\mathbf{HC}$ of $\mathbf{C}$.

(CO) A cylinder object for X is an object IX in $\mathbf{C}$ together with a diagram $X \amalg X \overset{\iota}{\rightarrowtail} IX \xrightarrow{\sim} X$ that factors the folding map $X \amalg X \to X$. Write $\begin{cases} i_0 : X \to IX \\ i_1 : X \to IX \end{cases}$ for the arrows $\begin{cases} \iota \circ \mathrm{in}_0 \\ \iota \circ \mathrm{in}_1 \end{cases}$. Since id_X factors as $\begin{cases} X \to IX \xrightarrow{\sim} X \\ X \to IX \xrightarrow{\sim} X \end{cases}$, $\begin{cases} i_0 \\ i_1 \end{cases}$ are weak equivalences. If X is in addition cofibrant, then $\begin{cases} i_0 \\ i_1 \end{cases}$ are cofibrations. Proof: $X \amalg X$ is defined by the

pushout square
$$\begin{tikzcd} \emptyset \arrow[r] \arrow[d] & X \arrow[d, "\mathrm{in}_1"] \\ X \arrow[r, "\mathrm{in}_0"'] & X \amalg X \end{tikzcd}$$
, so $\left\{\begin{array}{l} \mathrm{in}_0 \\ \mathrm{in}_1 \end{array}\right.$ are cofibrations and the class of cofibrations is composition closed.

(PO) A path object for X is an object PX in $\mathbf{C}$ together with a diagram $X \overset{\sim}{\to} PX \overset{\Pi}{\twoheadrightarrow} X \times X$ that factors the diagonal map $X \to X \times X$. Write $\left\{\begin{array}{l} p_0 : PX \to X \\ p_1 : PX \to X \end{array}\right.$ for the arrows $\left\{\begin{array}{l} \mathrm{pr}_0 \circ \Pi \\ \mathrm{pr}_1 \circ \Pi \end{array}\right.$. Since id_X factors as $\left\{\begin{array}{l} X \overset{\sim}{\to} PX \to X \\ X \overset{\sim}{\to} PX \to X \end{array}\right.$, $\left\{\begin{array}{l} p_0 \\ p_1 \end{array}\right.$ are weak equivalences. If X is in addition fibrant, then $\left\{\begin{array}{l} p_0 \\ p_1 \end{array}\right.$ are fibrations. Proof: $X \times X$ is defined by the pullback square
$$\begin{tikzcd} X \times X \arrow[r, "\mathrm{pr}_1"] \arrow[d, "\mathrm{pr}_0"'] & X \arrow[d] \\ X \arrow[r] & * \end{tikzcd}$$
, so $\left\{\begin{array}{l} \mathrm{pr}_0 \\ \mathrm{pr}_1 \end{array}\right.$ are fibrations and the class of fibrations is composition closed.

[Note: Cylinder objects and path objects exist (cf. MC–5).]

EXAMPLE Take $\mathbf{C} = \mathbf{TOP}$ (standard structure)—then a choice for IX is $X \times [0,1]$ (cf. p. 3–5) and a choice for PX is $C([0,1],X)$ (cf. p. 4–10).

EXAMPLE Take $\mathbf{C} = \mathbf{TOP}$ (singular structure)—then a choice for IX is $X \times [0,1]$ if X is a CW complex (but not in general). However, for any X, a choice for PX is $C([0,1],X)$.

[Note: Let X be the Warsaw circle—then the inclusion $i_0X \cup i_1X \to X \times [0,1]$ is not a cofibration in the singular structure. Thus consider
$$\begin{tikzcd} i_0X \cup i_1X \arrow[r, "f"] \arrow[d] & X \arrow[d] \\ X \times [0,1] \arrow[r] & * \end{tikzcd}$$
, where $\left\{\begin{array}{l} f(x,0) = x \\ f(x,1) = x_0 \end{array}\right.$. Since $X \to *$ is a Serre fibration and a weak homotopy equivalence, the existence of a filler for this diagram would mean that X is contractible which it isn't.]

LEMMA Let (K,L) be a relative CW complex, where K is a LCH space. Suppose that $X \to B$ is a Serre fibration—then the arrow $C(K,X) \to C(L,X) \times_{C(L,B)} C(K,B)$ is a Serre fibration which is a weak homotopy equivalence if this is the case of $L \to K$ or $X \to B$.

[Note: Dropping the assumption that (K,L) is a relative CW complex and supposing only that $L \to K$ is a closed cofibration (with K a LCH space), the result continues to hold if "Serre" is replaced by "Hurewicz" and weak homotopy equivalence by homotopy equivalence.]

Application: Let (K,L) be a relative CW complex, where K is a LCH space. Suppose that $A \to Y$ is a cofibration in the singular structure—then the arrow $L \times Y \cup K \times A \to K \times Y$ is a cofibration in the singular structure which is a weak homotopy equivalence if this is the case of $L \to K$ or $A \to Y$.

EXAMPLE Take $L = \{0,1\}$, $K = [0,1]$—then for any cofibration $A \rightarrow Y$ in the singular structure, the inclusion $i_0 Y \cup A \times [0,1] \cup i_1 Y \rightarrow Y \times [0,1]$ is a cofibration in the singular structure (cf. p. 3–6). In particular, $\forall$ cofibrant X, a choice for IX is $X \times [0,1]$.

(LH) Morphisms $f, g : X \rightarrow Y$ in $\mathbf{C}$ are said to be left homotopic if $\exists$ a cylinder object IX for X and a morphism $H : IX \rightarrow Y$ such that $H \circ i_0 = f$, $H \circ i_1 = g$. One calls H a left homotopy between f and g. Notation: $f \underset{l}{\simeq} g$. If Y is fibrant and if $f \underset{l}{\simeq} g$, then $\exists$ a cylinder object $I'X$ for X with $X \amalg X \overset{i'}{\rightarrowtail} I'X \overset{\sim}{\twoheadrightarrow} X$ and a left homotopy $H' : I'X \rightarrow Y$ between f and g. Proof: Factor $IX \xrightarrow{\sim} X$ as $IX \overset{\sim}{\rightarrowtail} I'X \overset{\sim}{\twoheadrightarrow} X$ and consider a filler $H' : I'X \rightarrow Y$ for the commutative diagram

$$\begin{tikzcd} IX \arrow[r, "H"] \arrow[d] & Y \arrow[d] \\ I'X \arrow[r] & * \end{tikzcd}.$$

[Note: Suppose that $f \underset{l}{\simeq} g$—then f is a weak equivalence iff g is a weak equivalence.]

(RH) Morphisms $f, g : X \rightarrow Y$ in $\mathbf{C}$ are said to be right homotopic if $\exists$ a path object PY for Y and a morphism $G : X \rightarrow PY$ such that $p_0 \circ G = f$, $p_1 \circ G = g$. One calls G a right homotopy between f and g. Notation: $f \underset{r}{\simeq} g$. If X is cofibrant and if $f \underset{r}{\simeq} g$, then $\exists$ a path object $P'Y$ for Y with $Y \overset{\sim}{\rightarrowtail} P'Y \overset{\Pi'}{\twoheadrightarrow} Y \times Y$ and a right homotopy $G' : X \rightarrow P'Y$ between f and g. Proof: Factor $Y \xrightarrow{\sim} PY$ as $Y \overset{\sim}{\rightarrowtail} P'Y \overset{\sim}{\twoheadrightarrow} PY$ and consider a filler $G' : X \rightarrow P'Y$ for the commutative diagram

$$\begin{tikzcd} \emptyset \arrow[r] \arrow[d] & P'Y \arrow[d] \\ X \arrow[r, "G"'] & PY \end{tikzcd}.$$

[Note: Suppose that $f \underset{r}{\simeq} g$—then f is a weak equivalence iff g is a weak equivalence.]

Notation: Given X, $Y \in \operatorname{Ob} \mathbf{C}$, let $\begin{cases} [X,Y]_l \\ [X,Y]_r \end{cases}$ be the set of equivalence classes in $\operatorname{Mor}(X,Y)$ under the equivalence relation generated by $\begin{cases} \text{left} \\ \text{right} \end{cases}$ homotopy.

[Note: The relations of $\begin{cases} \text{left} \\ \text{right} \end{cases}$ homotopy are reflexive and symmetric but not necessarily transitive. Elements of $\begin{cases} [X,Y]_l \\ [X,Y]_r \end{cases}$ are denoted by $\begin{cases} [f]_l \\ [f]_r \end{cases}$ and referred to as $\begin{cases} \text{left} \\ \text{right} \end{cases}$ homotopy classes of morphisms.]

Left homotopy is reflexive. Proof: Given $f : X \rightarrow Y$, take for H the composition $IX \xrightarrow{\sim} X \xrightarrow{f} Y$.

Left homotopy is symmetric. Proof: Given $f, g : X \rightarrow Y$ and $H : IX \rightarrow Y$ such that $H \circ i_0 = f$, $H \circ i_1 = g$, let $\top : X \amalg X \rightarrow X \amalg X$ be the interchange, note that $X \amalg X \overset{\iota \circ \top}{\rightarrowtail} IX \xrightarrow{\sim} X$ factors the folding map $X \amalg X \rightarrow X$, and $H \circ (\iota \circ \top) \circ \operatorname{in}_0 = g$, $H \circ (\iota \circ \top) \circ \operatorname{in}_1 = f$.

PROPOSITION 6 Left homotopy is an equivalence relation on $\operatorname{Mor}(X,Y)$ if X is cofibrant and right homotopy is an equivalence relation on $\operatorname{Mor}(X,Y)$ if Y is fibrant.

[To check transitivity in the case of left homotopy, suppose that $f \underset{l}{\simeq} g$ & $g \underset{l}{\simeq} h$, say $\begin{cases} H \circ i_0 = f \\ H \circ i_1 = g \end{cases}$ & $\begin{cases} H' \circ i'_0 = g \\ H' \circ i'_1 = h \end{cases}$. Define $I''X$ by the pushout square

$$\begin{tikzcd} X \arrow[r, "i'_0"] \arrow[d, "i_1"'] & I'X \arrow[d, "j_1"] \\ IX \arrow[r, "j'_0"'] & I''X \end{tikzcd}$$

then $I''X$ is a cylinder object for X (specify $\iota'' : X \amalg X \to I''X$ by $\begin{cases} \iota'' \circ \mathrm{in}_0 = j'_0 \circ i_0 \\ \iota'' \circ \mathrm{in}_1 = j_1 \circ i'_1 \end{cases}$). Moreover, $H \circ i_1 = H' \circ i'_0 \Rightarrow \exists\, H'' : I''X \to Y : \begin{cases} H'' \circ i''_0 = f \\ H'' \circ i''_1 = h \end{cases}$.]

[Note: Here is the verification that ι'' is a cofibration. Form the commutative diagram

$$\begin{tikzcd} X \arrow[d, "i_0"'] & \emptyset \arrow[l] \arrow[r] \arrow[d] & X \arrow[d, "i'_1"] \\ IX & X \arrow[l, "i_1"] \arrow[r, "i'_0"'] & I'X \end{tikzcd}$$

and apply Proposition 4.]

PROPOSITION 7 If X is cofibrant and $p : Y \to Z$ is an acyclic fibration, then the postcomposition arrow $p_* : [X,Y]_l \to [X,Z]_l$ is bijective, while if Z is fibrant and $i : X \to Y$ is an acyclic cofibration, then the precomposition arrow $i^* : [Y,Z]_r \to [X,Z]_r$ is bijective.

[In either case, the arrows are welldefined. That p_* is surjective follows from the fact that, generically,

$$\begin{tikzcd} \emptyset \arrow[r] \arrow[d] & Y \arrow[d, "p"] \\ X \arrow[r] & Z \end{tikzcd}$$

has a filler $X \to Y$. Assume now that $p \circ f \underset{l}{\simeq} p \circ g$, where $f, g \in \operatorname{Mor}(X,Y)$. Choose $H : IX \to Z$ with $\begin{cases} H \circ i_0 = p \circ f \\ H \circ i_1 = p \circ g \end{cases}$—then any filler $IX \to Y$ in

$$\begin{tikzcd} X \amalg X \arrow[r, "f \amalg g"] \arrow[d, "\iota"'] & Y \arrow[d, "p"] \\ IX \arrow[r, "H"'] & Z \end{tikzcd}$$

is a left homotopy between f and g. Therefore p_* is injective.]

FACT Suppose that $\begin{cases} Y \\ Z \end{cases}$ are fibrant and $p : Y \to Z$ is a weak equivalence—then for any X, the postcomposition arrow $p_* : [X,Y]_r \to [X,Z]_r$ is injective.

FACT Suppose that $\begin{cases} X \\ Y \end{cases}$ are cofibrant and $i : X \to Y$ is a weak equivalence—then for any Z, the precomposition arrow $i^* : [Y,Z]_l \to [X,Z]_l$ is injective.

LEMMA (LH) Let $f, g \in \operatorname{Mor}(X,Y)$ be left homotopic. Assume: Y is fibrant—then $\forall\, \phi : X' \to X$, $f \circ \phi \underset{l}{\simeq} g \circ \phi$.

[Since Y is fibrant, one can arrange that the left homotopy $H : IX \rightarrow Y$ between f and g is computed per $X \amalg X \overset{\iota}{\rightarrowtail} IX \overset{\sim}{\twoheadrightarrow} X$ (cf. LH). This said, form the commutative diagram
$$\begin{tikzcd} X' \amalg X' \arrow[r, "\phi \amalg \phi"] \arrow[d, "\iota'"'] & X \amalg X \arrow[r, "\iota"] & IX \arrow[d] \\ IX' \arrow[r] & X' \arrow[r, "\phi"'] & X \end{tikzcd}$$
, choose a filler $\Phi : IX' \rightarrow IX$, and note that $H \circ \Phi$ is a left homotopy between $f \circ \phi$ and $g \circ \phi$.]

PROPOSITION 8 (LH) Suppose that Y is fibrant—then composition in $\operatorname{Mor} \mathbf{C}$ induces a map $[X', X]_l \times [X, Y]_l \rightarrow [X', Y]_l$.

[The contention is that $[f]_l = [g]_l \ (f, g \in \operatorname{Mor}(X, Y))$ & $[\phi]_l = [\psi]_l \ (\phi, \psi \in \operatorname{Mor}(X', X))$ $\Rightarrow [f \circ \phi]_l = [g \circ \psi]_l$. From the definitions, $\exists\ f_1, \ldots, f_n \in \operatorname{Mor}(X, Y) : f_1 = f, f_n = g$ with $f_i \underset{l}{\simeq} f_{i+1}$, hence by the lemma, $f_i \circ \phi \underset{l}{\simeq} f_{i+1} \circ \phi \ \forall\ i \Rightarrow [f \circ \phi]_l = [g \circ \phi]_l$. But trivially, $[g \circ \phi]_l = [g \circ \psi]_l$.]

LEMMA (RH) Let $f, g \in \operatorname{Mor}(X, Y)$ be right homotopic. Assume: X is cofibrant—then $\forall\ \psi : Y \rightarrow Y'$, $\psi \circ f \underset{r}{\simeq} \psi \circ g$.

PROPOSITION 8 (RH) Suppose that X is cofibrant—then composition in $\operatorname{Mor} \mathbf{C}$ induces a map $[X, Y]_r \times [Y, Y']_r \rightarrow [X, Y']_r$.

FACT Let $f, g \in \operatorname{Mor}(X, Y)$ be left homotopic. Suppose that $\phi : X' \rightarrow X$ is an acyclic fibration—then $f \circ \phi \underset{l}{\simeq} g \circ \phi$.

FACT Let $f, g \in \operatorname{Mor}(X, Y)$ be right homotopic. Suppose that $\psi : Y \rightarrow Y'$ is an acyclic cofibration—then $\psi \circ f \underset{r}{\simeq} \psi \circ g$.

PROPOSITION 9 Let $f, g \in \operatorname{Mor}(X, Y)$—then (i) X cofibrant & $f \underset{l}{\simeq} g \Rightarrow f \underset{r}{\simeq} g$ and (ii) Y fibrant & $f \underset{r}{\simeq} g \Rightarrow f \underset{l}{\simeq} g$.

[We shall prove (i), the proof of (ii) being analogous. Choose a left homotopy $H : IX \rightarrow Y$ between f and g and let $p : IX \rightarrow X$ be the ambient weak equivalence. Fix a path object PY for Y and let $j : Y \rightarrow PY$ be the ambient weak equivalence. Since X is cofibrant, i_0 is an acyclic cofibration, thus the commutative diagram
$$\begin{tikzcd} X \arrow[r, "j \circ f"] \arrow[d, "i_0"'] & PY \arrow[d, "\Pi"] \\ IX \arrow[r, "(f \circ p, H)"'] & Y \times Y \end{tikzcd}$$
has a filler $\rho : IX \rightarrow PY$ and the composite $G = \rho \circ i_1$ is a right homotopy between f and g.]

Notation: Given a cofibrant X and a fibrant Y, write $\simeq$ for $\underset{l}{\simeq} = \underset{r}{\simeq}$, call this equivalence relation homotopy, and let $[X, Y]$ be the set of homotopy classes of morphisms in $\operatorname{Mor}(X, Y)$, a typical element being $[f]$.

[Note: If $f \simeq g$, then f is a weak equivalence iff g is a weak equivalence.]

Observation: Suppose that X is cofibrant and Y is fibrant. Let $f, g \in \operatorname{Mor}(X, Y)$—then the following conditions are equivalent: (1) f and g are left homotopic; (2) f and g are right homotopic with respect to a fixed choice of path object; (3) f and g are right homotopic; (4) f and g are left homotopic with respect to a fixed choice of cylinder object.

FACT Let
$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\phi"'] & Y \arrow[d, "\psi"] \\ W \arrow[r, "g"'] & Z \end{tikzcd}$$
be a diagram in $\mathbf{C}$, where X is cofibrant and Z is fibrant. Assume: $\psi \circ f \simeq g \circ \phi$—then if W is fibrant and g is a fibration, $\exists\, \widetilde{\phi} : X \to W$ such that $\phi \simeq \widetilde{\phi}$ & $g \circ \widetilde{\phi} = \psi \circ f$ and if Y is cofibrant and f is a cofibration, $\exists\, \widetilde{\psi} : Y \to Z$ such that $\psi \simeq \widetilde{\psi}$ & $\widetilde{\psi} \circ f = g \circ \phi$.

PROPOSITION 10 Suppose that $\begin{cases} X \\ Y \end{cases}$ are both cofibrant and fibrant. Let $f \in \operatorname{Mor}(X, Y)$—then f is a weak equivalence iff f has a homotopy inverse, i.e., iff there exists a $g \in \operatorname{Mor}(Y, X)$ such that $g \circ f \simeq \mathrm{id}_X$ & $f \circ g \simeq \mathrm{id}_Y$.

[Necessity: Write $f = p \circ i$, where $i : X \to Z$ is an acyclic cofibration and $p : Z \to Y$ is a fibration. Note that Z is both cofibrant and fibrant and p is a weak equivalence. Fix a filler $r : Z \to X$ for
$$\begin{tikzcd} X \arrow[r, equal] \arrow[d, "i"'] & X \arrow[d] \\ Z \arrow[r] & * \end{tikzcd}$$
. Since $i^*([i \circ r]) = [i \circ r \circ i] = [i] = [\mathrm{id}_Z \circ i] = i^*([\mathrm{id}_Z])$, it follows that $i \circ r \simeq \mathrm{id}_Z$ (cf. Proposition 7). Therefore r is a homotopy inverse for i. Similarly, p admits a homotopy inverse s. Put $g = r \circ s$—then $g : Y \to X$ is a homotopy inverse for f.

Sufficiency: Decompose f as above: $f = p \circ i$. Because i is a weak equivalence, one has only to prove that p is a weak equivalence. Let $g : Y \to X$ be a homotopy inverse for f. Fix a left homotopy $H : IY \to Y$ between $f \circ g$ and id_Y and choose a filler $H' : IY \to Z$ in
$$\begin{tikzcd} Y \arrow[r, "i \circ g"] \arrow[d, "i_0"'] & Z \arrow[d, "p"] \\ IY \arrow[r, "H"'] & Y \end{tikzcd}$$
. Set $s = H' \circ i_1$ ($\Rightarrow p \circ s = \mathrm{id}_Y$). If $r : Z \to X$ is a homotopy inverse for i, then $p \simeq f \circ r \Rightarrow s \circ p \simeq i \circ g \circ p \simeq i \circ g \circ f \circ r \simeq i \circ r \simeq \mathrm{id}_Z$, so $s \circ p$ is a weak equivalence. But p is a retract of $s \circ p$, hence it too is a weak equivalence.]

EXAMPLE Take $\mathbf{C} = \mathbf{TOP}$ (singular structure) and let X, Y be cofibrant, e.g., CW complexes—then Proposition 10 says that a weak homotopy equivalence $f : X \to Y$ is a homotopy equivalence, which,

when specialized to X, Y CW complexes, is the realization theorem.

[Note: Bear in mind that a cylinder object for a cofibrant X, Y is IX, IY (cf. p. 12–18).]

Notation: $\mathbf{C_c}$ is the full subcategory of $\mathbf{C}$ whose objects are cofibrant, $\mathbf{C_f}$ is the full subcategory of $\mathbf{C}$ whose objects are fibrant, and $\mathbf{C_{cf}}$ is the full subcategory of $\mathbf{C}$ whose objects are cofibrant and fibrant. $\mathbf{H_rC_c}$ is the category with $\mathrm{Ob}\,\mathbf{H_rC_c} = \mathrm{Ob}\,\mathbf{C_c}$ and $\mathrm{Mor}\,\mathbf{H_rC_c} =$ right homotopy classes of morphisms (cf. Proposition 8 (RH)), $\mathbf{H_lC_f}$ is the category with $\mathrm{Ob}\,\mathbf{H_lC_f} = \mathrm{Ob}\,\mathbf{C_f}$ and $\mathrm{Mor}\,\mathbf{H_lC_f} =$ left homotopy classes of morphisms (cf. Proposition 8 (LH)), and $\mathbf{HC_{cf}}$ is the category with $\mathrm{Ob}\,\mathbf{HC_{cf}} = \mathrm{Ob}\,\mathbf{C_{cf}}$ and $\mathrm{Mor}\,\mathbf{HC_{cf}} =$ homotopy classes of morphisms (cf. Proposition 9).

[Note: Write $\mathbf{HC_c}$ ($\mathbf{HC_f}$) for $\mathbf{HC_{cf}}$ if all objects are fibrant (cofibrant).]

Given $X \in \mathrm{Ob}\,\mathbf{C}$, use MC–5 to factor $\emptyset \to X$ as $\emptyset \rightarrowtail \mathcal{L}X \overset{\sim}{\twoheadrightarrow} X$ and $X \to *$ as $X \overset{\sim}{\rightarrowtail} \mathcal{R}X \twoheadrightarrow *$, thus $\pi_X : \mathcal{L}X \to X$ is an acyclic fibration and $\iota_X : X \to \mathcal{R}X$ is an acyclic cofibration.

[Note: $\mathcal{L}X$ is cofibrant and $\mathcal{R}X$ is fibrant. If X is cofibrant, take $\mathcal{L}X = X$ & $\pi_X = \mathrm{id}_X$ and if X is fibrant, take $\mathcal{R}X = X$ & $\iota_X = \mathrm{id}_X$.]

LEMMA $\mathcal{L}$ Fix $\left\{\begin{matrix} X \\ Y \end{matrix}\right. \in \mathrm{Ob}\,\mathbf{C}$ and let $f \in \mathrm{Mor}(X, Y)$—then there exists $\mathcal{L}f \in \mathrm{Mor}(\mathcal{L}X, \mathcal{L}Y)$ such that the diagram

$$\begin{tikzcd}
\mathcal{L}X \arrow[r, "\mathcal{L}f"] \arrow[d, "\pi_X"'] & \mathcal{L}Y \arrow[d, "\pi_Y"] \\
X \arrow[r, "f"'] & Y
\end{tikzcd}$$

commutes. $\mathcal{L}f$ is uniquely determined up to left homotopy and is a weak equivalence iff f is. Moreover, for fibrant Y, $\mathcal{L}f$ is uniquely determined up to left homotopy by $[f]_l$.

[To establish the existence of $\mathcal{L}f$, consider any filler $\mathcal{L}X \to \mathcal{L}Y$ for

$$\begin{tikzcd}
\emptyset \arrow[r] \arrow[d] & \mathcal{L}Y \arrow[d, "\pi_Y"] \\
\mathcal{L}X \arrow[r, "f \circ \pi_X"'] & Y
\end{tikzcd}.$$

Since $\mathcal{L}X$ is cofibrant and π_Y is an acyclic fibration, the postcomposition arrow $[\mathcal{L}X, \mathcal{L}Y]_l \to [\mathcal{L}X, Y]_l$ determined by π_Y is bijective (cf. Proposition 7). This implies that $\mathcal{L}f$ is unique up to left homotopy. The weak equivalence assertion is clear. Finally, if Y is fibrant, then composition in $\mathrm{Mor}\,\mathbf{C}$ induces a map $[\mathcal{L}X, X]_l \times [X, Y]_l \to [\mathcal{L}X, Y]_l$ (cf. Proposition 8 (LH)). Therefore $[f]_l = [g]_l \Rightarrow [f \circ \pi_X]_l = [g \circ \pi_X]_l \Rightarrow [\pi_Y \circ \mathcal{L}f]_l = [\pi_Y \circ \mathcal{L}g]_l \Rightarrow \mathcal{L}f \underset{l}{\simeq} \mathcal{L}g$ (cf. Proposition 7).]

Application: $\mathcal{L}\mathrm{id}_X \underset{l}{\simeq} \mathrm{id}_{\mathcal{L}X} \Rightarrow \mathcal{L}\mathrm{id}_X \underset{r}{\simeq} \mathrm{id}_{\mathcal{L}X}$ and $\mathcal{L}(g \circ f) \underset{l}{\simeq} \mathcal{L}g \circ \mathcal{L}f \Rightarrow \mathcal{L}(g \circ f) \underset{r}{\simeq} \mathcal{L}g \circ \mathcal{L}f$ (cf. Proposition 9), thus there is a functor $\mathcal{L} : \mathbf{C} \to \mathbf{H_rC_c}$ that takes X to $\mathcal{L}X$ and $f : X \to Y$ to $[\mathcal{L}f]_r \in [\mathcal{L}X, \mathcal{L}Y]_r$.

LEMMA $\mathcal{R}$ Fix $\begin{cases} X \\ Y \end{cases} \in \text{Ob}\,\mathbf{C}$ and let $f \in \text{Mor}(X,Y)$—then there exists $\mathcal{R}f \in \text{Mor}(\mathcal{R}X,\mathcal{R}Y)$ such that the diagram

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\iota_X"'] & Y \arrow[d, "\iota_Y"] \\ \mathcal{R}X \arrow[r, "\mathcal{R}f"'] & \mathcal{R}Y \end{tikzcd}$$

commutes. $\mathcal{R}f$ is uniquely determined up to right homotopy and is a weak equivalence iff f is. Moreover, for cofibrant X, $\mathcal{R}f$ is uniquely determined up to right homotopy by $[f]_r$.

Application: $\mathcal{R}\text{id}_X \underset{r}{\simeq} \text{id}_{\mathcal{R}X} \Rightarrow \mathcal{R}\text{id}_X \underset{l}{\simeq} \text{id}_{\mathcal{R}X}$ and $\mathcal{R}(g\circ f) \underset{r}{\simeq} \mathcal{R}g\circ\mathcal{R}f \Rightarrow \mathcal{R}(g\circ f) \underset{l}{\simeq} \mathcal{R}g\circ \mathcal{R}f$ (cf. Proposition 9), thus there is a functor $\mathcal{R} : \mathbf{C} \to \mathbf{H}_l\mathbf{C}_f$ that takes X to $\mathcal{R}X$ and $f : X \to Y$ to $[\mathcal{R}f]_l \in [\mathcal{R}X,\mathcal{R}Y]_l$.

REEDY'S LIFTING LEMMA Suppose that $\begin{cases} X \\ Y \end{cases}$ are cofibrant. Let $\phi \in \text{Mor}(X,Y)$—then ϕ is a weak equivalence iff given any commutative diagram

$$\begin{tikzcd} X \arrow[r, "u"] \arrow[d, "\phi"'] & U \arrow[d, "\Phi"] \\ Y \arrow[r, "v"'] & V \end{tikzcd}$$

, where Φ is a fibration, $\exists\, w : Y \to U$ & $H : IX \to U$ such that $\Phi\circ w = v$, $\begin{cases} H\circ i_0 = u \\ H\circ i_1 = w\circ\phi \end{cases}$, and $\Phi\circ H = v\circ\phi\circ p$, $p : IX \to X$ the projection.

[Necessity: Write $\phi = \eta\circ\xi$, where $\xi : X \to Z$ is an acyclic cofibration and $\eta : Z \to Y$ is an acyclic fibration. Define IZ by the pushout square

$$\begin{tikzcd} X \amalg X \arrow[r, "\iota"] \arrow[d] & IX \arrow[d] \\ Z \amalg Z \arrow[r] & IZ \end{tikzcd}$$

to get a cylinder object for Z compatible with that for X in the sense that there is a commutative diagram

$$\begin{tikzcd} IX \arrow[r, "p"] \arrow[d, "I\xi"'] & X \arrow[d, "\xi"] \\ IZ \arrow[r, "p"'] & Z \end{tikzcd}$$

. Since Y is cofibrant, one can find an $s : Y \to Z$ such that $\eta\circ s = \text{id}_Y$. Therefore $\eta\circ\text{id}_Z = \eta\circ(s\circ\eta) \Rightarrow \exists\, h : IZ \to Z$ such that $\begin{cases} h\circ i_0 = \text{id}_Z \\ h\circ i_1 = s\circ\eta \end{cases}$ and $\eta\circ h = \eta\circ p$:

$$\begin{tikzcd} IZ \arrow[r, "h"] \arrow[d, "p"'] & Z \arrow[d, "\eta"] \\ Z \arrow[r, "\eta"'] & Y \end{tikzcd}$$

(cf. Proposition 7 and its proof). Choose now a filler $\sigma : Z \to U$ for

$$\begin{tikzcd} X \arrow[r, "u"] \arrow[d, "\xi"'] & U \arrow[d, "\Phi"] \\ Z \arrow[r, "v\circ\eta"'] & V \end{tikzcd}$$

. Definition: $w = \sigma\circ s$ & $H = \sigma\circ h\circ I\xi$. So, e.g.,
$\Phi\circ H = \Phi\circ\sigma\circ h\circ I\xi = v\circ\eta\circ h\circ I\xi = v\circ\eta\circ p\circ I\xi = v\circ\eta\circ\xi\circ p = v\circ\phi\circ p$.

Sufficiency: If $\phi : X \to Y$ has the stated property, then for every fibrant Z, $\phi^* : [Y,Z]_l \to [X,Z]_l$

is surjective and $\phi^* : [Y, Z]_r \rightarrow [X, Z]_r$ is injective, hence $\phi^* : [Y, Z] \rightarrow [X, Z]$ is bijective. Because the horizontal arrows in the commutative diagram
$$\begin{tikzcd} {[\mathcal{R}Y, \mathcal{L}Z]} \arrow[r] \arrow[d] & {[Y, Z]} \arrow[d] \\ {[\mathcal{R}X, \mathcal{L}Z]} \arrow[r] & {[X, Z]} \end{tikzcd}$$
are bijective, $(\mathcal{R}\phi)^* : [\mathcal{R}Y, \mathcal{L}Z] \rightarrow [\mathcal{R}X, \mathcal{L}Z]$ is also bijective for every fibrant Z. Take $Z = \mathcal{RL}X : \mathcal{L}Z = \mathcal{LRL}X = \mathcal{RL}X = \mathcal{R}X \Rightarrow \exists\ \psi : \mathcal{R}Y \rightarrow \mathcal{R}X$ such that $(\mathcal{R}\phi)^*([\psi]) = [\mathrm{id}_{\mathcal{R}X}]$, i.e., $\psi \circ \mathcal{R}\phi \simeq \mathrm{id}_{\mathcal{R}X}$. Working next with $Z = \mathcal{RL}Y$, it follows that $\psi^* : [\mathcal{R}X, \mathcal{R}Y] \rightarrow [\mathcal{R}Y, \mathcal{R}Y]$ is the inverse to the bijection $(\mathcal{R}\phi)^* : [\mathcal{R}Y, \mathcal{R}Y] \rightarrow [\mathcal{R}X, \mathcal{R}Y]$, thus $(\mathcal{R}\phi)^*([\mathrm{id}_{\mathcal{R}Y}]) = [\mathcal{R}\phi] \Rightarrow \psi^*([\mathcal{R}\phi]) = [\mathrm{id}_{\mathcal{R}Y}] \Rightarrow \mathcal{R}\phi \circ \psi \simeq \mathrm{id}_{\mathcal{R}Y}$. In other words, $\mathcal{R}\phi$ has a homotopy inverse and this means that $\mathcal{R}\phi$ is a weak equivalence (cf. Proposition 10) or still, ϕ is a weak equivalence.]

The proof of Proposition 2 can be shortened by using Reedy's lifting lemma. Thus consider the pushout square
$$\begin{tikzcd} Z \arrow[r, "f"] \arrow[d, "g"'] & X \arrow[d, "\xi"] \\ Y \arrow[r, "\eta"'] & P \end{tikzcd}$$
, where f is a cofibration, g is a weak equivalence, and $\begin{cases} Z \\ Y \end{cases}$ are cofibrant—then the claim is that ξ is a weak equivalence. First define M_f by the pushout square
$$\begin{tikzcd} Z \arrow[r, "f"] \arrow[d, "i_0"'] & X \arrow[d] \\ IZ \arrow[r] & M_f \end{tikzcd}$$
(cf. p. 3–20) and construct a cylinder object IX for X with the property that the arrow $M_f \rightarrow IX$ is an acyclic cofibration. This done, fix a commutative diagram
$$\begin{tikzcd} X \arrow[r, "u"] \arrow[d, "\xi"'] & U \arrow[d, "\Phi"] \\ P \arrow[r, "v"'] & V \end{tikzcd}$$
(note that P is cofibrant). Since g is a weak equivalence, $\exists\ \overline{w} : Y \rightarrow U$ & $\overline{H} : IZ \rightarrow U$ such that $\Phi \circ \overline{w} = v \circ \eta$, $\begin{cases} \overline{H} \circ i_0 = u \circ f \\ \overline{H} \circ i_1 = \overline{w} \circ g \end{cases}$, and $\Phi \circ \overline{H} = v \circ \eta \circ g \circ p$, $p : IZ \rightarrow Z$ the projection. Choose a filler $H : IX \rightarrow U$ for
$$\begin{tikzcd} M_f \arrow[r, "{(\overline{H}, u)}"] \arrow[d] & U \arrow[d, "\Phi"] \\ IX \arrow[r, "v \circ \xi \circ p"'] & V \end{tikzcd}$$
$(p : IX \rightarrow X)$ and then determine $w : P \rightarrow U$ from the commutativity of
$$\begin{tikzcd} Z \arrow[r, "f"] \arrow[d, "g"'] & X \arrow[d, "H \circ i_1"] \\ Y \arrow[r, "\overline{w}"'] & U \end{tikzcd}$$
.

PROPOSITION 11 The restriction of the functor $\mathcal{L} : \mathbf{C} \rightarrow \mathbf{H_rC_c}$ to $\mathbf{C_f}$ induces a functor $H_{\mathcal{L}} : \mathbf{H_lC_f} \rightarrow \mathbf{HC_{cf}}$, while the restriction of the functor $\mathcal{R} : \mathbf{C} \rightarrow \mathbf{H_lC_f}$ to $\mathbf{C_c}$ induces a functor $H_{\mathcal{R}} : \mathbf{H_rC_c} \rightarrow \mathbf{HC_{cf}}$.

Definition: Let $\mathbf{C}$ be a model category—then the homotopy category $\mathbf{HC}$ of $\mathbf{C}$ is the category whose underlying object class is the same as that of $\mathbf{C}$, the morphism set $[X, Y]$

of X, Y being $[\mathcal{RL}X, \mathcal{RL}Y]$.

[Note: $[\mathcal{RL}X, \mathcal{RL}Y]$ is the morphism set of $H_{\mathcal{R}} \circ \mathcal{L}(X)$, $H_{\mathcal{R}} \circ \mathcal{L}(Y)$ in the category $\mathbf{HC_{cf}}$. Of course, the situation is symmetrical in that one could just as well work with $H_{\mathcal{L}} \circ \mathcal{R}$.]

Denote by Q the functor $\mathbf{C} \rightarrow \mathbf{HC}$ which is the identity on objects and sends $f : X \rightarrow Y$ to $H_{\mathcal{R}} \circ \mathcal{L}(f) = [\mathcal{RL}f]$.

FACT Let $f, g \in \operatorname{Mor}(X, Y)$—then $\mathcal{RL}f \simeq \mathcal{RL}g$ iff $\iota_Y \circ f \circ \pi_X \simeq \iota_Y \circ g \circ \pi_X$.

PROPOSITION 12 Let $f \in \operatorname{Mor}(X, Y)$—then Qf is an isomorphism iff f is a weak equivalence.

[This follows from Proposition 10 and the fact that f is a weak equivalence iff $\mathcal{RL}f$ is a weak equivalence.]

Application: Weakly equivalent objects in $\mathbf{C}$ are isomorphic in $\mathbf{HC}$.

PROPOSITION 13 The inclusion $\mathbf{HC_{cf}} \rightarrow \mathbf{HC}$ is an equivalence of categories.

[The inclusion is obviously full and faithful. On the other hand, a given $X \in \operatorname{Ob} \mathbf{C}$ is weakly equivalent to $\mathcal{RL}X : X \xleftarrow{\pi_X} \mathcal{L}X \xrightarrow{\iota_{\mathcal{L}X}} \mathcal{RL}X$, thus the inclusion has a representative image.]

LEMMA Let $\mathbf{C}$ be a model category. Suppose that $F : \mathbf{C} \rightarrow \mathbf{D}$ is a functor which sends weak equivalences to isomorphisms—then $\begin{cases} f \underset{l}{\simeq} g \\ f \underset{r}{\simeq} g \end{cases} \Rightarrow Ff = Fg$.

[Consider the case of left homotopy: $\begin{cases} H \circ i_0 = f \\ H \circ i_1 = g \end{cases}$ and let $p : IX \xrightarrow{\sim} X$ be the projection: $\begin{cases} p \circ i_0 \\ p \circ i_1 \end{cases} = \mathrm{id}_X \Rightarrow Fp \circ Fi_0 = Fp \circ Fi_1 \Rightarrow Fi_0 = Fi_1 \Rightarrow Ff = FH \circ Fi_0 = FH \circ Fi_1 = Fg$.]

Given a cofibrant X and a fibrant Y, the symbol $[X, Y]$ has two possible interpretations. If $\operatorname{Mor}(X, Y)/\simeq$ is the quotient of $\operatorname{Mor}(X, Y)$ modulo homotopy (the meaning of $[X, Y]$ on p. 12–21), then the lemma implies that Q induces a map $\operatorname{Mor}(X, Y)/\simeq \rightarrow [X, Y]$, which in fact is bijective.

FACT Let $p : Y \rightarrow Z$ be a weak equivalence, where $\begin{cases} Y \\ Z \end{cases}$ are fibrant—then for any cofibrant X and any $f : X \rightarrow Z$, $\exists$ a $g : X \rightarrow Y$ such that $p \circ g \simeq f$, g being unique up to homotopy.

THEOREM Q Let S be the class of weak equivalences—then $S^{-1}\mathbf{C} = \mathbf{HC}$, i.e., the pair $(\mathbf{HC}, Q)$ is a localization of $\mathbf{C}$ at S.

[Proposition 12 implies that Q sends weak equivalences to isomorphisms. Suppose now that $\mathbf{D}$ is a metacategory and $F : \mathbf{C} \to \mathbf{D}$ is a functor such that $\forall\ s \in S$, Fs is an isomorphism. Claim: There exists a unique functor $F' : \mathbf{HC} \to \mathbf{D}$ such that $F = F' \circ Q$. Thus take $F' = F$ on objects and given $[f] \in [X, Y]$, represent $[f]$ by $\phi \in \operatorname{Mor}(\mathcal{RL}X, \mathcal{RL}Y)$ and let $F'[f]$ be the filler $FX \to FY$ in the diagram

$$\begin{tikzcd}
F\mathcal{RL}X \arrow[d, "F\phi"'] & F\mathcal{L}X \arrow[l, "F\iota_{\mathcal{L}X}"'] \arrow[r, "F\pi_X"] & FX \arrow[d, dashed] \\
F\mathcal{RL}Y & F\mathcal{L}Y \arrow[l, "F\iota_{\mathcal{L}Y}"] \arrow[r, "F\pi_Y"'] & FY
\end{tikzcd} \quad .]$$

Example: Let $\mathbf{C}$ be a finitely complete and finitely cocomplete category—then $\mathbf{C}$ is a model category if weak equivalence=isomorphism, cofibration=any morphism, fibration=any morphism and $\mathbf{HC} = \mathbf{C}$.

Example: Consider the arrow category $\mathbf{C}(\to)$ of a model category $\mathbf{C}$—then $\mathbf{C}(\to)$ can be equipped with two distinct model category structures. Thus let $(\phi, \psi) : (X, f, Y) \to (X', f', Y')$ be a morphism in $\mathbf{C}(\to)$, so

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "\phi"'] & Y \arrow[d, "\psi"] \\
X' \arrow[r, "f'"'] & Y'
\end{tikzcd}$$

commutes. In the first structure, call (ϕ, ψ) a weak equivalence if ϕ & ψ are weak equivalences, a cofibration if ϕ and $X' \underset{X}{\sqcup} Y \to Y'$ are cofibrations, a fibration if ϕ & ψ are fibrations and, in the second structure, call (ϕ, ψ) a weak equivalence if ϕ & ψ are weak equivalences, a cofibration if ϕ & ψ are cofibrations, a fibration if ψ and $X \to X' \times_{Y'} Y$ are fibrations. The weak equivalences in either structure are the same, thus both lead to the same homotopy category $\mathbf{HC}(\to)$.

EXAMPLE Take $\mathbf{C} = \mathbf{TOP}$ (standard structure)—then $\mathbf{HTOP}$ "is" $\mathbf{HTOP}$ but the pointed situation is different. Thus let $\mathbf{TOP}_{*c}$ be the full subcategory of $\mathbf{TOP}_*$ whose objects are the (X, x_0) such that $* \to (X, x_0)$ is a closed cofibration, i.e., whose objects are cofibrant relative to the model category structure on $\mathbf{TOP}_*$ inherited from $\mathbf{TOP}$ (cf. p. 12–3). The corresponding homotopy category of $\mathbf{TOP}_*$ is equivalent to $\mathbf{HTOP}_{*c}$ (cf. Proposition 13). Here, the "H" has its usual interpretation since for X in $\mathbf{TOP}_{*c}$, the inclusion $X \vee X \to I(X, x_0)$ is a closed cofibration, so a homotopy between objects in $\mathbf{TOP}_{*c}$ preserves the base points. However, $\mathbf{HTOP}_{*c}$ is not equivalent to $\mathbf{HTOP}_*$ if this symbol is assigned its customary meaning. Reason: The isomorphism closure in $\mathbf{HTOP}_*$ of the objects in $\mathbf{TOP}_{*c}$ is the class of nondegenerate spaces, therefore the inclusion $\mathbf{HTOP}_{*c} \to \mathbf{HTOP}_*$ does not have a representative image. Of course the explanation is that the machine is rendering invertible not just pointed homotopy equivalences between pointed spaces but also homotopy equivalences between pointed spaces.

[Note: $\mathbf{TOP}_{*\mathbf{c}}$ itself satisfies all the axioms for a model category except the first.]

EXAMPLE Take $\mathbf{C} = \mathbf{TOP}$ (singular structure)—then $\mathbf{HC}$ is equivalent to $\mathbf{HCW}$.

Let $\mathbf{C}$ be a model category. Given a category $\mathbf{D}$ and a functor $F : \mathbf{C} \to \mathbf{D}$, a left derived functor for F is a pair (LF, l) consisting of a functor $LF : \mathbf{HC} \to \mathbf{D}$ and a natural transformation $l : LF \circ Q \to F$, (LF, l) being final among all pairs having this property, i.e., for any pair (F', Ξ'), where $F' \in \operatorname{Ob}[\mathbf{HC}, \mathbf{D}]$ & $\Xi' \in \operatorname{Nat}(F' \circ Q, F)$, there exists a unique natural transformation $\Xi : F' \to LF$ such that $\Xi' = l \circ \Xi Q$. Left derived functors, if they exist, are unique up to natural isomorphism.

[Note: A right derived functor for F is a pair (RF, r) consisting of a functor $RF : \mathbf{HC} \to \mathbf{D}$ and a natural transformation $r : F \to RF \circ Q$, (RF, r) being initial among all pairs having this property, i.e., for any pair (F', Ξ'), where $F' \in \operatorname{Ob}[\mathbf{HC}, \mathbf{D}]$ & $\Xi' \in \operatorname{Nat}(F, F' \circ Q)$, there exists a unique natural transformation $\Xi : RF \to F'$ such that $\Xi' = \Xi Q \circ r$.]

Example: Suppose that $F : \mathbf{C} \to \mathbf{D}$ sends weak equivalences to isomorphisms—then by Theorem Q, there exists a unique functor $F' : \mathbf{HC} \to \mathbf{D}$ with $F = F' \circ Q$, so one can take $LF = F'$ and $l = \mathrm{id}_F$.

FACT Let $\begin{cases} F \\ G \end{cases}$ be functors $\mathbf{HC} \to \mathbf{D}$. Suppose that $\Xi : F \circ Q \to G \circ Q$ is a natural transformation—then Ξ induces a natural transformation $F \to G$.

LEMMA Let $\mathbf{C}$ be a model category. Suppose that $F : \mathbf{C}_{\mathbf{c}} \to \mathbf{D}$ is a functor which sends acyclic cofibrations to isomorphisms—then $f \underset{r}{\simeq} g \Rightarrow Ff = Fg$.

[Fix a path object PY for Y with $Y \overset{\sim}{\rightarrowtail} PY \overset{\Pi}{\twoheadrightarrow} Y \times Y$ and a right homotopy $G : X \to PY$ between f and g (cf. RH (X is cofibrant)). Calling j the acyclic cofibration $Y \to PY$, Fj is an isomorphism. Therefore $\begin{cases} p_0 \circ j = \mathrm{id}_Y \\ p_1 \circ j = \mathrm{id}_Y \end{cases} \Rightarrow Fp_0 \circ Fj = Fp_1 \circ Fj \Rightarrow Fp_0 = Fp_1 \Rightarrow$ $Ff = Fp_0 \circ FG = Fp_1 \circ FG = Fg$.]

PROPOSITION 14 Let $\mathbf{C}$ be a model category. Given a category $\mathbf{D}$ and a functor $F : \mathbf{C} \to \mathbf{D}$, suppose that F sends weak equivalences between cofibrant objects to isomorphisms—then the left derived functor (LF, l) of F exists and $\forall$ cofibrant X, $l_X : LFX \to FX$ is an isomorphism.

[The lemma implies that F induces a functor $\overline{F} : \mathbf{H}_{\mathbf{r}}\mathbf{C}_{\mathbf{c}} \to \mathbf{D}$. In addition, there is a functor $\mathcal{L} : \mathbf{C} \to \mathbf{H}_{\mathbf{r}}\mathbf{C}_{\mathbf{c}}$ that takes X to $\mathcal{L}X$ and $f : X \to Y$ to $[\mathcal{L}f]_r \in [\mathcal{L}X, \mathcal{L}Y]_r$ (cf. p.

12–22). Since the composite $\overline{F}\circ\mathcal{L}$ sends weak equivalences to isomorphisms, it follows from Theorem Q that there exists a unique functor $LF : \mathbf{HC}\to\mathbf{D}$ such that $LF\circ Q = \overline{F}\circ\mathcal{L}$. Define a natural transformation $l : LF\circ Q\to F$ by assigning to each $X\in\operatorname{Ob}\mathbf{C}$ the element $l_X = F\pi_X\in\operatorname{Mor}(F\mathcal{L}X, FX)$—then X cofibrant $\Rightarrow\begin{cases}\mathcal{L}X = X\\ \pi_X = \mathrm{id}_X\end{cases}\Rightarrow l_X = F\mathrm{id}_X = \mathrm{id}_{FX}$. It remains to prove that the pair (LF, l) is final. So fix a pair (F', Ξ') as above. Define a natural transformation $\Xi : F'\to LF$ by assigning to each $X\in\operatorname{Ob}\mathbf{HC}$ the element $\Xi_X\in\operatorname{Mor}(F'X, LFX)$ determined from $F'X\xrightarrow{F'(Q\pi_X)^{-1}}F'\mathcal{L}X\xrightarrow{\Xi'_{\mathcal{L}X}}F\mathcal{L}X = LFX$. Bearing in mind that $\forall\, X$, $QX = X$ and $\mathcal{L}X$ is cofibrant, the commutativity of

$$\begin{tikzcd}
F'\mathcal{L}X \arrow[r, "\Xi_{\mathcal{L}X}"] \arrow[d, "F'Q\pi_X"'] & LF\mathcal{L}X \arrow[r, "l_{\mathcal{L}X}"] \arrow[d, equal] & F\mathcal{L}X \arrow[d, "F\pi_X"] \\
F'X \arrow[r, "\Xi_X"'] & LFX \arrow[r, "l_X"'] & FX
\end{tikzcd}$$

ensures the uniqueness of Ξ.]

Given model categories $\begin{cases}\mathbf{C}\\ \mathbf{D}\end{cases}$ and a functor $F:\mathbf{C}\to\mathbf{D}$, a total left derived functor for F is a functor $\mathbf{L}F : \mathbf{HC}\to\mathbf{HD}$ which is a left derived functor for the composite $Q\circ F : \mathbf{C}\to\mathbf{HD}$. Total left derived functors, if they exist, are unique up to natural isomorphism.

[Note: A total right derived functor for F is a functor $\mathbf{R}F : \mathbf{HC}\to\mathbf{HD}$ which is a right derived functor for the composite $Q\circ F : \mathbf{C}\to\mathbf{HD}$.]

Remark: The substitute for the failure of

$$\begin{tikzcd}
\mathbf{C} \arrow[r, "F"] \arrow[d] & \mathbf{D} \arrow[d] \\
\mathbf{HC} \arrow[r, "\mathbf{L}F"'] & \mathbf{HD}
\end{tikzcd}$$

to commute is the natural transformation $\boldsymbol{l} : \mathbf{L}F\circ Q\to Q\circ F$.

Example: Suppose that $F:\mathbf{C}\to\mathbf{D}$ sends weak equivalences between cofibrant objects to weak equivalences—then by Proposition 14, $\mathbf{L}F$ exists and $\forall$ cofibrant X, $\boldsymbol{l}_X : \mathbf{L}FX\to FX$ is an isomorphism.

LEMMA Let $F:\mathbf{C}\to\mathbf{D}$ be a functor between model categories. Suppose that F sends acyclic cofibrations between cofibrant objects to weak equivalences—then F preserves weak equivalences between cofibrant objects.

[Let $f : X\to Y$ be a weak equivalence, where X & Y are cofibrant. Factor $f\amalg\mathrm{id}_Y : X\amalg Y\to Y$ as $p\circ i$, where $i : X\amalg Y\to Z$ is a cofibration and $p : Z\to Y$ is an acyclic fibration. Since X & Y are cofibrant, the composites $\begin{cases} i\circ\mathrm{in}_0 : X\to Z\\ i\circ\mathrm{in}_1 : Y\to Z\end{cases}$ are cofibrations. In addition, $\begin{cases} p\circ i\circ\mathrm{in}_0\\ p\circ i\circ\mathrm{in}_1\end{cases}$ are weak equivalences, hence $\begin{cases} i\circ\mathrm{in}_0\\ i\circ\mathrm{in}_1\end{cases}$ are weak equivalences.

Therefore $\left\{\begin{array}{l} F(i \circ \mathrm{in}_0) \\ F(i \circ \mathrm{in}_1) \end{array}\right.$ are weak equivalences. But $Fp \circ F(i \circ \mathrm{in}_1) = \mathrm{id}_{FY}$, thus Fp is a weak equivalence and so $Ff = Fp \circ F(i \circ \mathrm{in}_0)$ is a weak equivalence.]

TDF THEOREM Let $\mathbf{C}$ and $\mathbf{D}$ be model categories. Suppose that $\left\{\begin{array}{l} F : \mathbf{C} \to \mathbf{D} \\ G : \mathbf{D} \to \mathbf{C} \end{array}\right.$ are functors and (F, G) is an adjoint pair. Assume: F preserves cofibrations and G preserves fibrations—then $\left\{\begin{array}{l} \mathbf{L}F : \mathbf{HC} \to \mathbf{HD} \\ \mathbf{R}G : \mathbf{HD} \to \mathbf{HC} \end{array}\right.$ exist and $(\mathbf{L}F, \mathbf{R}G)$ is an adjoint pair.

[The existence of $\mathbf{L}F$ follows from the fact that F preserves acyclic cofibrations (cf. p. 12–3 ff.), thus by the lemma, F preserves weak equivalences between cofibrant objects, and Proposition 14 is applicable (the argument for $\mathbf{R}G$ is dual). Because F is a left adjoint and G is a right adjoint, F preserves initial objects and G preserves final objects. Therefore F sends cofibrant objects to cofibrant objects and G sends fibrant objects to fibrant objects. Consider now the bijection of adjunction $\Xi_{X,Y} : \mathrm{Mor}\,(FX, Y) \to \mathrm{Mor}\,(X, GY)$ (cf. p. 0–14). If $\left\{\begin{array}{l} X \in \mathrm{Ob}\,\mathbf{C_c} \\ Y \in \mathrm{Ob}\,\mathbf{D_f} \end{array}\right.$, then $\Xi_{X,Y}$ respects the relation of homotopy and induces a bijection $[FX, Y] \to [X, GY]$. Using the definitions, for arbitrary $\left\{\begin{array}{l} X \in \mathrm{Ob}\,\mathbf{C} \\ Y \in \mathrm{Ob}\,\mathbf{D} \end{array}\right.$ this leads to functorial bijections $[\mathbf{L}FX, Y] \approx [F\mathcal{L}X, \mathcal{R}Y] \approx [\mathcal{L}X, G\mathcal{R}Y] \approx [X, \mathbf{R}GY]$.]

[Note: Suppose that $\forall \left\{\begin{array}{l} X \in \mathrm{Ob}\,\mathbf{C_c} \\ Y \in \mathrm{Ob}\,\mathbf{D_f} \end{array}\right.$, $\Xi_{X,Y}$ maps the weak equivalences in $\mathrm{Mor}\,(FX, Y)$ onto the weak equivalences in $\mathrm{Mor}\,(X, GY)$—then the pair $(\mathbf{L}F, \mathbf{R}G)$ is an adjoint equivalence of categories.]

Implicit in the proof of the TDF theorem is the fact that $\forall\ X$, $\mathbf{L}FX$ is isomorphic (in $\mathbf{HD}$) to FX', where X' is any cofibrant object which is weakly equivalent to X.

EXAMPLE (Pushouts) Fix a model category $\mathbf{C}$. Let $\mathbf{I}$ be the category $1 \bullet \xleftarrow{a} \underset{3}{\bullet} \xrightarrow{b} \bullet 2$ (cf. p. 0–9)—then the functor category $[\mathbf{I}, \mathbf{C}]$ is again a model category (cf. p. 12–4 ff.). Given a 2-source $X \xleftarrow{f} Z \xrightarrow{g} Y$, define P by the pushout square

$$\begin{tikzcd} Z \arrow[r, "g"] \arrow[d, "f"'] & Y \arrow[d, "\eta"] \\ X \arrow[r, "\xi"'] & P \end{tikzcd}$$

and put $\mathrm{colim}(X \xleftarrow{f} Z \xrightarrow{g} Y) = P$ to get a functor $\mathrm{colim} : [\mathbf{I}, \mathbf{C}] \to \mathbf{C}$ which is left adjoint to the constant diagram functor $K : \mathbf{C} \to [\mathbf{I}, \mathbf{C}]$. Since K preserves fibrations and acyclic fibrations, the hypotheses of the TDF theorem are satisfied (cf. p. 12–3 ff.). Therefore $\mathbf{L}\mathrm{colim}$ and $\mathbf{R}K$ exist and $(\mathbf{L}\mathrm{colim}, \mathbf{R}K)$ is an adjoint pair. Moreover, according to the theory, $\mathbf{L}\mathrm{colim}(X \xleftarrow{f} Z \xrightarrow{g} Y)$ is isomorphic (in $\mathbf{HC}$) to $\mathrm{colim}(X \xleftarrow{f} Z \xrightarrow{g} Y)$ whenever $X \xleftarrow{f} Z \xrightarrow{g} Y$ is cofibrant, i.e., whenever Z is cofibrant and $\left\{\begin{array}{l} f : Z \to X \\ g : Z \to Y \end{array}\right.$ are cofibrations. For instance, by way of illustration, let us take $\mathbf{C} = \mathbf{TOP}$ (standard structure). Claim: $\mathbf{L}\mathrm{colim}(X \xleftarrow{f} Z \xrightarrow{g} Y)$ and $M_{f,g}$ have the same homotopy

type. To see this, consider the 2-source $M_f \leftarrow Z \rightarrow M_g$. It is cofibrant and the vertical arrows in the commutative diagram

$$\begin{tikzcd} M_f \arrow[d] & Z \arrow[l] \arrow[r] \arrow[d, equal] & M_g \arrow[d] \\ X & Z \arrow[l] \arrow[r] & Y \end{tikzcd}$$

are homotopy equivalences (but $M_f \leftarrow Z \rightarrow M_g$ is not $\mathcal{L}(X \xleftarrow{f} Z \xrightarrow{g} Y)$), so $\mathbf{L}\mathrm{colim}(X \xleftarrow{f} Z \xrightarrow{g} Y) \approx \mathrm{colim}(M_f \leftarrow Z \rightarrow M_g) \approx M_{f,g}$ (cf. p. 3–23).

[Note: The story for pullbacks is analogous (work with Rlim).]

EXAMPLE Fix a model category C—then FIL(C) is again a model category (cf. p. 12–5). Assuming that C admits sequential colimits, there is a functor colim : FIL(C) → C which is left adjoint to the constant diagram functor K : C → FIL(C). Since K preserves fibrations and acyclic fibrations, the hypotheses of the TDF theorem are satisfied (cf. p. 12–3 ff.). Therefore Lcolim and RK exist and (Lcolim, RK) is an adjoint pair. Moreover, according to the theory, Lcolim(X, f) is isomorphic (in HC) to colim(X, f) whenever (X, f) is cofibrant, i.e., whenever X_0 is cofibrant and $\forall$ n, $f_n : X_n \rightarrow X_{n+1}$ is a cofibration. If C = TOP (standard structure), Lcolim(X, f) and tel(X, f) have the same homotopy type (cf. p. 3–21). In general, colim : FIL(C) → C preserves weak equivalences between cofibrant objects, a fact which specialized to the topological setting recovers Proposition 15 in §3 provided that the cofibrations are closed.

[Note: The story for TOW(C) is analogous (work with Rlim).]

The axioms defining a model category interlock cofibrations and fibrations in such a way that certain canonical examples are excluded. This difficulty can be circumvented by simply weakening the assumptions and concentrating on either the cofibrations or the fibrations.

Consider a category **C** equipped with two composition closed classes of morphisms termed weak equivalences (denoted $\xrightarrow{\sim}$) and cofibrations (denoted $\rightarrowtail$), each containing the isomorphisms of **C**. Agreeing to call a morphism which is both a weak equivalence and a cofibration an acyclic cofibration, **C** is said to be a cofibration category provided that the following axioms are satisfied.

(CC–1) **C** has an initial object $\emptyset$.

(CC–2) Given composable morphisms f, g, if any two of f, g, $g \circ f$ are weak equivalences, so is the third.

(CC–3) Every 2-source $X \xleftarrow{f} Z \xrightarrow{g} Y$, where f is a cofibration (acyclic cofibration), admits a pushout $X \xrightarrow{\xi} P \xleftarrow{\eta} Y$, where η is a cofibration (acyclic cofibration).

(CC–4) Every morphism can be written as the composite of a cofibration and a weak equivalence.

[Note: The axioms defining a fibration category are dual.]

Let **C** be a cofibration category—then an $X \in \operatorname{Ob} \mathbf{C}$ is said to be cofibrant if $\emptyset \rightarrow X$ is a cofibration and fibrant if every acyclic cofibration $X \rightarrow Y$ has a left inverse (cf. p. 12–2).

(Fibrant Embedding Axiom) (FEA) Given an object X in **C**, there is an acyclic cofibration $\iota_X : X \rightarrow \mathcal{R}X$, where $\mathcal{R}X$ is fibrant.

[Note: The FEA is trivially met if all objects are fibrant.]

Example: The cofibrant objects in a model category are the object class of a cofibration category satisfying the FEA.

EXAMPLE Take $\mathbf{C} = \mathbf{TOP}$—then **TOP** is a cofibration category if weak equivalence=homotopy equivalence, cofibration=cofibration. All objects are cofibrant and fibrant.

EXAMPLE Take $\mathbf{C} = \mathbf{TOP}_*$—then $\mathbf{TOP}_*$ is a cofibration category if weak equivalence=pointed homotopy equivalence, cofibration=pointed cofibration. All objects are cofibrant and fibrant.

[Note: This is the "internal" structure of a cofibration category on $\mathbf{TOP}_*$. An "external" structure is obtained by letting the weak equivalences be the pointed maps which are homotopy equivalences in **TOP** and the cofibrations be the pointed maps which are cofibrations in **TOP**. Here, all objects are fibrant and the cofibrant objects are the wellpointed spaces. Another "external" structure arises by requiring that the cofibrations be closed, which reduces the number of cofibrant objects.]

EXAMPLE Take for **C** the category whose objects are pairs (X, N_X), where X is a pointed connected CW space and N_X is a perfect normal subgroup of $\pi_1(X)$, and whose morphisms $f : (X, N_X) \rightarrow (Y, N_Y)$ are pointed continuous functions $f : X \rightarrow Y$ such that $f_*(N_X) \subset N_Y$. Stipulate that f is a weak equivalence if $f_* : \pi_1(X)/N_X \approx \pi_1(Y)/N_Y$ and $f_* : H_*(X; f^*\mathcal{G}) \approx H_*(Y; \mathcal{G})$ for every locally constant coefficient system $\mathcal{G}$ on Y arising from a $\pi_1(Y)/N_Y$-module. If by cofibration one understands a pointed continuous function which is a closed cofibration in **TOP**, then **C** is a cofibration category satisfying the FEA.

[CC–1, CC–2, and CC–4 are clear. As for CC–3, given a 2-source $X \xleftarrow{f} Z \xrightarrow{g} Y$, where f is a cofibration, define P by the pushout square

$$\begin{tikzcd} Z \arrow[r, "g"] \arrow[d, "f"'] & Y \arrow[d, "\eta"] \\ X \arrow[r, "\xi"'] & P \end{tikzcd}$$

and let N_P be the normal subgroup of $\pi_1(P) = \pi_1(X) *_{\pi_1(Z)} \pi_1(Y)$ generated by N_X & N_Y. To check the FEA assertion, fix a pair (X, N_X). Thanks to the plus construction, there is a pair $(X^+_{N_X}, 0)$ and a cofibration $(X, N_X) \rightarrow (X^+_{N_X}, 0)$ which is a weak equivalence (cf. §5, Proposition 22). Claim: $(X^+_{N_X}, 0)$ is fibrant. For suppose given $(X^+_{N_X}, 0) \overset{\sim}{\rightarrowtail} (Y, N_Y)$. Denote by f the composite $(X^+_{N_X}, 0) \overset{\sim}{\rightarrowtail} (Y, N_Y) \overset{\sim}{\rightarrowtail} (Y^+_{N_Y}, 0)$, so $f_* : \pi_1(X^+_{N_X}) \approx \pi_1(Y)/N_Y \approx \pi_1(Y^+_{N_Y})$. Since f is acyclic (as a map) and a cofibration, one may now invoke §5, Proposition 19 and §3, Proposition 5.]

EXAMPLE Take for $\mathbf{C}$ the category whose objects are the pointed connected CW spaces. Fix an abelian group G—then $\mathbf{C} = \mathbf{CONCWSP}_*$ is a cofibration category if weak equivalence=HG-equivalence, cofibration=closed cofibration in $\mathbf{TOP}$ and this structure satisfies the FEA.

[Note: The fibrant objects are the HG-local spaces.]

The formal "one sided" results in model category theory carry over to cofibration categories, e.g., Propositions 2, 3, and 4. Assuming in addition that $\mathbf{C}$ satisfies the FEA, one can also show that the inclusion $\mathbf{HC_{cf}} \to \mathbf{HC}$ is an equivalence of categories (cf. Proposition 13) and $S^{-1}\mathbf{C} = \mathbf{HC}$, where S is the class of weak equivalences (cf. Theorem Q).

EXAMPLE Take $\mathbf{C} = \mathbf{TOP}_*$—then $\mathbf{HTOP}_*$ "is" $\mathbf{HTOP}_*$ if $\mathbf{TOP}_*$ carries its "internal" structure of a cofibration category.

EXAMPLE The homotopy category of the cofibration category evolving from the plus construction is equivalent to $\mathbf{HCONCWSP}_*$.

Let $\mathbf{C}$ be a category. Suppose given a composition closed class $S \subset \operatorname{Mor}\mathbf{C}$ containing the isomorphisms of $\mathbf{C}$ such that for composable morphisms f, g, if any two of $f, g, g \circ f$ are in S, so is the third. Problem: Does $S^{-1}\mathbf{C}$ exist as a category? The assumption that S admits a calculus of left or right fractions does not suffice to resolve the issue. However, one strategy that will work is to somehow place on $\mathbf{C}$ the structure of a model category (or a cofibration category) in which S appears as the class of weak equivalences. For then $S^{-1}\mathbf{C}$ "is" $\mathbf{HC}$ and $\mathbf{HC}$ is a category.

EXAMPLE Let $\mathbf{C}$ be a model category. Assume: $\mathbf{C}$ is complete and cocomplete. Suppose that $\mathbf{I}$ is a small category and let $S \subset \operatorname{Mor}[\mathbf{I}, \mathbf{C}]$ be the class of levelwise weak equivalences—then it has been shown by Dwyer-Kan[†] that $S^{-1}[\mathbf{I}, \mathbf{C}]$ exists as a category even though $[\mathbf{I}, \mathbf{C}]$ need not carry the structure of a model category having S for its class of weak equivalences.

[Note: Given a functor $[\mathbf{I}, \mathbf{C}] \to \mathbf{C}$ or $\mathbf{C} \to [\mathbf{I}, \mathbf{C}]$, one can define in the obvious way its total left (right) derived functor. In particular: $\operatorname{colim} : [\mathbf{I}, \mathbf{C}] \to \mathbf{C}$ ($\lim : [\mathbf{I}, \mathbf{C}] \to \mathbf{C}$) is a left (right) adjoint for the constant diagram functor $K : \mathbf{C} \to [\mathbf{I}, \mathbf{C}]$. Moreover, $\mathbf{L}\operatorname{colim}$ and $\mathbf{R}K$ ($\mathbf{L}K$ and $\mathbf{R}\lim$) exist and $(\mathbf{L}\operatorname{colim}, \mathbf{R}K)$ $((\mathbf{L}K, \mathbf{R}\lim))$ is an adjoint pair (Dwyer-Kan (ibid.)).]

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§13. SIMPLICIAL SETS

It is possible to develop much of algebraic topology entirely within the context of simplicial sets. However, I shall not go down that road. Instead, the focus will be on the simplicial aspects of model categories which, for instance, is the homotopical basis of the algebraic K-theory of rings or spaces.

SISET$(= \widehat{\mathbf{\Delta}})$ is complete and cocomplete, wellpowered and cowellpowered, and cartesian closed (cf. p. 0–24).

[Note: **SISET** admits an involution $X \to X^{\text{OP}}$, where $d_i^{\text{OP}} = d_{n-i}$, $s_i^{\text{OP}} = s_{n-i}$. Example: $\forall$ small category $\mathbf{C}$, $\operatorname{ner} \mathbf{C}^{\text{OP}} = (\operatorname{ner} \mathbf{C})^{\text{OP}}$.]

Notation: $\emptyset$ stands for an initial object in **SISET** (e.g., $\dot{\Delta}[0]$) and $*$ stands for a final object in **SISET** (e.g., $\Delta[0]$).

The four exponential objects associated with $\emptyset$ and $*$ are $\emptyset^{\emptyset} = *$, $*^{\emptyset} = *$, $\emptyset^{*} = \emptyset$, $*^{*} = *$.

Let X be a simplicial set—then $|X|$ is a CW complex (cf. p. 5–7), thus is a compactly generated Hausdorff space. Therefore "geometric realization" can be viewed as a functor **SISET** $\to$ **CGH**.

$|?|$: **SISET** $\to$ **TOP** preserves colimits (being a left adjoint) and it is immaterial whether the colimit is taken in **TOP** or **CGH**. Reason: A colimit in **CGH** is calculated by taking the maximal Hausdorff quotient of the colimit calculated in **TOP**.

EXAMPLE The pushout square

$$\begin{tikzcd}
\dot{\Delta}[n] \arrow[r] \arrow[d] & \Delta[0] \arrow[d] \\
\Delta[n] \arrow[r] & \mathbf{S}[n]
\end{tikzcd}$$

defines the simplicial n-sphere $\mathbf{S}[n]$. Its geometric realization is homeomorphic to $\mathbf{S}^n$.

LEMMA $|?|$: **SISET** $\to$ **CGH** preserves equalizers.

[Let X and Y be simplicial sets; let $u, v : X \to Y$ be a pair of simplicial maps—then $Z = \operatorname{eq}(u, v)$ is a simplicial subset of X and $|Z|$ is a subcomplex of $|X|$ which is contained in $\operatorname{eq}(|u|, |v|)$. Take now a point $[x, t] \in \operatorname{eq}(|u|, |v|)$, say $x \in X_n^{\#}$ & $t \in \overset{\circ}{\Delta}{}^n$ (cf. p. 0–18). Write $\begin{cases} u(x) = (Y\alpha)y_u \\ v(x) = (Y\beta)y_v \end{cases}$, where $y_u, y_v \in Y$ are nondegenerate and $\alpha, \beta \in \operatorname{Mor} \mathbf{\Delta}$ are epimorphisms. By assumption, $|u|([x, t]) = |v|([x, t])$; moreover, $\begin{cases} |u|([x, t]) = [u(x), t] = [(Y\alpha)y_u, t] = [y_u, \Delta^{\alpha}(t)] \\ |v|([x, t]) = [v(x), t] = [(Y\beta)y_v, t] = [y_v, \Delta^{\beta}(t)] \end{cases}$, so $y_u = y_v$ and $\Delta^{\alpha}(t) = \Delta^{\beta}(t)$ (because the issue is one of epimorphisms, interior points go to interior points). But $\Delta^\alpha(t) = \Delta^\beta(t) \Rightarrow \alpha = \beta$, hence $u(x) = v(x)$ or still, $x \in Z \Rightarrow [x, t] \in |Z|$.]

LEMMA $|?| : \textbf{SISET} \to \textbf{CGH}$ preserves finite products.

[Let X and Y be simplicial sets. Write $\begin{cases} X = \operatorname{colim}_i \Delta[m_i] \\ Y = \operatorname{colim}_j \Delta[n_j] \end{cases}$ (cf. p. 0–20). Since **SISET** is cartesian closed, products commute with colimits. Therefore $|X \times Y| \approx |\operatorname{colim}_{i,j} \Delta[m_i] \times \Delta[n_j]|$, from which $|X \times Y| \approx \operatorname{colim}_{i,j} |\Delta[m_i] \times \Delta[n_j]| \approx \operatorname{colim}_{i,j}(|\Delta[m_i]| \times_k |\Delta[n_j]|)$, the arrow $|\Delta[m_i] \times \Delta[n_j]| \to |\Delta[m_i]| \times |\Delta[n_j]| \equiv |\Delta[m_i]| \times_k |\Delta[n_j]|$ being a homeomorphism (cf. p. 0–19). But **CGH** is also cartesian closed (cf. p. 1–32), thus once again products commute with colimits. This gives $|X \times Y| \approx \operatorname{colim}_i |\Delta[m_i]| \times_k \operatorname{colim}_j |\Delta[n_j]| \approx |X| \times_k |Y|$, i.e., the arrow $|X \times Y| \to |X| \times_k |Y|$ is a homeomorphism.]

[Note: While the arrow $|X \times Y| \to |X| \times |Y|$ is a set theoretic bijection, it need not be a homeomorphism when $|X| \times |Y|$ has the product topology.]

PROPOSITION 1 $|?| : \textbf{SISET} \to \textbf{CGH}$ preserves finite limits.

[This is implied by the lemmas.]

[Note: $|?| : \textbf{SISET} \to \textbf{CGH}$ does not preserve arbitrary limits. Example: The arrow $|\Delta[1]^\omega| \to |\Delta[1]|^\omega$ is not a homeomorphism.]

Example: The composite $|?| \circ \sin$ preserves homotopies ($f \simeq g \Rightarrow |\sin f| \simeq |\sin g|$).

[For any topological space X, $|\sin X| \times \Delta^1 \approx |\sin X| \times |\Delta[1]| \approx |\sin X \times \Delta[1]| \to |\sin X \times \sin|\Delta[1]|| \approx |\sin(X \times \Delta^1)|$, $\to$ being the geometric realization of $\mathrm{id}_{\sin X}$ times the arrow of adjunction $\Delta[1] \to \sin|\Delta[1]|$. So, if $H : X \times \Delta^1 \to Y$ is a homotopy, then $|\sin X| \times \Delta^1 \to |\sin(X \times \Delta^1)| \xrightarrow{|\sin H|} |\sin Y|$ is a homotopy.]

EXAMPLE Let G be a simplicial group—then $|G|$ is a compactly generated group.

[Note: $|G|$ is a topological group if $|G|$ is countable, i.e., if $\forall\, n$, $\#(G_n^\#) \leq \omega$.]

FACT Let X and Y be simplicial sets, ΠX and ΠY their fundamental groupoids—then $\Pi(X \times Y) \approx \Pi X \times \Pi Y$.

[Note: The functor $\Pi : \textbf{SISET} \to \textbf{GRD}$ does not preserve equalizers. Example: Define X by the pushout square

$$\begin{tikzcd} \dot\Delta[2] \arrow[r] \arrow[d] & \Delta[2] \arrow[d, "v"] \\ \Delta[2] \arrow[r, "u"'] & X \end{tikzcd}$$

: $\Pi\dot\Delta[2] = \Pi\,\mathrm{eq}(u, v) \neq \mathrm{eq}(\Pi u, \Pi v) = \Pi\Delta[2]$.]

Let $\langle 2n \rangle$ be the category whose objects are the integers in the interval $[0, 2n]$ and whose morphisms, apart from identities, are depicted by $\underset{0}{\bullet} \to \underset{1}{\bullet} \leftarrow \cdots \to \underset{2n-1}{\bullet} \leftarrow \underset{2n}{\bullet}$. Put

$I_{2n} = \mathrm{ner}\langle 2n\rangle : |I_{2n}|$ is homeomorphic to $[0,2n]$. Given a simplicial set X, a path in X is a simplicial map $\sigma : I_{2n} \to X$. One says that σ begins at $\sigma(0)$ and ends at $\sigma(2n)$. Write $\pi_0(X)$ for the quotient of X_0 with respect to the equivalence relation obtained by declaring that $x' \sim x''$ iff there exists a path in X which begins at x' and ends at x''—then the assignment $X \to \pi_0(X)$ defines a functor $\pi_0 : \mathbf{SISET} \to \mathbf{SET}$ which preserves finite products and is a left adjoint for the functor $\mathrm{si} : \mathbf{SET} \to \mathbf{SISET}$ that sends X to $\mathrm{si}X$, the constant simplicial set on X, i.e., $\mathrm{si}X([n]) = X$ & $\left\{\begin{array}{l} d_i = \mathrm{id}_X \\ s_i = \mathrm{id}_X \end{array}\right.$ $(\forall\, n)$.

[Note: The geometric realization of $\mathrm{si}X$ is X equipped with the discrete topology.]

Let X be a simplicial set, ΠX its fundamental groupoid—then there is a canonical surjection $\coprod\limits_{0}^{\infty} \mathrm{Nat}(I_{2n}, X) \to \mathrm{Mor}\,\Pi X$ compatible with the composition of morphisms. Thus fix n and call $\mathrm{in}_i : \Delta[1] \to I_{2n}$ the injection corresponding to i. Attach to $\sigma : I_{2n} \to X$ an element $x_i \in X_1$ by setting $x_i = \sigma \circ \mathrm{in}_i(\mathrm{id}_{[1]}) : \sigma \to \pi_\sigma \in \mathrm{Mor}\,\Pi X$, where $\pi_\sigma = x_{2n}^{-1} \circ x_{2n-1} \circ \cdots \circ x_2^{-1} \circ x_1$. Corollary: $\pi_0(X) \leftrightarrow \pi_0(\Pi X)$.

[Note: ΠX and $\Pi|X|$ are equivalent but, in general, not isomorphic.]

FACT Let X be a simplicial set; let $\left\{\begin{array}{l} d_1 : X_1 \to X_0 \\ d_0 : X_1 \to X_0 \end{array}\right.$ —then $\pi_0(X) \approx \mathrm{coeq}(d_1, d_0)$.

Given a simplicial set X, the decomposition of X_0 into equivalence classes determines a partition of X into simplicial subsets X_i. The X_i are called the components of X and X is connected if it has exactly one component.

[Note: $X = \coprod\limits_i X_i \Rightarrow |X| = \coprod\limits_i |X_i|$, $|X_i|$ running through the components of $|X|$, so $\pi_0(X) \leftrightarrow \pi_0(|X|)$.]

EXAMPLE A small category $\mathbf{C}$ is connected iff its nerve $\mathrm{ner}\,\mathbf{C}$ is connected or, equivalently, iff its classifying space $B\mathbf{C}$ is connected (=path connected).

Let B be a simplicial set. An object in $\mathbf{SISET}/B$ is a simplicial set X together with a simplicial map $p : X \to B$ called the projection. Given $b \in B_n$, define X_b by the pullback square

$$\begin{tikzcd} X_b \arrow[r] \arrow[d] & X \arrow[d, "p"] \\ \Delta[n] \arrow[r, "\Delta_b"'] & B \end{tikzcd}$$

—then X_b is the fiber of p over b if $b \in B_0$.

There is a functor $\mathbf{SISET} \to \mathbf{SISET}/B$ that sends a simplicial set T to $B \times T$ with projection $B \times T \to B$. An X in $\mathbf{SISET}/B$ is said to be trivial if there exists a T in $\mathbf{SISET}$ such that X is isomorphic over B to $B \times T$, locally trivial if $\forall\, n$ & $\forall\, b \in B_n$, X_b is trivial over $\Delta[n]$, say $X_b \approx \Delta[n] \times T_b$.

[Note: If for some T, $T_b \approx T \ \forall\ n \ \& \ \forall\ b \in B_n$, then X is said to be locally trivial with fiber T.]

Notation: Given $b \in B_n$, let $b_0, b_1, \ldots, b_n$ be its vertex set, i.e., $b_i = (B\epsilon_i)b$, $\epsilon_i : [0] \to [n]$ the ith vertex operator $(i = 0, 1, \ldots, n)$.

SUBLEMMA Let X be in **SISET**$/B$. Assume: X is locally trivial—then $\forall\ b \in B_n$, T_b is isomorphic to X_{b_i} $(i = 0, 1, \ldots, n)$.

[Take $i = 0$ and consider the commutative diagram

$$\begin{tikzcd} X_{b_0} \arrow[r] \arrow[d] & X_b \arrow[r] \arrow[d] & X \arrow[d, "p"] \\ \Delta[0] \arrow[r] & \Delta[n] \arrow[r, "\Delta_b"'] & B \end{tikzcd}.$$

Here,

$$\begin{tikzcd} X_{b_0} \arrow[r] \arrow[d] & X_b \arrow[d] \\ \Delta[0] \arrow[r] & \Delta[n] \end{tikzcd}$$

is a pullback square. But X_b, viewed as an object in **SISET**$/\Delta[n]$, is isomorphic to $\Delta[n] \times T_b$, so X_{b_0} is isomorphic to T_b.]

LEMMA Let X be in **SISET**$/B$. Assume: X is locally trivial and B is connected—then X is locally trivial with fiber T.

[The sublemma implies that $\forall \left\{\begin{array}{l} b' \in B_{n'} \\ b'' \in B_{n''} \end{array}\right., \left\{\begin{array}{l} T_{b'} \approx X_{b'_0} \\ T_{b''} \approx X_{b''_0} \end{array}\right.$ and $\forall\ b \in B_1, X_{b_0} \approx X_{b_1}$.]

The terms "trivial", "locally trivial", and "locally trivial with fiber T" as used in **TOP** are also used in **CGH**, the only difference being that products are taken in **CGH**.

PROPOSITION 2 Let X be a locally trivial object in **SISET**$/B$—then $|X|$ is a locally trivial object in **CGH** $/|B|$.

[There is no loss of generality in assuming that $|B|$ is connected, hence that B is connected. So, thanks to the lemma, X is locally trivial with fiber T and the contention is that $|X|$ is locally trivial with fiber $|T|$. Fix a point $[b, t] \in |B|$ with $b \in B_n^{\#}$, $t \in \mathring{\Delta}^n$—then the associated n-cell e_b is an open subset of $|B^{(n)}| = |B|^{(n)}$. Employing a standard collaring procedure, one can find an expanding sequence $e_b = O_n \subset O_{n+1} \subset \cdots$ of subsets of $|B|$ such that $O_\infty = \operatorname{colim} O_m$ is open in $|B|$ and contains e_b as a strong deformation retract. In this connection, recall that $O_{m-1} = |B^{(m-1)}| \cap O_m$, O_m is open in $|B^{(m)}|$, and there is a pushout square

$$\begin{tikzcd} \coprod\limits_{x \in B_m^{\#}} \dot{O}_x \arrow[r] \arrow[d] & O_{m-1} \arrow[d] \\ \coprod\limits_{x \in B_m^{\#}} O_x \arrow[r] & O_m \end{tikzcd}$$

, where $\forall\ x, \left\{\begin{array}{l} \dot{O}_x \subset \dot{\Delta}^m \\ O_x \subset \Delta^m \end{array}\right.$ and $\dot{O}_x \to O_x$ is a closed cofibration, thus $O_{m-1} \to O_m$ is a closed cofibration. It will, of course, be enough to prove that $|p|^{-1}(O_\infty) \approx O_\infty \times_k |T|$. One can go further. Indeed, $O_\infty \times_k |T| = \operatorname{colim}(O_m \times_k |T|)$

and $|p|^{-1}(O_\infty) = \operatorname{colim} |p|^{-1}(O_m)$, which reduces the problem to constructing a compatible sequence of homeomorphisms

$$\begin{tikzcd}
{|p|^{-1}(O_m)} \arrow[rr] \arrow[dr] & & {O_m \times_k |T|} \arrow[dl] \\
& O_m &
\end{tikzcd}.$$

$(m = n)$ Applying $|?|$ to the pullback square

$$\begin{tikzcd}
X_b \arrow[r] \arrow[d] & X \arrow[d, "p"] \\
\Delta[n] \arrow[r, "\Delta_b"'] & B
\end{tikzcd}$$

in **SISET** gives a pullback square

$$\begin{tikzcd}
{|X_b|} \arrow[r] \arrow[d] & {|X|} \arrow[d, "|p|"] \\
\Delta^n \arrow[r, "|\Delta_b|"'] & {|B|}
\end{tikzcd}$$

in **CGH** (cf. Proposition 1). On the other hand, $X_b \approx \Delta[n] \times T$ and $|\Delta_b| : \mathring{\Delta}^n \rightarrow e_b$ is a homeomorphism.

$(m > n)$ Suppose that the homeomorphism

$$\begin{tikzcd}
{|p|^{-1}(O_{m-1})} \arrow[rr] \arrow[dr] & & {O_{m-1} \times_k |T|} \arrow[dl] \\
& O_{m-1} &
\end{tikzcd}$$

has been constructed. There is a pushout square

$$\begin{tikzcd}
\coprod\limits_{x \in B_m^\#} \dot{O}_x \times_{|B|} |X| \arrow[r] \arrow[d] & {|p|^{-1}(O_{m-1})} \arrow[d] \\
\coprod\limits_{x \in B_m^\#} O_x \times_{|B|} |X| \arrow[r] & {|p|^{-1}(O_m)}
\end{tikzcd},$$

homeomorphisms

$$\begin{tikzcd}
\dot{O}_x \times_{|B|} |X| \arrow[rr] \arrow[dr] & & \dot{O}_x \times_k |T| \arrow[dl] \\
& \dot{O}_x &
\end{tikzcd}, \quad
\begin{tikzcd}
O_x \times_{|B|} |X| \arrow[rr] \arrow[dr] & & O_x \times_k |T| \arrow[dl] \\
& O_x &
\end{tikzcd}$$

and a commutative diagram

$$\begin{tikzcd}
\coprod\limits_{x \in B_m^\#} O_x \times_{|B|} |X| \arrow[d] & \coprod\limits_{x \in B_m^\#} \dot{O}_x \times_{|B|} |X| \arrow[l] \arrow[r] \arrow[d] & {|p|^{-1}(O_{m-1})} \arrow[d] \\
\coprod\limits_{x \in B_m^\#} O_x \times_k |T| & \coprod\limits_{x \in B_m^\#} \dot{O}_x \times_k |T| \arrow[l] \arrow[r] & O_{m-1} \times_k |T|
\end{tikzcd}$$

compatible with the projections. Accordingly, the induced map $|p|^{-1}(O_m) \rightarrow O_m \times_k |T|$ is a homeomorphism over O_m.]

Application: Let X be in **SISET**/B. Assume: X is locally trivial—then $|p| : |X| \rightarrow |B|$ is a **CG** fibration (cf. p. 4–11), thus is Serre (cf. p. 4–7).

The following lemma has been implicitly used in the proof of Proposition 2.

LEMMA Fix B in **CGH**, X in **CGH**$/B$, and let $\Delta : \mathbf{I} \to \mathbf{CGH}/B$ be a diagram. Assume: The colimit of Δ calculated in **TOP** is Hausdorff—then the arrow $\operatorname{colim}(\Delta_i \times_B X) \to (\operatorname{colim} \Delta_i) \times_B X$ is a homeomorphism of compactly generated Hausdorff spaces.

Let X be in **SISET**$/B$—then $p : X \to B$ is said to be a covering projection if X is locally trivial and $\forall\, b \in B_0$, X_b is discrete, i.e., $X_b = X_b^{(0)}$.

FACT A simplicial map $p : X \to B$ is a covering projection iff every commutative diagram

$$\begin{tikzcd}
\Delta[0] \arrow[r] \arrow[d] & X \arrow[d, "p"] \\
\Delta[n] \arrow[r] & B
\end{tikzcd}$$

has a unique filler.

EXAMPLE A covering projection in **SISET** is sent by $|?|$ to a covering projection in **TOP** and a covering projection in **TOP** is sent by sin to a covering projection in **SISET**.

EXAMPLE Let $\mathbf{C}$ be a small category—then the category of covering spaces of $B\mathbf{C}$ is equivalent to the functor category $[\pi_1(\mathbf{C}), \mathbf{SET}]$, $\pi_1(\mathbf{C})$ the fundamental groupoid of $\mathbf{C}$ (cf. p. 0–16).

PROPOSITION 3 Let $\Phi, \Psi : \Delta \to \mathbf{SISET}$ be functors; let $\Xi \in \operatorname{Nat}(\Phi, \Psi)$. Assume: $\forall\, n, |\Xi_{[n]}| : |\Phi[n]| \to |\Psi[n]|$ is a homotopy equivalence—then $\forall$ simplicial set X, the geometric realization of the arrow $\Gamma_\Phi X \to \Gamma_\Psi X$ is a homotopy equivalence provided that Γ_Φ, Γ_Ψ preserve injections.

[Γ_Φ, Γ_Ψ are the realization functors corresponding to Φ, Ψ, so $\Gamma_\Phi \circ \Delta = \Phi$, $\Gamma_\Psi \circ \Delta = \Psi$ (cf. p. 0–16), thus the assertion is true if $X = \Delta[n]$, thus too if $X = \coprod \Delta[n]$. In general there are pushout squares

$$\begin{tikzcd}
X_n^\# \cdot \Gamma_\Phi \dot{\Delta}[n] \arrow[r] \arrow[d] & \Gamma_\Phi X^{(n-1)} \arrow[d] \\
X_n^\# \cdot \Gamma_\Phi \Delta[n] \arrow[r] & \Gamma_\Phi X^{(n)}
\end{tikzcd}, \quad
\begin{tikzcd}
X_n^\# \cdot \Gamma_\Psi \dot{\Delta}[n] \arrow[r] \arrow[d] & \Gamma_\Psi X^{(n-1)} \arrow[d] \\
X_n^\# \cdot \Gamma_\Psi \Delta[n] \arrow[r] & \Gamma_\Psi X^{(n)}
\end{tikzcd},$$

where, by hypothesis, the vertical arrows on the left are injective simplicial maps. Consider now the commutative diagram

$$\begin{tikzcd}
X_n^\# \cdot |\Gamma_\Phi \Delta[n]| \arrow[d] & X_n^\# \cdot |\Gamma_\Phi \dot{\Delta}[n]| \arrow[l] \arrow[r] \arrow[d] & |\Gamma_\Phi X^{(n-1)}| \arrow[d] \\
X_n^\# \cdot |\Gamma_\Psi \Delta[n]| & X_n^\# \cdot |\Gamma_\Psi \dot{\Delta}[n]| \arrow[l] \arrow[r] & |\Gamma_\Psi X^{(n-1)}|
\end{tikzcd}.$$

Since the geometric realization of an injective simplicial map is a closed cofibration and since inductively the arrows $|\Gamma_\Phi \dot{\Delta}[n]| \rightarrow |\Gamma_\Psi \dot{\Delta}[n]|$, $|\Gamma_\Phi X^{(n-1)}| \rightarrow |\Gamma_\Psi X^{(n-1)}|$ are homotopy equivalences, the induced map $|\Gamma_\Phi X^{(n)}| \rightarrow |\Gamma_\Psi X^{(n)}|$ of pushouts is a homotopy equivalence (cf. p. 3–24 ff.). Finally, $\begin{cases} \Gamma_\Phi X = \operatorname{colim} \Gamma_\Phi X^{(n)} \\ \Gamma_\Psi X = \operatorname{colim} \Gamma_\Psi X^{(n)} \end{cases} \Rightarrow \begin{cases} |\Gamma_\Phi X| = \operatorname{colim} |\Gamma_\Phi X^{(n)}| \\ |\Gamma_\Psi X| = \operatorname{colim} |\Gamma_\Psi X^{(n)}| \end{cases}$, which leads to the desired conclusion (cf. §3, Proposition 15).]

EXAMPLE Let $\Phi : \mathbf{\Delta} \rightarrow \mathbf{SISET}$ be a functor such that $\forall\, n$, $|\Phi[n]|$ is contractible. Assume given a natural transformation $\Phi \rightarrow Y_{\mathbf{\Delta}}$—then $\forall$ simplicial set X, $|\Gamma_\Phi X| \rightarrow |X|$ is a homotopy equivalence whenever Γ_Φ preserves injections.

Let $M_{\mathbf{\Delta}}$ be the set of monomorphisms in $\operatorname{Mor} \mathbf{\Delta}$; let $E_{\mathbf{\Delta}}$ be the set of epimorphisms in $\operatorname{Mor} \mathbf{\Delta}$—then every $\alpha \in \operatorname{Mor} \mathbf{\Delta}$ can be written uniquely in the form $\alpha = \alpha^\sharp \circ \alpha^\flat$, where $\alpha^\sharp \in M_{\mathbf{\Delta}}$ and $\alpha^\flat \in E_{\mathbf{\Delta}}$.

[Note: Every $\alpha \in E_{\mathbf{\Delta}}$ has a "maximal" right inverse $\alpha^+ \in M_{\mathbf{\Delta}}$, viz. $\alpha^+(i) = \max \alpha^{-1}(i)$.]

Notation: $\mathbf{\Delta}_M$ is the category with $\operatorname{Ob} \mathbf{\Delta}_M = \operatorname{Ob} \mathbf{\Delta}$ and $\operatorname{Mor} \mathbf{\Delta}_M = M_{\mathbf{\Delta}}$, $\iota_M : \mathbf{\Delta}_M \rightarrow \mathbf{\Delta}$ being the inclusion and $\Delta_M : \mathbf{\Delta}_M \rightarrow \widehat{\mathbf{\Delta}}_M$ being the Yoneda embedding.

Write **SSISET** for the functor category $[\mathbf{\Delta}_M^{\mathrm{OP}}, \mathbf{SET}]$—then an object in **SSISET** is called a semisimplicial set and a morphism in **SSISET** is called a semisimplicial map.

There is a commutative triangle

$$\begin{tikzcd} \mathbf{\Delta}_M \arrow[r, "\Delta \circ \iota_M"] \arrow[d, "\Delta_M"'] & \widehat{\mathbf{\Delta}} \\ \widehat{\mathbf{\Delta}}_M \arrow[ur, "\Gamma_{\Delta \circ \iota_M}"'] & \end{tikzcd}$$

, where $\Gamma_{\Delta \circ \iota_M}$ is the realization functor corresponding to $\Delta \circ \iota_M$. It assigns to a semisimplicial set X a simplicial set PX, the prolongment of X. Explicitly, the elements of $(PX)_n$ are all pairs (x, ρ) with $x \in X_p$ and $\rho : [n] \rightarrow [p]$ an epimorphism, thus $(PX\alpha)(x, \rho) = ((X(\rho \circ \alpha)^\sharp)x, (\rho \circ \alpha)^\flat)$ if the codomain of α is $[n]$. And: P assigns to a semisimplicial map $f : X \rightarrow Y$ the simplicial map $Pf : \begin{cases} PX \rightarrow PY \\ (x, \rho) \rightarrow (f(x), \rho) \end{cases}$. The prolongment functor is a left adjoint for the forgetful functor $U : \widehat{\mathbf{\Delta}} \rightarrow \widehat{\mathbf{\Delta}}_M$ (the singular functor in this setup).

[Note: The Kan extension theorem implies that U is also a left adjoint. In particular: U preserves colimits.]

Definition: $|?|_M = |?| \circ P$. So, $(|?|_M, U \circ \sin)$ is an adjoint pair and $|?|_M$ is the realization functor determined by the composite $\Delta^? \circ \iota_M$.

[Note: $|?|_M : \mathbf{SSISET} \rightarrow \mathbf{CGH}$ does not preserve finite products.]

PROPOSITION 4 For any simplicial set X, the arrow $|UX|_M \rightarrow |X|$ is a homotopy equivalence.

[In the notation of Proposition 3, take $\Phi = P \circ U \circ \Delta$, $\Psi = \Delta$, and let $\Xi \in \mathrm{Nat}(\Phi, \Psi)$ be the natural transformation arising from the arrow of adjunction $P \circ U \to \mathrm{id}$ via precomposition. Because Γ_Φ, Γ_Ψ preserve injections, it need only be shown that $\forall\ n$, the arrow $|PU\Delta[n]| \to |\Delta[n]|$ is a homotopy equivalence or still, that $\forall\ n$, $|PU\Delta[n]|$ is contractible. Suppose first that $n = 0$. In this case, $|PU\Delta[0]| = \coprod\limits_n \Delta^n/\sim$, the equivalence relation being generated by writing $(t_0, \ldots, t_{i-1}, 0, t_{i+1}, \ldots, t_n) \sim (t_0, \ldots, t_{i-1}, t_{i+1}, \ldots, t_n)$. Therefore $|PU\Delta[0]|$ is the infinite dimensional "dunce hat" D. As such, it is contractible. For positive n, let $D * \cdots * D$ be the quotient of $D \times \cdots \times D \times \Delta^n$ with respect to the relations $(d'_0, \ldots, d'_n, (t_0, \ldots, t_n)) \sim (d''_0, \ldots, d''_n, (t_0, \ldots, t_n))$ iff $d'_i = d''_i$ when $t_i \neq 0$—then up to homeomorphism, $|PU\Delta[n]|$ is $D * \cdots * D$, a contractible space.]

Given n, let $\overline{\Delta}[n]$ be the simplicial set defined by the following conditions.

(Ob) $\overline{\Delta}[n]$ assigns to an object $[p]$ the set $\overline{\Delta}[n]_p$ of all finite sequences $\mu = (\mu_0, \ldots, \mu_p)$ of monomorphisms in $\mathbf{\Delta}$ having codomain $[n]$ such that $\forall\ i, j$ $(0 \leq i \leq j \leq p)$ there is a monomorphism μ_{ij} with $\mu_i = \mu_j \circ \mu_{ij}$.

(Mor) $\overline{\Delta}[n]$ assigns to a morphism $\alpha : [q] \to [p]$ the map $\overline{\Delta}[n]_p \to \overline{\Delta}[n]_q$ taking μ to $\mu \circ \alpha$, i.e., $(\mu_0, \ldots, \mu_p) \to (\mu_{\alpha(0)}, \ldots, \mu_{\alpha(q)})$.

Call $\overline{\Delta}$ the functor $\mathbf{\Delta} \to \widehat{\mathbf{\Delta}}$ that sends $[n]$ to $\overline{\Delta}[n]$ and $\alpha : [m] \to [n]$ to $\overline{\Delta}[\alpha] : \overline{\Delta}[m] \to \overline{\Delta}[n]$, where $\overline{\Delta}[\alpha]\nu = ((\alpha \circ \nu_0)^\sharp, \ldots, (\alpha \circ \nu_p)^\sharp)$. The associated realization functor $\Gamma_{\overline{\Delta}}$ is a functor $\mathbf{SISET} \to \mathbf{SISET}$ such that $\Gamma_{\overline{\Delta}} \circ \Delta = \overline{\Delta}$. It assigns to a simplicial set X a simplicial set $\mathrm{Sd}\, X = \int^{[n]} X_n \cdot \overline{\Delta}[n]$, the subdivision of X, and to a simplicial map $f : X \to Y$ a simplicial map $\mathrm{Sd}\, f : \mathrm{Sd}\, X \to \mathrm{Sd}\, Y$, the subdivision of f. In particular, $\mathrm{Sd}\, \Delta[n] = \overline{\Delta}[n]$ and $\mathrm{Sd}\, \Delta[\alpha] = \overline{\Delta}[\alpha]$. On the other hand, the realization functor Γ_Δ associated with the Yoneda embedding Δ is naturally isomorphic to the identity functor id on $\mathbf{SISET}$: $X = \int^{[n]} X_n \cdot \Delta[n]$. If $\mathrm{d}_n : \overline{\Delta}[n] \to \Delta[n]$ is the simplicial map that sends $\mu = (\mu_0, \ldots, \mu_p) \in \overline{\Delta}[n]_p$ to $\mathrm{d}_n\mu \in \Delta[n]_p : \mathrm{d}_n\mu(i) = \mu_i(m_i)$ $(\mu_i : [m_i] \to [n])$, then the d_n determine a natural transformation $\mathrm{d} : \overline{\Delta} \to \Delta$, which, by functoriality, leads to a natural transformation $\mathrm{d} : \Gamma_{\overline{\Delta}} \to \Gamma_\Delta$. Thus, $\forall\ X, Y$ and $\forall\ f : X \to Y$, there is a commutative diagram

$$\begin{tikzcd}
\mathrm{Sd}\, X \arrow[r, "\mathrm{d}_X"] \arrow[d, "\mathrm{Sd}\, f"'] & X \arrow[d, "f"] \\
\mathrm{Sd}\, Y \arrow[r, "\mathrm{d}_Y"'] & Y
\end{tikzcd}$$

. It will be shown below that $|\mathrm{d}_X| : |\mathrm{Sd}\, X| \to |X|$ is a homotopy equivalence (cf. Proposition 5).

Given n, write $\overline{\Delta}^n$ for $|\overline{\Delta}[n]|$ and $\overline{\Delta}^\alpha$ for $|\overline{\Delta}[\alpha]|$. The elements of $\overline{\Delta}^n$ are equivalence classes $[\mu, t]$. Any two representatives of $[\mu, t]$ are related by a finite chain of "elementary equivalences" involving omission of μ_i and t_i if $t_i = 0$ and replacement of t_i and t_{i+1} by $t_i + t_{i+1}$ if $\mu_{i+1} = \mu_i$. Every $[\mu, t]$ has a canonical representative, meaning that $[\mu, t]$ can be represented by a pair $(\mu, t) : \mu = (\mu_0, \ldots, \mu_n) \in \overline{\Delta}[n]_n$ with $\mu_i : [i] \to [n]$ $(0 \leq i \leq n)$ and $t = (t_0, \ldots, t_n) \in \Delta^n$. So, $\mu_n = \mathrm{id}_{[n]}$ and there exists a permutation π of $\{0, 1, \ldots, n\}$ such that $\forall\, i$, $\mu_i([i]) = \{\pi(0), \pi(1), \ldots, \pi(i)\}$.

Notation: Given $\alpha \in M_{\mathbf{\Delta}}$, say $\alpha : [m] \to [n]$, put $b(\alpha) = \dfrac{1}{m+1} \sum\limits_0^m e_{\alpha(i)} \in \mathbf{R}^{n+1}$.

LEMMA For each $n \geq 0$, the assignment $[\mu, t] \to \sum\limits_0^p t_i b(\mu_i)$ is a (welldefined) homeomorphism $h_n : \overline{\Delta}^n \to \Delta^n$.

[Note: Geometrically, $\overline{\Delta}^n$ is "barycentric subdivision" of Δ^n.]

The homeomorphisms h_n do not determine a natural transformation $|?| \circ \overline{\Delta} \to |?| \circ \Delta$. In fact, it is impossible for these functors to be naturally isomorphic. To see this, suppose to the contrary that there exists a natural isomorphism $\Xi : |?| \circ \overline{\Delta} \to |?| \circ \Delta$. There would then be homeomorphisms $\begin{cases} \Xi_m : \overline{\Delta}^m \to \Delta^m \\ \Xi_n : \overline{\Delta}^n \to \Delta^n \end{cases}$ such that for any $\alpha : [m] \to [n]$ the diagram

$$\begin{tikzcd} \overline{\Delta}^m \arrow[r, "\Xi_m"] \arrow[d, "\overline{\Delta}^\alpha"'] & \Delta^m \arrow[d, "\Delta^\alpha"] \\ \overline{\Delta}^n \arrow[r, "\Xi_n"'] & \Delta^n \end{tikzcd}$$

commutes. Take $m = 2$, $n = 1$ and trace the effect on the pair $(\mathrm{id}_{[2]}, 1)$ when α is in succession $\sigma_0 : [2] \to [1], \sigma_1 : [2] \to [1]$.

[Note: If $\alpha : [m] \to [n]$ is a monomorphism, then the diagram

$$\begin{tikzcd} \overline{\Delta}^m \arrow[r, "h_m"] \arrow[d, "\overline{\Delta}^\alpha"'] & \Delta^m \arrow[d, "\Delta^\alpha"] \\ \overline{\Delta}^n \arrow[r, "h_n"'] & \Delta^n \end{tikzcd}$$

commutes.]

SUBDIVISION THEOREM Let X be a simplicial set—then there is a homeomorphism $h_X : |\mathrm{Sd}\, X| \to |X|$.

[Before proceeding to the details, I shall first outline the argument. In order to define a continuous function $h_X : |\mathrm{Sd}\, X| \to |X|$, it is enough to define a continuous function $\coprod\limits_n X_n \times \overline{\Delta}^n \to \coprod\limits_n X_n \times \Delta^n$ that respects the relations defining $|\mathrm{Sd}\, X| = \int^{[n]} X_n \cdot \overline{\Delta}^n$ and $|X| = \int^{[n]} X_n \cdot \Delta^n$. This amounts to exhibiting a collection of continuous functions $h_x : \overline{\Delta}^n \to \Delta^n$ $(x \in X_n, n \geq 0)$ such that for all $\alpha : [m] \to [n]$, the diagram

$$\begin{tikzcd} \overline{\Delta}^m \arrow[r, "h_y"] \arrow[d, "\overline{\Delta}^\alpha"'] & \Delta^m \arrow[d, "\Delta^\alpha"] \\ \overline{\Delta}^n \arrow[r, "h_x"'] & \Delta^n \end{tikzcd}$$

commutes. Here, $y = (X\alpha)x$. To ensure that h_X is a homeomorphism, one need only arrange that if $x \in X_n^\#$ $(n \geq 0)$, then h_x restricts to a homeomorphism $h_n^{-1}(\mathring{\Delta}^n) \to \mathring{\Delta}^n$.

Let $x \in X_n$. Consider a pair (μ, t), with $\mu = (\mu_0, \ldots, \mu_p) \in \overline{\Delta}[n]_p$ and $t = (t_0, \ldots, t_p) \in \Delta^p$. Write $(X\mu_i)x = (X\alpha_i)x_i$, where α_i is an epimorphism and x_i is nondegenerate. Put $\gamma_{ij} = (\alpha_j \circ \mu_{ij})^{\flat}$, $b_{ij} = b(\mu_i \circ \gamma_{ij}^{+})$ $(0 \leq i \leq j \leq p)$. Definition:

$$h_x([\mu, t]) = t_p b_{pp} + \sum_{0 \leq i < p} t_i(1 - t_p - \cdots - t_{i+1}) b_{ii} + \sum_{0 \leq i < j \leq p} t_i t_j b_{ij}.$$

This expression is a convex combination of points in Δ^n, hence is in Δ^n. Moreover, its value depends only on the class $[\mu, t]$ and not on a specific representative (μ, t). Therefore $h_x : \overline{\Delta}^n \to \Delta^n$ makes sense. Because there exist finitely many nondegenerate μ such that $\bigcup\limits_{\mu} |\Delta_\mu|(\Delta^n) = \overline{\Delta}^n$, h_x is continuous. Turning to compatibility, fix $\alpha : [m] \to [n]$—then the claim is that $\Delta^\alpha \circ h_y = h_x \circ \overline{\Delta}^\alpha$. Given $\nu = (\nu_0, \ldots, \nu_p) \in \overline{\Delta}[m]_p$, let $\mu = \overline{\Delta}[\alpha]\nu \in \overline{\Delta}[n]_p$ and construct $\beta_i, y_i, \delta_{ij}$ per ν and y exactly like $\alpha_i, x_i, \gamma_{ij}$ are constructed per μ and x. From the definitions, $\alpha \circ \nu_i \circ \delta_{ij}^{+} = \mu_i \circ \gamma_{ij}^{+}$ and this implies that Δ^α matches the barycenters, which suffices.

Let $x \in X_n^{\#}$. Pick a canonical representative (μ, t) for $[\mu, t]$—then $\forall\, i$, $\gamma_{in} = \gamma_{in}^{+} = \mathrm{id}_{[i]}$ and $[\mu, t] \in h_n^{-1}(\mathring{\Delta}^n)$ iff $t_n > 0$. Since each of the coordinates of $h_x([\mu, t]) \in \Delta^n$ is bounded from below by $t_n/(n+1)$, it follows that $h_x(h_n^{-1}(\mathring{\Delta}^n)) \subset \mathring{\Delta}^n$. To address the issue of injectivity, suppose that $[\mu', t'], [\mu'', t''] \in h_n^{-1}(\mathring{\Delta}^n)$ and $h_x([\mu', t']) = (t_0, \ldots, t_n) = h_x([\mu'', t''])$. In terms of canonical representatives, one has to prove that $\forall\, i$, $t_i' = t_i''$ and $\mu_i' = \mu_i''$ if t_i' & t_i'' are > 0. This will be done by decreasing induction on i. Let $\left\{\begin{array}{l}\pi' \\ \pi''\end{array}\right.$ be the permutations attached to $\left\{\begin{array}{l}\mu' \\ \mu''\end{array}\right.$. Looking at $t_{\pi'(n)} = t_n'/(n+1)$ and $t_{\pi''(n)} = t_n''/(n+1)$ yields $t_n' = t_n''$, starting the induction. Assume that $k < n$ and that the assertion is true $\forall\, i > k$. Define $T' = (T_0', \ldots, T_n')$ by

$$\sum_{0 \leq i \leq k} t_i'(1 - t_n' - \cdots - t_{i+1}') b_{ii}' + \sum_{\substack{0 \leq i \leq k \\ i < j \leq n}} t_i' t_j' b_{ij}'.$$

Define $T'' = (T_0'', \ldots, T_n'')$ analogously—then, from the induction hypothesis, $T' = T''$. Case 1: $\mu_k' \neq \mu_k''$. Choose $l \in [n] : l \in \mu_k'([k])$ & $l \notin \mu_k''([k]) \Rightarrow t_k' t_n'/(k+1) \leq T_l' = T_l'' = 0 \Rightarrow t_k' = 0$. Similarly, $t_k'' = 0$. Case 2: $\mu_k' = \mu_k''$. Take T' and split off

$$(1 - t_n' - \cdots - t_{k+1}') b_{kk}' + \sum_{k < j \leq n} t_j' b_{kj}'$$

to get $S' = (S_0', \ldots, S_n')$. Do the same with T'' to get $S'' = (S_0'', \ldots, S_n'')$—then, from the induction hypothesis, $S' = S''$. Set $l = \pi'(k)$ and compute: $t_k' S_l' = T_l' = T_l'' \geq$

$t''_k S''_l = t''_k S'_l$. But $S'_l \geq t'_n/(k+1) > 0 \Rightarrow t'_k \geq t''_k$. Similarly, $t''_k \geq t'_k$. Thus the induction is complete. Owing to the theorem of invariance of domain, $h_x(h_n^{-1}(\mathring{\Delta}^n))$ is open in $\mathring{\Delta}^n$ and the restriction $h_n^{-1}(\mathring{\Delta}^n) \to h_x(h_n^{-1}(\mathring{\Delta}^n))$ is a homeomorphism. However, $h_x(\overline{\Delta}^n - h_n^{-1}(\mathring{\Delta}^n)) \subset \dot{\Delta}^n$, so $h_x(h_n^{-1}(\mathring{\Delta}^n)) = \mathring{\Delta}^n \cap h_x(\overline{\Delta}^n)$ is closed in $\mathring{\Delta}^n$. Being nonempty, $h_x(h_n^{-1}(\mathring{\Delta}^n))$ must be equal to $\mathring{\Delta}^n$.]

BARRATT'S LEMMA Let Δ be a simplex, Δ_1 a proper face of Δ, Δ_0 a proper face of Δ_1. Let $r : \Delta_1 \to \Delta_0$ be an affine retraction, i.e., a retraction induced by the composition of a linear map and a translation mapping vertexes onto vertexes. Define X by the pushout square

$$\begin{tikzcd}
\Delta_1 \arrow[r, "r"] \arrow[d] & \Delta_0 \arrow[d] \\
\Delta \arrow[r] & X
\end{tikzcd}$$

—then there exists a homeomorphism $\phi : X \to \Delta$ such that the triangle

$$\begin{tikzcd}
\Delta_0 \arrow[r] \arrow[d] & \Delta \\
X \arrow[ur, "\phi"'] &
\end{tikzcd}$$

commutes.

[Supposing that $n+1 = \dim \Delta$, normalize the situation as follows. Take for Δ the one point compactification of $\{(x_0, \ldots, x_n) : x_n \geq 0\}$, let Δ_1 be the convex hull of $\{0, e_0, \ldots, e_m\}$, let Δ_0 be the convex hull of $\{0, e_0, \ldots, e_k\}$, and let P be the orthogonal projection onto the span of $\{e_0, \ldots, e_k, e_{m+1}, \ldots, e_n\}$, so $P|\Delta_1 = r$ and $X = \Delta/\sim$, where $x \sim y$ iff $x = y \notin \Delta_1$ or $r(x) = r(y)$ $(x, y \in \Delta_1)$. Let $d(x)$ be the distance of x from Δ_1, $f(x) = \min\{1, d(x)\}$, and put $\phi(x) = f(x)x + (1 - f(x))P(x)$ (thus $\phi(\infty) = \infty$ and $\phi|\Delta_1 = r$).

Claim: $\phi : \Delta \to \Delta$ is surjective and $\phi|\Delta - \Delta_1$ is injective.

[Given $x = (x_0, \ldots, x_n)$, set $x(t) = (x_0, \ldots, x_k, tx_{k+1}, \ldots, tx_m, x_{m+1}, \ldots, x_n)$. Obviously, $x_{k+1} = \cdots = x_m = 0 \Rightarrow \phi(x) = x$. On the other hand, if some $x_i \neq 0$ $(k < i \leq m)$, then $t \to \infty \Rightarrow x(t) \to \infty \Rightarrow f(x(t)) = 1$ $(t >> 0)$. However, $\phi(x(t)) = (x_0, \ldots, x_k, tf(x(t))x_{k+1}, \ldots, tf(x(t))x_m, x_{m+1}, \ldots, x_n)$ and the intermediate value theorem guarantees that $\exists\, t : tf(x(t)) = 1$. Assume now that $x, y \in \Delta_1$ with $\phi(x) = \phi(y) : x_i = y_i$ $(i \leq k \ \&\ i > m)$, $f(x)x_i = f(y)y_i$ $(k < i \leq m) \Rightarrow y = x\left(\dfrac{f(x)}{f(y)}\right)$. But $t \to \phi(x(t))$ is one-to-one $(\Rightarrow x = y)$. To see this, it need only be shown that $t \to d(x(t))$ is nondecreasing. Proceeding by contradiction, suppose that $d(x(t')) < d(x(t))$ $(\exists\, t' > t)$ and choose $u : d(x(t')) < u < d(x(t)) \Rightarrow u > d(x(0))$, i.e., $x(0), x(t') \in d^{-1}([0,u])$, $x(t) \notin d^{-1}([0,u])$, an impossibility, $d^{-1}([0,u])$ being convex.]

Therefore ϕ determines a continuous bijection $X \to \Delta$ between compact Hausdorff spaces with the stated property.]

FACT Let X be a simplicial set—then $|\mathrm{Sd}\, X|$ is a polyhedron, hence $|X|$ can be triangulated.

[Using Barratt's lemma, apply the criterion on p. 5–13 to $|\mathrm{Sd}\, X|$, observing that $\forall$ nondegenerate x in $(\mathrm{Sd}\, X)_n$ there is a pushout square

$$\begin{tikzcd}
\Delta[n-1] \arrow[r] \arrow[d, "{\Delta[\delta_n]}"'] & \langle d_n x \rangle \arrow[d] \\
\Delta[n] \arrow[r] & \langle x \rangle
\end{tikzcd}$$

, where $\langle ? \rangle$ equals "generated simplicial subset".]

PROPOSITION 5 Let X be a simplicial set—then $|\mathrm{d}_X| : |\mathrm{Sd}\,X| \to |X|$ is a homotopy equivalence.

[One can define $|\mathrm{d}_X|$ by a collection of continuous functions $\mathrm{d}_x : \overline{\Delta}^n \to \Delta^n$ satisfying the same compatibility conditions as the $h_x : \overline{\Delta}^n \to \Delta^n$ that figure in the proof of the subdivision theorem. Introduce $H_x : \overline{\Delta}^n \times [0,1] \to \Delta^n$ by writing $H_x(u,t) = (1-t)h_x(u) + t\mathrm{d}_x(u)$—then, in total, the H_x define a homotopy $|\mathrm{Sd}\,X| \times [0,1] \to |X|$ between h_X and $|\mathrm{d}_X|$.]

[Note: h_X is not natural but is homotopic to $|\mathrm{d}_X|$ which is natural. The fact that $|\mathrm{d}_X|$ is a homotopy equivalence can also be seen directly. Proof: $\forall\, n$, $|\overline{\Delta}[n]| = \overline{\Delta}^n$ is contractible and $\Gamma_{\overline{\Delta}} = \mathrm{Sd}$ preserves injections, thus the example following Proposition 3 is applicable.]

EXAMPLE Let X be a simplicial set—then $|X|$ is homeomorphic to $B(c\mathrm{Sd}^2\,X)$ (Fritsch-Latch[†]). Therefore the geometric realization of a simplicial set is homeomorphic to the classifying space of a small category.

[Note: The homeomorphism is not natural.]

Sd is the realization functor $\Gamma_{\overline{\Delta}}$. The associated singular functor $S_{\overline{\Delta}}$ is denoted by Ex and referred to as extension. Since (Sd, Ex) is an adjoint pair, there is a bijective map $\Xi_{X,Y} : \mathrm{Nat}(\mathrm{Sd}\,X, Y) \to \mathrm{Nat}(X, \mathrm{Ex}\,Y)$ which is functorial in X and Y (cf. p. 0–14). Put $\mathrm{e}_X = \Xi_{X,X}(\mathrm{d}_X)$—then $\mathrm{e}_X : X \to \mathrm{Ex}\,X$ is the simplicial map given by $\mathrm{e}_X(x) = \Delta_x \circ \mathrm{d}_n$ $(x \in X_n)$, hence e_X is injective.

LEMMA For every simplicial set X, $|\mathrm{e}_X| : |X| \to |\mathrm{Ex}\,X|$ is a homotopy equivalence (cf. p. 13–29).

[Note: Since e_X is injective, $|X|$ can be considered as a strong deformation retract of $|\mathrm{Ex}\,X|$ (cf. §3, Proposition 5).]

Denote by Ex^∞ the colimit of $\mathrm{id} \to \mathrm{Ex} \to \mathrm{Ex}^2 \to \cdots$—then Ex^∞ is a functor **SISET** → **SISET** and for any simplicial set X, there is an arrow $\mathrm{e}_X^\infty : X \to \mathrm{Ex}^\infty X$. Claim: $|\mathrm{e}_X^\infty| : |X| \to |\mathrm{Ex}^\infty X|$ is a homotopy equivalence. In fact, $|\mathrm{Ex}^n X|$ embeds in $|\mathrm{Ex}^{n+1} X|$ as a strong deformation retract and $|\mathrm{Ex}^\infty X| = \mathrm{colim}\,|\mathrm{Ex}^n X|$. Therefore $|X|$ is a strong deformation retract of $|\mathrm{Ex}^\infty X|$ (cf. p. 3–20).

The subdivision functor can also be introduced in the semisimplicial setting. It is compatible with prolongment in that there is a commutative diagram

$$\begin{tikzcd}
\mathbf{SSISET} \arrow[r, "\mathrm{Sd}"] \arrow[d, "P"'] & \mathbf{SSISET} \arrow[d, "P"] \\
\mathbf{SISET} \arrow[r, "\mathrm{Sd}"'] & \mathbf{SISET}
\end{tikzcd}$$

[†] *Math. Zeit.* **177** (1981), 147–179.

and, in contradistinction to what happens in the simplicial setting, the homeomorphism $h_{PX} : |\mathrm{Sd}\, X|_M \to |X|_M$ is natural, as is the homotopy between h_{PX} and $|\mathrm{d}_{PX}|$.

Put $S = U \circ \sin$—then $S : \mathbf{TOP} \to \mathbf{SSISET}$ and $(|?|_M, S)$ is an adjoint pair. Given a topological space X, postcompose $h_{PSX} : |\mathrm{Sd}\, SX|_M \to |SX|_M$ with the arrow $|SX|_M \to X$ to get a continuous function $|\mathrm{Sd}\, SX|_M \to X$ which by adjointness corresponds to a semisimplicial map $g_{SX} : \mathrm{Sd}\, SX \to SX$. Definition: $b_X = |Pg_{SX}| \circ h_{PSX}^{-1} \in C(|SX|_M, |SX|_M)$. Using Proposition 4, one can check that b_X is naturally homotopic to $\mathrm{id}_{|SX|_M}$. In effect, the triangle

$$\begin{tikzcd} {|SX|_M} \arrow[rr, "b_X"] \arrow[dr] & & {|SX|_M} \arrow[dl] \\ & {|\sin X|} & \end{tikzcd}$$

commutes up to homotopy.

SIMPLICIAL EXCISION THEOREM Let X be a topological space. Suppose that $\begin{cases} X_1 \\ X_2 \end{cases}$ are subspaces of X with $X = \mathrm{int}\, X_1 \cup \mathrm{int}\, X_2$—then the geometric realization of $\sin X_1 \cup \sin X_2$ is a strong deformation retract of $|\sin X|$.

[The inclusion $|\sin X_1 \cup \sin X_2| \to |\sin X|$ is a closed cofibration, thus it will be enough to prove that it is a homotopy equivalence (cf. §3, Proposition 5). According to Proposition 4, the vertical arrows in the commutative diagram

$$\begin{tikzcd} {|SX_1 \cup SX_2|_M} \arrow[r] \arrow[d] & {|SX|_M} \arrow[d] \\ {|\sin X_1 \cup \sin X_2|} \arrow[r] & {|\sin X|} \end{tikzcd}$$

are homotopy equivalences, which reduces the problem to showing that the inclusion $|SX_1 \cup SX_2|_M \to |SX|_M$ is a homotopy equivalence or still, a weak homotopy equivalence. To this end, fix $n \geq 0$ and let $f : \mathbf{D}^n \to |SX|_M$ be a continuous function such that $f(\mathbf{S}^{n-1}) \subset |SX_1 \cup SX_2|_M$. Since the image of f is contained in the union of a finite number of cells of $|SX|_M$, $\exists\, k >> 0 : b_X^k \circ f$ factors through $|SX_1 \cup SX_2|_M$ (the "excisive" consequence of the assumption that $X = \mathrm{int}\, X_1 \cup \mathrm{int}\, X_2$). On the other hand, by naturality, $b_X(|SX_1 \cup SX_2|_M) \subset |SX_1 \cup SX_2|_M$ and the same is true of the homotopy between b_X and $\mathrm{id}_{|SX|_M}$, hence too for the kth iterate b_X^k. Therefore f is homotopic rel $\mathbf{S}^{n-1}$ to a continuous function $g : \mathbf{D}^n \to |SX|_M$ with $g(\mathbf{D}^n) \subset |SX_1 \cup SX_2|_M$. These considerations suffice to imply that the inclusion $|SX_1 \cup SX_2|_M \to |SX|_M$ is a weak homotopy equivalence (cf. p. 3–39).]

Let $\mathcal{C}$ be a class of topological spaces—then $\mathcal{C}$ is said to be homotopy cocomplete provided that the following conditions are satisfied.

(HOCO$_1$) If $X \in \mathcal{C}$ and if Y has the same homotopy type as X, then $Y \in \mathcal{C}$.

(HOCO$_2$) $\mathcal{C}$ is closed under the formation of coproducts.

(HOCO$_3$) If $X \xleftarrow{f} Z \xrightarrow{g} Y$ is a 2-source with $\begin{cases} X \\ Y \end{cases}$ & $Z \in \mathcal{C}$, then $M_{f,g} \in \mathcal{C}$.

Examples: (1) The class of CW spaces is homotopy cocomplete; (2) The class of numerably contractible spaces is homotopy cocomplete.

PROPOSITION 6 The class of topological spaces for which the arrow of adjunction $|\sin X| \to X$ is a homotopy equivalence is homotopy cocomplete.

[If $f : X \to Y$ is a homotopy equivalence, then $|\sin f| : |\sin X| \to |\sin Y|$ is a homotopy equivalence (cf. p. 13–2). Since the diagram

$$\begin{tikzcd} {|\sin X|} \arrow[r, "|\sin f|"] \arrow[d] & {|\sin Y|} \arrow[d] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

commutes, HOCO_1 obtains. That HOCO_2 holds is clear, so it remains to deal with HOCO_3. Viewing $M_{f,g}$ as a quotient of $X \amalg IZ \amalg Y$, let $\overline{X}$ be the image of $X \amalg Z \times [0, 2/3]$, let $\overline{Y}$ be the image of $Z \times [1/3, 1] \amalg Y$ and put $\overline{Z} = \overline{X} \cap \overline{Y}$—then $M_{f,g} = \operatorname{int} \overline{X} \cup \operatorname{int} \overline{Y}$ and there are homotopy equivalences $\overline{X} \to X$, $\overline{Y} \to Y$, $\overline{Z} \to Z$. Because X, Y, Z are in our class, the same is true of $\overline{X}$, $\overline{Y}$, $\overline{Z}$. To establish that the arrow $|\sin M_{f,g}| \to M_{f,g}$ is a homotopy equivalence, consider the commutative diagram

$$\begin{tikzcd} {|\sin \overline{X}|} \arrow[d] & {|\sin \overline{Z}|} \arrow[l] \arrow[r] \arrow[d] & {|\sin \overline{Y}|} \arrow[d] \\ \overline{X} & \overline{Z} \arrow[l] \arrow[r] & \overline{Y} \end{tikzcd}$$

. The horizontal arrows are closed cofibrations, hence the induced map of pushouts is a homotopy equivalence (cf. p. 3–24 ff.). The pushout arising from the 2-source on the bottom is $M_{f,g}$, while the pushout arising from the 2-source on the top is $|\sin \overline{X} \cup \sin \overline{Y}|$ which, by the simplicial excision theorem, is a strong deformation retract of $|\sin M_{f,g}|$. Inspection of the triangle

$$\begin{tikzcd} {|\sin \overline{X} \cup \sin \overline{Y}|} \arrow[r] \arrow[dr] & {|\sin M_{f,g}|} \arrow[d] \\ & M_{f,g} \end{tikzcd}$$

finishes the argument.]

[Note: $\forall\ X$, $|\sin X|$ is a CW complex, thus X is a CW space if the arrow of adjunction $|\sin X| \to X$ is a homotopy equivalence.]

Any homotopy cocomplete class of topological spaces that contains a one point space necessarily contains the class of CW spaces. But $\#(X) = 1 \Rightarrow \#(|\sin X|) = 1$, therefore the class of CW spaces is precisely the class of topological spaces for which the arrow of adjunction $|\sin X| \to X$ is a homotopy equivalence.

GIEVER-MILNOR THEOREM Let X be a topological space—then the arrow of adjunction $|\sin X| \to X$ is a weak homotopy equivalence.

[The adjoint pair $(|?|, \sin)$ determines a cotriple in **TOP** (cf. p. 0–28), which induces a cotriple in **HTOP** ($|?| \circ \sin$ preserves homotopies (cf. p. 13–2)). On general grounds, $\forall\ Y$, the postcomposition arrow $[|\sin Y|, |\sin X|] \to [|\sin Y|, X]$ is surjective. However here it is also injective. Reason: $\forall\ Z$, the arrow of adjunction $|\sin |\sin Z|| \to |\sin Z|$ is a homotopy equivalence, i.e., is an isomorphism in **HTOP**. It therefore follows that for every CW complex K, the postcomposition arrow $[K, |\sin X|] \to [K, X]$ is bijective and this means that the arrow of adjunction $|\sin X| \to X$ is a weak homotopy equivalence (cf. p. 5–15 ff.).]

Application: Let X be a simplicial set—then the geometric realization of the arrow of adjunction $X \to \sin|X|$ is a homotopy equivalence.

[The triangle
$$\begin{tikzcd} {|X|} \arrow[r] \arrow[dr, "\mathrm{id}_{|X|}"'] & {|\sin|X||} \arrow[d] \\ & {|X|} \end{tikzcd}$$
commutes.]

EXAMPLE Consider the adjoint situation (F, G, μ, ν), where $F = |?|$, $G = \sin$—then in the notation of p. 0–32, $\begin{cases} S^{-1}\mathbf{SISET} \\ T^{-1}\mathbf{TOP} \end{cases}$ are equivalent to **HCW**.

Given simplicial sets X and Y, write $\mathrm{map}(X, Y)$ in place of Y^X (cf. p. 0–23). The elements of $\mathrm{map}(X, Y)_0 \approx \mathrm{Nat}(X, Y)$ are the simplicial maps $X \to Y$, two such being termed homotopic if they belong to the same component of $\mathrm{map}(X, Y)$. In other words, simplicial maps $f, g \in \mathrm{Nat}(X, Y)$ are homotopic ($f \simeq g$) provided that $\exists\, n \geq 0$ and a simplicial map $H : X \times I_{2n} \to Y$ such that if $\begin{cases} H \circ i_0 \ : X \approx X \times \Delta[0] \xrightarrow{\mathrm{id}_X \times e_0} X \times I_{2n} \xrightarrow{H} Y \\ H \circ i_{2n} : X \approx X \times \Delta[0] \xrightarrow[\mathrm{id}_X \times e_{2n}]{} X \times I_{2n} \xrightarrow[H]{} Y \end{cases}$, then $\begin{cases} H \circ i_0 = f \\ H \circ i_{2n} = g \end{cases}$, where $\begin{cases} e_0 : \Delta[0] \to I_{2n} \\ e_{2n} : \Delta[0] \to I_{2n} \end{cases}$ are the vertex inclusions per $\begin{cases} 0 \\ 2n \end{cases}$.

[Note: Paths $I_{2n} \to \mathrm{map}(X, Y)$ correspond to homotopies $H : X \times I_{2n} \to Y$.]

Given simplicial sets X and Y, simplicial maps $f, g \in \mathrm{Nat}(X, Y)$ are said to be simplicially homotopic ($f \underset{s}{\simeq} g$) provided that $\exists$ a simplicial map $H : X \times \Delta[1] \to Y$ such that if $\begin{cases} H \circ i_0 : X \approx X \times \Delta[0] \xrightarrow{\mathrm{id}_X \times e_0} X \times \Delta[1] \xrightarrow{H} Y \\ H \circ i_1 : X \approx X \times \Delta[0] \xrightarrow[\mathrm{id}_X \times e_1]{} X \times \Delta[1] \xrightarrow[H]{} Y \end{cases}$, then $\begin{cases} H \circ i_0 = f \\ H \circ i_1 = g \end{cases}$, where $\begin{cases} e_0 : \Delta[0] \to \Delta[1] \\ e_1 : \Delta[0] \to \Delta[1] \end{cases}$ are the vertex inclusions per $\begin{cases} 0 \\ 1 \end{cases}$. The relation $\underset{s}{\simeq}$ is reflexive but it needn't be symmetric or transitive.

[Note: Elements of $\mathrm{map}(X, Y)_1$ correspond to simplicial homotopies $H : X \times \Delta[1] \to Y$.]

Example: Suppose that $\begin{cases} \mathbf{C} \\ \mathbf{D} \end{cases}$ are small categories. Let $F, G : \mathbf{C} \to \mathbf{D}$ be functors, $\Xi : F \to G$ a natural transformation—then Ξ defines a functor $\Xi_H : \mathbf{C} \times [1] \to \mathbf{D}$, hence $\mathrm{ner}\, \Xi_H : \mathrm{ner}(\mathbf{C} \times [1]) \to \mathrm{ner}\, \mathbf{D}$, i.e., $\mathrm{ner}\, \Xi_H : \mathrm{ner}\, \mathbf{C} \times \Delta[1] \to \mathrm{ner}\, \mathbf{D}$ is a simplicial homotopy between $\mathrm{ner}\, F$ and $\mathrm{ner}\, G$. So, e.g., $\begin{cases} B\mathbf{C} \\ B\mathbf{D} \end{cases}$ have the same homotopy type if there is a functor $\mathbf{C} \to \mathbf{D}$ which admits a left or right adjoint. In particular: The classifying space of a small category having either an initial or a final object is contractible. Example: $B\boldsymbol{\Delta}$ is contractible.

EXAMPLE Take $X = Y = \Delta[n]$ $(n > 0)$. Let $C_0 : \Delta[n] \to \Delta[n]$ be the projection of $\Delta[n]$ onto the 0th vertex, i.e., send $(\alpha_0, \ldots, \alpha_p) \in \Delta[n]_p$ to $(0, \ldots, 0) \in \Delta[n]_p$. Claim: $C_0 \underset{s}{\simeq} \mathrm{id}_{\Delta[n]}$. To see this, consider the simplicial map $H : \Delta[n] \times \Delta[1] \to \Delta[n]$ defined by $H((\alpha_0, \ldots, \alpha_p), (0, \ldots, 0, 1, \ldots, 1)) = (0, \ldots, 0, \alpha_{i+1}, \ldots, \alpha_p)$ so that $H((\alpha_0, \ldots, \alpha_p), (0, \ldots, 0)) = (0, \ldots, 0)$, $H((\alpha_0, \ldots, \alpha_p), (1, \ldots, 1)) = (\alpha_0, \ldots, \alpha_p)$—then H is a simplicial homotopy between C_0 and $\mathrm{id}_{\Delta[n]}$. On the other hand, there is no simplicial homotopy H between $\mathrm{id}_{\Delta[n]}$ and C_0. For suppose that $H((1,1),(0,1)) = (\mu, \nu) \in \Delta[n]_1$. Apply d_1 & d_0 to get $\mu = 1$ & $\nu = 0$, an impossibility.

[Note: Let $C_k : \Delta[n] \to \Delta[n]$ be the projection of $\Delta[n]$ onto the kth vertex, i.e., send $(\alpha_0, \ldots, \alpha_p) \in \Delta[n]_p$ to $(k, \ldots, k) \in \Delta[n]_p$ $(0 \leq k \leq n)$—then $\mathrm{id}_{\Delta[n]} \underset{s}{\simeq} C_n$ but $\mathrm{id}_{\Delta[n]} \underset{s}{\not\simeq} C_k$ $(0 \leq k < n)$. Still, $\forall$ k, $\exists$ a homotopy $H_k : \Delta[n] \times I_2 \to \Delta[n]$ such that $H_k \circ e_0 = \mathrm{id}_{\Delta[n]}$ and $H_k \circ e_2 = C_k$.]

FACT Suppose that $f, g : X \to Y$ are simplicially homotopic—then $\mathrm{Ex}\, f$, $\mathrm{Ex}\, g : \mathrm{Ex}\, X \to \mathrm{Ex}\, Y$ are simplicially homotopic.

[Ex is a right adjoint, hence preserves products.]

The equivalence relation generated by $\underset{s}{\simeq}$ is $\simeq$. Given simplicial sets X and Y, put $[X, Y]_0 = \mathrm{Nat}(X, Y)/\simeq$, so $[X, Y]_0 = \pi_0(\mathrm{map}(X, Y))$—then $\mathbf{H_0SISET}$ is the category whose objects are the simplicial sets and whose morphisms are the homotopy classes of simplicial maps.

[Note: The symbol **HSISET** is reserved for a different role (cf. p. 13–35).]

To check that the relation of homotopy is compatible with composition, let X, Y, and Z be simplicial sets. Define a simplicial map $C_{X,Y,Z} : \mathrm{map}(X, Y) \times \mathrm{map}(Y, Z) \to \mathrm{map}(X, Z)$ by assigning to a pair (f, g) in $\mathrm{map}(X, Y)_n \times \mathrm{map}(Y, Z)_n$ the composite $X \times \Delta[n] \xrightarrow{\mathrm{id} \times \mathrm{di}} X \times (\Delta[n] \times \Delta[n]) \xrightarrow{A} (X \times \Delta[n]) \times \Delta[n] \xrightarrow{f \times \mathrm{id}} Y \times \Delta[n] \xrightarrow{g} Z$ in $\mathrm{map}(X, Z)_n$. At level 0, $C_{X,Y,Z}$ is composition of simplicial maps. Since $\pi_0(\mathrm{map}(X, Y) \times \mathrm{map}(Y, Z)) \approx \pi_0(\mathrm{map}(X, Y)) \times \pi_0(\mathrm{map}(Y, Z))$, $C_{X,Y,Z}$ induces an arrow $[X, Y]_0 \times [Y, Z]_0 \to [X, Z]_0$ with the requisite properties.

[Note: $\mathbf{H_0SISET}$ has finite products. In addition, $\mathrm{map}(X \times Y, Z) \approx \mathrm{map}(X, \mathrm{map}(Y, Z)) \Rightarrow [X \times Y, Z]_0 \approx [X, \mathrm{map}(Y, Z)]_0$, so $\mathbf{H_0SISET}$ is cartesian closed.]

EXAMPLE Geometric realization preserves homotopies but $|?| : \mathbf{H_0SISET} \to \mathbf{HTOP}$ is not conservative.

[Take $X = \Delta[0]$, $Y = \mathrm{ner}(\infty)$, where (∞) is the zig-zag on the set of nonnegative integers: $0 < 1 > 2 < 3 > 4 \ldots$, and consider the inclusion $X \to Y$ corresponding to $0 \to 0$.]

Notation: Given a simplicial set X, write IX in place of $X \times \Delta[1]$.

The obvious composite $X \amalg X \rightarrow IX \rightarrow X$ factors the folding map $X \amalg X \rightarrow X$ and **SISET** carries the structure of a model category in which IX is a cylinder object (cf. p. 13–35).

A simplicial map $f : X \rightarrow Y$ is said to be a weak homotopy equivalence if its geometric realization $|f| : |X| \rightarrow |Y|$ is a weak homotopy equivalence (= homotopy equivalence). Example: $\forall\ X$, the projection $IX \rightarrow X$ is a weak homotopy equivalence.

[Note: A homotopy equivalence in **SISET** is a weak homotopy equivalence (but not conversely).]

EXAMPLE Suppose that $\begin{cases} X \\ Y \end{cases}$ are topological spaces and $f : X \rightarrow Y$ is a continuous function—then there is a commutative diagram

$$\begin{tikzcd}
{|\sin X|} \arrow[r, "{|\sin f|}"] \arrow[d] & {|\sin Y|} \arrow[d] \\
X \arrow[r, "f"'] & Y
\end{tikzcd}$$

, thus f is a weak homotopy equivalence iff $\sin f$ is a weak homotopy equivalence (Giever-Milnor theorem).]

EXAMPLE (Simplicial Groups) Given a simplicial group G, put $N_nG = \bigcap\limits_{i>0} \ker d_i$ $(n > 0)$ $(N_0G = G_0)$ and let $\partial_n : N_nG \rightarrow N_{n-1}G$ be the restriction $d_0|N_nG$ $(n > 0)$ $(\partial_0 : N_0G \rightarrow 0)$—then $\operatorname{im}\partial_{n+1}$ is a normal subgroup of $\ker \partial_n$. Definition: The homotopy groups of G are the quotients $\pi_n(G) = \ker \partial_n / \operatorname{im} \partial_{n+1}$. Justification: $\forall\ n \geq 0$, $\pi_n(G) \approx \pi_n(|G|, e)$. Since a homomorphism $f : G \rightarrow K$ of simplicial groups induces a morphism $Nf : NG \rightarrow NK$ of chain complexes, thus a homomorphism $\pi_*(f) : \pi_*(G) \rightarrow \pi_*(K)$ in homotopy, it follows that f is a weak homotopy equivalence iff $\pi_*(f)$ is bijective.

[Note: A short exact sequence $1 \rightarrow G' \rightarrow G \rightarrow G'' \rightarrow 1$ of simplicial groups gives rise to a short exact sequence $1 \rightarrow NG' \rightarrow NG \rightarrow NG'' \rightarrow 1$ of chain complexes and a long exact sequence $\cdots \rightarrow \pi_{n+1}(G'') \rightarrow \pi_n(G') \rightarrow \pi_n(G) \rightarrow \pi_n(G'') \rightarrow \pi_{n-1}(G') \rightarrow \cdots$ of homotopy groups.]

EXAMPLE (Simplex Categories) Let X be a simplicial set—then X is a cofunctor $\mathbf{\Delta} \rightarrow \mathbf{SET}$, thus one can form the Grothendieck construction $\mathrm{gro}_{\mathbf{\Delta}} X$ on X. So: The objects of $\mathrm{gro}_{\mathbf{\Delta}} X$ are the $([n], x)$ $(x \in X_n)$ and the morphisms $([n], x) \rightarrow ([m], y)$ are the $\alpha : [n] \rightarrow [m]$ such that $(X\alpha)y = x$. One calls $\mathrm{gro}_{\mathbf{\Delta}} X$ the simplex category of X. It is isomorphic to the comma category $|Y_{\mathbf{\Delta}}, K_X|$:

$$\begin{tikzcd}
\Delta[n] \arrow[rr] \arrow[dr] & & \Delta[m] \arrow[dl] \\
& X &
\end{tikzcd}$$

. There is a natural weak homotopy equivalence $\mathrm{ner}(\mathrm{gro}_{\mathbf{\Delta}} X) \rightarrow X$, viz. the rule $\mathrm{ner}_p(\mathrm{gro}_{\mathbf{\Delta}} X) \rightarrow X_p$ that sends $([n_0], x_0) \xrightarrow{\alpha_0} \cdots \xrightarrow{\alpha_{p-1}} ([n_p], x_p)$ to $(X\alpha)x_p$, where $\alpha : [p] \rightarrow [n_p]$ is defined by $\alpha(i) = \alpha_{p-1} \circ \cdots \circ \alpha_i(n_i)$ $(0 \leq i \leq p)$ $(\alpha(p) = n_p)$.

[First check the assertion when $X = \Delta[n]$.]

A simplicial map $f : X \to Y$ is said to be a cofibration if its geometric realization $|f| : |X| \to |Y|$ is a cofibration. Example: $\forall\, X$, the arrows $\begin{cases} i_0 : X \to IX \\ i_1 : X \to IX \end{cases}$ are cofibrations and weak homotopy equivalences.

LEMMA The cofibrations in **SISET** are the injective simplicial maps.

Example: Let X be a simplicial set—then the arrow of adjunction $X \to \sin|X|$ is a cofibration and a weak homotopy equivalence (cf. p. 13–15).

EXAMPLE Let X be a simplicial set—then $e_X : X \to \mathrm{Ex}\,X$ is a cofibration, as is $e_X^\infty : X \to \mathrm{Ex}^\infty X$ and both are weak homotopy equivalences (cf. p. 13–12).

PROPOSITION 7 Let $p : X \to B$ be a simplicial map—then p has the RLP w.r.t. the inclusions $\dot{\Delta}[n] \to \Delta[n]$ $(n \geq 0)$ iff p has the RLP w.r.t. all cofibrations.

[Let $i : A \to Y$ be an injective simplicial map. To construct a filler for

$$\begin{tikzcd} A \arrow[r, "u"] \arrow[d, "i"'] & X \arrow[d, "p"] \\ Y \arrow[r, "v"'] & B \end{tikzcd},$$

take i to be an inclusion and call $(Y, A)_n^\#$ the subset of $Y_n^\#$ consisting of those elements which do not belong to A—then $\forall\, n$, there is a pushout square

$$\begin{tikzcd} (Y,A)_n^\# \cdot \dot{\Delta}[n] \arrow[r] \arrow[d] & Y^{(n-1)} \cup A \arrow[d] \\ (Y,A)_n^\# \cdot \Delta[n] \arrow[r] & Y^{(n)} \cup A \end{tikzcd},$$

so one can construct the arrow $Y \to X$ by induction.]

Given $n \geq 1$, the kth-horn $\Lambda[k,n]$ of $\Delta[n]$ $(0 \leq k \leq n)$ is the simplicial subset of $\Delta[n]$ defined by the condition that $\Lambda[k,n]_m$ is the set of $\alpha : [m] \to [n]$ whose image does not contain the set $[n] - \{k\}$. So: $|\Lambda[k,n]| = \Lambda^{k,n}$ is the subset of $|\Delta[n]| = \Delta^n$ consisting of those $(t_0, \ldots, t_n) : t_i = 0$ $(\exists\, i \neq k)$, thus $\Lambda^{k,n}$ is a strong deformation retract of Δ^n.

Example: Let $\begin{cases} X \\ Y \end{cases}$ be topological spaces, $f : X \to Y$ a continuous function—then f is a Serre fibration iff f has the RLP w.r.t. the inclusions $\Lambda^{k,n} \to \Delta^n$ $(0 \leq k \leq n, n \geq 1)$.

The representation of $\dot{\Delta}[n]$ as a coequalizer can be modified to exhibit $\Lambda[k,n]$ as a coequalizer (in the notation of p. 0–18, replace $\coprod\limits_{0 \leq i \leq n} \Delta[n-1]_i$ by $\coprod\limits_{\substack{0 \leq i \leq n \\ i \neq k}} \Delta[n-1]_i$). A corollary is that for every simplicial set X, $\mathrm{Nat}(\Lambda[k,n], X)$ is in a one-to-one correspondence with the set of finite sequences $(x_0, \ldots, \widehat{x}_k, \ldots, x_n)$ of elements of X_{n-1} such that $d_i x_j = d_{j-1} x_i$ $(i < j$ & $i, j \neq k)$.

A retract invariant, composition closed class of injective simplicial maps is said to be replete if it contains the isomorphisms and is stable under the formation of coproducts, pushouts, and sequential colimits. The repletion of a set S_0 of injective simplicial maps is $\cap M$, M replete with $S_0 \subset M$.

Specialize to $S_0 = \{\Lambda[k,n] \rightarrow \Delta[n] \ (0 \leq k \leq n, n \geq 1)\}$—then the repletion of S_0 is the class of anodyne extensions. Examples: (1) The injections $\Delta[\delta_i] : \Delta[n-1] \rightarrow \Delta[n]$ are anodyne extensions; (2) The inclusions $\Delta[m] \times \Lambda[k,n] \cup \dot{\Delta}[m] \times \Delta[n] \rightarrow \Delta[m] \times \Delta[n]$ are anodyne extensions.

PROPOSITION 8 Let $f : X \rightarrow Y$ be an anodyne extension—then $|f|(|X|)$ is a strong deformation retract of $|Y|$.

[The class of injective simplicial maps with this property is replete (cf. §3, Proposition 3 and p. 3–20) and contains S_0.]

Application: Every anodyne extension is a weak homotopy equivalence.

PROPOSITION 9 Let $\begin{cases} A \\ B \end{cases}$ be a simplicial subset of $\begin{cases} X \\ Y \end{cases}$. Suppose that the inclusion $B \rightarrow Y$ is an anodyne extension—then the inclusion $X \times B \cup A \times Y \rightarrow X \times Y$ is an anodyne extension.

[The class of injective simplicial maps $B' \rightarrow Y'$ for which the arrow $X \times B' \underset{A \times B'}{\sqcup} A \times Y' \rightarrow X \times Y'$ is an anodyne extension is replete. On the other hand, an induction shows that the inclusions $X \times \Lambda[k,n] \cup A \times \Delta[n] \rightarrow X \times \Delta[n]$ are anodyne.]

EXAMPLE The inclusion $\operatorname{Sd} \Lambda[k,n] \rightarrow \operatorname{Sd} \Delta[n]$ is an anodyne extension.

[Note: In general, Sd preserves anodyne extensions (cf. p. 13–34).]

FACT The class of homotopy classes of anodyne extensions admits a calculus of left fractions.

[The point is to show that if $f, g : X \rightarrow Y$ are simplicial maps and if $s : X' \rightarrow X$ is an anodyne extension with $f \circ s \simeq g \circ s$, then $\exists$ an anodyne extension $t : Y \rightarrow Y'$ with $t \circ f \simeq t \circ g$.]

Let $p : X \rightarrow B$ be a simplicial map—then p is said to be a Kan fibration if it has the RLP w.r.t. the inclusions $\Lambda[k,n] \rightarrow \Delta[n]$ $(0 \leq k \leq n, n \geq 1)$.

[Note: Let $p : X \rightarrow B$ be a Kan fibration—then for any component A of X, $p(A)$ is a component of B and $A \rightarrow p(A)$ is a Kan fibration. Therefore $p(X)$ is a union of components of B. So, if B is connected and X is nonempty, then p is surjective.]

Example: Let $\begin{cases} X \\ Y \end{cases}$ be topological spaces, $f : X \rightarrow Y$ a continuous function—then f is a Serre fibration iff $\sin f : \sin X \rightarrow \sin Y$ is a Kan fibration.

In "parameters", the condition that p be a Kan fibration is equivalent to requiring that if $(x_0, \ldots, \widehat{x}_k, \ldots, x_n)$ is a finite sequence of elements of X_{n-1} such that $d_i x_j = d_{j-1} x_i$ $(i < j \ \& \ i, j \neq k)$ and $p(x_i) = d_i b$ $(b \in B_n)$, then $\exists\, x \in X_n : d_i x = x_i$ $(i \neq k)$ with $p(x) = b$.

PROPOSITION 10 Let $p : X \rightarrow B$ be a simplicial map—then p is a Kan fibration iff it has the RLP w.r.t. every anodyne extension.

[The class of injective simplicial maps that have the LLP w.r.t. p is replete.]

Application: Let A be a simplicial subset of Y. Suppose that $p : X \rightarrow B$ is a Kan fibration—then every commutative diagram

$$\begin{tikzcd} i_0 Y \cup IA \arrow[r, "F"] \arrow[d] & X \arrow[d, "p"] \\ IY \arrow[r, "h"'] & B \end{tikzcd}$$

has a filler $H : IY \rightarrow X$ (cf. §4, Proposition 12).

[The vertex inclusion $e_0 : \Delta[0] \rightarrow \Delta[1]$ is anodyne.]

FACT Let $p : X \rightarrow B$ be a Kan fibration—then $\mathrm{Ex}\, p : \mathrm{Ex}\, X \rightarrow \mathrm{Ex}\, B$ is a Kan fibration.

A simplicial set X is said to be fibrant if the arrow $X \rightarrow *$ is a Kan fibration. The fibrant objects are therefore those X such that every simplicial map $f : \Lambda[k, n] \rightarrow X$ can be extended to a simplicial map $F : \Delta[n] \rightarrow X$ $(0 \leq k \leq n, n \geq 1)$.

[Note: The components of a fibrant X are fibrant.]

Example: Let X be a topological space—then $\sin X$ is fibrant.

LEMMA Suppose that X is fibrant. Assume: $\exists\, n_0 \geq 1$ such that $\#(X^{\#}_{n_0}) \geq 1$—then $\forall\, n \geq n_0$, $\#(X^{\#}_{n}) \geq 1$.

[Fix $x \in X^{\#}_{n_0}$ and choose $y \in X_{n_0+1}$ such that $d_0 y = x$, $d_1 y = s_0 d_0 x$. Claim: $y \in X^{\#}_{n_0+1}$. Suppose not, so $y = s_i z$ $(\exists\, i)$. Case 1: $i \geq 1 : x = d_0 y = d_0 s_i z = s_{i-1} d_0 z$, an impossibility. Case 2: $i = 0 : x = d_0 y = d_0 s_0 z = z \Rightarrow x = z \Rightarrow y = s_0 x \Rightarrow d_1 y = d_1 s_0 x \Rightarrow x = d_1 s_0 x = s_0 d_0 x$, an impossibility.]

Application: $\Delta[n]$ $(n \geq 1)$ is not fibrant.

Remark: Let Y be a simplicial set—then the arrow $Y \rightarrow *$ is a homotopy fibration.

Proof: Take any commutative diagram

$$\begin{tikzcd} X' \times Y \arrow[r, "\Phi"] \arrow[d] & X \times Y \arrow[r] \arrow[d] & Y \arrow[d] \\ X' \arrow[r, "\phi"'] & X \arrow[r] & * \end{tikzcd}$$

, where ϕ is a

weak homotopy equivalence, and apply $|?|$ to get a commutative diagram

$$\begin{tikzcd}
{|X'| \times_k |Y|} \arrow[r, "|\Phi|"] \arrow[d] & {} \\
{|X'|} \arrow[r, "|\phi|"'] & {}
\end{tikzcd}$$

$$\begin{tikzcd}
{|X| \times_k |Y|} \arrow[r] \arrow[d] & {|Y|} \arrow[d] \\
{|X|} \arrow[r] & *
\end{tikzcd}$$

in **CGH** (cf. Proposition 1). Since the projection $|X| \times_k |Y| \rightarrow |X|$ is a **CG** fibration and $|\phi|$ is a homotopy equivalence, $|\Phi|$ is a homotopy equivalence (cf. p. 4–24), i.e., Φ is a weak homotopy equivalence.

[Note: See p. 13–32 for the model category structure on **SISET**.]

EXAMPLE The underlying simplicial set of a simplicial group G is fibrant.

[Let $(x_0, \ldots, \widehat{x}_k, \ldots, x_n)$ be a finite sequence of elements of G_{n-1} such that $d_i x_j = d_{j-1} x_i$ $(i < j$ & $i, j \neq k)$. Claim: $\exists$ elements $g_{-1}, g_0, \ldots \in G_n$ such that $d_i g_r = x_i$ $(i \leq r, i \neq k)$. Thus put $g_{-1} = e \in G_n$ and assume that $g_{r-1} \in G_n$ has been constructed. Case 1: $r = k$. Take $g_r = g_{r-1}$. Case 2: $r \neq k$. Take $g_r = g_{r-1}(s_r h_r)^{-1}$, where $h_r = x_r^{-1}(d_r g_{r-1})$.]

[Note: A homomorphism $f : G \rightarrow K$ of simplicial groups is a Kan fibration iff $N_n f : N_n G \rightarrow N_n K$ is surjective $\forall$ $n > 0$. Therefore a surjective homomorphism of simplicial groups is a Kan fibration.]

EXAMPLE Let $\mathbf{C}$ be a small category—then $\operatorname{ner} C$ is fibrant iff $\mathbf{C}$ is a groupoid.

[Note: It is a corollary that $\Delta[n]$ $(n \geq 1)$ is not fibrant.]

LEMMA Put $\mathrm{d}_{k,n} = \mathrm{d}_{\Lambda[k,n]}$ $(0 \leq k \leq n, n \geq 1)$—then there is a simplicial map $\mathrm{D}_{k,n} : \mathrm{Sd}^2 \Delta[n] \rightarrow \mathrm{Sd}\, \Lambda[k,n]$ such that $\mathrm{D}_{k,n} | \mathrm{Sd}^2 \Lambda[k,n] = \mathrm{Sd}\, \mathrm{d}_{k,n}$.

FACT For any simplicial set X, $\mathrm{Ex}^\infty X$ is fibrant.

[Suppose given a simplicial map $f : \Lambda[k,n] \rightarrow \mathrm{Ex}^\infty X$. Choose an r such that f factors through $\mathrm{Ex}^r X$ and let g be the composite $\Lambda[k,n] \rightarrow \mathrm{Ex}^r X \rightarrow \mathrm{Ex}\, \mathrm{Ex}^r X$—then, under $\mathrm{Nat}(\Lambda[k,n], \mathrm{Ex}\, \mathrm{Ex}^r X) \approx \mathrm{Nat}(\mathrm{Sd}\, \Lambda[k,n], \mathrm{Ex}^r X)$, g corresponds to $h : \mathrm{Sd}\, \Lambda[k,n] \rightarrow \mathrm{Ex}^r X$ and an extension $F : \Delta[n] \rightarrow \mathrm{Ex}^\infty X$ of f can be constructed by working with the "double adjoint" of $h \circ \mathrm{D}_{k,n}$ (it being a simplicial map from $\Delta[n]$ to $\mathrm{Ex}^2\, \mathrm{Ex}^r X$).]

The class of Kan fibrations is pullback stable. In particular: The fibers of a Kan fibration are fibrant objects.

PROPOSITION 11 Let $p : X \rightarrow B$ be a Kan fibration—then B fibrant $\Rightarrow$ X fibrant and X fibrant $+$ p surjective $\Rightarrow$ B fibrant.

PROPOSITION 12 Suppose that $L \to K$ is an inclusion of simplicial sets and $X \to B$ is a Kan fibration—then the arrow $\operatorname{map}(K,X) \to \operatorname{map}(L,X) \times_{\operatorname{map}(L,B)} \operatorname{map}(K,B)$ is a Kan fibration.

[Pass from

$$\begin{tikzcd}
\Lambda[k,n] \arrow[r] \arrow[d] & \operatorname{map}(K,X) \arrow[d] \\
\Delta[n] \arrow[r] & \operatorname{map}(L,X) \times_{\operatorname{map}(L,B)} \operatorname{map}(K,B)
\end{tikzcd},$$

to

$$\begin{tikzcd}
\Lambda[k,n] \times K \cup \Delta[n] \times L \arrow[r] \arrow[d, "i"'] & X \arrow[d] \\
\Delta[n] \times K \arrow[r] & B
\end{tikzcd}$$

and note that i is anodyne (cf. Proposition 9).]

[Note: Compare this result with its topological analog on p. 12–17.]

Application: Let $p : X \to B$ be a Kan fibration—then for any simplicial set Y, the postcomposition arrow $p_* : \operatorname{map}(Y,X) \to \operatorname{map}(Y,B)$ is a Kan fibration (cf. §4, Proposition 5).

[Note: Take $B = *$ to see that X fibrant $\Rightarrow \operatorname{map}(Y,X)$ fibrant $\forall\, Y$.]

Application: Let $i : A \to X$ be a cofibration—then for any fibrant Y, the precomposition arrow $i^* : \operatorname{map}(X,Y) \to \operatorname{map}(A,Y)$ is a Kan fibration (cf. §4, Proposition 6).

FACT Let $L \to K$ be an anodyne extension—then $\forall$ fibrant Z, the arrow $[K,Z]_0 \to [L,Z]_0$ is bijective.

[Since Z is fibrant, the arrow $[K,Z]_0 \to [L,Z]_0$ is surjective, hence bijective (cf. p. 13–19).]

Application: Let $L \to K$ be an anodyne extension—then $\forall$ fibrant Z, the arrow $\operatorname{map}(K,Z) \to \operatorname{map}(L,Z)$ is a homotopy equivalence.

[For any simplicial set X, the inclusion $X \times L \to X \times K$ is anodyne (cf. Proposition 9). But $[X, \operatorname{map}(K,Z)]_0 \to [X, \operatorname{map}(L,Z)]_0$ is bijective iff $[X \times K, Z]_0 \to [X \times L, Z]_0$ is bijective.]

PROPOSITION 13 Let $p : X \to B$ be a Kan fibration. Suppose that $b', b'' \in B_0$ are in the same component of B—then the fibers $X_{b'}, X_{b''}$ have the same homotopy type.

[Note: Compare this result with its topological analog on p. 4–13.]

LEMMA For any fibrant X, simplicial homotopy of simplicial maps $\Delta[0] \to X$ is an equivalence relation.

[The relation is reflexive: $\forall\ x \in X_0$, $d_1 s_0 x = x = d_0 s_0 x$.

The relation is transitive. For suppose that $x \underset{s}{\simeq} y$ & $y \underset{s}{\simeq} z$ $(x, y, z \in X_0)$, say $\begin{cases} d_1 u = x \\ d_0 u = y \end{cases}$ $(u \in X_1)$ & $\begin{cases} d_1 v = y \\ d_0 v = z \end{cases}$ $(v \in X_1)$. The pair (v, u) determines a simplicial map $\Lambda[1,2] \rightarrow X$. Extend it to a simplicial map $F : \Delta[2] \rightarrow X$ and put $w = d_1 F \in X_1 : d_1 w = d_1 d_1 F = d_1 d_2 F = x$ & $d_0 w = d_0 d_1 F = d_0 d_0 F = z$ $(F \leftrightarrow F(\mathrm{id}_{[2]}))$.

The relation is symmetric. For suppose that $x \underset{s}{\simeq} y$ $(x, y \in X_0)$, say $\begin{cases} d_1 u = x \\ d_0 u = y \end{cases}$ $(u \in X_1)$. The pair $(s_0 x, u)$ determines a simplicial map $\Lambda[0,2] \rightarrow X$. Extend it to a simplicial map $G : \Delta[2] \rightarrow X$ and put $v = d_0 G : d_1 v = d_1 d_0 G = d_0 d_2 G = y$ & $d_0 v = d_0 d_0 G = d_0 d_1 G = x$ $(G \leftrightarrow G(\mathrm{id}_{[2]}))$.]

[Note: It is a corollary that $\Delta[n]$ $(n \geq 1)$ is not fibrant.]

Application: For any fibrant X and any Y, simplicial homotopy of simplicial maps $Y \rightarrow X$ is an equivalence relation, so homotopy=simplicial homotopy in this situation.

[In fact, X fibrant $\Rightarrow$ $\mathrm{map}(Y, X)$ fibrant $\forall\ Y$ (cf. supra).]

Denote by ι_n the inclusion $\dot{\Delta}[n] \rightarrow \Delta[n]$. Given a Kan fibration $p : X \rightarrow B$, put $\mathrm{map}(\iota_n, p) = \mathrm{map}(\dot{\Delta}[n], X) \times_{\mathrm{map}(\dot{\Delta}[n], B)} \mathrm{map}(\Delta[n], B)$ and let ι_n/p be the arrow map $(\Delta[n], X) \rightarrow \mathrm{map}(\iota_n, p)$—then ι_n/p is a Kan fibration (cf. Proposition 12). Definition: Elements $x', x'' \in X_n$ are said to be p-connected $(x' \underset{p}{\simeq} x'')$ if $\Delta_{x'}, \Delta_{x''} \in \mathrm{map}(\Delta[n], X)_0$ belong to the same component of the same fiber of ι_n/p. Since an element of $\mathrm{map}(\iota_n, p)_0$ is a pair (f, F), where $f : \dot{\Delta}[n] \rightarrow X$, $F : \Delta[n] \rightarrow B$ and $p \circ f = F \circ \iota_n$, an element $\Delta_x \in \mathrm{map}(\Delta[n], X)_0$ lies on the fiber $\mathrm{map}(\Delta[n], X)_{(f,F)}$ of ι_n/p over (f, F) if $\begin{cases} p \circ \Delta_x = F \\ \Delta_x \circ \iota_n = f \end{cases}$. Accordingly, elements $x', x'' \in X_n$ with $\begin{cases} p \circ \Delta_{x'} \\ p \circ \Delta_{x''} \end{cases} = F$ & $\begin{cases} \Delta_{x'} \circ \iota_n \\ \Delta_{x''} \circ \iota_n \end{cases} = f$ are p-connected if $\exists\ H : I\Delta[n] \rightarrow X : H \circ i_0 = \Delta_{x'}$, $H \circ i_1 = \Delta_{x''}$, $p \circ H = F \circ \mathrm{pr}$, $H | I\dot{\Delta}[n] = f \circ \mathrm{pr}$ or still, if $\exists\ H', H'' : I\Delta[n] \rightarrow X : \begin{cases} H' \circ i_0 = \Delta_{x'} \\ H'' \circ i_0 = \Delta_{x''} \end{cases}$ & $H' \circ i_1 = H'' \circ i_1$ (or $\begin{cases} H' \circ i_1 = \Delta_{x'} \\ H'' \circ i_1 = \Delta_{x''} \end{cases}$ & $H' \circ i_0 = H'' \circ i_0$), $p \circ H' = p \circ H''$, $H' | I\dot{\Delta}[n] = H'' | I\dot{\Delta}[n]$.

[Note: The relation $\underset{p}{\simeq}$ is an equivalence relation on X_n.]

LEMMA Let X be a simplicial set. Suppose that $x', x'' \in X_n$ are degenerate—then $d_i x' = d_i x''$ $(0 \leq i \leq n) \Rightarrow x' = x''$.

[Write $x' = s_k y'$, $x'' = s_l y''$. Case 1: $k = l$. Here, $y' = d_k x' = d_k x'' = y'' \Rightarrow x' = x''$. Case 2: $k \neq l$, say $k < l$. (1) $y' = d_k x' = d_k x'' = d_k s_l y'' = s_{l-1} d_k y''$; (2) $x' = s_k y' = s_k s_{l-1} d_k y'' = s_l s_k d_k y''$; (3) $y'' = d_l x'' = d_l x' = d_l s_l s_k d_k y'' = s_k d_k y''$; (4) $x' = s_l y'' = x''$.]

Application: Given a Kan fibration $p : X \to B$, degenerate elements $x', x'' \in X_n$ are p-connected iff they are equal.

A Kan fibration $p : X \to B$ is said to be minimal if $\forall\, n$, $\forall\, x', x'' \in X_n : x' \underset{p}{\simeq} x'' \Rightarrow x' = x''$.

[Note: A fibrant X is minimal when $X \to *$ is minimal.]

FACT Suppose that X is fibrant—then X is minimal iff $\forall\, n$, $\forall\, x', x'' \in X_n : d_i x' = d_i x''$ $(\forall\, i \neq j) \Rightarrow d_j x' = d_j x''$ $(0 \leq i, j \leq n)$.

EXAMPLE Let G be a simplicial group—then G is minimal iff the chain complex (NG, ∂) is minimal, i.e., iff $\forall\, n, \partial_n : N_n G \to N_{n-1} G$ is the zero homomorphism.

The class of minimal Kan fibrations is pullback stable. In particular: The fibers of a minimal Kan fibration are minimal fibrant objects.

PROPOSITION 14 A minimal Kan fibration $p : X \to B$ is locally trivial.

[The claim is that $\forall\, n$ & $\forall\, b \in B_n$, X_b is trivial over $\Delta[n]$. Therefore it will be enough to prove that every minimal Kan fibration $p : X \to \Delta[n]$ is trivial. To this end, let $C_0 : \Delta[n] \to \Delta[n]$ be the projection onto the 0th vertex and choose a simplicial homotopy $H : I\Delta[n] \to \Delta[n]$ between C_0 and $\text{id}_{\Delta[n]}$ (cf. p. 13–16). Call A the fiber of p over the 0th vertex—then there is a retraction $r : X \to A$ and a simplicial homotopy $\overline{H} : IX \to X$ between $X \xrightarrow{r} A \to X$ and id_X with $p \circ \overline{H} = H \circ (p \times \text{id}_{\Delta[1]})$. Define a simplicial map $f : X \to \Delta[n] \times A$ over $\Delta[n]$ by $f(x) = (p(x), r(x))$. To establish that f is an isomorphism, we shall proceed by induction on k, taking $X_{-1} = \emptyset$ and assuming $f|X_l$ is bijective $(l < k, k \geq 0)$.

Injectivity: Suppose that $f(x') = f(x'')$, where $x', x'' \in X_k$. Put $H'(\alpha, t) = \overline{H}((X\alpha)x', t)$, $H''(\alpha, t) = \overline{H}((X\alpha)x'', t)$ to get simplicial homotopies $H', H'' : I\Delta[k] \to X$ such that $\begin{cases} H' \circ i_1 = \Delta_{x'} \\ H'' \circ i_1 = \Delta_{x''} \end{cases}$ & $H' \circ i_0 = H'' \circ i_0$, $p \circ H' = p \circ H''$, $H'|I\dot{\Delta}[k] = H''|I\dot{\Delta}[k]$, thus $x' \underset{p}{\simeq} x''$, so minimality forces $x' = x''$.

Surjectivity: Let $(\alpha_0, a_0) \in (\Delta[n] \times A)_k$. The induction hypothesis, coupled with the injectivity of f, ensures the existence of a simplicial map $g : \dot{\Delta}[k] \to X$ such that $\forall\, \alpha \in \dot{\Delta}[k]$, $f \circ g(\alpha) = (\alpha_0 \circ \alpha, (X\alpha)a_0)$. In addition, one can find a simplicial homotopy $G : I\Delta[k] \to X$ satisfying $G \circ i_0 = \Delta_{a_0}$, $G|I\dot{\Delta}[k] = \overline{H} \circ (g \times \text{id}_{\Delta[1]})$, $p \circ G(\alpha, t) = H(\alpha_0 \circ \alpha, t)$. Write $\overline{a}_0 = r(x_k)$ $(x_k = G(\text{id}_{[k]}, 1))$ and set $\overline{G} = \overline{H} \circ (G \circ i_1 \times \text{id}_{\Delta[1]})$—then $\begin{cases} G \circ i_0 = \Delta_{a_0} \\ \overline{G} \circ i_0 = \Delta_{\overline{a}_0} \end{cases}$

& $G \circ i_1 = \overline{G} \circ i_1$, $p \circ G = p \circ \overline{G}$, $G|I\dot{\Delta}[k] = \overline{G}|I\dot{\Delta}[k]$. Therefore $a_0 \underset{p}{\simeq} \overline{a}_0 \Rightarrow a_0 = \overline{a}_0 \Rightarrow f(x_k) = (\alpha_0, a_0)$.]

Application: The geometric realization of a minimal Kan fibration is a Serre fibration (cf. p. 13–5).

Let $p : X \to B$ be a Kan fibration; let A be a simplicial subset of X, $i : A \to X$ the inclusion.

(DR) A is said to be a deformation retract of X over B if there is a simplicial map $r : X \to A$ over B and a simplicial homotopy $H : IX \to X$ over B such that $r \circ i = \mathrm{id}_A$ and $H \circ i_0 = i \circ r$, $H \circ i_1 = \mathrm{id}_X$.

(SDR) A is said to be a strong deformation retract of X over B if there is a simplicial map $r : X \to A$ over B and a simplicial homotopy $H : IX \to X$ over B such that $r \circ i = \mathrm{id}_A$ and $H \circ i_0 = i \circ r$, $H(a,t) = a$ $(a \in A)$, $H \circ i_1 = \mathrm{id}_X$.

[Note: Taking $B = *$ leads to the corresponding absolute notions for fibrant objects.]

If $p : X \to B$ is Kan and $A \subset X$ is a retract of X over B, then the restriction $p_A = p|A$ is Kan.

FACT Let $p : X \to B$ be a Kan fibration. Suppose that $A \subset X$ is a deformation retract of X over B—then p has the RLP w.r.t. every cofibration that has the LLP w.r.t. p_A.

PROPOSITION 15 Let $p : X \to B$ be a Kan fibration—then there is a simplicial subset $A \subset X$ which is a strong deformation retract of X over B such that p_A is a minimal Kan fibration.

[Let E be a set of representatives for the equivalence classes per $\underset{p}{\simeq}$ containing the degenerate elements of X (cf. p. 13–24). Choose a simplicial subset $A \subset X$ maximal with respect to $A \subset E$: p_A will be minimal if it is Kan. Consider the set $\mathcal{Y}$ of all pairs (Y, G), where $A \subset Y \subset X$ and $G : IY \to X$ is a simplicial homotopy over B such that $G(i_0(Y)) \subset A$, $G(a,t) = a$ $(a \in A)$, $G \circ i_1 = Y \to X$. Example: $(A, IA \xrightarrow{\mathrm{pr}} A \to X) \in \mathcal{Y}$. Order $\mathcal{Y}$ by stipulating that $(Y', G') \leq (Y'', G'')$ iff $Y' \subset Y''$ & $G''|IY' = G'$. Every chain in $\mathcal{Y}$ has an upper bound, so by Zorn, $\mathcal{Y}$ has a maximal element (Y_0, G_0). Claim: $Y_0 = X$. Supposing this is false, take $x \in X_n : x \notin Y_0$, with n minimal. Note that x is nondegenerate. Call Y_x the smallest simplicial subset of $X : Y_0 \subset Y_x$ & $x \in Y_x$. Since $\Delta_x|\dot{\Delta}[n]$ factors through Y_0, there is a pushout square

$$\begin{tikzcd}
\dot{\Delta}[n] \arrow[r] \arrow[d] & Y_0 \arrow[d] \\
\Delta[n] \arrow[r, "\Delta_x"'] & Y_x
\end{tikzcd}$$

. Fix a simplicial homotopy $H_x : I\Delta[n] \to X$ over

B such that $H_x \circ i_1 = \Delta_x$ and $H_x|I\dot{\Delta}[n] = G_0 \circ (\Delta_x|\dot{\Delta}[n] \times \mathrm{id}_{\Delta[1]})$. Put $x'' = H_x(\mathrm{id}_{[n]}, 0)$ and define $x' \in E$ via $x' \underset{p}{\simeq} x'' : d_i x'' \in A$ $(0 \leq i \leq n) \Rightarrow x' \in A$. Fix a simplicial homotopy $H : I\Delta[n] \to X$ rel $\dot{\Delta}[n]$ over B such that $H \circ i_0 = \Delta_{x'}, H \circ i_1 = \Delta_{x''}$. Determine a simplicial map $K : I^2\Delta[n] \to X$ satisfying $p \circ K(\alpha, t, T) = p((X\alpha)x)$, $\begin{cases} K(\alpha, t, 1) = H_x(\alpha, t) \\ K(\alpha, 0, T) = H(\alpha, T) \end{cases}$, $K(\alpha, t, T) = G_0((X\alpha)x, t)$ $(\alpha \in \dot{\Delta}[n])$, and $K(\alpha, 1, T) = (X\alpha)x$. Extend G_0 to a simplicial homotopy $G_x : IY_x \to X$ $(G_x(x, t) = K(\mathrm{id}_{[n]}, t, 0)) : (Y_x, G_x) \in \mathcal{Y}$. Contradiction.]

LEMMA Let $f, g : X \to Y$ be simplicial maps, where $f \underset{s}{\simeq} g$ & $\begin{cases} X \\ Y \end{cases}$ are fibrant. Assume: f is an isomorphism and Y is minimal—then g is an isomorphism.

Application: A simplicial homotopy equivalence between minimal fibrant objects is an isomorphism.

Consequently, if X is fibrant and if $\begin{cases} A' \\ A'' \end{cases}$ are deformation retracts of X that are minimal, then $\begin{cases} A' \\ A'' \end{cases}$ are isomorphic.

A simplicial map $p : X \to B$ which has the RLP w.r.t. the inclusions $\dot{\Delta}[n] \to \Delta[n]$ $(n \geq 0)$ is a Kan fibration (cf. Proposition 7). Moreover, p is a simplicial homotopy equivalence. Proof: p admits a section $s : B \to X$ and

$$\begin{tikzcd} X \times \dot{\Delta}[1] \arrow[r, "u"] \arrow[d] & X \arrow[d, "p"] \\ X \times \Delta[1] \arrow[r] & B \end{tikzcd}$$

admits a filler $H : X \times \Delta[1] \to X$. Here, $u(x, 0) = s(p(x))$, $u(x, 1) = x$.

PROPOSITION 16 Let $p : X \to B$ be a Kan fibration—then p can be written as the composite of a simplicial map which has the RLP w.r.t. the inclusions $\dot{\Delta}[n] \to \Delta[n]$ $(n \geq 0)$ and a minimal Kan fibration.

[Using the notation of Proposition 15, write $p = p_A \circ r$, $r : X \to A$ the retraction. Suppose given a commutative diagram

$$\begin{tikzcd} \dot{\Delta}[n] \arrow[r, "u"] \arrow[d] & X \arrow[d, "r"] \\ \Delta[n] \arrow[r, "v"'] & A \end{tikzcd}$$

. Since A is a strong deformation retract of X over B, there is a simplicial homotopy $H : IX \to X$ over B such that $H \circ i_0 = i \circ r$, $H(a, t) = a$ $(a \in A)$, $H \circ i_1 = \mathrm{id}_X$. Choose a simplicial homotopy $G : I\Delta[n] \to X$ subject to $G(\alpha, 0) = v(\alpha)$, $G|I\dot{\Delta}[n] = H \circ (u \times \mathrm{id}_{\Delta[1]})$, $p \circ G(\alpha, t) = p(v(\alpha))$. Let $\overline{G}(\alpha, t) = H((X\alpha)\overline{x}, t)$, where $\overline{x} = G(\mathrm{id}_{[n]}, 1)$. Put $\begin{cases} a' = v(\mathrm{id}_{[n]}) \\ a'' = r(\overline{x}) \end{cases}$, $\begin{cases} G' = r \circ G \\ G'' = r \circ \overline{G} \end{cases}$ —

then $\begin{cases} G'\circ i_0 = \Delta_{a'} \\ G''\circ i_0 = \Delta_{a''} \end{cases}$ & $G'\circ i_1 = G''\circ i_1$, $p_A \circ G' = p_A \circ G''$, $G'|I\dot{\Delta}[n] = G''|I\dot{\Delta}[n]$. So: $a' \underset{p_A}{\simeq} a'' \Rightarrow a' = a''$ (by minimality), hence $\Delta_{\overline{x}} : \Delta[n] \to X$ is our filler.]

LEMMA Suppose that $p : X \to B$ has the RLP w.r.t. the inclusions $\dot{\Delta}[n] \to \Delta[n]$ ($n \geq 0$)—then $|p| : |X| \to |B|$ is a **CG** fibration, thus is Serre (cf. p. 4–7).

[Consider a filler $X \times B \to X$ for

$$\begin{tikzcd} X \arrow[r, "\mathrm{id}_X"] \arrow[d] & X \arrow[d, "p"] \\ X \times B \arrow[r] & B \end{tikzcd}$$

, bearing in mind that $|X \times B| \approx |X| \times_k |B|$.]

PROPOSITION 17 The geometric realization of a Kan fibration is a Serre fibration.

[This follows from Proposition 16, the lemma, and the fact that the geometric realization of a minimal Kan fibration is a Serre fibration (cf. p. 13–25).]

[Note: The argument proves more: The geometric realization of a Kan fibration is a **CG** fibration.]

For instance, suppose that $p : X \to B$ is Kan and a weak homotopy equivalence. Let $B' \to B$ be a simplicial map and define X' by the pullback square

$$\begin{tikzcd} X' \arrow[r] \arrow[d, "p'"'] & X \arrow[d, "p"] \\ B' \arrow[r] & B \end{tikzcd}$$

—then p' is Kan and a weak homotopy equivalence.

Suppose that X is fibrant—then X is said to be simplicially contractible if the projection $X \to *$ is a simplicial homotopy equivalence.

EXAMPLE Let X be fibrant—then $\operatorname{Ex} X$ is fibrant (cf. p. 13–20) and is simplicially contractible if this is so of X.

[Recall that Ex preserves simplicial homotopy equivalences (cf. p. 13–16).]

PROPOSITION 18 A fibrant X is simplicially contractible iff every simplicial map $f : \dot{\Delta}[n] \to X$ can be extended to a simplicial map $F : \Delta[n] \to X$ ($n \geq 0$).

[The stated extension property implies that X is fibrant and simplicially contractible (cf. p. 13–26). To deal with the converse, fix a section $s : \Delta[0] \to X$ for $p : X \to \Delta[0]$ and a simplicial homotopy $H : IX \to X$ between $s \circ p$ and id_X. Given $f : \dot{\Delta}[n] \to X$, choose $G : I\Delta[n] \to X$ such that $G \circ i_0 = s \circ (\Delta[n] \to \Delta[0])$, $G|I\dot{\Delta}[n] = H \circ (f \times \mathrm{id}_{\Delta[1]})$ and put $F = G \circ i_1$—then $F|\dot{\Delta}[n] = f$.]

A simplicial pair is a pair (X, A), where X is a simplicial set and $A \subset X$ is a simplicial subset. Example: Fix $x_0 \in X_0$ and, in an abuse of notation, let x_0 be the simplicial subset of X generated by x_0 so that $(x_0)_n = \{s_{n-1} \cdots s_0 x_0\}$ $(n \geq 1)$—then (X, x_0) is a simplicial pair.

A pointed simplicial set is a simplicial pair (X, x_0). A pointed simplicial map is a base point preserving simplicial map $f : X \rightarrow Y$, i.e., a simplicial map $f : X \rightarrow Y$ for which the triangle

$$\begin{tikzcd} & \Delta[0] \arrow[dl, "\Delta_{x_0}"'] \arrow[dr, "\Delta_{y_0}"] & \\ X \arrow[rr, "f"'] & & Y \end{tikzcd}$$

commutes or, in brief, $f(x_0) = y_0$.

$\mathbf{SISET}_*$ is the category whose objects are the pointed simplicial sets and whose morphisms are the pointed simplicial maps. Thus $\mathbf{SISET}_* = [\mathbf{\Delta}^{\mathrm{OP}}, \mathbf{SET}_*]$ and the forgetful functor $\mathbf{SISET}_* \rightarrow \mathbf{SISET}$ has a left adjoint that sends a simplicial set X to the pointed simplicial set $X_+ = X \amalg *$.

[Note: The vertex inclusion $e_0 : \Delta[0] \rightarrow \Delta[1]$ defines the base point of $\Delta[1]$, hence of $\dot{\Delta}[1]$.]

$\Delta[0]$ is a zero object in $\mathbf{SISET}_*$ and $\mathbf{SISET}_*$ has the obvious products and coproducts. In addition, the pushout square

$$\begin{tikzcd} X \vee Y \arrow[r] \arrow[d] & \Delta[0] \arrow[d] \\ X \times Y \arrow[r] & X \# Y \end{tikzcd}$$

defines the smash product $X \# Y$. Therefore $\mathbf{SISET}_*$ is a closed category if $X \otimes Y = X \# Y$ and $e = \dot{\Delta}[1]$. Here, the internal hom functor sends (X, Y) to $\mathrm{map}_*(X, Y)$, the simplicial subset of $\mathrm{map}(X, Y)$ whose elements in degree n are the $f : X \times \Delta[n] \rightarrow Y$ with $f(x_0 \times \Delta[n]) = y_0$, i.e., the pointed simplicial maps $X \# \Delta[n]_+ \rightarrow Y$, the zero morphism 0_{XY} being the base point.

FACT Let $i : A \rightarrow X$ be a pointed cofibration—then for any pointed fibrant Y, the precomposition arrow $i^* : \mathrm{map}_*(X, Y) \rightarrow \mathrm{map}_*(A, Y)$ is a Kan fibration.

[Consider the pullback square

$$\begin{tikzcd} \mathrm{map}_*(X, Y) \arrow[r] \arrow[d] & \mathrm{map}(X, Y) \arrow[d] \\ \mathrm{map}_*(A, Y) \arrow[r] & \mathrm{map}(A, Y) \end{tikzcd}$$

, recalling that the arrow $\mathrm{map}(X, Y) \rightarrow \mathrm{map}(A, Y)$ is a Kan fibration (cf. p. 13–22).]

Application: Fix a pointed fibrant Y—then $\forall$ pointed X, $\mathrm{map}_*(X, Y)$ is fibrant.

Suppose that X is fibrant. Fix $x_0 \in X_0$—then the mapping space ΘX of the pointed simplicial set (X, x_0) is defined by the pullback square

$$\begin{tikzcd} \Theta X \arrow[r] \arrow[d] & \mathrm{map}(\Delta[1], X) \arrow[d, "e_0^*"] & \\ \Delta[0] \arrow[r, "\Delta_{x_0}"'] & \mathrm{map}(\Delta[0], X) & \approx X \end{tikzcd}$$

.

Since X is fibrant, e_0^* is a Kan fibration (cf. p. 13–22), hence ΘX is fibrant. Furthermore, the composite $\Theta X \to \operatorname{map}(\Delta[1],X) \xrightarrow{e_1^*} \operatorname{map}(\Delta[0],X) \approx X$ is a Kan fibration, call it p_1.

Proof: Consider the pullback square

$$\begin{tikzcd}
\Theta X \arrow[r] \arrow[d, "p_1"'] & \operatorname{map}(\Delta[1],X) \arrow[d, "i^*"] \\
X \arrow[r, bend right, "(\Delta_{x_0}, \mathrm{id}_X)"'] & \operatorname{map}(\dot\Delta[1],X) \approx X \times X
\end{tikzcd}$$

, noting that i^* is a Kan fibration (cf. p. 13–22).

ΘX can be identified with $\operatorname{map}_*(\Delta[1],X)$, thus is a pointed simplicial set. The fiber of $p_1 : \Theta X \to X$ over the base point is the loop space ΩX, i.e., $\operatorname{map}_*(\mathrm{S}[1],X)$, $\mathrm{S}[1] = \Delta[1]/\dot\Delta[1]$ the simplicial circle. Example: $\forall$ pointed topological space X, there are natural isomorphisms $\Theta(\sin X) \approx \sin \Theta X$, $\Omega(\sin X) \approx \sin \Omega X$.

LEMMA $e_0^* : \operatorname{map}(\Delta[1],X) \to \operatorname{map}(\Delta[0],X)$ has the RLP w.r.t. the inclusions $\dot\Delta[n] \to \Delta[n]$ $(n \geq 0)$.

[Convert

$$\begin{tikzcd}
\dot\Delta[n] \arrow[r] \arrow[d] & \operatorname{map}(\Delta[1],X) \arrow[d, "e_0^*"] \\
\Delta[n] \arrow[r] & \operatorname{map}(\Delta[0],X)
\end{tikzcd}$$

to

$$\begin{tikzcd}
\Delta[n] \times \Delta[0] \cup \dot\Delta[n] \times \Delta[1] \arrow[r] \arrow[d] & X \arrow[d] \\
\Delta[n] \times \Delta[1] \arrow[r] & *
\end{tikzcd}$$

, bearing in mind that $e_0 : \Delta[0] \to \Delta[1]$ is anodyne.]

PROPOSITION 19 Suppose that X is fibrant—then ΘX is simplicially contractible.

[In view of the lemma, this is a consequence of Proposition 18.]

LEMMA For every simplicial set X, $|\mathrm{e}_X| : |X| \to |\operatorname{Ex} X|$ is a homotopy equivalence (cf. p. 13–12).

[Show that $|\mathrm{e}_X|$ is bijective on π_0 and π_1 and, using an acyclic models argument, that $|\mathrm{e}_X|$ is a homology equivalence. To handle the higher homotopy groups, define ΘX by the pullback square

$$\begin{tikzcd}
\Theta X \arrow[r] \arrow[d] & \Theta \sin |X| \arrow[d, "p_1"] \\
X \arrow[r] & \sin |X|
\end{tikzcd}$$

. Since $X \to \sin|X|$ is a weak homotopy equivalence (cf. p. 13–15), the same is true of $\Theta X \to \Theta \sin|X|$ (cf. p. 13–33). But $\Theta \sin|X|$ is simplicially contractible (cf. Proposition 19), thus $\Theta X \to *$ is a weak homotopy equivalence and so $\operatorname{Ex} \Theta X \to *$ is a weak homotopy equivalence. In addition: $\Theta X \to X$ Kan $\Rightarrow$ $\operatorname{Ex} \Theta X \to \operatorname{Ex} X$ Kan (cf. p. 13–20). Compare the homotopy sequences of the associated Serre fibrations and use induction.]

SIMPLICIAL EXTENSION THEOREM Let (K,L) be a simplicial pair, $p : X \to B$ a Kan fibration. Suppose given a commutative diagram

$$\begin{tikzcd}
L \arrow[r, "g"] \arrow[d] & X \arrow[d, "p"] \\
K \arrow[r, "f"'] & B
\end{tikzcd}$$

—then $\forall\ \phi \in$

$C(|K|,|X|)$ such that $\phi|\,|L| = |g|$ and $|p|\circ\phi = |f|$ there is a simplicial map $F : K \to X$ with $F|L = g$, $p\circ F = f$, and $|F| \underset{|B|}{\simeq} \phi \operatorname{rel} |L|$.

[It will be enough to consider the case when $\left\{\begin{array}{l} K = \Delta[n] \\ L = \dot{\Delta}[n] \end{array}\right.$ $(n \geq 0)$. Identify $\left\{\begin{array}{l} X \\ B \end{array}\right.$ with its image in $\left\{\begin{array}{l} \sin|X| \\ \sin|B| \end{array}\right.$ under the arrow of adjunction $\left\{\begin{array}{l} X \to \sin|X| \\ B \to \sin|B| \end{array}\right.$, so that $\phi \in C(\Delta^n, |X|) = \sin_n|X|$, $d_i\phi \in X$ $(0 \leq i \leq n)$, $b_\phi = |p|\circ\phi \in B_n$. The assertion can thus be recast: $\exists\, x \in X_n$ such that $x \underset{\sin|p|}{\simeq} \phi$. This being clear if $n = 0$, take $n > 0$, write $b_\phi = (B\beta)b$, where β is an epimorphism and b is nondegenerate, and argue inductively on n and on the finite set of epimorphisms having domain $[n]$ (viz., $\beta' \leq \beta''$ iff $\forall\, i$, $\beta'(i) \leq \beta''(i)$).

[Note: p Kan $\Rightarrow$ $|p|$ Serre (cf. Proposition 17) $\Rightarrow$ $\sin|p|$ Kan.]

(I) $\beta : [n] \to [0]$. Here, $b \in B_0$ and $d_i\phi \in X_b$ $(0 \leq i \leq n)$. View X_b (which is fibrant) as a pointed simplicial set with base point ϕ_0 (the 0th element in the vertex set of ϕ (cf. p. 13–4)). Put $Y = X_b$, $W = \Theta Y$, $q = p_1$, and choose a finite sequence $(w_0, \ldots, w_{n-1}, \widehat{w}_n)$ of elements of W_{n-1} such that $d_iw_j = d_{j-1}w_i$ $(i < j$ & $i, j \neq n)$ with $q(w_i) = d_i\phi$ $(0 \leq i \leq n-1)$ (q maps W surjectively onto the component of Y containing the base point). Encode the data in the commutative diagram

$$\begin{tikzcd} \Lambda[n,n] \arrow[r] \arrow[d] & \sin|W| \arrow[d] \\ \Delta[n] \arrow[r, "\Delta_\phi"'] & \sin|Y| \end{tikzcd}$$

to produce a $\psi \in \sin_n|W| : \sin|q|(\psi) = \phi$. The induction hypothesis furnishes a $w_n \in W_{n-1} : w_n \underset{\sin|q|}{\simeq} d_n\psi$. On the other hand, W is simplicially contractible (cf. Proposition 19), so one can find a $w \in W_n : d_iw = w_i$ $(0 \leq i \leq n)$ (cf. Proposition 18). Claim: $x \underset{\sin|p|}{\simeq} \phi$, where $x = q(w)$. To see this, fix a simplicial homotopy $H : I\Delta[n-1] \to \sin|W| \operatorname{rel} \dot{\Delta}[n-1]$ over $\sin|Y|$ such that $H\circ i_0 = \Delta_{w_n}$, $H\circ i_1 = \Delta_{d_n\psi}$. Define a simplicial map $\overline{H} : \Delta[n]\times\dot{\Delta}[1] \cup \dot{\Delta}[n]\times\Delta[1] \to \sin|W|$ by the recipe $\overline{H}\circ i_0 = \Delta_w$, $\overline{H}\circ i_1 = \Delta_\psi$, $\overline{H}(d_i\mathrm{id}_{[n]}, t) = w_i$ $(0 \leq i \leq n-1)$, $\overline{H}(d_n\mathrm{id}_{[n]}, t) = H(\mathrm{id}_{[n-1]}, t)$. Using the fact that $(|?|, \sin)$ is an adjoint pair, $\overline{H}$ determines a continuous function $\overline{G} : i_0\Delta^n \cup i_1\Delta^n \cup I\dot{\Delta}^n \to |W|$ which can then be extended to a continuous function $\widetilde{G} : I\Delta^n \to |W|$ ($|W|$ is contractible). Pass back to get a simplicial homotopy $\widetilde{H} : I\Delta[n] \to \sin|W|$ extending $\overline{H}$. Consider the composite $\sin|q|\circ\widetilde{H}$ followed by the inclusion $\sin|Y| \to \sin|X|$.

(II) $\beta : [n] \to [m]$ $(m > 0)$. Let $k = \min\limits_{0\leq i\leq n} i : \beta(i) \neq \beta(i+1)$. Choose $\overline{x} \in X_n : d_i\overline{x} = d_i\phi$ $(0 \leq i \leq n-1)$ with $p(\overline{x}) = b_\phi$ and choose $\psi \in \sin_{n+1}|X| : d_k\psi = \overline{x}$, $d_{k+1}\psi = \phi$, $d_i\psi = d_is_k\phi$ $(0 \leq i \leq n,\ i \neq k,\ k+1)$ with $|p|\circ\psi = s_kb_\phi$—then $\exists\, \overline{y} \in X_n : \overline{y} \underset{\sin|p|}{\simeq} d_{n+1}\psi$ (induction). Choose $\overline{w} \in X_{n+1} : d_k\overline{w} = \overline{x}$, $d_{n+1}\overline{w} = \overline{y}$, $d_i\overline{w} = d_is_k\phi$ $(0 \leq i \leq n,\ i \neq k,\ k+1)$ with $p(\overline{w}) = s_kb_\phi$. Fix a simplicial homotopy $H : I\Delta[n] \to \sin|X| \operatorname{rel} \dot{\Delta}[n]$ over $\sin|B|$ such that $H \circ i_0 = \Delta_{\overline{y}}$, $H \circ i_1 = \Delta_{d_{n+1}\psi}$ and incorporate the choices into a simplicial homotopy $\overline{H} : I\Delta[n+1] \rightarrow \sin|X|$ satisfying $\overline{H} \circ i_0 = \Delta_{\overline{w}}$, $\overline{H} \circ i_1 = \Delta_{\psi}$, $\overline{H}(d_i \mathrm{id}_{[n+1]}, t) = d_i \overline{w}$ $(0 \leq i \leq n, i \neq k+1)$, $\overline{H}(d_{n+1}\mathrm{id}_{[n+1]}, t) = H(\mathrm{id}_{[n]}, t), |p| \circ \overline{H}(\mathrm{id}_{[n+1]}, t) = s_k b_{\phi}$. Put $x = d_{k+1}\overline{w}$ and examine $\overline{H} \circ (\Delta[\delta_{k+1}] \times \mathrm{id}_{\Delta[1]}) : I\Delta[n] \rightarrow \sin|X|$ to conclude that $x \underset{\sin|p|}{\simeq} \phi$.]

Specialized to $B = *$, one can say that if (K, L) is a simplicial pair and X is fibrant, then given a simplicial map $g : L \rightarrow X$ and a continuous extension $\phi : |K| \rightarrow |X|$ of $|g|$, there exists a simplicial extension $F : K \rightarrow X$ of g such that $|F| \simeq \phi \operatorname{rel} |L|$. Conversely, every simplicial set X with this property is fibrant. Proof: The geometric realization of a simplicial map $\Lambda[k, n] \rightarrow X$ can be extended to a continuous function $\Delta^n \rightarrow |X|$.

Example: Suppose that X is fibrant—then X is a strong deformation retract of $\sin|X|$.

[Apply the simplicial extension theorem to the commutative diagram

$$\begin{tikzcd}
X \arrow[r, "\mathrm{id}_X"] \arrow[d] & X \arrow[d] \\
\sin|X| \arrow[r] & *
\end{tikzcd}$$

, taking for $\phi \in C(|\sin|X||, |X|)$ the arrow of adjunction $|\sin|X|| \rightarrow |X|$.]

EXAMPLE Let $\begin{cases} X \\ Y \end{cases}$ be simplicial sets. Assume: Y is fibrant—then there is a weak homotopy equivalence $|\mathrm{map}(X, Y)| \rightarrow \mathrm{map}(|X|, |Y|)$.

[Since Y is fibrant, the arrow of adjunction $Y \rightarrow \sin|Y|$ is a simplicial homotopy equivalence, thus the arrow $\mathrm{map}(X, Y) \rightarrow \mathrm{map}(X, \sin|Y|)$ is a simplicial homotopy equivalence. But $\mathrm{map}(X, \sin|Y|) \approx \sin \mathrm{map}(|X|, |Y|)$ and the arrow of adjunction $|\sin \mathrm{map}(|X|, |Y|)| \rightarrow \mathrm{map}(|X|, |Y|)$ is a weak homotopy equivalence (Giever-Milnor theorem).]

PROPOSITION 20 Let $\begin{cases} X \\ Y \end{cases}$ be fibrant—then a simplicial map $f : X \rightarrow Y$ is a simplicial homotopy equivalence iff its geometric realization $|f| : |X| \rightarrow |Y|$ is a homotopy equivalence.

[In general, geometric realization takes simplicial homotopy equivalences to homotopy equivalences. The fibrancy of X & Y is used to go the other way. Thus fix a homotopy inverse $g : |Y| \rightarrow |X|$ for $|f|$ and let $r : \sin|X| \rightarrow X$ be a simplicial homotopy inverse for $X \rightarrow \sin|X|$ (cf. supra)—then the composite $Y \rightarrow \sin|Y| \xrightarrow{\sin g} \sin|X| \xrightarrow{r} X$ is a simplicial homotopy inverse for f.]

[Note: It is a corollary that a fibrant X is simplicially contractible iff $|X|$ is contractible.]

Application: Suppose that $\begin{cases} X \\ Y \end{cases}$ are topological spaces and $f : X \rightarrow Y$ is a continuous function—then f is a weak homotopy equivalence iff $\sin f : \sin X \to \sin Y$ is a simplicial homotopy equivalence.

[If f is a weak homotopy equivalence, then $|\sin f|$ is a weak homotopy equivalence (cf. p. 13–17) or still, a homotopy equivalence. But this means that $\sin f$ is a simplicial homotopy equivalence, $\left\{\begin{array}{l}\sin X\\ \sin Y\end{array}\right.$ being fibrant.]

A simplicial set X is said to be finite if $|X|$ is finite.

[Note: A finite simplicial set is a simplicial object in the category of finite sets (but not conversely).]

SIMPLICIAL APPROXIMATION THEOREM Let $\left\{\begin{array}{l}X\\ Y\end{array}\right.$ be simplicial sets with X finite. Fix $\phi \in C(|X|,|Y|)$—then $\exists\ n > 0$ and a simplicial map $f : \mathrm{Sd}^n X \to Y$ such that $|f| \simeq \phi \circ |\mathrm{d}_X^n|$.

[Since $\mathrm{Ex}^\infty Y$ is fibrant (cf. p. 13–21), it follows from the simplicial extension theorem that there exists a simplicial map $F : X \to \mathrm{Ex}^\infty Y$ such that $|F| \simeq |\mathrm{e}_Y^\infty| \circ \phi$. But X is finite, so F factors through $\mathrm{Ex}^n Y$ for some n.]

[Note: The natural transformations $\mathrm{d}^n : \mathrm{Sd}^n \to \mathrm{id}$ are defined inductively by $\mathrm{d}_X^0 = \mathrm{id}_X$, $\mathrm{d}_X^{n+1} = \mathrm{d}_X^n \circ \mathrm{d}_{\mathrm{Sd}^n X}$.]

PROPOSITION 21 Let $p : X \to B$ be a simplicial map—then p is a Kan fibration and a weak homotopy equivalence iff p has the RLP w.r.t. the inclusions $\dot{\Delta}[n] \to \Delta[n]$ $(n \geq 0)$.

[That the condition is sufficient has been noted on p. 13–26. As for the necessity, one can assume that p is minimal (cf. Proposition 16). To construct a filler $\Delta[n] \to X$ for

$$\begin{tikzcd}
\dot{\Delta}[n] \arrow[r] \arrow[d] & X \arrow[d, "p"] \\
\Delta[n] \arrow[r, "\Delta_b"'] & B
\end{tikzcd}$$

$(b \in B_n)$, it suffices to construct a filler $\Delta[n] \to X_b$ for

$$\begin{tikzcd}
\dot{\Delta}[n] \arrow[r] \arrow[d] & X_b \arrow[d] \\
\Delta[n] \arrow[r, equal] & \Delta[n]
\end{tikzcd}$$

. But the projection $X_b \to \Delta[n]$ is a weak homotopy equivalence (cf. p. 13–27) and X_b is trivial over $\Delta[n]$ (cf. Proposition 14), say $X_b \approx \Delta[n] \times T_b$, where T_b is fibrant. Therefore $|T_b|$ is contractible, hence T_b is simplicially contractible. Now quote Proposition 18.]

Recall that **CGH** in its singular structure is a proper model category (cf. p. 12–12).

FUNDAMENTAL THEOREM OF SIMPLICIAL HOMOTOPY THEORY **SISET** is a proper model category if weak equivalence=weak homotopy equivalence, cofibration=injective simplicial map, fibration=Kan fibration. Every object is cofibrant and the fibrant objects are the fibrant simplicial sets.

[Axioms MC–1, MC–2, and MC–3 are immediate.

Claim: Every simplicial map $f : X \to Y$ can be written as a composite $f_\omega \circ i_\omega$, where $i_\omega : X \to X_\omega$ is an anodyne extension and $f_\omega : X_\omega \to Y$ is a Kan fibration.

[In the small object argument, take $S_0 = \{\Lambda[k,n] \to \Delta[n] \ (0 \leq k \leq n, \ n \geq 1)\}$.]

Claim: Every simplicial map $f : X \to Y$ can be written as a composite $f_\omega \circ i_\omega$, where $i_\omega : X \to X_\omega$ is a cofibration and $f_\omega : X_\omega \to Y$ is both a weak homotopy equivalence and a Kan fibration.

[In the small object argument, take $S_0 = \{\dot{\Delta}[n] \to \Delta[n] \ (n \geq 0)\}$.]

Combining the claims gives MC–5. Turning to MC–4, consider a commutative diagram

$$\begin{tikzcd} A \arrow[r, "u"] \arrow[d, "i"'] & X \arrow[d, "p"] \\ Y \arrow[r, "v"'] & B \end{tikzcd}$$

, where i is a cofibration and p is a Kan fibration. If p is a weak homotopy equivalence, then the existence of a filler $w : Y \to X$ is implied by Proposition 7 and Proposition 21. On the other hand, if i is a weak homotopy equivalence, then by the first claim $i = q \circ j$, where $j : A \to Z$ is anodyne and $q : Z \to Y$ is a Kan fibration which is necessarily a weak homotopy equivalence, so $\exists \ f : Y \to Z$ such that $f \circ i = j$, $q \circ f = \mathrm{id}_Y$. Consequently, i is a retract of j, thus is itself anodyne.

There remains the verification of PMC. Since all objects are cofibrant, half of this is automatic (cf. §12, Proposition 5). Employing the usual notation, consider a pullback square

$$\begin{tikzcd} P \arrow[r, "\eta"] \arrow[d, "\xi"'] & Y \arrow[d, "g"] \\ X \arrow[r, "f"'] & Z \end{tikzcd}$$

in **SISET**. Assume: g is a Kan fibration and f is a weak homotopy equivalence—then η is a weak homotopy equivalence. Proof:

$$\begin{tikzcd} {|P|} \arrow[r, "|\eta|"] \arrow[d, "|\xi|"'] & {|Y|} \arrow[d, "|g|"] \\ {|X|} \arrow[r, "|f|"'] & {|Z|} \end{tikzcd}$$

is a pullback square in **CGH** (cf. Proposition 1), $|g|$ is a Serre fibration (cf. Proposition 17), and $|f|$ is a weak homotopy equivalence. Therefore $|\eta|$ is a weak homotopy equivalence.]

[Note: It is a corollary that **SISET**$_*$ ($= \Delta[0]\backslash$**SISET**) is a proper model category.]

EXAMPLE (Simplicial Groups) The free group functor F_{gr} : **SET** $\to$ **GR** extends to a functor F_{gr} : **SISET** $\to$ **SIGR** which is left adjoint to the forgetful functor U : **SIGR** $\to$ **SISET**. Call a homomorphism $f : G \to K$ of simplicial groups a weak equivalence if Uf is a weak homotopy equivalence, a fibration if Uf is a Kan fibration, and a cofibration if f has the LLP w.r.t. acyclic fibrations—then with these choices, **SIGR** is a model category. Here the point is that $f : G \to K$ is a fibration (acyclic fibration) iff it has the RLP w.r.t. the arrows $F_{\mathrm{gr}}\Lambda[k,n] \to F_{\mathrm{gr}}\Delta[n]$ $(0 \leq k \leq n, n \geq 1)$ $(F_{\mathrm{gr}}\dot{\Delta}[n] \to F_{\mathrm{gr}}\Delta[n]$ $(n \geq 0))$. Since F_{gr} preserves cofibrations and U preserves fibrations, the TDF theorem implies that $\mathbf{L}F_{\mathrm{gr}} : \mathbf{HSISET} \to \mathbf{HSIGR}$ and $\mathbf{R}U : \mathbf{HSIGR} \to \mathbf{HSISET}$ exist and constitute an adjoint pair.

[Note: Every object in **SIGR** is fibrant (cf. p. 13–21) but not every object in **SIGR** is cofibrant. Definition: A simplicial group G is said to be free if $\forall\, n$, G_n is a free group with a specified basis B_n such that $s_i B_n \subset B_{n+1}$ $(0 \leq i \leq n)$. Every free simplicial group is cofibrant and every cofibrant simplicial group is the retract of a free simplicial group.]

EXAMPLE (Groupoids) **GRD** acquires the structure of a model category when one stipulates that a functor F is a weak equivalence if F is an equivalence of categories, a cofibration if F is injective on objects, and a fibration if $\operatorname{ner} F$ is a Kan fibration. All objects are cofibrant and fibrant.

EXAMPLE (G-Sets) Fix a group G. Denote by $\mathbf{G}$ the groupoid having a single object $*$ with $\operatorname{Mor}(*,*) = G$—then the category $\mathbf{SET}_G$ of right G-sets is the functor category $[\mathbf{G}^{\mathrm{OP}}, \mathbf{SET}]$ and the category of simplicial right G-sets $\mathbf{SISET}_G$ is the functor category $[\mathbf{\Delta}^{\mathrm{OP}}, [\mathbf{G}^{\mathrm{OP}}, \mathbf{SET}]] \approx [(\mathbf{\Delta} \times \mathbf{G})^{\mathrm{OP}}, \mathbf{SET}]$. Claim: $\mathbf{SISET}_G$ is a model category. Thus let $U : \mathbf{SISET}_G \to \mathbf{SISET}$ be the forgetful functor and declare that a morphism $f : X \to Y$ of simplicial right G-sets is a weak equivalence if Uf is a weak homotopy equivalence, a fibration if Uf is a Kan fibration, and a cofibration if f has the LLP w.r.t. acyclic fibrations.

(CO) An object X in $\mathbf{SISET}_G$ is cofibrant iff $\forall\, n$, X_n is a free G-set.

Fix a cofibrant XG in $\mathbf{SISET}_G$ such that $XG \to *$ is an acyclic fibration. Put $BG = XG/G$—then XG is simplicially contractible and locally trivial with fiber G (i.e., $\mathrm{si}G$), the projection $XG \to BG$ is a Kan fibration, BG is fibrant, and $|BG|$ is a $K(G,1)$. Explicit models for (XG, BG) can be found, e.g., in the notation of p. 0–45, $XG = \mathrm{bar}(*; \mathbf{G}; G)$ $(\approx \operatorname{ner} \operatorname{tran} G)$, $BG = \mathrm{bar}(*; \mathbf{G}; *)$ $(\approx \operatorname{ner} \mathbf{G})$.

[Note: U has a left adjoint F_G which sends X to $X \times \mathrm{si}G$. And, thanks to the TDF theorem, $(\mathbf{L}F_G, \mathbf{R}U)$ is an adjoint pair.]

Remark: The class of anodyne extensions is precisely the class of acyclic cofibrations.

Claim: Sd preserves anodyne extensions. For suppose that $f : X \to Y$ is anodyne and form the commutative diagram

$$\begin{tikzcd} \mathrm{Sd}\, X \arrow[r, "\mathrm{Sd}\, f"] \arrow[d, "\mathrm{d}_X"'] & \mathrm{Sd}\, Y \arrow[d, "\mathrm{d}_Y"] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

. Since Sd preserves injections, $\mathrm{Sd}\, f$ is a cofibration. But d_X & d_Y are weak homotopy equivalences (cf. Proposition 5), thus $\mathrm{Sd}\, f$ is an acyclic cofibration, i.e., is anodyne.

PROPOSITION 22 Suppose that $L \to K$ is an inclusion of simplicial sets and $X \to B$ is a Kan fibration—then the arrow $\mathrm{map}(K, X) \to \mathrm{map}(L, X) \times_{\mathrm{map}(L,B)} \mathrm{map}(K, B)$ is a

Kan fibration (cf. Proposition 12) which is a weak homotopy equivalence if this is the case of $L \to K$ or $X \to B$.

[Owing to Proposition 21, the problem is to produce a filler $\Delta[n] \times K \to X$ for

$$\begin{tikzcd}
\dot{\Delta}[n] \times K \cup \Delta[n] \times L \arrow[r] \arrow[d, "i"'] & X \arrow[d] \\
\Delta[n] \times K \arrow[r] & B
\end{tikzcd}$$

. If $L \to K$ is an acyclic cofibration, then, as pointed out above, $L \to K$ is anodyne. Therefore i is anodyne (cf. Proposition 9) and the filler exists. If $X \to B$ is an acyclic Kan fibration, then the existence of the filler is guaranteed by MC–4.]

HSISET is the homotopy category of **SISET** (cf. p. 12–24 ff.). In this situation, $IX = X \times \Delta[1]$ serves as a cylinder object while $PX = \text{map}(\Delta[1], X)$ is a path object when X is fibrant but not in general (Berger[†]). Since all objects are cofibrant, $\mathcal{L}X = X \ \forall X$ and there are canonical choices for $\mathcal{R}X$, e.g., $\sin|X|$ or $\text{Ex}^\infty X$. If X is cofibrant and Y is fibrant, then left homotopy=right homotopy or still, simplicial homotopy: $[X, Y] \approx [X, Y]_0$. **HSISET** has finite products. And: **HSISET** is cartesian closed. Proof: $[X \times Y, Z] \approx [X \times Y, \sin|Z|] \approx [X \times Y, \sin|Z|]_0 \approx [X, \text{map}(Y, \sin|Z|)]_0 \approx [X, \text{map}(Y, \sin|Z|)]$.

[Note: Recall too that the inclusion $\mathbf{HSISET}_\mathbf{f} \to$ **HSISET** is an equivalence of categories (cf. §12, Proposition 13).]

Example: X and X^{OP} are naturally isomorphic in **HSISET**.

FACT Let $S \subset \text{Mor}\,\mathbf{H_0SISET}$ be the class of homotopy classes of anodyne extensions—then $S^{-1}\mathbf{H_0SISET}$ is equivalent to **HSISET**.

COMPARISON THEOREM The adjoint pair $(|?|, \sin)$ induces an adjoint equivalence of categories between **HSISET** and **HTOP** (singular structure).

[In the TDF theorem, take $F = |?|$, $G = \sin$—then F preserves cofibrations and G preserves fibrations, thus $\begin{cases} \mathbf{L}F \\ \mathbf{R}G \end{cases}$ exist and $(\mathbf{L}F, \mathbf{R}G)$ is an adjoint pair. Consider now the bijection of adjunction $\Xi_{X,Y} : C(|X|, Y) \to \text{Nat}(X, \sin Y)$, so $\Xi_{X,Y} f$ is the composition $X \to \sin|X| \xrightarrow{\sin f} \sin Y$. Since the arrow $X \to \sin|X|$ is a weak homotopy equivalence (cf. p. 13–15), $\Xi_{X,Y} f$ is a weak homotopy equivalence iff $\sin f$ is a weak homotopy equivalence, i.e., iff f is a weak homotopy equivalence (cf. p. 13–17). Therefore the pair $(\mathbf{L}F, \mathbf{R}G)$ is an adjoint equivalence of categories (cf. p. 12–29).]

Application: **HSISET** is equivalent to **HCW**.

[†] *Bull. Soc. Math. France* **123** (1995), 1–32.

[Note: Analogously, **HSISET**$_*$ is equivalent to **HCW**$_*$.]

Are there other model categories **C** whose associated homotopy category **HC** is equivalent to **HCW**? The answer is "yes".

EXAMPLE Take **C** = **CAT** and call a morphism f a weak equivalence if $\mathrm{Ex}^2 \circ \mathrm{ner}\, f$ is a weak homotopy equivalence, a fibration if $\mathrm{Ex}^2 \circ \mathrm{ner}\, f$ is a Kan fibration, and a cofibration if f has the LLP w.r.t. all fibrations that are weak equivalences—then Thomason[†] has shown that **CAT** is a proper model category. Put $\begin{cases} F = c \circ \mathrm{Sd}^2 \\ G = \mathrm{Ex}^2 \circ \mathrm{ner} \end{cases}$: (F, G) is an adjoint pair with the property that F preserves cofibrations and G preserves fibrations, thus $\begin{cases} \mathbf{L}F \\ \mathbf{R}G \end{cases}$ exist and $(\mathbf{L}F, \mathbf{R}G)$ is an adjoint pair (TDF theorem). Moreover, the arrow $X \rightarrow \mathrm{Ex}^2 \circ \mathrm{ner} \circ c \circ \mathrm{Sd}^2 X$ is a weak homotopy equivalence of simplicial sets, so the pair $(\mathbf{L}F, \mathbf{R}G)$ is an adjoint equivalence of categories. It therefore follows that **HSISET**, **HCAT**, and **HCW** are equivalent.

[Note: Latch[‡] proved that ner : **CAT** → **SISET** induces an equivalence **HCAT** → **HSISET** (but the adjoint pair (c, ner) does not induce an adjoint equivalence).]

EXAMPLE The category of simplicial groupoids is a model category and its homotopy category is equivalent to **HSISET**, hence to **HCW** (Dwyer-Kan[||]).

[Note: A simplicial groupoid **G** is a category object (M, O) in **SISET**, where O is a constant simplicial set, equipped with a simplicial map $\chi : M \rightarrow M$ such that $s = t \circ \chi$, $t = s \circ \chi$, $c \circ (\chi \times \mathrm{id}_M) = e \circ s$, $c \circ (\mathrm{id}_M \times \chi) = e \circ t$. So, $\forall\, n$, $\mathbf{G}_n$ is a groupoid and $\mathrm{Ob}\, \mathbf{G}_n = \mathrm{Ob}\, \mathbf{G}_0$. Introducing the obvious notion of morphism, the simplicial groupoids are seen to constitute a category which is complete and cocomplete. Its model category structure is derived from (1)–(3) below.

(1) A morphism $F : \mathbf{G} \rightarrow \mathbf{K}$ of simplicial groupoids is a weak equivalence if F restricts to a bijection on components and $\forall\, X \in O$, the induced morphism $\mathbf{G}(X) \rightarrow \mathbf{K}(FX)$ of simplicial groups is a weak equivalence.

(2) A morphism $F : \mathbf{G} \rightarrow \mathbf{K}$ of simplicial groupoids is a fibration if $F_0 : \mathbf{G}_0 \rightarrow \mathbf{K}_0$ is a fibration of groupoids and $\forall\, X \in O$, the induced morphism $\mathbf{G}(X) \rightarrow \mathbf{K}(FX)$ of simplicial groups is a fibration.

(3) A morphism $F : \mathbf{G} \rightarrow \mathbf{K}$ of simplicial groupoids is a cofibration if it has the LLP w.r.t. acyclic fibrations.]

[†] *Cahiers Topologie Géom. Différentielle* **21** (1980), 305–324.

[‡] *J. Pure Appl. Algebra* **9** (1977), 221–237.

[||] *Nederl. Akad. Wetensch. Indag. Math.* **46** (1984), 379–385; see also Heller, *Illinois J. Math.* **24** (1980), 576–605.

Fix a small category $\mathbf{I}$—then the functor category $[\mathbf{I}, \mathbf{SISET}]$ admits two proper model category structures. However, the weak equivalences in either structure are the same, so both give rise to the same homotopy category $\mathbf{H}[\mathbf{I}, \mathbf{SISET}]$.

(L) Given functors $F, G : \mathbf{I} \rightarrow \mathbf{SISET}$, call $\Xi \in \operatorname{Nat}(F, G)$ a weak equivalence if $\forall\ i, \Xi_i : Fi \rightarrow Gi$ is a weak homotopy equivalence, a fibration if $\forall\ i, \Xi_i : Fi \rightarrow Gi$ is a Kan fibration, a cofibration if Ξ has the LLP w.r.t. acyclic fibrations.

(R) Given functors $F, G : \mathbf{I} \rightarrow \mathbf{SISET}$, call $\Xi \in \operatorname{Nat}(F, G)$ a weak equivalence if $\forall\ i, \Xi_i : Fi \rightarrow Gi$ is a weak homotopy equivalence, a cofibration if $\forall\ i : \Xi_i : Fi \rightarrow Gi$ is an injective simplicial map, a fibration if Ξ has the RLP w.r.t. acyclic cofibrations.

In practice, both structures are used but for theoretical work, structure L is generally the preferred choice.

[Note: When $\mathbf{I}$ is discrete, structure L=structure R (all data is levelwise).]

Since the arguments are dual, it will be enough to outline the proof in the case of structure L.

Notation: Let $f : X \rightarrow Y$ be a simplicial map—then f admits a functorial factorization $X \xrightarrow{i_f} \mathcal{L}_f \xrightarrow{\pi_f} Y$, where i_f is a cofibration and π_f is an acyclic Kan fibration, and a functorial factorization $X \xrightarrow{\iota_f} \mathcal{R}_f \xrightarrow{p_f} Y$, where ι_f is an acyclic cofibration and p_f is a Kan fibration.

Observation: These factorizations extend levelwise to factorizations of $\Xi : F \rightarrow G$, viz. $F \xrightarrow{i_\Xi} \mathcal{L}_\Xi \xrightarrow{\pi_\Xi} G$ and $F \xrightarrow{\iota_\Xi} \mathcal{R}_\Xi \xrightarrow{p_\Xi} G$.

Write $\mathbf{I}_{\text{dis}}$ for the discrete category underlying $\mathbf{I}$—then the forgetful functor $U : [\mathbf{I}, \mathbf{SISET}] \rightarrow [\mathbf{I}_{\text{dis}}, \mathbf{SISET}]$ has a left adjoint that sends X to $\operatorname{fr}X$, where $\operatorname{fr}Xj = \coprod\limits_{i \in \operatorname{Ob}\mathbf{I}} \operatorname{Mor}(i, j) \cdot Xi$.

LEMMA Fix an F in $[\mathbf{I}, \mathbf{SISET}]$. Suppose that $\Phi : UF \rightarrow X$ is a cofibration in $[\mathbf{I}_{\text{dis}}, \mathbf{SISET}]$ and

$$\begin{tikzcd}
\operatorname{fr}UF \arrow[r, "\operatorname{fr}\Phi"] \arrow[d, "\nu_F"'] & \operatorname{fr}X \arrow[d, "u"] \\
F \arrow[r] & G
\end{tikzcd}$$

is a pushout square in $[\mathbf{I}, \mathbf{SISET}]$—then the composite $Uu \circ \mu_X : X \xrightarrow{\mu_X} U\operatorname{fr}X \xrightarrow{Uu} UG$ is a cofibration in $[\mathbf{I}_{\text{dis}}, \mathbf{SISET}]$.

[The commutative diagram

$$\begin{tikzcd}
 & Xj \arrow[r, equal] \arrow[d] & Xj \arrow[d, "(\mu_X)_j"] \\
\Big(\coprod\limits_{\substack{i \xrightarrow{\delta} j \\ \delta \neq \mathrm{id}_j}} Fi\Big) \amalg Fj \arrow[r] \arrow[d] & \Big(\coprod\limits_{\substack{i \xrightarrow{\delta} j \\ \delta \neq \mathrm{id}_j}} Fi\Big) \amalg Xj \arrow[r] \arrow[d] & \Big(\coprod\limits_{\substack{i \xrightarrow{\delta} j \\ \delta \neq \mathrm{id}_j}} Xi\Big) \amalg Xj \arrow[d, "u_j"] \\
Fj \arrow[r, "\Phi_j"'] & Xj \arrow[r, "u_j \circ (\mu_X)_j"'] & Gj
\end{tikzcd}$$

tells the tale. Indeed, the middle row is a factorization of $(\mathrm{fr}\Phi)_j$ (suppression of "U"), the bottom square on the right is a pushout, and a coproduct of cofibrations is a cofibration.]

[Note: As usual, $\begin{cases} \mu \\ \nu \end{cases}$ are the ambient arrows of adjunction.]

Consider any $\Xi : F \to G$. Claim: Ξ can be written as the composite of a cofibration and an acyclic fibration. Thus define F_1 by the pushout square

$$\begin{tikzcd} \mathrm{fr}UF \arrow[r, "\mathrm{fr}Ui_\Xi"] \arrow[d, "\nu_F"'] & \mathrm{fr}U\mathcal{L}_\Xi \arrow[d] \\ F \arrow[r] & F_1 \end{tikzcd}$$

—then there is a commutative diagram

$$\begin{tikzcd} \mathrm{fr}UF \arrow[r, "\mathrm{fr}Ui_\Xi"] \arrow[d, "\nu_F"'] & \mathrm{fr}U\mathcal{L}_\Xi \arrow[r, "\mathrm{fr}U\pi_\Xi"] \arrow[d] & \mathrm{fr}UG \arrow[d, "\nu_G"] \\ F \arrow[r] \arrow[dr] & F_1 \arrow[r] \arrow[d] & G \\ & \mathcal{L}_\Xi \arrow[ur] & \end{tikzcd}$$

in which $\mathrm{fr}U\mathcal{L}_\Xi \to F_1 \to \mathcal{L}_\Xi$ is $\nu_{\mathcal{L}_\Xi}$. Putting $F_0 = F$ (and $\Xi_0 = \Xi$), iterate the construction to obtain a sequence $F = F_0 \to F_1 \to \cdots \to F_\omega$ of objects in $[\mathbf{I}, \mathbf{SISET}]$, taking $F_\omega = \operatorname{colim} F_n$. This leads to a commutative triangle

$$\begin{tikzcd} F \arrow[rr, "i_\omega"] \arrow[dr, "\Xi"'] & & F_\omega \arrow[dl, "\Xi_\omega"] \\ & G & \end{tikzcd}$$

. Here, i_ω is a cofibration (since the $F_n \to F_{n+1}$ are). Moreover, i_ω is a weak equivalence whenever Ξ is a weak equivalence and in that situation, i_ω has the LLP w.r.t. all fibrations. To see that Ξ_ω is an acyclic fibration, look at the interpolation

$$\begin{tikzcd} UF_0 \arrow[r] \arrow[d] & U\mathcal{L}_{\Xi_0} \arrow[r] \arrow[d] & UF_1 \arrow[r] \arrow[d] & U\mathcal{L}_{\Xi_1} \arrow[r] \arrow[d] & \cdots \\ UG \arrow[r, equal] & UG \arrow[r, equal] & UG \arrow[r, equal] & UG \arrow[r, equal] & \cdots \end{tikzcd}$$

in $[\mathbf{I}_{\mathrm{dis}}, \mathbf{SISET}]$. Thanks to the lemma, the horizontal arrows in the top row are cofibrations. On the other hand, the arrows $U\mathcal{L}_{\Xi_n} \to UG$ are acyclic fibrations. But then $U\Xi_\omega$ is an acyclic fibration per $[\mathbf{I}_{\mathrm{dis}}, \mathbf{SISET}]$, i.e., Ξ_ω is an acyclic fibration per $[\mathbf{I}, \mathbf{SISET}]$. Hence the claim.

To finish the verification of MC–5, one has to establish that Ξ can be written as the composite of an acyclic cofibration and a fibration. This, however, is immediate: Apply the claim to ι_Ξ. MC–4 is equally clear. For if Ξ is a cofibration, then Ξ is a retract of i_ω, so if Ξ is an acyclic cofibration, then Ξ has the LLP w.r.t. all fibrations. PMC is obvious.

EXAMPLE Definition: A functor $F : \mathbf{I} \to \mathbf{SISET}$ is said to be free if $\exists$ functors $B_n : \mathbf{I}_{\mathrm{dis}} \to \mathbf{SET}$ $(n \geq 0)$ such that $\forall\ j \in \mathrm{Ob}\,\mathbf{I} : B_n j \subset (Fj)_n$ & $s_i B_n j \subset B_{n+1} j$ $(0 \leq i \leq n)$, with $\mathrm{fr} B_n \approx F_n$ $(F_n j = (Fj)_n)$. Every free functor is cofibrant in structure L and every cofibrant functor in structure L is the retract of a free functor. Example: $\mathrm{ner}(\mathbf{I}/\text{—})$ is a free functor, hence is cofibrant in structure L.

Fix an abelian group G. Let $f : X \to Y$ be a simplicial map—then f is said to be an HG-equivalence if $\forall\ n \geq 0$, $|f|_* : H_n(|X|;G) \to H_n(|Y|;G)$ is an isomorphism. Agreeing that an HG-cofibration is an injective simplicial map, an HG-fibration is a simplicial map which has the RLP w.r.t. all HG-cofibrations that are HG-equivalences. Every HG-fibration is a Kan fibration. Proof: $\Lambda^{k,n}$ is a strong deformation retract of Δ^n.

PROPOSITION 23 Let $p : X \to B$ be a simplicial map—then p is an HG-fibration and an HG-equivalence iff p is a Kan fibration and a weak homotopy equivalence.

[Necessity: Write $p = q \circ j$, where $j : X \to Y$ is a cofibration and $q : Y \to B$ is an acyclic Kan fibration. Since p is an HG-equivalence, the same is true of j, thus the commutative diagram

$$\begin{tikzcd} X \arrow[r, equal] \arrow[d, "j"'] & X \arrow[d, "p"] \\ Y \arrow[r, "q"'] & B \end{tikzcd}$$

admits a filler $g : Y \to X$. Therefore p is a retract of q, hence is an acyclic Kan fibration.

Sufficiency: Apply Proposition 7 and Proposition 21.]

Notation: Given a simplicial set X, write $\#(X)$ for $\#(\mathcal{E})$, the cardinality of the set of cells in $|X|$.

[Note: $\forall$ set X, $\#(\mathrm{si}X) = \#(X)$, the cardinality of X.]

PROPOSITION 24 Let $p : X \to B$ be a simplicial map which has the RLP w.r.t. every inclusion $A \to Y$, where $H_*(|Y|,|A|;G) = 0$ and $\#(Y)$ is $\leq \#(G)$ if $\#(G)$ is infinite and $\leq \omega$ if $\#(G)$ is finite—then p is an HG-fibration.

[It suffices to prove that p has the RLP w.r.t. every inclusion $L \to K$ ($L \neq K$) with $H_*(|K|,|L|;G) = 0$. This can be established by using Zorn's lemma. Indeed, $\exists$ a simplicial subset $A \subset K$ ($A \not\subset L$) such that $H_*(|A|,|A \cap L|;G) = 0$ subject to the restriction that $\#(A)$ is $\leq \#(G)$ if $\#(G)$ is infinite and $\leq \omega$ if $\#(G)$ is finite (cf. p. 9–25).]

PREFACTORIZATION LEMMA Suppose that κ is an infinite cardinal. Let $f : X \to Y$ be a simplicial map—then f can be written as a composite $f = p_f \circ i_f$, where $i_f : X \to X_f$ is an injection with $H_*(|X_f|,|X|;G) = 0$, such that every commutative diagram

$$\begin{tikzcd} L \arrow[r] \arrow[d] & X \arrow[r, "i_f"] & X_f \arrow[d, "p_f"] \\ K \arrow[rr] & & Y \end{tikzcd}$$

has a filler $K \to X_f$, (K,L) being any simplicial pair with $\#(K) \leq \kappa$ and $H_*(|K|,|L|;G) = 0$.

[Choose a set of simplicial pairs (K_i,L_i) with $\#(K_i) \leq \kappa$ and $H_*(|K_i|,|L_i|;G) = 0$ which contains up to isomorphism all such simplicial pairs. Consider the set of pairs

of morphisms (g,h) such that the diagram
$$\begin{tikzcd} L_i \arrow[r, "g"] \arrow[d] & X \arrow[d, "f"] \\ K_i \arrow[r, "h"'] & Y \end{tikzcd}$$
commutes, define X_f by the pushout square
$$\begin{tikzcd} \coprod\limits_i \coprod\limits_{(g,h)} L_i \arrow[r] \arrow[d] & X \arrow[d, "i_f"] \\ \coprod\limits_i \coprod\limits_{(g,h)} K_i \arrow[r] & X_f \end{tikzcd}$$
, and let $p_f : X_f \rightarrow Y$ be the induced simplicial map.]

HOMOLOGICAL MODEL CATEGORY THEOREM Fix an abelian group G—then **SISET** is a model category if weak equivalence=HG-equivalence, cofibration=HG-cofibration, fibration=HG-fibration.

[On the basis of Proposition 23, one has only to show that every simplicial map $f : X \rightarrow Y$ can be written as a composite $p \circ i$, where i is an acyclic HG-cofibration and p is an HG-fibration. This can be done by a transfinite lifting argument, using the prefactorization lemma with κ a regular cardinal $> \#(G)$ (cf. Proposition 24).]

[Note: The fibrant objects in this structure are the $\underline{HG\text{-local}}$ objects, i.e., those X such that $X \rightarrow *$ is an HG-fibration.]

PROPOSITION 25 Suppose that $L \rightarrow K$ is an inclusion of simplicial sets and $X \rightarrow B$ is an HG-fibration—then the arrow $\text{map}(K,X) \rightarrow \text{map}(L,X) \times_{\text{map}(L,B)} \text{map}(K,B)$ is an HG-fibration which is an HG-equivalence if this is the case of $L \rightarrow K$ or $X \rightarrow B$.

EXAMPLE The model category structure on **SISET** provided by the homological model category theorem is generally not proper. Thus factor $X \rightarrow *$ as $X \rightarrow X_{HG} \rightarrow *$, where $X \rightarrow X_{HG}$ is an acyclic HG-cofibration and $X_{HG} \rightarrow *$ is an HG-fibration. Assuming that X is fibrant and connected, define E_{HG} by the pullback square
$$\begin{tikzcd} E_{HG} \arrow[r] \arrow[d] & \Theta X_{HG} \arrow[d] \\ X \arrow[r] & X_{HG} \end{tikzcd}$$
—then the arrow $E_{HG} \rightarrow \Theta X_{HG}$ is not necessarily an HG-equivalence.

FACT Suppose given simplicial maps $f : X \rightarrow Y$, $g : Y \rightarrow Z$, where f is a Kan fibration and g, $g \circ f$ are HG-fibrations—then f is an HG-fibration.

Application: If $f : X \rightarrow Y$ is a Kan fibration and $\begin{cases} X \\ Y \end{cases}$ are HG-local, then f is an HG-fibration.

EXAMPLE The HG-local objects in **SISET** are closed under the formation of products and $\text{map}(X,Y)$ is HG-local $\forall$ X provided that Y is HG-local. Given a 2-sink $X \xrightarrow{f} Z \xleftarrow{g} Y$ of HG-local objects with f a Kan fibration, the pullback $X \times_Z Y$ is HG-local. Finally, for any tower $X_0 \leftarrow X_1 \leftarrow \cdots$ of Kan fibrations and HG-local X_n, the limit $\lim X_n$ is HG-local.

A simplicial category is a **SISET**-category. So, to specify a simplicial category one must specify a class of objects O and a function that assigns to each ordered pair $X, Y \in O$ a simplicial set $\mathrm{HOM}(X,Y)$ plus simplicial maps $C_{X,Y,Z} : \mathrm{HOM}(X,Y) \times \mathrm{HOM}(Y,Z) \rightarrow \mathrm{HOM}(X,Z)$, $I_X : \Delta[0] \rightarrow \mathrm{Hom}(X,X)$ satisfying **SISET**-cat$_1$ and **SISET**-cat$_2$ (cf. p. 0–40). Here is an equivalent description. Fix a class O. Consider the metacategory $\mathcal{CAT}_O$ whose objects are the categories with object class O, the morphisms being the functors which are the identity on objects—then a simplicial category with object class O is a simplicial object in $\mathcal{CAT}_O$.

A category object (M,O) in **SISET**, where O is a constant simplicial set, is a simplicial category. In particular: A simplicial groupoid is a simplicial category (cf. p. 13–36).

EXAMPLE There is a functor $\mathbf{\Delta}^{\mathrm{OP}} \rightarrow \mathbf{SISET}$ which sends $[n]$ to $\Delta[1]^n$ and $\begin{cases} \delta_i \text{ to } d_i \\ \sigma_i \text{ to } s_i \end{cases}$, where

$$d_i(\alpha_1, \ldots, \alpha_n) = \begin{cases} (\alpha_2, \ldots, \alpha_n) & (i = 0) \\ (\alpha_1, \ldots, \max(\alpha_{i+1}, \alpha_i), \ldots, \alpha_n) & (0 < i < n) \\ (\alpha_1, \ldots, \alpha_{n-1}) & (i = n) \end{cases},$$

$s_i(\alpha_1, \ldots, \alpha_n) = (\alpha_1, \ldots, \alpha_i, 0, \alpha_{i+1}, \ldots, \alpha_n)$. Now fix a small category $\mathbf{C}$. Given $X, Y \in \mathrm{Ob}\,\mathbf{C}$, let $C = C(X,Y)$ be the cosimplicial set defined by taking for $C(X,Y)^n$ the set of all functors $F : [n+1] \rightarrow \mathbf{C}$ with $F_0 = X$, $F_{n+1} = Y$ and letting $C\delta_i : C^n \rightarrow C^{n+1}$, $C\sigma_i : C^n \rightarrow C^{n-1}$ be the assignments $(f_0, \ldots, f_n) \rightarrow (f_0, \ldots, f_{i-1}, \mathrm{id}, f_i, \ldots, f_n)$, $(f_0, \ldots, f_n) \rightarrow (f_0, \ldots, f_{i+1} \circ f_i, \ldots, f_n)$. Definition: $\mathrm{HOM}(X,Y) = \int^{[n]} \Delta[1]^n \times C(X,Y)^n$. Since $\mathrm{HOM}(X,Y)_m = \int^{[n]} \Delta[1]^n_m \times C(X,Y)^n$, one can introduce a "composition" rule and a "unit" rule satisfying the axioms. The upshot, therefore, is a simplicial category $\mathbf{FRC}$ with $O = \mathrm{Ob}\,\mathbf{C}$.

[Note: The abstract interpretation of **FRC** is this. Observe first that the forgetful functor from **CAT** to the category of small graphs with distinguished loops at the vertexes has a left adjoint. Consider the associated cotriple in **CAT**—then the standard resolution of **C** is **FRC** and the underlying category **UFRC** is the free category on $\mathrm{Ob}\,\mathbf{C}$ having one generator for each nonidentity morphism in **C**.]

Let $\mathbf{C}$ be a category. Suppose that X, Y are simplicial objects in $\mathbf{C}$ and let K be a simplicial set—then a formality $f : X \square K \rightarrow Y$ is a collection of morphisms $f_n(k) : X_n \rightarrow Y_n$ in $\mathbf{C}$, one for each $n \geq 0$ and $k \in K_n$, such that $Y\alpha \circ f_n(k) = f_m((K\alpha)k) \circ X\alpha$ $(\alpha : [m] \rightarrow [n])$. Notation: $\mathrm{For}(X \square K, Y)$. Example: $\mathrm{For}(X \square \Delta[0], Y)$ can be identified with $\mathrm{Nat}(X,Y)$.

[Note: As it stands, $X\square K$ is just a symbol, not an object in **SIC** (but see below).]

PROPOSITION 26 Let **C** be a category—then the class of simplicial objects in **C** is the object class of a simplicial category **SIMC**.

[Define $\mathrm{HOM}(X,Y)$ by letting $\mathrm{HOM}(X,Y)_n$ be $\mathrm{For}(X\square\Delta[n],Y)$.]

[Note: **SIC** is isomorphic to the underlying category of **SIMC**.]

A simplicial functor is a **SISET**-functor. Example: If $\begin{cases}\mathbf{C}\\ \mathbf{D}\end{cases}$ are categories and $F:\mathbf{C}\to\mathbf{D}$ is a functor, then F extends to a simplicial functor $SF:\mathbf{SIMC}\to\mathbf{SIMD}$.

EXAMPLE **CAT** is cartesian closed, hence can be viewed as a **CAT**-category. Since ner : **CAT** → **SISET** is a morphism of symmetric monoidal categories, $\mathrm{ner}_*\mathbf{CAT}$ is a simplicial category whose object class is the class of small categories, $\mathrm{HOM}(\mathbf{C},\mathbf{D})$ being $\mathrm{ner}[\mathbf{C},\mathbf{D}]$ (cf. p. 0–41). One may therefore interpret ner as a simplicial functor $\mathrm{ner}_*\mathbf{CAT}\to\mathbf{SISET}$ (for $\mathrm{ner}[\mathbf{C},\mathbf{D}]\approx\mathrm{map}(\mathrm{ner}\,\mathbf{C},\mathrm{ner}\,\mathbf{D})$).

Given a category **C**, a simplicial action on **C** is a functor $\square:\mathbf{C}\times\mathbf{SISET}\to\mathbf{C}$, together with natural isomorphisms R and A, where $R_X:X\square\Delta[0]\to X$, $A_{X,K,L}:X\square(K\times L)\to(X\square K)\square L$, subject to the following assumptions.

(SA$_1$) The diagram

$$\begin{tikzcd}
X\square(K\times(L\times M)) \arrow[r, "A"] \arrow[d, "\mathrm{id}\square A"'] & (X\square K)\square(L\times M) \arrow[r, "A"] & ((X\square K)\square L)\square M \\
X\square((K\times L)\times M) \arrow[rr, "A"'] & & (X\square(K\times L))\square M \arrow[u, "A\square\mathrm{id}"']
\end{tikzcd}$$

commutes.

(SA$_2$) The diagram

$$\begin{tikzcd}
X\square(\Delta[0]\times K) \arrow[r, "A"] \arrow[d, "\mathrm{id}\square L"'] & (X\square\Delta[0])\square K \arrow[d, "R\square\mathrm{id}"] \\
X\square K \arrow[r, equal] & X\square K
\end{tikzcd}$$

commutes.

[Note: Every category admits a simplicial action, viz. the trivial simplicial action.]

It is automatic that the diagram

$$\begin{tikzcd}
X\square(K\times\Delta[0]) \arrow[r, "A"] \arrow[d, "\mathrm{id}\square R"'] & (X\square K)\square\Delta[0] \arrow[d, "R"] \\
X\square K \arrow[r, equal] & X\square K
\end{tikzcd}$$

commutes.

EXAMPLE If $\square$ is a simplicial action on $\mathbf{C}$, then for every small category $\mathbf{I}$, the composition $[\mathbf{I},\mathbf{C}]\times\mathbf{SISET}\rightarrow[\mathbf{I},\mathbf{C}]\times[\mathbf{I},\mathbf{SISET}]\approx[\mathbf{I},\mathbf{C}\times\mathbf{SISET}]\xrightarrow{[\mathbf{I},\square]}[\mathbf{I},\mathbf{C}]$ is a simplicial action on $[\mathbf{I},\mathbf{C}]$.

PROPOSITION 27 Let $\mathbf{C}$ be a category. Assume: $\mathbf{C}$ admits a simplicial action $\square$—then there is a simplicial category $\square\mathbf{C}$ such that $\mathbf{C}$ is isomorphic to the underlying category $\mathbf{U}\square\mathbf{C}$.

[Put $O = \mathrm{Ob}\,\mathbf{C}$ and assign to each ordered pair $X, Y \in O$ the simplicial set $\mathrm{HOM}(X,Y)$ defined by $\mathrm{HOM}(X,Y)_n = \mathrm{Mor}\,(X\square\Delta[n],Y)$ $(n\geq 0)$.

(Composition) Given X,Y,Z, let $C_{X,Y,Z}$: $\mathrm{HOM}(X,Y)\times\mathrm{HOM}(Y,Z)\rightarrow\mathrm{HOM}(X,Z)$ be the simplicial map that sends $\begin{cases} f : X\square\Delta[n]\rightarrow Y \\ g : Y\square\Delta[n]\rightarrow Z\end{cases}$ to the composite $X\square\Delta[n]\xrightarrow{\mathrm{id}\square\mathrm{di}}X\square(\Delta[n]\times\Delta[n])\xrightarrow{A}(X\square\Delta[n])\square\Delta[n]\xrightarrow{f\square\mathrm{id}}Y\square\Delta[n]\xrightarrow{g}Z$.

(Unit) Given X, let $I_X : \Delta[0]\rightarrow\mathrm{HOM}(X,X)$ be the simplicial map that sends $[n]\rightarrow[0]$ to $X\square\Delta[n]\rightarrow X\square\Delta[0]\xrightarrow{R}X$.

Call $\square\mathbf{C}$ the simplicial category arising from this data. That $\mathbf{C}$ is isomorphic to the underlying category $\mathbf{U}\square\mathbf{C}$ can be seen by considering the functor which is the identity on objects and sends a morphism $f : X\rightarrow Y$ in $\mathbf{C}$ to $X\square\Delta[0]\xrightarrow{R}X\xrightarrow{f}Y$, an element of $\mathrm{Mor}\,(X\square\Delta[0],Y) = \mathrm{HOM}(X,Y)_0\approx\mathrm{Nat}(\Delta[0],\mathrm{HOM}(X,Y))$.]

[Note: $\mathrm{HOM} : \mathbf{C}^{\mathrm{OP}}\times\mathbf{C}\rightarrow\mathbf{SISET}$ is a functor and the simplicial set $\mathrm{HOM}(X,Y)$ is called the simplicial mapping space between X and Y. Example: Take for $\square$ the trivial simplicial action—then in this case, $\mathrm{HOM}(X,Y) = \mathrm{siMor}(X,Y)$.]

Examples: (1) **SISET** admits a simplicial action: $K\square L = K\times L$ (so $\mathrm{HOM}(K,L) = \mathrm{map}(K,L)$); (2) **CGH** admits a simplicial action: $X\square K = X\times_k|K|$ (so $\mathrm{HOM}(X,Y)_n =$ all continuous functions $X\times_k\Delta^n\rightarrow Y$); (3) $\mathbf{SISET}_*$ admits a simplicial action: $K\square L = K\#L_+$ (so $\mathrm{HOM}(K,L) = \mathrm{map}_*(K,L)$); (4) $\mathbf{CGH}_*$ admits a simplicial action: $X\square K = X\#_k|K|_+$ (so $\mathrm{HOM}(X,Y)_n =$ all pointed continuous functions $X\#_k\Delta^n_+\rightarrow Y$).

[Note: If X,Y are in **CGH**, then $\mathrm{HOM}(X,Y)\approx\sin(\mathrm{map}(X,Y))$ and if X,Y are in $\mathbf{CGH}_*$, then $\mathrm{HOM}(X,Y)\approx\sin(\mathrm{map}_*(X,Y))$. In either situation, $\mathrm{HOM}(X,Y)$ is fibrant.]

Neither **TOP** nor $\mathbf{TOP}_*$ fits into the preceding framework (products or smash products are preserved in general only if the compactly generated category is used). This difficulty can be circumvented by restricting the definition of simplicial action to the full subcategory of **SISET** whose objects are the finite simplicial sets. It is therefore still the case that **TOP** ($\mathbf{TOP}_*$) is isomorphic to the underlying category of a simplicial category with $\mathrm{HOM}(X,Y)_n$ = all continuous functions $X \times \Delta^n \rightarrow Y$ (all pointed continuous functions $X \# \Delta^n_+ \rightarrow Y$).

Example: Let **C** be a category. Assume: **C** has coproducts—then **SIC** admits a simplicial action $\square$ such that $\square$**SIC** is isomorphic to **SIMC** (cf. Proposition 26).

[Define $X \square K$ by $(X \square K)_n = K_n \cdot X_n$ (thus for $\alpha : [m] \rightarrow [n]$, $K_n \cdot X_n \xrightarrow{X\alpha} K_n \cdot X_m \xrightarrow{K\alpha} K_m \cdot X_m$). The symbol $X \square K$ also has another connotation (cf. p. 13–41). To reconcile the ambiguity, note that there is a formality $\mathrm{in} : X \square K \rightarrow X \square K$, where $\mathrm{in}_n(k) : X_n \rightarrow (X \square K)_n$ is the injection from X_n to $K_n \cdot X_n$ corresponding to $k \in K_n$ (cf. p. 0–8). Moreover, $\mathrm{in}^* : \mathrm{Nat}(X \square K, Y) \rightarrow \mathrm{For}(X \square K, Y)$ is bijective and functorial. Therefore $\square$**SIC** and **SIMC** are isomorphic.]

[Note: $\square$ is the canonical simplicial action on **SIC**.]

EXAMPLE Let **I** be a small category—then there is an induced simplicial action on [**I**, **SISET**] ($(F \square K)i = Fi \times K$ (cf. p. 13–43)). And: $\mathrm{HOM}(F,G) \approx \int_i \mathrm{map}(Fi, Gi)$. In fact, $\mathrm{HOM}(F,G)_n \approx \mathrm{Nat}(F \square \Delta[n], G) \approx \int_i \mathrm{Nat}(Fi \times \Delta[n], Gi) \approx \int_i \mathrm{Nat}(\Delta[n], \mathrm{map}(Fi, Gi)) \approx \mathrm{Nat}(\Delta[n], \int_i \mathrm{map}(Fi, Gi)) \approx (\int_i \mathrm{map}(Fi, Gi))_n$.

A simplicial action $\square$ on a category **C** is said to be cartesian if $\forall\, X \in \mathrm{Ob}\,\mathbf{C}$, the functor $X \square \text{—} : \mathbf{SISET} \rightarrow \mathbf{C}$ has a right adjoint.

Example: Let **C** be a category. Assume: **C** has coproducts—then the canonical simplicial action $\square$ on **SIC** is cartesian.

[Let $\mathrm{HOM}(X,Y)$ be the simplicial set figuring in the definition of **SIMC**, so $\mathrm{HOM}(X,Y)_n = \mathrm{For}(X \square \Delta[n], Y)$ (cf. Proposition 26). Define $\mathrm{ev} \in \mathrm{For}(X \square \mathrm{HOM}(X,Y), Y)$ by $\mathrm{ev}_n(f) = f_n(\mathrm{id}_{[n]}) : X_n \rightarrow Y_n$. Viewing ev as "evaluation", there is an induced functorial bijection $\mathrm{Nat}(K, \mathrm{HOM}(X,Y)) \rightarrow \mathrm{For}(X \square K, Y)$. However, $\mathrm{For}(X \square K, Y) \approx \mathrm{Nat}(X \square K, Y)$ (cf. supra), hence $\square$ is cartesian.]

PROPOSITION 28 Suppose that the simplicial action $\square$ on **C** is cartesian—then $\forall\, X \in \mathrm{Ob}\,\mathbf{C}$, $\mathrm{HOM}(X, \text{—}) : \mathbf{C} \rightarrow \mathbf{SISET}$ is a right adjoint for $X \square \text{—}$.

[Given a simplicial set K, write $K = \mathrm{colim}_i\, \Delta[n_i]$: $\mathrm{Mor}(X \square K, Y) \approx \lim_i \mathrm{Mor}(X \square \Delta[n_i], Y) \approx \lim_i \mathrm{HOM}(X,Y)_{n_i} \approx \lim_i \mathrm{Nat}(\Delta[n_i], \mathrm{HOM}(X,Y)) \approx \mathrm{Nat}(K, \mathrm{HOM}(X,Y))$.]

A simplicial action $\square$ on a category **C** is said to be closed provided that it is cartesian and each of the functors $\text{—} \square K : \mathbf{C} \rightarrow \mathbf{C}$ has a right adjoint $X \rightarrow \textsc{hom}(K,X)$, so $\mathrm{Mor}(X \square K, Y) \approx \mathrm{Mor}(X, \textsc{hom}(K,Y))$.

[Note: The above defined simplicial actions on **SISET**, **CGH**, $\mathbf{SISET}_*$, and $\mathbf{CGH}_*$ are closed.]

If $\mathbf{C}$ admits a closed simplicial action, then $\mathbf{C}^{\text{OP}}$ admits a closed simplicial action.

Example: **GRD** admits a closed simplicial action: $\mathbf{G}\,\square\, K = \mathbf{G} \times \Pi K(\textsc{hom}(K,\mathbf{G}) = [\Pi K, \mathbf{G}])$.

[Note: Recall that $\Pi : \mathbf{SISET} \to \mathbf{GRD}$ preserves finite products (cf. p. 13–2).]

EXAMPLE If $\square$ is a closed simplicial action on $\mathbf{C}$, then for every small category $\mathbf{I}$, the composition $[\mathbf{I},\mathbf{C}] \times \mathbf{SISET} \to [\mathbf{I},\mathbf{C}] \times [\mathbf{I},\mathbf{SISET}] \approx [\mathbf{I},\mathbf{C} \times \mathbf{SISET}] \xrightarrow{[\mathbf{I},\square]} [\mathbf{I},\mathbf{C}]$ is a closed simplicial action on $[\mathbf{I},\mathbf{C}]$.

PROPOSITION 29 Suppose that the simplicial action $\square$ on $\mathbf{C}$ is closed—then $\text{HOM}(X\,\square\, K, Y) \approx \text{map}(K, \text{HOM}(X,Y)) \approx \text{HOM}(X, \textsc{hom}(K,Y))$.

FACT Let $\left\{\begin{array}{l}\mathbf{C}\\ \mathbf{D}\end{array}\right.$ be categories equipped with closed simplicial actions. Suppose that $\left\{\begin{array}{l}F:\mathbf{C}\to\mathbf{D}\\ G:\mathbf{D}\to\mathbf{C}\end{array}\right.$ are functors and (F,G) is an adjoint pair. Assume: $\forall\ X, \forall\ K : F(X\,\square\, K) \approx FX\,\square\, K$—then $\text{HOM}(FX,Y) \approx \text{HOM}(X,GY)$ and $G\textsc{hom}(K,Y) \approx \textsc{hom}(K,GY)$.

Notation: Given a category $\mathbf{C}$ and a simplicial object X in $\mathbf{C}$, write h_X for the cofunctor $\mathbf{C} \to \mathbf{SISET}$ defined by $(h_X A)_n = \text{Mor}\,(A, X_n)$.

[Note: For all X, Y in **SIC**, $\text{Nat}(X,Y) \approx \text{Nat}(h_X, h_Y)$ (simplicial Yoneda).]

PROPOSITION 30 Let $\mathbf{C}$ be a category. Assume: $\mathbf{C}$ has coproducts and is complete—then the canonical simplicial action $\square$ on **SIC** is closed ($\square$ is necessarily cartesian (cf. p. 13–44)).

[Given a simplicial set K, write $K \times \Delta[n] = \text{colim}_i\, \Delta[n_i] : \text{Nat}(K \times \Delta[n], h_Y A) \approx \lim_i \text{Nat}(\Delta[n_i], h_Y A) \approx \lim_i \text{Mor}\,(A, Y_{n_i}) \approx \text{Mor}\,(A, \lim_i Y_{n_i}) \approx \text{Mor}\,(A, \textsc{hom}(K,Y)_n)$, where by definition $\textsc{hom}(K,Y)_n = \lim_i Y_{n_i}$. In other words, $\textsc{hom}(K,Y)_n$ represents $A \to \text{Nat}(K \times \Delta[n], h_Y A)$. Varying n yields a simplicial object $\textsc{hom}(K,Y)$ in $\mathbf{C}$ with $h_{\textsc{hom}(K,Y)} \approx \text{map}(K, h_Y)$. Agreeing to let $h_X \,\square\, K$ be the cofunctor $\mathbf{C} \to \mathbf{SISET}$ that sends A to $h_X A \times K$, we have $\text{Nat}(X\,\square\, K, Y) \approx \text{Nat}(h_{X\square K}, h_Y) \approx \text{Nat}(h_X \,\square\, K, h_Y) \approx \text{Nat}(h_X, \text{map}(K, h_Y)) \approx \text{Nat}(h_X, h_{\textsc{hom}(K,Y)}) \approx \text{Nat}(X, \textsc{hom}(K,Y))$, which proves that $\square$ is closed.]

Example: The canonical simplicial action $\square$ on **SIGR** or **SIAB** is closed.

EXAMPLE (G-Sets) Fix a group G—then $\mathbf{SISET}_G$ admits a canonical simplicial action $\square$, viz. $X\,\square\, K = X \times K$, with trivial operations on K. In addition, $\square$ is closed, $\textsc{hom}(K,X)$ being $\text{map}(K,X)$ (operations in the target). Obviously, $F_G(X\,\square\, K) \approx F_G(X)\,\square\, K$.

A simplicial model category is a model category $\mathbf{C}$ equipped with a closed simplicial action $\square$ satisfying

(SMC) Suppose that $A \rightarrow Y$ is a cofibration and $X \rightarrow B$ is a fibration—then the arrow $\mathrm{HOM}(Y, X) \rightarrow \mathrm{HOM}(A, X) \times_{\mathrm{HOM}(A,B)} \mathrm{HOM}(Y, B)$ is a Kan fibration which is a weak homotopy equivalence if $A \rightarrow Y$ or $X \rightarrow B$ is acyclic.

Observation: It is clear that SMC $\Rightarrow$ MC–4. Indeed, the commutative diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d] & X \arrow[d] \\ Y \arrow[r] & B \end{tikzcd}$$

is a vertex of $\mathrm{HOM}(A, X) \times_{\mathrm{HOM}(A,B)} \mathrm{HOM}(Y, B)$, a filler $Y \rightarrow X$ is a preimage in $\mathrm{HOM}(Y, X)_0$, and acyclic Kan fibrations are surjective.

Example: **SISET**, **CGH**, $\mathbf{SISET}_*$, $\mathbf{CGH}_*$ are simplicial model categories.

[Note: **CGH** and $\mathbf{CGH}_*$ are taken in their singular structures (cf. p. 12–11).]

EXAMPLE Fix a small category $\mathbf{I}$—then the functor category $[\mathbf{I}, \mathbf{SISET}]$ is a simplicial model category (use structure L (cf. p. 13–37)).

EXAMPLE Fix an abelian group G and take **SISET** in the model category structure furnished by the homological model category theorem. Since every HG-fibration is a Kan fibration, it follows from Propositions 23 and 25 that **SISET** is a simplicial model category.

EXAMPLE (G-Sets) Fix a group G—then $\mathbf{SISET}_G$ is a simplicial model category (cf. p. 13–34).

In a simplicial model category $\mathbf{C}$: (1) $X \square \Delta[0] \approx X$; (2) $\mathrm{hom}(\Delta[0], X) \approx X$; (3) $\emptyset \square K \approx \emptyset$; (4) $\mathrm{hom}(K, *) \approx *$; (5) $\mathrm{HOM}(\emptyset, X) \approx \Delta[0]$; (6) $\mathrm{HOM}(X, *) \approx \Delta[0]$; (7) $X \square \emptyset \approx \emptyset$; (8) $\mathrm{hom}(\emptyset, X) \approx *$.

PROPOSITION 31 Suppose that $\square$ is a closed simplicial action on a model category $\mathbf{C}$—then $\mathbf{C}$ is a simplicial model category iff whenever $A \rightarrow Y$ is a cofibration in $\mathbf{C}$ and $L \rightarrow K$ is an inclusion of simplicial sets, the arrow $A \square K \underset{A \square L}{\sqcup} Y \square L \rightarrow Y \square K$ is a cofibration which is acyclic if $A \rightarrow Y$ or $L \rightarrow K$ is acyclic.

Application: Let $\mathbf{C}$ be a simplicial model category.

(i) Suppose that $A \rightarrow Y$ is a cofibration in $\mathbf{C}$—then for every simplicial set K, the arrow $A \square K \rightarrow Y \square K$ is a cofibration which is acyclic if $A \rightarrow Y$ is acyclic.

(ii) Suppose that Y is cofibrant and $L \rightarrow K$ is an inclusion of simplicial sets—then the arrow $Y \square L \rightarrow Y \square K$ is a cofibration which is acyclic if $L \rightarrow K$ is acyclic.

[Note: In particular, Y cofibrant $\Rightarrow Y \square K$ cofibrant.]

FACT Suppose that $\Box$ is a closed simplicial action on a model category **C**—then **C** is a simplicial model category iff whenever $A \to Y$ is a cofibration in **C**, the arrows $A\Box\Delta[n] \underset{A\Box\dot{\Delta}[n]}{\sqcup} Y\Box\dot{\Delta}[n] \to Y\Box\Delta[n]$ $(n \geq 0)$ are cofibrations which are acyclic if $A \to Y$ is acyclic and the arrows $A\Box\Delta[1] \underset{A\Box\Lambda[i,1]}{\sqcup} Y\Box\Lambda[i,1] \to Y\Box\Delta[1]$ $(i = 0, 1)$ are acyclic cofibrations.

PROPOSITION 32 Suppose that $\Box$ is a closed simplicial action on a model category **C**—then **C** is a simplicial model category iff whenever $L \to K$ is an inclusion of simplicial sets and $X \to B$ is a fibration in **C**, the arrow $\mathrm{HOM}(K,X) \to \mathrm{HOM}(L,X) \times_{\mathrm{HOM}(L,B)} \mathrm{HOM}(K,B)$ is a fibration which is acyclic if $L \to K$ or $X \to B$ is acyclic.

Application: Let **C** be a simplicial model category.

(i) Suppose that $L \to K$ is an inclusion of simplicial sets and X is fibrant—then the arrow $\mathrm{HOM}(K,X) \to \mathrm{HOM}(L,X)$ is a fibration which is acyclic if $L \to K$ is acyclic.

(ii) Suppose that $X \to B$ is a fibration in **C**—then for every simplicial set K, the arrow $\mathrm{HOM}(K,X) \to \mathrm{HOM}(K,B)$ is a fibration which is acyclic if $X \to B$ is acyclic.

[Note: In particular, X fibrant $\Rightarrow \mathrm{HOM}(K,X)$ fibrant.]

FACT Suppose that $\Box$ is a closed simplicial action on a model category **C**—then **C** is a simplicial model category iff whenever $X \to B$ is a fibration in **C**, the arrows $\mathrm{HOM}(\Delta[n],X) \to \mathrm{HOM}(\dot{\Delta}[n],X) \times_{\mathrm{HOM}(\dot{\Delta}[n],B)} \mathrm{HOM}(\Delta[n],B)$ $(n \geq 0)$ are fibrations which are acyclic if $X \to B$ is acyclic and the arrows $\mathrm{HOM}(\Delta[1],X) \to \mathrm{HOM}(\Lambda[i,1],X) \times_{\mathrm{HOM}(\Lambda[i,1],B)} \mathrm{HOM}(\Delta[1],B)$ $(i = 0,1)$ are acyclic fibrations.

Example: Let **C** be a category. Assume: **C** is complete and cocomplete and there is an adjoint pair (F,G), where $\begin{cases} F : \mathbf{SISET} \to \mathbf{SIC} \\ G : \mathbf{SIC} \to \mathbf{SISET} \end{cases}$, subject to the requirement that G preserves filtered colimits. Call a morphism $f : X \to Y$ a weak equivalence if Gf is a weak homotopy equivalence, a fibration if Gf is a Kan fibration, and a cofibration if f has the LLP w.r.t. acyclic fibrations—then **SIC** is a model category provided that every cofibration with the LLP w.r.t. fibrations is a weak equivalence (cf. infra). Claim: **SIC** is a simplicial model category ($\Box$ = canonical simplicial action (cf. Proposition 30)). To see this, note first that $F(X \times K) \approx FX\Box K$, hence $G\mathrm{HOM}(K,Y) \approx \mathrm{map}(K,GY)$ (cf. p. 13–45). Let now $L \to K$ be an inclusion of simplicial sets and $X \to B$ a fibration in **SIC**. Apply G to the arrow $\mathrm{HOM}(K,X) \to \mathrm{HOM}(L,X) \times_{\mathrm{HOM}(L,B)} \mathrm{HOM}(K,B)$ to get $G\mathrm{HOM}(K,X) \to G\mathrm{HOM}(L,X) \times_{G\mathrm{HOM}(L,B)} G\mathrm{HOM}(K,B)$ or still, $\mathrm{map}(K,GX) \to \mathrm{map}(L,GX) \times_{\mathrm{map}(L,GB)} \mathrm{map}(K,GB)$. Taking into account Proposition 22 and the definitions, the claim thus follows from Proposition 32.

[Note: Typically, such a setup is realized in "algebraic" situations (consider, e.g., **C** = **GR**). Consult Crans[†] for a variation on the overall procedure with applications to simplicial sheaves.]

The model category structure on **SIC** is produced by a small object argument. Thus one works with the $F\dot{\Delta}[n] \rightarrow F\Delta[n]$ $(n \geq 0)$ to show that every f can be written as the composite of a cofibration and an acyclic fibration and one works with the $F\Lambda[k,n] \rightarrow F\Delta[n]$ $(0 \leq k \leq n, n \geq 1)$ to show that every f can be written as the composite of a cofibration that has the LLP w.r.t. fibrations and a fibration. This leads to MC–5 under the assumption that every cofibration with the LLP w.r.t. fibrations is a weak equivalence, which is also needed to establish the nontrivial half of MC–4. In practice, this condition can be forced.

SUBLEMMA Let $\begin{cases} X \\ Y \end{cases}$ be topological spaces, $f : X \rightarrow Y$ a continuous function; let $\phi : X' \rightarrow X$, $\psi : Y \rightarrow Y'$ be continuous functions. Assume: $f \circ \phi$, $\psi \circ f$ are weak homotopy equivalences—then f is a weak homotopy equivalence.

LEMMA Suppose that there is a functor $T : \mathbf{SIC} \rightarrow \mathbf{SIC}$ and a natural transformation $\epsilon : \mathrm{id}_{\mathbf{SIC}} \rightarrow T$ such that $\forall\ X$, $\epsilon_X : X \rightarrow TX$ is a weak equivalence and $TX \rightarrow *$ is a fibration—then every cofibration with the LLP w.r.t. fibrations is a weak equivalence.

[Let $i : A \rightarrow Y$ be a cofibration with the stated properties. Fix a filler $w : Y \rightarrow TA$ for

$$\begin{tikzcd} A \arrow[r, "\epsilon_A"] \arrow[d, "i"'] & TA \arrow[d] \\ Y \arrow[r] & * \end{tikzcd}$$

. Consider the commutative diagram

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "i"'] & \textsc{hom}(\Delta[1], TY) \arrow[d, "\Pi"] \\ Y \arrow[r, "g"'] & \textsc{hom}(\dot{\Delta}[1], TY) \end{tikzcd}$$

, where f is the arrow $A \xrightarrow{i} Y \xrightarrow{\epsilon_Y} TY \approx \textsc{hom}(\Delta[0], TY) \rightarrow \textsc{hom}(\Delta[1], TY)$ and g is the arrow $\begin{cases} Y \xrightarrow{\epsilon_Y} TY \\ Y \xrightarrow{w} TA \xrightarrow{Ti} TY \end{cases}$ $(\textsc{hom}(\dot{\Delta}[1], TY) \approx TY \times TY)$. Since GTY is fibrant and $\begin{cases} G\textsc{hom}(\Delta[1], TY) \approx \mathrm{map}(\Delta[1], GTY) \\ G\textsc{hom}(\dot{\Delta}[1], TY) \approx \mathrm{map}(\dot{\Delta}[1], GTY) \end{cases}$, Π is a fibration (cf. p. 13–22), thus our diagram admits a filler $Y \rightarrow \textsc{hom}(\Delta[1], TY)$. This in turn implies that $Ti \circ w$ is a weak equivalence, i.e., $|GTi| \circ |Gw|$ is a weak homotopy equivalence. Assemble the data: $|GA| \xrightarrow{|Gi|} |GY| \xrightarrow{|Gw|} |GTA| \xrightarrow{|GTi|} |GTY|$. Because $|Gw| \circ |Gi| = |G\epsilon_A|$ is a weak homotopy equivalence, one can apply the sublemma and conclude that $|Gw|$ is a weak homotopy equivalence. Therefore $|Gi|$ is a weak homotopy equivalence which means by definition that i is a weak equivalence.]

EXAMPLE The hypotheses of the lemma are trivially met if $\forall\ X$, $X \rightarrow *$ is a fibration. So, for instance, **SIC** is a simplicial model category when **C** = **GR**, **AB**, or A-**MOD**, G being the forgetful functor.

[†] *J. Pure Appl. Algebra* **101** (1995), 35–57.

Retaining the supposition that **C** is complete and cocomplete, let us assume in addition that **C** has a set of separators and is cowellpowered. Given a simplicial object X in **C**, the cofunctor $\mathbf{C} \rightarrow \mathbf{SET}$ defined by $A \rightarrow (\mathrm{Ex}\,\mathrm{HOM}(A, X))_n$ $(n \geq 0)$ is representable (view A as a constant simplicial object). Indeed, $\mathrm{HOM}(—, X)$ converts colimits into limits and Ex preserves limits. The assertion is then a consequence of the special adjoint functor theorem. Accordingly, $\exists$ an object $(\mathrm{Ex}\,X)_n$ in **C** and a natural isomorphism $\mathrm{Mor}\,(A, (\mathrm{Ex}\,X)_n) \approx (\mathrm{Ex}\,\mathrm{HOM}(A, X))_n$. Thus there is a functor $\mathrm{Ex} : \mathbf{SIC} \rightarrow \mathbf{SIC}$, where $\forall\, X$, $\mathrm{Ex}\,X([n]) = (\mathrm{Ex}\,X)_n$ $(n \geq 0)$, with $\mathrm{HOM}(A, \mathrm{Ex}\,X) \approx \mathrm{Ex}\,\mathrm{HOM}(A, X)$ (since $\mathrm{HOM}(A, \mathrm{Ex}\,X)_n \approx \mathrm{Nat}(A\square\Delta[n], \mathrm{Ex}\,X) \approx \mathrm{Mor}\,(A, (\mathrm{Ex}\,X)_n) \approx (\mathrm{Ex}\,\mathrm{HOM}(A, X))_n$). Iterate to arrive at $\mathrm{Ex}^\infty : \mathbf{SIC} \rightarrow \mathbf{SIC}$ and $\epsilon^\infty : \mathrm{id}_{\mathbf{SIC}} \rightarrow \mathrm{Ex}^\infty$.

SMALL OBJECT CONSTRUCTION Fix a $P \in \mathrm{Ob}\,\mathbf{C}$ such that $\mathrm{Mor}\,(P, —) : \mathbf{C} \rightarrow \mathbf{SET}$ preserves filtered colimits. Viewing P as a constant simplicial object, define $G : \mathbf{SIC} \rightarrow \mathbf{SISET}$ by $GX = \mathrm{HOM}(P, X)$—then G has a left adjoint F, viz. $FK = P\square K$, and G preserves filtered colimits (for $(G\,\mathrm{colim}\,X_i)_n \approx \mathrm{HOM}(P, \mathrm{colim}\,X_i)_n \approx \mathrm{Nat}(P\square\Delta[n], \mathrm{colim}\,X_i) \approx \mathrm{Mor}\,(P, (\mathrm{colim}\,X_i)_n) \approx \mathrm{Mor}\,(P, \mathrm{colim}(X_i)_n) \approx \mathrm{colim}\,\mathrm{Mor}\,(P, (X_i)_n) \approx \mathrm{colim}\,\mathrm{Nat}(P\square\Delta[n], X_i) \approx \mathrm{colim}\,\mathrm{HOM}(P, X_i)_n \approx (\mathrm{colim}\,GX_i)_n$). In the lemma, take $T = \mathrm{Ex}^\infty$, $\epsilon = \epsilon^\infty$. Because $\mathrm{HOM}(P, \mathrm{Ex}^\infty X) \approx \mathrm{HOM}(P, \mathrm{colim}\,\mathrm{Ex}^n X)$ $\approx \mathrm{colim}\,\mathrm{HOM}(P, \mathrm{Ex}^n X) \approx \mathrm{Ex}^\infty \mathrm{HOM}(P, X)$, it follows that $\forall\, X$, $\epsilon_X^\infty : X \rightarrow \mathrm{Ex}^\infty X$ is a weak equivalence (cf. p. 13–12) and $\mathrm{Ex}^\infty X \rightarrow *$ is a fibration (cf. p. 13–21). Therefore **SIC** admits the structure of a simplicial model category in which a morphism $f : X \rightarrow Y$ is a weak equivalence or a fibration if this is the case of the simplicial map $f_* : \mathrm{HOM}(P, X) \rightarrow \mathrm{HOM}(P, Y)$.

EXAMPLE In the small object construction, take $\mathbf{C} = \mathbf{SISET}$—then every finite simplicial set P determines a simplicial model category structure on $[\mathbf{\Delta}^{\mathrm{OP}}, \mathbf{SISET}]$.

PROPOSITION 33 Let X, Y, and Z be objects in a simplicial model category **C**.

(i) If $f : X \rightarrow Y$ is an acyclic cofibration and Z is fibrant, then $f^* : \mathrm{HOM}(Y, Z) \rightarrow \mathrm{HOM}(X, Z)$ is a weak homotopy equivalence.

(ii) If $g : Y \rightarrow Z$ is an acyclic fibration and X is cofibrant, then $g_* : \mathrm{HOM}(X, Y) \rightarrow \mathrm{HOM}(X, Z)$ is a weak homotopy equivalence.

PROPOSITION 34 Let X, Y, and Z be objects in a simplicial model category **C**.

(i) If $f : X \rightarrow Y$ is a weak equivalence between cofibrant objects and Z is fibrant, then $f^* : \mathrm{HOM}(Y, Z) \rightarrow \mathrm{HOM}(X, Z)$ is a weak homotopy equivalence.

(ii) If $g : Y \rightarrow Z$ is a weak equivalence between fibrant objects and X is cofibrant, then $g_* : \mathrm{HOM}(X, Y) \rightarrow \mathrm{HOM}(X, Z)$ is a weak homotopy equivalence.

[Use Proposition 33 and the lemma prefacing the proof of the TDF theorem.]

EXAMPLE Take $\mathbf{C} = \mathbf{CGH}$ (singular structure)—then all objects are fibrant, so if $g : Y \rightarrow Z$ is a weak homotopy equivalence and X is cofibrant, $g_* : \mathrm{HOM}(X, Y) \rightarrow \mathrm{HOM}(X, Z)$ is a weak homotopy equivalence. But $\mathrm{HOM}(X,Y) \approx \sin(\mathrm{map}(X,Y))$, $\mathrm{HOM}(X,Z) \approx \sin(\mathrm{map}(X,Z))$, thus $g_* : \mathrm{map}(X,Y) \to \mathrm{map}(X,Z)$ is a weak homotopy equivalence (cf. p. 13–17).

[Note: Contrast this approach with that used on p. 9–39.]

Let $i : A \to Y$, $p : X \to B$ be morphisms in a simplicial model category **C**. Assume: i is a cofibration and p is a fibration—then i is said to have the homotopy left lifting property with respect to p (HLLP w.r.t. p) and p is said to have the homotopy right lifting property with respect to i (HRLP w.r.t. i) if the arrow $\mathrm{HOM}(Y,X) \to \mathrm{HOM}(A,X) \times_{\mathrm{HOM}(A,B)} \mathrm{HOM}(Y,B)$ is a weak homotopy equivalence.

FACT Given a cofibration $i : A \to Y$ and a fibration $p : X \to B$ in a simplicial model category **C**, each of the following conditions is equivalent to i having the HLLP w.r.t p and p having the HRLP w.r.t. i.

(1) If $L \to K$ is an inclusion of simplicial sets, then p has the RLP w.r.t. the arrow $A \square K \underset{A \square L}{\sqcup} Y \square L \to Y \square K$.

(2) The fibration p has the RLP w.r.t. the arrows $A \square \Delta[n] \underset{A \square \dot{\Delta}[n]}{\sqcup} Y \square \dot{\Delta}[n] \to Y \square \Delta[n]$ $(n \geq 0)$.

(3) If $L \to K$ is an inclusion of simplicial sets, then i has the LLP w.r.t. the arrow $\textsc{hom}(K,X) \to \textsc{hom}(L,X) \times_{\textsc{hom}(L,B)} \textsc{hom}(K,B)$.

(4) The cofibration i has the LLP w.r.t. the arrows $\textsc{hom}(\Delta[n],X) \to \textsc{hom}(\dot{\Delta}[n],X) \times_{\textsc{hom}(\dot{\Delta}[n],B)} \textsc{hom}(\Delta[n],B)$ $(n \geq 0)$.

Let **C** be a simplicial model category. Agreeing to identify $\mathrm{Mor}(X,Y)$ and $\mathrm{HOM}(X,Y)_0$, one may transfer from **SISET** to **C** the notions of homotopic $(f \simeq g)$ and simplicially homotopic $(f \underset{s}{\simeq} g)$ leading thereby to $\mathbf{H_0C}$ (thus $[X,Y]_0 = \mathrm{Mor}(X,Y)/\simeq\ (\equiv \pi_0(\mathrm{HOM}(X,Y)))$.

[Note: $\mathrm{Mor}(X \square I_{2n}, Y) \approx \mathrm{Nat}(I_{2n}, \mathrm{HOM}(X,Y)) \approx \mathrm{Mor}(X, \textsc{hom}(I_{2n},Y))$ and $\mathrm{Mor}(X \square \Delta[1], Y) \approx \mathrm{Nat}(\Delta[1], \mathrm{HOM}(X,Y)) \approx \mathrm{Mor}(X, \textsc{hom}(\Delta[1],Y))$.]

Example: Suppose that $i : A \to Y$ is a cofibration and $p : X \to B$ is a fibration. Assume: i has the HLLP w.r.t. p—then every commutative diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d] & X \arrow[d] \\ Y \arrow[r] & B \end{tikzcd}$$

has a filler and any two such are homotopic.

PROPOSITION 35 Let **C** be a simplicial model category. Suppose that $f \simeq g$—then f, g are left homotopic and right homotopic.

[Note: Therefore $Qf = Qg$ (cf. p. 12–25). Corollary: A homotopy equivalence in **C** is a weak equivalence (but not conversely).]

PROPOSITION 36 Let **C** be a simplicial model category. Assume: X is cofibrant and Y is fibrant—then the relations of homotopy, simplicial homotopy, left homotopy, and right homotopy on $\operatorname{Mor}(X, Y)$ coincide and are equivalence relations. Therefore "homotopy is homotopy" and $[X, Y]_0 \leftrightarrow [X, Y]$.

[Note: $\mathrm{HOM}(X, Y)$ is necessarily fibrant (cf. SMC).]

EXAMPLE Under the assumption that X is cofibrant and Y is fibrant, $[X \square K, Y] \approx [K, \mathrm{HOM}(X, Y)] \approx [X, \mathrm{hom}(K, Y)]$.

[Note: Bear in mind that $X \square K$ is cofibrant (cf. p. 13–46) and $\mathrm{hom}(K, Y)$ is fibrant (cf. p. 13–47).]

PROPOSITION 37 Let X, Y, and Z be objects in a simplicial model category **C**.

(i) Let $f \in \operatorname{Mor}(X, Y)$—then the homotopy class of the precomposition arrow $f^* : \mathrm{HOM}(Y, Z) \to \mathrm{HOM}(X, Z)$ depends only on the homotopy class of f.

[Note: Thus f^* is a homotopy equivalence of simplicial sets if f is a homotopy equivalence.]

(ii) Let $g \in \operatorname{Mor}(Y, Z)$—then the homotopy class of the postcomposition arrow $g_* : \mathrm{HOM}(X, Y) \to \mathrm{HOM}(X, Z)$ depends only on the homotopy class of g.

[Note: Thus g_* is a homotopy equivalence of simplicial sets if g is a homotopy equivalence.]

PROPOSITION 38 Suppose that **C** is a simplicial model category. Let $f \in \operatorname{Mor}(X, Y)$. Assume: The precomposition arrows $\begin{cases} \mathrm{HOM}(Y, X) \to \mathrm{HOM}(X, X) \\ \mathrm{HOM}(Y, Y) \to \mathrm{HOM}(X, Y) \end{cases}$ are weak homotopy equivalences—then f is a homotopy equivalence.

[Note: The result can also be formulated in terms of the postcomposition arrows $\begin{cases} \mathrm{HOM}(X, X) \to \mathrm{HOM}(X, Y) \\ \mathrm{HOM}(Y, X) \to \mathrm{HOM}(Y, Y) \end{cases}$.]

PROPOSITION 39 Let **C** be a simplicial model category—then a morphism $f : X \to Y$ is a weak equivalence if $\forall$ fibrant Z, the precomposition arrow $f^* : \mathrm{HOM}(Y, Z) \to \mathrm{HOM}(X, Z)$ is a weak homotopy equivalence.

[Using the notation of Lemma $\mathcal{R}$ (cf. p. 12–23), consider the commutative diagram

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\iota_X"'] & Y \arrow[d, "\iota_Y"] \\ \mathcal{R}X \arrow[r, "\mathcal{R}f"'] & \mathcal{R}Y \end{tikzcd}$$

and apply $\mathrm{HOM}(—, Z)$ to get

$$\begin{tikzcd} \mathrm{HOM}(X, Z) & \mathrm{HOM}(Y, Z) \arrow[l] \\ \mathrm{HOM}(\mathcal{R}X, Z) \arrow[u] & \mathrm{HOM}(\mathcal{R}Y, Z) \arrow[l] \arrow[u] \end{tikzcd}$$

(Z fibrant). Since $\begin{cases} \iota_X \\ \iota_Y \end{cases}$ are acyclic cofibrations, the vertical arrows are weak homotopy equivalences (cf. Proposition 33). Taking into account the hypothesis, it follows that

$(\mathcal{R}f)^*: \text{HOM}(\mathcal{R}Y, Z) \to \text{HOM}(\mathcal{R}X, Z)$ is a weak homotopy equivalence. But $\left\{\begin{matrix}\mathcal{R}X \\ \mathcal{R}Y\end{matrix}\right.$ are fibrant, so one can let $Z = \mathcal{R}X, \mathcal{R}Y$ and conclude that $\mathcal{R}f$ is a homotopy equivalence (cf. Proposition 38), hence a weak equivalence (cf. Proposition 35). Therefore f is a weak equivalence (cf. Lemma $\mathcal{R}$).]

[Note: The result can also be formulated in terms of the postcomposition arrows $f_*: \text{HOM}(Z, X) \to \text{HOM}(Z, Y)$ (Z cofibrant).]

Application: Let **C** be a simplicial model category. Suppose that $f: X \to Y$ is a weak equivalence between cofibrant objects—then $\forall\, K$, $f \square \text{id}_K : X \square K \to Y \square K$ is a weak equivalence between cofibrant objects (cf. p. 13–46).

[Take any fibrant Z and consider the arrow $\text{HOM}(Y \square K, Z) \to \text{HOM}(X \square K, Z)$ or still, the arrow $\text{HOM}(Y, \textsc{hom}(K, Z)) \to \text{HOM}(X, \textsc{hom}(K, Z))$. Because $\textsc{hom}(K, Z)$ is fibrant (cf. p. 13–47), the latter is a weak homotopy equivalence (cf. Proposition 34), so by the above, the arrow $X \square K \to Y \square K$ is a weak equivalence.]

EXAMPLE Fix a small category **I** and view the functor category $[\mathbf{I}^{\text{OP}}, \mathbf{SISET}]$ as a simplicial model category (cf. p. 13–46). Suppose that $L \to K$ is a weak equivalence, where $L, K : \mathbf{I}^{\text{OP}} \to \mathbf{SISET}$ are cofibrant—then $\forall\, F: \mathbf{I} \to \mathbf{SISET}$, the induced map $\int^i Fi \times Li \to \int^i Fi \times Ki$ of simplicial sets is a weak homotopy equivalence. To see this, use Proposition 39. Thus take any fibrant Z and consider the arrow $\text{map}(\int^i Fi \times Ki, Z) \to \text{map}(\int^i Fi \times Li, Z)$, i.e., the arrow $\int_i \text{map}(Fi \times Ki, Z) \to \int_i \text{map}(Fi \times Li, Z)$, i.e., the arrow $\int_i \text{map}(Ki, \text{map}(Fi, Z)) \to \int_i \text{map}(Li, \text{map}(Fi, Z))$, i.e., the arrow $\text{HOM}(K, \text{map}(F, Z)) \to \text{HOM}(L, \text{map}(F, Z))$ (cf. p. 13–44), which is a weak homotopy equivalence (cf. Proposition 34).

[Note: Here, $\text{map}(F, Z)$ is the functor $\mathbf{I}^{\text{OP}} \to \mathbf{SISET}$ defined by $i \to \text{map}(Fi, Z)$, thus $\text{map}(F, Z)$ is a fibrant object in $[\mathbf{I}^{\text{OP}}, \mathbf{SISET}]$.]

Let $\rho: A \to B$ be an inclusion of simplicial sets—then a fibrant object Z in **SISET** is said to be ρ-local if $\rho^*: \text{map}(B, Z) \to \text{map}(A, Z)$ is a weak homotopy equivalence.

[Note: Since Z is fibrant, ρ^* is actually a simplicial homotopy equivalence (cf. Proposition 20).]

Imitating the (A, B) construction in §9 (cf. p. 9–43 ff.), one can show that there is a functor $L_\rho : \mathbf{SISET} \to \mathbf{SISET}$ and a natural transformation $\text{id} \to L_\rho$, where $\forall\, X$, $L_\rho X$ is ρ-local and $l_\rho : X \to L_\rho X$ is a cofibration such that for all ρ-local Z, the arrow $\text{map}(L_\rho X, Z) \to \text{map}(X, Z)$ is a weak homotopy equivalence. Consequently, the full subcategory of $\mathbf{H_0SISET}$ whose objects are ρ-local is reflective.

[Note: Observe that it is necessary to work not only with the $A \times \Delta[n] \underset{A \times \dot{\Delta}[n]}{\sqcup} B \times \dot{\Delta}[n] \rightarrow B \times \Delta[n]$ $(n \geq 0)$ but also with the $\Lambda[k,n] \rightarrow \Delta[n]$ $(0 \leq k \leq n, n \geq 1)$ (this to ensure that $L_\rho X$ is fibrant).]

LEMMA Let $f : X \rightarrow Y$ be a cofibration in **SISET**. Assume: $\forall$ ρ-local Z, $f^* : \operatorname{map}(Y,Z) \rightarrow \operatorname{map}(X,Z)$ is a weak homotopy equivalence—then $L_\rho f : L_\rho X \rightarrow L_\rho Y$ is a homotopy equivalence.

[Pass from

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d] & Y \arrow[d] \\ L_\rho X \arrow[r, "L_\rho f"'] & L_\rho Y \end{tikzcd}$$

to

$$\begin{tikzcd} \operatorname{map}(X,Z) & \operatorname{map}(Y,Z) \arrow[l] \\ \operatorname{map}(L_\rho X,Z) \arrow[u] & \operatorname{map}(L_\rho Y,Z) \arrow[l] \arrow[u] \end{tikzcd}$$

(Z ρ-local), take $Z = L_\rho X, L_\rho Y$, and quote Proposition 38.]

Application: Suppose that $f : X \rightarrow Y$ is an acyclic cofibration—then $L_\rho f : L_\rho X \rightarrow L_\rho Y$ is a homotopy equivalence.

[Note: Therefore L_ρ : **SISET** $\rightarrow$ **SISET** preserves weak homotopy equivalences (cf. p. 12–28) (all objects are cofibrant), hence $\mathbf{L}L_\rho$: **HSISET** $\rightarrow$ **HSISET** exists (cf. §12, Proposition 14).]

EXAMPLE Fix an inclusion $\rho : A \rightarrow B$ of simplicial sets. Let $f : X \rightarrow Y$ be a simplicial map—then f is said to be a ρ-equivalence if $L_\rho f : L_\rho X \rightarrow L_\rho Y$ is a homotopy equivalence (or just a weak homotopy equivalence (cf. Proposition 20)). Agreeing that a ρ-cofibration is an injective simplicial map, a ρ-fibration is a simplicial map which has the RLP w.r.t. all ρ-cofibrations that are ρ-equivalences. Every ρ-fibration is a Kan fibration (cf. supra). This said, **SISET** acquires the structure of a simplicial model category by letting weak equivalence = ρ-equivalence, cofibration = ρ-cofibration, fibration = ρ-fibration.

[Note: The fibrant objects in this structure are the ρ-local objects.]

Let **C** be a complete and cocomplete category—then in the notation of p. 0–18, the truncation $\operatorname{tr}^{(n)}$: **SIC** $\rightarrow$ $\mathbf{SIC}_n$ has a left adjoint $\operatorname{sk}^{(n)}$: $\mathbf{SIC}_n \rightarrow$ **SIC**, where $\forall$ X in $\mathbf{SIC}_n$, $(\operatorname{sk}^{(n)} X)_m = \underset{\substack{[m] \rightarrow [k] \\ k \leq n}}{\operatorname{colim}} X_k$, and a right adjoint $\operatorname{cosk}^{(n)}$: $\mathbf{SIC}_n \rightarrow$ **SIC**, where $\forall$ X in $\mathbf{SIC}_n$, $(\operatorname{cosk}^{(n)} X)_m = \lim\limits_{\substack{[k] \rightarrow [m] \\ k \leq n}} X_k$.

[Note: The colimit and limit are taken over a comma category.]

EXTENSION PRINCIPLE (OBJECTS) Let X be an object in $\mathbf{SIC}_n$—then a factorization $(\operatorname{sk}^{(n)} X)_{n+1} \rightarrow X_{n+1} \rightarrow (\operatorname{cosk}^{(n)} X)_{n+1}$ of the arrow $(\operatorname{sk}^{(n)} X)_{n+1} \rightarrow (\operatorname{cosk}^{(n)} X)_{n+1}$ determines an extension of X to an object in $\mathbf{SIC}_{n+1}$.

EXTENSION PRINCIPLE (MORPHISMS) Let $\begin{cases} X \\ Y \end{cases}$ be objects in $\mathbf{SIC}_{n+1}$; let $f : X|\Delta_n^{\text{OP}} \rightarrow Y|\Delta_n^{\text{OP}}$ be a morphism—then an arrow $X_{n+1} \rightarrow Y_{n+1}$ determines an extension $F : X \rightarrow Y$ of f provided that

$$\begin{tikzcd}
(\mathrm{sk}^{(n)}X)_{n+1} \arrow[r] \arrow[d] & X_{n+1} \arrow[r] \arrow[d] & (\mathrm{cosk}^{(n)}X)_{n+1} \arrow[d] \\
(\mathrm{sk}^{(n)}Y)_{n+1} \arrow[r] & Y_{n+1} \arrow[r] & (\mathrm{cosk}^{(n)}Y)_{n+1}
\end{tikzcd}$$

commutes in $\mathbf{C}$.

Let X be a simplicial object in $\mathbf{C}$. Recall that $sk^{(n)}X = \mathrm{sk}^{(n)}(\mathrm{tr}^{(n)}X)$ and $cosk^{(n)}X = \mathrm{cosk}^{(n)}\,(\mathrm{tr}^{(n)}X)$ (cf. p. 0–18).

(L) The latching object of X at $[n]$ is $L_nX = (sk^{(n-1)}X)_n$ and the latching morphism is the arrow $L_nX \rightarrow X_n$.

(M) The matching object of X at $[n]$ is $M_nX = (cosk^{(n-1)}X)_n$ and the matching morphism is the arrow $X_n \rightarrow M_nX$.

[Note: The connecting morphism of X at $[n]$ is the composite $L_nX \rightarrow X_n \rightarrow M_nX$.]

In particular: L_0X is an initial object in $\mathbf{C}$ and M_0X is a final object in $\mathbf{C}$.

PROPOSITION 40 Let $\mathbf{C}$ be a complete and cocomplete model category. Suppose that $f : X \rightarrow Y$ is a morphism in $\mathbf{SIC}$ such that $\forall\ n$, the arrow $X_n \underset{L_nX}{\sqcup} L_nY \rightarrow Y_n$ is a cofibration (acyclic cofibration) in $\mathbf{C}$—then $\forall\ n$, $L_nf : L_nX \rightarrow L_nY$ is a cofibration (acyclic cofibration) in $\mathbf{C}$.

[One checks by induction that L_nf has the LLP w.r.t. acyclic fibrations (fibrations) in $\mathbf{C}$.]

[Note: There is a parallel statement for fibrations (acyclic fibrations) involving the arrows $X_n \rightarrow M_nX \times_{M_nY} Y_n$.]

PROPOSITION 41 Let $\mathbf{C}$ be a complete and cocomplete model category. Suppose that $f : X \rightarrow Y$ is a morphism in $\mathbf{SIC}$ such that $\forall\ n$, the arrow $X_n \underset{L_nX}{\sqcup} L_nY \rightarrow Y_n$ $(X_n \rightarrow M_nX \times_{M_nY} Y_n)$ is a cofibration (fibration) in $\mathbf{C}$—then $\forall\ n$, $f_n : X_n \rightarrow Y_n$ is a cofibration (fibration) in $\mathbf{C}$.

[Consider the pushout square

$$\begin{tikzcd}
L_nX \arrow[r] \arrow[d] & L_nY \arrow[d] \\
X_n \arrow[r] & X_n \underset{L_nX}{\sqcup} L_nY
\end{tikzcd}$$

. Owing to Proposition 40, the arrow $L_nX \rightarrow L_nY$ is a cofibration. Therefore the arrow $X_n \rightarrow X_n \underset{L_nX}{\sqcup} L_nY$ is a cofibration. But f_n is the composite $X_n \rightarrow X_n \underset{L_nX}{\sqcup} L_nY \rightarrow Y_n$.]

PROPOSITION 42 Let $\mathbf{C}$ be a complete and cocomplete model category. Suppose that $f : X \rightarrow Y$ is a morphism in $\mathbf{SIC}$ such that $\forall\ n$, $f_n : X_n \rightarrow Y_n$ is a weak equivalence in **C** and the arrow $X_n \underset{L_nX}{\sqcup} L_nY \rightarrow Y_n$ is a cofibration in **C**—then $\forall\ n$, the arrow $X_n \underset{L_nX}{\sqcup} L_nY \rightarrow Y_n$ is an acyclic cofibration in **C**.

[One checks by induction that L_nf has the LLP w.r.t. fibrations in **C**.]

[Note: There is a parallel statement for fibrations involving the arrows $X_n \rightarrow M_nX \times_{M_nY} Y_n$.]

Let **C** be a complete and cocomplete model category. Given a morphism $f : X \rightarrow Y$ in **SIC**, call f a weak equivalence if $\forall\ n$, $f_n : X_n \rightarrow Y_n$ is a weak equivalence in **C**, a cofibration if $\forall\ n$, the arrow $X_n \underset{L_nX}{\sqcup} L_nY \rightarrow Y_n$ is a cofibration in **C**, a fibration if $\forall\ n$, the arrow $X_n \rightarrow M_nX \times_{M_nY} Y_n$ is a fibration in **C**. This structure is the Reedy structure on **SIC**.

REEDY MODEL CATEGORY THEOREM Let **C** be a complete and cocomplete (proper) model category—then **SIC** in the Reedy structure is a (proper) model category.

[The crux of the matter is the verification of MC–4 and MC–5. However, due to the extension principle, the requisite liftings and factorizations can be constructed via induction, using Propositions 40, 41, and 42.]

[Note: Suppose further that **C** is a simplicial model category—then **SIC** is a simplicial model category. In fact, **SIC** admits a closed simplicial action derived from that on **C** (cf. p. 13–45), so it suffices to verify that SMC holds. For this, it is convenient to employ Proposition 31. Thus let $X \rightarrow Y$ be a cofibration in **SIC** and $L \rightarrow K$ an inclusion of simplicial sets. Claim: The arrow $X \square K \underset{X \square L}{\sqcup} Y \square L \rightarrow Y \square K$ is a cofibration which is acyclic if $X \rightarrow Y$ or $L \rightarrow K$ is acyclic. Fix n and consider the arrow $(X \square K \underset{X \square L}{\sqcup} Y \square L)_n \sqcup_{L_n(X \square K \underset{X \square L}{\sqcup} Y \square L)} L_n(Y \square K) \rightarrow (Y \square K)_n$ or, equivalently, the arrow $(X_n \underset{L_nX}{\sqcup} L_nY) \square K \sqcup_{(X_n \underset{L_nX}{\sqcup} L_nY) \square L} Y_n \square L \rightarrow Y_n \square K$. On the other hand, the canonical simplicial action $\square$ on **SIC** need not be compatible with the Reedy structure on **SIC**. Thus let $X \rightarrow Y$ be a cofibration in **SIC** and consider the arrows $X \square \Delta[1] \underset{X \square \Lambda[i,1]}{\sqcup} Y \square \Lambda[i,1] \rightarrow Y \square \Delta[1]$ $(i = 0, 1)$ (cf. p. 13–47). While cofibrations, they need not be weak equivalences.]

EXAMPLE Take **C** = **TOP**$_*$ (singular structure)—then according to Dwyer-Kan-Stover[†] there is a model category structure on **SITOP**$_*$ having for its weak equivalences those $f : X \rightarrow Y$ such that $\forall\ n \geq 1$, $f_* : \pi_n(X) \rightarrow \pi_n(Y)$ is a weak equivalence of simplicial groups. Obviously, every weak equivalence in the Reedy structure is a weak equivalence in this structure (but not conversely).

[†] *J. Pure Appl. Algebra* **90** (1993), 137–152; see also *J. Pure Appl. Algebra* **103** (1995), 167–188.

The functor category $[\Delta^{OP}, \mathbf{SISET}]$ carries two other proper model category structures (cf. p. 13–37). Every cofibration in the Reedy structure is a cofibration in structure R and every fibration in the Reedy structure is a fibration in structure L (cf. Proposition 41). Therefore every fibration in structure R is a fibration in the Reedy structure and every cofibration in structure L is a cofibration in the Reedy structure.

[Note: In reality, the cofibrations in the Reedy structure are precisely the levelwise injective simplicial maps, thus the Reedy structure is structure R.]

$\mathbf{\Gamma}$ is the category whose objects are the finite sets $\mathbf{n} \equiv \{0, 1, \ldots, n\}$ $(n \geq 0)$ with base point 0 and whose morphisms are the base point preserving maps.

[Note: Suppose that $\gamma : \mathbf{m} \to \mathbf{n}$ is a morphism in $\mathbf{\Gamma}$—then the partition $\coprod\limits_{0 \leq j \leq n} \gamma^{-1}(j) = \mathbf{m}$ of $\mathbf{m}$ determines a permutation $\theta : \mathbf{m} \to \mathbf{m}$ such that $\gamma \circ \theta$ is order preserving. Therefore γ has a unique factorization of the form $\alpha \circ \sigma$, where $\alpha : \mathbf{m} \to \mathbf{n}$ is order preserving and $\sigma : \mathbf{m} \to \mathbf{m}$ is a base point preserving permutation which is order preserving in the fibers of γ.]

Notation: Write $\mathbf{\Gamma SISET_*}$ for the full subcategory of $[\mathbf{\Gamma}, \mathbf{SISET_*}]$ whose objects are the $X : \mathbf{\Gamma} \to \mathbf{SISET_*}$ such that $X_0 = *$ $(X_n = X(\mathbf{n}))$.

EXAMPLE Let G be an abelian semigroup with unit. Using additive notation, view G^n as the set of base point preserving functions $\mathbf{n} \to G$—then the rule $X_n = \mathrm{si}G^n$ defines an object in $\mathbf{\Gamma SISET_*}$. Here the arrow $G^m \to G^n$ attached to $\gamma : \mathbf{m} \to \mathbf{n}$ sends $(g_1, \ldots, g_m)$ to $(\bar{g}_1, \ldots, \bar{g}_n)$, where $\bar{g}_j = \sum\limits_{\gamma(i)=j} g_i$ if $\gamma^{-1}(j) \neq \emptyset$, $\bar{g}_j = 0$ if $\gamma^{-1}(j) = \emptyset$.

Let $S_n(\mathbf{SISET_*})$ be the category whose objects are the pointed simplicial left S_n-sets—then $S_n(\mathbf{SISET_*})$ is a simplicial model category (cf. p. 13–46).

[Note: The group of base point preserving permutations $\mathbf{n} \to \mathbf{n}$ is S_n and for any X in $\mathbf{\Gamma SISET_*}$, X_n is a pointed simplicial left S_n-set.]

Let $\mathbf{\Gamma}_n$ be the full subcategory of $\mathbf{\Gamma}$ whose objects are the $\mathbf{m}$ $(m \leq n)$. Assigning to the symbol $\mathbf{\Gamma}_n\mathbf{SISET_*}$ the obvious interpretation, one can follow the usual procedure and introduce $\mathrm{tr}^{(n)} : \mathbf{\Gamma SISET_*} \to \mathbf{\Gamma}_n\mathbf{SISET_*}$ and its left (right) adjoint $\mathrm{sk}^{(n)}$ $(\mathrm{cosk}^{(n)})$ (cf. p. 0–18). Put $sk^{(n)} = \mathrm{sk}^{(n)} \circ \mathrm{tr}^{(n)}$ (the n-skeleton), $cosk^{(n)} = \mathrm{cosk}^{(n)} \circ \mathrm{tr}^{(n)}$ (the n-coskeleton).

EXTENSION PRINCIPLE (OBJECTS) Let X be an object in $\mathbf{\Gamma}_n\mathbf{SISET_*}$—then a factorization $(\mathrm{sk}^{(n)}X)_{n+1} \to X_{n+1} \to (\mathrm{cosk}^{(n)}X)_{n+1}$ of the arrow $(\mathrm{sk}^{(n)}X)_{n+1} \to (\mathrm{cosk}^{(n)}X)_{n+1}$ in $S_{n+1}(\mathbf{SISET_*})$ determines an extension of X to an object in $\mathbf{\Gamma}_{n+1}\mathbf{SISET_*}$.

EXTENSION PRINCIPLE (MORPHISMS) Let $\begin{cases} X \\ Y \end{cases}$ be objects in $\mathbf{\Gamma}_{n+1}\mathbf{SISET_*}$; let $f : X|\mathbf{\Gamma}_n \to Y|\mathbf{\Gamma}_n$ be a morphism—then an S_{n+1}-equivariant arrow $X_{n+1} \to Y_{n+1}$ determines an extension $F : X \rightarrow Y$ of f provided that

$$\begin{tikzcd}(\mathrm{sk}^{(n)}X)_{n+1} \arrow[r] \arrow[d] & X_{n+1} \arrow[r] \arrow[d] & (\mathrm{cosk}^{(n)}X)_{n+1} \arrow[d] \\ (\mathrm{sk}^{(n)}Y)_{n+1} \arrow[r] & Y_{n+1} \arrow[r] & (\mathrm{cosk}^{(n)}Y)_{n+1}\end{tikzcd}$$

commutes in $S_{n+1}(\mathbf{SISET}_*)$.

Given an X in $\mathbf{\Gamma SISET}_*$, write $L_nX = (sk^{(n-1)}X)_n$, $M_nX = (cosk^{(n-1)}X)_n$ for the latching, matching objects of X at $\mathbf{n}$ (cf. p. 13–54).

Given a morphism $f : X \rightarrow Y$, call f a weak equivalence if $\forall\ n \geq 1$, $f_n : X_n \rightarrow Y_n$ is a weak equivalence in $S_n(\mathbf{SISET}_*)$, a cofibration if $\forall\ n \geq 1$, the arrow $X_n \underset{L_nX}{\sqcup} L_nY \rightarrow Y_n$ is a cofibration in $S_n(\mathbf{SISET}_*)$, a fibration if $\forall\ n \geq 1$, the arrow $X_n \rightarrow M_nX \times_{M_nY} Y_n$ is a fibration in $S_n(\mathbf{SISET}_*)$. This structure is the Reedy structure on $\mathbf{\Gamma SISET}_*$.

BOUSFIELD-FRIEDLANDER MODEL CATEGORY THEOREM $\mathbf{\Gamma SISET}_*$ in the Reedy structure is a proper simplicial model category.

Observation: The opposite of a model category is a model category (cf. p. 12–3). So, if $\mathbf{C}$ is a complete and cocomplete model category, then by the above $[\mathbf{\Delta}^{\mathrm{OP}}, \mathbf{C}^{\mathrm{OP}}]$ is a model category. Therefore $[\mathbf{\Delta}^{\mathrm{OP}}, \mathbf{C}^{\mathrm{OP}}]^{\mathrm{OP}}$ is a model category, i.e., **COSIC** is a model category (Reedy structure).

EXAMPLE Take $\mathbf{C} = \mathbf{SISET}$—then the class of weak equivalences in $[\mathbf{\Delta}, \mathbf{SISET}]$ (Reedy structure) is the same as the class of weak equivalences in $[\mathbf{\Delta}, \mathbf{SISET}]$ (structure L (cf. p. 13–37)) but the class of cofibrations is larger. Example: $Y_{\mathbf{\Delta}} \equiv \Delta$ (cf. p. 0–17) is a cosimplicial object in $\widehat{\mathbf{\Delta}}$ which is cofibrant in the Reedy structure but not in structure L.

PROPOSITION 43 Let $\mathbf{C}$ be a complete and cocomplete model category. Equip **SIC** with its Reedy structure—then the functor $L_n : \mathbf{SIC} \rightarrow \mathbf{C}$ preserves weak equivalences between cofibrant objects.

[Inspect the proof of Proposition 42 and quote the lemma on p. 12–28.]

Let $\mathbf{C}$ be a simplicial model category. Assume: $\mathbf{C}$ is complete and cocomplete. Given an X in **SIC**, put $|X| = \int^{[n]} X_n \square \Delta[n]$—then $|X|$ is the realization of X and the assignment $X \rightarrow |X|$ is a functor $\mathbf{SIC} \rightarrow \mathbf{C}$. $|?|$ is a left adjoint for $\sin : \mathbf{C} \rightarrow \mathbf{SIC}$, where $\sin_n Y = \mathrm{HOM}(\Delta[n], Y)$. In fact, $\mathrm{Mor}(|X|, Y) \approx \mathrm{Mor}(\int^{[n]} X_n \square \Delta[n], Y) \approx \int_{[n]} \mathrm{Mor}(X_n \square \Delta[n], Y) \approx \int_{[n]} \mathrm{Mor}(X_n, \mathrm{HOM}(\Delta[n], Y)) \approx \int_{[n]} \mathrm{Mor}(X_n, \sin_n Y) \approx \mathrm{Nat}(X, \sin Y)$.

EXAMPLE Take $\mathbf{C} = \mathbf{SISET}$ and let X be a simplicial object in $\mathbf{C}$. One can fix $[m]$ and form $|X^h_m|$, the geometric realization of $[n] \to X([n],[m])$ and one can fix $[n]$ and form $|X^v_n|$, the geometric realization of $[m] \to X([n],[m])$. The assignments $\left\{\begin{array}{l}[m] \to |X^h_m| \\ [n] \to |X^v_n|\end{array}\right.$ define simplicial objects $\left\{\begin{array}{l}X^h \\ X^v\end{array}\right.$ in **CGH** and their realizations $\left\{\begin{array}{l}|X^h| \\ |X^v|\end{array}\right.$ are homeomorphic to the geometric realization of $|X|$.

LEMMA Let X be a simplicial object in $\mathbf{C}$—then $|X| \approx \operatorname{colim} |X|_n$, where $|X|_n = \int^{[k]} X_k \square \Delta[k]^{(n)}$. Moreover, $\forall\ n > 0$ there is a pushout square

$$\begin{tikzcd}
L_nX \square \Delta[n] \underset{L_nX \square \dot{\Delta}[n]}{\sqcup} X_n \square \dot{\Delta}[n] \arrow[r] \arrow[d] & {|X|_{n-1}} \arrow[d] \\
X_n \square \Delta[n] \arrow[r] & {|X|_n}
\end{tikzcd}$$

[The functors $X_n \square$— are left adjoints, hence preserve colimits, so $|X| = \int^{[n]} X_n \square \Delta[n]$ $\approx \int^{[n]} X_n \square \operatorname{colim}_k \Delta[n]^{(k)} \approx \int^{[n]} \operatorname{colim}_k X_n \square \Delta[n]^{(k)} \approx \operatorname{colim}_n \int^{[k]} X_k \square \Delta[k]^{(n)} =$ $\operatorname{colim}_n |X|_n$. And: Relative to the inclusion $\mathbf{\Delta}_n \to \mathbf{\Delta}$, the left Kan extension of $[m] \to \Delta[m]$ $(m \leq n)$ is $[k] \to \Delta[k]^{(n)}$, thus $|X|_n$ can be identified with $\int^{[m]} X_m \square \Delta[m]$ $(m \leq n)$.]

If X is a cofibrant object in **SIC** (Reedy structure), then the latching morphism $L_nX \to X_n$ is a cofibration in $\mathbf{C}$. Therefore the arrow $L_nX \square \Delta[n] \underset{L_nX \square \dot{\Delta}[n]}{\sqcup} X_n \square \dot{\Delta}[n] \to$ $X_n \square \Delta[n]$ is a cofibration in $\mathbf{C}$ (cf. Proposition 31). Consequently, the arrow $|X|_{n-1} \to$ $|X|_n$ is a cofibration in $\mathbf{C}$.

[Note: It follows from Proposition 40 that L_nX is a cofibrant object in $\mathbf{C}$, hence X_n is a cofibrant object in $\mathbf{C}$. This means that $L_nX \square \dot{\Delta}[n]$, $L_nX \square \Delta[n]$, and $X_n \square \dot{\Delta}[n]$ are cofibrant objects in $\mathbf{C}$, so $L_nX \square \Delta[n] \underset{L_nX \square \dot{\Delta}[n]}{\sqcup} X_n \square \dot{\Delta}[n]$ is a cofibrant object in $\mathbf{C}$ (cf. p. 13–46).]

LEMMA Let $\mathbf{C}$ be a simplicial model category. Assume: $\mathbf{C}$ is complete and cocomplete. Suppose that $\left\{\begin{array}{l}X \\ Y\end{array}\right.$ are cofibrant objects in **SIC** (Reedy structure) and $f : X \to Y$ is a weak equivalence—then the arrow $L_nX \square \Delta[n] \underset{L_nX \square \dot{\Delta}[n]}{\sqcup} X_n \square \dot{\Delta}[n] \to L_nY \square \Delta[n]$ $\underset{L_nY \square \dot{\Delta}[n]}{\sqcup} Y_n \square \dot{\Delta}[n]$ is a weak equivalence in $\mathbf{C}$.

[Consider the commutative diagram

$$\begin{tikzcd}
L_nX\square\Delta[n] \arrow[d] & L_nX\square\dot{\Delta}[n] \arrow[l] \arrow[r] \arrow[d] & X_n\square\dot{\Delta}[n] \arrow[d] \\
L_nY\square\Delta[n] & L_nY\square\dot{\Delta}[n] \arrow[l] \arrow[r] & Y_n\square\dot{\Delta}[n]
\end{tikzcd}.$$

The horizontal arrows are cofibrations (cf. p. 13–46) and the vertical arrows are weak equivalences (cf. Proposition 43 and p. 13–52). Therefore Proposition 3 in §12 is applicable.]

PROPOSITION 44 Let **C** be a simplicial model category. Assume: **C** is complete and cocomplete. Suppose that $\begin{cases} X \\ Y \end{cases}$ are cofibrant objects in **SIC** (Reedy structure) and $f : X \rightarrow Y$ is a weak equivalence—then $|f| : |X| \rightarrow |Y|$ is a weak equivalence.

[Since $\begin{cases} |X|_0 = X_0 \\ |Y|_0 = Y_0 \end{cases}$ and $\forall\ n$, $\begin{cases} |X|_n \rightarrow |X|_{n+1} \\ |Y|_n \rightarrow |Y|_{n+1} \end{cases}$ is a cofibration in **C**, one may view $\begin{cases} \{|X|_n : n \geq 0\} \\ \{|Y|_n : n \geq 0\} \end{cases}$ as cofibrant objects in **FIL(C)** (cf. p. 12–5). So, to prove that $|f| : |X| \rightarrow |Y|$ is a weak equivalence, it need only be shown that $\forall\ n$, $|f|_n : |X|_n \rightarrow |Y|_n$ is a weak equivalence (cf. p. 12–30). For this, work with

$$\begin{tikzcd}
X_n\square\Delta[n] \arrow[d] & L_nX\square\Delta[n] \underset{L_nX\square\dot{\Delta}[n]}{\sqcup} X_n\square\dot{\Delta}[n] \arrow[l] \arrow[r] \arrow[d] & {|X|_{n-1}} \arrow[d] \\
Y_n\square\Delta[n] & L_nY\square\Delta[n] \underset{L_nY\square\dot{\Delta}[n]}{\sqcup} Y_n\square\dot{\Delta}[n] \arrow[l] \arrow[r] & {|Y|_{n-1}}
\end{tikzcd}$$

and use induction (cf. §12, Proposition 3).]

EXAMPLE Take **C** = **SISET** and suppose that $f : X \rightarrow Y$ is a weak equivalence, i.e., $\forall\ n$, $f_n : X_n \rightarrow Y_n$ is a weak homotopy equivalence—then $|f| : |X| \rightarrow |Y|$ is a weak homotopy equivalence.

[All simplicial objects in $\widehat{\mathbf{\Delta}}$ are cofibrant in the Reedy structure.]

[Note: Fix an abelian group G and consider **SISET** in the homological model category structure determined by G—then **SISET** is a simplicial model category (cf. p. 13–46), hence $|f| : |X| \rightarrow |Y|$ is an HG-equivalence if $\forall\ n$, $f_n : X_n \rightarrow Y_n$ is an HG-equivalence.]

EXAMPLE Suppose that **C** is a simplicial model category which is complete and cocomplete. Let X be a cofibrant object in **SIC** (Reedy structure). Assume: $\forall\ \alpha$, $X\alpha$ is a weak equivalence—then the arrow $|X|_0 \rightarrow |X|$ is a weak equivalence.

Let **C** be a simplicial model category. Assume: **C** is complete and cocomplete. Given an X in **COSIC**, put $\text{tot}\, X = \int_{[n]} \text{HOM}(\Delta[n], X_n)$—then $\text{tot}\, X$ is the totalization of X and the assignment $X \to \operatorname{tot} X$ is a functor $\mathbf{COSIC} \to \mathbf{C}$. tot is a right adjoint for $\operatorname{cosin} : \mathbf{C} \to \mathbf{COSIC}$, where $\operatorname{cosin}_n Y = Y_n \square \Delta[n]$. In fact, $\operatorname{Mor}(Y, \operatorname{tot} X) \approx \operatorname{Mor}(Y, \int_{[n]} \operatorname{HOM}(\Delta[n], X_n)) \approx \int_{[n]} \operatorname{Mor}(Y, \operatorname{HOM}(\Delta[n], X_n)) \approx \int_{[n]} \operatorname{Mor}(Y \square \Delta[n], X_n) \approx \int_{[n]} \operatorname{Mor}(\operatorname{cosin}_n Y, X_n) \approx \operatorname{Nat}(\operatorname{cosin} Y, X)$.

Example: Take $\mathbf{C} = \mathbf{SISET}$—then $\operatorname{tot} X = \operatorname{HOM}(Y_{\mathbf{\Delta}}, X)$ (cf. p. 13–44).

Example: Let X be a simplicial set. Given a cosimplicial object Y in $\widehat{\mathbf{\Delta}}$, the functor $\mathbf{\Delta} \to \mathbf{SISET}$ that sends $[n]$ to $\operatorname{map}(X, Y_n)$ defines another cosimplicial object in $\widehat{\mathbf{\Delta}}$, call it $\operatorname{map}(X, Y)$. And: $\operatorname{tot} \operatorname{map}(X, Y) \approx \int_{[n]} \operatorname{map}(\Delta[n], \operatorname{map}(X, Y_n)) \approx \int_{[n]} \operatorname{map}(X, \operatorname{map}(\Delta[n], Y_n)) \approx \operatorname{map}(X, \int_{[n]} \operatorname{map}(\Delta[n], Y_n)) \approx \operatorname{map}(X, \operatorname{tot} Y)$.

EXAMPLE Given a simplicial set K and a compactly generated Hausdorff space X, let X^K be the cosimplicial object in **CGH** with $(X^K)_n = X^{K_n}$—then $\operatorname{map}(|K|, X) \approx \operatorname{tot} X^K$.

EXAMPLE Fix a prime p—then there is a forgetful functor from the category of simplicial vector spaces over $\mathbf{F}_p$ to **SISET**. It has a left adjoint, thus this data determines a triple in **SISET**. Write $\operatorname{res}_p X$ for the standard resolution of X : $\operatorname{res}_p X$ is therefore a cosimplicial object in $\widehat{\mathbf{\Delta}}$ and $\operatorname{tot} \operatorname{res}_p X$ is the $\mathbf{F}_p$-completion $\mathbf{F}_p X$ of X (Bousfield-Kan[†]).

PROPOSITION 45 Let $\mathbf{C}$ be a simplicial model category. Assume: $\mathbf{C}$ is complete and cocomplete. Suppose that $\left\{\begin{matrix} X \\ Y \end{matrix}\right.$ are fibrant objects in **COSIC** (Reedy structure) and $f : X \to Y$ is a weak equivalence—then $\operatorname{tot} f : \operatorname{tot} X \to \operatorname{tot} Y$ is a weak equivalence.

[The proof is dual to that of Proposition 44. Of course, $\operatorname{tot} X \approx \lim \operatorname{tot}_n X$ (obvious notation).]

The simplex category $\operatorname{gro}_{\mathbf{\Delta}} K$ of a simplicial set K can be viewed as a comma category:

$$\begin{tikzcd}
\Delta[n] \arrow[rr] \arrow[dr] & & \Delta[m] \arrow[dl] \\
& K &
\end{tikzcd}$$

(cf. p. 13–17). Call this interpretation ΔK, $\Delta^{\mathrm{OP}} K$ being its opposite. There is a forgetful functor $\Delta K : \Delta K \to \mathbf{SISET}$ and $K \approx \operatorname{colim} \Delta K$ (cf. p. 0–20).

FACT The fundamental groupoid of $\mathbf{\Delta} K$ is equivalent to the fundamental groupoid of K.

[†] *SLN* **304** (1972).

Given a category $\mathbf{C}$, write K-**SIC** for the functor category $[\mathbf{\Delta}^{\mathrm{OP}}K, \mathbf{C}]$ and K-**COSIC** for the functor category $[\mathbf{\Delta}K, \mathbf{C}]$—then by definition, a K-simplicial object in $\mathbf{C}$ is an object in K-**SIC** and a K-cosimplicial object in $\mathbf{C}$ is an object in K-**COSIC**.

[Note: Take $K = \Delta[0]$ to recover **SIC** and **COSIC**.]

The preceding results can now be generalized. Thus if $\mathbf{C}$ is a complete and cocomplete model category, one can again introduce latching objects and matching objects and use them to equip K-**SIC** (dually, K-**COSIC**) with the structure of a model category (Reedy structure). Assuming in addition that $\mathbf{C}$ is a simplicial model category, there is a realization functor $|?|_K : K\text{-}\mathbf{SIC} \to \mathbf{C}$ that sends X to $|X|_K = \int^{\mathbf{\Delta}K} X \square \Delta K$, where $X \square \Delta K : \mathbf{\Delta}^{\mathrm{OP}}K \times \mathbf{\Delta}K \to \mathbf{C}$ is the composite $\mathbf{\Delta}^{\mathrm{OP}}K \times \mathbf{\Delta}K \xrightarrow{X \times \Delta K} \mathbf{C} \times \mathbf{SISET} \xrightarrow{\square} \mathbf{C}$. So, in the notation of the Kan extension theorem, $|?|_K = |?| \circ \mathrm{lan}$, i.e., the diagram

$$\begin{tikzcd} K\text{-}\mathbf{SIC} \arrow[r, "\mathrm{lan}"] \arrow[dr, "|?|_K"'] & \mathbf{SIC} \arrow[d, "|?|"] \\ & \mathbf{C} \end{tikzcd}$$

commutes. Here, lan is computed from the arrow $\mathbf{\Delta}^{\mathrm{OP}}K \to \mathbf{\Delta}^{\mathrm{OP}}$ induced by the projection $K \to \Delta[0]$. $|?|_K$ is a left adjoint for $\sin_K : \mathbf{C} \to K\text{-}\mathbf{SIC}$. On the other hand, there is a totalization functor $\mathrm{tot}_K : K\text{-}\mathbf{COSIC} \to \mathbf{C}$ that sends X to $\mathrm{tot}_K X = \int_{\mathbf{\Delta}K} \mathrm{HOM}(\Delta K, X)$, where $\mathrm{HOM}(\Delta K, X) : \mathbf{\Delta}^{\mathrm{OP}}K \times \mathbf{\Delta}K \to \mathbf{C}$ is the composite $\mathbf{\Delta}^{\mathrm{OP}}K \times \mathbf{\Delta}K \xrightarrow{\Delta^{\mathrm{OP}}K \times X} \mathbf{SISET}^{\mathrm{OP}} \times \mathbf{C} \xrightarrow{\mathrm{HOM}} \mathbf{C}$. So, in the notation of the Kan extension theorem, $\mathrm{tot}_K = \mathrm{tot} \circ \mathrm{ran}$, i.e., the diagram

$$\begin{tikzcd} K\text{-}\mathbf{COSIC} \arrow[r, "\mathrm{ran}"] \arrow[dr, "\mathrm{tot}_K"'] & \mathbf{COSIC} \arrow[d, "\mathrm{tot}"] \\ & \mathbf{C} \end{tikzcd}$$

commutes. Here, ran is computed from the arrow $\mathbf{\Delta}K \to \mathbf{\Delta}$ induced by the projection $K \to \Delta[0]$. tot_K is a right adjoint for $\mathrm{cosin}_K : \mathbf{C} \to K\text{-}\mathbf{COSIC}$.

To check the claimed factorization of $|?|_K$, represent $|X|_K$ as the coequalizer of the diagram $\coprod\limits_{k \to l} X_l \square \Delta K k \rightrightarrows \coprod\limits_{k} X_k \square \Delta K k$. Noting that $(\mathrm{lan}\, X)_n = \coprod\limits_{k \in K_n} X_k$, we have $\coprod\limits_{k \to l} X_l \square \Delta K k \approx \coprod\limits_{n,m \geq 0} \coprod\limits_{[n] \to [m]} \coprod\limits_{l \in K_m} X_l \square \Delta[n] \approx \coprod\limits_{n,m \geq 0} \coprod\limits_{[n] \to [m]} (\mathrm{lan}\, X)_m \square \Delta[n]$ and $\coprod\limits_{k} X_k \square \Delta K k \approx \coprod\limits_{n \geq 0} \coprod\limits_{k \in K_n} X_k \square \Delta[n] \approx \coprod\limits_{n \geq 0} (\mathrm{lan}\, X)_n \square \Delta[n]$, i.e., $|X|_K$ is naturally isomorphic to the coequalizer of the diagram $\coprod\limits_{n,m \geq 0} \coprod\limits_{[n] \to [m]} (\mathrm{lan}\, X)_m \square \Delta[n] \rightrightarrows \coprod\limits_{n \geq 0} (\mathrm{lan}\, X)_n \square \Delta[n]$, i.e., to $|\mathrm{lan}\, X|$.

Example: Take $\mathbf{C} = \mathbf{SISET}$—then $| * |_K = K$.

EXAMPLE Let B be a simplicial set. Fix an X in $\mathbf{SISET}/B$—then $\forall\, n$ & $\forall\, b \in B_n$, there is a

pullback square
$$\begin{tikzcd} X_b \arrow[r] \arrow[d] & X \arrow[d, "p"] \\ \Delta[n] \arrow[r, "\Delta_b"'] & B \end{tikzcd}$$
(cf. p. 13–3). This data thus determines a B-cosimplicial object X_B in **SISET**. One has $X \approx \operatorname{colim} X_B$ and X_B is cofibrant in the Reedy structure.

PROPOSITION 44 (K) Let **C** be a simplicial model category. Assume: **C** is complete and cocomplete. Suppose that $\begin{cases} X \\ Y \end{cases}$ are cofibrant objects in K-**SIC** (Reedy structure) and $f : X \to Y$ is a weak equivalence—then $|f|_K : |X|_K \to |Y|_K$ is a weak equivalence.

PROPOSITION 45 (K) Let **C** be a simplicial model category. Assume: **C** is complete and cocomplete. Suppose that $\begin{cases} X \\ Y \end{cases}$ are fibrant objects in K-**COSIC** (Reedy structure) and $f : X \to Y$ is a weak equivalence—then $\operatorname{tot}_K f : \operatorname{tot}_K X \to \operatorname{tot}_K Y$ is a weak equivalence.

FACT $\sin_K$ preserves fibrations and acyclic fibrations.

[Note: Therefore $|?|_K$ preserves cofibrations and acyclic cofibrations (cf. p. 12–3 ff.).]

FACT cosin_K preserves cofibrations and acyclic cofibrations.

[Note: Therefore tot_K preserves fibrations and acyclic fibrations (cf. p. 12–3 ff.).]

Notation: Let **I** be a small category. Put $\mathbf{\Delta I} = \mathbf{\Delta} \operatorname{ner} \mathbf{I}$ and call it the simplex category of **I**—then $\mathbf{\Delta I}$ is isomorphic to the comma category $|\iota, K_{\mathbf{I}}|$:
$$\begin{tikzcd} {[n]} \arrow[rr] \arrow[dr] & & {[m]} \arrow[dl] \\ & \mathbf{I} & \end{tikzcd}$$
($\iota : \mathbf{\Delta} \to \mathbf{CAT}$).

There is a projection $\pi_{\mathbf{I}} : \mathbf{\Delta I} \to \mathbf{I}$ that sends an object $[n] \xrightarrow{f} \mathbf{I}$ to $fn \in \operatorname{Ob} \mathbf{I}$. Example: $\mathbf{\Delta 1} = \mathbf{\Delta}$.

[Note: $\mathbf{\Delta}^{\mathrm{OP}}\mathbf{I}$ is the opposite of $\mathbf{\Delta I}$. Example: $\mathbf{\Delta}^{\mathrm{OP}}\mathbf{1} = \mathbf{\Delta}^{\mathrm{OP}}$. Replacing **I** by $\mathbf{I}^{\mathrm{OP}}$, there is a projection $\pi_{\mathbf{I}}^{\mathrm{OP}} : \mathbf{\Delta}^{\mathrm{OP}}\mathbf{I}^{\mathrm{OP}} \to \mathbf{I}$ that sends an object $[n] \xrightarrow{f} \mathbf{I}^{\mathrm{OP}}$ to $fn \in \operatorname{Ob} \mathbf{I}$.]

EXAMPLE Let **C** be a complete and cocomplete model category. Suppose that $F : \mathbf{I} \to \mathbf{C}$ is a functor such that $\forall\ i$, Fi is cofibrant (fibrant)—then $F \circ \pi_{\mathbf{I}}^{\mathrm{OP}}$ ($F \circ \pi_{\mathbf{I}}$) is a cofibrant (fibrant) object in $[\mathbf{\Delta}^{\mathrm{OP}}\mathbf{I}^{\mathrm{OP}}, \mathbf{C}]$ ($[\mathbf{\Delta I}, \mathbf{C}]$) (Reedy structure).

Let **I** be a small category and **C** a simplicial model category. Assume: **C** is complete and cocomplete—then the functor $\operatorname{colim} : [\mathbf{I}, \mathbf{C}] \to \mathbf{C}$ ($\lim : [\mathbf{I}, \mathbf{C}] \to \mathbf{C}$) need not preserve levelwise weak equivalences between levelwise cofibrant (fibrant) objects. To remedy this defect, one introduces the notion of homotopy colimit (limit). Thus define a functor $\text{hocolim}_{\mathbf{I}} : [\mathbf{I}, \mathbf{C}] \rightarrow \mathbf{C}$ by $\text{hocolim}_{\mathbf{I}} F$ (or $\text{hocolim}\, F$) $= \int^{\mathbf{I}^{\text{OP}}} F \,\square\, \text{ner}(-\backslash \mathbf{I})^{\text{OP}}$ and define a functor $\text{holim}_{\mathbf{I}} : [\mathbf{I}, \mathbf{C}] \rightarrow \mathbf{C}$ by $\text{holim}_{\mathbf{I}} F$ (or $\text{holim}\, F$) $= \int_{\mathbf{I}} \text{HOM}(\text{ner}(\mathbf{I}/-), F)$.

[Note: One has $\text{HOM}(\text{hocolim}_{\mathbf{I}} F, Y) \approx \text{HOM}(\int^{i} Fi \,\square\, \text{ner}(i\backslash \mathbf{I})^{\text{OP}}, Y) \approx \int_{i} \text{HOM}(Fi \,\square\, \text{ner}(i\backslash \mathbf{I})^{\text{OP}}, Y) \approx \int_{i} \text{map}(\text{ner}(i\backslash \mathbf{I})^{\text{OP}}, \text{HOM}(Fi, Y)) \approx \int_{i} \text{map}(\text{ner}(\mathbf{I}^{\text{OP}}/i),$ $\text{HOM}(Fi, Y)) \approx \text{holim}_{\mathbf{I}^{\text{OP}}} \text{HOM}(F, Y)$, where $\text{HOM}(F, Y) : \mathbf{I}^{\text{OP}} \rightarrow \mathbf{SISET}$ sends i to $\text{HOM}(Fi, Y)$.]

Remark: The functor hocolim has a right adjoint, viz. $\text{HOM}(\text{ner}(-\backslash \mathbf{I})^{\text{OP}}, -)$, and the functor holim has a left adjoint, viz. $- \,\square\, \text{ner}(\mathbf{I}/-)$.

Remark: There are natural transformations hocolim → colim, lim → holim.

[Note: It can be shown that Lhocolim and Rholim exist and that there are natural isomorphisms Lhocolim → Lcolim, Rlim → Rholim (Dwyer-Kan[†]) (cf. p. 12–32).]

Example: Take $\mathbf{C} = \mathbf{SISET}, \mathbf{CGH}, \mathbf{SISET}_*, \mathbf{CGH}_*$—then $Fi \,\square\, \text{ner}(i\backslash \mathbf{I})^{\text{OP}} = Fi \times \text{ner}(i\backslash \mathbf{I})^{\text{OP}}, Fi \times_k B(i\backslash \mathbf{I})^{\text{OP}}, Fi \# \text{ner}(i\backslash \mathbf{I})^{\text{OP}}_+, Fi \#_k B(i\backslash \mathbf{I})^{\text{OP}}_+$ and $\text{HOM}(\text{ner}(\mathbf{I}/i), Fi) = \text{map}(\text{ner}(\mathbf{I}/i), Fi), \text{map}(B(\mathbf{I}/i), Fi), \text{map}_*(\text{ner}(\mathbf{I}/i)_+, Fi), \text{map}_*(B(\mathbf{I}/i)_+, Fi)$.

[Note: Consider $\int^{i} Fi \,\square\, \text{ner}(i\backslash \mathbf{I})$ and $\int^{i} Fi \,\square\, \text{ner}(i\backslash \mathbf{I})^{\text{OP}}$. When $\mathbf{C} = \mathbf{SISET}$ or $\mathbf{SISET}_*$, they are simplicial opposites of one another (cf. p. 13–1), hence are naturally weakly equivalent, and when $\mathbf{C} = \mathbf{CGH}$ or $\mathbf{CGH}_*$, they are related by a natural homeomorphism (since $\forall\ i$, $B(i\backslash \mathbf{I}) \approx B(i\backslash \mathbf{I})^{\text{OP}}$ (cf. p. 0–19)).]

Place on $[\mathbf{I}^{\text{OP}}, \mathbf{SISET}]$ and $[\mathbf{I}, \mathbf{SISET}]$ structure L (cf. p. 13–37)—then $i \rightarrow \text{ner}(i\backslash \mathbf{I})^{\text{OP}}$ is a cofibrant object in $[\mathbf{I}^{\text{OP}}, \mathbf{SISET}]$ and $i \rightarrow \text{ner}(\mathbf{I}/i)$ is a cofibrant object in $[\mathbf{I}, \mathbf{SISET}]$ (cf. p. 13–38). Observe too that $\forall\ i \in \text{Ob}\,\mathbf{I}$, the classifying spaces $B(i\backslash \mathbf{I})^{\text{OP}}$ and $B(\mathbf{I}/i)$ are contractible (cf. p. 13–15).

EXAMPLE Let F be the functor $\mathbf{I} \rightarrow \mathbf{SISET}$ that sends $i \in \text{Ob}\, I$ to $Fi = \Delta[0]$—then $\text{hocolim}\, F \approx \text{ner}\, \mathbf{I}^{\text{OP}}$, i.e., $\int^{i} \Delta[0] \times \text{ner}(i\backslash \mathbf{I})^{\text{OP}} \approx \text{ner}\, \mathbf{I}^{\text{OP}}$ or still, $\int^{i} \Delta[0] \times \text{ner}(\mathbf{I}^{\text{OP}}/i) \approx \text{ner}\, \mathbf{I}^{\text{OP}}$. Similarly, $\int^{i} \Delta[0] \times \text{ner}(i\backslash \mathbf{I}) \approx \text{ner}\, \mathbf{I}$ and $\int^{i} \Delta[0] \times \text{ner}(i\backslash \mathbf{I}^{\text{OP}}) \approx \text{ner}\, \mathbf{I}^{\text{OP}}$. In addition, $\int^{i} \Delta[0] \times \text{ner}(i\backslash \mathbf{I}^{\text{OP}})^{\text{OP}} \approx \text{ner}\, \mathbf{I}$ or still, $\int^{i} \Delta[0] \times \text{ner}(\mathbf{I}/i) \approx \text{ner}\, \mathbf{I}$.

EXAMPLE Let $U : \mathbf{CGH}_* \rightarrow \mathbf{CGH}$ be the forgetful functor and consider a functor $F : \mathbf{I} \rightarrow \mathbf{CGH}_*$. Question: What is the relation between $\text{hocolim}\, F$ & $\text{hocolim}\, U \circ F$ and $\text{holim}\, F$ & $\text{holim}\, U \circ F$? The answer for homotopy limits is that there is essentially no difference (since $\mathrm{map}_*(X_+, Y) \approx \mathrm{map}(X, UY)$). Turning to homotopy colimits, assume that $\forall\ i$, Fi is cofibrant—then there is a cofibration $B\mathbf{I}^{\mathrm{OP}} \rightarrow \mathrm{hocolim}\, U \circ F$ and a homeomorphism $\mathrm{hocolim}\, U \circ F / B\mathbf{I}^{\mathrm{OP}} \rightarrow \mathrm{hocolim}\, F$.

[†] *Model Categories and General Abstract Homotopy Theory,*

[Note: If $B\mathbf{I}^{\mathrm{OP}}$ is contractible, the projection $\mathrm{hocolim}\, U \circ F \rightarrow \mathrm{hocolim}\, F$ is a weak homotopy equivalence. Proof: Consider the pushout square

$$\begin{tikzcd} B\mathbf{I}^{\mathrm{OP}} \arrow[r] \arrow[d] & * \arrow[d] \\ \mathrm{hocolim}\, U \circ F \arrow[r] & \mathrm{hocolim}\, F \end{tikzcd}$$

, bearing in mind that **CGH** (singular structure) is a proper model category.]

LEMMA Let $X \rightarrow B$ be a simplicial map. Suppose that for every commutative diagram

$$\begin{tikzcd} X_{b'} \arrow[r] \arrow[d] & X_b \arrow[r] \arrow[d] & X \arrow[d, "p"] \\ \Delta[n'] \arrow[r] & \Delta[n] \arrow[r, "\Delta_b"'] & B \end{tikzcd}$$

, the arrow $X_{b'} \rightarrow X_b$ is a weak homotopy equivalence—then p is a homotopy fibration.

FACT Let $F : \mathbf{I} \rightarrow \mathbf{SISET}$ be a functor—then the arrow $\mathrm{hocolim}\, F \rightarrow \mathrm{ner}\, \mathbf{I}^{\mathrm{OP}}$ is a homotopy fibration iff $\forall\ \delta \in \mathrm{Mor}\, \mathbf{I}$, $F\delta$ is a weak homotopy equivalence.

PROPOSITION 46 Fix $F \in \mathrm{Ob}[\mathbf{I}, \mathbf{C}]$—then

$$\mathrm{hocolim}\, F \approx \int^{\Delta \mathbf{I}^{\mathrm{OP}}} F \circ \pi_{\mathbf{I}}^{\mathrm{OP}} \square \Delta\, \mathrm{ner}\, \mathbf{I}^{\mathrm{OP}} (= |F \circ \pi_{\mathbf{I}}^{\mathrm{OP}}|_{\mathrm{ner}\, \mathbf{I}^{\mathrm{OP}}})$$

and

$$\mathrm{holim}\, F \approx \int_{\Delta \mathbf{I}} \mathrm{HOM}(\Delta\, \mathrm{ner}\, \mathbf{I}, F \circ \pi_{\mathbf{I}}) \qquad (= \mathrm{tot}_{\mathrm{ner}\, \mathbf{I}} F \circ \pi_{\mathbf{I}}).$$

Application: Let $F, G : \mathbf{I} \rightarrow \mathbf{C}$ be functors and let $\Xi : F \rightarrow G$ be a natural transformation. Assume: $\forall\ i$, $\Xi_i : Fi \rightarrow Gi$ is a weak equivalence—then $\mathrm{hocolim}\, \Xi : \mathrm{hocolim}\, F \rightarrow \mathrm{hocolim}\, G$ is a weak equivalence provided that $\forall\ i$, $\left\{\begin{matrix} Fi \\ Gi \end{matrix}\right.$ is cofibrant and $\mathrm{holim}\, \Xi : \mathrm{holim}\, F \rightarrow \mathrm{holim}\, G$ is a weak equivalence provided that $\forall\ i$, $\left\{\begin{matrix} Fi \\ Gi \end{matrix}\right.$ is fibrant.

[In view of the above result and the example on p. 13–62, this follows from Propositions 44(K) and 45(K).]

EXAMPLE Let $F : \mathbf{I} \rightarrow \mathbf{SISET}$ be a functor—then there is a natural homeomorphism $|\mathrm{hocolim}\, F| \rightarrow \mathrm{hocolim}\, |F|$ of compactly generated Hausdorff spaces.

[Geometric realization is a left adjoint, hence preserves colimits.]

EXAMPLE Let $F : \mathbf{I} \rightarrow \mathbf{CGH}$ be a functor such that $\forall\ i$, Fi is cofibrant—then there is a natural weak homotopy equivalence $\operatorname{hocolim} \sin F \rightarrow \sin \operatorname{hocolim} F$.

[Consider the natural transformation $|\sin F| \rightarrow F$. Thanks to the Giever-Milnor theorem, $\forall\ i$, $|\sin Fi| \rightarrow Fi$ is a weak homotopy equivalence, thus the arrow $\operatorname{hocolim} |\sin F| \rightarrow \operatorname{hocolim} F$ is a weak homotopy equivalence (cf. supra). But from the preceding example, $|\operatorname{hocolim} \sin F| \approx \operatorname{hocolim} |\sin F|$, so taking adjoints leads to the conclusion.]

EXAMPLE Let $F : \mathbf{I} \rightarrow \mathbf{CGH}$ be a functor—then there is a natural isomorphism $\sin \operatorname{holim} F \rightarrow \operatorname{holim} \sin F$ of simplicial sets.

EXAMPLE Let $F : \mathbf{I} \rightarrow \mathbf{SISET}$ be a functor such that $\forall\ i$, Fi is fibrant—then there is a natural weak homotopy equivalence $|\operatorname{holim} F| \rightarrow \operatorname{holim} |F|$.

Another corollary to Proposition 46 is the fact that $\operatorname{hocolim} F \approx |\operatorname{lan} F \circ \pi_{\mathbf{I}}^{\mathrm{OP}}|$ and $\operatorname{holim} F \approx \operatorname{tot} \operatorname{ran} F \circ \pi_{\mathbf{I}}$.

SIMPLICIAL REPLACEMENT LEMMA Fix $F \in \operatorname{Ob}[\mathbf{I}, \mathbf{C}]$. Define $\coprod F$ in **SIC** by $(\coprod F)_n = \coprod\limits_{[n] \xrightarrow{f} \mathbf{I}^{\mathrm{OP}}} Ffn$—then $\coprod F \approx \operatorname{lan} F \circ \pi_{\mathbf{I}}^{\mathrm{OP}}$.

COSIMPLICIAL REPLACEMENT LEMMA Fix $F \in \operatorname{Ob}[\mathbf{I}, \mathbf{C}]$. Define $\prod F$ in **COSIC** by $(\prod F)_n = \prod\limits_{[n] \xrightarrow{f} \mathbf{I}} Ffn$—then $\prod F \approx \operatorname{ran} F \circ \pi_{\mathbf{I}}$.

FACT Let $F, G : \mathbf{I} \rightarrow \mathbf{SISET}$ be functors and let $\Xi : F \rightarrow G$ be a natural transformation. Assume: $\forall\ i$, $\Xi_i : Fi \rightarrow Gi$ is a Kan fibration—then $\operatorname{holim} \Xi : \operatorname{holim} F \rightarrow \operatorname{holim} G$ is a Kan fibration.

[The arrow $\prod \Xi : \prod F \rightarrow \prod G$ is a fibration in $[\mathbf{\Delta}, \mathbf{SISET}]$ (Reedy structure). But $\operatorname{tot} : [\mathbf{\Delta}, \mathbf{SISET}] \rightarrow \mathbf{SISET}$ preserves fibrations (cf. p. 13–62).]

Application: Let $F : \mathbf{I} \rightarrow \mathbf{SISET}$ be a functor. Assume: $\forall\ i$, Fi is fibrant—then $\operatorname{holim} F$ is fibrant.

EXAMPLE Let $\rho : A \rightarrow B$ be an inclusion of simplicial sets. Suppose that $F : \mathbf{I} \rightarrow \mathbf{SISET}$ is a functor such that $\forall\ i$, Fi is ρ-local—then $\operatorname{holim} F$ is ρ-local.

[Each Fi is fibrant, so $\operatorname{holim} F$ is fibrant. Denote by $\begin{cases} \operatorname{map}(A, F) \\ \operatorname{map}(B, F) \end{cases}$ the functor $\mathbf{I} \rightarrow \mathbf{SISET}$ that sends i to $\begin{cases} \operatorname{map}(A, Fi) \\ \operatorname{map}(B, Fi) \end{cases}$, which are fibrant (cf. p. 13–22). Since $\begin{cases} \operatorname{map}(A, \operatorname{holim} F) \approx \operatorname{holim} \operatorname{map}(A, F) \\ \operatorname{map}(B, \operatorname{holim} F) \approx \operatorname{holim} \operatorname{map}(B, F) \end{cases}$ and each Fi is ρ-local, the arrow $\operatorname{map}(B, \operatorname{holim} F) \rightarrow \operatorname{map}(A, \operatorname{holim} F)$ is a weak homotopy equivalence (cf. p. 13–64).]

EXAMPLE Let $\rho : A \rightarrow B$ be an inclusion of simplicial sets. Suppose that $F, G : \mathbf{I} \rightarrow \mathbf{SISET}$ are functors and $\Xi : F \rightarrow G$ is a natural transformation. Assume: $\forall\ i$, $\Xi_i : Fi \rightarrow Gi$ is a ρ-equivalence—then $\operatorname{hocolim} \Xi : \operatorname{hocolim} F \rightarrow \operatorname{hocolim} G$ is a ρ-equivalence.

[It is a question of proving that the arrow $\operatorname{map}(\operatorname{hocolim} G, Z) \rightarrow \operatorname{map}(\operatorname{hocolim} F, Z)$ is a weak homotopy equivalence $\forall$ ρ-local Z or still, that the arrow $\operatorname{holim} \operatorname{map}(G, Z) \rightarrow \operatorname{holim} \operatorname{map}(F, Z)$ is a weak homotopy equivalence, which is true (cf. p. 13–64).]

PROPOSITION 47 For any cofibrant object F in $[\mathbf{I}, \mathbf{SISET}]$ (structure L), the arrow $\operatorname{hocolim} F \rightarrow \operatorname{colim} F$ is a weak homotopy equivalence.

[It suffices to show that $\forall$ fibrant Z, the arrow $\operatorname{map}(\operatorname{colim} F, Z) \rightarrow \operatorname{map}(\operatorname{hocolim} F, Z)$ is a weak homotopy equivalence (cf. Proposition 39). Since $\operatorname{hocolim} F \rightarrow \operatorname{colim} F$ is induced by the projection $\operatorname{ner}(\text{—}\backslash \mathbf{I})^{\mathrm{OP}} \rightarrow *$, one need only consider the arrow $\operatorname{HOM}(F, \operatorname{map}(*, Z)) \rightarrow \operatorname{HOM}(F, \operatorname{map}(\operatorname{ner}(\text{—}\backslash \mathbf{I})^{\mathrm{OP}}, Z))$. But F is a cofibrant object in $[\mathbf{I}, \mathbf{SISET}]$ and $\operatorname{map}(*, Z) \rightarrow \operatorname{map}(\operatorname{ner}(\text{—}\backslash \mathbf{I})^{\mathrm{OP}}, Z)$ is a weak equivalence between fibrant objects in $[\mathbf{I}, \mathbf{SISET}]$, thus the assertion is a consequence of Proposition 34.]

FACT Suppose that $\mathbf{I}$ is filtered—then $\forall$ F in $[\mathbf{I}, \mathbf{SISET}]$, the arrow $\operatorname{hocolim} F \rightarrow \operatorname{colim} F$ is a weak homotopy equivalence.

EXAMPLE $\forall$ F in $\mathbf{FIL}(\mathbf{SISET})$, the arrow $\operatorname{hocolim} F \rightarrow \operatorname{colim} F$ is a weak homotopy equivalence. Therefore $|\operatorname{colim} F|$ is contractible if $\forall$ n, $|Fn|$ is contractible.

[The arrow $\operatorname{hocolim} F \rightarrow \operatorname{ner}[\mathbf{N}]^{\mathrm{OP}}$ is a weak homotopy equivalence. And: $[\mathbf{N}]^{\mathrm{OP}}$ has a final object, hence $B[\mathbf{N}]^{\mathrm{OP}}$ is contractible (cf. p. 13–15).]

LEMMA If X is a cofibrant K-simplicial (K-cosimplicial) object in $\mathbf{SISET}$, then $\forall$ fibrant Y in $\mathbf{SISET}$, $\operatorname{map}(X, Y)$ is a fibrant K-cosimplicial (K-simplicial) object in $\mathbf{SISET}$.

PROPOSITION 48 For any cofibrant K-simplicial (K-cosimplicial) object X in $\mathbf{SISET}$, the arrow $\operatorname{hocolim} X \rightarrow \operatorname{colim} X$ is a weak homotopy equivalence.

EXAMPLE Let B be a simplicial set. Fix an X in $\mathbf{SISET}/B$ and determine the cofibrant B-cosimplicial object X_B in $\mathbf{SISET}$ as on p. 13–61 ff.—then the arrow $\operatorname{hocolim} X_B \rightarrow \operatorname{colim} X_B$ ($\approx X$) is a weak homotopy equivalence.

[Note: Suppose given
$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "p"'] & & Y \arrow[dl, "q"] \\ & B & \end{tikzcd}$$
such that $\forall$ n & $\forall$ $b \in B_n$, $X_b \rightarrow Y_b$ is a weak homotopy equivalence—then $\operatorname{hocolim} X_B \rightarrow \operatorname{hocolim} Y_B$ is a weak homotopy equivalence (cf. p. 13–64). Since there

is a commutative diagram $$\begin{tikzcd} \operatorname{hocolim} X_B \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ \operatorname{hocolim} Y_B \arrow[r] & Y \end{tikzcd}$$, it follows that f is a weak homotopy equivalence. Example: p is a weak homotopy equivalence if the $|X_b|$ are contractible.]

Given a category $\mathbf{C}$, write **BISIC** for the functor category $[(\mathbf{\Delta} \times \mathbf{\Delta})^{\mathrm{OP}}, \mathbf{C}]$ (i.e., $[\mathbf{\Delta}^{\mathrm{OP}}, \mathbf{SIC}]$)—then by definition, a bisimplicial object in $\mathbf{C}$ is an object in **BISIC** (i.e., a simplicial object in **SIC**). Example: Assuming that $\mathbf{C}$ has finite products, if $\begin{cases} X \\ Y \end{cases}$ are simplicial objects in $\mathbf{C}$, the assignment $([n],[m]) \to X_n \times Y_m$ defines a bisimplicial object $X \underline{\times} Y$ in $\mathbf{C}$.

Specialize to $\mathbf{C} = \mathbf{SET}$—then an object in **BISISET** $(= \widehat{\mathbf{\Delta} \times \mathbf{\Delta}})$ is called a bisimplicial set and a morphism in **BISISET** is called a bisimplicial map. Given a bisimplicial set X, put $X_{n,m} = X([n],[m])$ $(= X_n([m]))$—then there are horizontal operators $\begin{cases} d_i^h : X_{n,m} \to X_{n-1,m} \\ s_i^h : X_{n,m} \to X_{n+1,m} \end{cases}$ $(0 \leq i \leq n)$ and vertical operators $\begin{cases} d_j^v : X_{n,m} \to X_{n,m-1} \\ s_j^v : X_{n,m} \to X_{n,m+1} \end{cases}$ $(0 \leq j \leq m)$. The horizontal operators commute with the vertical operators, the simplicial identities are satisfied horizontally and vertically, and thanks to the Yoneda lemma, $\operatorname{Nat}(\Delta[n,m], X) \approx X_{n,m}$, where $\Delta[n,m] = \Delta[n] \underline{\times} \Delta[m]$.

[Note: Every simplicial set X can be regarded as a bisimplicial set by trivializing its structure in either the horizontal or vertical direction, i.e., $X_{n,m} = X_m$ or $X_{n,m} = X_n$.]

EXAMPLE Any functor $T : \mathbf{\Delta} \to \mathbf{CAT}$ gives rise to a functor $X_T : \mathbf{CAT} \to \mathbf{BISISET}$ by writing $X_T\mathbf{I}([n],[m]) = \operatorname{ner}_n([T[m], \mathbf{I}])$ $(\approx \operatorname{Nat}([n], [T[m], \mathbf{I}]) \approx \operatorname{Nat}(T[m], [[n], \mathbf{I}]) \approx (S_T[[n], \mathbf{I}])_m$, S_T the singular functor (cf. p. 0–16)).

EXAMPLE Let $\mathbf{C}$ be a double category, i.e., a category object in **CAT**—then $\operatorname{ner} \mathbf{C}$ is a simplicial object in **CAT**, hence $\operatorname{ner}(\operatorname{ner} \mathbf{C})$ is a bisimplicial set.

Viewing $[n]$ as a small category, one may form its simplex category $\mathbf{\Delta}[n]$ $(= \mathbf{\Delta} \operatorname{ner}[n] = \mathbf{\Delta}\Delta[n] = \mathbf{\Delta}/[n])$. The assignments $[n] \to \operatorname{ner} \mathbf{\Delta}[n]$, $[n] \to \Delta[n]$ define cosimplicial objects $\mathbf{Y}_{\mathbf{\Delta}}, Y_{\mathbf{\Delta}}$ in **SISET** which are cofibrant in the Reedy structure and there is a weak equivalence $\mathbf{Y}_{\mathbf{\Delta}} \to Y_{\mathbf{\Delta}}$ (cf. p. 13–17).

Let X be a bisimplicial set—then $\operatorname{hocolim} X = \int^{[n]} X_n \times \operatorname{ner}([n] \backslash \mathbf{\Delta}^{\mathrm{OP}})^{\mathrm{OP}} = \int^{[n]} X_n \times \operatorname{ner}(\mathbf{\Delta}/[n]) = \int^{[n]} X_n \times \operatorname{ner} \mathbf{\Delta}[n] \to \int^{[n]} X_n \times \Delta[n] = |X|$.

PROPOSITION 49 The arrow $\operatorname{hocolim} X \to |X|$ is a weak homotopy equivalence.

[Bearing in mind Proposition 39, take a fibrant Z and consider the arrow $\text{map}(|X|, Z) \rightarrow \text{map}(\text{hocolim}\, X, Z)$ or still, the arrow $\text{HOM}(X, \text{map}(Y_{\Delta}, Z)) \rightarrow \text{HOM}(X, \text{map}(\mathbf{Y}_{\Delta}, Z))$. In the Reedy structure, X is necessarily cofibrant while $\text{map}(Y_{\Delta}, Z) \rightarrow \text{map}(\mathbf{Y}_{\Delta}, Z)$ is a weak equivalence between fibrant objects (see the lemma prefacing Proposition 48). One may therefore quote Proposition 34.]

Using the notation of the Kan extension theorem, take $\mathbf{C} = \mathbf{\Delta}^{\text{OP}}$, $\mathbf{D} = \mathbf{\Delta}^{\text{OP}} \times \mathbf{\Delta}^{\text{OP}}$, $\mathbf{S} = \mathbf{SET}$, and let K be the diagonal $\mathbf{\Delta}^{\text{OP}} \rightarrow \mathbf{\Delta}^{\text{OP}} \times \mathbf{\Delta}^{\text{OP}}$—then the functor $[K, \mathbf{S}] \equiv \text{di} : \mathbf{BISISET} \rightarrow \mathbf{SISET}$ has both a right and left adjoint. One calls di the diagonal: $(\text{di}X)_n = X([n],[n])$, the operators being $\begin{cases} d_i = d_i^h d_i^v = d_i^v d_i^h \\ s_i = s_i^h s_i^v = s_i^v s_i^h \end{cases}$. Example: $\text{di}(X \underline{\times} Y) = X \times Y$ ($\Rightarrow \text{di}\Delta[n,m] = \Delta[n] \times \Delta[m]$).

PROPOSITION 50 Up to natural isomorphism, di and $|?|$ are the same.

[It suffices to prove that di is a left adjoint for sin : $\text{Nat}(\text{di}X, Y) \approx \text{Nat}(X, \sin Y)$. But $X \approx \text{colim}_{i,j}\, \Delta[n_i, m_j]$ and one has $\text{Nat}(\Delta[n,m], \sin Y) \approx \text{map}(\Delta[n], Y)_m \approx \text{Nat}(\Delta[n] \times \Delta[m], Y) \approx \text{Nat}(\text{di}\Delta[n,m], Y)$.]

Application: $\forall$ bisimplicial set X, there is a weak homotopy equivalence $\text{hocolim}\, X \rightarrow \text{di}X$.

EXAMPLE Let $F : \mathbf{I} \rightarrow \mathbf{SISET}$—then in the notation of the simplicial replacement lemma, F determines a bisimplicial set $\coprod F$ by the rule $(\coprod F)_n = \displaystyle\coprod_{[n] \xrightarrow{f} \mathbf{I}^{\text{OP}}} Ffn$. And: $\text{hocolim}\, F \approx |\coprod F| \approx \text{di} \coprod F$.

EXAMPLE Place on $\mathbf{CGH}$ its singular structure and equip $[\mathbf{\Delta}^{\text{OP}}, \mathbf{CGH}]$ with the corresponding Reedy structure. Take an X in $\mathbf{SICGH}$ which is both Reedy fibrant and Reedy cofibrant and let UX be the simplicial set obtained from X by forgetting the topologies—then the arrow $|UX| \rightarrow |X|$ is a weak homotopy equivalence. To see this, let $\sin X$ be the bisimplicial set defined by $(\sin X)_n = \sin X_n$ and write $\sin^{\top} X$ for the "transpose" of $\sin X$, i.e., $(\sin^{\top} X)_{n,m} = (\sin X)_{m,n}$ ($\Rightarrow (\sin^{\top} X)_{0,*} \approx UX$). Since $\sin X$ is Reedy fibrant, $\forall\, \alpha$, $\sin^{\top} X(\alpha)$ is a weak homotopy equivalence. Therefore the arrow $|\sin^{\top} X|_0 \rightarrow |\sin^{\top} X|$ is a weak homotopy equivalence (cf. p. 13–59). Write $|\sin X|$ for the simplicial object in $\mathbf{CGH}$ with $|\sin X|_n = |\sin X_n|$. Because $|\sin X|$ is Reedy cofibrant, in view of the Giever-Milnor theorem, the arrow $\|\sin X\| \rightarrow |X|$ is a weak homotopy equivalence (cf. Proposition 44). So, putting everything together gives $|UX| \approx \| \sin^{\top} X|_0| \xrightarrow{\sim} \|\sin^{\top} X\| \approx |\text{disin}^{\top} X| = |\text{disin}\, X| \approx \|\sin X\| \xrightarrow{\sim} |X|$.

PROPOSITION 51 Suppose that $f : X \to Y$ is a bisimplicial map. Assume: $\forall\ n$, $f_n : X_n \to Y_n$ is a weak homotopy equivalence—then $\text{di}f : \text{di}X \to \text{di}Y$ is a weak homotopy equivalence.

[Since all simplicial objects in $\widehat{\mathbf{\Delta}}$ are cofibrant in the Reedy structure, this is a consequence of Propositions 44 and 50.]

[Note: In both the statement and the conclusion, one can replace "weak homotopy equivalence" by "HG-equivalence" (cf. p. 13–59).]

Let $X \xrightarrow{f} Z \xleftarrow{g} Y$ by a 2-sink in **SISET**—then a commutative diagram

$$\begin{tikzcd} W \arrow[r] \arrow[d] & Y \arrow[d, "g"] \\ X \arrow[r, "f"'] & Z \end{tikzcd}$$

is said to be a pullback up to homotopy if the arrow $W \to X \times_Z Y$ is a weak homotopy equivalence. Example:

$$\begin{tikzcd} \dot{\Delta}[1] \arrow[r] \arrow[d] & \Lambda[1,2] \arrow[d] \\ \Delta[1] \arrow[r] & \Delta[2] \end{tikzcd}$$

is not a homotopy pullback but is a pullback up to homotopy.

FACT Let $f : X \to Y$ be a bisimplicial map. Assume: $\forall\ m, n\ \&\ \forall\ \alpha : [m] \to [n]$, the commutative diagram

$$\begin{tikzcd} X_n \arrow[r, "X\alpha"] \arrow[d, "f_n"'] & X_m \arrow[d, "f_m"] \\ Y_n \arrow[r, "Y\alpha"'] & Y_m \end{tikzcd}$$

is a pullback up to homotopy—then $\forall\ n$,

$$\begin{tikzcd} X_n \times \Delta[n] \arrow[r] \arrow[d] & \text{di}X \arrow[d] \\ Y_n \times \Delta[n] \arrow[r] & \text{di}Y \end{tikzcd}$$

is a pullback up to homotopy.

PROPOSITION 52 **BISISET** carries a proper model category structure in which a bisimplicial map $f : X \to Y$ is a weak equivalence if $\text{di}f$ is a weak homotopy equivalence, a fibration if $\text{di}f$ is a Kan fibration, and a cofibration if f has the LLP w.r.t. acyclic fibrations.

[This is an instance of the generalities on p. 13–47, the essential point being that di (which plays the role of "G") has both a right and left adjoint. In particular: di preserves filtered colimits. The stage is thus set for a small object argument. Let D be the left adjoint of di normalized by the condition $D\Delta[n] = \Delta[n,n]$. Put $\dot{\Delta}[n,n] = D\dot{\Delta}[n], \Lambda[k,n,n] = D\Lambda[k,n]$—then the arrow $\dot{\Delta}[n,n] \to \Delta[n,n]$ is a cofibration and the arrow $\Lambda[k,n,n] \to \Delta[n,n]$ is an acyclic cofibration ($|\text{di}\Lambda[k,n,n]|$ is contractible). The requisite factorizations can therefore be established in the usual way. Let us note only that every f admits a decomposition of the form $f = p \circ i$, where p is a fibration and i is an acyclic cofibration that has the LLP w.r.t. fibrations (specifically, i is a sequential colimit of pushouts of coproducts of inclusions $\Lambda[k,n,n] \to \Delta[n,n]$). As for properness, the part of PMC concerning pullbacks is obvious while the part concerning pushouts follows from the observation that a cofibration is necessarily an injective bisimplicial map.]

FACT Take **BISISET** in the model category structure supplied by Proposition 52—then the adjoint pair (D, di) induces an adjoint equivalence of categories between **HSISET** and **HBISISET**.

For certain purposes, it is technically more convenient to use a modification of the homotopy colimit in order to minimize the proliferation of opposites. Definition: Given $F \in \operatorname{Ob}[\mathbf{I}, \mathbf{C}]$, put $\overline{\operatorname{hocolim}_{\mathbf{I}}}\, F$ (or $\overline{\operatorname{hocolim}}\, F$) $= \int^{\mathbf{I}^{\mathrm{OP}}} F \square \operatorname{ner}(-\backslash \mathbf{I})$. The formal properties of $\overline{\operatorname{hocolim}}$ are the same as those of hocolim, the primary difference being that $\overline{\operatorname{hocolim}}\, F \approx |\coprod F|$, where now $(\coprod F)_n = \coprod\limits_{[n] \xrightarrow{f} \mathbf{I}} Ff0$.

EXAMPLE Let $F : \mathbf{I} \to \mathbf{CAT}$ be a functor—then the Grothendieck construction on F is the category $\operatorname{gro}_{\mathbf{I}} F$ whose objects are the pairs (i, X), where $i \in \operatorname{Ob}\mathbf{I}$ and $X \in \operatorname{Ob} Fi$, and whose morphisms are the arrows $(\delta, f) : (i, X) \to (j, Y)$, where $\delta \in \operatorname{Mor}(i, j)$ and $f \in \operatorname{Mor}((F\delta)X, Y)$ (composition is given by $(\delta', f') \circ (\delta, f) = (\delta' \circ \delta, f' \circ (F\delta')f))$. Put $NF = \operatorname{ner} \circ F$, so $NF : \mathbf{I} \to \mathbf{SISET}$. One can thus form $\overline{\operatorname{hocolim}}\, NF$ and Thomason[†] has shown that there is a natural weak homotopy equivalence $\eta : \overline{\operatorname{hocolim}}\, NF \to \operatorname{ner} \operatorname{gro}_{\mathbf{I}} F$. The situation for homotopy limits is simpler. Indeed, $\operatorname{holim} NF \approx \int_i \operatorname{map}(\operatorname{ner}(\mathbf{I}/i), (\operatorname{ner} \circ F)i) \approx \int_i \operatorname{ner}[\mathbf{I}/i, Fi] \approx \operatorname{ner}(\int_i [\mathbf{I}/i, Fi])$.

[Note: Here is the definition of η. Representing $\overline{\operatorname{hocolim}}\, NF$ as $\mathrm{di} \coprod NF$, fix n and consider a typical string $(i_0 \xrightarrow{\delta_0} i_1 \to \cdots \to i_{n-1} \xrightarrow{\delta_{n-1}} i_n, X_0 \to X_1 \to \cdots \to X_{n-1} \to X_n)$, where the $X_k \in \operatorname{Ob} Fi_0$ $(0 \leq k \leq n)$—then η_n takes it to the element of $\operatorname{ner}_n \operatorname{gro}_{\mathbf{I}} F$ given by $(i_0, X_0) \to (i_1, (F\delta_0)X_1) \to \cdots \to (i_n, (F\delta_{n-1} \circ \cdots \circ F\delta_0)X_n)$.]

Let $\mathbf{I}$ and $\mathbf{J}$ be small categories, $\nabla : \mathbf{J} \to \mathbf{I}$ a functor.

Notation: Given $i \in \operatorname{Ob}\mathbf{I}$, write $i\backslash\nabla$ for the comma category $|K_i, \nabla|$.

[Note: Dually, ∇/i stands for the comma category $|\nabla, K_i|$.]

Observation: The commutative diagram
$$\begin{tikzcd} i\backslash\nabla \arrow[r] \arrow[d] & \mathbf{J} \arrow[d] \\ i\backslash\mathbf{I} \arrow[r] & \mathbf{I} \end{tikzcd}$$
is a pullback square in **CAT**.

[Note: The fiber of ∇ over i is defined by the pullback square
$$\begin{tikzcd} \nabla^{-1}(i) \arrow[r] \arrow[d] & \mathbf{J} \arrow[d, "\nabla"] \\ \mathbf{1} \arrow[r, "K_i"'] & \mathbf{I} \end{tikzcd}$$
So:

[†] *Math. Proc. Cambridge Philos. Soc.* **85** (1979), 91–109; see also Heggie, *Cahiers Topologie Géom. Différentielle Catégoriques* **34** (1993), 13–36.

$\nabla^{-1}(i)$ is the subcategory of $\mathbf{J}$ having objects j such that $\nabla j = i$, morphisms δ such that $\nabla\delta = \mathrm{id}_i$, and there is a commutative diagram

$$\begin{tikzcd} \nabla^{-1}(i) \arrow[r] \arrow[d] \arrow[dr] & i\backslash\nabla \arrow[d] \\ \nabla/i \arrow[r] & \mathbf{J} \end{tikzcd}$$

.]

EXAMPLE The arrow $\mathrm{colim}\,\mathrm{ner}(\text{—}\backslash\nabla) \to \mathrm{ner}\,\mathbf{J}$ is an isomorphism. Viewed as an object in $[\mathbf{I}^{\mathrm{OP}}, \mathbf{SISET}]$ (structure L), $\mathrm{ner}(\text{—}\backslash\nabla)$ is free, hence cofibrant (cf. p. 13–38). Therefore the arrow $\overline{\mathrm{hocolim}}\,\mathrm{ner}(\text{—}\backslash\nabla) \to \mathrm{colim}\,\mathrm{ner}(\text{—}\backslash\nabla)$ ($\approx \mathrm{ner}\,\mathbf{J}$) is a weak homotopy equivalence (cf. Proposition 47).

[Note: Take $\mathbf{I} = \mathbf{J}$ and $\nabla = \mathrm{id}_{\mathbf{I}}$—then the arrow $\overline{\mathrm{hocolim}}\,\mathrm{ner}(\text{—}\backslash\mathbf{I}) = \int^i \mathrm{ner}(i\backslash\mathbf{I}) \times \mathrm{ner}(i\backslash\mathbf{I}^{\mathrm{OP}}) \to \int^i \mathrm{ner}(i\backslash\mathbf{I}) \times \Delta[0] \approx \mathrm{ner}\,\mathbf{I}$ is a weak homotopy equivalence, as is the arrow $\overline{\mathrm{hocolim}}\,\mathrm{ner}(\text{—}\backslash\mathbf{I}) = \int^i \mathrm{ner}(i\backslash\mathbf{I}) \times \mathrm{ner}(i\backslash\mathbf{I}^{\mathrm{OP}}) \to \int^i \Delta[0] \times \mathrm{ner}(i\backslash\mathbf{I}^{\mathrm{OP}}) \approx \mathrm{ner}\,\mathbf{I}^{\mathrm{OP}}$.]

LEMMA Let $\mathbf{I}$ and $\mathbf{J}$ be small categories, $\nabla : \mathbf{J} \to \mathbf{I}$ a functor—then $\forall\ F$ in $[\mathbf{I}^{\mathrm{OP}}, \mathbf{SISET}]$, $\int_i \mathrm{map}(\mathrm{ner}(i\backslash\nabla), Fi) \approx \int_j \mathrm{map}(\mathrm{ner}(j\backslash\mathbf{J}), (F \circ \nabla^{\mathrm{OP}})j)$, i.e., $\mathrm{HOM}(\mathrm{ner}(\text{—}\backslash\nabla), F) \approx \mathrm{HOM}(\mathrm{ner}(\text{—}\backslash\mathbf{J}), F \circ \nabla^{\mathrm{OP}})$.

[The left Kan extension of $\mathrm{ner}(\text{—}\backslash\mathbf{J})$ along ∇^{OP} is $\mathrm{ner}(\text{—}\backslash\nabla)$.]

PROPOSITION 53 Let $\mathbf{I}$ and $\mathbf{J}$ be small categories, $\nabla : \mathbf{J} \to \mathbf{I}$ a functor—then $\forall\ F$ in $[\mathbf{I}, \mathbf{SISET}]$, the arrow $\int^j (F \circ \nabla)j \times \mathrm{ner}(j\backslash\mathbf{J}) \to \int^i Fi \times \mathrm{ner}(i\backslash\nabla)$ is a weak homotopy equivalence.

[This is yet another application of Proposition 39. Thus fix a fibrant Z and pass to $\mathrm{map}(\int^i Fi \times \mathrm{ner}(i\backslash\nabla), Z) \to \mathrm{map}(\int^j (F \circ \nabla)j \times \mathrm{ner}(j\backslash\mathbf{J}), Z)$, i.e., to $\int_i \mathrm{map}(\mathrm{ner}(i\backslash\nabla), \mathrm{map}(Fi, Z)) \to \int_j \mathrm{map}(\mathrm{ner}(j\backslash\mathbf{J}), \mathrm{map}((F \circ \nabla)j, Z))$, i.e., to $\mathrm{HOM}(\mathrm{ner}(\text{—}\backslash\nabla), \mathrm{map}(F, Z)) \to \mathrm{HOM}(\mathrm{ner}(\text{—}\backslash\mathbf{J}), \mathrm{map}(F, Z) \circ \nabla^{\mathrm{OP}})$, which by the lemma is an isomorphism, hence a fortiori, a weak homotopy equivalence.]

A small category is contractible if its classifying space is contractible. Example: Every filtered category is contractible.

EXAMPLE Let $\mathbf{C}$ be a small category—then the cone $\Gamma\mathbf{C}$ of $\mathbf{C}$ is the small category with $\mathrm{Ob}\,\Gamma\mathbf{C} = \mathrm{Ob}\,\mathbf{C} \coprod \{\emptyset\}$, where $\emptyset$ is an adjoined initial object. Example: $\Gamma\mathbf{0} = \mathbf{1}$. So, $\Gamma\mathbf{C}$ is contractible (cf. p. 13–15) and $B\Gamma\mathbf{C} \approx \Gamma B\mathbf{C}$.

[Note: Given small categories $\begin{cases}\mathbf{C}\\ \mathbf{D}\end{cases}$, their join $C * D$ is the full subcategory of $\Gamma\mathbf{C} \times \Gamma\mathbf{D}$ with $\operatorname{Ob}\mathbf{C} * \mathbf{D} = \operatorname{Ob}\mathbf{C} \times \operatorname{Ob}\mathbf{D} \coprod \operatorname{Ob}\mathbf{C} \times \{\emptyset\} \coprod \{\emptyset\} \times \operatorname{Ob}\mathbf{D}$. Under the join, **CAT** is a symmetric monoidal category ($\mathbf{0}$ is the unit). One has $B(\mathbf{C} * \mathbf{D}) \approx B\mathbf{C} *_k B\mathbf{D}$.]

Given small categories $\begin{cases}\mathbf{I}\\ \mathbf{J}\end{cases}$, a functor $\nabla : \mathbf{J} \rightarrow \mathbf{I}$ is said to be strictly final provided that for every $i \in \operatorname{Ob}\mathbf{I}$, the comma category $|K_i, \nabla|$ is contractible. A strictly final functor is final. In particular: $\nabla : \mathbf{J} \rightarrow \mathbf{I}$ strictly final $\Rightarrow \operatorname{colim} \Delta \circ \nabla \approx \operatorname{colim} \Delta$, where $\Delta : \mathbf{I} \rightarrow \mathbf{SISET}$ (cf. p. 0–11).

[Note: A subcategory of a small category is strictly final if the inclusion is a strictly final functor.]

PROPOSITION 54 Let $\mathbf{I}$ and $\mathbf{J}$ be small categories, $\nabla : \mathbf{J} \rightarrow \mathbf{I}$ a strictly final functor—then $\forall\, F$ in $[\mathbf{I}, \mathbf{SISET}]$, the arrow $\overline{\operatorname{hocolim}}\, F \circ \nabla \rightarrow \overline{\operatorname{hocolim}}\, F$ is a weak homotopy equivalence.

[According to Proposition 53, the arrow $\overline{\operatorname{hocolim}}\, F \circ \nabla = \int^j (F \circ \nabla) j \times \operatorname{ner}(j \backslash \mathbf{J}) \rightarrow \int^i Fi \times \operatorname{ner}(i \backslash \nabla)$ is a weak homotopy equivalence. Claim: The arrow $\int^i Fi \times \operatorname{ner}(i \backslash \nabla) \rightarrow \int^i Fi \times \operatorname{ner}(i \backslash \mathbf{I}) = \overline{\operatorname{hocolim}}\, F$ is a weak homotopy equivalence. Indeed: $\operatorname{ner}(\text{—} \backslash \nabla)$, $\operatorname{ner}(\text{—} \backslash \mathbf{I})$ are cofibrant objects in $[\mathbf{I}^{\mathrm{OP}}, \mathbf{SISET}]$ and since ∇ is strictly final, the arrow $\operatorname{ner}(\text{—} \backslash \nabla) \rightarrow \operatorname{ner}(\text{—} \backslash \mathbf{I})$ is a weak equivalence. Therefore one may appeal to the example on p. 13–52.]

FACT Let $\mathbf{I}$ and $\mathbf{J}$ be small categories, $\nabla : \mathbf{J} \rightarrow \mathbf{I}$ a functor. Assume: $\operatorname{ner}\nabla : \operatorname{ner}\mathbf{J} \rightarrow \operatorname{ner}\mathbf{I}$ is a weak homotopy equivalence. Suppose that $F : \mathbf{I} \rightarrow \mathbf{SISET}$ sends the morphisms in $\mathbf{I}$ to weak homotopy equivalences—then the arrow $\overline{\operatorname{hocolim}}\, F \circ \nabla \rightarrow \overline{\operatorname{hocolim}}\, F$ is a weak homotopy equivalence.

[The commutative diagram
$$\begin{tikzcd} \coprod F \circ \nabla \arrow[r] \arrow[d] & \coprod F \arrow[d] \\ \coprod * \arrow[r] & \coprod * \end{tikzcd}$$
of bisimplicial sets is a pullback square, therefore the commutative diagram
$$\begin{tikzcd} \operatorname{di} \coprod F \circ \nabla \arrow[r] \arrow[d] & \operatorname{di} \coprod F \arrow[d] \\ \operatorname{di} \coprod * \arrow[r] & \operatorname{di} \coprod * \end{tikzcd}$$
of simplicial sets is a pullback square (di is a right adjoint). Accordingly, in **SISET**, the commutative diagram
$$\begin{tikzcd} \overline{\operatorname{hocolim}}\, F \circ \nabla \arrow[r] \arrow[d] & \overline{\operatorname{hocolim}}\, F \arrow[d] \\ \operatorname{ner}\mathbf{J} \arrow[r] & \operatorname{ner}\mathbf{I} \end{tikzcd}$$
is a

pullback square. The result thus follows from the fact that the arrow $\overline{\text{hocolim}}\, F \rightarrow \text{ner}\, \mathbf{I}$ is a homotopy fibration (cf. p. 13–64).]

EXAMPLE If $\mathbf{I}$ is contractible and if $F : \mathbf{I} \rightarrow \mathbf{SISET}$ sends the morphisms in $\mathbf{I}$ to weak homotopy equivalences, then $\forall\, i \in \text{Ob}\, \mathbf{I}$, the arrow $Fi \rightarrow \overline{\text{hocolim}}\, F$ is a weak homotopy equivalence.

FACT (Homotopy Pushdowns) Let $\mathbf{I}$ and $\mathbf{J}$ be small categories, $\nabla : \mathbf{J} \rightarrow \mathbf{I}$ a functor. Given a functor $G : \mathbf{J} \rightarrow \mathbf{SISET}$, define an object $\overline{\text{hocolim}}_\nabla\, G$ in $[\mathbf{I}, \mathbf{SISET}]$ by $(\overline{\text{hocolim}}_\nabla\, G)i = \overline{\text{hocolim}}_{\nabla/i}\, G \circ U_i$, where $U_i : \nabla/i \rightarrow \mathbf{J}$ is the forgetful functor—then the arrow $\overline{\text{hocolim}}_{\mathbf{I}}\, \overline{\text{hocolim}}_\nabla\, G \rightarrow \overline{\text{hocolim}}_{\mathbf{J}}\, G$ is a weak homotopy equivalence.

QUILLEN'S THEOREM A Suppose that $\mathbf{I}$ and $\mathbf{J}$ are small categories and $\nabla : \mathbf{J} \rightarrow \mathbf{I}$ is a strictly final functor—then $\text{ner}\, \nabla : \text{ner}\, \mathbf{J} \rightarrow \text{ner}\, \mathbf{I}$ is a weak homotopy equivalence, hence $B\nabla : B\mathbf{J} \rightarrow B\mathbf{I}$ is a homotopy equivalence.

[In Proposition 54, let F be the functor $\mathbf{I} \rightarrow \mathbf{SISET}$ that sends $i \in \text{Ob}\, \mathbf{I}$ to $Fi = \Delta[0]$.]

[Note: The same conclusion obtains if ∇ is "strictly initial".]

EXAMPLE Let X be a topological space, $\sin X$ its singular set—then $\sin X$ can be regarded as a category:

$$\begin{tikzcd}
\Delta^m \arrow[rr, "\Delta^\alpha"] \arrow[dr] & & \Delta^n \arrow[dl] \\
& X &
\end{tikzcd}$$

($\alpha \in \text{Mor}([m], [n])$) (cf. p. 4–38). This category is isomorphic to $\Delta/X \equiv \text{gro}_{\mathbf{\Delta}}\, \sin X$ and there is a natural weak homotopy equivalence $\text{ner}\Delta/X \rightarrow \sin X$ (cf. p. 13–17), which thus gives a natural weak homotopy equivalence $B\Delta/X \rightarrow X$ (Giever-Milnor theorem). Let $\mathbf{C}$ be any small full subcategory of $\mathbf{TOP}/X$ containing Δ/X as a subcategory. Assume: $\forall\, Y \rightarrow X$ in $\mathbf{C}$, Y is homotopically trivial—then the arrow $B\iota : B\Delta/X \rightarrow B\mathbf{C}$ induced by the inclusion $\iota : \Delta/X \rightarrow \mathbf{C}$ is a homotopy equivalence. To see this, one can suppose that X is nonempty and appeal to Quillen's theorem A. Claim: ι is a strictly initial functor, i.e., $\forall\, Y \rightarrow X$ in $\mathbf{C}$, the comma category $\iota/Y \rightarrow X$ is contractible. Indeed, $\iota/Y \rightarrow X$ is simply Δ/Y and the arrow $B\Delta/Y \rightarrow *$ is a weak homotopy equivalence, hence a homotopy equivalence.

Let $\mathbf{C}$ be a category—then the twisted arrow category $\mathbf{C}(\rightsquigarrow)$ of $\mathbf{C}$ is the category whose objects are the arrows $f : X \rightarrow Y$ of $\mathbf{C}$ and whose morphisms $f \rightarrow f'$ are the pairs $(\phi, \psi) : \begin{cases} \phi \in \text{Mor}(X', X) \\ \psi \in \text{Mor}(Y, Y') \end{cases}$ for which the square

$$\begin{tikzcd}
X \arrow[r, "f"] & Y \arrow[d, "\psi"] \\
X' \arrow[u, "\phi"] \arrow[r, "f'"'] & Y'
\end{tikzcd}$$

commutes. Denote by $\begin{cases} s \\ t \end{cases}$ the canonical projections $\begin{cases} \mathbf{C}(\rightsquigarrow) \rightarrow \mathbf{C}^{\text{OP}} \\ \mathbf{C}(\rightsquigarrow) \rightarrow \mathbf{C} \end{cases}$.

EXAMPLE Suppose that $\mathbf{C}$ is a small category—then $\operatorname{ner} s : \operatorname{ner}\mathbf{C}(\rightsquigarrow) \rightarrow \operatorname{ner}\mathbf{C}^{\mathrm{OP}}$, $\operatorname{ner} t : \operatorname{ner}\mathbf{C}(\rightsquigarrow) \rightarrow \operatorname{ner}\mathbf{C}$ are weak homotopy equivalences.

[To discuss $\operatorname{ner} s$, observe that $\forall\ X$, the functor $X\backslash\mathbf{C} \rightarrow s/X$ that sends $X \xrightarrow{f} Y$ to $(X \xrightarrow{f} Y, \mathrm{id}_X)$ (so $s(X \xrightarrow{f} Y) \xrightarrow{\mathrm{id}_X} X$) has a left adjoint. Since $X\backslash\mathbf{C}$ is contractible, s/X must be too (cf. p. 13–15), i.e., s is strictly initial, thus by Quillen's theorem A, $\operatorname{ner} s$ is a weak homotopy equivalence.]

[Note: It is a corollary that $\operatorname{ner}\mathbf{C}$ and $\operatorname{ner}\mathbf{C}^{\mathrm{OP}}$ are naturally weakly equivalent.]

Let $\mathbf{I}$ and $\mathbf{J}$ be small categories, $\nabla : \mathbf{J} \rightarrow \mathbf{I}$ a functor—then by $\nabla(\rightsquigarrow)$ we shall understand the category whose objects are the triples (i, δ, j), where $\delta : i \rightarrow \nabla j$, and whose morphisms $(i, \delta, j) \rightarrow (i', \delta', j')$ are the pairs $(\phi, \psi) : \begin{cases} \phi \in \operatorname{Mor}(i', i) \\ \psi \in \operatorname{Mor}(j, j') \end{cases}$ for which the square

$$\begin{tikzcd} i \arrow[r, "\delta"] & \nabla j \arrow[d, "\nabla\psi"] \\ i' \arrow[u, "\phi"] \arrow[r, "\delta'"'] & \nabla j' \end{tikzcd}$$

commutes. Example: $\mathrm{id}_{\mathbf{I}}(\rightsquigarrow) = \mathbf{I}(\rightsquigarrow)$.

QUILLEN'S THEOREM B Suppose that $\mathbf{I}$ and $\mathbf{J}$ are small categories and $\nabla : \mathbf{J} \rightarrow \mathbf{I}$ is a functor with the property that for every morphism $i' \rightarrow i''$ in $\mathbf{I}$, the arrow $\operatorname{ner}(i''\backslash\nabla) \rightarrow \operatorname{ner}(i'\backslash\nabla)$ is a weak homotopy equivalence—then $\forall\ i \in \operatorname{Ob}\mathbf{I}$, the pullback square

$$\begin{tikzcd} \operatorname{ner}(i\backslash\nabla) \arrow[r] \arrow[d] & \operatorname{ner}\mathbf{J} \arrow[d] \\ \operatorname{ner}(i\backslash\mathbf{I}) \arrow[r] & \operatorname{ner}\mathbf{I} \end{tikzcd}$$

is a homotopy pullback.

[Each of the squares in the commutative diagram

$$\begin{tikzcd} i\backslash\nabla \arrow[r] \arrow[d] & \nabla(\rightsquigarrow) \arrow[r] \arrow[d] & \mathbf{J} \arrow[d] \\ i\backslash\mathbf{I} \arrow[r] \arrow[d] & \mathbf{I}(\rightsquigarrow) \arrow[r] \arrow[d] & \mathbf{I} \\ \mathbf{1} \arrow[r] & \mathbf{I}^{\mathrm{OP}} & \end{tikzcd}$$

are pullback squares in **CAT**, hence each of the squares in the commutative diagram

$$\begin{tikzcd} \operatorname{ner}(i\backslash\nabla) \arrow[r] \arrow[d] & \operatorname{ner}\nabla(\rightsquigarrow) \arrow[r] \arrow[d] & \operatorname{ner}\mathbf{J} \arrow[d] \\ \operatorname{ner}(i\backslash\mathbf{I}) \arrow[r] \arrow[d] & \operatorname{ner}\mathbf{I}(\rightsquigarrow) \arrow[r] \arrow[d] & \operatorname{ner}\mathbf{I} \\ \Delta[0] \arrow[r] & \operatorname{ner}\mathbf{I}^{\mathrm{OP}} & \end{tikzcd}$$

are pullback squares in **SISET** (ner is a right adjoint). And, from the definitions, $\overline{\operatorname{hocolim}}\operatorname{ner}(-\backslash\nabla) \approx \operatorname{ner}\nabla(\rightsquigarrow)$, $\overline{\operatorname{hocolim}}\operatorname{ner}(-\backslash\mathbf{I}) \approx \operatorname{ner}\mathbf{I}(\rightsquigarrow)$. Since the arrows $\overline{\operatorname{hocolim}}\operatorname{ner}(-\backslash\mathbf{I}) \rightarrow \operatorname{ner}\mathbf{I}^{\mathrm{OP}}$, $\operatorname{ner}(i\backslash\mathbf{I}) \rightarrow \Delta[0]$ are weak homotopy equivalences, the commutative diagram

$$\begin{tikzcd} \operatorname{ner}(i\backslash\mathbf{I}) \arrow[r] \arrow[d] & \overline{\operatorname{hocolim}}\operatorname{ner}(-\backslash\mathbf{I}) \arrow[d] \\ \Delta[0] \arrow[r] & \operatorname{ner}\mathbf{I}^{\mathrm{OP}} \end{tikzcd}$$

is a homotopy pullback (cf. p. 12–

14); since the arrows $\overline{\text{hocolim}}\,\text{ner}(—\backslash\nabla) \to \text{ner}\,\mathbf{J}$, $\overline{\text{hocolim}}\,\text{ner}(—\backslash\mathbf{I}) \to \text{ner}\,\mathbf{I}$ are weak homotopy equivalences, the commutative diagram

$$\begin{tikzcd} \overline{\text{hocolim}}\,\text{ner}(—\backslash\nabla) \arrow[r] \arrow[d] & \text{ner}\,\mathbf{J} \arrow[d] \\ \overline{\text{hocolim}}\,\text{ner}(—\backslash\mathbf{I}) \arrow[r] & \text{ner}\,\mathbf{I} \end{tikzcd}$$

is a homotopy pullback (cf. p. 12–14). Owing to our assumption on ∇, the arrow $\overline{\text{hocolim}}\,\text{ner}(—\backslash\nabla) \to \text{ner}\,\mathbf{I}^{\text{OP}}$ is a homotopy fibration (cf. p. 13–64). Accordingly, the pullback square

$$\begin{tikzcd} \text{ner}(i\backslash\nabla) \arrow[r] \arrow[d] & \overline{\text{hocolim}}\,\text{ner}(—\backslash\nabla) \arrow[d] \\ \Delta[0] \arrow[r] & \text{ner}\,\mathbf{I}^{\text{OP}} \end{tikzcd}$$

is a homotopy pullback (cf. p. 12–16). The composition lemma therefore implies that the commutative diagram

$$\begin{tikzcd} \text{ner}(i\backslash\nabla) \arrow[r] \arrow[d] & \overline{\text{hocolim}}\,\text{ner}(—\backslash\nabla) \arrow[d] \\ \text{ner}(i\backslash\mathbf{I}) \arrow[r] & \overline{\text{hocolim}}\,\text{ner}(—\backslash\mathbf{I}) \end{tikzcd}$$

is a homotopy pullback. Finally, then, by another application of the composition lemma, one concludes that the commutative diagram

$$\begin{tikzcd} \text{ner}(i\backslash\nabla) \arrow[r] \arrow[d] & \text{ner}\,\mathbf{J} \arrow[d] \\ \text{ner}(i\backslash\mathbf{I}) \arrow[r] & \text{ner}\,\mathbf{I} \end{tikzcd}$$

is a homotopy pullback.]

[Note: One can also formulate the result in terms of the ∇/i.]

LEMMA If

$$\begin{tikzcd} W \arrow[r] \arrow[d] & Y \arrow[d, "g"] \\ X \arrow[r, "f"'] & Z \end{tikzcd}$$

is a homotopy pullback in **SISET**, then

$$\begin{tikzcd} {|W|} \arrow[r] \arrow[d] & {|Y|} \arrow[d, "|g|"] \\ {|X|} \arrow[r, "|f|"'] & {|Z|} \end{tikzcd}$$

is a homotopy pullback in **CGH** (singular structure) and the arrow $|W| \to W_{|f|,|g|}$ is a homotopy equivalence (compactly generated double mapping track).

[In the notation of p. 12–13, write $Y \xrightarrow{\sim} \overline{Y} \twoheadrightarrow Z$—then $\overline{Y} \to Z$ Kan $\Rightarrow$ $|\overline{Y}| \to |Z|$ Serre and $W \to X \times_{\overline{Y}} Z$ goes to $|W| \to |X \times_{\overline{Y}} Z| = |X| \times_{|\overline{Y}|} |Z|$ (cf. Proposition 1), so

$$\begin{tikzcd} {|W|} \arrow[r] \arrow[d] & {|Y|} \arrow[d, "|g|"] \\ {|X|} \arrow[r, "|f|"'] & {|Z|} \end{tikzcd}$$

is a homotopy pullback in **CGH**. The double mapping track of the 2-sink $|X| \xrightarrow{|f|} |Z| \xleftarrow{|g|} Y$ calculated in **TOP** is a CW space (cf. §6, Proposition 8). Its image under k is $W_{|f|,|g|}$, thus $W_{|f|,|g|}$ is a CW space. Therefore the arrow $|W| \to W_{|f|,|g|}$, which is a priori a weak homotopy equivalence, is actually a homotopy equivalence.]

Consequently, under the conditions of Quillen's theorem B, $\forall\ i \in \text{Ob}\,\mathbf{I} : \nabla^{-1}(i) \neq \mathbf{0}$, there is a homotopy equivalence $B(i\backslash\nabla) \to E_{B\nabla}$ (compactly generated mapping fiber), so $\forall\ j \in \nabla^{-1}(i)$, there is an exact sequence $\cdots \to \pi_{q+1}(B\mathbf{I}, i) \to \pi_q(B(i\backslash\nabla), (j, \text{id}_i)) \to \pi_q(B\mathbf{J}, j) \to \pi_q(B\mathbf{I}, i) \to \cdots$.

Remark: It is thus a corollary that theorem B $\Rightarrow$ theorem A.

Waldhausen[†] has extended Quillen's theorems A and B from **CAT** to $[\mathbf{\Delta}^{\mathrm{OP}}, \mathbf{CAT}]$.

Fix an abelian group G—then a commutative diagram
$$\begin{tikzcd} W \arrow[r] \arrow[d] & Y \arrow[d, "g"] \\ X \arrow[r, "f"'] & Z \end{tikzcd}$$
of simplicial sets is said to be an HG-pullback if for some factorization $Y \xrightarrow{\sim} \overline{Y} \twoheadrightarrow Z$ of g, the induced simplicial map $W \to X \times_Z \overline{Y}$ is an HG-equivalence. Here, the factorization of g is in the usual model category structure on **SISET** and not in that of the homological model category theorem, hence the choice of the factorization of g is immaterial and one can work with either g or f. Example: A homotopy pullback is an HG-pullback.

[Note: When $G = \mathbf{Z}$, the term is homology pullback.]

Example: A commutative diagram
$$\begin{tikzcd} W \arrow[r] \arrow[d] & Y \arrow[d, "g"] \\ X \arrow[r, "f"'] & Z \end{tikzcd}$$
of simplicial sets, where f is a weak homotopy equivalence, is an HG-pullback iff the arrow $W \to Y$ is an HG-equivalence.

COMPOSITION LEMMA Consider the commutative diagram
$$\begin{tikzcd} \bullet \arrow[r] \arrow[d] & \bullet \arrow[r] \arrow[d] & \bullet \arrow[d] \\ \bullet \arrow[r] & \bullet \arrow[r] & \bullet \end{tikzcd}$$
in **SISET**. Assume: The square on the right is a homotopy pullback—then the rectangle is an HG-pullback iff the square on the left is an HG-pullback.

Rappel: **SISET** is a topos, so $\forall\ B$, $\mathbf{SISET}/B$ is a topos (MacLane-Moerdijk[‡]), thus is cartesian closed.

[Note: Similar remarks apply to **BISISET**.]

PROPOSITION 55 Let $F : \mathbf{I} \to \mathbf{SISET}$ be a functor. Assume: $\forall\ \delta \in \operatorname{Mor} \mathbf{I}$, $F\delta$ is an HG-equivalence—then $\forall\ i \in \operatorname{Ob} I$, the pullback square
$$\begin{tikzcd} Fi \arrow[r] \arrow[d] & \overline{\operatorname{hocolim}} F \arrow[d] \\ \Delta[0] \arrow[r, "\Delta_i"'] & \operatorname{ner} \mathbf{I} \end{tikzcd}$$
is an HG-pullback.

[†] *CMS Conf. Proc.* **2** (1982), 141–184.

[‡] *Sheaves in Geometry and Logic*, Springer Verlag (1992), 190.

[Factor $\Delta[0] \xrightarrow[\Delta_i]{} \operatorname{ner}\mathbf{I}$ as $\Delta[0] \overset{}{\underset{\Delta_x}{\rightarrowtail}} X \twoheadrightarrow \operatorname{ner}\mathbf{I}$, where Δ_x is a weak homotopy equivalence, the claim being that the arrow $Fi \to X \times_{\operatorname{ner}\mathbf{I}} \overline{\operatorname{hocolim}} F$ is an HG-equivalence. In view of the small object argument, one can suppose that Δ_x is a sequential colimit of pushouts of coproducts of inclusions $\Lambda[k,n] \to \Delta[n]$. Because of this and the fact that the functor $- \times_{\operatorname{ner}\mathbf{I}} \overline{\operatorname{hocolim}} F$ preserves colimits, it is obviously enough to prove that every diagram of the form

$$\begin{tikzcd}
 & & \overline{\operatorname{hocolim}} F \arrow[d] \\
\Lambda[k,n] \arrow[r] & \Delta[n] \arrow[r, "\Delta_f"'] & \operatorname{ner}\mathbf{I}
\end{tikzcd}$$

leads to an HG-equivalence $\Lambda[k,n] \times_{\operatorname{ner}\mathbf{I}} \overline{\operatorname{hocolim}} F \to \Delta[n] \times_{\operatorname{ner}\mathbf{I}} \overline{\operatorname{hocolim}} F$. To begin with, $\Delta[n] \times_{\operatorname{ner}\mathbf{I}} \overline{\operatorname{hocolim}} F \approx \overline{\operatorname{hocolim}} F \circ f$ $(f : [n] \to \mathbf{I})$. Furthermore, the initial object $0 \in [n]$ defines a natural transformation $F \circ f(0) \to F \circ f$, so there is a commutative diagram

$$\begin{tikzcd}
\coprod\limits_{\substack{\alpha_0 \to \cdots \to \alpha_m \\ \in \Lambda[k,n]}} F \circ f(0) \arrow[r] \arrow[d] & \coprod\limits_{\substack{\alpha_0 \to \cdots \to \alpha_m \\ \in \Delta[n]}} F \circ f(0) \arrow[d] \\
\coprod\limits_{\substack{\alpha_0 \to \cdots \to \alpha_m \\ \in \Lambda[k,n]}} F \circ f(\alpha_0) \arrow[r] & \coprod\limits_{\substack{\alpha_0 \to \cdots \to \alpha_m \\ \in \Delta[n]}} F \circ f(\alpha_0)
\end{tikzcd}$$

of bisimplicial sets. The hypothesis on F, in conjunction with the appended note to Proposition 51, implies that the diagonal of either vertical arrow is an HG-equivalence. But the diagonal of the top horizontal arrow is the weak homotopy equivalence $\Lambda[k,n] \times F \circ f(0) \to \Delta[n] \times F \circ f(0)$, therefore the diagonal of the bottom horizontal arrow is an HG-equivalence, i.e., $\Lambda[k,n] \times_{\operatorname{ner}\mathbf{I}} \overline{\operatorname{hocolim}} F \to \Delta[n] \times_{\operatorname{ner}\mathbf{I}} \overline{\operatorname{hocolim}} F$ is an HG-equivalence.]

PROPOSITION 56 Suppose that $\mathbf{I}$ and $\mathbf{J}$ are small categories and $\nabla : \mathbf{J} \to \mathbf{I}$ is a functor with the property that for every morphism $i' \to i''$ in $\mathbf{I}$, the arrow $\operatorname{ner}(i''\backslash\nabla) \to \operatorname{ner}(i'\backslash\nabla)$ is an HG-equivalence—then $\forall\, i \in \operatorname{Ob}\mathbf{I}$, the pullback square

$$\begin{tikzcd}
\operatorname{ner}(i\backslash\nabla) \arrow[r] \arrow[d] & \operatorname{ner}\mathbf{J} \arrow[d] \\
\operatorname{ner}(i\backslash\mathbf{I}) \arrow[r] & \operatorname{ner}\mathbf{I}
\end{tikzcd}$$

is an HG-pullback.

[One has only to trace the proof of Quillen's theorem B, using Proposition 55 to establish that the pullback square

$$\begin{tikzcd}
\operatorname{ner}(i\backslash\nabla) \arrow[r] \arrow[d] & \overline{\operatorname{hocolim}}\, \operatorname{ner}(-\backslash\nabla) \arrow[d] \\
\Delta[0] \arrow[r] & \operatorname{ner}\mathbf{I}^{\mathrm{OP}}
\end{tikzcd}$$

is an HG-pullback.]

[Note: It follows that $\forall\ i \in \operatorname{Ob}\mathbf{I} : \nabla^{-1}(i) \neq \mathbf{0}$, the arrow $B(i\backslash\nabla) \to E_{B\nabla}$ is an HG-equivalence (compactly generated mapping fiber).]

Proposition 56 is the homological analog of Quillen's theorem B. The same style of argument can also be used for it (in Proposition 55, replace "HG-equivalence" by "weak homotopy equivalence" and "HG-pullback" by "homotopy pullback").

Let (M, O) be a category object in **SISET**. Suppose that Y is a left $\mathbf{M}$-object and $\mathrm{tran}Y$ is the associated translation category—then the projection $T : Y \to O$ gives rise to an internal functor $\mathrm{tran}Y \to \mathbf{M}$ from which a morphism $\mathrm{ner}\,\mathrm{tran}Y \to \mathrm{ner}\,\mathbf{M}$ of simplicial objects in $\widehat{\mathbf{\Delta}}$ or still, a bisimplicial map. Each $x \in O_0$ determines a pullback square

$$\begin{tikzcd} Y_x \arrow[r] \arrow[d] & Y \arrow[d, "T"] \\ \Delta[0] \arrow[r, "\Delta_x"'] & O \end{tikzcd}$$

in **SISET** and through $e : O \to M$, arrows $\Delta[0] \xrightarrow[\Delta_x]{} \mathrm{ner}_n \mathbf{M}$, thus there is a pullback square

$$\begin{tikzcd} Y_x \arrow[r] \arrow[d] & \mathrm{ner}\,\mathrm{tran}Y \arrow[d] \\ \Delta[0] \arrow[r, "\Delta_x"'] & \mathrm{ner}\,\mathbf{M} \end{tikzcd}$$

in **BISISET** (abuse of notation).

[Note: $\forall\, f \in M_0$, $\left\{\begin{matrix} sf \\ tf \end{matrix}\right. \in O_0$ and $\lambda : M \times_O Y \to Y$ defines an arrow $Y_{sf} \to Y_{tf}$.]

PROPOSITION 57 If $\forall\, f \in M_0$, the arrow $Y_{sf} \to Y_{tf}$ is an HG-equivalence, then the pullback square

$$\begin{tikzcd} Y_x \arrow[r] \arrow[d] & {|\mathrm{ner}\,\mathrm{tran}Y|} \arrow[d] \\ \Delta[0] \arrow[r, "|\Delta_x|"'] & {|\mathrm{ner}\,\mathbf{M}|} \end{tikzcd}$$

(cf. Proposition 50) is an HG-pullback provided that O is a constant simplicial set.

[Use the model category structure on **BISISET** furnished by Proposition 52 to factor $\Delta[0] \xrightarrow[\Delta_x]{} \mathrm{ner}\,\mathbf{M}$ as $p \circ i$, where p is a fibration and i is an acyclic cofibration representable as a sequential colimit of pushouts of coproducts of inclusions $\Lambda[k, n, n] \to \Delta[n, n]$. Reasoning as in the proof of Proposition 55, it suffices to show that for any diagram of the form

$$\begin{tikzcd} & & \mathrm{ner}\,\mathrm{tran}Y \arrow[d] \\ \Lambda[k,n,n] \arrow[r] & \Delta[n,n] \arrow[r, "\Delta_f"'] & \mathrm{ner}\,\mathbf{M} \end{tikzcd}$$

, $|\Lambda[k, n, n]| \times_{|\mathrm{ner}\,\mathbf{M}|} |\mathrm{ner}\,\mathrm{tran}Y| \to |\Delta[n, n]| \times_{|\mathrm{ner}\,\mathbf{M}|} |\mathrm{ner}\,\mathrm{tran}Y|$ is an HG-equivalence. The arrow $\Delta_f : \Delta[n, n] \to \mathrm{ner}\,\mathbf{M}$ corresponds to $x_0 \xrightarrow{f_0} x_1 \to \cdots \to x_{n-1} \xrightarrow{f_{n-1}} x_n$, where the $x_i \in O_n$ $(= O)$ and the $f_i \in M_n$. This said, consider the commutative diagram

$$\begin{tikzcd} \Lambda[k,n,n] \times Y_{x_0} \arrow[r] \arrow[d] & \Delta[n,n] \times Y_{x_0} \arrow[d] \\ \Lambda[k,n,n] \times_{\mathrm{ner}\,\mathbf{M}} \mathrm{ner}\,\mathrm{tran}Y \arrow[r] & \Delta[n,n] \times_{\mathrm{ner}\,\mathbf{M}} \mathrm{ner}\,\mathrm{tran}Y \end{tikzcd}$$

which results from piecing together the definitions. The diagonal of the top horizontal arrow is an HG-equivalence ($|\mathrm{di}\Lambda[k,n,n]|$ is contractible), as is the diagonal of the two vertical arrows.]

[Note: Changing the assumption to "weak homotopy equivalence" changes the conclusion to "homotopy pullback".]

EXAMPLE Let (M,O) be a category object in **SISET** with $O \approx \Delta[0]$. So: M is a simplicial monoid or, equivalently, M is a simplicial object in $\mathbf{MON}_{\mathbf{SET}}$. Let Y be a left $\mathbf{M}$-object. Assume: $\forall\ m \in M_0$, $m_* : H_*(|Y|;G) \to H_*(|Y|;G)$ is an isomorphism—then the pullback square

$$\begin{tikzcd} Y \arrow[r] \arrow[d] & {|\mathrm{bar}(*;\mathbf{M};Y)|} \arrow[d] \\ \Delta[0] \arrow[r] & {|\mathrm{bar}(*;\mathbf{M};*)|} \end{tikzcd}$$

is an HG-pullback.

Let
$$\begin{tikzcd} W \arrow[r] \arrow[d] & Y \arrow[d] \\ X \arrow[r] & Z \end{tikzcd}$$
be a commutative diagram of bisimplicial sets. Problem: Find conditions which ensure that
$$\begin{tikzcd} \mathrm{di}W \arrow[r] \arrow[d] & \mathrm{di}Y \arrow[d] \\ \mathrm{di}X \arrow[r] & \mathrm{di}Z \end{tikzcd}$$
is a homotopy pullback. To this end, assume that $\forall\ n$,
$$\begin{tikzcd} W_n \arrow[r] \arrow[d] & Y_n \arrow[d] \\ X_n \arrow[r] & Z_n \end{tikzcd}$$
is a homotopy pullback. Using the Reedy structure on $[\mathbf{\Delta}^{\mathrm{OP}},\mathbf{SISET}]$, construct a commutative diagram
$$\begin{tikzcd} W \arrow[r] \arrow[d] & Y \arrow[r] \arrow[d] & \overline{Y} \arrow[d] \\ X \arrow[r] & Z \arrow[r] & \overline{Z} \end{tikzcd}$$
, where $\left\{\begin{array}{l} Y \to \overline{Y} \\ Z \to \overline{Z} \end{array}\right.$ are levelwise weak homotopy equivalences, $\left\{\begin{array}{l} \overline{Y} \\ \overline{Z} \end{array}\right.$ are Reedy fibrant, and $\overline{Y} \to \overline{Z}$ is a Reedy fibration—then $\forall\ n$,
$$\begin{tikzcd} W_n \arrow[r] \arrow[d] & \overline{Y}_n \arrow[d] \\ X_n \arrow[r] & \overline{Z}_n \end{tikzcd}$$
is a homotopy pullback. Form the commutative diagram
$$\begin{tikzcd} \mathrm{di}W \arrow[r] \arrow[d] & \mathrm{di}Y \arrow[r] \arrow[d] & \mathrm{di}\overline{Y} \arrow[d] \\ \mathrm{di}X \arrow[r] & \mathrm{di}Z \arrow[r] & \mathrm{di}\overline{Z} \end{tikzcd}$$
. The square
$$\begin{tikzcd} \mathrm{di}Y \arrow[r] \arrow[d] & \mathrm{di}\overline{Y} \arrow[d] \\ \mathrm{di}Z \arrow[r] & \mathrm{di}\overline{Z} \end{tikzcd}$$
is a homotopy pullback (cf. Proposition 51), so by the composition lemma,
$$\begin{tikzcd} \mathrm{di}W \arrow[r] \arrow[d] & \mathrm{di}Y \arrow[d] \\ \mathrm{di}X \arrow[r] & \mathrm{di}Z \end{tikzcd}$$
will be a homotopy pullback if this is the case of
$$\begin{tikzcd} \mathrm{di}W \arrow[r] \arrow[d] & \mathrm{di}\overline{Y} \arrow[d] \\ \mathrm{di}X \arrow[r] & \mathrm{di}\overline{Z} \end{tikzcd}$$
. Since $\forall\ n, \overline{Y}_n \to \overline{Z}_n$ is a Kan fibration (cf. Proposition

41), the induced map $W \to X \times_{\overline{Z}} \overline{Y}$ of bisimplicial sets is a levelwise weak homotopy equivalence, thus $\mathrm{di}W \to \mathrm{di}X \times_{\mathrm{di}\overline{Z}} \mathrm{di}\overline{Y}$ is a weak homotopy equivalence (cf. Proposition 51). Therefore the central issue is whether $\mathrm{di}\overline{Y} \to \mathrm{di}\overline{Z}$ is a Kan fibration. However it is definitely not automatic that di takes Reedy fibrations to Kan fibrations, meaning that conditions have to be imposed.

EXAMPLE Let $\left\{\begin{array}{l} X \\ Y \end{array}\right.$ be simplicial sets, $f : X \to Y$ a simplicial map. Extend $\left\{\begin{array}{l} X \\ Y \end{array}\right.$ to bisimplicial sets by rendering them trivial in the vertical direction—then the associated bisimplicial map is a fibration in the Reedy structure and its diagonal is f but, of course, f need not be Kan.

PROPOSITION 58 Let $\left\{\begin{array}{l} X \\ Y \end{array}\right.$ be bisimplicial sets, $f : X \to Y$ a Reedy fibration. Assume: $\forall\, m$, the arrow $X_{*,m} \to Y_{*,m}$ is a Kan fibration—then $\mathrm{di}f : \mathrm{di}X \to \mathrm{di}Y$ is a Kan fibration.

[Convert the lifting problem

$$\begin{tikzcd} \Lambda[k,n] \arrow[r] \arrow[d] & \mathrm{di}X \arrow[d] \\ \Delta[n] \arrow[r] \arrow[ur, dashed] & \mathrm{di}Y \end{tikzcd}$$

to the lifting problem

$$\begin{tikzcd} \Lambda[k,n,n] \arrow[r] \arrow[d] & X \arrow[d] \\ \Delta[n,n] \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd}$$

(notation as in the proof of Proposition 52) and factor the inclusion $\Lambda[k,n,n] \to \Delta[n,n]$ as $\Lambda[k,n,n] \to \Lambda[k,n] \underline{\times} \Delta[n] \to \Delta[n,n]$. Since f is Reedy, it has the RLP w.r.t. the first inclusion and since f is horizontally Kan, it has the RLP w.r.t. the second inclusion.]

Let K be a simplicial set. Given a bisimplicial set X, the matching space of X at K is the simplicial set $M_K X$ defined by the end $\int_{[n]} X_n^{K_n}$. So: $M_K X([m]) \approx \mathrm{Nat}(\Delta[m], \int_{[n]} X_n^{K_n})$ $\approx \int_{[n]} \mathrm{Nat}(\Delta[m], X_n^{K_n}) \approx \int_{[n]} \mathrm{Nat}(\Delta[m], X_n)^{K_n} \approx \int_{[n]} X_{n,m}^{K_n} \approx \int_{[n]} \mathrm{Mor}(K_n, X_{n,m}) \approx$ $\mathrm{Nat}(K, X_{*,m})$. Obviously, $M_K X$ is functorial, covariant in X and contravariant in K.

[Note: The functor $X \to M_K X$ is a right adjoint for the functor $L \to K \underline{\times} L$.]

Examples: (1) $M_{\Delta[n]}X([m]) \approx \mathrm{Nat}(\Delta[n], X_{*,m}) \approx X_{n,m} \Rightarrow M_{\Delta[n]}X \approx X_{n,*} (\equiv X_n)$; (2) $M_{\dot{\Delta}[n]}X([m]) \approx \mathrm{Nat}(\dot{\Delta}[n], X_{*,m}) \approx \mathrm{Nat}(sk^{(n-1)}\Delta[n], X_{*,m}) \approx (cosk^{(n-1)}X)_n \Rightarrow$ $M_{\dot{\Delta}[n]}X \approx M_n X$.

[Note: The inclusion $\dot{\Delta}[n] \to \Delta[n]$ leads to an arrow $M_{\Delta[n]}X \to M_{\dot{\Delta}[n]}X$ or still, to an arrow $X_n \to M_n X$, which is precisely the matching morphism.]

One can use an analogous definition for the matching space of X at K if X is a simplicial set rather than a bisimplicial set: $M_K X = \int_{[n]} X_n^{K_n}$ $(\approx \mathrm{Nat}(K, X))$.

[Note: Suppose that X is a bisimplicial set—then $M_K X_{*,m} \approx (M_K X)_m$.]

Put $M_{k,n} X = M_{\Lambda[k,n]} X$ $(0 \leq k \leq n, n \geq 1)$. Because $\Lambda[k,n] \subset \dot{\Delta}[n]$, there are arrows $X_n \to M_n X \to M_{k,n} X$ natural in X.

LEMMA A simplicial map $K \to L$ is a Kan fibration iff the arrows $K_n \to M_{k,n}K \times_{M_{k,n}L} L_n$ are surjective $(0 \leq k \leq n, n \geq 1)$.

[Note: A simplicial map $K \to L$ is a Kan fibration and a weak homotopy equivalence iff the arrows $K_n \to M_n K \times_{M_n L} L_n$ are surjective $(n \geq 0)$.]

PROPOSITION 59 Let $\begin{cases} X \\ Y \end{cases}$ be bisimplicial sets, $f : X \to Y$ a Reedy fibration. Suppose that the arrows $\pi_0(X_{n,*}) \to \pi_0(M_{k,n}X \times_{M_{k,n}Y} Y_{n,*})$ arising from the squares

$$\begin{tikzcd} X_{n,*} \arrow[r] \arrow[d] & Y_{n,*} \arrow[d] \\ M_{k,n}X \arrow[r] & M_{k,n}Y \end{tikzcd}$$

are surjective $(0 \leq k \leq n, n \geq 1)$—then di$f$ is a Kan fibration.

[Since **SISET** satisfies SMC, so does **BISISET** (Reedy structure) (cf. p. 13–55). Applying this to the cofibration $\Lambda[k,n]\underline{\times}\Delta[0] \to \Delta[n]\underline{\times}\Delta[0]$, it follows that the arrow $\text{HOM}(\Delta[n]\underline{\times}\Delta[0], X) \to \text{HOM}(\Lambda[k,n]\underline{\times}\Delta[0], X) \times_{\text{HOM}(\Lambda[k,n]\underline{\times}\Delta[0], Y)} \text{HOM}(\Delta[n]\underline{\times}\Delta[0], Y)$ is a Kan fibration. Therefore the arrow $X_{n,*} \to M_{k,n}X \times_{M_{k,n}Y} Y_{n,*}$ is a Kan fibration. It is surjective by the assumption on π_0. The lemma thus implies that f is horizontally Kan, from which the assertion (cf. Proposition 58).]

Convention: The homotopy groups of a pointed simplicial set are those of its geometric realization.

Homotopy groups commute with finite products. Homotopy groups also commute with infinite products if the data is fibrant but not in general (consider $\pi_1(\text{S}[1]^\omega)$).

Let X be a bisimplicial set—then for $n, q \geq 1$ and $x \in X_{n,0}$, there are homomorphisms $(d_i^h)_* : \pi_q(X_{n,*}, x) \to \pi_q(X_{n-1,*}, d_i^h x)$ $(0 \leq i \leq n)$.

(π_q) X satisfies the π_q-Kan condition at $x \in X_{n,0}$ if for every finite sequence $(\alpha_0, \ldots, \widehat{\alpha}_k, \ldots, \alpha_n)$, where $\alpha_i \in \pi_q(X_{n-1,*}, d_i^h x)$ and $(d_i^h)_*\alpha_j = (d_{j-1}^h)_*\alpha_i$ $(i < j$ & $i, j \neq k)$, $\exists$ $\alpha \in \pi_q(X_{n,*}, x) : (d_i^h)_*\alpha = \alpha_i$ $(i \neq k)$.

[Note: If $x', x'' \in X_{n,0}$ are in the same component of X_n, then X satisfies the π_q-Kan condition at x' iff X satisfies the π_q-Kan condition at x''.]

Definition: A bisimplicial set X satisfies the π_*-Kan condition if $\forall$ $n, q \geq 1$, X satisfies the π_q-Kan condition at each $x \in X_{n,0}$.

Example: Bisimplicial groups satisfy the π_*-Kan condition.

EXAMPLE Let X be a bisimplicial set such that $\forall$ n, X_n is connected—then X satisfies the π_*-Kan condition.

[Consider the π_q-Kan condition at $x = s_{n-1}^h \cdots s_0^h x_0$ $(x_0 \in X_{0,0})$.]

LEMMA Let $\begin{cases} X \\ Y \end{cases}$ be bisimplicial sets, $f : X \to Y$ a bisimplicial map. Assume: f is a levelwise weak homotopy equivalence—then X satisfies the π_*-Kan condition iff Y satisfies the π_*-Kan condition.

One can describe $\dot{\Delta}[n]$ as the simplicial subset of $\Delta[n]$ generated by the $d_i \mathrm{id}_{[n]}$ $(0 \leq i \leq n)$ and one can describe $\Lambda[k,n]$ as the simplicial subset of $\Delta[n]$ generated by the $d_i \mathrm{id}_{[n]}$ $(0 \leq i \leq n, i \neq k)$. In general, if $t_0, \ldots, t_r$ are integers such that $0 \leq t_0 < \cdots < t_r \leq n$, let $\Delta_n^{(t_0,\ldots,t_r)}$ be the simplicial subset of $\Delta[n]$ generated by the $d_{t_0}\mathrm{id}_{[n]}, \ldots, d_{t_r}\mathrm{id}_{[n]}$—then there is a pushout square

$$\begin{tikzcd}
\Delta_{n-1}^{(t_0,\ldots,t_{r-1})} \arrow[r, "\Delta[\delta^n_{t_{r-1}}]"] \arrow[d] & \Delta_n^{(t_0,\ldots,t_{r-1})} \arrow[d] \\
\Delta[n-1] \arrow[r, "\Delta[\delta^n_{t_r}]"'] & \Delta_n^{(t_0,\ldots,t_r)}
\end{tikzcd} .$$

[Note: $\Delta_n^{(t_0,\ldots,t_r)}$ is a simplicial subset of $\Lambda[k,n]$ provided that $k \neq t_i$ $(i = 0, \ldots, r)$.]

Given a bisimplicial set X, write $M_n^{(t_0,\ldots,t_r)}X$ for the matching space of X at $\Delta_n^{(t_0,\ldots,t_r)}$. There are arrows $X_n \to M_nX \to M_n^{(t_0,\ldots,t_r)}X$ natural in X. Example: $M_n^{(0,\ldots,\widehat{k},\ldots n)}X = M_{k,n}X$.

[Note: $M_n^{(t_0,\ldots,t_r)}X([m])$ consists of the set of finite sequences $(x_{t_0}, \ldots, x_{t_r})$ of elements of $X_{n-1,m}$ such that $d_i^h x_j = d_{j-1}^h x_i$ for all $i < j$ in $\{t_0, \ldots, t_r\}$ (cf. p. 13–18). Moreover, the arrow $X_n \to M_n^{(t_0,\ldots,t_r)}X$ sends $x \in X_{n,m}$ to $(d_{t_0}^h x, \ldots, d_{t_r}^h x)$ and it is Kan if X is Reedy fibrant.]

LEMMA Let X be a bisimplicial set. Assume: X is Reedy fibrant and satisfies the π_*-Kan condition. Suppose that $x = (x_{t_0}, \ldots, x_{t_r}) \in M_n^{(t_0,\ldots,t_r)}X([0])$—then $\forall\, q \geq 1$, the map $\pi_q(M_n^{(t_0,\ldots,t_r)}X, x) \to \pi_q(X_{n-1,*}, x_{t_0}) \times \cdots \times \pi_q(X_{n-1,*}, x_{t_r})$ is injective and its range is the set of finite sequences $(\alpha_{t_0}, \ldots, \alpha_{t_r})$ in the product such that $(d_i^h)_*\alpha_j = (d_{j-1}^h)_*\alpha_i$ for all $i < j$ in $\{t_0, \ldots, t_r\}$.

[Work inductively with the pullback squares

$$\begin{tikzcd}
M_n^{(t_0,\ldots,t_r)}X \arrow[r] \arrow[d] & X_{n-1,*} \arrow[d] \\
M_n^{(t_0,\ldots,t_{r-1})}X \arrow[r] & M_{n-1}^{(t_0,\ldots,t_{r-1})}X
\end{tikzcd} .]$$

[Note: The result also holds for $q = 0$.]

Given a bisimplicial set X, define a simplicial set $\pi_0(X)$ by $\pi_0(X)_n = \pi_0(X_n)$ $(= \pi_0(X_{n,*}))$. Example: Suppose that X is Reedy fibrant and satisfies the π_*-Kan condition—then $\pi_0(M_{k,n}X) \approx M_{k,n}\pi_0(X)$.

EXAMPLE Let X be a bisimplicial set such that $\forall\ n$, the path components of $|X_n|$ are abelian. Write $[\mathbf{S}^q, X]$ for the simplicial set with $[\mathbf{S}^q, X]_n = [\mathbf{S}^q, |X_n|]$—then X satisfies the π_*-Kan condition if the simplicial map $[\mathbf{S}^q, X] \rightarrow \pi_0(X)$ is a Kan fibration $\forall\ q \geq 1$.

FACT Let X be a bisimplicial set such that $\forall\ n$, X_n is connected—then $\mathrm{di}X$ is connected. [There is a coequalizer diagram $\pi_0(X_1) \underset{d_0}{\overset{d_1}{\rightrightarrows}} \pi_0(X_0) \rightarrow \pi_0(\mathrm{di}X)$.]

PROPOSITION 60 Let $\begin{cases} X \\ Y \end{cases}$ be bisimplicial sets, $f : X \rightarrow Y$ a Reedy fibration with $f_* : \pi_0(X) \rightarrow \pi_0(Y)$ a Kan fibration. Assume: $\begin{cases} X \\ Y \end{cases}$ are Reedy fibrant and satisfy the π_*-Kan condition—then $\mathrm{di}f$ is a Kan fibration.

[According to Proposition 59, it suffices to show that the arrows $\pi_0(X_{n,*}) \rightarrow \pi_0(M_{k,n}X \times_{M_{k,n}Y} Y_{n,*})$ are surjective $(0 \leq k \leq n, n \geq 1)$. Consider the square

$$\begin{tikzcd} X_{n,*} \arrow[r] \arrow[d] & Y_{n,*} \arrow[d] \\ M_{k,n}X \arrow[r] & M_{k,n}Y \end{tikzcd}$$

—then $\pi_0(M_{k,n}X \times_{M_{k,n}Y} Y_{n,*}) \approx \pi_0(M_{k,n}X) \times_{\pi_0(M_{k,n}Y)} \pi_0(Y_{n,*})$. In fact, $Y_{n,*} \rightarrow M_{k,n}Y$ is a Kan fibration and the lemma implies that $\forall\ y \in Y_{n,0}, Y_{n,*} \rightarrow M_{k,n}Y$ induces a surjection of fundamental groups (cf. infra). But $\pi_0(M_{k,n}X) \times_{\pi_0(M_{k,n}Y)} \pi_0(Y_{n,*}) \approx M_{k,n}\pi_0(X) \times_{M_{k,n}\pi_0(Y)} \pi_0(Y_{n,*})$ and $\pi_0(X_{n,*}) \rightarrow M_{k,n}\pi_0(X) \times_{M_{k,n}\pi_0(Y)} \pi_0(Y_{n,*})$ is surjective, $\pi_0(X) \rightarrow \pi_0(Y)$ being Kan by assumption (cf. p. 13–81).]

LEMMA Let

$$\begin{tikzcd} X' \arrow[r] \arrow[d] & X \arrow[d, "p"] \\ B' \arrow[r] & B \end{tikzcd}$$

be a pullback square of topological spaces, where $p : X \rightarrow B$ is a Serre fibration. Assume: $\forall\ x \in X$, the homomorphism $\pi_1(X, x) \rightarrow \pi_1(B, p(x))$ is surjective—then the arrow $\pi_0(X') \rightarrow \pi_0(B') \times_{\pi_0(B)} \pi_0(X)$ is bijective.

[Injectivity is a consequence of the π_1-hypothesis.]

THEOREM OF BOUSFIELD-FRIEDLANDER Let

$$\begin{tikzcd} W \arrow[r] \arrow[d] & Y \arrow[d] \\ X \arrow[r] & Z \end{tikzcd}$$

be a commutative diagram of bisimplicial sets such that $\forall\ n$,

$$\begin{tikzcd} W_n \arrow[r] \arrow[d] & Y_n \arrow[d] \\ X_n \arrow[r] & Z_n \end{tikzcd}$$

is a homotopy pullback. Assume: $\pi_0(Y) \rightarrow \pi_0(Z)$ is a Kan fibration and Y, Z satisfy the π_*-Kan condition—then

$$\begin{tikzcd} \mathrm{di}W \arrow[r] \arrow[d] & \mathrm{di}Y \arrow[d] \\ \mathrm{di}X \arrow[r] & \mathrm{di}Z \end{tikzcd}$$

is a homotopy pullback.

[Proceed as on p. 13–79 ff.: $\mathrm{di}\overline{Y} \rightarrow \mathrm{di}\overline{Z}$ is a Kan fibration (cf. Proposition 60).]

[Proceed as on p. 13–79 ff.: $\mathrm{di}\overline{Y} \to \mathrm{di}\overline{Z}$ is a Kan fibration (cf. Proposition 60).]

[Note: When Y_n, Z_n are connected $\forall\ n$, $\pi_0(Y) \to \pi_0(Z)$ is trivially Kan and Y, Z necessarily satisfy the π_*-Kan condition (cf. p. 13–81).]

Let K be a simplicial set. Given a bisimplicial set X, define a bisimplicial set $\mathrm{map}(K, X)$ by $\mathrm{map}(K, X)_n = \mathrm{map}(K, X_n)$.

LEMMA There is a canonical arrow $|\mathrm{map}(K, X)| \to \mathrm{map}(K, |X|)$.

[The evaluation $K \times \mathrm{map}(K, X_n) \to X_n$ defines a bisimplicial map $K \times \mathrm{map}(K, X) \to X$ or still, a simplicial map $|K \times \mathrm{map}(K, X)| \to |X|$. However multiplication by K in **BISISET** is a left adjoint, hence $|K \times \mathrm{map}(K, X)| \approx K \times |\mathrm{map}(K, X)|$.]

A bisimplicial set X is said to be pointed if an $x \in X_{0,0}$ has been fixed and each X_n is equipped with the base point $s^h_{n-1} \cdots s^h_0 x_0$.

EXAMPLE Let X be a Reedy fibrant pointed bisimplicial set such that $\forall\ n$, X_n is connected—then X is π_*-Kan, thus $|X| \approx \mathrm{di}X$ is fibrant (cf. Proposition 60). Denote by ΘX (ΩX) the bisimplicial set which takes $[n]$ to ΘX_n (ΩX_n) (it follows from Proposition 41 that $\forall\ n$, X_n is fibrant). Specializing the lemma to $K = \Delta[1]$ provides us with canonical arrows $|\Theta X| \to \Theta|X|$ ($|\Omega X| \to \Omega|X|$) ($|?|$ preserves pullbacks) and a commutative diagram

$$\begin{tikzcd} {|\Omega X|} \arrow[r] \arrow[d] & {|\Theta X|} \arrow[r] \arrow[d] & {|X|} \arrow[d, equal] \\ \Omega|X| \arrow[r] & \Theta|X| \arrow[r] & {|X|} \end{tikzcd}$$

. On the other hand, the theorem of Bousfield-Friedlander says that

$$\begin{tikzcd} {|\Omega X|} \arrow[r] \arrow[d] & {|\Theta X|} \arrow[d] \\ \Delta[0] \arrow[r] & {|X|} \end{tikzcd}$$

is a homotopy pullback. Because the geometric realization of $|\Theta X|$ is contractible (cf. Proposition 51), the conclusion is that the canonical arrow $|\Omega X| \to \Omega|X|$ is a weak homotopy equivalence.

EXAMPLE Let X be a pointed bisimplicial set such that $\forall\ n$, $|X_n|$ is simply connected—then $|\mathrm{di}X|$ is simply connected.

[For this, one can suppose that X is Reedy fibrant. On general grounds, $|\mathrm{di}X|$ is path connected (cf. p. 13–83) and by the preceding example, $\pi_0(\mathrm{di}\Omega X) \approx \pi_0(\Omega\mathrm{di}X)$. But $\forall\ n$, ΩX_n is connected, thus $\mathrm{di}\Omega X$ is connected (cf. p. 13–83) and so $|\mathrm{di}X|$ is simply connected.

[Note: It is clear that the argument can be iterated: $|X_n|$ k-connected $\forall\ n \Rightarrow |\mathrm{di}X|$ k-connected.]

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§14. SIMPLICIAL SPACE

After working through the foundations of the theory, various applications will be given, e.g., the James construction and infinite symmetric products. I have also included some material on operads and delooping procedures.

A simplicial space is a simplicial object in **TOP** and a simplicial map is a morphism of simplicial spaces. **TOP**, in its standard structure, is a model category, thus **SITOP** is a model category (Reedy structure) (cf. p. 13–55). This fact notwithstanding, it will be simplest to proceed from first principles.

There is a forgetful functor **SITOP** → **SISET** and it has a left and right adjoint (cf. p. 0–15).

[Note: The purely set theoretic properties of simplicial spaces are the same as those of simplicial sets.]

Given an X in **SITOP**, put $|X| = \int^{[n]} X_n \times \Delta^n$—then $|X|$ is the geometric realization of X and the assignment $X \rightarrow |X|$ is a functor **SITOP** → **TOP**. $|?|$ has a right adjoint **TOP** → **SITOP** (compact open topology on the singular set).

EXAMPLE (Star Construction) Let X be a nonempty topological space. Define a simplicial space ΛX by the prescription $(\Lambda X)_n = X \times \cdots \times X$ ($n+1$ factors) with $d_i(x_0, \ldots, x_n) = (x_0, \ldots, \widehat{x_i}, \ldots, x_n)$, $s_i(x_0, \ldots, x_n) = (x_0, \ldots, x_i, x_i, \ldots, x_n)$. Represent Δ^n as the set of points $(t_1, \ldots, t_n)$ in $\mathbf{R}^n$ such that $0 \leq t_1 \leq \cdots \leq t_n \leq 1$ (which entails a change in the formulas defining the simplicial operators). Form X^* as on p. 1–28 and let $\lambda_n : X^{n+1} \times \Delta^n \rightarrow X^*$ be the continuous function that sends $((x_0, \ldots, x_n), (t_1, \ldots, t_n))$ to the right continuous step function $[0, 1[\rightarrow X$ which is equal to x_i on $[t_i, t_{i+1}[$ $(t_0 = 0, t_{n+1} = 1)$—then the λ_n combine to give a continuous bijection $\lambda : |\Lambda X| \rightarrow X^*$. Since X $T_2 \Rightarrow X^*$ T_2, $|\Lambda X|$ is Hausdorff whenever X is and in this situation, the composite $X \rightarrow X \times \Delta^0 \rightarrow |\Lambda X|$ is a closed embedding.

[Note: Like X^*, $|\Lambda X|$ is contractible (cf. p. 14–17).]

A simplicial space X is said to be Hausdorff, compactly generated ... if $\forall$ n, X_n is Hausdorff, compactly generated ... , i.e., if X is a simplicial object in **HAUS**, **CG** .... On general grounds, the geometric realization of a compactly generated simplicial space is automatically compactly generated but there is no a priori guarantee that the geometric realization of a Hausdorff simplicial space is Hausdorff.

Observation: If X is a simplicial space and if $\alpha : [m] \rightarrow [n]$ is an epimorphism, then $X\alpha : X_n \rightarrow X_m$ is an embedding and $(X\alpha)X_n$ is a retract of X_m.

Let X be a simplicial space—then X is said to satisfy the embedding condition if $\forall\ n$ & $\forall\ i$, $s_i : X_{n-1} \rightarrow X_n$ is a closed embedding. Examples: (1) A Hausdorff simplicial space satisfies the embedding condition; (2) A Δ-separated compactly generated simplicial space satisfies the embedding condition.

LEMMA Suppose given a diagram

$$\begin{tikzcd} X' \arrow[d, "p'"'] & X \arrow[d, "p"] \\ B' \arrow[r, "i"'] & B \end{tikzcd}$$

of topological spaces and continuous functions, where p is quotient and i is one-to-one. Assume: $\exists$ a neighborhood finite collection $\{A_j\}$ of closed subsets of X and continuous functions $f_j : A_j \rightarrow X'$ such that $p^{-1}(i(B')) = \bigcup\limits_j A_j$ with $p|A_j = i \circ p' \circ f_j\ \forall\ j$—then p' is quotient and i is a closed embedding.

If X is a simplicial space, then $|X|$ can be identified with the quotient $\coprod\limits_n X_n \times \Delta^n/\sim$, the equivalence relation being generated by writing $((X\alpha)x, t) \sim (x, \Delta^\alpha t)$. Let $p : \coprod\limits_n X_n \times \Delta^n \rightarrow |X|$ be the projection and put $|X|_n = p(\coprod\limits_{m \leq n} X_m \times \Delta^m)$.

PROPOSITION 1 Let X be a simplicial space. Assume: X satisfies the embedding condition—then $\forall\ n$, $|X|_n$ is a closed subspace of $|X|$ and $|X| = \operatorname{colim} |X|_n$.

[Fix n' and consider

$$\begin{tikzcd} \coprod\limits_{m \leq n'} X_m \times \Delta^m \arrow[d, "p'"'] & \coprod\limits_n X_n \times \Delta^n \arrow[d, "p"] \\ {|X|_{n'}} \arrow[r, "i"'] & {|X|} \end{tikzcd}$$

. For each $m \leq n'$ and n, there are but finitely many diagrams of the form $[m] \xleftarrow{\beta} [k] \xrightarrow{\alpha} [n]$, where α is a monomorphism and β is an epimorphism. Put $A_{\alpha,\beta} = (X\alpha)^{-1}(X\beta)X_m \times \Delta^\alpha\Delta^k \subset X_n \times \Delta^n$, define $f_{\alpha,\beta} : A_{\alpha,\beta} \rightarrow X_m \times \Delta^m$ by $f_{\alpha,\beta}(x,t) = (y, \Delta^\beta u)$ ($t = \Delta^\alpha u$ ($\exists!\ u \in \Delta^k$), $(X\alpha)x = (X\beta)y$ ($\exists!\ y \in X_m$)), and apply the lemma.]

FACT Suppose that X is a simplicial space satisfying the embedding condition. Define a simplicial set $\pi_0(X)$ by $\pi_0(X)_n = \pi_0(X_n)$—then $\pi_0(|X|) \approx \pi_0|\pi_0(X)|$.

[Every point in $|X|$ can be joined by a path in $|X|$ to a point in $X_0 = |X|_0$. On the other hand, given $x \in X_1$, $\sigma(t) = [x, (1-t, t)]$ $(0 \leq t \leq 1)$ is a path in $|X|$ which begins at d_1x and ends at d_0x.]

[Note: Therefore $|X|$ is path connected if X_0 is path connected.]

Notation: Given an X in **SITOP**, write sX_{n-1} for the union $s_0X_{n-1} \cup \cdots \cup s_{n-1}X_{n-1}$.

PROPOSITION 2 Let X be a simplicial space. Assume: X satisfies the embedding condition—then $\forall\, n$, there is a pushout square

$$\begin{tikzcd} X_n \times \dot{\Delta}^n \cup sX_{n-1} \times \Delta^n \arrow[r] \arrow[d] & {|X|_{n-1}} \arrow[d] \\ X_n \times \Delta^n \arrow[r] & {|X|_n} \end{tikzcd}.$$

[The arrow $X_n \times \Delta^n \to |X|_n$ is quotient. To see this, form

$$\begin{tikzcd} X_n \times \Delta^n \arrow[d] & \coprod\limits_{m \leq n} X_m \times \Delta^m \arrow[d] \\ {|X|_n} \arrow[r, equal] & {|X|_n} \end{tikzcd}.$$

Taking into account the lemma, let $f_n : X_n \times \Delta^n \to X_n \times \Delta^n$ be the identity. To define $f_m : X_m \times \Delta^m \to X_n \times \Delta^n$ if $m < n$, fix a monomorphism $\alpha : [m] \to [n]$, an epimorphism $\beta : [n] \to [m]$ such that $\beta \circ \alpha = \mathrm{id}_{[m]}$, and put $f_m(x,t) = ((X\beta)x, \Delta^{\alpha} t)$.]

Application: Suppose that X is a Δ-separated compactly generated simplicial space—then $|X|$ is a Δ-separated compactly generated space.

[$|X|_n$ is a Δ-separated compactly generated space (AD_6 (cf. p. 3–1)), thus the assertion follows from the fact that $|X| = \operatorname{colim} |X|_n$ (cf. p. 1–36).]

Let X be a simplicial space—then X is said to satisfy the cofibration condition if $\forall\, n$ & $\forall\, i$, $s_i : X_{n-1} \to X_n$ is a closed cofibration. Since the commutative diagram

$$\begin{tikzcd} X_{n-1} \arrow[r, "s_i"] \arrow[d, "s_j"'] & X_n \arrow[d, "s_{j+1}"] \\ X_n \arrow[r, "s_i"'] & X_{n+1} \end{tikzcd}$$

is a pullback square $(0 \leq i \leq j \leq n-1)$, one can use Proposition 8 in §3 to see that the cofibration condition implies that the $sX_{n-1} \to X_n$ are closed cofibrations.

Example: Given a topological space X, denote by $\mathrm{si}X$ the constant simplicial set on X, i.e., $\mathrm{si}X([n]) = X$ & $\begin{cases} d_i = \mathrm{id}_X \\ s_i = \mathrm{id}_X \end{cases}$ $(\forall\, n)$—then $\mathrm{si}X$ satisfies the cofibration condition and $|\mathrm{si}X| \approx X$.

Since $L_n X$ can be identified with sX_{n-1}, every X which satisfies the cofibration condition is necessarily cofibrant (Reedy structure).

FACT Suppose that X is a simplicial space satisfying the embedding condition—then X satisfies the cofibration condition iff X is Reedy cofibrant.

PROPOSITION 3 Let X be a simplicial space. Assume: X satisfies the cofibration condition—then $\forall\, n$, the arrow $|X|_{n-1} \to |X|_n$ is a closed cofibration.

[The arrow $X_n \times \dot{\Delta}^n \cup sX_{n-1} \times \Delta^n \to X_n \times \Delta^n$ is a closed cofibration (cf. §3, Proposition 7). Now quote Proposition 2 (cf. §3, Proposition 2).]

Application: Let X be a compactly generated simplicial space satisfying the cofibration condition. Assume: $\forall$ n, X_n is Hausdorff—then $|X|$ is a compactly generated Hausdorff space.

[This follows from the lemma on p. 3–8 and condition B on p. 1–29.]

Application: Let X be a simplicial space satisfying the cofibration condition. Assume: $\forall$ n, X_n is numerably contractible—then $|X|$ is numerably contractible.

[It suffices to show that the $|X|_n$ are numerably contractible (cf. p. 3–13). But inductively, the double mapping cylinder of the 2-source $X_n \times \Delta^n \leftarrow X_n \times \dot{\Delta}^n \cup sX_{n-1} \times \Delta^n \to |X|_{n-1}$ is numerably contractible and numerable contractibility is a homotopy type invariant (cf. p. 3–13).]

EXAMPLE Let X be a Hausdorff simplicial space. Assume: $\forall$ n, the inclusion $\Delta_{X_n} \to X_n \times X_n$ is a cofibration—then X satisfies the cofibration condition.

[$\forall$ i, $s_i X_{n-1}$ is a retract of X_n, hence the inclusion $s_i X_{n-1} \to X_n$ is a closed cofibration (cf. p. 3–15).]

LEMMA Let

$$\begin{tikzcd} X^0 \arrow[r] \arrow[d] & X^1 \arrow[r] \arrow[d] & \cdots \\ Y^0 \arrow[r] & Y^1 \arrow[r] & \cdots \end{tikzcd}$$

be a commutative ladder connecting two expanding sequences of topological spaces. Assume: $\forall$ n, the inclusions $\begin{cases} X^n \to X^{n+1} \\ Y^n \to Y^{n+1} \end{cases}$ are closed cofibrations,

$$\begin{tikzcd} X^n \arrow[r] \arrow[d] & X^{n+1} \arrow[d] \\ Y^n \arrow[r] & Y^{n+1} \end{tikzcd}$$

is a pullback square, and the vertical arrows $\phi^n : X^n \to Y^n$ are closed cofibrations—then the induced map $\phi^\infty : X^\infty \to Y^\infty$ is a closed cofibration.

[Take any arrow $Z \to B$ which is both a homotopy equivalence and a Hurewicz fibration and construct a filler $Y^\infty \to Z$ for

$$\begin{tikzcd} X^\infty \arrow[r] \arrow[d] & Z \arrow[d] \\ Y^\infty \arrow[r] & B \end{tikzcd}$$

via induction, noting that $Y^n \underset{X^n}{\sqcup} X^{n+1} \to Y^{n+1}$ is a closed cofibration (cf. §3, Proposition 8).]

Application: Let $X^0 \subset X^1 \subset \cdots$ be an expanding sequence of topological spaces. Assume: $\forall$ n, X^n is in **Δ-CG**, $X^n \to X^{n+1}$ is a cofibration, and $\Delta_{X^n} \to X^n \times_k X^n$ is a cofibration—then $\Delta_{X^\infty} \to X^\infty \times_k X^\infty$ is a cofibration.

EXAMPLE Let X be a Δ-separated compactly generated simplicial space. Assume: $\forall\, n$, $\Delta_{X_n} \to X_n \times_k X_n$ is a cofibration—then X satisfies the cofibration condition (cf. p. 3–15) and $\Delta_{|X|_n} \to |X|_n \times_k |X|_n$ is a cofibration (cf. p. 3–16). Therefore $\Delta_{|X|} \to |X| \times_k |X|$ is a cofibration.

FACT Let $\left\{\begin{array}{l} X \\ Y \end{array}\right.$ be Δ-separated compactly generated simplicial spaces satisfying the cofibration condition. Suppose that $f : X \to Y$ is a simplicial map such that $\forall\, n, f_n : X_n \to Y_n$ is a cofibration—then $|f| : |X| \to |Y|$ is a cofibration.

[Use the lemma on p. 3–15 ff. to conclude that $\forall\, n$, $|f|_n : |X|_n \to |Y|_n$ is a cofibration. And:

$$\begin{tikzcd} {|X|_{n-1}} \arrow[r] \arrow[d] & {|X|_n} \arrow[d] \\ {|Y|_{n-1}} \arrow[r] & {|Y|_n} \end{tikzcd}$$

is a pullback square.]

PROPOSITION 4 Suppose that $\left\{\begin{array}{l} X \\ Y \end{array}\right.$ are simplicial spaces satisfying the cofibration condition and let $f : X \to Y$ be a simplicial map. Assume: $\forall\, n$, $f_n : X_n \to Y_n$ is a homotopy equivalence—then $|f| : |X| \to |Y|$ is a homotopy equivalence.

[Since $\left\{\begin{array}{l} |X| = \operatorname{colim} |X|_n \\ |Y| = \operatorname{colim} |Y|_n \end{array}\right.$ and the $\left\{\begin{array}{l} |X|_{n-1} \to |X|_n \\ |Y|_{n-1} \to |Y|_n \end{array}\right.$ are closed cofibrations, it need only be shown that the $|X|_n \to |Y|_n$ are homotopy equivalences (cf. §3, Proposition 15). This is done by induction, the point being that $sX_{n-1} \to sY_{n-1}$ is a homotopy equivalence.]

EXAMPLE Let X be a simplicial space such that $\forall\, n$, X_n has the homotopy type of a compactly generated space—then the arrow $|kX| \to |X|$ is a homotopy equivalence if X satisfies the cofibration condition.

[$\forall\, n$, $kX_n \to X_n$ is a homotopy equivalence and kX satisfies the cofibration condition (cf. p. 3–8).]

Given an X in **SITOP**, the homotopic realization of X is the quotient $\mathrm{HR}X = \coprod_n X_n \times \Delta^n/\sim$, where $\sim$ is restricted to the monomorphisms in Δ, i.e., $((X\alpha)x, t) \sim (x, \Delta^\alpha t)$ $(\alpha \in M_\Delta)$. Write $(\mathrm{HR}X)_n$ for the image of $\coprod_{m \le n} X_m \times \Delta^m$ under the projection $\coprod_n X_n \times \Delta^n/\sim \to \mathrm{HR}X$.

Example: Viewing a simplicial set X as a "discrete" simplicial space, $\mathrm{HR}X = |UX|_M$ (cf. p. 13–7).

Example: $|*| = *$ but $\mathrm{HR}* =$ "a large contractible space".

PROPOSITION 5 Let X be a simplicial space—then $\forall\, n$, $(\mathrm{HR}X)_n$ is a closed subspace of $\mathrm{HR}X$ and $\mathrm{HR}X = \operatorname{colim}(\mathrm{HR}X)_n$.

PROPOSITION 6 Let X be a simplicial space—then $\forall\, n$, there is a pushout square

$$\begin{tikzcd} X_n \times \dot{\Delta}^n \arrow[r] \arrow[d] & (\mathrm{HR}X)_{n-1} \arrow[d] \\ X_n \times \Delta^n \arrow[r] & (\mathrm{HR}X)_n \end{tikzcd}$$

and the arrow $(\mathrm{HR}X)_{n-1} \rightarrow (\mathrm{HR}X)_n$ is a closed cofibration.

FACT Let X be a simplicial space. Assume: X_0 is numerably contractible—then $\mathrm{HR}X$ is numerably contractible.

[It suffices to show that the $(\mathrm{HR}X)_n$ are numerably contractible (cf. p. 3–13). This is done by induction on n, starting from $(\mathrm{HR}X)_0 = X_0$. Suppose, therefore, that n is positive and $(\mathrm{HR}X)_{n-1}$ is numerably contractible. Choose distinct points $u, v \in \overset{\circ}{\Delta}{}^n$. Because the arrow $X \times \Delta^n \rightarrow (\mathrm{HR}X)_n$ is surjective, $(\mathrm{HR}X)_n = U \cup V$, where $U = \mathrm{im}(X_n \times \Delta^n - \{u\})$, $V = \mathrm{im}(X_n \times \Delta^n - \{v\})$. But $\{U, V\}$ is a numerable covering of $(\mathrm{HR}X)_n$ and the retractions $\Delta^n - \{u\} \rightarrow \dot{\Delta}^n$, $\Delta^n - \{v\} \rightarrow \dot{\Delta}^n$ induce homotopy equivalences $U \rightarrow (\mathrm{HR}X)_{n-1}$, $V \rightarrow (\mathrm{HR}X)_{n-1}$.]

It follows from Propositions 5 and 6 that the homotopic realization of a Hausdorff simplicial space is a Hausdorff space and the homotopic realization of a (Δ-separated, Hausdorff) compactly generated simplicial space is a (Δ-separated, Hausdorff) compactly generated space.

[Note: Another corollary is that if $\forall\, n$, X_n is a CW space, then $\mathrm{HR}X$ is a CW space (cf. §5, Propositions 7 and 8).]

Notation: UW is the semisimplicial set defined by $\mathrm{UW}_n = \{(i_0, \ldots, i_n) : i_j \in \mathbf{Z}_{\geq 0}\ \&\ i_0 < \cdots < i_n\}$, where $d_j : \mathrm{UW}_n \rightarrow \mathrm{UW}_{n-1}$ sends $(i_0, \ldots, i_n)$ to $(i_0, \ldots, \widehat{i_j}, \ldots, i_n)$.

Let X be a simplicial space—then the unwinding $\mathrm{UW}X$ is the "homotopic realization" of the cofunctor $\mathbf{\Delta}_M \rightarrow \mathbf{TOP}$ which takes $[n]$ to $X_n \times \mathrm{UW}_n$ $(= \coprod_{i_0 < \cdots < i_n} X_n)$. Example: UW$*$ is the "infinite dimensional simplex" (Whitehead topology).

EXAMPLE Let G be a topological group, $\mathbf{G}$ the topological groupoid having a single object $*$ with $\mathrm{Mor}(*, *) = G$—then $\mathrm{ner}\,\mathbf{G}$ is a simplicial space and there is a canonical continuous bijection $\mathrm{UWner}\,\mathbf{G} \rightarrow B_G^\infty$.

[Note: This arrow is not a homeomorphism (consider $G = *$) but it is a homotopy equivalence.]

FACT For every simplicial space X, the projection $\mathrm{UW}X \rightarrow \mathrm{HR}X$ is a homotopy equivalence.

PROPOSITION 7 Let X be a simplicial space. Assume: X satisfies the cofibration condition—then the arrow $\mathrm{HR}X \rightarrow |X|$ is a homotopy equivalence.

[The argument is similar to that used in the proof of Proposition 4 in §13.]

Application: Let X be a simplicial space. Assume: $\forall\, n$, X_n is a CW space—then $|X|$ is a CW space whenever X satisfies the cofibration condition.

EXAMPLE Let X be a simplicial space satisfying the cofibration condition. Assume: X_0 is numerably contractible—then $|X|$ is numerably contractible (cf. p. 14–4).

[HRX is numerably contractible (cf. p. 14–6) and numerable contractibility is a homotopy type invariant (cf. p. 3–13).]

FACT Equip **TOP** with its standard structure. Let $f : X \to Y$ be a simplicial map. Assume: $\forall\, m,n$ & $\forall\, \alpha : [m] \to [n]$, the commutative diagram

$$\begin{tikzcd} X_n \arrow[r, "X\alpha"] \arrow[d, "f_n"'] & X_m \arrow[d, "f_m"] \\ Y_n \arrow[r, "Y\alpha"'] & Y_m \end{tikzcd}$$

is a homotopy pullback—then $\forall\, n$,

$$\begin{tikzcd} X_n \times \Delta^n \arrow[r] \arrow[d] & \text{HR}X \arrow[d] \\ Y_n \times \Delta^n \arrow[r] & \text{HR}Y \end{tikzcd}$$

is a homotopy pullback.

[One first shows by induction that $\forall\, n$,

$$\begin{tikzcd} X_n \times \Delta^n \arrow[r] \arrow[d] & (\text{HR}X)_n \arrow[d] \\ Y_n \times \Delta^n \arrow[r] & (\text{HR}Y)_n \end{tikzcd}$$

is a homotopy pullback. To carry out the passage from $n-1$ to n, observe that the squares in the commutative diagram

$$\begin{tikzcd} X_n \times \Delta^n \arrow[d] & X_n \times \dot{\Delta}^n \arrow[l] \arrow[r] \arrow[d] & (\text{HR}X)_{n-1} \arrow[d] \\ Y_n \times \Delta^n & Y_n \times \dot{\Delta}^n \arrow[l] \arrow[r] & (\text{HR}Y)_{n-1} \end{tikzcd}$$

are homotopy pullbacks, thus the squares in the commutative diagram

$$\begin{tikzcd} X_n \times \Delta^n \arrow[r] \arrow[d] & (\text{HR}X)_n \arrow[d] & (\text{HR}X)_{n-1} \arrow[l] \arrow[d] \\ Y_n \times \Delta^n \arrow[r] & (\text{HR}Y)_n & (\text{HR}Y)_{n-1} \arrow[l] \end{tikzcd}$$

are homotopy pullbacks (cf. p. 12–15). So, $\forall\, n$,

$$\begin{tikzcd} (\text{HR}X)_n \arrow[r] \arrow[d] & \text{HR}X \arrow[d] \\ (\text{HR}Y)_n \arrow[r] & \text{HR}Y \end{tikzcd}$$

is a homotopy pullback (cf. p. 12–15). Accordingly, both the squares in the commutative diagram

$$\begin{tikzcd} X_n \times \Delta^n \arrow[r] \arrow[d] & (\text{HR}X)_n \arrow[r] \arrow[d] & \text{HR}X \arrow[d] \\ Y_n \times \Delta^n \arrow[r] & (\text{HR}Y)_n \arrow[r] & \text{HR}Y \end{tikzcd}$$

are homotopy pullbacks, hence by the composition lemma,

$$\begin{tikzcd} X_n \times \Delta^n \arrow[r] \arrow[d] & \text{HR}X \arrow[d] \\ Y_n \times \Delta^n \arrow[r] & \text{HR}Y \end{tikzcd}$$

is a homotopy pullback.]

It follows from Proposition 7 that this result remains valid if $\left\{\begin{matrix}\mathrm{HR}X\\ \mathrm{HR}Y\end{matrix}\right.$ are replaced by $\left\{\begin{matrix}|X|\\ |Y|\end{matrix}\right.$ provided that $\left\{\begin{matrix}X\\ Y\end{matrix}\right.$ satisfy the cofibration condition. Proof: Consider the commutative diagram

$$\begin{tikzcd} X_n \times \Delta^n \arrow[r] \arrow[d] & {} \\ Y_n \times \Delta^n \arrow[r] & {} \end{tikzcd}$$

$$\begin{tikzcd} \mathrm{HR}X \arrow[r] \arrow[d] & {|X|} \arrow[d] \\ \mathrm{HR}Y \arrow[r] & {|Y|} \end{tikzcd}.$$

PROPOSITION 8 Suppose that $\left\{\begin{matrix}X\\ Y\end{matrix}\right.$ are simplicial spaces and let $f : X \to Y$ be a simplicial map. Assume: $\forall\ n$, $f_n : X_n \to Y_n$ is a homotopy equivalence—then $\mathrm{HR}f : \mathrm{HR}X \to \mathrm{HR}Y$ is a homotopy equivalence.

PROPOSITION 9 Suppose that $\left\{\begin{matrix}X\\ Y\end{matrix}\right.$ are simplicial spaces and let $f : X \to Y$ be a simplicial map. Assume: $\forall\ n$, $f_n : X_n \to Y_n$ is a weak homotopy equivalence—then $\mathrm{HR}f : \mathrm{HR}X \to \mathrm{HR}Y$ is a weak homotopy equivalence.

[If the vertical arrows in the commutative diagram

$$\begin{tikzcd} X_n \times \Delta^n \arrow[d] & X_n \times \dot{\Delta}^n \arrow[l] \arrow[r] \arrow[d] & (\mathrm{HR}X)_{n-1} \arrow[d] \\ Y_n \times \Delta^n & Y_n \times \dot{\Delta}^n \arrow[l] \arrow[r] & (\mathrm{HR}Y)_{n-1} \end{tikzcd}$$

are weak homotopy equivalences, then the induced map $(\mathrm{HR}X)_n \to (\mathrm{HR}Y)_n$ is a weak homotopy equivalence (cf. p. 4–51). Pass now to colimits via the result on p. 4–48.]

Application: Let $\left\{\begin{matrix}X\\ Y\end{matrix}\right.$ be simplicial spaces satisfying the cofibration condition. Suppose that $f : X \to Y$ is a simplicial map such that $\forall\ n$, $f_n : X_n \to Y_n$ is a weak homotopy equivalence—then $|f| : |X| \to |Y|$ is a weak homotopy equivalence.

Example: Let X be a simplicial space satisfying the cofibration condition. Consider the commutative triangle

$$\begin{tikzcd} k|X| \arrow[r] & {|X|} \\ & {|kX|} \arrow[ul] \arrow[u] \end{tikzcd}.$$

By the above, $|kX| \to |X|$ is a weak homotopy equivalence. Since the same is true of $k|X| \to |X|$, it follows that the arrow $|kX| \to k|X|$ is a weak homotopy equivalence.

EXAMPLE Given an X in **SITOP**, denote by $|\sin X|$ the simplicial space which takes $[n]$ to $|\sin X_n|$. Thanks to the Giever-Milnor theorem, the arrow of adjunction $|\sin X_n| \to X_n$ is a weak homotopy equivalence. On the other hand, $|\sin X|$ satisfies the cofibration condition. Consequently, the arrow $\|\sin X\| \to |X|$ is a weak homotopy equivalence if X satisfies the cofibration condition.

[Note: $\sin X$ is a bisimplicial set and $|\mathrm{disin}\, X| \approx \|\sin X\|$.]

EXAMPLE (Homotopy Pullbacks) Equip **CG** with its singular structure and suppose given a commutative diagram
$$\begin{tikzcd} W \arrow[r] \arrow[d] & Y \arrow[d, "g"] \\ X \arrow[r, "f"'] & Z \end{tikzcd}$$
of compactly generated simplicial spaces such that
$$\begin{tikzcd} W_n \arrow[r] \arrow[d] & Y_n \arrow[d] \\ X_n \arrow[r] & Z_n \end{tikzcd}$$
is a homotopy pullback in **CG** $\forall\ n$, where Y_n, Z_n are path connected. The associated commutative diagram
$$\begin{tikzcd} \sin W \arrow[r] \arrow[d] & \sin Y \arrow[d] \\ \sin X \arrow[r] & \sin Z \end{tikzcd}$$
of bisimplicial sets then has the property that $\forall\ n$,
$$\begin{tikzcd} \sin W_n \arrow[r] \arrow[d] & \sin Y_n \arrow[d] \\ \sin X_n \arrow[r] & \sin Z_n \end{tikzcd}$$
is a homotopy pullback in **SISET** with $\sin Y_n$, $\sin Z_n$ connected. Accordingly,
$$\begin{tikzcd} \mathrm{disin}\, W \arrow[r] \arrow[d] & \mathrm{disin}\, Y \arrow[d] \\ \mathrm{disin}\, X \arrow[r] & \mathrm{disin}\, Z \end{tikzcd}$$
is a homotopy pullback in **SISET** (theorem of Bousfield-Friedlander), so
$$\begin{tikzcd} |\mathrm{disin}\, W| \arrow[r] \arrow[d] & |\mathrm{disin}\, Y| \arrow[d] \\ |\mathrm{disin}\, X| \arrow[r] & |\mathrm{disin}\, Z| \end{tikzcd}$$
is a homotopy pullback in **CG** (cf. p. 13–75). Therefore
$$\begin{tikzcd} {|W|} \arrow[r] \arrow[d] & {|Y|} \arrow[d, "|g|"] \\ {|X|} \arrow[r, "|f|"'] & {|Z|} \end{tikzcd}$$
is a homotopy pullback in **CG** if W, X, Y, Z satisfy the cofibration condition.

[Note: Equip **TOP** with its singular structure and suppose given a commutative diagram
$$\begin{tikzcd} W \arrow[r] \arrow[d] & Y \arrow[d, "g"] \\ X \arrow[r, "f"'] & Z \end{tikzcd}$$
of simplicial spaces such that
$$\begin{tikzcd} W_n \arrow[r] \arrow[d] & Y_n \arrow[d] \\ X_n \arrow[r] & Z_n \end{tikzcd}$$
is a homotopy pullback in **TOP** $\forall\ n$, where Y_n, Z_n are path connected—then
$$\begin{tikzcd} {|W|} \arrow[r] \arrow[d] & {|Y|} \arrow[d, "|g|"] \\ {|X|} \arrow[r, "|f|"'] & {|Z|} \end{tikzcd}$$
is a homotopy pullback in **TOP** if W, X, Y, Z satisfy the cofibration condition. To see this, observe that
$$\begin{tikzcd} {|kW|} \arrow[r] \arrow[d] & {|kY|} \arrow[d, "|kg|"] \\ {|kX|} \arrow[r, "|kf|"'] & {|kZ|} \end{tikzcd}$$
is a homotopy pullback in **CG**, thus the arrow

$|kW| \to W_{|kf|,|kg|}$ is a weak homotopy equivalence. In the commutative diagram

$$\begin{tikzcd} {|W|} \arrow[r] & W_{|f|,|g|} \\ k|W| \arrow[u] \arrow[r] & W_{k|f|,k|g|} \arrow[u] \end{tikzcd},$$

the vertical arrow on the left is a weak homotopy equivalence, as is the vertical arrow on the right. Therefore $|W| \to W_{|f|,|g|}$ is a weak homotopy equivalence iff $k|W| \to W_{k|f|,k|g|}$ is a weak homotopy equivalence. Working in the compactly generated category, form

$$\begin{tikzcd} {|kX|} \arrow[r, "|kf|"] \arrow[d] & {|kZ|} \arrow[d] & {|kY|} \arrow[l, "|kg|"'] \arrow[d] \\ k|X| \arrow[r, "k|f|"'] & k|Z| & k|Y| \arrow[l, "k|g|"] \end{tikzcd}.$$

The vertical arrows are weak homotopy equivalences (cf. p. 14–8), so $W_{|kf|,|kg|} \to W_{k|f|,k|g|}$ is a weak homotopy equivalence (cf. p. 4–48). Examination of

$$\begin{tikzcd} W_{|kf|,|kg|} \arrow[r] & W_{k|f|,k|g|} \\ {|kW|} \arrow[u] \arrow[r] & k|W| \arrow[u] \end{tikzcd}$$

then implies that $k|W| \to W_{k|f|,k|g|}$ is a weak homotopy equivalence.]

PROPOSITION 10 If $\begin{cases} X \\ Y \end{cases}$ are Hausdorff simplicial spaces and if $f : X \to Y$ is a simplicial map such that $\forall\, n$, $f_n : X_n \to Y_n$ is a homology equivalence, then $\text{HR}f : \text{HR}X \to \text{HR}Y$ is a homology equivalence, thus so is $|f| : |X| \to |Y|$ subject to the cofibration condition on $\begin{cases} X \\ Y \end{cases}$.

[By Mayer-Vietoris and the five lemma, the arrow $(\text{HR}X)_n \to (\text{HR}Y)_n$ is a homology equivalence $\forall\, n$.]

[Note: The Hausdorff assumption can be replaced by Δ-separated and compactly generated.]

Notation: Given an X in **SITOP**, put $IX = X \times \text{si}[0,1]$, so $\forall\, n$, $(IX)_n = IX_n$.

LEMMA For every simplicial space X, $|IX| \approx I|X|$.

[The functor $— \times [0,1] : \textbf{TOP} \to \textbf{TOP}$ has a right adjoint, thus preserves colimits, in particular, coends.]

Application: Let X, Y be simplicial spaces, $H : IX \to Y$ a simplicial map—then $|H \circ i_0| \simeq |H \circ i_1|$.

Example: Suppose that X is a simplicial space. Define simplicial spaces ΓX, ΣX by $(\Gamma X)_n = \Gamma X_n$, $(\Sigma X)_n = \Sigma X_n$—then $|\Gamma X| \approx \Gamma|X|$, $|\Sigma X| \approx \Sigma|X|$.

[The diagrams

$$\begin{tikzcd} X_n \arrow[r] \arrow[d] & * \arrow[d] \\ IX_n \arrow[r] & \Gamma X_n \end{tikzcd}, \quad \begin{tikzcd} X_n \amalg X_n \arrow[r] \arrow[d] & * \amalg * \arrow[d] \\ IX_n \arrow[r] & \Sigma X_n \end{tikzcd}$$

determine pushout squares

in $[\mathbf{\Delta}^{\mathrm{OP}}, \mathbf{TOP}]$, thus the diagrams

$$\begin{tikzcd}
{|X|} \arrow[r] \arrow[d] & * \arrow[d] \\
{|IX|} \arrow[r] & {|\Gamma X|}
\end{tikzcd}, \quad
\begin{tikzcd}
{|X| \amalg |X|} \arrow[r] \arrow[d] & * \amalg * \arrow[d] \\
{|IX|} \arrow[r] & {|\Sigma X|}
\end{tikzcd}$$

are pushout squares in **TOP** and, from the lemma, $|IX| \approx I|X|$.]

[Note: When dealing with a pointed simplicial space X, one can work with either its unpointed geometric realization $\int^{[n]} X_n \times \Delta^n$ or its pointed geometric realization $\int^{[n]} X_n \# \Delta_+^n$. However, both give the "same" result (consider right adjoints). Therefore if one defines pointed simplicial spaces ΓX, ΣX by $(\Gamma X)_n = \Gamma X_n$, $(\Sigma X)_n = \Sigma X_n$ (pointed cone, pointed suspension), then it is still the case that $|\Gamma X| \approx \Gamma |X|$, $|\Sigma X| \approx \Sigma |X|$ (unpointed geometric realization).]

EXAMPLE Let X be a pointed simplicial space satisfying the cofibration condition (give $|X|$ the base point $x_0 \in X_0 = |X|_0$). Assume: $\forall\, n$, X_n is path connected. Denote by ΘX (ΩX) the simplicial space which takes $[n]$ to ΘX_n (ΩX_n)—then ΘX (ΩX) satisfies the cofibration condition (inspect the proof of Proposition 6 in §3), hence

$$\begin{tikzcd}
{|\Omega X|} \arrow[r] \arrow[d] & {|\Theta X|} \arrow[d] \\
\{x_0\} \arrow[r] & {|X|}
\end{tikzcd}$$

is a homotopy pullback in **TOP** (singular structure). Because there is a commutative diagram

$$\begin{tikzcd}
{|\Omega X|} \arrow[r] \arrow[d] & {|\Theta X|} \arrow[r] \arrow[d] & {|X|} \arrow[d, equal] \\
\Omega |X| \arrow[r] & \Theta |X| \arrow[r] & {|X|}
\end{tikzcd}$$

and $|\Theta X|$ is contractible, it follows that the arrow $|\Omega X| \to \Omega |X|$ is a weak homotopy equivalence.

FACT Let X be a pointed simplicial space satisfying the cofibration condition (give $|X|$ the base point $x_0 \in X_0 = |X|_0$)—then X_0 n-connected, X_1 $(n-1)$-connected, $\ldots$, X_{n-1} 1-connected $\Rightarrow$ $|X|$ n-connected.

[If $n = 1$, one can suppose that $\forall\, m > 1$, X_m is path connected, thus $|\Omega X|$ is path connected and $* = \pi_0(|\Omega X|) \approx \pi_0(\Omega |X|) \approx \pi_1(|X|)$. If $n > 1$, show that $H_q(|X|) = 0$ $(q \leq n)$ and quote Hurewicz.]

Recall that if X is a locally compact space and $g : Y \to Z$ is quotient, then $\mathrm{id}_X \times g : X \times Y \to X \times Z$ is quotient (cf. §2, Proposition 1 (X is cartesian)). Here is a variant in which X is allowed to be arbitrary.

WHITEHEAD LEMMA Let $g : Y \to Z$ be quotient. Assume: $\forall\, z \in Z$ and $\forall$ neighborhood V of z, there exists an open subset $U \subset Y$ with $\overline{U}$ compact and contained in $g^{-1}(V)$ such that $g(U)$ is a neighborhood of z—then for any X, $\mathrm{id}_X \times g : X \times Y \to X \times Z$ is quotient.

[Writing $p = \mathrm{id}_X \times g$, the claim is that a subset $O \subset X \times Z$ having the property that $p^{-1}(O)$ is open in $X \times Y$ is itself open in $X \times Z$. Fix $(x_0, z_0) \in O$ and choose an open $Y_0 \subset Y : \{x_0\} \times Y_0 = (\{x_0\} \times Y) \cap p^{-1}(O)$. If $V_0 = g(Y_0)$, then $Y_0 = g^{-1}(V_0)$, so V_0 is open in Z. Per z_0 & V_0, take U_0 as in the assumption and let $X_0 = \{x : \{x\} \times \overline{U}_0 \subset p^{-1}(O)\}$. Since X_0 is open in X and $(x_0, z_0) \in X_0 \times g(U_0) \subset O$, it follows that O is open in $X \times Z$.]

[Note: The argument goes through for any arrow $X \to W$ which is quotient.]

Application: For every topological space X, $|\mathrm{si}X \times \Delta[1]| \approx X \times [0,1]$.

LEMMA For every simplicial space X, $|X \times \Delta[1]| \approx |X| \times [0,1]$.

$$[|X \times \Delta[1]| \approx \int^{[n]}\int^{[m]} X_n \times \Delta[1]_m \times \Delta^n \times \Delta^m \approx \int^{[n]} (\int^{[m]} X_n \times \Delta[1]_m \times \Delta^m) \times \Delta^n \approx$$
$$\int^{[n]} X_n \times [0,1] \times \Delta^n \approx (\int^{[n]} X_n \times \Delta^n) \times [0,1] \approx |X| \times [0,1].]$$

FACT Let X, Y be simplicial spaces and let $f, g : X \to Y$ be simplicial maps. Suppose that $\forall\, n$, there are continuous functions $h_i : X_n \to Y_{n+1}$ $(0 \leq i \leq n)$ such that $d_0 \circ h_0 = f_n$, $d_{n+1} \circ h_n = g_n$ and

$$d_i \circ h_j = \begin{cases} h_{j-1} \circ d_i & (i < j) \\ d_i \circ h_{i-1} & (i = j > 0) \\ h_j \circ d_{i-1} & (i > j+1) \end{cases}, \qquad s_i \circ h_j = \begin{cases} h_{j+1} \circ s_i & (i \leq j) \\ h_j \circ s_{i-1} & (i > j) \end{cases}.$$

Then $|f| = |g|$ in the homotopy category.

EXAMPLE Given a triple $\mathbf{T} = (T, m, \epsilon)$ in **TOP**, $\forall$ **T**-algebra X, $|\mathrm{bar}(T; \mathbf{T}; X)|$ and X have the same homotopy type (cf. p. 0–46 ff.).]

EXAMPLE Let X be a simplicial space—then the translate TX of X is the simplicial space with $T_nX = X_{n+1}$, where if $\alpha : [m] \to [n]$, $TX(\alpha) : T_nX \to T_mX$ is $X(T\alpha) : X_{n+1} \to X_{m+1}$, $T\alpha : [m+1] \to [n+1]$ being the rule that sends 0 to 0 and i to $\alpha(i-1)+1$ $(i > 0)$. There are simplicial maps $\mathrm{si}X_0 \to TX$, $TX \to \mathrm{si}X_0$, viz. $s_0^{n+1} : X_0 \to X_{n+1}$, $d_1^{n+1} : X_{n+1} \to X_0$, and the composition $\mathrm{si}X_0 \to TX \to \mathrm{si}X_0$ is the identity. On the other hand, if $h_i : T_nX \to T_{n+1}X$ is defined by $h_i = s_0^{i+1} \circ d_1^i$ $(0 \leq i \leq n)$, then $d_1 \circ h_0 = \mathrm{id}$, $d_{n+2} \circ h_n = s_0^{n+1} \circ d_1^{n+1}$ and

$$d_{i+1} \circ h_j = \begin{cases} h_{j-1} \circ d_{i+1} & (i < j) \\ d_{i+1} \circ h_{i-1} & (i = j > 0) \\ h_j \circ d_i & (i > j+1) \end{cases}, \qquad s_{i+1} \circ h_j = \begin{cases} h_{j+1} \circ s_{i+1} & (i \leq j) \\ h_j \circ s_i & (i > j) \end{cases}.$$

Therefore $|TX|$ and X_0 have the same homotopy type. In particular: X_0 contractible $\Rightarrow$ $|TX|$ contractible.

While the general theory of simplicial spaces does not require a compactly generated hypothesis, one can say more with it than without it. A key point here is that CG admits a closed simplicial action, viz. $X \square K = X \times_k |K|$, relative to which **CG** satisfies SMC in either its standard or singular model category structure. Note, however, that the formal definition of, e.g., $\overline{\text{hocolim}}_{\mathbf{I}} : [\mathbf{I}, \mathbf{CG}] \to \mathbf{CG}$ depends only on $\square$($\overline{\text{hocolim}}_{\mathbf{I}} \text{—} = \int^i \text{—}i \times_k B(i\backslash\mathbf{I})$ (cf. p. 13–70)) and not on the underlying simplicial model category structure.

LEMMA Let $F, G : \mathbf{I} \to \mathbf{CG}$ be functors and let $\Xi : F \to G$ be a natural transformation. Assume: $\forall\ i$, $\Xi_i : Fi \to Gi$ is a weak homotopy equivalence—then $\overline{\text{hocolim}}\ \Xi : \overline{\text{hocolim}}\ F \to \overline{\text{hocolim}}\ G$ is a weak homotopy equivalence.

[One has $\begin{cases} \overline{\text{hocolim}}\ F \approx |\coprod F| \\ \overline{\text{hocolim}}\ G \approx |\coprod G| \end{cases}$ (cf. p. 13–70) and $\begin{cases} \coprod F \\ \coprod G \end{cases}$ satisfy the cofibration condition. In addition, $\forall\ n$, $(\coprod \Xi)_n : (\coprod F)_n \to (\coprod G)_n$ is a weak homotopy equivalence. Therefore $|\coprod \Xi| : |\coprod F| \to |\coprod G|$ is a weak homotopy equivalence (cf. p. 14–8).]

[Note: Changing the assumption to "homotopy equivalence" changes the conclusion to "homotopy equivalence" (cf. Proposition 4).]

EXAMPLE For any compactly generated simplicial space X, $\overline{\text{hocolim}}\ X$ and $\text{HR}X$ have the same weak homotopy type. To see this, consider $|\sin X|$ (cf. p. 14–8 ff.)—then the arrow $\overline{\text{hocolim}}\ |\sin X| \to \overline{\text{hocolim}}\ X$ is a weak homotopy equivalence (by the lemma) and the arrow $\text{HR}|\sin X| \to \text{HR}X$ is a weak homotopy equivalence (cf. Proposition 9). But $|\overline{\text{hocolim}}\ \sin X|$ is homeomorphic to $\overline{\text{hocolim}}\ |\sin X|$ (cf. p. 13–64) and the homotopy type of $|\overline{\text{hocolim}}\ \sin X|$ is the same as that of $|\text{disin}\, X| \approx \|\sin X\|$ (cf. p. 13–68), the homotopy type of the latter being that of $\text{HR}|\sin X|$ (cf. Proposition 7).

[Note: More is true: $\overline{\text{hocolim}}\ X$ and $\text{HR}X$ have the same homotopy type. Thus take **CG** in its standard structure and equip **SICG** with the corresponding Reedy structure—then $\forall$ Reedy cofibrant X, the arrow $\overline{\text{hocolim}}\ X \to |X|$ is a homotopy equivalence (cf. §13, Proposition 49) and $|X|$ has the same homotopy type as $\text{HR}X$ (cf. Proposition 7). To handle an arbitrary X, pass to $\mathcal{L}X$ (cf. p. 12–22). Because the arrow $\mathcal{L}X \to X$ is a levelwise homotopy equivalence, $\overline{\text{hocolim}}\ \mathcal{L}X$ and $\overline{\text{hocolim}}\ X$ have the same homotopy type (cf. supra). However $\mathcal{L}X$ is Reedy cofibrant, so $\overline{\text{hocolim}}\ \mathcal{L}X$ has the same homotopy type as $\text{HR}\mathcal{L}X$, i.e., as $\text{HR}X$ (cf. Proposition 8).]

Let $\begin{cases} \mathbf{C} \\ \mathbf{D} \end{cases}$ and $\mathbf{I}$ be small categories.

($\otimes_{\mathbf{I}}$) This is the functor $[\mathbf{C} \times \mathbf{I}^{\text{OP}}, \mathbf{CG}] \times [\mathbf{I} \times \mathbf{D}, \mathbf{CG}] \to [\mathbf{C} \times \mathbf{D}, \mathbf{CG}]$ given by $(F \otimes_{\mathbf{I}} G)_{X,Y} = \int^i F(X,i) \times_k G(i,Y)$.

($\text{Hom}_{\mathbf{I}}$) This is the functor $[\mathbf{C} \times \mathbf{I}, \mathbf{CG}]^{\text{OP}} \times [\mathbf{I} \times \mathbf{D}, \mathbf{CG}] \to [\mathbf{C}^{\text{OP}} \times \mathbf{D}, \mathbf{CG}]$ given by $\text{Hom}_{\mathbf{I}}(F, G)_{X,Y} = \int_i G_{i,Y}^{F_{X,i}}$.

[Note: In either situation one can, of course, take $\begin{cases}\mathbf{C}\\ \mathbf{D}\end{cases} = \mathbf{1}$. Special cases: $* \otimes_{\mathbf{I}} — \approx \mathrm{colim}_{\mathbf{I}}\,—$, $\mathrm{Hom}_{\mathbf{I}}(*, —) \approx \lim_{\mathbf{I}}\,—$.]

Examples: (1) $(F \otimes_{\mathbf{I}} G) \otimes_{\mathbf{J}} H \approx F \otimes_{\mathbf{I}} (G \otimes_{\mathbf{J}} H)$; (2) $\mathrm{Hom}_{\mathbf{J}}(F \otimes_{\mathbf{I}} G, H) \approx \mathrm{Hom}_{\mathbf{I}^{\mathrm{OP}}}(F, \mathrm{Hom}_{\mathbf{J}}(G, H))$.

Example: Suppose that X is a compactly generated simplicial space—then $X \otimes_{\mathbf{\Delta}} \Delta^? = |X|$.

[Note: $\Delta^? : \mathbf{\Delta} \to \mathbf{CG}$ sends $[n]$ to Δ^n.]

Example: Suppose that X is a compactly generated simplicial space—then $X_M \otimes_{\mathbf{\Delta}_M} \Delta_M^? = \mathrm{HR}X$.

[Note: X_M is the restriction of X to $\mathbf{\Delta}_M$ and $\Delta_M^? : \mathbf{\Delta}_M \to \mathbf{CG}$ sends $[n]$ to Δ^n.]

Given Y, Z in $[\mathbf{I}, \mathbf{CG}]$, put $Z^Y = \mathrm{Hom}_{\mathbf{I}}(\mathrm{Mor} \times Y, Z)$, where $\mathrm{Mor} \times Y : \mathbf{I}^{\mathrm{OP}} \times \mathbf{I} \to \mathbf{CG}$ sends (j, i) to $\mathrm{Mor}(j, i) \times Y_i$. So, e.g., $\mathrm{Hom}_{\mathbf{I}}(\mathrm{Mor}, Z)_j = \int_i Z_i^{\mathrm{Mor}(j,i)} = Z_j$ (integral Yoneda).

FACT The functor category $[\mathbf{I}, \mathbf{CG}]$ is cartesian closed.

[Let X, Y, Z be in $[\mathbf{I}, \mathbf{CG}]$—then $\mathrm{Nat}(X \times Y, Z) \approx \mathrm{Nat}(X \otimes_{\mathbf{I}^{\mathrm{OP}}} (\mathrm{Mor} \times Y), Z) \approx \mathrm{Nat}(X, \mathrm{Hom}_{\mathbf{I}}(\mathrm{Mor} \times Y, Z)) \approx \mathrm{Nat}(X, Z^Y)$.]

LEMMA Let $\mathbf{I}$ and $\mathbf{J}$ be small categories, $\nabla : \mathbf{J} \to \mathbf{I}$ a functor—then $F \circ \nabla^{\mathrm{OP}} \otimes_{\mathbf{J}} G \approx F \otimes_{\mathbf{I}} \mathrm{lan}\, G$.

Notation: Given a small category $\mathbf{I}$ and functors $\begin{cases}F\\ G\end{cases} : \mathbf{I} \to \mathbf{CG}$, write $F \underline{\times}_k G$ for the functor $\mathbf{I} \times \mathbf{I} \to \mathbf{CG}$ that sends (i, j) to $Fi \times_k Gj$.

LEMMA Relative to the diagonal $\mathbf{\Delta} \to \mathbf{\Delta} \times \mathbf{\Delta}$, $\mathrm{lan}\, \Delta^? \approx \Delta^? \underline{\times}_k \Delta^?$.

PROPOSITION 11 If X and Y are compactly generated simplicial spaces, then $|X \times_k Y| \approx |X| \times_k |Y|$.

[One has $|X \times_k Y| \approx (X \times_k Y) \otimes_{\mathbf{\Delta}} \Delta^? \approx (X \underline{\times}_k Y) \otimes_{\mathbf{\Delta} \times \mathbf{\Delta}} \Delta^? \underline{\times}_k \Delta^? \approx (X \otimes_{\mathbf{\Delta}} \Delta^?) \times_k (Y \otimes_{\mathbf{\Delta}} \Delta^?) \approx |X| \times_k |Y|$.]

[Note: Therefore $|?|$ preserves finite products so long as one works in $[\mathbf{\Delta}^{\mathrm{OP}}, \mathbf{CG}]$.]

It is not true that HR preserves finite products. However $\overline{\mathrm{hocolim}}(X \times_k Y)$ and $\overline{\mathrm{hocolim}}\, X \times_k \overline{\mathrm{hocolim}}\, Y$ are homeomorphic, thus $\mathrm{HR}(X \times_k Y)$ and $\mathrm{HR}X \times_k \mathrm{HR}Y$ have the same homotopy type (cf. p. 14–13).

FACT Let X be a simplicial object in $\mathbf{CG}/B$; let Y be an object in $\mathbf{CG}/B$. Assume: B is Δ-separated—then $|X \times_{\mathrm{si}B} \mathrm{si}Y| \approx |X| \times_B Y$.

[Since B is Δ-separated, the functor $- \times_B Y$ has a right adjoint (cf. p. 1–35).]

FACT $|?| : [\mathbf{\Delta}^{\mathrm{OP}}, \mathbf{\Delta\text{-}CG}] \rightarrow \mathbf{\Delta\text{-}CG}$ preserves finite limits.

[It suffices to deal with equalizers. For this, let $u, v : X \rightarrow Y$ be a pair of simplicial maps—then $|\mathrm{eq}(u,v)|$ is closed in $|X|$, which is enough.]

Let $\mathbf{C}$ be a small category—then $\mathbf{C}$ is said to be compactly generated if $O = \mathrm{Ob}\,\mathbf{C}$ and $M = \mathrm{Mor}\,\mathbf{C}$ are compactly generated topological spaces and the four structure functions $s : M \rightarrow O$, $t : M \rightarrow O$, $e : O \rightarrow M$, $c : M \times_O M \rightarrow M$ are continuous. One appends the terms Δ-separated or Hausdorff when O and M are, in addition, Δ-separated or Hausdorff. Example: Every compactly generated semigroup with unit (= monoid in **CG**) determines a compactly generated category.

[Note: Any small category can be regarded as a compactly generated category by equipping its objects and morphisms with the discrete topology.]

If **C**, **D** are compactly generated categories, then a functor $F : \mathbf{C} \rightarrow \mathbf{D}$ is said to be continuous provided that the functions $\begin{cases} \mathrm{Ob}\,\mathbf{C} \rightarrow \mathrm{Ob}\,\mathbf{D} \\ X \rightarrow FX \end{cases}$, $\begin{cases} \mathrm{Mor}\,\mathbf{C} \rightarrow \mathrm{Mor}\,\mathbf{D} \\ f \rightarrow Ff \end{cases}$ are continuous.

If **C**, **D** are compactly generated categories and if $F, G : \mathbf{C} \rightarrow \mathbf{D}$ are continuous functors, then a natural transformation $\Xi : F \rightarrow G$ is said to be continuous provided that the function $\begin{cases} \mathrm{Ob}\,\mathbf{C} \rightarrow \mathrm{Mor}\,\mathbf{D} \\ X \rightarrow \Xi_X \end{cases}$ is continuous.

In other words, per **CG**, compactly generated category = internal category, continuous functor = internal functor, continuous natural transformation = internal natural transformation.

[Note: If (M, O) is a category object in **SISET**, then $(|M|, |O|)$ is a category object in **CG**. Conversely, if (M, O) is a category object in **CG**, then $(\sin M, \sin O)$ is a category object in **SISET**.]

Let $\mathbf{C}$ be a compactly generated category—then $\mathrm{ner}\,\mathbf{C}$ is a compactly generated simplicial space: $\mathrm{ner}_0\,\mathbf{C} = O, \mathrm{ner}_1\,\mathbf{C} = M, \ldots, \mathrm{ner}_n\,\mathbf{C} = M \times_O \cdots \times_O M$ (n factors) (fiber product in **CG**), an n-tuple $(f_{n-1}, \ldots, f_0)$ corresponding to $X_0 \xrightarrow{f_0} X_1 \rightarrow \cdots \rightarrow X_{n-1} \xrightarrow{f_{n-1}} X_n$. Thus one can form either the geometric realization or the homotopic realization of $\mathrm{ner}\,\mathbf{C}$. These two spaces are necessarily compactly generated and they have the same homotopy type if $\mathrm{ner}\,\mathbf{C}$ satisfies the cofibration condition (cf. Proposition 7).

[Note: Meyer† has established versions of Quillen's theorems A and B for compactly generated categories.]

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† *Israel J. Math.* 48 (1984), 331–339.

EXAMPLE Let $\mathbf{C}$ be a compactly generated category, where O has the discrete topology—then $\mathbf{C}$ is a $\mathbf{CG}$-category and $\forall\ X,Y, \mathrm{Mor}(X,Y)$ is a clopen subset of M, so $\mathrm{ner}\,\mathbf{C}$ satisfies the cofibration condition provided that $\forall\ X$, the inclusion $\{\mathrm{id}_X\} \to \mathrm{Mor}(X,X)$ is a closed cofibration.

EXAMPLE Let $\mathbf{C}$ be a compactly generated category. View M as an object in $\mathbf{CG}/O \times_k O$ via $\begin{cases} s : M \to O \\ t : M \to O \end{cases}$. Assume: The $\mathbf{CG}$ embedding $e : O \to M$ is a closed cofibration over $O \times_k O$—then $\mathrm{ner}\,\mathbf{C}$ satisfies the cofibration condition.

Example: Given an internal category $\mathbf{M}$ in $\mathbf{CG}$, and a right $\mathbf{M}$-object X and a left $\mathbf{M}$-object Y, consider $\mathrm{bar}(X;\mathbf{M};Y)$, the bar construction on (X,Y). So: $\mathrm{bar}(X;\mathbf{M};Y) \approx \mathrm{ner}\,\mathbf{M}_{X,Y}$, where $\mathbf{M}_{X,Y} = \mathrm{tran}(X,Y)$, the translation category of (X,Y).

[Note: Suppose that $\mathbf{I}$ is a small category. Let $F : \mathbf{I}^{\mathrm{OP}} \to \mathbf{CG}$, $G : \mathbf{I} \to \mathbf{CG}$ be functors—then F determines a right $\mathbf{I}$-object X_F, G determines a left $\mathbf{I}$-object Y_G, and there is a canonical arrow $|\mathrm{bar}(X_F;\mathbf{I};Y_G)| \to F \otimes_{\mathbf{I}} G$.]

To simplify the notation, write $\mathrm{bar}(F;\mathbf{I};G)$ in place of $\mathrm{bar}(X_F;\mathbf{I};Y_G)$.

Examples: (1) The assignment $j \to |\mathrm{bar}(\mathrm{Mor}(—,j);\mathbf{I};G)|$ defines a functor $PG : \mathbf{I} \to \mathbf{CG}$ and the arrow of evaluation $(PG)j \to Gj$ is a homotopy equivalence; (2) The assignment $i \to |\mathrm{bar}(F;\mathbf{I};\mathrm{Mor}(i,—))|$ defines a functor $PF : \mathbf{I}^{\mathrm{OP}} \to \mathbf{CG}$ and the arrow of evaluation $(PF)i \to Fi$ is a homotopy equivalence.

Observation: $|\mathrm{bar}(F;\mathbf{I};G)| \approx PF \otimes_{\mathbf{I}} G \approx F \otimes_{\mathbf{I}} PG$.

EXAMPLE $\overline{\mathrm{hocolim}}\, G \approx B(—\backslash\mathbf{I}) \otimes_{\mathbf{I}} G \approx P* \otimes_{\mathbf{I}} G \approx * \otimes_{\mathbf{I}} PG \approx \mathrm{colim}\, PG$.

Working with the unit interval, one can define a notion of homotopy ($\simeq$) in the functor category $[\mathbf{I},\mathbf{CG}]$ that formally extends the special case $\mathbf{I} = \mathbf{1}$. This leads to a quotient category $[\mathbf{I},\mathbf{CG}]/\simeq$. Agreeing to call a morphism in $[\mathbf{I},\mathbf{CG}]$ a homotopy equivalence if its image in $[\mathbf{I},\mathbf{CG}]/\simeq$ is an isomorphism, it is seen by the usual argument that $[\mathbf{I},\mathbf{CG}]/\simeq$ is the localization of $[\mathbf{I},\mathbf{CG}]$ at the class of homotopy equivalences.

[Note: The functor $P : [\mathbf{I},\mathbf{CG}] \to [\mathbf{I},\mathbf{CG}]$ respects the homotopy congruence.]

LEMMA Let $G', G'' : \mathbf{I} \to \mathbf{CG}$ be functors and let $\Xi : G' \to G''$ be a natural transformation. Assume: $\forall\ j$, $\Xi_j : G'j \to G''j$ is a homotopy equivalence—then $P\Xi : PG' \to PG''$ is a homotopy equivalence.

Application: $\forall\ G$, the arrow of evaluation $PPG \to PG$ is a homotopy equivalence.

Application: Assume: $\forall\ j$, $G'j, G''j$ are contractible—then there is a homotopy equivalence $PG' \to PG''$.

[The arrows $PG' \to P*$, $PG'' \to P*$ are homotopy equivalences.]

[Note: There is only one homotopy class of arrows $PG' \to PG''$. Thus suppose that Φ, Ψ : $PG' \to PG''$ are not homotopic and form the commutative diagrams

$$\begin{tikzcd} PPG' \arrow[r, "P\Phi"] \arrow[d] & PPG'' \arrow[r, "PT"] \arrow[d] & P* \arrow[d] \\ PG' \arrow[r, "\Phi"'] & PG'' \arrow[r, "T"'] & * \end{tikzcd},$$

$$\begin{tikzcd} PPG' \arrow[r, "P\Psi"] \arrow[d] & PPG'' \arrow[r, "PT"] \arrow[d] & P* \arrow[d] \\ PG' \arrow[r, "\Psi"'] & PG'' \arrow[r, "T"'] & * \end{tikzcd}.$$

Since the vertical arrows in the squares on the left are homotopy equivalences, $P\Phi$, $P\Psi$ are not homotopic. On the other hand, $T \circ \Phi = T \circ \Psi \Rightarrow PT \circ P\Phi = PT \circ P\Psi \Rightarrow P\Phi \simeq P\Psi$ (T is a levelwise homotopy equivalence, hence PT is a homotopy equivalence). Contradiction.]

PROPOSITION 12 Suppose that $\begin{cases} \mathbf{C} \\ \mathbf{D} \end{cases}$ are compactly generated categories. Let $F, G : \mathbf{C} \to \mathbf{D}$ be continuous functors, $\Xi : F \to G$ a continuous natural transformation—then $|\text{ner}\, F|$, $|\text{ner}\, G| : |\text{ner}\, \mathbf{C}| \to |\text{ner}\, \mathbf{D}|$ are homotopic via $|\text{ner}\, \Xi_H|$ (cf. p. 13–15).

[Note: A topological category is a category object in **TOP**. And: The analog of Proposition 12 is true in this setting as well (since $|? \times \Delta[1]| \approx |?| \times [0,1]$ (cf. p. 14–12)).]

EXAMPLE Let X be a nonempty compactly generated space. View $\text{grd}X$ as a compactly generated category—then $|\text{ner}\,\text{grd}X|$ is contractible.

[Note: For any nonempty topological space X, $\text{grd}X$ is a topological category and $|\text{ner}\,\text{grd}X|$ ($= |\Lambda X|$ (cf. p. 14–1)) is contractible.]

Given a monoid G in **CG** with the property that the inclusion $\{e\} \to G$ is a closed cofibration, write $\mathbf{G}$ for the associated compactly generated category and put $XG = |\text{bar}(*; \mathbf{G}; G)|$ $((XG)_n = |\text{bar}(*; \mathbf{G}; G)|_n)$, $BG = |\text{bar}(*; \mathbf{G}; *)|$ $((BG)_n = |\text{bar}(*; \mathbf{G}; *)|_n)$—then there are projections $XG \to BG((XG)_n \to (BG)_n)$ and closed cofibrations $G \to XG$, $\{e\} \to BG$.

[Note: The assumption on G implies that $\text{bar}(*; \mathbf{G}; G)$, $\text{bar}(*; \mathbf{G}; *)$ satisfy the cofibration condition.]

EXAMPLE $\text{bar}(*; \mathbf{G}; G)$ is isomorphic to $T\text{bar}(*; \mathbf{G}; *)$, the translate of $\text{bar}(*; \mathbf{G}; *)$ (cf. p. 14–12).

[Use the transposition $\text{bar}(*; \mathbf{G}; G) \xrightarrow{\mathrm{T}} T\text{bar}(*; \mathbf{G}; *)$ defined by $\text{bar}_n(*; \mathbf{G}; G) \xrightarrow{\mathrm{T}_n} T_n\text{bar}(*; \mathbf{G}; *)$, where $\mathrm{T}_n(g_0, \ldots, g_{n-1}, g_n) = (g_n, g_0, \ldots, g_{n-1})$.]

LEMMA XG is contractible.

[Consider the compactly generated category tran$G$. It has an initial object, viz. e (the unique morphism from e to g is (g, e)). But the assignment $\begin{cases} G \to G \times_k G \\ g \to (g, e) \end{cases}$ is continuous. Therefore $|\text{bar}(*; \mathbf{G}; G)|$ is contractible (cf. Proposition 12).]

[Note: XG is a right G-space.]

LEMMA BG is path connected (cf. p. 14–2) and numerably contractible (cf. p. 14–7).

[Note: BG is called the classifying space of G but I shall pass in silence on just what BG classifies (for an abstract approach to this question, see Moerdijk[†]).]

Remark: XG and BG are abelian monoids in **CG** provided that G is abelian.

The formation of $|\text{bar}(X; \mathbf{G}; Y)|$ is functorial in the sense that if $\phi : G \to G'$ is a continuous homomorphism and $\begin{cases} X \to X' \\ Y \to Y' \end{cases}$ are ϕ-equivariant, then there is an arrow $|\text{bar}(X; \mathbf{G}; Y)| \to |\text{bar}(X'; \mathbf{G}'; Y')|$. In particular: ϕ induces arrows $XG \to XG'$, $BG \to BG'$.

The formation of $|\text{bar}(X; \mathbf{G}; Y)|$ is product preserving in the sense that the projections define a natural homeomorphism $|\text{bar}(X \times_k X'; \mathbf{G} \times_k \mathbf{G}'; Y \times_k Y')| \to |\text{bar}(X; \mathbf{G}; Y)| \times_k |\text{bar}(X'; \mathbf{G}'; Y')|$.

[Note: In the compactly generated category, $B(G \times_k G') \approx BG \times_k BG'$ but in the topological category all one can say is that the arrow $B(G \times G') \to BG \times BG'$ is a homotopy equivalence (Vogt[‡]).]

EXAMPLE Let G be a compactly generated group with $\{e\} \to G$ a closed cofibration—then XG is a compactly generated group containing G as a closed subgroup, the action $XG \times_k G \to XG$ agrees with the product in XG, BG is the homogeneous space XG/G, and XG is a numerable G-bundle over BG (in the compactly generated category).

A cofibered monoid is a monoid G in **CG** for which the inclusion $\{e\} \to G$ is a closed cofibration.

LEMMA Let G, K be cofibered monoids in **CG**, $f : G \to K$ a continuous homomorphism. Assume: f is a weak homotopy equivalence—then $Bf : BG \to BK$ is a weak homotopy equivalence.

[Apply the criterion on p. 14–8 to bar f : $\text{bar}(*; \mathbf{G}; *) \to \text{bar}(*; \mathbf{K}; *)$.]

[†] *SLN* **1616** (1995).

[‡] *Math. Zeit.* **153** (1977), 59–82.

Let G be a monoid in **CG**. If the inclusion $\{e\} \to G$ is not a closed cofibration, consider $\check{G}$ (cf. p. 3–33)—then by construction, the inclusion $\{\check{e}\} \to \check{G}$ is a closed cofibration. Moreover, $\check{G}$ is a monoid in **CG**: Take for the product in $[0,1]$ the usual product and extend the product in G by writing $gt = g = tg$ $(g \in G, 0 \leq t \leq 1)$. The retraction $r : \check{G} \to G$ is a morphism of monoids and a homotopy equivalence.

EXAMPLE (Wreath Products) Let G be a cofibered monoid in **CG**—then the wreath product $S_n \int G$ is the cofibered monoid in **CG** with $S_n \int G = S_n \times G^n$ as a set, multiplication being $(\sigma, (g_1, \ldots, g_n)) \cdot (\tau, (h_1, \ldots, h_n)) = (\sigma\tau, (g_{\tau(1)} h_1, \ldots, g_{\tau(n)} h_n))$ (so $(\mathrm{id}, (e, \ldots e))$ is the unit). Generalizing the fact that $BS_n \approx XS_n/S_n$, one has $B(S_n \int G) \approx XS_n \times_{S_n} (BG)^n$.

[Note: Embedding S_n in S_{n+1} as the subgroup fixing the last letter and embedding G^n in G^{n+1} as $G^n \times \{e\}$ serves to fix an embedding of $S_n \int G$ in $S_{n+1} \int G$ and $S_\infty \int G$ is by definition $\bigcup\limits_n S_n \int G$ (colimit topology). Another point is that if X is a compactly generated space on which G operates to the right, then X^n is a compactly generated space on which $S_n \int G$ operates to the right: $(x_1, \ldots, x_n) \cdot (\sigma, (g_1, \ldots, g_n)) = (x_{\sigma(1)} \cdot g_1, \ldots, x_{\sigma(n)} \cdot g_n)$.]

A discrete monoid is a monoid G in **SET** equipped with the discrete topology. If G is a discrete monoid, then G is a cofibered monoid and $BG = B\mathbf{G}$. Example: Suppose that G is a discrete group—then BG is a $K(G,1)$.

EXAMPLE Let G be a discrete monoid; let $\phi, \psi : G \to G$ be homomorphisms—then ϕ, ψ correspond to functors $\Phi, \Psi : \mathbf{G} \to \mathbf{G}$ and there exists a natural transformation $\Xi : \Phi \to \Psi$ iff ϕ, ψ are semiconjugate in the sense that $\xi\phi = \psi\xi$ for some $\xi \in G$. Semiconjugate homomorphisms lead to homotopic maps at the classifying space level (cf. p. 13–15). To illustrate, suppose that X is an infinite set and let M_X be the monoid of one-to-one functions $X \to X$. Fix $\iota \in M_X : \#(\iota(X)) = \#(X - \iota(X))$. Define a homomorphism $\phi : M_X \to M_X$ by $\phi(f)(x) = \iota(f(\iota^{-1}(x)))$ if $x \in \iota(X), \phi(f)(x) = x$ if $x \notin \iota(X)$. Obviously, $\iota \mathrm{id}_{M_X} = \phi\iota$. Fix an injection $i : X \to X - \iota(X)$ and let $C_{\mathrm{id}_X} : \begin{cases} M_X \to M_X \\ f \to \mathrm{id}_X \end{cases}$, so $iC_{\mathrm{id}_X} = \phi i$. Conclusion: BM_X is contractible.

EXAMPLE Every nonempty path connected topological space has the weak homotopy type of the classifying space of a discrete monoid (McDuff[†]). Consequently, if G is a discrete monoid, then the $\pi_q(BG)$ can be anything at all.

[Note: Compare this result with the Kan-Thurston theorem.]

PROPOSITION 13 Let G be a cofibered monoid in **CG**. Assume: G admits a homotopy inverse—then the sequence $G \to XG \to BG$ is a fibration up to homotopy (per **CG** (standard structure)).

[†] *Topology* **18** (1979), 313–320.

[The fact that G has a homotopy inverse implies that $\forall\, m, n$ & $\forall\, \alpha : [m] \rightarrow [n]$, the commutative diagram

$$\begin{tikzcd} G^{n+1} \arrow[r] \arrow[d] & G^{m+1} \arrow[d] \\ G^{n} \arrow[r] & G^{m} \end{tikzcd}$$

is a homotopy pullback, which suffices (cf. p. 14–7 ff.).]

[Note: If the inclusion $\{e\} \rightarrow G$ is a closed cofibration, $\pi_0(G)$ is a group, and G is numerably contractible, then G admits a homotopy inverse (cf. p. 4–27).]

Notation: Given a pointed compactly generated space X, put $\Theta_k X = X^{[0,1]}, \Omega_k X = X^{\mathbf{S}^1}$ (pointed exponential objects in $\mathbf{CG}_*$) (dispense with the "sub k" if there is no question as to the context).

Returning to G, there is a morphism of H spaces $G \rightarrow \Omega BG$ which sends g to the loop $\sigma_g : [0,1] \rightarrow BG$ defined by $\sigma_g(t) = [g,(1-t,t)]$ $(0 \leq t \leq 1)$.

[Note: The base point of BG is $[e, 1]$.]

PROPOSITION 14 Let G be a cofibered monoid in $\mathbf{CG}$. Assume: G admits a homotopy inverse—then the arrow $G \rightarrow \Omega BG$ is a pointed homotopy equivalence.

[There is an arrow $XG \rightarrow \Theta BG$ and a commutative diagram

$$\begin{tikzcd} G \arrow[r] \arrow[d] & XG \arrow[r] \arrow[d] & BG \arrow[d, equal] \\ \Omega BG \arrow[r] & \Theta BG \arrow[r] & BG \end{tikzcd}$$

. Since XG is contractible, the arrow from the compactly generated mapping fiber of $XG \rightarrow BG$ to the compactly generated mapping fiber of $\Theta BG \rightarrow BG$, i.e., to ΩBG, is a homotopy equivalence. Therefore by Proposition 13, the arrow $G \rightarrow \Omega BG$ is a homotopy equivalence or still, a pointed homotopy equivalence, both spaces being wellpointed.]

Example: Let G be an abelian group—then BG is an abelian compactly generated group, so $B^{(2)}G \equiv BBG$ is a $K(G,2)$ and by iteration, $B^{(n)}G$ is a $K(G,n)$.

Let X be a pointed compactly generated simplicial space. Given $n \geq 1$, there are maps $\pi_i : [1] \rightarrow [n]$ $(i = 1, \ldots, n)$, where $\pi_i(0) = i - 1$, $\pi_i(1) = i$. Definition: X is said to be monoidal if $X_0 = *$ and $\forall\, n \geq 1$, the arrow $X_n \rightarrow X_1 \times_k \cdots \times_k X_1$ determined by the π_i is a pointed homotopy equivalence. Example: Let G be a monoid in $\mathbf{CG}$—then ner $\mathbf{G}$ is monoidal.

EXAMPLE There is a functor sp : $\mathbf{CG}_* \rightarrow [\mathbf{\Delta}^{\mathrm{OP}}, \mathbf{CG}_*]$ that assigns to each pointed compactly generated space (X, x_0) a monoidal compactly generated simplicial space sp$X$, where, suitably topologized,

$\mathrm{sp}_n X$ is the set of continuous functions $\Delta^n \to X$ which carry the vertexes v_i of Δ^n to the base point x_0 of X. In particular: $\mathrm{sp}_1 X = \Omega X$.

[Consider $[0, n]$ as the segmented interval consisting of the edges of Δ^n connecting the vertexes $v_0, \ldots, v_n$—then $[0, n]$ is a strong deformation retract of Δ^n and a continuous function $f : [0, n] \to X$ such that $f(v_i) = x_0$ can be identified with a sequence of n loops in X.]

[Note: $\mathrm{sp}X$ generally does not satisfy the cofibration condition.]

If X is monoidal, then X_1 is a homotopy associative H space: $X_1 \times_k X_1 \to X_2 \xrightarrow{d_1} X_1$ (relative to some choice of a homotopy inverse for $X_2 \to X_1 \times_k X_1$), thus $\pi_0(X_1)$ is a monoid. Moreover, one has an arrow $\Sigma X_1 \to |X|$ (Σ = pointed suspension), hence, by adjunction, an arrow $X_1 \to \Omega|X|$ (which is a morphism of H spaces).

FACT Let X be a monoidal compactly generated simplicial space. Assume: X satisfies the cofibration condition and X_1 admits a homotopy inverse—then the arrow $X_1 \to \Omega|X|$ is a pointed homotopy equivalence.

[The role of XG in the above is played here by the contractible space $|TX|$, where TX is the translate of X (cf. p. 14–12), and the sequence $X_1 \to |TX| \to |X|$ is a fibration up to homotopy (per **CG** (standard structure)).]

[Note: The $d_0 : X_{n+1} \to X_n$ define a simplicial map $TX \to X$.]

Remark: If **C** is a pointed category with finite products and if X is a monoidal simplicial object in **C** (obvious definition), then X_1 is a monoid object in **C**.

DOLD-LASHOF THEOREM Let G be a cofibered monoid in **CG**—then the arrow $G \to \Omega BG$ is a weak homotopy equivalence iff $\pi_0(G)$ is a group.

[The necessity is clear. To establish the sufficiency, note that $|\sin G|$ is a cofibered monoid in **CG**. Form now the commutative diagram

$$\begin{tikzcd} {|\sin G|} \arrow[r] \arrow[d] & \Omega B|\sin G| \arrow[d] \\ G \arrow[r] & \Omega BG \end{tikzcd}$$

. Thanks to the Giever-Milnor theorem, the arrow of adjunction $|\sin G| \to G$ is a weak homotopy equivalence. Because $\pi_0(|\sin G|)$ is a group and $|\sin G|$ is a CW complex, hence numerably contractible (cf. p. 5–10 (TCW_4)), the arrow $|\sin G| \to \Omega B|\sin G|$ is, in particular, a weak homotopy equivalence (cf. supra). Finally, $B|\sin G| \to BG$ is a weak homotopy equivalence (cf. p. 14–8), thus $\Omega B|\sin G| \to \Omega BG$ is a weak homotopy equivalence (cf. p. 9–39). Therefore the arrow $G \to \Omega BG$ is a weak homotopy equivalence.]

Example: Let G, K be path connected cofibered monoids in **CG**, $f : G \to K$ a continuous homomorphism. Assume: $Bf : BG \to BK$ is a weak homotopy equivalence—then f is a weak homotopy equivalence.

[Consider the commutative diagram $\begin{tikzcd} G \arrow[r] \arrow[d] & \Omega BG \arrow[d] \\ K \arrow[r] & \Omega BK \end{tikzcd}$ and apply Dold-Lashof.]

Modulo obvious changes in the definitions, Propositions 13 and 14 are valid for cofibered monoids in **TOP**. The same holds for the Dold-Lashof theorem. Indeed, if G is a cofibered monoid in **TOP**, then kG is a cofibered monoid in **CG** and the arrow $kG \rightarrow G$ is a weak homotopy equivalence. Suppose in addition that $\pi_0(G)$ is a group—then $\pi_0(kG)$ is a group, so the arrow $kG \rightarrow \Omega_k BkG$ is a weak homotopy equivalence. On the other hand, $BkG \rightarrow BG$ is a weak homotopy equivalence (cf. p. 14–8), thus $\Omega BkG \rightarrow \Omega BG$ is a weak homotopy equivalence, as is $\Omega_k BkG \rightarrow \Omega BkG$. Since the diagram $\begin{tikzcd} kG \arrow[r] \arrow[d] & \Omega_k BkG \arrow[d] \\ G \arrow[r] & \Omega BG \end{tikzcd}$ commutes, it follows that the arrow $G \rightarrow \Omega BG$ is a weak homotopy equivalence.

EXAMPLE (The Moore Loop Space) Let (X, x_0) be a pointed topological space—then $\Omega_M X$ is a monoid in **TOP**. As such, it admits a homotopy inverse and there is a canonical arrow $B\Omega_M X \rightarrow X$ such that the composite $\Omega X \rightarrow \Omega_M X \rightarrow \Omega B\Omega_M X \rightarrow \Omega X$ is the identity. Assume now that X is path connected, numerably contractible, and the inclusion $\{x_0\} \rightarrow X$ is a closed cofibration (so $\Omega_M X$ is cofibered (cf. §3, Proposition 21)). Owing to Proposition 14, the arrow $\Omega_M X \rightarrow \Omega B\Omega_M X$ is a homotopy equivalence. But the retraction $\Omega_M X \rightarrow \Omega X$ is a homotopy equivalence. Therefore the arrow $\Omega B\Omega_M X \rightarrow \Omega X$ is a homotopy equivalence. Since $B\Omega_M X$ is numerably contractible (cf. p. 14–7), the delooping criterion on p. 4–27 then says that the arrow $B\Omega_M X \rightarrow X$ is a homotopy equivalence.

[Note: The same reasoning shows that $B\Omega_M X \rightarrow X$ is a weak homotopy equivalence provided that X is path connected and the inclusion $\{x_0\} \rightarrow X$ is a closed cofibration.]

LEMMA Let M be a simplicial monoid, Y a left $\mathbf{M}$-object—then $|Y|$ is a left $|\mathbf{M}|$-object and the geometric realization of $|\mathrm{bar}(*; \mathbf{M}; Y)|$ can be identified with $|\mathrm{bar}(*; |\mathbf{M}|; |Y|)|$.

[One has $\mathrm{bar}_n(*; \mathbf{M}; Y) = M \times \cdots \times M \times Y$. The geometric realization of $[m] \rightarrow \mathrm{bar}_n(*; \mathbf{M}; Y)_m = M_m \times \cdots \times M_m \times Y_m$ is $|M|^n \times_k |Y| = \mathrm{bar}_n(*; |\mathbf{M}|; |Y|)$, which, when realized with respect to $[n]$, gives $|\mathrm{bar}(*; |\mathbf{M}|; |Y|)|$.]

[Note: As a special case, $\|\mathrm{bar}(*; \mathbf{M}; *)\|$ $(= \|\mathrm{ner}\,\mathbf{M}\|) \approx B|M|$. Alternatively, $|[m] \rightarrow |[n] \rightarrow \mathrm{bar}_n(*; \mathbf{M}; *)_m|| \approx |[m] \rightarrow |\mathrm{ner}\,\mathbf{M}_m|| \approx |[m] \rightarrow B\mathbf{M}_m| \approx B|M|$.]

EXAMPLE (Algebraic K-Theory) Let A be a ring with unit. Put $M(A) = \coprod_{n \geq 0} \mathrm{ner}\,\mathbf{GL}(n, A)$ $(\mathrm{ner}\,\mathbf{GL}(0, A) = \Delta[0])$—then $M(A)_k = \coprod_{n \geq 0} \mathbf{GL}(n, A)^k$, thus $M(A)$ acquires the structure of a simplicial monoid from matrix addition, i.e., if $\begin{cases} (g_1, \ldots, g_k) \in \mathbf{GL}(n, A)^k \\ (h_1, \ldots, h_k) \in \mathbf{GL}(m, A)^k \end{cases}$, $(g_1, \ldots, g_k) \cdot (h_1, \ldots, h_k) =$

$(g_1 \oplus h_1, \cdots, g_k \oplus h_k)$, where $g_i \oplus h_i = \begin{pmatrix} g_i & 0 \\ 0 & h_i \end{pmatrix} \in \mathbf{GL}(n+m, A)$ $(i = 1, \ldots, k)$. Right multiplication by the vertex $1 \in \operatorname{ner}_0 \mathbf{GL}(1, A)$ determines a simplicial map $- \oplus 1 : M(A) \to M(A)$ whose restriction to $\operatorname{ner} \mathbf{GL}(n, A)$ is the arrow $\operatorname{ner} \mathbf{GL}(n, A) \to \operatorname{ner} \mathbf{GL}(n+1, A)$ induced by the canonical inclusion $\mathbf{GL}(n, A) \to \mathbf{GL}(n+1, A)$. The colimit of the diagram $M(A) \xrightarrow{-\oplus 1} M(A) \xrightarrow{-\oplus 1} \cdots$ is isomorphic to the simplicial set $Y(A) = \coprod_{\mathbf{Z}} \operatorname{ner} \mathbf{GL}(A)$. It is a left $\mathbf{M(A)}$-object and the pullback square

$$\begin{tikzcd} Y(A) \arrow[r] \arrow[d] & {|\operatorname{bar}(*; \mathbf{M(A)}; Y(A))|} \arrow[d] \\ \Delta[0] \arrow[r] & {|\operatorname{bar}(*; \mathbf{M(A)}; *)|} \end{tikzcd}$$

is a homology pullback (cf. p. 13–79). In fact, left multiplication by a vertex $n \in M(A)$ shifts the vertexes of $Y(A)$ (the term indexed by $z \in \mathbf{Z}$ is sent to the term indexed by $n + z$) and the corresponding map of simplicial sets $\operatorname{ner} \mathbf{GL}(A) \to \operatorname{ner} \mathbf{GL}(A)$ is induced by the homomorphism $\begin{cases} \mathbf{GL}(A) \to \mathbf{GL}(A) \\ g \to I_n \oplus g \end{cases}$ (I_n = rank n identity matrix), so $n_* : H_*(|Y(A)|) \to H_*(|Y(A)|)$ is an isomorphism. But $\operatorname{bar}(*; \mathbf{M(A)}; Y(A)) \approx \operatorname{colim}_{[\mathbf{N}]} \operatorname{bar}(*; \mathbf{M(A)}; M(A)) \Rightarrow |\operatorname{bar}(*; \mathbf{M(A)}; Y(A))| \approx \operatorname{colim}_{[\mathbf{N}]} |\operatorname{bar}(*; \mathbf{M(A)}; M(A))|$ and, by the lemma, the geometric realization of $|\operatorname{bar}(*; \mathbf{M(A)}; M(A))|$ is $|\operatorname{bar}(*; |\mathbf{M(A)}|; |M(A)|)| = X|M(A)|$, a contractible space. Therefore the geometric realization of $|\operatorname{bar}(*; \mathbf{M(A)}; Y(A))|$ is contractible (cf. p. 13–66). Consequently, $|Y(A)| = \coprod_{\mathbf{Z}} B\mathbf{GL}(A)$ has the homology of $\Omega B|M(A)|$ $(|M(A)| = \coprod_{n \geq 0} B\mathbf{GL}(n, A))$ and a model for $B\mathbf{GL}(A)^+$ is the path component of $\Omega B|M(A)|$ containing the constant loop.

[Note: An analogous discussion can be given for the simplicial monoid $M_\infty = \coprod_{n \geq 0} \operatorname{ner} \mathbf{S}_n$ that one obtains from the symmetric groups S_n. Spelled out, if S_∞ is as on p. 5–28, $\coprod_{\mathbf{Z}} BS_\infty$ has the homology of $\Omega B|M_\infty|$ $(|M_\infty| = \coprod_{n \geq 0} BS_n)$ and a model for BS_∞^+ is the path component of $\Omega B|M_\infty|$ containing the constant loop.]

A left $\mathbf{G}$-object Y is a compactly generated space on which G operates to the left and there is a commutative diagram

$$\begin{tikzcd} Y \arrow[r] \arrow[d] & {|\operatorname{bar}(*; \mathbf{G}; Y)|} \arrow[d] \\ * \arrow[r] & {|\operatorname{bar}(*; \mathbf{G}; *)| = BG} \end{tikzcd}.$$

PROPOSITION 15 Let G be a cofibered monoid in **CG**. Let Y be a left $\mathbf{G}$-object such that $\forall\, g \in G$, the arrow $y \to g \cdot y$ is a weak homotopy equivalence—then the sequence $Y \to |\operatorname{bar}(*; \mathbf{G}; Y)| \to BG$ is a fibration up to homotopy (per **CG** (singular structure)).

[Pass to the simplicial monoid $\sin G$, noting that $\sin Y$ is a left $\mathbf{sin}\,\mathbf{G}$-object. Since every $g \in \sin_0 G$ induces a weak homotopy equivalence $\sin Y \to \sin Y$, the pullback square

$$\begin{tikzcd} \sin Y \arrow[r] \arrow[d] & {|\operatorname{bar}(*; \mathbf{sin}\,\mathbf{G}; \sin Y)|} \arrow[d] \\ \Delta[0] \arrow[r] & {|\operatorname{bar}(*; \mathbf{sin}\,\mathbf{G}; *)|} \end{tikzcd}$$

is a homotopy pullback (cf. p. 13–79). Therefore, taking into account the lemma, the sequence $|\sin Y| \to |\operatorname{bar}(*; |\mathbf{sin}\,\mathbf{G}|; |\sin Y|) \to B|\sin G|$

is a fibration up to homotopy (per **CG** (singular structure)) (cf. p. 13–75). The obvious comparison then implies that the same is true of the sequence $Y \to |\mathrm{bar}(*;\mathbf{G};Y)| \to BG$.]

[Note: Similar methods lead to a homological version of this proposition.]

EXAMPLE Given a cofibered monoid G in **CG**, let UG be the associated discrete monoid—then the mapping fiber of the arrow $BUG \to BG$ at the base point has the weak homotopy type of $|\mathrm{bar}(*;\mathbf{UG};G)|$ whenever $\pi_0(G)$ is a group.

The forgetful functor from the category of groups to the category of monoids has a left adjoint that sends a monoid G to its group completion $\overline{G}$. Example: Let G be any monoid with a zero element ($0g = g0 = 0 \ \forall\, g \in G$), e.g., $G = \mathbf{Z}_2^{\times}$—then $\overline{G} = *$, the trivial group.

[Note: G abelian $\Rightarrow \overline{G}$ abelian.]

LEMMA The functor $G \to \overline{G}$ preserves finite products.

EXAMPLE Suppose that G is a discrete abelian monoid. In this situation, a model for $\overline{G}$ is the quotient of $G \times G$ by the equivalence relation $(g', h') \sim (g'', h'')$ iff $\exists\ k', k'' \in G$ such that $(g'k', h'k') = (g''k'', h''k'')$, the morphism $G \to \overline{G}$ being induced by $g \to (g, e)$. Let G operate on $G \times G$ via the diagonal and form $|\mathrm{bar}(*;\mathbf{G};G{\times}G)|$—then $\pi_0(|\mathrm{bar}(*;\mathbf{G};G{\times}G)|)$ is the coequalizer of $\begin{cases} d_1 : G \times (G \times G) \to G \times G \\ d_0 : G \times (G \times G) \to G \times G \end{cases}$ (cf. p. 13–3), which, from the definitions, is precisely $\overline{G}$.

[Note: There is an arrow $|\mathrm{bar}(*;*;G)| \to |\mathrm{bar}(*;\mathbf{G};G \times G)|$ corresponding to $(*,*,g) \to (*,e,(g,e))$ and $G \approx \pi_0(G) \approx \pi_0(|\mathrm{bar}(*;*;G)|)$.]

FACT Let M be a simplicial monoid, $\overline{M}$ its simplicial group completion—then the arrow $\pi_0(M) \to \pi_0(\overline{M})$ is a morphism of monoids and $\overline{\pi_0(M)} \approx \pi_0(\overline{M})$.

[Representing $\pi_0(M)$ as $\mathrm{coeq}(d_1, d_0)$ (cf. p. 13–3), one has $\overline{\pi_0(M)} = \overline{\mathrm{coeq}(d_1, d_0)} \approx \mathrm{coeq}(\overline{d}_1, \overline{d}_0) = \pi_0(\overline{M})$.]

LEMMA Let X be a pointed simplicial set. Assume: $X_0 = *$—then cX is a monoid and $\pi_1(X) \approx \overline{cX}$.

Application: Let M be a simplicial monoid—then $\mathrm{c}|\mathrm{ner}\,\mathbf{M}| \approx \pi_0(M)$, hence $\pi_1(|\mathrm{ner}\,\mathbf{M}|) \approx \overline{\pi_0(M)}$ or still, $\pi_1(B|M|) \approx \overline{\pi_0(M)}$.

PROPOSITION 16 Let G be a cofibered monoid in **CG**—then $\pi_1(BG) \approx \overline{\pi_0(G)}$.

[In the above, take $M = \sin G$ to get $\pi_1(B|\sin G|) \approx \overline{\pi_0(\sin G)}$.]

[Note: If G is a discrete monoid, then $\pi_1(BG) \approx \overline{G} \approx \pi_1(B\overline{G})$.]

Let M be a simplicial monoid, $\overline{M}$ its simplicial group completion—then $\overline{\pi_0(M)} \approx \pi_0(\overline{M})$, so $\pi_1(B|M|) \approx \pi_1(B|\overline{M}|)$. When $\pi_0(M)$ is a group, $|M|$ and $|\overline{M}|$ admit a homotopy inverse (cf. p. 4–27) (CW complexes are numerably contractible (cf. p. 5–10 (TCW$_4$))), thus the rows in the commutative diagram

$$\begin{tikzcd}
{|M|} \arrow[r] \arrow[d] & X|M| \arrow[r] \arrow[d] & B|M| \arrow[d] \\
{|\overline{M}|} \arrow[r] & X|\overline{M}| \arrow[r] & B|\overline{M}|
\end{tikzcd}$$

are fibrations up to homotopy per **CG** (standard structure) (cf. Proposition 13). Therefore the arrow $|M| \rightarrow |\overline{M}|$ is a pointed homotopy equivalence iff the arrow $B|M| \rightarrow B|\overline{M}|$ is a pointed homotopy equivalence, i.e., iff the arrow $B|M| \rightarrow B|\overline{M}|$ is acyclic (cf. §5, Proposition 19). Of course, the arrow $|M| \rightarrow |\overline{M}|$ cannot be a pointed homotopy equivalence if $\pi_0(M)$ is not a group. Since the fundamental groups of $B|M|$ and $B|\overline{M}|$ are isomorphic, the general question is whether the arrow $B|M| \rightarrow B|\overline{M}|$ is acyclic and for this one has the criterion provided by Proposition 22 in §5.

EXAMPLE Suppose that G is a discrete monoid—then the arrow $BG \rightarrow B\overline{G}$ is a pointed homotopy equivalence iff $\mathrm{Tor}_*^{\mathbf{Z}[G]}(\mathbf{Z}, \mathbf{Z}[\overline{G}]) \approx \mathrm{Tor}_*^{\mathbf{Z}[\overline{G}]}(\mathbf{Z}, \mathbf{Z}[\overline{G}])$, i.e., iff $\mathrm{Tor}_q^{\mathbf{Z}[G]}(\mathbf{Z}, \mathbf{Z}[\overline{G}]) = 0 \ \forall\ q \geq 1$ and $\mathbf{Z} \otimes_{\mathbf{Z}[G]} \mathbf{Z}[\overline{G}] \approx \mathbf{Z}$. For instance, this will be true if G is abelian. It also holds when G is free (Cartan-Eilenberg[†]).

[Note: $\mathrm{Tor}_0^{\mathbf{Z}[G]}(\mathbf{Z}, \mathbf{Z}[\overline{G}]) \approx \mathbf{Z} \otimes_{\mathbf{Z}[G]} \mathbf{Z}[\overline{G}] \approx (\mathbf{Z}[G]/I[G]) \otimes_{\mathbf{Z}[G]} \mathbf{Z}[\overline{G}] \approx \mathbf{Z}[\overline{G}]/I[G] \cdot \mathbf{Z}[\overline{G}] \approx \mathbf{Z}$, $I[G] \cdot \mathbf{Z}[\overline{G}]$ being $I[\overline{G}]$.]

FACT Let M be a simplicial monoid, $\overline{M}$ its simplicial group completion. Suppose that $\forall\ n$, the arrow $BM_n \rightarrow B\overline{M}_n$ is a pointed homotopy equivalence—then the arrow $B|M| \rightarrow B|\overline{M}|$ is a pointed homotopy equivalence.

[Given a $\pi_0(\overline{M})$-module $\overline{A}$, compare the spectral sequence $E^1_{n,m} \approx \mathrm{Tor}_m^{\mathbf{Z}[M_n]}(\mathbf{Z}, \overline{A}) \Rightarrow H_{n+m}(B|M|, \overline{A})$ with the spectral sequence $E^1_{n,m} \approx \mathrm{Tor}_m^{\mathbf{Z}[\overline{M}_n]}(\mathbf{Z}, \overline{A}) \Rightarrow H_{n+m}(B|\overline{M}|, \overline{A})$.]

Application: If $\forall\ n$, M_n is abelian or free, then the arrow $B|M| \rightarrow B|\overline{M}|$ is a pointed homotopy equivalence.

According to the Dold-Lashof theorem, for a cofibered monoid G in **CG**, the arrow $G \rightarrow \Omega BG$ is a weak homotopy equivalence iff $\pi_0(G)$ is a group. What happens in general? To give an answer, one replaces "homotopy" by "homology", the point being that the arrow $G \rightarrow \Omega BG$ is a morphism of H spaces, thus the arrow $H_*(G) \rightarrow H_*(\Omega BG)$ is a morphism of

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[†] *Homological Algebra*, Princeton University Press (1956), 192.

Pontryagin rings. Viewing $\pi_0(G)$ as a multiplicative subset of $H_*(G)$, the image of $\pi_0(G)$ in $H_*(\Omega BG)$ consists of units (since $\pi_0(\Omega BG)$ is a group) and under certain conditions, $H_*(\Omega BG)$ represents the localization of $H_*(G)$ at $\pi_0(G)$.

GROUP COMPLETION THEOREM Let G be a cofibered monoid in **CG**. Assume: $\pi_0(G)$ is in the center of $H_*(G)$—then $H_*(G)[\pi_0(G)^{-1}] \approx H_*(\Omega BG)$.

[Note: The diagram

$$\begin{tikzcd} \mathbf{Z}[\pi_0(G)] \arrow[r] \arrow[d] & \mathbf{Z}[\overline{\pi_0(G)}] \arrow[d] \\ H_*(G) \arrow[r] & H_*(\Omega BG) \end{tikzcd}$$

is therefore a pushout square in the category of graded associative **Z**-algebras.]

EXAMPLE The group completion theorem is false for an arbitrary cofibered monoid in **CG**. Thus choose a discrete monoid G whose classifying space BG has the weak homotopy type of $\mathbf{S}^n (n > 1)$ (cf. p. 14–19)—then if the group completion theorem held for G, one would have $H_*(\Omega\mathbf{S}^n) \approx H_0(\Omega\mathbf{S}^n)$, an absurdity.

To eliminate topological technicalities, we shall work with $|\sin G|$ and argue simplicially.

LEMMA Let A be a ring with unit. Suppose that S is a countable multiplicative subset of A which is contained in the center of A—then $A[S^{-1}]$ is isomorphic as a (left or right) A-module to the colimit of $A \xrightarrow{\rho_{s_1}} A \xrightarrow{\rho_{s_2}} \cdots$, where ρ_{s_i} is right multiplication by s_i and $\{s_i\}$ is an enumeration of the elements of S, each element being repeated infinitely often.

PROPOSITION 17 Let M be a simplicial monoid—then $H_*(|M|)[\pi_0(|M|)^{-1}] \approx H_*(\Omega B|M|)$ provided that $\pi_0(|M|)$ is contained in the center of $H_*(|M|)$.

[As functors of M, both sides of the purported relation commute with filtered colimits. Because M can be written as a filtered colimit of countable simplicial submonoids M_k such that $\pi_0(|M_k|)$ is contained in the center of $H_*(|M_k|)$, one can assume that M is countable. Pick a vertex in each component of M and, with an eye to the lemma, arrange them in a sequence $\{m_i\}$ subject to the proviso that every choice appears an infinity of times. Consider $M \xrightarrow{\rho_{m_1}} M \xrightarrow{\rho_{m_2}} \cdots$, where $\rho_{m_i} : M \to M$ is right multiplication by m_i. This sequence defines an object in **FIL(SISET)**. Form its colimit to get a left **M**-object Y such that the geometric realization of $|\text{bar}(*; \mathbf{M}; Y)|$ is contractible (compare the discussion in the example preceding Proposition 15). By construction, $H_*(|Y|) \approx H_*(|M|)[\pi_0(|M|)^{-1}]$, hence $\forall\ m \in M_0$, $m_* : H_*(|Y|) \to H_*(|Y|)$ is an isomorphism. This means that the pullback square

$$\begin{tikzcd} Y \arrow[r] \arrow[d] & {|\mathrm{bar}(*;\mathbf{M};Y)|} \arrow[d] \\ \Delta[0] \arrow[r] & {|\mathrm{bar}(*;\mathbf{M};*)|} \end{tikzcd}$$

is a homology pullback (cf. p. 13–79), so the arrow from $|Y|$ to the mapping fiber E of $|\mathrm{bar}(*;|\mathbf{M}|;|Y|)| \to B|M|$ over the base point is a homology equivalence. Working with the standard model category structure on **CG** (cf. p. 12–2), factor the projection $X|M| \to B|M|$ into an acyclic closed cofibration $X|M| \to X$ followed by a **CG** fibration $X \to B|M|$ to get the commutative diagram

$$\begin{tikzcd} {|M|} \arrow[r] \arrow[d] & X|M| \arrow[r] \arrow[d] & B|M| \arrow[d, equal] \\ F \arrow[r] & X \arrow[r] & B|M| \end{tikzcd}$$

, F denoting the fiber. Choose a filler $X \to \Theta B|M|$ for

$$\begin{tikzcd} X|M| \arrow[r] \arrow[d] & \Theta B|M| \arrow[d] \\ X \arrow[r] & B|M| \end{tikzcd}$$

—then

$$\begin{tikzcd} {|M|} \arrow[r] \arrow[d] & X|M| \arrow[r] \arrow[d] & B|M| \arrow[d, equal] \\ F \arrow[r] \arrow[d] & X \arrow[r] \arrow[d] & B|M| \arrow[d, equal] \\ \Omega B|M| \arrow[r] & \Theta B|M| \arrow[r] & B|M| \end{tikzcd}$$

commutes, the composite $|M| \to F \to \Omega B|M|$ being our morphism of H spaces. There is also a commutative diagram

$$\begin{tikzcd} {|M|} \arrow[r] \arrow[d] & X|M| \arrow[r] \arrow[d] & B|M| \arrow[d, equal] \\ {|Y|} \arrow[r] & Y|M| \arrow[r] & B|M| \end{tikzcd}$$

, where $Y|M| = |\mathrm{bar}(*;|\mathbf{M}|;|Y|)|$. Putting everything together leads finally to the commutative diagram

$$\begin{tikzcd} & {|M|} \arrow[r] \arrow[d] \arrow[dl] & {|Y|} \arrow[d] \\ \Omega B|M| & F \arrow[l] \arrow[r] & E \end{tikzcd}$$

. Since the arrows $\Omega B|M| \leftarrow F \to E$ are homotopy equivalences, the result then falls out by applying H_*.]

Upon forming the commutative diagram

$$\begin{tikzcd} {|\sin G|} \arrow[r] \arrow[d] & \Omega B|\sin G| \arrow[d] \\ G \arrow[r] & \Omega BG \end{tikzcd}$$

, the group completion theorem is seen to follow from Proposition 17.

[Note: The centrality hypothesis on $\pi_0(G)$ is automatic if G is homotopy commutative.]

The group completion theorem remains in force when $\mathbf{Z}$ is replaced by any commutative ring $\mathbf{k}$ with unit as long as $\pi_0(G)$ is in the center of $H_*(G;\mathbf{k})$.

EXAMPLE (Strict Monoidal Categories) **CAT** is a monoidal category ($\otimes = \times, e = \mathbf{1}$) and a monoid therein is a strict monoidal category (strict in the sense that multiplication is literally associative (not just up to natural isomorphism) and the unit is a two sided identity). A strict monoidal category is therefore a category object in **CAT** with object element $\mathbf{1}$. When considered as a discrete category, every monoid in **SET** becomes a strict monoidal category. Fix now a strict monoidal category $\mathbf{M}$. Viewing $\mathbf{M}$ as an internal category in **CAT**, one can form $\text{bar}(1;\mathbf{M};1)$ (cf. p. 0–45), which is a simplicial object in **CAT**. On the other hand, viewing $\mathbf{M}$ as a small category (= internal category in **SET**), one can form $\text{ner}\,\mathbf{M}$ (a simplicial monoid) and $B\mathbf{M}$ (a cofibered monoid in **CG**). Bearing in mind that $\text{bar}(1;\mathbf{M};1) : \mathbf{\Delta}^{\text{OP}} \to \mathbf{CAT}$, put $\mathbf{GM} = \text{gro}_{\mathbf{\Delta}^{\text{OP}}}\text{bar}(1;\mathbf{M};1)$—then there is a weak homotopy equivalence $\overline{\text{hocolim}}\,N\text{bar}(1;\mathbf{M};1) \to \text{ner}\,\mathbf{GM}$ (cf. p. 13–70). But there is also a weak homotopy equivalence $\overline{\text{hocolim}}\,N\text{bar}(1;\mathbf{M};1) \to |N\text{bar}(1;\mathbf{M};1)|$ (cf. §13, Proposition 49). Since $N\text{bar}(1;\mathbf{M};1) \approx \text{bar}(*;\text{ner}\,\mathbf{M};*)$ and $\|\text{bar}(*;\text{ner}\,\mathbf{M};*)\| \approx B|\text{ner}\,\mathbf{M}|$, it follows that $B|\text{ner}\,\mathbf{M}|$ and $B\mathbf{GM}$ have the same homotopy type. Therefore $H_*(B\mathbf{M})[\pi_0(B\mathbf{M})^{-1}] \approx H_*(\Omega B\mathbf{GM})$ if $\mathbf{M}$ is in addition symmetric (for this condition implies that $B\mathbf{M}$ is homotopy commutative).

[Note: A symmetric strict monoidal category is said to be permutative. Every small symmetric monoidal category is equivalent to a permutative category (Isbell[†]). Examples: (1) $\mathbf{\Gamma}$ is a permutative category under wedge sum. Thus $\mathbf{m}\vee\mathbf{n} = \mathbf{m}+\mathbf{n}$ in blocks (the empty wedge sum is $\mathbf{0}$) and for $\begin{cases}\gamma:\mathbf{m}\to\mathbf{n}\\ \gamma':\mathbf{m}'\to\mathbf{n}'\end{cases}$, $(\gamma\vee\gamma')(k) = 0$ if $\gamma(k) = 0$ or $\gamma'(k) = 0$, otherwise $(\gamma\vee\gamma')(k) = \begin{cases}\gamma(k) & (1\leq k\leq m)\\ \gamma'(k-m)+n & (m<k\leq m+m')\end{cases}$; (2) $\mathbf{\Gamma}$ is a permutative category under smash product. Thus $\mathbf{m}\#\mathbf{n} = \mathbf{mn}$ via lexicographic ordering of pairs (the empty smash product is $\mathbf{1}$) and for $\begin{cases}\gamma:\mathbf{m}\to\mathbf{n}\\ \gamma':\mathbf{m}'\to\mathbf{n}'\end{cases}$, $(\gamma\#\gamma')((i-1)m'+i') = 0$ if $\gamma(i) = 0$ or $\gamma'(i') = 0$, otherwise $(\gamma\#\gamma')((i-1)m'+i') = (\gamma(i)-1)n'+\gamma'(i')$ $(1\leq i\leq m, 1\leq i'\leq m')$.]

EXAMPLE (Algebraic K-Theory) Let A be a ring with unit. Denote by $\mathbf{M}(A)$ the category whose objects are the A^n $(n\geq 0)$, there being no morphism from A^n to A^m unless $n = m$, in which case $\text{Mor}(A^n,A^n) = \mathbf{GL}(n,A)$—then $\mathbf{M}(A)$ is a permutative category and $\text{ner}\,\mathbf{M}(A) = M(A) = \coprod_{n\geq 0}\text{ner}\,\mathbf{GL}(n,A)$ (cf. p. 14–22 ff.). Here, $\mathbf{Z}_{\geq 0} \approx \pi_0(B\mathbf{M}(A))$, $\mathbf{Z} \approx \overline{\pi_0(B\mathbf{M}(A))} \approx \pi_0(\Omega B|M(A)|)$, and $H_*(B\mathbf{M}(A))[\pi_0(B\mathbf{M}(A))^{-1}] \approx H_*(\Omega B|M(A)|)$.

[Note: Write $\mathbf{M}_\infty$ for the category whose objects are the finite sets $\mathbf{n} \equiv \{0,1,\ldots,n\}$ $(n\geq 0)$ with base point 0, there being no morphism from $\mathbf{n}$ to $\mathbf{m}$ unless $n = m$, in which case $\text{Mor}(\mathbf{n},\mathbf{n}) = S_n$ (thus $\mathbf{M}_\infty = \text{iso}\,\mathbf{\Gamma}$ (cf. p. 0–16)). Again, $\mathbf{M}_\infty$ is permutative and the discussion above can be paralleled (cf. p. 14–23).]

The compactly generated analog of the "free topological group" on X $((X,x_0))$ is meaningful on purely formal grounds (cf. p. 1–37) but the situation is simpler since one has a direct description of the topology on $F_{\text{gr}}X$ $(F_{\text{gr}}(X,x_0))$, the free compactly generated group on X $((X,x_0))$. To be specific, consider an (X,x_0) in $\mathbf{CG}_*$. Let (X^{-1},x_0^{-1}) be a copy of (X,x_0). Put $\overline{X} = X\vee X^{-1}$, $\overline{X}^n = \overline{X}\times_k\cdots\times_k\overline{X}$ (n factors)—then with

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[†] *J. Algebra* **13** (1969), 299–307.

$F_{\mathrm{gr}}(X,x_0)$ the free group on $X - \{x_0\}$, there is a surjection $p : \coprod_n \overline{X}^n \to F_{\mathrm{gr}}(X,x_0)$ sending $\overline{X}^n$ to $F^n_{\mathrm{gr}}(X,x_0)$, the subset of $F_{\mathrm{gr}}(X,x_0)$ consisting of those words of length at most n, and $F_{\mathrm{gr}}(X,x_0)$ is equipped with the quotient topology derived from p. When X is Δ-separated, $F_{\mathrm{gr}}(X,x_0)$ is Δ-separated, the arrow of adjunction $X \to F_{\mathrm{gr}}(X,x_0)$ is a closed embedding, $F^n_{\mathrm{gr}}(X,x_0)$ is closed, $p_n : \overline{X}^n \to F^n_{\mathrm{gr}}(X,x_0)$ is quotient ($p_n = p|\overline{X}^n$), $F_{\mathrm{gr}}(X,x_0) = \operatorname{colim} F^n_{\mathrm{gr}}(X,x_0)$, and the commutative diagram

$$\begin{tikzcd}
\overline{X}^{n-1}_* \arrow[r] \arrow[d] & F^{n-1}_{\mathrm{gr}}(X,x_0) \arrow[d] \\
\overline{X}^n \arrow[r] & F^n_{\mathrm{gr}}(X,x_0)
\end{tikzcd}$$

is a pushout square ($\overline{X}^{n-1}_* = p_n^{-1}(F^{n-1}_{\mathrm{gr}}(X,x_0))$).

[Note: A reference for this material is La Martin[†]. Incidentally, it is false in general that k applied to the free topological group on (X,x_0) is the free compactly generated group on (X,x_0) but if X is the colimit of an expanding sequence of compact Hausdorff spaces, then the free compactly generated group on (X,x_0) is a topological group, hence is the free topological group on (X,x_0).]

EXAMPLE The structure of $F_{\mathrm{gr}}(X,x_0)$ definitely depends on whether one is working in the topological category or the compactly generated category. This can be seen by taking $X = \mathbf{Q}$. For the free topological group on $(\mathbf{Q},0)$ is not compactly generated and its topology is not the quotient topology associated with the projection $\coprod_n \overline{\mathbf{Q}}^n \to F_{\mathrm{gr}}(\mathbf{Q},0)$. Moreover, $F_{\mathrm{gr}}(\mathbf{Q},0)$ is not the colimit of the $F^n_{\mathrm{gr}}(\mathbf{Q},0)$. Still, $\forall\ n$, $F^n_{\mathrm{gr}}(\mathbf{Q},0)$ is closed in $F_{\mathrm{gr}}(\mathbf{Q},0)$ and every compact subset of $F_{\mathrm{gr}}(\mathbf{Q},0)$ is contained in some $F^n_{\mathrm{gr}}(\mathbf{Q},0)$. Nevertheless, $p_n : \overline{\mathbf{Q}}^n \to F^n_{\mathrm{gr}}(\mathbf{Q},0)$ is not quotient if $n >> 0$.

[Note: Details can be found in Fay-Ordman-Thomas[‡].]

The intent of the preceding remarks is motivational, our main concern being with free compactly generated monoids, not free compactly generated groups. Thus fix (X,x_0) in $\mathbf{CG}_*$, call JX the free monoid on $X - \{x_0\}$, and give JX the quotient topology coming from $\coprod_n X^n \xrightarrow{p} JX$. Letting π be the multiplication in JX, consider the commutative diagram

$$\begin{tikzcd}
\coprod_n X^n \times_k \coprod_n X^n \arrow[r, "p \times_k p"] \arrow[d] & JX \times_k JX \arrow[d, "\pi"] \\
\coprod_n X^n \arrow[r, "p"'] & JX
\end{tikzcd}$$

. Since $\pi \circ (p \times_k p)$ is continuous and $p \times_k p$ is quotient, π is continuous. Therefore JX is a monoid in $\mathbf{CG}$. Suppose now that G is a monoid in $\mathbf{CG}$ and $f : X \to G$ is a pointed continuous function. On algebraic grounds, there exists a unique morphism of monoids $J_f : JX \to G$ rendering the triangle

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[†] *Dissertationes Math.* **146** (1977), 1–36; see also Ordman, *General Topology Appl.* **5** (1975), 205–219.

[‡] *General Topology Appl.* **10** (1979), 33–47.

$$\begin{tikzcd} X \arrow[r] \arrow[dr, "f"'] & JX \arrow[d, "J_f"] \\ & G \end{tikzcd}$$

commutative. Claim: J_f is continuous. Indeed, there is a continuous function $p_f : \coprod_n X^n \to G$ with $J_f \circ p = p_f$. But p is quotient, so J_f is continuous. Therefore JX is the free compactly generated monoid on (X, x_0).

[Note: JX is the James construction on (X, x_0).]

JX can be represented as a coend, viz. $JX \approx \int^{\mathbf{n}} X^n \times_k J\mathbf{n}$, $J\mathbf{n}$ the James construction on the pointed finite set $\mathbf{n} = \{0, 1, \ldots, n\}$ (cf. p. 13–56).

LEMMA Let X be a pointed compactly generated simplicial space. Define a simplicial space JX by $(JX)_n = JX_n$—then $|JX| \approx J|X|$.

[In fact, $|JX| = \int^{n} JX_n \times \Delta^n \approx \int^{n} (\int^{\mathbf{m}} (X_n)^m \times_k J\mathbf{m}) \times \Delta^n \approx \int^{\mathbf{m}} (\int^{n} (X_n)^m \times \Delta^n) \times_k J\mathbf{m} \approx \int^{\mathbf{m}} |X|^m \times_k J\mathbf{m} \approx J|X|$.]

Put $J^nX = p(X^n)$ and consider $p^{-1}(J^nX) \cap X^m$. Obviously, $m < n \Rightarrow p^{-1}(J^nX) \cap X^m = X^m$. On the other hand, $n < m \Rightarrow p^{-1}(J^nX) \cap X^m = \bigcup_S X_S^m$, where for $S \subset \{1, \ldots, m\} : \#(S) = m - n$, $X_S^m = \{(x_1, \ldots, x_m) : x_i = x_0 \ (i \in S)\}$. Consequently, J^nX is closed in JX if $\{x_0\}$ is closed in X.

LEMMA Assume: $\{x_0\}$ is closed in X. Let A be a subset of J^nX such that $p^{-1}(A) \cap X^n$ is closed in X^n—then A is closed in JX.

[Case 1: $m < n$. Denoting by $i_{m,n}$ the insertion $X^m \to X^n$ that sends $(x_1, \ldots, x_m)$ to $(x_1, \ldots, x_m, x_0, \ldots, x_0)$, one has $p^{-1}(A) \cap X^m = i_{m,n}^{-1}\,(p^{-1}(A) \cap X^n)$. Case 2: $n < m$. Write $p^{-1}(A) \cap X^m = \bigcup_S (p_S^{-1}(p^{-1}(A) \cap X^n))$, $p_S : X_S^m \to X^n$ the striking map (i.e., $p_S(x_1, \ldots, x_m)$ retains only those x_i, where $i \notin S$).]

Accordingly, when $\{x_0\} \subset X$ is closed, the arrow $X^n \to J^nX$ is quotient and the commutative diagram

$$\begin{tikzcd} X_*^n \arrow[r] \arrow[d] & J^{n-1}X \arrow[d] \\ X^n \arrow[r] & J^nX \end{tikzcd}$$

is a pushout square ($X_*^n = \bigcup_S X_S^n$ $(\#(S) = 1) \Rightarrow X^n/X_*^n \approx X \#_k \cdots \#_k X$ (n factors)). It therefore follows that if X is Δ-separated, then each J^nX is Δ-separated (AD_6 (cf. p. 3–1)), hence $JX = \operatorname{colim} J^nX$ is Δ-separated (cf. p. 1–36).

[Note: The arrow of adjunction $X \to JX$ is a closed embedding. Reason: The continuous bijection $X \to J^1X$ is quotient.]

PROPOSITION 18 Let (X, x_0) be a wellpointed compactly generated space with $\{x_0\} \subset X$ closed—then (JX, x_0) is a wellpointed compactly generated space with $\{x_0\} \subset JX$ closed, thus is a cofibered monoid in **CG**.

[In fact, by the above, $\forall\, n$, $J^{n-1}X \to J^nX$ is a closed cofibration.]

LEMMA If (X, x_0) is a wellpointed compactly generated Hausdorff space, then (JX, x_0) is a wellpointed compactly generated Hausdorff space.

[$\forall\, n$, J^nX is Hausdorff (cf. p. 3–8) and condition B on p. 1–29 can be applied.]

FACT Suppose that (X, x_0) is a pointed CW complex—then (JX, x_0) is a pointed CW complex.

If X is a wellpointed compactly generated space with $\{x_0\} \subset X$ closed, then the pointed cone ΓX and the pointed suspension ΣX are wellpointed compactly generated spaces with closed basepoints.

Define E by the pushout square

$$\begin{tikzcd} X \times_k JX \arrow[r] \arrow[d] & JX \arrow[d] \\ \Gamma X \times_k JX \arrow[r] & E \end{tikzcd}$$

, where $X \times_k JX \to JX$ is multiplication.

LEMMA E is contractible.

[Letting E_n be the image of $\Gamma X \times_k J^nX$ in E, there is a pushout square

$$\begin{tikzcd} \Gamma X \times_k J^{n-1}X \cup \{x_0\} \times_k J^nX \arrow[r] \arrow[d] & E_{n-1} \arrow[d] \\ \Gamma X \times_k J^nX \arrow[r] & E_n \end{tikzcd},$$

so the arrow $E_{n-1} \to E_n$ is a closed cofibration. But $E_n/E_{n-1} \approx \Gamma X \#_k (J^nX/J^{n-1}X)$, hence E_n/E_{n-1} is contractible. Since $E_0 \approx \Gamma X$, it follows by induction that E_n is contractible (cf. p. 3–24). Therefore $E = \operatorname{colim} E_n$ is contractible (cf. p. 3–20).]

Notation: Given a pointed compactly generated space X, let $\Theta_{kM}X$ ($\Omega_{kM}X$) be the compactly generated Moore mapping (loop) space of X (dispense with the "sub k" if there is no question as to the context).

There are two ways to place a compactly generated topology on $\Theta_M X$ ($\Omega_M X$).

(1) View $\Theta_M X$ ($\Omega_M X$) as a subset of $C(\mathbf{R}_{\geq 0}, X) \times \mathbf{R}_{\geq 0}$ (cf. p. 3–31 ff.) and take the "k-ification" of the induced topology.

(2) Form $kC(\mathbf{R}_{\geq 0}, X) \times_k \mathbf{R}_{\geq 0} = kC(\mathbf{R}_{\geq 0}, X) \times \mathbf{R}_{\geq 0}$, equip $\Theta_M X$ ($\Omega_M X$) with the induced topology, and pass to its "k-ification".

Both procedures yield the same compactly generated topology on $\Theta_M X$ ($\Omega_M X$), from which $\Theta_{kM} X$ ($\Omega_{kM} X$).

EXAMPLE Let X be a pointed compactly generated space. Write mo$X$ for the nerve of the category associated with the compactly generated monoid $\Omega_M X$—then there is a canonical arrow mo$X \to$ sp$X$ which is a levelwise homotopy equivalence.

Let X be a wellpointed compactly generated space with $\{x_0\} \subset X$ closed. Choose a continuous function $\phi : X \to [0,1]$ such that $\phi^{-1}(0) = \{x_0\}$ (cf. §3, Proposition 21)—then the meridian map $m : X \to \Omega_M \Sigma X$ is the pointed continuous function specified by the rule $m(x)(t) = [x, t/\phi(x)]$ $(0 \leq t \leq \phi(x))$, where $[x_0, 0/0]$ is the base point of ΣX. Since $\Omega_M \Sigma X$ is a monoid in **CG**, m extends to JX :

$$\begin{tikzcd} X \arrow[r] \arrow[dr, "m"'] & JX \arrow[d, "J_m"] \\ & \Omega_M \Sigma X \end{tikzcd}$$

, J_m being the arrow of James.

[Note: The composite $X \xrightarrow{m} \Omega_M \Sigma X \to \Omega \Sigma X$ is $x \to [x, —]$.]

Ostensibly, the meridian map depends on ϕ, call it m_ϕ. Suppose, however, that m_ψ is the meridian map corresponding to another continuous function $\psi : X \to [0,1]$ such that $\psi^{-1}(0) = \{x_0\}$—then $m_\phi \simeq m_\psi$.

[Let $H : IX \to \Omega_M \Sigma X$ be the homotopy given by $H(x,t) : [0, (1-t)\phi(x) + t\psi(x)] \to \Sigma X$, where $H(x,t)(T) = [x, T/((1-t)\phi(x) + t\psi(x))]$. Write $G : X \to (\Omega_M \Sigma X)^{[0,1]}$ for its adjoint, view $(\Omega_M \Sigma X)^{[0,1]}$ as a monoid in **CG**, determine $\overline{G}$ via the commutative triangle

$$\begin{tikzcd} X \arrow[r] \arrow[dr, "G"'] & JX \arrow[d, "\overline{G}"] \\ & (\Omega_M \Sigma X)^{[0,1]} \end{tikzcd}$$

, and consider its adjoint $\overline{H} : IJX \to \Omega_M \Sigma X$.]

Let $\Gamma_m : \Gamma X \to \Theta_M \Sigma X$ be the continous function defined by the prescription $\Gamma_m([x,t])(T) = [x, T/\phi(x)]$ $(0 \leq T \leq t\phi(x))$—then there is an arrow $\Gamma X \times_k JX \xrightarrow{\Gamma_m \times_k J_m} \Theta_M \Sigma X \times_k \Omega_M \Sigma X \xrightarrow{+} \Theta_M \Sigma X$ and a commutative diagram

$$\begin{tikzcd} X \times_k JX \arrow[r] \arrow[d] & JX \arrow[d] \arrow[dr] & \\ & & \Omega_M \Sigma X \arrow[dl] \\ \Gamma X \times_k JX \arrow[r] & \Theta_M \Sigma X & \end{tikzcd}$$

. This leads in turn to an arrow $E \to \Theta_M \Sigma X$ and a commutative triangle

$$\begin{tikzcd} E \arrow[r] \arrow[dr] & \Theta_M \Sigma X \arrow[d] \\ & \Sigma X \end{tikzcd}$$

($\Theta_M \Sigma X \to \Sigma X$ is the CG fibration that evaluates a Moore path at its free end).

PROPOSITION 19 Let (X, x_0) be a wellpointed compactly generated space with $\{x_0\} \subset X$ closed. Assume: X is path connected and numerably contractible—then the arrow of James $JX \rightarrow \Omega_M \Sigma X$ is a pointed homotopy equivalence.

[In the commutative diagram

$$\begin{tikzcd}
\Gamma X \times_k JX \arrow[d] & X \times_k JX \arrow[l] \arrow[r] \arrow[d] & JX \arrow[d] \\
\Gamma X & X \arrow[l] \arrow[r] & *
\end{tikzcd}$$

, the arrows $X \times_k JX \rightarrow \Gamma X \times_k JX$, $X \rightarrow \Gamma X$ are closed cofibrations and

$$\begin{tikzcd}
\Gamma X \times_k JX \arrow[d] & X \times_k JX \arrow[l] \arrow[d] \\
\Gamma X & X \arrow[l]
\end{tikzcd}$$

is a homotopy pullback, as is

$$\begin{tikzcd}
X \times_k JX \arrow[r] \arrow[d] & JX \arrow[d] \\
X \arrow[r] & *
\end{tikzcd}$$

(the shearing map sh : $\begin{cases} X \times_k JX \rightarrow X \times_k JX \\ (x, y) \rightarrow (x, xy) \end{cases}$ is a homotopy equivalence (cf. p. 4–27)). Consequently, the sequence $JX \rightarrow E \rightarrow \Sigma X$ is a fibration up to homotopy (per **CG** (standard structure) (cf. p. 12–15)). Since E is contractible, it remains only to consider the commutative diagram

$$\begin{tikzcd}
JX \arrow[r] \arrow[d] & E \arrow[r] \arrow[d] & \Sigma X \arrow[d, equal] \\
\Omega_M \Sigma X \arrow[r] & \Theta_M \Sigma X \arrow[r] & \Sigma X
\end{tikzcd}$$

.]

Application: Under the hypotheses of Proposition 19, the composite $JX \xrightarrow{J_M} \Omega_M \Sigma X \rightarrow \Omega \Sigma X$ is a pointed homotopy equivalence.

PROPOSITION 20 Let (X, x_0) be a wellpointed compactly generated space with $\{x_0\} \subset X$ closed. Assume: X is path connected—then the arrow of James $JX \rightarrow \Omega_M \Sigma X$ is a weak homotopy equivalence.

[Thanks to the cone construction (cf. p. 4–56 ff.), the arrow $E \rightarrow \Sigma X$ is a quasifibration. Work with

$$\begin{tikzcd}
JX \arrow[r] \arrow[d] & E \arrow[r] \arrow[d] & \Sigma X \arrow[d, equal] \\
\Omega_M \Sigma X \arrow[r] & \Theta_M \Sigma X \arrow[r] & \Sigma X
\end{tikzcd}$$

and compare the long exact sequences of homotopy groups.]

[Note: In the case at hand, ΣX is simply connected.]

Application: Under the hypotheses of Proposition 20, the composite $JX \xrightarrow{J_M} \Omega_M \Sigma X \rightarrow \Omega \Sigma X$ is a weak homotopy equivalence.

EXAMPLE Let X be the broom pointed at $(0, 0)$—then X is path connected. But JX and $\Omega \Sigma X$ do not have the same weak homotopy type (ΣX is not simply connected).

PROPOSITION 21 Let $\begin{cases} (X, x_0) \\ (Y, y_0) \end{cases}$ be wellpointed compactly generated spaces with $\begin{cases} \{x_0\} \subset X \\ \{y_0\} \subset Y \end{cases}$ closed; let $f : X \rightarrow Y$ be a pointed continuous function. Assume: f is a homotopy equivalence (weak homotopy equivalence)—then $Jf : JX \to JY$ is a homotopy equivalence (weak homotopy equivalence).

[Arguing by induction from

$$\begin{tikzcd} X^n \arrow[d] & X^n_* \arrow[l] \arrow[r] \arrow[d] & J^{n-1}X \arrow[d] \\ Y^n & Y^n_* \arrow[l] \arrow[r] & J^{n-1}Y \end{tikzcd}$$

, one finds that $\forall\ n$, $J^nX \to J^nY$ is a homotopy equivalence (cf. p. 3–24 ff.) (weak homotopy equivalence (cf. p. 4–51)), hence $JX \to JY$ is a homotopy equivalence (cf. §3, Proposition 15) (weak homotopy equivalence (cf. p. 4–48)).]

Convention: Given a cofibered monoid G in **CG**, $\Sigma G \to BG$ is the adjoint of $G \to \Omega BG$ (cf. p. 14–20).

LEMMA Let (X, x_0) be a wellpointed compactly generated space with $\{x_0\} \subset X$ closed. Assume: X is discrete—then the composite $\Sigma X \to \Sigma JX \to BJX$ is a weak homotopy equivalence.

[Since $X = \bigvee\limits_{X-\{x_0\}} \mathbf{S}^0$, $JX = \coprod\limits_{X-\{x_0\}} J\mathbf{S}^0$ ($\coprod$ the coproduct in the category of monoids), where $J\mathbf{S}^0 = \mathbf{Z}_{\geq 0}$, thus it suffices to consider

$$\begin{tikzcd} & \Sigma\mathbf{Z}_{\geq 0} \arrow[r] & B\mathbf{Z}_{\geq 0} \arrow[d] \\ \Sigma\mathbf{S}^0 \arrow[ur] \arrow[dr] & & \\ & \Sigma\mathbf{Z} \arrow[r] & B\mathbf{Z} \end{tikzcd}$$

.]

PROPOSITION 22 Let (X, x_0) be a wellpointed compactly generated space with $\{x_0\} \subset X$ closed—then the composite $\Sigma X \to \Sigma JX \to BJX$ is a weak homotopy equivalence.

[The lemma implies that $\forall\ n$, the composite $\Sigma \sin_n X \to \Sigma J \sin_n X \to BJ \sin_n X$ is a weak homotopy equivalence ($\sin_n X$ being supplied with the discrete topology), thus the composite $|n \to \Sigma \sin_n X| \to |n \to \Sigma J \sin_n X| \to |n \to BJ \sin_n X|$ is a weak homotopy equivalence (cf. p. 14–8). But $|n \to \Sigma \sin_n X| \approx \Sigma|\sin X|$ (cf. p. 14–10 ff.), $|n \to \Sigma J \sin_n X| \approx \Sigma|n \to J \sin_n X| \approx \Sigma J|\sin X|$ (cf. p. 14–30), $|n \to BJ \sin_n X| \approx B|n \to J \sin_n X|$ (cf. p. 14–22) $\approx BJ|\sin X|$ and there is a commutative diagram

$$\begin{tikzcd} \Sigma|\sin X| \arrow[r] \arrow[d] & \Sigma J|\sin X| \arrow[r] \arrow[d] & BJ|\sin X| \arrow[d] \\ \Sigma X \arrow[r] & \Sigma JX \arrow[r] & BJX \end{tikzcd}$$

. The arrow $\Sigma|\sin X| \to \Sigma X$ is a weak homotopy equivalence (cf. infra). According to Proposition 21, the same holds for the arrow $J|\sin X| \to JX$ or still, for the arrows $\Sigma J|\sin X| \to \Sigma JX$, $BJ|\sin X| \to BJX$ (cf. p. 14–18). Combining these facts yields the assertion.]

LEMMA Let $\begin{cases} (X, x_0) \\ (Y, y_0) \end{cases}$, (Z, z_0) be wellpointed compactly generated spaces with $\begin{cases} \{x_0\} \subset X \\ \{y_0\} \subset Y \end{cases}$,

$\{z_0\} \subset Z$ closed and let $f : X \to Y$ be a pointed continuous function. Assume: f is a weak homotopy equivalence—then $f\#_k \mathrm{id}_Z : X\#_k Z \to Y\#_k Z$ is a weak homotopy equivalence.

Application: Let $\begin{cases} (X, x_0) \\ (Y, y_0) \end{cases}$ be wellpointed compactly generated spaces with $\begin{cases} \{x_0\} \subset X \\ \{y_0\} \subset Y \end{cases}$ closed and let $f : X \to Y$ be a pointed continuous function. Assume: f is a weak homotopy equivalence—then $\Sigma f : \Sigma X \to \Sigma Y$ is a weak homotopy equivalence.

[Note: Recall too that $\Omega f : \Omega X \to \Omega Y$ is a weak homotopy equivalence (cf. p. 9–39).]

LEMMA Let (X, x_0) be a wellpointed compactly generated space with $\{x_0\} \subset X$ closed—then there is a canonical arrow $B\Omega_M X \to X$ and a commutative diagram

$$\begin{tikzcd}
\Sigma\Omega_M X \arrow[r] \arrow[d] & B\Omega_M X \arrow[dl] \\
X &
\end{tikzcd}$$

[Note: $B\Omega_M X \to X$ is a weak homotopy equivalence provided that X is path connected (cf. 14–22).]

PROPOSITION 23 Let (X, x_0) be a wellpointed compactly generated space with $\{x_0\} \subset X$ closed—then the arrow of James $J_m : JX \to \Omega_M \Sigma X$ induces a weak homotopy equivalence $BJ_m : BJX \to B\Omega_M\Sigma X$.

[The composite $\Sigma X \xrightarrow{\Sigma m} \Sigma\Omega_M\Sigma X \to \Sigma X$ is $\mathrm{id}_{\Sigma X}$. Proof: $[x, t] \to [m(x), t] \to m(x)(t\phi(x)) = [x, t\phi(x)/\phi(x)] = [x, t]$. With this in mind, the commutative diagram

$$\begin{tikzcd}
 & \Sigma JX \arrow[r] \arrow[d] & BJX \arrow[d, "BJ_m"] \\
\Sigma X \arrow[ur] \arrow[r] \arrow[dr] & \Sigma\Omega_M\Sigma X \arrow[r] \arrow[d] & B\Omega_M\Sigma X \arrow[dl] \\
 & \Sigma X &
\end{tikzcd}$$

shows that $\Sigma X \to \Sigma JX \to BJX \xrightarrow{BJ_m} B\Omega_M\Sigma X \to \Sigma X$ is also $\mathrm{id}_{\Sigma X}$. On account of Proposition 22, the composite $\Sigma X \to \Sigma JX \to BJX$ is a weak homotopy equivalence. However ΣX is path connected, hence $B\Omega_M\Sigma X \to \Sigma X$ is a weak homotopy equivalence. Therefore $BJ_m : BJX \to B\Omega_M\Sigma X$ is a weak homotopy equivalence.]

[Note: One can view Proposition 23 as the $\pi_0(X) \neq *$ analog of Proposition 20.]

FACT Let (X, x_0) be a wellpointed compactly generated space with $\{x_0\} \subset X$ closed. Assume: X is Δ-separated and $\Delta_X \to X \times_k X$ is a cofibration. Put $GX = F_{\mathrm{gr}}(X, x_0)$ (cf. p. 14–28)—then the arrow $BJX \to BGX$ is a weak homotopy equivalence.

[Note: It follows that the arrow $JX \to GX$ is a weak homotopy equivalence whenever X is path connected (cf. p. 14–21).]

It is also of interest to consider the free abelian compactly generated monoid on (X, x_0), denoted by $SP^\infty X$ and referred to as the infinite symmetric product on (X, x_0). Like JX, $SP^\infty X$ carries the quotient topology coming from $\coprod_n X^n \to SP^\infty X$. Put $SP^n X = p(X^n)$—then if $\{x_0\}$ is closed in X, $SP^n X$ is closed in $SP^\infty X$ and the arrow $X^n \to SP^n X$ is quotient, hence $SP^\infty X = \operatorname{colim} SP^n X$ and $X^n/S_n \approx SP^n X$. Example: $SP^\infty \mathbf{S}^0 \approx \mathbf{Z}_{\geq 0}$.

Under certain conditions, it is possible to identify X^n/S_n. For instance, $\mathbf{S}^2/S_n$ is homeomorphic to $\mathbf{P}^n(\mathbf{C})$, therefore $SP^\infty \mathbf{S}^2$ is homeomorphic to $\mathbf{P}^\infty(\mathbf{C})$, a $K(\mathbf{Z}, 2)$ (cf. p. 14–38).

[Note: A survey of this aspect of the theory has been given by Wagner[†].]

EXAMPLE Let X be a compact metric space with $\dim X < \infty$. Assume: X is an ANR—then X^n/S_n is an ANR (Floyd[‡]).

PROPOSITION 24 Let (X, x_0) be a wellpointed compactly generated space with $\{x_0\} \subset X$ closed—then $(SP^\infty X, x_0)$ is a wellpointed compactly generated space with $\{x_0\} \subset SP^\infty X$ closed, thus is an abelian cofibered monoid in **CG**.

LEMMA If (X, x_0) is a wellpointed compactly generated Hausdorff space, then $(SP^\infty X, x_0)$ is a wellpointed compactly generated Hausdorff space.

FACT Suppose that (X, x_0) is a pointed CW complex—then $(SP^\infty X, x_0)$ is a pointed CW complex.

[It is enough to place a CW structure on each $SP^n X$ in such a way that $SP^{n-1} X$ is a subcomplex of $SP^n X$ (cf. p. 5–25). For this, it is necessary to alter the CW structure on X^n in order to reflect the action of S_n.]

PROPOSITION 25 Let (X, x_0) be a wellpointed compactly generated space with $\{x_0\} \subset X$ closed—then there is an isomorphism $BSP^\infty X \approx SP^\infty \Sigma X$ of abelian monoids in **CG**.

[†] *Dissertationes Math.* **182** (1980), 1–52.

[‡] *Duke Math. J.* **22** (1955), 33–38.

[Analogously, $XSP^\infty X \approx SP^\infty \Gamma X$ and the diagram

$$\begin{tikzcd}
SP^\infty X \arrow[r] \arrow[d, equal] & XSP^\infty X \arrow[r] \arrow[d] & BSP^\infty X \arrow[d] \\
SP^\infty X \arrow[r] & SP^\infty \Gamma X \arrow[r] & SP^\infty \Sigma X
\end{tikzcd}$$

commutes.]

PROPOSITION 26 Let (X, x_0) be a wellpointed compactly generated space with $\{x_0\} \subset X$ closed. Assume: X is path connected and numerably contractible—then the arrow $SP^\infty X \to \Omega BSP^\infty X$ is a pointed homotopy equivalence.

[$\forall\, n$, $SP^n X$ is numerably contractible, so $SP^\infty X = \operatorname{colim} SP^n X$ is numerably contractible (cf. p. 3–13). Since the inclusion $\{x_0\} \to SP^\infty X$ is a closed cofibration and $SP^\infty X$ is path connected, it follows that $SP^\infty X$ admits a homotopy inverse (cf. p. 4–27). Therefore the arrow $SP^\infty X \to \Omega BSP^\infty X$ is a pointed homotopy equivalence (cf. Proposition 14).]

Application: Under the hypotheses of Proposition 26, the composite $SP^\infty X \to \Omega BSP^\infty X \to \Omega SP^\infty \Sigma X$ is a pointed homotopy equivalence.

DOLD-THOM THEOREM Suppose that (X, x_0) is a pointed connected CW complex —then $\forall\, n > 0$, $\pi_n(SP^\infty X) \approx H_n(X)$.

[There are pointed homotopy equivalences $|SP^\infty \sin X| \to SP^\infty |\sin X|$, $SP^\infty |\sin X| \to SP^\infty X$. One has $\widetilde{H}_*(|\sin X|) \approx \widetilde{H}_*(X)$ and, in the notation of p. 13–17, $\pi_*(F_{\text{ab}}(\sin X, x_0)) \approx \widetilde{H}_*(|\sin X|)$ (Weibel[†]). But $\overline{SP^\infty \sin X} = F_{\text{ab}}(\sin X, x_0)$, thus the arrow $|SP^\infty \sin X| \to |F_{\text{ab}}(\sin X, x_0)|$ is a pointed homotopy equivalence (cf. p. 14–25). Accordingly, $\pi_*(|SP^\infty \sin X|) \approx \pi_*(|F_{\text{ab}}(\sin X, x_0)|) \approx \pi_*(F_{\text{ab}}(\sin X, x_0))$, from which the assertion.]

EXAMPLE Dold-Thom can fail if X is not a CW complex. Example: Take for X the Hawaiian earring pointed at $(0,0)$, form its cone ΓX and consider $\Gamma X \vee \Gamma X$—then $H_1(\Gamma X \vee \Gamma X) \neq 0$, so either $\pi_1(SP^\infty \Gamma X) \neq H_1(\Gamma X)$ or $\pi_1(SP^\infty(\Gamma X \vee \Gamma X)) \neq H_1(\Gamma X \vee \Gamma X)$.

Remark: If (X, x_0) is a pointed connected CW complex, then $(SP^\infty X, x_0)$ is a pointed connected CW complex (cf. p. 14–36) and $SP^\infty X \approx (w) \prod\limits_1^\infty K(\pi_n(SP^\infty X), n))$ (cf. p. 5–43) or still, by the Dold-Thom theorem, $SP^\infty X \approx (w) \prod\limits_1^\infty K(H_n(X), n)$.

[†] *An Introduction to Homological Algebra*, Cambridge University Press (1994), 266–267.

EXAMPLE Let π be an abelian group and let $X = M(\pi, n)$ (realized as a pointed connected CW complex)—then $SP^\infty M(\pi, n)$ is a $K(\pi, n)$. In particular: $SP^\infty \mathbf{S}^n$ is a $K(\mathbf{Z}, n)$.

$\mathbf{\Gamma}_{\text{in}}$ is the category whose objects are the finite sets $\mathbf{n} \equiv \{0, 1, \ldots, n\}$ $(n \geq 0)$ with base point 0 and whose morphisms are the base point preserving injective maps.

Example: Let (X, x_0) be a wellpointed compactly generated space with $\{x_0\} \subset X$ closed. Viewing X^n as the space of base point preserving continuous functions $\mathbf{n} \rightarrow X$, define a functor $\text{pow}\,X : \mathbf{\Gamma}_{\text{in}} \rightarrow \mathbf{CG}_*$ by writing $\text{pow}_n X = X^n$, stipulating that the arrow $X^m \rightarrow X^n$ attached to $\gamma : \mathbf{m} \rightarrow \mathbf{n}$ sends $(x_1, \ldots, x_m)$ to $(\overline{x}_1, \ldots, \overline{x}_n)$, where $\overline{x}_j = x_{\gamma^{-1}(j)}$ if $\gamma^{-1}(j) \neq \emptyset$, $\overline{x}_j = x_0$ if $\gamma^{-1}(j) = \emptyset$.

[Note: colim pow X can be identified with $SP^\infty X$.]

EXAMPLE For $n > 0$, colim pow $\mathbf{n} \approx SP^\infty \mathbf{n} \approx \mathbf{Z}_{\geq 0} \times \cdots \times \mathbf{Z}_{\geq 0}$ (n factors). On the other hand, $\overline{\text{hocolim}}$ pow $\mathbf{n}$ has the homotopy type of $B\mathbf{M}_\infty \times_k \cdots \times_k B\mathbf{M}_\infty$ (n factors), $\mathbf{M}_\infty$ the permutative category of p. 14–28 (so $B\mathbf{M}_\infty = \coprod\limits_{n \geq 0} BS_n$).

[Note: colim pow $\mathbf{0} \approx SP^\infty \mathbf{0} \approx \{0\}$ while $\overline{\text{hocolim}}$ pow $\mathbf{0} \approx B\mathbf{\Gamma}_{\text{in}}$, a contractible space (cf. p. 13–15).]

Definition: A creation operator is a functor $\mathcal{C} : \mathbf{\Gamma}_{\text{in}}^{\text{OP}} \rightarrow \mathbf{CG}$ such that $\mathcal{C}_0 = *$.

[Note: $\forall\, n$, $\mathcal{C}_n$ is a right S_n-space.]

EXAMPLE Every nonempty compactly generated Hausdorff space Y gives rise to a creation operator $\mathcal{C}Y$ whose n^{th} space is Y^n $(Y^0 = *)$, the arrow $Y^n \rightarrow Y^m$ determined by $\gamma : \mathbf{m} \rightarrow \mathbf{n}$ being the map $(y_1, \ldots, y_n) \rightarrow (y_{\gamma(1)}, \ldots, y_{\gamma(m)})$.

If $\mathcal{C}$ is a creation operator and if (X, x_0) is a wellpointed compactly generated space with $\{x_0\} \subset X$ closed, then the realization $\mathcal{C}[X]$ of $\mathcal{C}$ at X is $\int^{\mathbf{n}} \mathcal{C}_n \times_k X^n$ $(= \mathcal{C} \otimes_{\mathbf{\Gamma}_{\text{in}}} \text{pow}\,X)$. Example: Suppose that $\mathcal{C}_n = * \;\forall\, n$—then $\mathcal{C}[X] = * \otimes_{\mathbf{\Gamma}_{\text{in}}} \text{pow}\,X \approx \text{colim pow}\,X \approx SP^\infty X$.

EXAMPLE Let $\mathcal{C}_n = S_n \;\forall\, n$. Given a morphism $\gamma : \mathbf{m} \rightarrow \mathbf{n}$ in $\mathbf{\Gamma}_{\text{in}}$, specify $\mathcal{C}\gamma : S_n \rightarrow S_m$ as follows: $\forall\, \sigma \in S_n$, there exists a unique order preserving injection $\gamma' : \mathbf{m} \rightarrow \mathbf{n}$ such that $\gamma'(\mathbf{m}) = (\sigma \circ \gamma)(\mathbf{m})$ and $(\mathcal{C}\gamma)\sigma \in S_m$ is the permutation for which $\gamma' \circ (\mathcal{C}\gamma)\sigma = \sigma \circ \gamma$. This data thus defines a creation operator and $\forall\, X$, $\mathcal{C}[X] \approx JX$.

PROPOSITION 27 Suppose that (X, x_0) is a wellpointed compactly generated space with $\{x_0\} \subset X$ closed and let $\mathcal{C}$ be a creation operator. Denote by $\mathcal{C}_n[X]$ the image of $\coprod\limits_{m \leq n} \mathcal{C}_m \times_k X^m$ in $\mathcal{C}[X]$—then $\mathcal{C}_n[X]$ is a closed subspace of $\mathcal{C}[X]$ and $\mathcal{C}[X] = \text{colim}\; \mathcal{C}_n[X]$.

In addition, the commutative diagram

$$\begin{tikzcd}
\mathcal{C}_n \times_{S_n} X^n_* \arrow[r] \arrow[d] & \mathcal{C}_{n-1}[X] \arrow[d] \\
\mathcal{C}_n \times_{S_n} X^n \arrow[r] & \mathcal{C}_n[X]
\end{tikzcd}$$

is a pushout square and the arrow $\mathcal{C}_{n-1}[X] \rightarrow \mathcal{C}_n[X]$ is a closed cofibration.

[Note: The base point of $\mathcal{C}[X]$ is $[*, x_0]$ and the inclusion $\{[*, x_0]\} \rightarrow \mathcal{C}[X]$ is a closed cofibration.]

Remark: X Δ-separated $+\mathcal{C}_n$ Δ-separated $\forall\, n \Rightarrow \mathcal{C}[X]$ Δ-separated.

The validation of the above remark depends on Proposition 27 and the following lemma.

LEMMA Let G be a compact Hausdorff topological group. Suppose that X is a Δ-separated right G-space—then X/G is Δ-separated.

[It is a matter of proving that $\{(x, x \cdot g) : x \in X, g \in G\}$ is closed in $X \times_k X$ (cf. p. 1–35). However, G acts to the right on $X \times_k X$, viz. $(x, y) \cdot g = (x, y \cdot g)$, and Δ_X is closed in $X \times_k X$, hence $\Delta_X \cdot G$ is closed in $X \times_k X$, G being compact Hausdorff.]

FACT Let $\begin{cases} (X, x_0) \\ (Y, y_0) \end{cases}$ be wellpointed compactly generated spaces with $\begin{cases} \{x_0\} \subset X \\ \{y_0\} \subset Y \end{cases}$ closed; let $f : X \rightarrow Y$ be a pointed continuous function. Assume: f is a closed cofibration—then $\forall$ creation operator $\mathcal{C}$, the induced map $\mathcal{C}[X] \rightarrow \mathcal{C}[Y]$ is a closed cofibration.

[Use the lemma on p. 3–15 ff. and the lemma on p. 14–4.]

[Note: The conclusion of the lemma on p. 3–15 ff. is "closed cofibration" rather than just "cofibration" provided that this is so of the vertical arrow on the right in the hypothesis. To see this, observe that the argument there can be repeated, testing against any arrow $Z \rightarrow B$ which is both a homotopy equivalence and a Hurewicz fibration (cf. p. 4–22).]

PROPOSITION 28 Let $\phi : \mathcal{C} \rightarrow \mathcal{D}$ be a morphism of creation operators. Assume: $\forall\, n$, $\phi_n : \mathcal{C}_n \rightarrow \mathcal{D}_n$ is an S_n-equivariant homotopy equivalence—then ϕ induces a homotopy equivalence $\mathcal{C}[X] \rightarrow \mathcal{D}[X]$.

By definition, $\overline{\text{hocolim}}\, \text{pow}\, X \approx B(—\backslash\mathbf{\Gamma}_{\text{in}}) \otimes_{\mathbf{\Gamma}_{\text{in}}} \text{pow}\, X$. Problem: Exhibit models for $\overline{\text{hocolim}}\, \text{pow}\, X$ in the homotopy category.

[Note: Strictly speaking, $B(—\backslash\mathbf{\Gamma}_{\text{in}})$ is not a creation operator (since $B(\mathbf{0}\backslash\mathbf{\Gamma}_{\text{in}}) \neq *$).]

A compactly generated paracompact Hausdorff space X is said to be S_n-universal if it is a contractible free right S_n-space. The covering projection $X \rightarrow X/S_n$ is then a closed map, hence X/S_n is a compactly generated paracompact Hausdorff space. Therefore X/S_n is a classifying space for S_n (in the sense of p. 4–60). Examples: (1) $X^{\infty}_{S_n}$ is S_n-universal; (2) $B(\mathbf{n}\backslash\mathbf{\Gamma}_{\text{in}})$ is S_n-universal; (3) XS_n is S_n-universal.

A creation operator $\mathcal{C}$ is said to be universal if $\forall\, n$, $\mathcal{C}_n$ is S_n-universal.

Example: Let $\mathcal{C}$ be a universal creation operator—then for any cofibered monoid G in **CG**, $\mathcal{C}_n \times_{S_n} (BG)^n$ has the same homotopy type as $B(S_n \textstyle\int G)$ (cf. p. 14–19).

PROPOSITION 29 Suppose that $\mathcal{C}$ is a universal creation operator—then there exists an arrow $B(\text{—}\backslash\mathbf{\Gamma}_{\text{in}}) \to \mathcal{C}$ such that $\forall n$, $B(\mathbf{n}\backslash\mathbf{\Gamma}_{\text{in}}) \to \mathcal{C}_n$ is an S_n-equivariant homotopy equivalence.

[In the notation of p. 14–16 ff., compose the homotopy equivalence $B(\text{—}\backslash\mathbf{\Gamma}_{\text{in}}) \to P\mathcal{C}$ and the arrow of evaluation $P\mathcal{C} \to \mathcal{C}$.]

Application: Let (X, x_0) be a wellpointed compactly generated space with $\{x_0\} \subset X$ closed—then $\forall$ universal creation operator $\mathcal{C}$, $\mathcal{C}[X]$ and $\overline{\operatorname{hocolim}}\operatorname{pow} X$ have the same homotopy type.

FACT Let $\phi : \mathcal{C} \to \mathcal{D}$ be a morphism of creation operators. Assume: $\mathcal{C}$ and $\mathcal{D}$ are universal—then ϕ induces a homotopy equivalence $\mathcal{C}[X] \to \mathcal{D}[X]$.

Given a nonempty compactly generated Hausdorff space Y, let $F(Y,n)$ be the subspace of Y^n consisting of those n-tuples $(y_1, \ldots, y_n)$ such that $i \neq j \Rightarrow y_i \neq y_j$—then $F(Y,n)$ is open in Y^n, hence is a compactly generated Hausdorff space, and S_n operates freely to the right by permuting coordinates.

[Note: $F(Y,n)$ is the configuration space of n-tuples of distinct points in Y. Consult Cohen[†] for additional information and references.]

Notation: $\operatorname{con} Y$ is the creation operator that sends n to $F(Y,n)$, the arrow $F(Y,n) \to F(Y,m)$ determined by $\gamma : \mathbf{m} \to \mathbf{n}$ being the map $(y_1, \ldots, y_n) \to (y_{\gamma(1)}, \ldots, y_{\gamma(m)})$.

[Note: Therefore $\operatorname{con} Y$ is a subfunctor of $\mathcal{C}Y$ (cf. p. 14–38).]

Observation: The points of $\operatorname{con} Y[X]$ are equivalence classes of pairs (S, f), where $S \subset Y$ is a finite subset of Y, $f : S \to X$ is a function, and $(S, f) \sim (S - \{y\}, f|S - \{y\})$ iff $f(y) = x_0$.

[Note: All pairs (S, f), where $f(S) = \{x_0\}$, are identified with $(\emptyset, \emptyset)$.]

Examples: (1) $\operatorname{con} \mathbf{R}^0[X] \approx X$; (2) $\operatorname{con} Y[\mathbf{S}^0] \approx \{S \subset Y : \#(S) < \omega\}$.

LEMMA $F(\mathbf{R}^\infty, n)$ is S_n-universal.

[$(\mathbf{R}^\infty)^n$ is a polyhedron. But $F(\mathbf{R}^\infty, n)$ is an open subset of $(\mathbf{R}^\infty)^n$, thus it too is a polyhedron (cf. p. 5–3). Therefore $F(\mathbf{R}^\infty, n)$ is a compactly generated paracompact

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[†] *J. Pure Appl. Algebra* **100** (1995), 19–42.

Hausdorff space. Contractibility is clear if $n = 0$ or 1, so take $n \geq 2$ and represent $F(\mathbf{R}^\infty, n)$ as colim $F(\mathbf{R}^q, n)$. Since for $q >> 0$, $F(\mathbf{R}^q, n)$ is the complement in $\mathbf{R}^{qn}$ of certain hyperplanes of codimension q, $F(\mathbf{R}^q, n)$ is $(q-2)$-connected, and this implies that $F(\mathbf{R}^\infty, n)$ is contractible.]

PROPOSITION 30 $\operatorname{con}\mathbf{R}^\infty$ is a universal creation operator.

Application: Let (X, x_0) be a wellpointed compactly generated space with $\{x_0\} \subset X$ closed—then $\overline{\text{hocolim}}\,\text{pow}\,X$ and $\operatorname{con}\mathbf{R}^\infty[X]$ have the same homotopy type.

EXAMPLE $\operatorname{con}\mathbf{R}^\infty[\mathbf{S}^0] \approx \coprod\limits_{n\geq 0} F(\mathbf{R}^\infty, n)/S_n \approx \coprod\limits_{n\geq 0} BS_n$, which agrees with the fact that the homotopy type of $\overline{\text{hocolim}}\,\text{pow}\,\mathbf{S}^0$ is $B\mathbf{M}_\infty$ (cf. p. 14–38).

A q-dimensional rectangle in $[0,1]^q$ is a product of the form $R = [a_1, b_1] \times \cdots \times [a_q, b_q]$, where $0 \leq a_i < b_i \leq 1$. Call $R(q)$ the set of such and topologize it as a subspace of $[0,1]^{2q}$. Note that there is a closed embedding $R(q) \to R(q+1)$ defined by multiplication on the right by $[0,1]$ and put $R(\infty) = \operatorname{colim} R(q)$. Let $\mathrm{BV}(R(q), n)$ be the subspace of $F(R(q), n)$ consisting of those n-tuples $(R_1, \ldots, R_n)$ with the property that the interior of R_i does not meet the interior of R_j if $i \neq j$—then there is a closed embedding $\mathrm{BV}(R(q), n) \to \mathrm{BV}(R(q+1), n)$ and $\mathrm{BV}(R(\infty), n) = \operatorname{colim} \mathrm{BV}(R(q), n)$ is a free right S_n-space.

Notation: BV^∞ is the creation operator that sends n to $\mathrm{BV}(R(\infty), n)$.

LEMMA $\mathrm{BV}(R(\infty), n)$ is S_n-universal.

[It follows from condition C on p. 1–29 that $\mathrm{BV}(R(\infty), n)$ is a compactly generated paracompact Hausdorff space. Since the closed embedding $\mathrm{BV}(R(q), n) \to \mathrm{BV}(R(q+1), n)$ is a cofibration, one need only establish that it is also inessential in order to conclude that $\mathrm{BV}(R(\infty), n)$ is contractible (cf. p. 3–20). To define $H : I\mathrm{BV}(R(q), n) \to \mathrm{BV}(R(q+1), n)$, represent an n-tuple $(R_1, \ldots, R_n)$ by a $2n$-tuple $(A_1, B_1, \ldots, A_n, B_n)$ of points in $[0,1]^q$. Here $R_k \leftrightarrow (A_k, B_k)$ and $A_k = (a_{k1}, \ldots, a_{kq})$, $B_k = (b_{k1}, \ldots, b_{kq})$ $(1 \leq k \leq n)$. Now write $H((A_1, B_1, \ldots, A_n, B_n), t) = (A_1(t), B_1(t), \ldots, A_n(t), B_n(t))$, where

$$A_k(t) = \begin{cases} (a_{k1}, \ldots, a_{kq}, 2t(k-1)/n) & (0 \leq t \leq 1/2) \\ ((2-2t)a_{k1}, \ldots, (2-2t)a_{kq}, (k-1)/n) & (1/2 \leq t \leq 1) \end{cases}$$

and

$$B_k(t) = \begin{cases} (b_{k1}, \ldots, b_{kq}, 1 - 2t(1 - k/n)) & (0 \leq t \leq 1/2) \\ (2t - 1 + (2-2t)b_{k1}, \ldots, 2t - 1 + (2-2t)b_{kq}, k/n) & (1/2 \leq t \leq 1).] \end{cases}$$

[Note: At the opposite extreme, each path component of $\mathrm{BV}(R(1), n)$ is contractible and $\pi_0(\mathrm{BV}(R(1), n)) \approx S_n$.]

PROPOSITION 31 BV^∞ is a universal creation operator.

Application: Let (X, x_0) be a wellpointed compactly generated space with $\{x_0\} \subset X$ closed—then $\overline{\text{hocolim}}\operatorname{pow} X$ and $\mathrm{BV}^\infty[X]$ have the same homotopy type.

Let BV^q be the creation operator that sends n to $\mathrm{BV}(R(q), n)$—then $\mathrm{BV}^\infty = \operatorname{colim} \mathrm{BV}^q$ $\Rightarrow \mathrm{BV}^\infty[X] = \operatorname{colim} \mathrm{BV}^q[X]$.

FACT The arrow $\mathrm{BV}^q[X] \to \mathrm{BV}^{q+1}[X]$ is a closed cofibration.

PROPOSITION 32 The map $\mathrm{BV}(R(q), n) \to F(\mathbf{R}^q, n)$ which takes $(R_1, \ldots, R_n)$ to its center is an S_n-equivariant homotopy equivalence, hence induces a homotopy equivalence $\mathrm{BV}^q[X] \to \operatorname{con} \mathbf{R}^q[X]$.

The elements of $R(q)$ are in a one-to-one correspondence with the functions $[0,1]^q \to [0,1]^q$ of the form $R = r_1 \times \cdots \times r_q$, where $r_i(t) = (b_i - a_i)t + a_i$ $(0 \leq a_i < b_i \leq 1)$. Thus $R(q)$ can be viewed as a subspace of $C([0,1]^q, [0,1]^q)$ (compact open topology), there being no ambiguity in so doing since the two interpretations are homeomorphic.

Representing $\mathbf{S}^q$ as $[0,1]^q / \operatorname{fr}[0,1]^q$, adjust the definitions of $\Sigma^q X$ and $\Omega^q \Sigma^q X$ correspondingly—then the arrow of May is the continuous function $m_q : \mathrm{BV}^q[X] \to \Omega^q \Sigma^q X$ specified by the rule

$$m_q[(R_1, \ldots, R_n), x_1, \ldots, x_n](s) = \begin{cases} [x_i, t] & \text{if } R_i(t) = s \\ * & \text{if } s \notin \bigcup\limits_i \operatorname{im} R_i \end{cases}.$$

MAY'S APPROXIMATION THEOREM Let (X, x_0) be a wellpointed compactly generated space with $\{x_0\} \subset X$ closed. Assume: X is path connected—then $m_q : \mathrm{BV}^q[X] \to \Omega^q \Sigma^q X$ is a weak homotopy equivalence.

[Note: If X has the pointed homotopy type of a pointed connected CW complex, then $\mathrm{BV}^q[X]$ is a pointed CW space, as is $\Omega^q \Sigma^q X$ (loop space theorem), thus under these circumstances the arrow of May is a pointed homotopy equivalence.]

The proof of this result is fairly lengthy and will be omitted. In principle, the argument is an elaboration of that used in Proposition 20 and can be summarized in a sentence: There is a commutative diagram

$$\begin{tikzcd}
\mathrm{BV}^q[X] \arrow[r] \arrow[d, "m_q"'] & E^q(\Gamma X, X) \arrow[r] \arrow[d] & \mathrm{BV}^{q-1}[\Sigma X] \arrow[d, "m_{q-1}"] \\
\Omega\Omega^{q-1}\Sigma^q X \arrow[r] & \Theta\Omega^{q-1}\Sigma^q X \arrow[r] & \Omega^{q-1}\Sigma^q X
\end{tikzcd},$$

where $E^q(\Gamma X, X) \rightarrow \mathrm{BV}^{q-1}[\Sigma X]$ is a quasifibration with fiber $\mathrm{BV}^q[X]$ and $E^q(\Gamma X, X)$ is contractible, thus one may proceed by induction. Details are in May[†].

[Note: When $q = 1$, $\mathrm{BV}^0[\Sigma X] = \Sigma X$ and m_0 is the identity map.]

Notation: Given a pointed Δ-separated compactly generated space X, let $\Omega^\infty\Sigma^\infty X = \operatorname{colim} \Omega^q\Sigma^q X$.

[Note: The reason for imposing the Δ-separation condition is that it ensures the validity of the repetition principle: $\Omega\Omega^\infty\Sigma^\infty\Sigma X \approx \Omega^\infty\Sigma^\infty X$. Proof: $(\Omega^\infty\Sigma^\infty\Sigma X)^{\mathbf{S}^1} \approx (\operatorname{colim} \Omega^q\Sigma^q\Sigma X)^{\mathbf{S}^1} \approx \operatorname{colim}(\Omega^q\Sigma^q\Sigma X)^{\mathbf{S}^1} \approx \operatorname{colim} \Omega^{q+1}\Sigma^{q+1} X \approx \Omega^\infty\Sigma^\infty X$.]

The arrow $\Omega^q\Sigma^q X \rightarrow \Omega^{q+1}\Sigma^{q+1} X$ is the result of applying Ω^q to the arrow of adjunction $\Sigma^q X \rightarrow \Omega\Sigma\Sigma^q X$. It is a closed embedding but it need not be a closed cofibration even if X is wellpointed (in which case, of course, $\Omega^q\Sigma^q X$ is wellpointed $\forall\, q$).

EXAMPLE Suppose that X and Y are pointed finite CW complexes—then $\Omega^\infty\Sigma^\infty X$ and $\Omega^\infty\Sigma^\infty Y$ are homotopy equivalent iff $\Sigma^q X$ and $\Sigma^q Y$ are homotopy equivalent for some $q >> 0$ (Bruner-Cohen-McGibbon[‡]).

Notation: Given a wellpointed Δ-separated compactly generated space X, put $m_\infty = \operatorname{colim} m_q : \mathrm{BV}^\infty[X] \rightarrow \Omega^\infty\Sigma^\infty X$.

[Note: $\mathrm{BV}^\infty[X]$ is wellpointed (since $\forall\, q$, the arrow $\mathrm{BV}^q[X] \rightarrow \mathrm{BV}^{q+1}[X]$ is a closed cofibration (cf. p. 14–42)) but it is problematic whether this is true of $\Omega^\infty\Sigma^\infty X$ without additional assumptions on X.]

PROPOSITION 33 Let (X, x_0) be a wellpointed compactly generated space with $\{x_0\} \subset X$ closed. Assume: X is Δ-separated and path connected—then $m_\infty : \mathrm{BV}^\infty[X] \rightarrow \Omega^\infty\Sigma^\infty X$ is a weak homotopy equivalence.

[†] *SLN* **271** (1972), 50–68.

[‡] *Quart. J. Math.* **46** (1995), 11–20.

[In the commutative ladder $$\begin{tikzcd} \mathrm{BV}^1[X] \arrow[r] \arrow[d] & \mathrm{BV}^2[X] \arrow[r] \arrow[d] & \cdots \\ \Omega\Sigma X \arrow[r] & \Omega^2\Sigma^2 X \arrow[r] & \cdots \end{tikzcd}$$, the vertical arrows are weak homotopy equivalences and the spaces are T_1, so the generality on p. 4–48 can be quoted.]

A compactly generated space X is said to be Δ-cofibered if the inclusion $\Delta_X \to X \times_k X$ is a closed cofibration.

[Note: It is automatic that $\forall\ x_0 \in X$, $\{x_0\} \to X$ is a closed cofibration (cf. p. 3–15).]

FACT Let K be a pointed compact Hausdorff space. Suppose that X is pointed and Δ-cofibered—then the pointed exponential object X^K is Δ-cofibered.

Example: Let (X, x_0) be a pointed compactly generated space. Assume: X is Δ-cofibered—then ΣX is Δ-cofibered (cf. p. 3–16), as is ΩX.

LEMMA Let (X, x_0) be a pointed compactly generated space. Assume: X is Δ-cofibered—then the arrow of adjunction $X \to \Omega\Sigma X$ is a closed cofibration.

Application: Let (X, x_0) be a pointed compactly generated space. Assume: X is Δ-cofibered—then $\forall\ q$, the arrow $\Omega^q\Sigma^q X \to \Omega^{q+1}\Sigma^{q+1} X$ is a closed cofibration.

[Note: It is a corollary that $\Omega^\infty\Sigma^\infty X$ is Δ-cofibered (cf. p. 14–4).]

LEMMA Let (X, x_0) be a pointed compactly generated space. Assume: X is Δ-cofibered—then for every pointed Δ-cofibered compact Hausdorff space $K \neq *$, the arrow $X \to (X \#_k K)^K$ adjoint to the identity $X \#_k K \to X \#_k K$ is a closed cofibration.

[Note: Specialize and take $K = \mathbf{S}^1$ to see that the arrow of adjunction $X \to \Omega\Sigma X$ is a closed cofibration.]

FACT Let $\begin{cases} (X, x_0) \\ (Y, y_0) \end{cases}$ be pointed compactly generated spaces. Assume: X is Δ-cofibered and Y is Δ-separated—then for every pointed Δ-cofibered compact Hausdorff space $K \neq *$, the arrow $X \to Y^K$ adjoint to a closed cofibration $X \#_k K \to Y$ is a closed cofibration.

[Factor the arrow $X \to Y^K$ as the composite $X \to (X \#_k K)^K \to Y^K$.]

FACT Suppose that $A \to X$ is a closed cofibration, where X is Δ-cofibered—then A is Δ-cofibered (cf. §3, Proposition 11) and the arrow $\Omega^\infty\Sigma^\infty A \to \Omega^\infty\Sigma^\infty X$ is a closed cofibration.

[All the arrows in the pullback square

$$\begin{tikzcd} \Omega^q\Sigma^q A \arrow[r] \arrow[d] & \Omega^{q+1}\Sigma^{q+1} A \arrow[d] \\ \Omega^q\Sigma^q X \arrow[r] & \Omega^{q+1}\Sigma^{q+1} X \end{tikzcd}$$

are closed cofibrations, so one can appeal to the lemma on p. 14–4.]

PROPOSITION 34 Let (X, x_0) be a pointed compactly generated space. Assume: X is Δ-cofibered and has the pointed homotopy type of a pointed connected CW complex—then $m_\infty : \mathrm{BV}^\infty[X] \to \Omega^\infty\Sigma^\infty X$ is a pointed homotopy equivalence.

[In the commutative ladder

$$\begin{tikzcd} \mathrm{BV}^1[X] \arrow[r] \arrow[d] & \mathrm{BV}^2[X] \arrow[r] \arrow[d] & \cdots \\ \Omega\Sigma X \arrow[r] & \Omega^2\Sigma^2 X \arrow[r] & \cdots \end{tikzcd}$$

, the horizontal arrows are closed cofibrations and the vertical arrows are pointed homotopy equivalences. Now cite Proposition 15 in §3.]

HOMOTOPY COLIMIT THEOREM Let (X, x_0) be a pointed connected CW complex or a pointed connected ANR—then $\overline{\mathrm{hocolim}}\,\mathrm{pow}\,X$ and $\Omega^\infty\Sigma^\infty X$ have the same homotopy type.

[One has only to recall that $\overline{\mathrm{hocolim}}\,\mathrm{pow}\,X$ and $\mathrm{BV}^\infty[X]$ have the same homotopy type (cf. p. 14–42).]

[Note: For the validity of the condition on the diagonal, cf. p. 3–14 & p. 6–14.]

EXAMPLE **Connectedness is essential here. For example, the homotopy type of $\overline{\mathrm{hocolim}}\,\mathrm{pow}\,\mathbf{S}^0$ is represented by $B\mathbf{M}_\infty$ (cf. p. 14–38) but the homotopy type of $\Omega^\infty\Sigma^\infty\mathbf{S}^0$ is represented by $\Omega B|M_\infty|$ (cf. p. 14–61) ($|M_\infty| = B\mathbf{M}_\infty = \coprod_{n\geq 0} BS^n$).**

Given a cofunctor $\mathcal{C} : \mathrm{iso}\,\mathbf{\Gamma} \to \mathbf{CG}$, let $\widehat{\mathcal{C}}(\mathbf{m},\mathbf{n}) = \coprod_{\gamma:\mathbf{m}\to\mathbf{n}} \prod_{1\leq j\leq n} \mathcal{C}(\#(\gamma^{-1}(j)))$ (here γ ranges over the morphisms $\mathbf{m} \to \mathbf{n}$ in $\mathbf{\Gamma}$ and $\widehat{\mathcal{C}}(\mathbf{m},\mathbf{0})$ is a point indexed by the unique arrow $\mathbf{m} \to \mathbf{0}$)—then with the obvious choice for the unit, $[(\mathrm{iso}\,\mathbf{\Gamma})^{\mathrm{OP}}, \mathbf{CG}]$ acquires the structure of a monoidal category by writing $\mathcal{C} \circ \mathcal{D}(\mathbf{m}) = \coprod_{n\geq 0} \mathcal{C}(\mathbf{n}) \times_{S_n} \widehat{\mathcal{D}}(\mathbf{m},\mathbf{n})$.

LEMMA The functor $-\circ\mathcal{D}$ has a right adjoint $\mathrm{HOM}(\mathcal{D}, -)$, where $\mathrm{HOM}(\mathcal{D},\mathcal{E})(\mathbf{n}) = \prod_{m\geq 0} \mathrm{hom}(\widehat{\mathcal{D}}(\mathbf{m},\mathbf{n}), \mathcal{E}(\mathbf{m}))^{S_m}$ ($\mathrm{hom} = kC_k$, the internal hom functor in $\mathbf{CG}$ (cf. p. 1–33)), so $\mathrm{Nat}(\mathcal{C}\circ\mathcal{D}, \mathcal{E}) \approx \mathrm{Nat}(\mathcal{C}, \mathrm{HOM}(\mathcal{D},\mathcal{E}))$.

An operad $\mathcal{O}$ in $\mathbf{CG}$ is a monoid in the monoidal category $[(\mathrm{iso}\,\mathbf{\Gamma})^{\mathrm{OP}}, \mathbf{CG}]$. Examples: (1) Let $\mathcal{O}_n = * \ \forall\, n$; (2) Let $\mathcal{O}_n = S_n \ \forall\, n$.

The definition of an operad makes sense if **CG** is replaced by any symmetric monoidal category **C** which is complete and cocomplete.

[Note: Agreeing to write $\mathbf{OPER_C}$ for $\mathbf{MON}_{[(\text{iso}\,\mathbf{\Gamma})^{\text{OP}},\mathbf{C}]}$, one can show that $\mathbf{OPER_C}$ is complete and cocomplete and that the forgetful functor $\mathbf{OPER_C} \rightarrow [(\text{iso}\,\mathbf{\Gamma})^{\text{OP}}, \mathbf{C}]$ has a left adjoint, the free operad functor (Getzler-Jones[†]).]

Equivalently, an operad $\mathcal{O}$ in **CG** consists of compactly generated spaces $\mathcal{O}_n$ equipped with a right action of S_n, a point $1 \in \mathcal{O}_1$ (the unit), and for each sequence $j_1, \ldots, j_n$ of nonnegative integers, a continuous function $\Lambda : \mathcal{O}_n \times_k (\mathcal{O}_{j_1} \times_k \cdots \times_k \mathcal{O}_{j_n}) \rightarrow \mathcal{O}_{j_1+\cdots+j_n}$ satisfying the following conditions.

(OPER$_1$) Given $\sigma \in S_n$, $\sigma_k \in S_{j_k}$ $(k = 1, \ldots, n)$, and $f \in \mathcal{O}_n$, $g_k \in \mathcal{O}_{j_k}$, one has $\Lambda(f \cdot \sigma; g_1, \ldots, g_n) = \Lambda(f; g_{\sigma^{-1}(1)}, \ldots, g_{\sigma^{-1}(n)}) \cdot \sigma(j_1, \ldots, j_n)$ $(\sigma(j_1, \ldots, j_n)$ the permutation in $S_{j_1+\cdots+j_n}$ that permutes the n blocks of j_k successive integers per σ, the order within each block staying fixed) and $\Lambda(f; g_1 \cdot \sigma_1, \ldots, g_n \cdot \sigma_n) = \Lambda(f; g_1, \ldots, g_n) \cdot (\sigma_1 \amalg \cdots \amalg \sigma_n)$ $(\sigma_1 \amalg \cdots \amalg \sigma_n$ the permutation in $S_{j_1+\cdots+j_n}$ that leaves the n blocks invariant and which restricts to σ_k on the kth block).

(OPER$_2$) Given $f \in \mathcal{O}_n$, $g_k \in \mathcal{O}_{j_k}$ $(k = 1, \ldots, n)$, $h_{kl} \in \mathcal{O}_{i_{kl}}$ $(l = 1, \ldots, j_k)$, one has $\Lambda(f; \Lambda(g_k; h_{kl})) = \Lambda(\Lambda(f; g_k); h_{kl})$.

(OPER$_3$) Given $f \in \mathcal{O}_n$, one has $\Lambda(f; 1, \ldots, 1) = f$ and given $g \in \mathcal{O}_j$, one has $\Lambda(1; g) = g$.

Example: BV^q is an operad in **CG**. Thus with $\mathcal{O}_n = \text{BV}(R(q), n)$, write $(R_1, \ldots, R_n) \cdot \sigma = (R_{\sigma(1)}, \ldots, R_{\sigma(n)})$ $(\sigma \in S_n)$, take for $1 \in \text{BV}(R(q), 1)$ the identity function, and let $\Lambda : \text{BV}(R(q), n) \times_k (\text{BV}(R(q), j_1) \times_k \cdots \times_k \text{BV}(R(q), j_k)) \rightarrow \text{BV}(R(q), j_1 + \cdots + j_n)$ be defined on elements via composition $j_1 \cdot [0,1]^q \coprod \cdots \coprod j_n \cdot [0,1]^q \rightarrow n \cdot [0,1]^q \rightarrow [0,1]$.

[Note: $\text{con}\,\mathbf{R}^q$ is not an operad in **CG**.]

EXAMPLE Let $\mathcal{O}$ be an operad in **CG** such that $\forall\, n$, $\mathcal{O}_n \neq \emptyset$. Definition: $\text{grd}\mathcal{O}$ is the operad in **CG** with $\text{grd}_n\mathcal{O} = |\text{ner}\,\text{grd}\mathcal{O}_n|$ (cf. p. 14–17). To specify the right action of S_n, note that there is a simplicial map $\text{si}S_n \rightarrow \text{ner}\,\text{grd}S_n$, hence $|\text{ner}\,\text{grd}\mathcal{O}_n| \times S_n \rightarrow |\text{ner}\,\text{grd}\mathcal{O}_n| \times |\text{ner}\,\text{grd}S_n| \approx |\text{ner}\,\text{grd}(\mathcal{O}_n \times S_n)| \rightarrow |\text{ner}\,\text{grd}\mathcal{O}_n|$. Next, $\mathcal{O}_1 = |\text{ner}\,\text{grd}\mathcal{O}_1|_0$, so the choice for 1 is clear. Finally, Λ is defined by $|\text{ner}\,\text{grd}\mathcal{O}_n| \times_k (|\text{ner}\,\text{grd}\mathcal{O}_{j_1}| \times_k \cdots \times_k |\text{ner}\,\text{grd}\mathcal{O}_{j_n}|) \approx |\text{ner}\,\text{grd}(\mathcal{O}_n \times_k (\mathcal{O}_{j_1} \times_k \cdots \times_k \mathcal{O}_{j_n}))| \rightarrow |\text{ner}\,\text{grd}(\mathcal{O}_{j_1+\cdots+j_n})|$. Example: Let $\mathcal{O}_n = S_n$ $\forall\, n$—then $\text{grd}_n\mathcal{O} \approx |\text{ner}\,\text{tran}S_n|$ (cf. p. 0–45 ff.), i.e., $\text{grd}_n\mathcal{O} \approx XS_n$.

[†] *Operads, Homotopy Algebra, and Iterated Integrals for Double Loop Spaces*, Preprint.

In terms of the Λ, a morphism $\mathcal{O} \to \mathcal{P}$ of operads in **CG** is a sequence of S_n-equivariant continuous functions $\mathcal{O}_n \to \mathcal{P}_n$ such that the diagrams

$$\begin{tikzcd}
\mathcal{O}_n \times_k (\mathcal{O}_{j_1} \times_k \cdots \times_k \mathcal{O}_{j_n}) \arrow[r] \arrow[d] & \mathcal{O}_{j_1+\cdots+j_n} \arrow[d] \\
\mathcal{P}_n \times_k (\mathcal{P}_{j_1} \times_k \cdots \times_k \mathcal{P}_{j_n}) \arrow[r] & \mathcal{P}_{j_1+\cdots+j_n}
\end{tikzcd}$$

commute and $\mathcal{O}_1 \to \mathcal{P}_1$ sends 1 to 1.

Example: $\forall\, q$, the arrow $\mathrm{BV}^q \to \mathrm{BV}^{q+1}$ is a morphism of operads in **CG**.

EXAMPLE If $\mathcal{O}$ is an operad in **CG**, then $\sin \mathcal{O}$ is an operad in **SISET**. Its geometric realization $|\sin \mathcal{O}|$ is an operad in **CG** and the arrow $|\sin \mathcal{O}| \to \mathcal{O}$ is a morphism of operads in **CG**.

An operad $\mathcal{O}$ in **CG** is said to be reduced if $\mathcal{O}_0 = *$.

PROPOSITION 35 Let $\mathcal{O}$ be a reduced operad in **CG**—then $\mathcal{O}$ extends to a creation operator $\mathbf{\Gamma}_{\mathbf{in}}^{\mathrm{OP}} \to \mathbf{CG}$.

[It suffices to define $\mathcal{O}$ on the order preserving injections (cf. p. 13–56) or still, for each n, on the $n+1$ elementary order preserving injections $\sigma_i : \mathbf{n} \to \mathbf{n+1}$, where $\begin{cases} j \to j & (j \leq i) \\ j \to j+1 & (j > i) \end{cases}$ $(0 \leq i \leq n)$, the requisite arrows $\mathcal{O}_{n+1} \to \mathcal{O}_n$ thus being the assignments $f \to \Lambda(f; (1^i, *, 1^{n-i}))$.]

Notation: $\mathbf{CG}_{*\mathbf{c}}$ is the full subcategory of $\mathbf{CG}_*$ whose objects are the (X, x_0) such that $* \to (X, x_0)$ is a closed cofibration.

[Note: The standard model category structure on $\mathbf{CG}_*$ is that inherited from the standard model category structure on **CG** (cf. p. 12–3) and the cofibrant objects therein are the objects of $\mathbf{CG}_{*\mathbf{c}}$.]

Observation: For any creation operator $\mathcal{C}$, $\mathcal{C}[?]$ is a functor $\mathbf{CG}_{*\mathbf{c}} \to \mathbf{CG}_{*\mathbf{c}}$ (cf. Proposition 27).

PROPOSITION 36 Let $\mathcal{O}$ be a reduced operad in **CG**—then $\mathcal{O}$ determines a triple $\mathbf{T}_{\mathcal{O}} = (T_{\mathcal{O}}, m, \epsilon)$ in $\mathbf{CG}_{*\mathbf{c}}$.

[Take $T_{\mathcal{O}} = \mathcal{O}[?]$ and for each X, define $m_X : \mathcal{O}^2[X] \to \mathcal{O}[X]$, $\epsilon_X : X \to \mathcal{O}[X]$ by the formulas $m_X[f, [g_1, x_1], \ldots, [g_n, x_n]] = [\Lambda(f; g_1, \ldots, g_n), x_1, \ldots, x_n]$ $(f \in \mathcal{O}_n, g_k \in \mathcal{O}_{j_k}$ & $x_k \in X^{j_k}$ $(1 \leq k \leq n))$, $\epsilon_X(x) = [1, x]$ $(x \in X)$.]

[Note: A morphism $\mathcal{O} \to \mathcal{P}$ of reduced operads in **CG** leads to a morphism $\mathbf{T}_{\mathcal{O}} \to \mathbf{T}_{\mathcal{P}}$ of triples in $\mathbf{CG}_{*\mathbf{c}}$.]

Examples: (1) With $\mathcal{O}_n = * \;\forall\, n$, $T_{\mathcal{O}}X = SP^{\infty}X$; (2) With $\mathcal{O}_n = S_n \;\forall\, n$, $T_{\mathcal{O}}X = JX$.

FACT Let X be a pointed compactly generated simplicial space satisfying the cofibration condition such that $\forall\, n$, X_n is in $\mathbf{CG}_{*c}$. Given a reduced operad $\mathcal{O}$ in $\mathbf{CG}$, define a pointed compactly generated simplicial space $\mathcal{O}[X]$ by $\mathcal{O}[X]_n = \mathcal{O}[X_n]$—then $|\mathcal{O}[X]| \approx \mathcal{O}[|X|]$.

[Work with the arrow $[[f, x_1, \ldots, x_k], t] \rightarrow [f, [x_1, t], \ldots, [x_k, t]]$, where $f \in \mathcal{O}_k$, $x_j \in X_n$ $(1 \leq j \leq k)$, $t \in \Delta^n$.]

[Note: The diagrams

$$\begin{tikzcd}
{|\mathcal{O}^2[X]|} \arrow[r] \arrow[d, "|m_X|"'] & \mathcal{O}^2[|X|] \arrow[d, "m_{|X|}"] \\
{|\mathcal{O}[X]|} \arrow[r] & \mathcal{O}[|X|]
\end{tikzcd}, \quad
\begin{tikzcd}
 & {|\mathcal{O}[X]|} \arrow[d] \\
{|X|} \arrow[ur, "|\epsilon_X|"] \arrow[r, "\epsilon_{|X|}"'] & \mathcal{O}[|X|]
\end{tikzcd}$$

commute. Consequently, if X is a simplicial $\mathbf{T}_{\mathcal{O}}$-algebra, then $|X|$ is a $\mathbf{T}_{\mathcal{O}}$-algebra (by the composite $\mathcal{O}[|X|] \rightarrow |\mathcal{O}[X]| \rightarrow |X|$).]

Let $\mathcal{O}$ be a reduced operad in $\mathbf{CG}$—then an $\mathcal{O}$-space is an object (X, x_0) in $\mathbf{CG}_{*c}$ and continuous functions $\theta_n : \mathcal{O}_n \times_k X^n \rightarrow X$ $(n \geq 0)$ subject to the following assumptions.

($\mathcal{O}$-SP_1) Given $\sigma \in S_n$, $f \in \mathcal{O}_n$, and $x_k \in X$ $(k = 1, \ldots, n)$, one has $\theta_n(f \cdot \sigma, x_1, \ldots, x_n) = \theta_n(f, x_{\sigma^{-1}(1)}, \ldots, x_{\sigma^{-1}(n)})$.

($\mathcal{O}$-SP_2) Given $f \in \mathcal{O}_n$, $g_k \in \mathcal{O}_{j_k}$ $(k = 1, \ldots, n)$, $x_{kl} \in X$ $(l = 1, \ldots, j_k)$, one has $\theta_{j_1 + \cdots + j_n}(\Lambda(f; g_1, \ldots, g_n), x_{11}, \ldots, x_{1j_1}, \ldots, x_{n1}, \ldots, x_{nj_n}) = \theta_n(f, \theta_{j_1}(g_1; x_{11}, \ldots, x_{1j_1}), \ldots, \theta_{j_n}(g_n; x_{n1}, \ldots, x_{nj_n}))$.

($\mathcal{O}$-SP_3) $\theta_0(*) = x_0$ and $\theta_1(1, x) = x\ \forall\, x \in X$.

[Note: In practice, one sometimes encounters objects in $\mathbf{CG}_*$ satisfying all the assumptions that define an $\mathcal{O}$-space but, strictly speaking, are not $\mathcal{O}$-spaces because they may not be in $\mathbf{CG}_{*c}$. Up to homotopy equivalence, this is not a problem. Thus let X be an $\mathcal{O}$-space in $\mathbf{CG}_*$ and consider $\check{X}$ (cf. p. 3–33). Define $\check{\theta}_n : \mathcal{O}_n \times_k \check{X}^n \rightarrow \check{X}$ by $\check{\theta}_n(f, \check{x}_1, \ldots, \check{x}_n) = \begin{cases} \theta_n(f, r(\check{x}_1), \ldots, r(\check{x}_n)) & \text{if } \check{x}_i \notin [0,1] - \{0\}\ (\exists\, i) \\ \check{x}_1 \cdots \check{x}_n & \text{if } \check{x}_i \in [0,1]\ (\forall\, i) \end{cases}$—then $\check{X}$ is an $\mathcal{O}$-space in $\mathbf{CG}_{*c}$ and the retraction $r : \check{X} \rightarrow X$ is a morphism of $\mathcal{O}$-spaces.]

Examples: (1) If $\mathcal{O}_n = *\ \forall\, n$, then the $\mathcal{O}$-spaces are the abelian cofibered monoids in $\mathbf{CG}$; (2) If $\mathcal{O}_n = S_n\ \forall\, n$, then the $\mathcal{O}$-spaces are the cofibered monoids in $\mathbf{CG}$.

Example: $\forall\, X$ in $\mathbf{CG}_{*c}$, $\Omega^q X$ is a BV^q-space.

[Define $\theta_n : \mathrm{BV}(R(q), n) \times_k (\Omega^q X)^n \rightarrow \Omega^q X$ by sending $((R_1, \ldots, R_n), f_1, \ldots, f_n)$ to that element of $\Omega^q X$ which at s is $f_i(t)$ if $R_i(t) = s$ lies in the interior of R_i and is x_0 otherwise.]

EXAMPLE Let $\mathcal{S}$ be the operad in $\mathbf{CAT}$ with $\mathcal{S}_n = \mathrm{tran}S_n\ \forall\, n$—then in suggestive terminology, a permutative category $\mathbf{C}$ is an $\mathcal{S}$-category, thus its classifying space $B\mathbf{C}$ is a $B\mathcal{S}$-space.

[Note: $B\mathcal{S}_n = B\mathrm{tran}S_n = |\mathrm{ner}\,\mathrm{tran}S_n| = |\mathrm{bar}(*; \mathbf{S}_n; S_n)| = XS_n$.]

$\mathcal{O}$-$\mathbf{SP}$ is the category whose objects are the $\mathcal{O}$-spaces and whose morphisms $X \rightarrow Y$

are the pointed continuous functions $X \to Y$ such that the diagrams

$$\begin{tikzcd} \mathcal{O}_n \times_k X^n \arrow[r] \arrow[d] & \mathcal{O}_n \times_k Y^n \arrow[d] \\ X \arrow[r] & Y \end{tikzcd}$$

commute.

Example: $\mathcal{O}\text{-}\mathbf{SP} = \mathbf{CG}_{*c}$ if $\mathcal{O}_0 = *$, $\mathcal{O}_1 = \{1\}$, $\mathcal{O}_n = \emptyset$ $(n > 1)$.

EXAMPLE If X is an $\mathcal{O}$-space, then so are ΩX and ΘX. Moreover, the inclusion $\Omega X \to \Theta X$ is a morphism of $\mathcal{O}$-spaces, as is the CG fibration $\Theta X \to X$.

PROPOSITION 37 Let $\mathcal{O}$ be a reduced operad in **CG**—then the categories $\mathcal{O}$-**SP** and $\mathbf{T}_{\mathcal{O}}$-**ALG** are canonically isomorphic.

[There is a one-to-one correspondence between the $\mathcal{O}$-space structures on X and the $\mathbf{T}_{\mathcal{O}}$-algebra structures on X, encapsulated in the commutativity of the diagrams

$$\begin{tikzcd} \mathcal{O}_n \times_k X^n \arrow[r] \arrow[drr, "\theta_n"'] & \mathcal{O}_n[X] \arrow[r] & \mathcal{O}[X] \arrow[d, "\theta"] \\ & & X \end{tikzcd}$$

for all n, i.e., the θ_n combine to define an arrow $\theta : \mathcal{O}[X] \to X$ satisfying TA_1 and TA_2 (cf. p. 0–27 ff.) and vice versa).]

[Note: The endomorphism operad End X of X is defined by $(\text{End}\, X)_n = X^{X^n}$ (pointed exponential object in $\mathbf{CG}_*$), supplied with the evident operations. Taking adjoints, the $\mathbf{T}_{\mathcal{O}}$-algebra structures on X correspond bijectively to morphisms of operads $\mathcal{O} \to \text{End}\, X$ in **CG**.]

Example: $\forall\, X$, $\mathcal{O}[X]$ is a $\mathbf{T}_{\mathcal{O}}$-algebra, hence is an $\mathcal{O}$-space.

EXAMPLE The functors $\Sigma^q : \mathbf{CG}_* \to \mathbf{CG}_*$, $\Omega^q : \mathbf{CG}_* \to \mathbf{CG}_*$ both respect $\mathbf{CG}_{*c}$ and (Σ^q, Ω^q) is an adjoint pair, thus $\forall\, X$, there is an arrow of adjunction $X \to \Omega^q\Sigma^q X$. As noted above, $\Omega^q\Sigma^q X$ is a BV^q-space or still, is a $\mathbf{T}_{\text{BV}^q}$-algebra. The composite $\text{BV}^q[X] \to \text{BV}^q[\Omega^q\Sigma^q X] \to \Omega^q\Sigma^q X$ is m_q, the arrow of May. It is a morphism of $\mathbf{T}_{\text{BV}^q}$-algebras. On the other hand, $\forall\, X$, there is an arrow of adjunction $\Sigma^q\Omega^q X \to X$, from which $\Omega^q\Sigma^q\Omega^q X \to \Omega^q X$. Viewing the BV^q-space $\Omega^q X$ as a $\mathbf{T}_{\text{BV}^q}$-algebra, its structural morphism $\text{BV}^q[\Omega^q X] \to \Omega^q X$ is the composite $\text{BV}^q[\Omega^q X] \xrightarrow{m_q} \Omega^q\Sigma^q\Omega^q X \to \Omega^q X$.

FACT Let X be a pointed compactly generated simplicial space satisfying the cofibration condition such that $\forall\, n$, X_n is in $\mathbf{CG}_{*c}$—then the arrow $|\Omega^q X| \to \Omega^q|X|$ is a morphism of $\mathbf{T}_{\text{BV}^q}$-algebras.

[The structural morphism $\text{BV}^q[|\Omega^q X|] \to |\Omega^q X|$ is the composite $\text{BV}^q[|\Omega^q X|] \to |\text{BV}^q[\Omega^q X]| \to$

$|\Omega^q X|$ (cf. p. 14–48), thus one has to check that the diagram

$$\begin{tikzcd}
\mathrm{BV}^q[|\Omega^q X|] \arrow[rr] \arrow[d] & & \mathrm{BV}^q[\Omega^q|X|] \arrow[d] \\
|\mathrm{BV}^q[\Omega^q X]| \arrow[r] & |\Omega^q X| \arrow[r] & \Omega^q|X|
\end{tikzcd}$$

commutes.]

FACT Let X be a pointed compactly generated simplicial space satisfying the cofibration condition such that $\forall\, n$, X_n is in $\mathbf{CG}_{*\mathbf{c}}$—then the diagram

$$\begin{tikzcd}
|\mathrm{BV}^q[X]| \arrow[r] \arrow[d, "|m_q|"'] & \mathrm{BV}^q[|X|] \arrow[d, "m_q"] \\
|\Omega^q\Sigma^q X| \arrow[r] & \Omega^q\Sigma^q|X|
\end{tikzcd}$$

commutes.

Let $\mathcal{O}$ be a reduced operad in $\mathbf{CG}$, $F : \mathbf{CG}_{*\mathbf{c}} \to \mathbf{CG}_{*\mathbf{c}}$ a right $\mathbf{T}_{\mathcal{O}}$-functor—then for any $\mathbf{T}_{\mathcal{O}}$-algebra X, $\mathrm{bar}(F; \mathbf{T}_{\mathcal{O}}; X)$ is a simplicial object in $\mathbf{CG}_{*\mathbf{c}}$ (cf. p. 0–46) and one writes $B(F; \mathcal{O}; X)$ for its geometric realization (or just $B(\mathcal{O}; \mathcal{O}; X)$ if $F = T_{\mathcal{O}} = \mathcal{O}[?]$).

PROPOSITION 38 Let $\mathcal{O}$ be a reduced operad in $\mathbf{CG}$ such that $\{1\} \to \mathcal{O}_1$ is a closed cofibration. Suppose that $F : \mathbf{CG}_{*\mathbf{c}} \to \mathbf{CG}_{*\mathbf{c}}$ is a right $\mathbf{T}_{\mathcal{O}}$-functor which preserves closed cofibrations—then $\forall$ $\mathcal{O}$-space X, $\mathrm{bar}(F; \mathbf{T}_{\mathcal{O}}; X)$ satisfies the cofibration condition, hence $B(F; \mathcal{O}; X)$ is in $\mathbf{CG}_{*\mathbf{c}}$.

[On general grounds, $\mathcal{O}[?]$ preserves closed cofibrations (cf. p. 14–39). Moreover the assumption on the unit of $\mathcal{O}$ implies that $\epsilon_X : X \to \mathcal{O}[X]$ is a closed cofibration $\forall$ X, so the conclusion follows from the definition of the s_i and the fact that F preserves closed cofibrations.]

EXAMPLE Σ is a right $\mathbf{T}_{\mathrm{BV}^1}$-functor and preserves closed cofibrations. If G is a cofibered monoid in $\mathbf{CG}$, then G acquires the structure of a $\mathbf{T}_{\mathrm{BV}^1}$-algebra via the composite $\mathrm{BV}^1[G] \to JG \to G$. Thus it is meaningful to form $\mathrm{bar}(\Sigma; \mathbf{T}_{\mathrm{BV}^1}; G)$. Since $\{1\} \to \mathrm{BV}(R(1), 1)$ is a closed cofibration, $\mathrm{bar}(\Sigma; \mathbf{T}_{\mathrm{BV}^1}; G)$ satisfies the cofibration condition (cf. Proposition 38) and its geometric realization $B(\Sigma; \mathrm{BV}^1; G)$ is the classifying space of G in the sense of May. It is true but not obvious that $B(\Sigma; \mathrm{BV}^1; G)$ and BG have the same weak homotopy type (Thomason[†]).

EXAMPLE Suppose that X is a path connected BV^q-space—then X has the weak homotopy type of a q-fold loop space. In fact, Σ^q is a right $\mathbf{T}_{\mathrm{BV}^q}$-functor, as is $\Omega^q\Sigma^q$, so one can form $B(\Sigma^q; \mathrm{BV}^q; X)$ and $B(\Omega^q\Sigma^q; \mathrm{BV}^q; X)$, where now X is viewed as a $\mathbf{T}_{\mathrm{BV}^q}$-algebra. Consider the following diagram in the category of $\mathbf{T}_{\mathrm{BV}^q}$-algebras: $X \leftarrow B(\mathrm{BV}^q;\mathrm{BV}^q;X) \rightarrow B(\Omega^q\Sigma^q;\mathrm{BV}^q;X) \rightarrow \Omega^q B(\Sigma^q;\mathrm{BV}^q;X)$. Owing to the generalities on p. 0–46 ff., the arrow $X \leftarrow B(\mathrm{BV}^q;\mathrm{BV}^q;X)$ is a homotopy equivalence (cf. p. 14–12). Next, according to May's approximation theorem, $\forall\ n$, $m_q : \mathrm{BV}^q[(\mathrm{BV}^q)^n[X]] \rightarrow \Omega^q\Sigma^q(\mathrm{BV}^q)^n[X]$ is a weak homotopy equivalence. Therefore, on account of Proposition 38, the arrow $B(\mathrm{BV}^q;\mathrm{BV}^q;X) \rightarrow B(\Omega^q\Sigma^q;\mathrm{BV}^q;X)$ is a weak homotopy equivalence (cf. p. 14–8). As for the arrow $B(\Omega^q\Sigma^q;\mathrm{BV}^q;X) \rightarrow \Omega^q B(\Sigma^q;\mathrm{BV}^q;X)$, it too is a weak homotopy equivalence. Indeed, all data is path connected and $\mathrm{bar}(\Omega^q\Sigma^q;\mathbf{T}_{\mathrm{BV}^q};X) = \Omega^q\mathrm{bar}(\Sigma^q;\mathbf{T}_{\mathrm{BV}^q};X)$, thus $|\Omega^q\mathrm{bar}(\Sigma^q;\mathbf{T}_{\mathrm{BV}^q};X) \rightarrow \Omega^q|\mathrm{bar}(\Sigma^q;\mathbf{T}_{\mathrm{BV}^q};X)|$ is a weak homotopy equivalence (cf. p. 14–11).

[†] *Duke Math. J.* **46** (1979), 217–252; see also Fiedorowicz, *Amer. J. Math.* **106** (1984), 301–350.

[Note: The composite $X \rightarrow B(\mathrm{BV}^q;\mathrm{BV}^q;X) \rightarrow B(\Omega^q\Sigma^q;\mathrm{BV}^q;X) \rightarrow \Omega^q B(\Sigma^q;\mathrm{BV}^q;X)$ is the adjoint of $\Sigma^q X \rightarrow B(\Sigma^q;\mathrm{BV}^q;X)$ but it is not a morphism of $\mathbf{T}_{\mathrm{BV}^q}$-algebras and one cannot expect to always find a morphism $X \rightarrow \Omega^q Y$ of $\mathbf{T}_{\mathrm{BV}^q}$-algebras which is a weak homotopy equivalence. Take, e.g., $q = 1$ and let X be a path connected cofibered monoid in **CG** (thought of as a $\mathbf{T}_{\mathrm{BV}^1}$-algebra). Claim: The only morphism $X \rightarrow \Omega Y$ of $\mathbf{T}_{\mathrm{BV}^1}$-algebras is the constant map $X \rightarrow j(y_0)$. Proof: Inspect the commutative diagram

$$\begin{tikzcd}
\mathrm{BV}(R(1),1) \times_k X \arrow[r] \arrow[d] & \mathrm{BV}(R(1),1) \times_k \Omega Y \arrow[d] \\
X \arrow[r] & \Omega Y
\end{tikzcd}$$

.]

EXAMPLE Let $\mathcal{O}$ be a reduced operad in **CG** such that $\{1\} \rightarrow \mathcal{O}_1$ is a closed cofibration. Assume: $\forall\ n$, $\mathcal{O}_n \rightarrow *$ is an S_n-equivariant homotopy equivalence—then every $\mathcal{O}$-space X has the homotopy type of an abelian cofibered monoid in **CG**. Indeed, X and $B(\mathcal{O};\mathcal{O};X)$ have the same homotopy type. Moreover, $\forall\ n$, the arrow $\mathcal{O}[\mathcal{O}^n[X]] \rightarrow SP^\infty\mathcal{O}^n[X]$ is a homotopy equivalence (cf. Proposition 28), so the arrow $B(\mathcal{O};\mathcal{O};X) \rightarrow B(SP^\infty;\mathcal{O};X)$ is a homotopy equivalence (cf. Proposition 4 and Proposition 38). But $B(SP^\infty;\mathcal{O};X)$ is an abelian cofibered monoid in **CG**.

Let $\mathcal{O}$ be a reduced operad in **CG**—then $\mathcal{O}$ is said to be an E_∞ operad if $\forall\ n$, $\mathcal{O}_n$ is a contractible compactly generated Hausdorff space, the action of S_n is free, and the inclusion $\{1\} \rightarrow \mathcal{O}_1$ is a closed cofibration.

Example: $\mathrm{BV}^\infty = \mathrm{colim}\ \mathrm{BV}^q$ is an E_∞ operad, the Boardman-Vogt operad.

[In view of Proposition 31, the only thing that has to be checked is the cofibration condition on the unit. However, by definition, $\mathrm{BV}(R(\infty),1) = \mathrm{colim}\ \mathrm{BV}(R(q),1)$ and $\mathrm{BV}(R(q),1) \rightarrow \mathrm{BV}(R(q+1),1)$ is a closed cofibration. In addition, the diagonal embedding $\mathrm{BV}(R(q),1) \rightarrow \mathrm{BV}(R(q),1) \times_k \mathrm{BV}(R(q),1)$ is a closed cofibration ($\mathrm{BV}(R(q),1)$ is a polyhedron), thus the diagonal embedding $\mathrm{BV}(R(\infty),1) \rightarrow \mathrm{BV}(R(\infty),1) \times_k \mathrm{BV}(R(\infty),1)$ is a closed cofibration (cf. p. 14–4). Therefore the inclusion $\{1\} \rightarrow \mathrm{BV}(R(\infty),1)$ is a closed cofibration (cf. p. 3–15).]

EXAMPLE Let $\mathcal{O}_n = S_n$ $\forall\ n$—then $\mathrm{grd}\mathcal{O}$ is an E_∞ operad (cf. p. 14–46), the permutation operad PER.

[Note: In the notation of p. 14–49, PER $\approx BS$.]

Given two real inner product spaces $\begin{cases} U \\ V \end{cases}$ with $\begin{cases} \dim U \leq \omega \\ \dim V \leq \omega \end{cases}$, each equipped with the finite topology, let $\mathcal{I}(U,V)$ be the set of linear isometries $U \to V$. Endow $\mathcal{I}(U,V)$ with the structure of a compactly generated Hausdorff space by relativising the compact open topology on $C(U,V)$ and taking its "k-ification".

LEMMA Fix a real inner product space V with $\dim V = \omega$—then $\forall$ real inner product space U with $\dim U \leq \omega$, $\mathcal{I}(U,V)$ is contractible.

[Let $\{u_i\}$, $\{v_j\}$ be orthonormal bases for U, V and let $\begin{cases} i_1, i_2 : U \to U \oplus U \\ j_1, j_2 : V \to V \oplus V \end{cases}$ be the inclusions onto the first, second summands. Choose a homotopy F through isometries between i_1 and i_2 and choose a homotopy Φ through isometries between id_V and $\phi : V \to V$, where $\phi(v_j) = v_{2j}$. Let $h : V \to V \oplus V$ be the isometry $\begin{cases} h(v_{2j}) = (v_j, 0) \\ h(v_{2j-1}) = (0, v_j) \end{cases}$, fix $f_0 \in \mathcal{I}(U,V)$, and define $H : I\mathcal{I}(U,V) \to \mathcal{I}(U,V)$ by $H(f,t) = \begin{cases} \Phi(2t) \circ f & (0 \leq t \leq 1/2) \\ h^{-1} \circ (f \oplus f_0) \circ F(2t-1) & (1/2 \leq t \leq 1) \end{cases}$—then $H(f,0) = f$, $H(f,1/2) = \phi \circ f = h^{-1} \circ h \circ \phi \circ f = h^{-1} \circ j_1 \circ f = h^{-1} \circ (f \oplus f_0) \circ i_1$, and $H(f,1) = h^{-1} \circ (f \oplus f_0) \circ i_2 = h^{-1} \circ (f_0 \oplus f_0) \circ i_2$, which is independent of f.]

FACT Suppose that $\dim U < \omega$ and $\dim V = \omega$—then $\mathcal{I}(U,V)$ is a CW complex, hence the diagonal embedding $\mathcal{I}(U,V) \to \mathcal{I}(U,V) \times_k \mathcal{I}(U,V)$ is a closed cofibration (and, by the lemma, a homotopy equivalence).

LEMMA Fix a real inner product space V with $\dim V = \omega$—then the diagonal embedding $\mathcal{I}(V,V) \to \mathcal{I}(V,V) \times_k \mathcal{I}(V,V)$ is a closed cofibration.

[Write $V = \operatorname{colim} V_n$, where $\forall\, n$, $\dim V_n = n$ and $V_n \subset V_{n+1} \subset V$. Consider the commutative diagram

$$\begin{tikzcd}
\mathcal{I}(V_n, V) \arrow[d] & \mathcal{I}(V_{n+1}, V) \arrow[l] \arrow[d] \\
\mathcal{I}(V_n, V) \times_k \mathcal{I}(V_n, V) & \mathcal{I}(V_{n+1}, V) \times_k \mathcal{I}(V_{n+1}, V) \arrow[l]
\end{tikzcd}.$$

Here, the horizontal arrows are CG fibrations and the vertical arrows are closed cofibrations and homotopy equivalences. Since $\mathcal{I}(V,V) = \lim \mathcal{I}(V_n, V)$, the assertion is a consequence of the generality infra.]

Application: The inclusion $\{\mathrm{id}_V\} \to \mathcal{I}(V,V)$ is a closed cofibration.

LEMMA Let
$$\begin{tikzcd} X_0 & X_1 \arrow[l] & \cdots \arrow[l] \\ Y_0 \arrow[from=u, d] & Y_1 \arrow[l] \arrow[from=u, d] & \cdots \arrow[l] \end{tikzcd}$$
be a commutative ladder of compactly generated spaces. Assume: $\forall\, n$, the horizontal arrows are CG fibrations and the vertical arrows are closed cofibrations and homotopy equivalences—then the induced map $\lim X_n \to \lim Y_n$ is a closed cofibration and a homotopy equivalence.

Example: Let V be a real inner product space with $\dim V = \omega$ and write V^n for the orthogonal direct sum of n copies of V—then the assignment $n \to \mathcal{L}_n = \mathcal{I}(V^n, V)$ defines an E_∞ operad $\mathcal{L}$, the linear isometries operad.

[The left action of S_n on V^n by permutations induces a free right action of S_n on $\mathcal{L}_n$, the unit $1 \in \mathcal{L}_1$ is the identity map $V \to V$, and $\Lambda : \mathcal{L}_n \times_k (\mathcal{L}_{j_1} \times_k \cdots \times_k \mathcal{L}_{j_n}) \to \mathcal{L}_{j_1 + \cdots + j_n}$ sends $(f; g_1, \ldots, g_n)$ to $f \circ (g_1 \oplus \cdots \oplus g_n)$.]

EXAMPLE Take $V = \mathbf{R}^\infty$—then $\Omega^\infty\Sigma^\infty\mathbf{S}^0$ is an $\mathcal{L}$-space. Indeed, $\Omega^\infty\Sigma^\infty\mathbf{S}^0 \approx \operatorname{colim} \Omega^n\mathbf{S}^n = \operatorname{colim}(\mathbf{S}^n)^{\mathbf{S}^n}$ and $\forall\, m, n$, there is a smash product pairing $(\mathbf{S}^m)^{\mathbf{S}^m} \times_k (\mathbf{S}^n)^{\mathbf{S}^n} \to (\mathbf{S}^m \#_k \mathbf{S}^n)^{\mathbf{S}^m \#_k \mathbf{S}^n}$, where $\mathbf{S}^m \#_k \mathbf{S}^n = \mathbf{S}^{m+n}$ (cf. p. 3–28).]

[Note: Boardman-Vogt[†] have given a systematic procedure for generating various classes of examples of $\mathcal{L}$-spaces.]

LEMMA Let G be a finite group and let X be a right G-space. Assume: Each $x \in X$ has a neighborhood U with the property that $U \cdot g \cap U = \emptyset\ \forall\, g \neq e$—then the projection $X \to X/G$ is a covering projection.

Application: Let G be a finite group and let X be a right G-space. Assume: The action of G is free and X is Hausdorff—then the projection $X \to X/G$ is a covering projection.

[Note: Subject to these conditions on X, given any other right G-space Y, the product $X \times Y$ satisfies the hypotheses of the lemma, as does $X \times_k Y$, hence the projection $X \times Y \to (X \times Y)/G$ is a covering projection, as is $X \times_k Y \to (X \times_k Y)/G$.]

PROPOSITION 39 Let $\mathcal{O} \to \mathcal{P}$ be a morphism of E_∞ operads—then $\forall\, X$, the induced map $\mathcal{O}[X] \to \mathcal{P}[X]$ is a weak homotopy equivalence.

[Consider the commutative diagram

$$\begin{tikzcd} \mathcal{O}_n \times_{S_n} X^n \arrow[d] & \mathcal{O}_n \times_{S_n} X^n_* \arrow[l] \arrow[r] \arrow[d] & \mathcal{O}_{n-1}[X] \arrow[d] \\ \mathcal{P}_n \times_{S_n} X^n & \mathcal{P}_n \times_{S_n} X^n_* \arrow[l] \arrow[r] & \mathcal{P}_{n-1}[X] \end{tikzcd} .$$

[†] *SLN* **347** (1973), 207–217; see also May, *SLN* **577** (1977), 9–24.

Arguing inductively, the arrow $\mathcal{O}_{n-1}[X] \to \mathcal{P}_{n-1}[X]$ is a weak homotopy equivalence. But the same is also true of the other two vertical arrows (compare the long exact sequences in homotopy of the relevant covering projections). Therefore, since the horizontal arrows on the left are closed cofibrations, it follows that $\mathcal{O}_n[X] \to \mathcal{P}_n[X]$ is a weak homotopy equivalence (cf. p. 4–51), thus $\mathcal{O}[X] \to \mathcal{P}[X]$ is a weak homotopy equivalence (cf. p. 4–48).]

Example: Let $\begin{cases} \mathcal{O}' \\ \mathcal{O}'' \end{cases}$ be uce_∞ operads—then their product $\mathcal{O}' \times \mathcal{O}''$ is an E_∞ operad and $\forall\, X$, the arrows $(\mathcal{O}' \times \mathcal{O}'')[X] \to \begin{cases} \mathcal{O}'[X] \\ \mathcal{O}''[X] \end{cases}$ induced by the projections $\mathcal{O}' \times \mathcal{O}'' \to \begin{cases} c\mathcal{O}' \\ \mathcal{O}'' \end{cases}$ are weak homotopy equivalences.

Example: Let $\mathcal{O}$ be an E_∞ operad—then $|\sin \mathcal{O}|$ is an E_∞ operad (cf. p. 14–47) and $\forall\, X$, the arrow $|\sin \mathcal{O}|[X] \to \mathcal{O}[X]$ is a weak homotopy equivalence.

[Note: Viewed as a creation operator, $\mathcal{O}$ need not be universal (but $|\sin \mathcal{O}|$ is).]

FACT Let $\begin{cases} \mathcal{C} \\ \mathcal{D} \end{cases}$ be creation operators, where $\forall\, n$, $\begin{cases} \mathcal{C}_n \\ \mathcal{D}_n \end{cases}$ is a compactly generated Hausdorff space and the action of S_n is free. Suppose given an arrow $\phi : \mathcal{C} \to \mathcal{D}$ such that $\forall\, n$, $\phi_n : \mathcal{C}_n \to \mathcal{D}_n$ is a weak homotopy equivalence—then $\forall\, X$, ϕ induces a weak homotopy equivalence $\mathcal{C}[X] \to \mathcal{D}[X]$.

[Note: By the same token, if $f : X \to Y$ is a weak homotopy equivalence, then $\mathcal{C}f : \mathcal{C}[X] \to \mathcal{C}[Y]$ is a weak homotopy equivalence provided that $\forall\, n$, $\mathcal{C}_n$ is a compactly generated Hausdorff space and the action of S_n is free.]

PROPOSITION 40 Let $\mathcal{O}$ be an E_∞ operad—then every $\mathcal{O}$-space X is a homotopy associative, homotopy commutative H-space.

[To define the product, fix $f_2 \in \mathcal{O}_2$ and consider $\theta_2(f_2, \text{—}|) : X^2 \to X$ (up to homotopy, the product is independent of the choice of $f_2 \in \mathcal{O}_2$).]

[Note: If $X \to Y$ is a morphism of $\mathcal{O}$-spaces, then $X \to Y$ is a morphism of H-spaces.]

EXAMPLE Let $\mathcal{O} = \mathrm{PER} \approx BS$ and take $f_2 = e \in S_2 \subset XS_2$—then with this choice for the product, every $\mathcal{O}$-space is a homotopy commutative cofibered monoid in **CG**.

Working in the compactly generated category, let X be a homotopy associative, homotopy commutative H-space—then a group completion of X is a morphism $X \to Y$ of H-spaces, where Y is homotopy associative and $\pi_0(Y)$ is a group, such that $\overline{\pi_0(X)} \approx \pi_0(Y)$ and $H_*(X;\mathbf{k})[\pi_0(X)^{-1}] \approx H_*(Y;\mathbf{k})$ for every commutative ring $\mathbf{k}$ with unit.

Example: Let G be a cofibered monoid in **CG**. Assume: G is homotopy commutative—then according to Proposition 16 and the group completion theorem, the arrow $G \to \Omega BG$ is a group completion.

EXAMPLE Take $X = \mathbf{Q}$ (discrete topology), $Y = \mathbf{Q}$ (usual topology)—then the identity map $X \to Y$ is a group completion but it is not a homotopy equivalence.

[Note: Suppose that $X \to Y$ is a group completion, where $\begin{cases} X \\ Y \end{cases}$ are pointed compactly generated CW spaces—then $X \to Y$ is a weak homotopy equivalence if $\pi_0(X)$ is a group. Proof: One has $\pi_0(X) \approx \overline{\pi_0(X)} \approx \pi_0(Y)$ and there are homotopy equivalences $\begin{cases} X \to X_0 \times \pi_0(X) \\ Y \to Y_0 \times \pi_0(Y) \end{cases}$, where $\begin{cases} X_0 \\ Y_0 \end{cases}$ is the path component of the identity element, thus the assertion follows from Dror's Whitehead theorem.]

EXAMPLE Given a permutative category $\mathbf{C}$, let C^+ be the simplicial object in **CAT** defined by $C_n^+ = \prod\limits_1^{n+2} \mathbf{C}$, where

$$d_i(X_0, X_0', X_1, \ldots, X_n) = \begin{cases} (X_0 \otimes X_1, X_0' \otimes X_1, X_2, \ldots, X_n) & (i = 0) \\ (X_0, X_0', X_1, \ldots, X_i \otimes X_{i+1}, \ldots, X_n) & (0 < i < n) \\ (X_0, X_0', X_1, \ldots, X_{n-1}) & (i = n) \end{cases},$$

$s_i(X_0, X_0', X_1, \ldots, X_n) = (X_0, X_0', X_1, \ldots, X_i, e, X_{i+1}, \ldots, X_n)$—then there is a functor $\mathbf{C} \to \mathrm{gro}_{\mathbf{\Delta}^{\mathrm{OP}}} C^+$ and Thomason[†] has shown that the arrow $B\mathbf{C} \to B(\mathrm{gro}_{\mathbf{\Delta}^{\mathrm{OP}}} C^+)$ is a group completion.

EXAMPLE Let X be a monoidal compactly generated simplicial space. Assume: X satisfies the cofibration condition and X_1 is homotopy commutative—then the arrow $X_1 \to \Omega|X|$ is a group completion (Quillen[‡]).

LEMMA Let X be a homotopy associative, homotopy commutative H space. Suppose that $X \to Y$ is a morphism of H spaces, where Y is homotopy associative and $\pi_0(Y)$ is a group, such that $\overline{\pi_0(X)} \approx \pi_0(Y)$ and $H_*(X; \mathbf{k})[\pi_0(X)^{-1}] \approx H_*(Y; \mathbf{k})$ for all prime fields $\mathbf{k}$—then the arrow $X \to Y$ is a group completion.

SUBLEMMA Let $\begin{cases} K \\ L \end{cases}$ be pointed CW complexes, $f : K \to L$ a pointed continuous function. Assume: f is a pointed homology equivalence—then $\Sigma f : \Sigma K \to \Sigma L$ is a pointed homotopy equivalence.

[Given (X, x_0) in $\mathbf{CW}_*$, let X_{i_0}, X_i $(i \in I)$ be its set of path components, where $x_0 \in X_{i_0}$. Choose a vertex x_i in each X_i—then up to pointed homotopy, $\Sigma X = \bigvee\limits_i \Sigma X_i \vee \Sigma \pi_0(X)$.]

LEMMA Let $\begin{cases} X \\ Y \end{cases}$, Z be Δ-separated pointed CW spaces in $\mathbf{CG}_{*c}$, $f : X \to Y$ a pointed homology equivalence. Suppose that Z is a homotopy associative H space such that $\pi_0(Z)$ is a group—then the precomposition arrow $f^* : [Y, Z] \to [X, Z]$ is bijective.

[†] *Math. Proc. Cambridge Philos. Soc.* **85** (1979), 91–109.

[‡] *Memoirs Amer. Math. Soc.* **529** (1994), 89–105.

[Take Z path connected and fix a retraction $JZ \to Z$. Since $[\Sigma Y, \Sigma Z] \approx [\Sigma X, \Sigma Z]$, the arrow $[Y, \Omega\Sigma Z] \to [X, \Omega\Sigma Z]$ is bijective, so the assertion is true for JZ (cf. Proposition 19). Now use the commutative diagram

$$\begin{tikzcd}
{[Y,JZ]} \arrow[r] \arrow[d] & {[X,JZ]} \arrow[d] \\
{[Y,Z]} \arrow[r] & {[X,Z]}
\end{tikzcd}$$

to see that the assertion is true for Z.]

[Note: To define a retraction $JZ \to Z$, make a choice for associating iterated products. Continuity is ensured if the homotopy unit is a strict unit, which can always be arranged (since $Z \vee Z \to Z \times_k Z$ is a closed cofibration (cf. p. 3–27)).]

FACT Let X, $\begin{cases} Y_1 \\ Y_2 \end{cases}$ be Δ-separated pointed CW spaces in $\mathbf{CG}_{*c}$. Assume: $\pi_0(X) = \mathbf{Z}_{\geq 0}$ and $\begin{cases} X \to Y_1 \\ X \to Y_2 \end{cases}$ are group completions—then $\exists$ a pointed homotopy equivalence $Y_1 \to Y_2$.

MAY'S GROUP COMPLETION THEOREM Let (X, x_0) be a wellpointed compactly generated space with $\{x_0\} \subset X$ closed. Assume: X is Δ-separated—then $m_\infty : \mathrm{BV}^\infty[X] \to \Omega^\infty\Sigma^\infty X$ is a group completion.

[Note: When specialized to a path connected X, one recovers Proposition 33.]

Homological calculations of this sort have their origins in the work of Dyer-Lashof[†]. Details are in May[‡].

Example: X Δ-cofibered $\Rightarrow$ $\Omega^\infty\Sigma^\infty X$ Δ-cofibered (cf. p. 14–44). And: $\Omega^\infty\Sigma^\infty X$ is a BV^∞-space. The composite $\mathrm{BV}^\infty[X] \to \mathrm{BV}^\infty[\Omega^\infty\Sigma^\infty X] \to \Omega^\infty\Sigma^\infty X$ is m_∞, the arrow of May. It is a morphism of $\mathrm{T}_{\mathrm{BV}^\infty}$-algebras.

PROPOSITION 41 Let $\mathcal{O}$ be an E_∞ operad—then there is a functor $G : \mathcal{O}\text{-}\mathbf{SP} \to \mathbf{CG}_{*c}$ and a natural transformation id $\to G$ such that for every $\mathcal{O}$-space X, the arrow $X \to GX$ is a group completion.

[The product $\mathcal{O} \times \mathrm{PER}$ is an E_∞ operad and X is an $\mathcal{O} \times \mathrm{PER}$-space (through the projection $\mathcal{O} \times \mathrm{PER} \to \mathcal{O}$). Consider the arrows $X \leftarrow B(\mathcal{O} \times \mathrm{PER}; \mathcal{O} \times \mathrm{PER}; X) \to B(\mathrm{PER}; \mathcal{O} \times \mathrm{PER}; X)$ in the category of $\mathbf{T}_{\mathcal{O} \times \mathrm{PER}}$-algebras. The generalities on p. 0–46 ff. imply that the arrow $X \leftarrow B(\mathcal{O} \times \mathrm{PER}; \mathcal{O} \times \mathrm{PER}; X)$ is a homotopy equivalence (cf. p. 14–12) and Propositions 38 and 39 imply that the arrow $B(\mathcal{O} \times \mathrm{PER}; \mathcal{O} \times \mathrm{PER}; X) \to B(\mathrm{PER}; \mathcal{O} \times \mathrm{PER}; X)$ is a weak homotopy equivalence (cf. p. 14–8). Since $B(\mathrm{PER}; \mathcal{O} \times \mathrm{PER}; X)$ is a PER-space, it is a homotopy commutative cofibered monoid in $\mathbf{CG}$ (cf.

[†] *Amer. J. Math.* **84** (1962), 35–88.

[‡] *SLN* **533** (1976), 39–59.

p. 14–54). Put $GX = \Omega BB(\mathrm{PER}; \mathcal{O} \times \mathrm{PER}; X)$ and let $X \to GX$ be the composite $X \to B(\mathcal{O} \times \mathrm{PER}; \mathcal{O} \times \mathrm{PER}; X) \to B(\mathrm{PER}; \mathcal{O} \times \mathrm{PER}; X) \to GX$.]

FACT Let $\mathcal{O}$ be an E_∞ operad. Suppose that $A \to X$ is a closed cofibration, where A, X are Δ-separated $\mathcal{O}$-spaces—then $GA \to GX$ is a closed cofibration.

[The arrow $B(\mathrm{PER}; \mathcal{O} \times \mathrm{PER}; A) \to B(\mathrm{PER}; \mathcal{O} \times \mathrm{PER}; X)$ is a closed cofibration (cf. p. 14–5 & p. 14–39).]

PROPOSITION 42 Let $\mathcal{O}$ be an E_∞ operad such that $\forall\, n$, $\mathcal{O}_n$ is an S_n-CW complex—then $\forall$ Δ-cofibered X, $\mathcal{O}[X]$ is Δ-cofibered.

[By induction, $\forall\, n$, $\mathcal{O}_n[X]$ is Δ-cofibered (cf. p. 3–16). Therefore $\mathcal{O}[X] = \operatorname{colim} \mathcal{O}_n[X]$ is Δ-cofibered (cf. p. 14–4).]

[Note: If $\mathcal{O}$ is an E_∞ operad, then $|\sin \mathcal{O}|$ is an E_∞ operad such that $\forall\, n$, $|\sin \mathcal{O}_n|$ is an S_n-CW complex.]

Given an E_∞ operad $\mathcal{O}$, put $\mathcal{O}^\infty = \mathcal{O} \times \mathrm{BV}^\infty$—then every $\mathcal{O}$-space X is an $\mathcal{O}^\infty$-space. On the other hand, $|\sin X|$ is a $|\sin \mathcal{O}|$-space, hence is a $|\sin \mathcal{O}^\infty|$-space. The arrows $|\sin \mathcal{O}^\infty|[|\sin X|] \to |\sin \mathrm{BV}^\infty|[|\sin X|]$, $|\sin \mathrm{BV}^\infty|[|\sin X|] \to \mathrm{BV}^\infty[|\sin X|]$ are weak homotopy equivalences (cf. Proposition 39), thus the composite $|\sin \mathcal{O}^\infty|[|\sin X|] \to \Omega^\infty \Sigma^\infty |\sin X|$ is a group completion.

[Note: The diagram

$$\begin{tikzcd}
{|\sin X|} \arrow[r] \arrow[d] & B(|\sin \mathcal{O}^\infty|; |\sin \mathcal{O}^\infty|; |\sin X|) \arrow[d] \\
X \arrow[r] & B(\mathcal{O}^\infty; \mathcal{O}^\infty; X)
\end{tikzcd}$$

commutes. Here, the horizontal arrows are homotopy equivalences and the vertical arrows are weak homotopy equivalences.]

PROPOSITION 43 Let $\mathcal{O}$ be an E_∞ operad. Suppose that X is an $\mathcal{O}$-space—then the arrow $B(|\sin \mathcal{O}^\infty|; |\sin \mathcal{O}^\infty|; |\sin X|) \to B(\Omega^\infty \Sigma^\infty; |\sin \mathcal{O}^\infty|; |\sin X|)$ is a morphism of $|\sin \mathcal{O}^\infty|$-spaces (cf. p. 14–48) and a group completion.

[Consider the commutative diagram

$$\begin{tikzcd}
{|\sin X|} \arrow[d] & B(|\sin \mathcal{O}^\infty|; |\sin \mathcal{O}^\infty|; |\sin X|) \arrow[l] \arrow[r] \arrow[d] & B(\Omega^\infty \Sigma^\infty; |\sin \mathcal{O}^\infty|; |\sin X|) \arrow[d] \\
G|\sin X| & B(G|\sin \mathcal{O}^\infty|; |\sin \mathcal{O}^\infty|; |\sin X|) \arrow[l] \arrow[r] & B(G\Omega^\infty \Sigma^\infty; |\sin \mathcal{O}^\infty|; |\sin X|)
\end{tikzcd}$$

The arrow $|\sin X| \leftarrow B(|\sin \mathcal{O}^\infty|;|\sin \mathcal{O}^\infty|;|\sin X|)$ is a homotopy equivalence, as is the arrow $G|\sin X| \leftarrow B(G|\sin \mathcal{O}^\infty|;|\sin \mathcal{O}^\infty|;|\sin X|)$. But $|\sin X| \rightarrow G|\sin X|$ is a group completion, so $B(|\sin \mathcal{O}^\infty|;|\sin \mathcal{O}^\infty|;|\sin X|) \rightarrow B(G|\sin \mathcal{O}^\infty|;|\sin \mathcal{O}^\infty|;|\sin X|)$ is a group completion. Since $\Omega^\infty\Sigma^\infty$ preserves closed cofibrations between Δ-cofibered objects (cf. p. 14–44), Proposition 42 implies that $\mathrm{bar}(\Omega^\infty\Sigma^\infty;\mathbf{T}_{|\sin \mathcal{O}^\infty|};|\sin X|)$ satisfies the cofibration condition (see the proof of Proposition 38). Analogous remarks apply to $\mathrm{bar}(G\Omega^\infty\Sigma^\infty;\mathbf{T}_{|\sin \mathcal{O}^\infty|};|\sin X|)$ and $\mathrm{bar}(G|\sin \mathcal{O}^\infty|;\mathbf{T}_{|\sin \mathcal{O}^\infty|};|\sin X|)$. Therefore the arrows $B(\Omega^\infty\Sigma^\infty;|\sin \mathcal{O}^\infty|;|\sin X|) \rightarrow B(G\Omega^\infty\Sigma^\infty;|\sin \mathcal{O}^\infty|;|\sin X|)$, $B(G|\sin \mathcal{O}^\infty|;|\sin \mathcal{O}^\infty|;|\sin X|) \rightarrow B(G\Omega^\infty\Sigma^\infty;|\sin \mathcal{O}^\infty|;|\sin X|)$ induce isomorphisms in homology $\forall$ $\mathbf{k}$ (cf. Proposition 10) and the assertion follows.]

Maintaining the preceding assumptions, put $\mathcal{O}^q = \mathcal{O} \times \mathrm{BV}^q$.

LEMMA Let $\mathcal{O}$ be an E_∞ operad. Suppose that X is a Δ-separated $\mathcal{O}$-space—then the arrow $B(\Omega^\infty\Sigma^\infty;|\sin \mathcal{O}^\infty|;|\sin X|) \rightarrow B(\Omega^\infty\Sigma^\infty;\mathcal{O}^\infty;X)$ is a weak homotopy equivalence.

[Since $B(\Omega^\infty\Sigma^\infty;|\sin \mathcal{O}^\infty|;|\sin X|) \approx \operatorname{colim} B(\Omega^q\Sigma^q;|\sin \mathcal{O}^q|;|\sin X|)$, $B(\Omega^\infty\Sigma^\infty;\mathcal{O}^\infty;X) \approx \operatorname{colim} B(\Omega^q\Sigma^q;\mathcal{O}^q;X)$, where $B(\Omega^q\Sigma^q;|\sin \mathcal{O}^q|;|\sin X|) \rightarrow B(\Omega^{q+1}\Sigma^{q+1};|\sin \mathcal{O}^{q+1}|;|\sin X|)$, $B(\Omega^q\Sigma^q;\mathcal{O}^q;X) \rightarrow B(\Omega^{q+1}\Sigma^{q+1};\mathcal{O}^{q+1};X)$ are closed embeddings, it will be enough to show that $\forall$ q, the arrow $B(\Omega^q\Sigma^q;|\sin \mathcal{O}^q|;|\sin X|) \rightarrow B(\Omega^q\Sigma^q;\mathcal{O}^q;X)$ is a weak homotopy equivalence (cf. p. 4–48). However, bearing in mind Proposition 38, $\forall$ n, $|\sin \mathcal{O}^q|^n[|\sin X|] \rightarrow (\mathcal{O}^q)^n[X]$ is a weak homotopy equivalence (cf. p. 14–54), hence $\forall$ n, $\Omega^q\Sigma^q|\sin \mathcal{O}^q|^n[|\sin X|] \rightarrow \Omega^q\Sigma^q(\mathcal{O}^q)^n[X]$ is a weak homotopy equivalence (cf. p. 14–34 ff.), so the generality on p. 14–8 is applicable.]

[Note: While $B(\Omega^\infty\Sigma^\infty;|\sin \mathcal{O}^\infty|;|\sin X|)$ is in $\mathbf{CG}_{*c}$, this is not a priori the case of $B(\Omega^\infty\Sigma^\infty;\mathcal{O}^\infty;X)$ (both spaces are, of course, Δ-separated). Still, $B(\Omega^\infty\Sigma^\infty;\mathcal{O}^\infty;X)$ is an $\mathcal{O}^\infty$-space in $\mathbf{CG}_*$ (see the remarks on p. 14–48).]

PROPOSITION 44 Let $\mathcal{O}$ be an E_∞ operad. Suppose that X is a Δ-separated $\mathcal{O}$-space—then the arrow $B(\mathcal{O}^\infty;\mathcal{O}^\infty;X) \rightarrow B(\Omega^\infty\Sigma^\infty;\mathcal{O}^\infty;X)$ is a morphism of $\mathcal{O}^\infty$-spaces (cf. p. 14–48) and a group completion.

[In the commutative diagram

$$\begin{tikzcd}
B(|\sin \mathcal{O}^\infty|;|\sin \mathcal{O}^\infty|;|\sin X|) \arrow[r] \arrow[d] & B(\Omega^\infty\Sigma^\infty;|\sin \mathcal{O}^\infty|;|\sin X|) \arrow[d] \\
B(\mathcal{O}^\infty;\mathcal{O}^\infty;X) \arrow[r] & B(\Omega^\infty\Sigma^\infty;\mathcal{O}^\infty;X)
\end{tikzcd},$$

the vertical arrows are weak homotopy equivalences and, by Proposition 43, the top horizontal arrow is a group completion.]

[Note: When X is path connected, the arrow $B(\mathcal{O}^\infty;\mathcal{O}^\infty;X) \rightarrow B(\Omega^\infty\Sigma^\infty;\mathcal{O}^\infty;X)$ is a weak homotopy equivalence (cf. Proposition 33).]

A spectrum $\mathbf{X}$ is a sequence of pointed Δ-separated compactly generated spaces X_q and pointed homeomorphisms $X_q \xrightarrow{\sigma_q} \Omega X_{q+1}$. **SPEC** is the category whose objects are the spectra and whose morphisms $\mathbf{f}: \mathbf{X} \rightarrow \mathbf{Y}$ are sequences of pointed continuous functions $f_q : X_q \rightarrow Y_q$ such that the diagram

$$\begin{tikzcd} X_q \arrow[r, "f_q"] \arrow[d] & Y_q \arrow[d] \\ \Omega X_{q+1} \arrow[r, "\Omega f_{q+1}"'] & \Omega Y_{q+1} \end{tikzcd}$$

commutes $\forall\, q$.

[Note: The indexing begins at 0.]

There is a functor $\mathbf{U}^\infty : \mathbf{SPEC} \rightarrow \mathbf{\Delta}\text{-}\mathbf{CG}_*$ that sends $\mathbf{X} = \{X_q\}$ to X_0. It has a left adjoint $\mathbf{Q}^\infty : \mathbf{\Delta}\text{-}\mathbf{CG}_* \rightarrow \mathbf{SPEC}$ defined by $(\mathbf{Q}^\infty X)_q = \Omega^\infty\Sigma^\infty\Sigma^q X$.

[Note: The repetition principle implies that $\Omega\Omega^\infty\Sigma^\infty\Sigma^{q+1}X \approx \Omega\Omega^\infty\Sigma^\infty\Sigma\Sigma^q X \approx \Omega^\infty\Sigma^\infty\Sigma^q X$.]

An infinite loop space is a pointed Δ-separated compactly generated space in the image of $\mathbf{U}^\infty$. Example: $\forall\, X$, $\Omega^\infty\Sigma^\infty X$ is an infinite loop space. Every infinite loop space is a BV^∞-space (in the extended sense of the word (cf. p. 14–48)).

EXAMPLE If $\mathbf{X} = \{X_q\}$ is a spectrum such that X_0 is wellpointed, then $\forall\, q$, there is an arrow $\Omega^q\Sigma^q\Omega^q X_q \rightarrow \Omega^q X_q$, from which an arrow $\Omega^\infty\Sigma^\infty X_0 \rightarrow X_0$. Viewing the BV^∞-space X_0 as a $\mathrm{T}_{\mathrm{BV}^\infty}$-algebra, its structural morphism $\mathrm{BV}^\infty[X_0] \rightarrow X_0$ is the composite $\mathrm{BV}^\infty[X_0] \xrightarrow{m_\infty} \Omega^\infty\Sigma^\infty X_0 \rightarrow X_0$.

A spectrum $\mathbf{X}$ is said to be connective if X_1 is path connected and X_q is $(q-1)$-connected $(q > 1)$.

Example: Given an E_∞ operad $\mathcal{O}$ and a Δ-separated $\mathcal{O}$-space X, the assignment $q \rightarrow B_q X = \operatorname{colim} \Omega^n B(\Sigma^{n+q};\mathcal{O}^{n+q};X)$ specifies a connective spectrum $\mathbf{B}X$.

[To check that $B_q X$ is Δ-separated, it need only be shown that the arrow $\Omega^n B(\Sigma^{n+q};\mathcal{O}^{n+q};X) \rightarrow \Omega^{n+1}B(\Sigma^{n+1+q};\mathcal{O}^{n+1+q};X)$ is a closed embedding (cf. p. 1–36). To see this, note that $\Sigma B(\Sigma^{n+q};\mathcal{O}^{n+q};X) \approx B(\Sigma^{n+1+q};\mathcal{O}^{n+q};X)$ (cf. p. 14–11) and $B(\Sigma^{n+1+q};\mathcal{O}^{n+q};X) \rightarrow B(\Sigma^{n+1+q};\mathcal{O}^{n+1+q};X)$ is a closed embedding (in fact, a closed cofibration). Therefore $B(\Sigma^{n+q};\mathcal{O}^{n+q};X) \rightarrow \Omega B(\Sigma^{n+1+q};\mathcal{O}^{n+1+q};X)$ is a closed embedding. And: Ω^n preserves closed embeddings.]

[Note: That $\mathbf{B}X$ is connective is implied by the generalities on p. 14–11.]

Remark: The arrow $\operatorname{colim} B(\Omega^q\Sigma^q;\mathcal{O}^q;X) \rightarrow \operatorname{colim} \Omega^q B(\Sigma^q;\mathcal{O}^q;X)$ is a morphism of Δ-separated $\mathcal{O}^\infty$-spaces (cf. p. 14–49 ff.) and a weak homotopy equivalence.

[In fact, $\mathrm{bar}(\Omega^q\Sigma^q;\mathbf{T}_{\mathcal{O}^q};X) = \Omega^q\mathrm{bar}(\Sigma^q;\mathbf{T}_{\mathcal{O}^q};X)$, so $|\Omega^q\mathrm{bar}(\Sigma^q;\mathbf{T}_{\mathcal{O}^q};X)| \rightarrow \Omega^q|\mathrm{bar}(\Sigma^q;\mathbf{T}_{\mathcal{O}^q};X)|$ is a weak homotopy equivalence (cf. p. 14–11).]

PROPOSITION 45 Let $\mathcal{O}$ be an E_∞ operad. Suppose that X is a Δ-separated $\mathcal{O}$-space—then the composite $X \rightarrow B(\mathcal{O}^\infty;\mathcal{O}^\infty;X) \rightarrow B(\Omega^\infty\Sigma^\infty;\mathcal{O}^\infty;X) \rightarrow B_0X$ is a group completion.

[Taking into account Proposition 44, this follows from what has been said above.]

[Note: It is not claimed that B_0X is wellpointed.]

Therefore every Δ-separated $\mathcal{O}$-space X group completes to an infinite loop space.

[Note: Consequently, if X is path connected, then X has the weak homotopy type of an infinite loop space.]

Remark: Proposition 45 is true for any Δ-separated $\mathcal{O}^\infty$-space (same argument).

[Note: Observe that every BV^∞-space is an $\mathcal{O}^\infty$-space.]

EXAMPLE Specializing to $\mathcal{O} = \mathrm{PER}$, one sees that the classifying space $B\mathbf{C}$ of a permutative category $\mathbf{C}$ group completes to an infinite loop space.

PROPOSITION 46 Let $\mathcal{O}$ be an E_∞ operad. Suppose that $\mathbf{X} = \{X_q\}$ is a spectrum such that X_0 is wellpointed—then there is a morphism $\mathbf{b} : \mathbf{B}X_0 \rightarrow \mathbf{X}$ in **SPEC** such that the diagram

$$\begin{tikzcd}
B(\mathcal{O}^\infty;\mathcal{O}^\infty;X_0) \arrow[r] \arrow[d] & B(\Omega^\infty\Sigma^\infty;\mathcal{O}^\infty;X_0) \arrow[d] \\
X_0 & B_0X_0 \arrow[l, "b_0"']
\end{tikzcd}$$

commutes.

[Proceeding formally, use the arrow $B(\Sigma^{n+q};\mathcal{O}^{n+q};\Omega^{n+q}X_{n+q}) \rightarrow X_{n+q}$ to define $b_q : B_qX_0 \rightarrow X_q$.]

[Note: It is a corollary that the composite $X_0 \rightarrow B_0X_0 \xrightarrow{b_0} X_0$ is the identity. Another corollary is that b_0 is a weak homotopy equivalence provided that X_0 is path connected.]

PROPOSITION 47 Let $\mathcal{O}$ be an E_∞ operad—then $\forall$ Δ-cofibered X in $\mathbf{CG}_*$, there is a morphism $\mathbf{f}: \mathbf{B}\mathcal{O}^\infty[X] \rightarrow \mathbf{Q}^\infty X$ of spectra such that $\forall\, q$, $f_q : B_q\mathcal{O}^\infty[X] \rightarrow \Omega^\infty\Sigma^\infty\Sigma^q X$ is a pointed homotopy equivalence.

[The arrow $B(\Sigma^{n+q};\mathcal{O}^{n+q};\mathcal{O}^{n+q}[X]) \rightarrow \Sigma^{n+q}X$ is a pointed homotopy equivalence (cf. p. 0–46 ff.). Apply Ω^n and let $n \rightarrow \infty$. In this connection, the assumption that X is Δ-cofibered guarantees that $\Omega^n\Sigma^{n+q}X \rightarrow \Omega^{n+1}\Sigma^{n+1+q}X$ is a closed cofibration (cf. p. 14–44), so Proposition 15 in §3 is applicable.]

[Note: Working through the definitions, one finds that $\mathbf{f}$ is equal to the composite $\mathbf{B}\mathcal{O}^\infty[X] \rightarrow \mathbf{B}\mathrm{BV}^\infty[X] \rightarrow \mathbf{B}\Omega^\infty\Sigma^\infty X \xrightarrow{\mathbf{b}} \mathbf{Q}^\infty X$.]

EXAMPLE Take $\mathcal{O} = \text{PER} \approx BS$ and let $X = \mathbf{S}^0$—then $\mathcal{O}[\mathbf{S}^0] \approx |M_\infty| = \coprod\limits_{n \geq 0} BS_n$ and the projection $\mathcal{O}^\infty[\mathbf{S}^0] \rightarrow \mathcal{O}[\mathbf{S}^0]$ is a weak homotopy equivalence. On the other hand, the composite $\mathcal{O}^\infty[\mathbf{S}^0] \rightarrow B_0\mathcal{O}^\infty[\mathbf{S}^0] \rightarrow \Omega^\infty\Sigma^\infty\mathbf{S}^0$ is a group completion (cf. Propositions 45 and 47), as is the arrow $|M_\infty| \rightarrow \Omega B|M_\infty|$. Therefore $\Omega^\infty\Sigma^\infty\mathbf{S}^0$ and $\Omega B|M_\infty|$ have the same pointed homotopy type (cf. p. 14–56). The homotopy groups π_*^s of $\Omega^\infty\Sigma^\infty\mathbf{S}^0$ are the stable homotopy groups of spheres. Since $\Omega B|M_\infty| \approx \mathbf{Z} \times BS_\infty^+$, it follows that $\pi_*^s \approx \pi_*(BS_\infty^+)$. Example: $\pi_1^s \approx \pi_1(BS_\infty^+) = S_\infty/A_\infty \approx \mathbf{Z}/2\mathbf{Z}$. There is also a connection with algebraic K-theory. Thus $S_\infty \subset \mathbf{GL}(\mathbf{Z})$, $A_\infty \subset \mathbf{E}(\mathbf{Z})$, so there is an arrow $BS_\infty^+ \rightarrow B\mathbf{GL}(\mathbf{Z})^+$. The associated homomorphism $\pi_n^s \rightarrow K_n(\mathbf{Z})$ $(= \pi_n(B\mathbf{GL}(\mathbf{Z})^+))$ can be bijective (e.g., if $n = 1$) but in general is neither injective nor surjective (see Mitchell[†] for a discussion and more information).

[Note: Let $\mathbf{C} = \mathbf{M}_\infty = \text{iso}\,\mathbf{\Gamma}$—then another model for $\Omega^\infty\Sigma^\infty\mathbf{S}^0$ is $B(\text{gro}_{\mathbf{\Delta}^{\text{OP}}} C^+)$ (cf. p. 14–55).]

EXAMPLE Given a discrete group G, form $S_\infty \int G$ (cf. p. 14–19)—then a model for the plus construction on $BS_\infty \int G$ is the path component of $\Omega^\infty\Sigma^\infty BG_+$ containing the constant loop. E.g.: When $G = *$, $\Omega^\infty\Sigma^\infty BG_+$ is $\Omega^\infty\Sigma^\infty\mathbf{S}^0$ and when $G = \mathbf{Z}/2\mathbf{Z}$, $\Omega^\infty\Sigma^\infty BG_+$ is $\Omega^\infty\Sigma^\infty\mathbf{P}^\infty(\mathbf{R})_+$.

$\mathbf{\Pi}$ is the category whose objects are the finite sets $\mathbf{n} \equiv \{0, 1, \ldots, n\}$ $(n \geq 0)$ with base point 0 and whose morphisms are the base point preserving maps $\gamma : \mathbf{m} \rightarrow \mathbf{n}$ such that $\#(\gamma^{-1}(j)) \leq 1$ $(1 \leq j \leq n)$. So: $\mathbf{\Gamma}_{\text{in}}$ is a subcategory of $\mathbf{\Pi}$ and $\mathbf{\Pi}$ is a subcategory of $\mathbf{\Gamma}$.

[Note: Let (X, x_0) be a wellpointed compactly generated space with $\{x_0\} \subset X$ closed—then the formulas that define $\text{pow}\,X$ as a functor $\mathbf{\Gamma}_{\text{in}} \rightarrow \mathbf{CG}_*$ serve to define $\text{pow}\,X$ as a functor $\mathbf{\Pi} \rightarrow \mathbf{CG}_*$.]

A category of operators is a compactly generated category $\mathbf{C}$ such that $\text{Ob}\,\mathbf{C} \rightarrow \text{Mor}\,\mathbf{C}$ is a closed cofibration, where $\text{Ob}\,\mathbf{C} = \text{Ob}\,\mathbf{\Gamma}$ (discrete topology), subject to the requirement that $\mathbf{C}$ contains $\mathbf{\Pi}$ and admits an augmentation $\epsilon : \mathbf{C} \rightarrow \mathbf{\Gamma}$ which restricts to the inclusion $\mathbf{\Pi} \rightarrow \mathbf{\Gamma}$. One writes $\mathbf{C}(\mathbf{m}, \mathbf{n})$ for the set of morphisms $\mathbf{m} \rightarrow \mathbf{n}$. Example: $\mathbf{\Gamma}$ is a category of operators, as is $\mathbf{\Pi}$.

Every category of operators is a $\mathbf{CG}$-category.

[Note: A morphism of categories of operators is a continuous functor $F : \mathbf{C} \rightarrow \mathbf{D}$ such that $F\mathbf{n} = \mathbf{n}$ for all $\mathbf{n}$ and

$$\begin{tikzcd}
& \mathbf{\Gamma} & \\
\mathbf{C} \arrow[ur] \arrow[rr, "F"] & & \mathbf{D} \arrow[ul] \\
& \mathbf{\Pi} \arrow[ul] \arrow[ur] &
\end{tikzcd}$$

commutes.]

[†] In: *Algebraic Topology and its Applications*, G. Carlsson et al. (ed.), Springer Verlag (1994), 163–240 (cf. 182–183).

FACT Let $\mathbf{C}$ be a category of operators. Suppose that X is a right $\mathbf{C}$-object and Y is a left $\mathbf{C}$-object—then $\text{bar}(X;\mathbf{C};Y)$ satisfies the cofibration condition.

A cofibered operad in $\mathbf{CG}$ is a reduced operad $\mathcal{O}$ in $\mathbf{CG}$ for which the inclusion $\{1\} \to \mathcal{O}_1$ is a closed cofibration. Example: Every E_∞ operad is a cofibered operad in $\mathbf{CG}$.

Notation: Given morphisms $\gamma : \mathbf{m} \to \mathbf{n}$, $\delta : \mathbf{n} \to \mathbf{p}$ in $\mathbf{\Gamma}$, let $\sigma_k(\delta,\gamma)$ be the permutation on $\#((\delta\circ\gamma)^{-1}(k))$ letters which converts the natural ordering of $(\delta\circ\gamma)^{-1}(k)$ to the ordering associated with $\bigcup\limits_{\delta(j)=k} \gamma^{-1}(j)$ (all elements of $\gamma^{-1}(j)$ precede all elements of $\gamma^{-1}(j')$ if $j < j'$ and each $\gamma^{-1}(j)$ has its natural ordering).

PROPOSITION 48 Let $\mathcal{O}$ be a cofibered operad in $\mathbf{CG}$—then $\mathcal{O}$ determines a category of operators $\widehat{\mathcal{O}}$.

[Put $\widehat{\mathcal{O}}(\mathbf{m},\mathbf{n}) = \coprod\limits_{\gamma:\mathbf{m}\to\mathbf{n}} \prod\limits_{1\leq j\leq n} \mathcal{O}(\#(\gamma^{-1}(j)))$ (cf. p. 14–45). Here composition $\widehat{\mathcal{O}}(\mathbf{m},\mathbf{n}) \times \widehat{\mathcal{O}}(\mathbf{n},\mathbf{p}) \to \widehat{\mathcal{O}}(\mathbf{m},\mathbf{p})$ is the rule $(\delta; g_1,\ldots,g_p)\circ(\gamma; f_1,\ldots,f_n) = (\delta\circ\gamma; h_1,\ldots,h_p)$, h_k being $\Lambda(g_k; f_j(\delta(j)=k))\cdot\sigma_k(\delta,\gamma)$, and $(\mathrm{id}_\mathbf{n}; 1,\ldots,1)$ is the identity element in $\widehat{\mathcal{O}}(\mathbf{n},\mathbf{n})$. The augmentation $\epsilon : \widehat{\mathcal{O}} \to \mathbf{\Gamma}$ is obvious, viz. $\epsilon(\gamma; f_1,\ldots,f_n) = \gamma$. To define the inclusion $\mathbf{\Pi} \to \widehat{\mathcal{O}}$, send $\gamma : \mathbf{m}\to\mathbf{n}$ to $(\gamma; f_1,\ldots,f_n)$, where $\begin{cases} f_j = 1 & (j \in \operatorname{im}\gamma) \\ f_j = * & (j \notin \operatorname{im}\gamma) \end{cases}$.]

Examples: (1) Let $\mathcal{O}_n = * \ \forall\, n$—then $\widehat{\mathcal{O}} = \mathbf{\Gamma}$; (2) Let $\mathcal{O}_0 = *$, $\mathcal{O}_1 = \{1\}$, $\mathcal{O}_n = \emptyset$ $(n > 1)$—then $\widehat{\mathcal{O}} = \mathbf{\Pi}$.

A $\mathbf{\Pi}$-space is a functor $X : \mathbf{\Pi} \to \mathbf{CG}_*$ and a $\mathbf{\Pi}$-map is a natural transformation $f : X \to Y$.

Given $n \geq 1$, there are projections $\pi_i : \mathbf{n} \to \mathbf{1}$ $(i = 1,\ldots,n)$, where $\pi_i(j) = \begin{cases} 1 & (i=j) \\ 0 & (i\neq j) \end{cases}$. A $\mathbf{\Pi}$-space X is said to be special if $X_0 = *$ and $\forall\, n \geq 1$, the arrow $X_n \to X_1 \times_k \cdots \times_k X_1$ determined by the π_i is a weak homotopy equivalence.

Given an injection $\gamma : \mathbf{m}\to\mathbf{n}$, let S_γ be the subgroup of S_n consisting of those σ such that $\sigma(\operatorname{im}\gamma) = \operatorname{im}\gamma$. A $\mathbf{\Pi}$-space X is said to be proper if $X_0 = *$ and $\forall\,\gamma : \mathbf{m}\to\mathbf{n}$ in $\mathbf{\Gamma}_{\text{in}}$, $X\gamma : X_m \to X_n$ is a closed S_γ-cofibration (cf. infra). In particular: $* \to X_n$ is a closed S_n-cofibration, so $\forall\, n$, X_n is in $\mathbf{CG}_{*c}$.

[Note: Associated with each $\sigma \in S_\gamma$ is a permutation $\widetilde{\sigma} \in S_m$ such that $\sigma\circ\gamma = \gamma\circ\widetilde{\sigma}$ and the assignment $\sigma \to \widetilde{\sigma}$ is a homomorphism $S_\gamma \to S_m$. Thus X_m and X_n are left S_γ-spaces and $X\gamma : X_m \to X_n$ is equivariant.]

Example: $\forall\, X$ in $\mathbf{CG}_{*c}$, $\text{pow}\,X$ is a proper special $\mathbf{\Pi}$-space.

Let G be a finite group. Let A and X be left G-spaces—then an equivariant continuous function $i : A \to X$ is said to be a G-cofibration if it has the following property: Given any left G-space Y and any pair (F, h) of equivariant continuous functions $\begin{cases} F : X \to Y \\ h : IA \to Y \end{cases}$ such that $F \circ i = h \circ i_0$, there is an equivariant continuous function $H : IX \to Y$ such that $F = H \circ i_0$ and $H \circ Ii = h$.

[Note: Every G-cofibration is an embedding and the induced map $G\backslash A \to G\backslash X$ is a cofibration.]

The theory set forth in §3 has an equivariant analog (Boardman-Vogt[†]). For example, Proposition 1 in §3 becomes: Let A be an invariant subspace of X—then the inclusion $A \to X$ is a G-cofibration iff $i_0 X \cup IA$ is an equivariant retract of IX. The notion of an equivariant Strøm structure on (X, A) is clear and there is a G-cofibration characterization theorem.

[Note: A G-cofibration is thus a cofibration.]

EXAMPLE Suppose that (X, x_0) is in $\mathbf{CG}_{*c}$—then the inclusion $X_*^n \to X^n$ is a closed S_n-cofibration.

LEMMA Let A be an invariant subspace of the left G-space X. Suppose that $A = A_1 \cup \cdots \cup A_n$, where each A_i is closed in X, and suppose that G operates on $\{1, \ldots, n\}$ in such a way that $g \cdot A_i = A_{g \cdot i}$. Put $A_S = \bigcap\limits_{i \in S} A_i$ $(S \subset \{1, \ldots, n\})$—then $A \to X$ is a closed G-cofibration if $\forall\, S \neq \emptyset$, $A_S \to X$ is a closed G_S-cofibration, $G_S \subset G$ the stabilizer of S.

[Note: Take for G the trivial group to recover Proposition 8 in §3 (with 2 replaced by n).]

EXAMPLE Let X be a proper $\mathbf{\Pi}$-space. Put $sX_{n-1} = s_0 X_{n-1} \cup \cdots \cup s_{n-1} X_{n-1}$, where $s_i = X\sigma_i$ and $\sigma_i(j) = \begin{cases} j & (j \leq i) \\ j+1 & (j > i) \end{cases}$ $(0 \leq i < n)$—then the inclusion $sX_{n-1} \to X_n$ is a closed S_n-cofibration.

Notation: $\mathbf{ps\,\Pi\text{-}SP}$ is the category of proper special $\mathbf{\Pi}$-spaces.

PROPOSITION 49 Let L be the functor from $\mathbf{ps\,\Pi\text{-}SP}$ to $\mathbf{CG}_{*c}$ that sends X to X_1 and let R be the functor from $\mathbf{CG}_{*c}$ to $\mathbf{ps\,\Pi\text{-}SP}$ that sends X to $\operatorname{pow} X$—then (L, R) is an adjoint pair.

[Note: The arrow of adjunction $LRX \to X$ is the identity and the arrow of adjunction $X \to RLX$ has for its components the map induced by the π_i.]

Let $\mathbf{C}$ be a category of operators—then a $\mathbf{C}$-space is a continuous functor $X : \mathbf{C} \to \mathbf{CG}_*$ and a $\mathbf{C}$-map is a natural transformation $f : X \to Y$.

[†] *SLN* **347** (1973), 231–239.

Continuity in this context means that $\forall m, n$, the arrow $\mathbf{C}(\mathbf{m},\mathbf{n}) \times_k X_m \rightarrow X_n$ is continuous. To clarify the matter, let $E = X_n^{X_m}$ (exponential object in **CG**), $E_* = X_n^{X_m}$ (pointed exponential object in $\mathbf{CG}_*$)—then there is a commutative triangle

$$\begin{tikzcd} \mathbf{C}(\mathbf{m},\mathbf{n}) \arrow[r] \arrow[dr] & E_* \arrow[d] \\ & E \end{tikzcd}$$

, where $E_* \rightarrow E$ is a **CG**-embedding.

Thus the arrow $\mathbf{C}(\mathbf{m},\mathbf{n}) \rightarrow E_*$ is continuous iff the arrow $\mathbf{C}(\mathbf{m},\mathbf{n}) \rightarrow E$ is continuous or still, iff the arrow $\mathbf{C}(\mathbf{m},\mathbf{n}) \times_k X_m \rightarrow X_n$ is continuous.

A **C**-space is said to be special or proper if its restriction to $\mathbf{\Pi}$ is special or proper.

Example: A $\mathbf{\Gamma}$-space is an $\widehat{\mathcal{O}}$-space, where $\mathcal{O}_n = * \ \forall\, n$. Every abelian monoid G in **CG** gives rise to a special $\mathbf{\Gamma}$-space (cf. p. 13–56), the $\mathbf{\Gamma}$-nerve of $\mathbf{G}$: $\mathbf{\Gamma}\text{-}\mathrm{ner}\,\mathbf{G}$ (which is proper if G is cofibered).

LEMMA Let $\mathcal{O}$ be a cofibered operad in **CG**—then an $\widehat{\mathcal{O}}$-space with underlying space $\mathrm{pow}\, X$ determines and is determined by an $\mathcal{O}$-space structure on X.

[To specify an $\mathcal{O}$-space structure on X is to specify a morphism $\mathcal{O} \rightarrow \mathrm{End}\, X$ of operads in **CG** (cf. p. 14–49), from which an $\widehat{\mathcal{O}}$-space $\widehat{\mathcal{O}} \rightarrow \mathbf{CG}_*$ with underlying space $\mathrm{pow}\, X$. Conversely, let $\gamma_n : \mathbf{n} \rightarrow \mathbf{1}$ be the arrow $j \rightarrow 1$ $(1 \leq j \leq n)$ and view $\mathcal{O}_n$ as the component of γ_n in $\widehat{\mathcal{O}}(\mathbf{n},\mathbf{1})$. Per an $\widehat{\mathcal{O}}$-space with underlying space $\mathrm{pow}\, X$, restriction of $\widehat{\mathcal{O}}(\mathbf{n},\mathbf{1}) \rightarrow X^{X^n}$ to $\mathcal{O}_n$ defines a morphism $\mathcal{O} \rightarrow \mathrm{End}\, X$ of operads in **CG**.]

Let $\mathcal{O}$ be a cofibered operad in **CG**—then by restriction, $\widehat{\mathcal{O}}(—,\mathbf{n})$ defines a functor $\mathbf{\Pi}^{\mathrm{OP}} \rightarrow \mathbf{CG}\ \forall\, n \geq 0$. Given a $\mathbf{\Pi}$-space X, put $\widehat{\mathcal{O}}_n[X] = \widehat{\mathcal{O}}(—,\mathbf{n}) \otimes_{\mathbf{\Pi}} X$ (so $\widehat{\mathcal{O}}_0[X] = X_0$) and call $\widehat{\mathcal{O}}[X]$ the $\mathbf{\Pi}$-space which takes $\mathbf{n}$ to $\widehat{\mathcal{O}}_n[X]$. Composition in $\widehat{\mathcal{O}}$ leads to maps $\widehat{\mathcal{O}}(\mathbf{m},\mathbf{n}) \times \widehat{\mathcal{O}}_m[X] \rightarrow \widehat{\mathcal{O}}_n[X]$ or still, to an arrow $m_X : \widehat{\mathcal{O}}^2[X] \rightarrow \widehat{\mathcal{O}}[X]$, while the identities in $\widehat{\mathcal{O}}$ induce an arrow $\epsilon_X : X \rightarrow \widehat{\mathcal{O}}[X]$. Both arrows are natural in X and with $T_{\widehat{\mathcal{O}}} = \widehat{\mathcal{O}}[?]$, it is seen that $\mathbf{T}_{\widehat{\mathcal{O}}} = (T_{\widehat{\mathcal{O}}}, m, \epsilon)$ is a triple in $[\mathbf{\Pi}, \mathbf{CG}_*]$.

Notation: Let $\mathcal{E}(\mathbf{m},\mathbf{n})$ be the set of base point preserving maps $\epsilon : \mathbf{m} \rightarrow \mathbf{n}$ such that $\epsilon^{-1}(0) = \{0\}$ and $i \leq i' \Rightarrow \epsilon(i) \leq \epsilon(i')$. Put $S_\epsilon = S_{\epsilon_1} \times \cdots \times S_{\epsilon_n} \subset S_m$, where $\epsilon_j = \#(\epsilon^{-1}(j))$.

[Note: Let $\sigma \in S_m$—then $\epsilon \circ \sigma \in \mathcal{E}(\mathbf{m},\mathbf{n})$ iff $\sigma \in S_\epsilon$.]

PROPOSITION 50 Suppose that X is a proper $\mathbf{\Pi}$-space. Denote by $\widehat{\mathcal{O}}_{m,n}[X]$ the image of $\coprod_{m' \leq m} \widehat{\mathcal{O}}(\mathbf{m}',\mathbf{n}) \times_k X_{m'}$ in $\widehat{\mathcal{O}}_n[X]$—then $\widehat{\mathcal{O}}_{m,n}[X]$ is a closed subspace of $\widehat{\mathcal{O}}_n[X]$

and $\widehat{\mathcal{O}}_n[X] = \operatorname{colim} \widehat{\mathcal{O}}_{m,n}[X]$. In addition, the commutative diagram

$$\begin{tikzcd}
\coprod\limits_{\epsilon \in \mathcal{E}(\mathrm{m},\mathrm{n})} \prod\limits_{1 \leq j \leq n} \mathcal{O}_{\epsilon_j} \times_{S_\epsilon} sX_{m-1} \arrow[r] \arrow[d] & \widehat{\mathcal{O}}_{m-1,n}[X] \arrow[d] \\
\coprod\limits_{\epsilon \in \mathcal{E}(\mathrm{m},\mathrm{n})} \prod\limits_{1 \leq j \leq n} \mathcal{O}_{\epsilon_j} \times_{S_\epsilon} X_m \arrow[r] & \widehat{\mathcal{O}}_{m,n}[X]
\end{tikzcd}$$

is a pushout square and the arrow $\widehat{\mathcal{O}}_{m-1,n}[X] \rightarrow \widehat{\mathcal{O}}_{m,n}[X]$ is a closed cofibration.

[Note: For the definition of "s", see p. 14–63.]

Remark: X_n Δ-separated $\forall\, n$ + $\mathcal{O}_n$ Δ-separated $\forall\, n \Rightarrow \widehat{\mathcal{O}}_n[X]$ Δ-separated $\forall\, n$ (cf. p. 14–39).

FACT If X is a proper $\mathbf{\Pi}$-space, then $\widehat{\mathcal{O}}[X]$ is a proper $\mathbf{\Pi}$-space and $\epsilon_X : X \rightarrow \widehat{\mathcal{O}}[X]$ is a levelwise closed cofibration.

PROPOSITION 51 Fix an X in $\mathbf{CG}_{*c}$—then $L\widehat{\mathcal{O}}[RX]$ ($= \widehat{\mathcal{O}}_1[\operatorname{pow} X]$ (cf. Proposition 49)) $\approx \mathcal{O}[X]$ and $\widehat{\mathcal{O}}[RX] \approx R\mathcal{O}[X]$.

LEMMA Let $\mathcal{O}$ be a cofibered operad in $\mathbf{CG}$. Assume: $\forall\, n$, $\mathcal{O}_n$ is a compactly generated Hausdorff space and the action of S_n is free. Suppose given a $\mathbf{\Pi}$-map $f : X \rightarrow Y$ such that $\forall\, n$, $f_n : X_n \rightarrow Y_n$ is a weak homotopy equivalence—then $\forall\, n$, $\widehat{\mathcal{O}}_n f : \widehat{\mathcal{O}}_n[X] \rightarrow \widehat{\mathcal{O}}_n[Y]$ is a weak homotopy equivalence provided that X and Y are proper.

[This is a variant on the argument used in the proof of Proposition 39.]

PROPOSITION 52 Let $\mathcal{O}$ be a cofibered operad in $\mathbf{CG}$. Assume: $\forall\, n$, $\mathcal{O}_n$ is a compactly generated Hausdorff space and the action of S_n is free. Suppose that X is a proper special $\mathbf{\Pi}$-space—then $\widehat{\mathcal{O}}[X]$ is a proper special $\mathbf{\Pi}$-space.

[$\widehat{\mathcal{O}}[X]$ is necessarily proper (cf. supra). To check that $\widehat{\mathcal{O}}[X]$ is special, consider the commutative diagram

$$\begin{tikzcd}
\widehat{\mathcal{O}}_n[X] \arrow[r] \arrow[d] & \widehat{\mathcal{O}}_n[RLX] \arrow[d] \\
(\widehat{\mathcal{O}}_1[X])^n \arrow[r] & (\widehat{\mathcal{O}}_1[RLX])^n
\end{tikzcd}$$

, bearing in mind the lemma and the fact that the arrow of adjunction $X \rightarrow RLX$ is a levelwise weak homotopy equivalence (Proposition 51 supplies an identification $\widehat{\mathcal{O}}_n[RLX] \approx (\widehat{\mathcal{O}}_1[RLX])^n$).]

Application: Let $\mathcal{O}$ be an E_∞ operad—then the triple $\mathbf{T}_{\widehat{\mathcal{O}}} = (T_{\widehat{\mathcal{O}}}, m, \epsilon)$ in $[\mathbf{\Pi}, \mathbf{CG}_*]$ restricts to a triple in $\mathbf{ps}\,\mathbf{\Pi}$-$\mathbf{SP}$ and its associated category of algebras is canonically isomorphic to the category $\mathbf{ps}\,\widehat{\mathcal{O}}$-$\mathbf{SP}$ of proper special $\widehat{\mathcal{O}}$-spaces (cf. Proposition 37).

Suppose that X is a simplicial $\mathbf{\Pi}$-space—then the realization $|X|$ of X is the $\mathbf{\Pi}$-space defined by $|X|(\mathbf{n}) = |[m] \rightarrow X_m(\mathbf{n})|$.

Example: If $\mathcal{O}$ is an E_∞ operad and if X is a proper special $\widehat{\mathcal{O}}$-space, then the realization $B(\widehat{\mathcal{O}};\widehat{\mathcal{O}};X)$ of $\mathrm{bar}(T_{\widehat{\mathcal{O}}};\mathbf{T}_{\widehat{\mathcal{O}}};X)$ is a proper special $\widehat{\mathcal{O}}$-space.

LEMMA Suppose that $F:\mathbf{CG}_{*c} \rightarrow \mathbf{V}$ is a right $\mathbf{T}_{\mathcal{O}}$-functor—then $F \circ L$:$\mathbf{ps\,\Pi\text{-}SP} \rightarrow \mathbf{V}$ is a right $\mathbf{T}_{\widehat{\mathcal{O}}}$-functor.

[The relevant natural transformation $F \circ L \circ T_{\widehat{\mathcal{O}}} \rightarrow F \circ L$ is the composite $FL\widehat{\mathcal{O}}[X] \rightarrow FL\widehat{\mathcal{O}}[RLX] = FLR\mathcal{O}[LX] = F\mathcal{O}[LX] \xrightarrow{\rho_{LX}} FLX$.]

Let $\mathcal{O}$ be an E_∞ operad, $F:\mathbf{CG}_{*c} \rightarrow \mathbf{CG}_{*c}$ a right $\mathbf{T}_{\mathcal{O}}$-functor—then for any $\mathbf{T}_{\widehat{\mathcal{O}}}$-algebra X, $\mathrm{bar}(F \circ L;\mathbf{T}_{\widehat{\mathcal{O}}};X)$ is a simplicial object in $\mathbf{CG}_{*c}$ and one writes $B(F \circ L;\widehat{\mathcal{O}};X)$ for its geometric realization.

[Note: It is clear that there is a version of Proposition 38 applicable to this situation.]

PROPOSITION 53 Let $\mathcal{O}$ be an E_∞ operad—then there is a functor U from $\mathbf{ps}\,\widehat{\mathcal{O}}\textbf{-SP}$ to $\mathbf{ps}\,\widehat{\mathcal{O}}\textbf{-SP}$ and a functor V from $\mathbf{ps}\,\widehat{\mathcal{O}}\textbf{-SP}$ to $\mathcal{O}\textbf{-SP}$ plus $\widehat{\mathcal{O}}$-maps $X \leftarrow UX \rightarrow RVX$ natural in X such that $X \leftarrow UX$ is a levelwise homotopy equivalence and $UX \rightarrow RVX$ is a lefelwise weak homotopy equivalence.

[Put $UX = B(\widehat{\mathcal{O}};\widehat{\mathcal{O}};X)$ and $VX = B(T_{\mathcal{O}} \circ L;\widehat{\mathcal{O}};X)$. So, in obvious notation, $RVX = B(R \circ T_{\mathcal{O}} \circ L;\widehat{\mathcal{O}};X)$ and the arrow $UX \rightarrow RVX$ is defined in terms of the arrows $\widehat{\mathcal{O}}_n[X] \rightarrow (\mathcal{O}[X_1])^n$, hence is a levelwise weak homotopy equivalence (see the proof of Proposition 52).]

[Note: Suppose that X is an $\mathcal{O}$-space—then $B(\widehat{\mathcal{O}};\widehat{\mathcal{O}};RX) \approx RB(\mathcal{O};\mathcal{O};X)$ $(\Rightarrow LB(\widehat{\mathcal{O}};\widehat{\mathcal{O}};RX) \approx B(\mathcal{O};\mathcal{O};X))$ and $VRX \approx B(\mathcal{O};\mathcal{O};X)$ (cf. Proposition 51).]

Remarks: (1) X Δ-separated $\Rightarrow UX, VX$ Δ-separated; (2) $X \rightarrow UX$ is not an $\widehat{\mathcal{O}}$-map (but it is a $\mathbf{\Pi}$-map).

FACT Let $\mathcal{O}$ be an E_∞ operad, $\epsilon:\widehat{\mathcal{O}} \rightarrow \mathbf{\Gamma}$ the augmentation—then there are functors ϵ^* : $\mathbf{ps\,\Gamma\text{-}SP} \rightarrow \mathbf{ps}\,\widehat{\mathcal{O}}\textbf{-SP}$, ϵ_* : $\mathbf{ps}\,\widehat{\mathcal{O}}\textbf{-SP} \rightarrow \mathbf{ps\,\Gamma\text{-}SP}$ respecting the Δ-separation condition and an $\widehat{\mathcal{O}}$-map $UX \rightarrow \epsilon^*\epsilon_*X$ natural in X which is a levelwise weak homotopy equivalence.

Let $\mathcal{O}$ be an E_∞ operad—then there is a functor $\mathbf{B}$ from the category of Δ-separated $\mathcal{O}$-spaces to the category of connective spectra (cf. p. 14–59) and this functor can be extended to the category of Δ-separated proper special $\widehat{\mathcal{O}}$-spaces by writing $B_qX = \mathrm{colim}\,\Omega^n B(\Sigma^{n+q}L;\widehat{\mathcal{O}}^{n+q};X)$. To see that this prescription really is an extension, consider

any Δ-separated $\mathcal{O}$-space $X : B(\Sigma^{n+q}L; \widehat{\mathcal{O}}^{n+q}; RX) \approx B(\Sigma^{n+q}; \mathcal{O}^{n+q}; X)$ (cf. Proposition 51) $\Rightarrow \mathbf{B}RX \approx \mathbf{B}X$.

PROPOSITION 54 Let $\mathcal{O}$ be an E_∞ operad. Suppose that X is a Δ-separated proper special $\widehat{\mathcal{O}}$-space—then the composite $B(T_{\mathcal{O}^\infty} \circ L; \widehat{\mathcal{O}}^\infty; X) \rightarrow B(\Omega^\infty\Sigma^\infty L; \widehat{\mathcal{O}}^\infty; X) \rightarrow B_0X$ is a group completion.

[Rework the discussion leading up to Proposition 45.]

Let $\mathcal{O}$ be a cofibered operad in **CG**—then an infinite loop space machine on $\widehat{\mathcal{O}}$ consists of a functor $\mathbf{B}$ from the category of Δ-separated proper special $\widehat{\mathcal{O}}$-spaces to the category of connective spectra, a functor K from the category of Δ-separated proper special $\widehat{\mathcal{O}}$-spaces to the category of homotopy associative, homotopy commutative H spaces, a natural transformation $L \rightarrow K$ such that $\forall\, X$, the arrow $LX \rightarrow KX$ is a weak homotopy equivalence, and a natural transformation $K \rightarrow B_0$ such that $\forall\, X$, $KX \rightarrow B_0X$ is a group completion.

PROPOSITION 55 Let $\mathcal{O}$ be an E_∞ operad—then there exists an infinite loop space machine on $\widehat{\mathcal{O}}$, the May machine.

[Take $\mathbf{B}$ as above and put $KX = B(T_{\mathcal{O}^\infty} \circ L;\ \widehat{\mathcal{O}}^\infty; X)$. The composite $X \rightarrow B(\widehat{\mathcal{O}}^\infty; \widehat{\mathcal{O}}^\infty; X) \rightarrow RB(T_{\mathcal{O}^\infty} \circ L; \widehat{\mathcal{O}}^\infty; X)$ is a levelwise weak homotopy equivalence, hence $LX \rightarrow KX$ is a weak homotopy equivalence. On the other hand, thanks to Proposition 54, the composite $KX \rightarrow B(\Omega^\infty\Sigma^\infty L; \widehat{\mathcal{O}}^\infty; X) \rightarrow B_0X$ is a group completion.]

Let $\mathcal{O}$ be an E_∞ operad—then, using the augmentation $\epsilon : \widehat{\mathcal{O}} \rightarrow \mathbf{\Gamma}$, a Δ-separated proper special $\mathbf{\Gamma}$-space can be regarded as a Δ-separated proper special $\widehat{\mathcal{O}}$-space. Therefore an infinite loop space machine on $\widehat{\mathcal{O}}$ defines an infinite loop space machine on $\mathbf{\Gamma}$. However, there is another ostensibly very different method for generating connective spectra from Δ-separated proper special $\mathbf{\Gamma}$-spaces which is completely internal and makes no reference to operads. The question then arises: Are the spectra thereby produced in some sense the "same"? As we shall see, the answer is "yes" (cf. Proposition 62), a corollary being that infinite loop machines associated with distinct E_∞ operads $\mathcal{O}$ and $\mathcal{P}$ attach the "same" spectra to a Δ-separated proper special $\mathbf{\Gamma}$-space.

LEMMA $\mathbf{\Delta}^{\mathrm{OP}}$ is isomorphic to the category whose objects are the $\mathbf{n}_+$ $(j < *, 0 \leq j \leq n)$ and whose morphisms are the order preserving maps $\alpha : \mathbf{m}_+ \rightarrow \mathbf{n}_+$ such that $\alpha(0) = 0$ and $\alpha(*) = *$.

The composite $[n] \rightarrow \mathbf{n}_+ \rightarrow \mathbf{n}_+/0 \sim * \equiv \mathbf{n}$ defines a functor $\mathbf{S}[1] : \mathbf{\Delta}^{\mathrm{OP}} \rightarrow \mathbf{\Gamma}$.

[Note: To justify the notation, observe that the pointed simplicial set $\Delta^{\mathrm{OP}} \rightarrow \mathbf{\Gamma} \subset \mathbf{SET}_*$ thus displayed is in fact a model for the simplicial circle (cf. p. 13–29).]

EXAMPLE Suppose that $\alpha : [n] \rightarrow [m]$ is a morphism in $\mathbf{\Delta}$. Put $\gamma = \mathbf{S}[1]\alpha$ (so $\gamma : \mathbf{m} \rightarrow \mathbf{n}$ is a morphism in $\mathbf{\Gamma}$)—then γ is given by $\gamma^{-1}(j) = \{i : \alpha(j-1) < i \leq \alpha(j)\}$ $(1 \leq j \leq n)$, $\gamma^{-1}(0) = \mathbf{m} - \bigcup\limits_{j=1}^{n} \gamma^{-1}(j)$. Examples: (1) The $\sigma_i : [n+1] \rightarrow [n]$ of p. 0–16 are sent by $\mathbf{S}[1]$ to the $\sigma_i : \mathbf{n} \rightarrow \mathbf{n+1}$ of p. 14–47 $(n \geq 0, 0 \leq i \leq n)$; (2) The $\pi_i : [1] \rightarrow [n]$ of p. 14–20 are sent by $\mathbf{S}[1]$ to the $\pi_i : \mathbf{n} \rightarrow \mathbf{1}$ of p. 14–62 $(n \geq 1, 1 \leq i \leq n)$.

Notation: Call $\overline{\mathrm{pow}}\, X$ the functor $\mathbf{\Gamma}^{\mathrm{OP}} \rightarrow \mathbf{CG}_*$ corresponding to a cofibrant X in $\mathbf{CG}_*$ (standard model category structure).

EXAMPLE Let $Y : \mathbf{\Gamma} \rightarrow \mathbf{CG}$ be a functor—then $\forall\ X$, one can form $\mathrm{bar}(\overline{\mathrm{pow}}\, X; \mathbf{\Gamma}; Y)$ and denoting by $B(X;\mathbf{\Gamma};Y)$ its geometric realization, there is a canonical arrow $B(X;\mathbf{\Gamma};Y) \rightarrow \overline{\mathrm{pow}}\, X \otimes_{\mathbf{\Gamma}} Y$ (cf. p. 14–16). Example: $\forall\ n$, $(PY)\mathbf{n} \approx B(\mathbf{n};\mathbf{\Gamma};Y)$, $Y(\mathbf{n}) \approx \overline{\mathrm{pow}}\, \mathbf{n} \otimes_{\mathbf{\Gamma}} Y$ and the arrow of evaluation $(PY)\mathbf{n} \rightarrow Y(\mathbf{n})$ is a homotopy equivalence.

EXAMPLE Let $\zeta : \mathbf{\Gamma}_{\mathrm{in}}^{\mathrm{OP}} \rightarrow \mathbf{\Gamma}$ be the functor which is the identity on objects and sends $\gamma : \mathbf{m} \rightarrow \mathbf{n}$ to $\zeta\gamma : \mathbf{n} \rightarrow \mathbf{m}$, where $\zeta\gamma(j) = \gamma^{-1}(j)$ if $\gamma^{-1}(j) \neq \emptyset$, $\zeta\gamma(j) = 0$ if $\gamma^{-1}(j) = \emptyset$—then for any X in $\mathbf{CG}_{*c}$, $\overline{\mathrm{pow}}\, X \circ \zeta^{\mathrm{OP}} = \mathrm{pow}\, X$. The assignment $\mathbf{n} \rightarrow \overline{\mathrm{hocolim}}\, \mathrm{pow}\, \mathbf{n}$ defines a functor $\gamma^{\infty} : \mathbf{\Gamma} \rightarrow \mathbf{CG}$. And: $\overline{\mathrm{hocolim}}\, \mathrm{pow}\, X \approx \overline{\mathrm{pow}}\, X \otimes_{\mathbf{\Gamma}} \gamma^{\infty}$.

[The left Kan extension of $B(—\backslash\mathbf{\Gamma}_{\mathrm{in}})$ along ζ is γ^{∞}, hence $\overline{\mathrm{hocolim}}\, \mathrm{pow}\, X \approx B(—\backslash\mathbf{\Gamma}_{\mathrm{in}}) \otimes_{\mathbf{\Gamma}_{\mathrm{in}}} \mathrm{pow}\, X \approx \mathrm{pow}\, X \otimes_{\mathbf{\Gamma}_{\mathrm{in}}^{\mathrm{OP}}} B(—\backslash\mathbf{\Gamma}_{\mathrm{in}}) \approx \overline{\mathrm{pow}}\, X \circ \zeta^{\mathrm{OP}} \otimes_{\mathbf{\Gamma}_{\mathrm{in}}^{\mathrm{OP}}} B(—\backslash\mathbf{\Gamma}_{\mathrm{in}}) \approx \overline{\mathrm{pow}}\, X \otimes_{\mathbf{\Gamma}} \gamma^{\infty}$.]

[Note: Let X be a pointed connected CW complex or a pointed connected ANR—then the homotopy colimit theorem says that $\overline{\mathrm{hocolim}}\, \mathrm{pow}\, X$ and $\Omega^{\infty}\Sigma^{\infty}X$ have the same homotopy type, thus by the above, $\overline{\mathrm{pow}}\, X \otimes_{\mathbf{\Gamma}} \gamma^{\infty}$ and $\Omega^{\infty}\Sigma^{\infty}X$ have the same homotopy type.]

LEMMA Relative to $\mathbf{S}[1]^{\mathrm{OP}} : \mathbf{\Delta} \rightarrow \mathbf{\Gamma}^{\mathrm{OP}}$, $\mathrm{lan}\, \Delta^{?} \approx \overline{\mathrm{pow}}\, \mathbf{S}^1$.

Let $X : \mathbf{\Gamma} \rightarrow \mathbf{CG}$ be a functor—then the realization $|X|_{\mathbf{\Gamma}}$ of X is by definition $|X \circ \mathbf{S}[1]|$, the geometric realization of $X \circ \mathbf{S}[1]$. And: $|X \circ \mathbf{S}[1]| = X \circ \mathbf{S}[1] \otimes_{\mathbf{\Delta}} \Delta^{?} \approx X \otimes_{\mathbf{\Gamma}^{\mathrm{OP}}} \mathrm{lan}\, \Delta^{?} \approx X \otimes_{\mathbf{\Gamma}^{\mathrm{OP}}} \overline{\mathrm{pow}}\, \mathbf{S}^1 \approx \overline{\mathrm{pow}}\, \mathbf{S}^1 \otimes_{\mathbf{\Gamma}} X$.

Example: Let G be an abelian cofibered monoid in $\mathbf{CG}$—then $(\mathbf{\Gamma}\text{-}\mathrm{ner}\, \mathbf{G}) \circ \mathbf{S}[1] = \mathrm{ner}\, \mathbf{G} \Rightarrow |\mathbf{\Gamma}\text{-}\mathrm{ner}\, \mathbf{G}|_{\mathbf{\Gamma}} = BG$.

Given an abelian cofibered monoid G in $\mathbf{CG}$, let $SP^{\infty}(?;G)$ be the functor $\mathbf{CG}_{*c} \rightarrow \mathbf{CG}_{*c}$ that sends X to $\overline{\mathrm{pow}}\, X \otimes_{\mathbf{\Gamma}} \mathbf{\Gamma}\text{-}\mathrm{ner}\, \mathbf{G}$—then $SP^{\infty}(X;G)$ is an abelian cofibered monoid in $\mathbf{CG}$, the infinite symmetric product on (X, x_0) with coefficients in G. Example: Take $G = \mathbf{Z}_{\geq 0}$ to see that $SP^\infty X \approx \int^{\mathbf{n}} X^n \times_k SP^\infty \mathbf{n} \approx SP^\infty(X; \mathbf{Z}_{\geq 0})$ (the choice $G = \mathbf{Z}$ leads to the free abelian compactly generated group on (X, x_0)).

LEMMA $\forall\, X, Y, SP^\infty(X \#_k Y; G) \approx SP^\infty(X; SP^\infty(Y; G))$ (isomorphism of abelian monoids in **CG**).

EXAMPLE Let G be an abelian cofibered monoid in **CG**—then $SP^\infty(\mathbf{S}^0; G) \approx G$, $SP^\infty(\mathbf{S}^1; G) \approx BG$, and in general, $SP^\infty(\mathbf{S}^{n+1}; G) \approx B^{(n+1)}G$, where $B^{(n+1)}G = B(B^{(n)}G)$.

[Representing $\mathbf{S}^{n+1}$ as the smash product $\mathbf{S}^n \#_k \mathbf{S}^1$, the lemma implies that $SP^\infty(\mathbf{S}^{n+1}; G) \approx SP^\infty(\mathbf{S}^n; BG)$).]

Let X be a proper special $\mathbf{\Gamma}$-space—then $X \circ \mathrm{S}[1]$ satisfies the cofibration condition. Moreover, if $X \circ \mathrm{S}[1]$ is monoidal, then X_1 is a homotopy associative, homotopy commutative H space and the arrow $X_1 \to \Omega|X|_{\mathbf{\Gamma}}$ is a group completion (cf. p. 14–55).

[Note: $\sin X$ is an object in $\mathbf{\Gamma}\mathbf{SISET}_*$ (cf. p. 13–56) and $|\sin X|$ is a proper special $\mathbf{\Gamma}$-space. The simplicial space $|\sin X| \circ \mathrm{S}[1]$ is monoidal and there is a commutative diagram

$$\begin{tikzcd} {|\sin X_1|} \arrow[r] \arrow[d] & \Omega\|\sin X\|_{\mathbf{\Gamma}} \arrow[d] \\ X_1 \arrow[r] & \Omega|X|_{\mathbf{\Gamma}} \end{tikzcd}$$

. Since the vertical arrows are weak homotopy equivalences (Giever-Milnor (cf. p. 14–8 ff.)) and since the arrow $|\sin X_1| \to \Omega\|\sin X\|_{\mathbf{\Gamma}}$ is a group completion, it follows that the arrow $X_1 \to \Omega|X|_{\mathbf{\Gamma}}$ is a weak group completion (X_1 is not necessarily an H space) (but $\forall\, \mathbf{k}$, $\pi_0(X_1)$ is a central submonoid of $H_*(X_1; \mathbf{k})$ and $H_*(X_1; \mathbf{k})[\pi_0(X_1)^{-1}] \approx H_*(\Omega|X|_{\mathbf{\Gamma}}; \mathbf{k})$.]

Remark: If **C** is a pointed category with finite products and if X is a special $\mathbf{\Gamma}$-object in **C** (obvious definition), then X_1 is an abelian monoid object in **C** (cf. p. 14–21).

FACT Let X be a proper special $\mathbf{\Gamma}$-space. Assume: $\forall\, n \geq 1$, the arrow $X_n \to X_1 \times_k \cdots \times_k X_1$ determined by the π_i is an S_n-equivariant homotopy equivalence—then there exists an abelian cofibered monoid G in **CG** and a levelwise homotopy equivalence $X \to \mathbf{\Gamma}\text{-}\mathrm{ner}\, \mathbf{G}$.

LEMMA Let X be a proper special $\mathbf{\Gamma}$-space—then X_1 path connected $\Rightarrow |X|_{\mathbf{\Gamma}}$ simply connected and X_1 n-connected $\Rightarrow |X|_{\mathbf{\Gamma}}$ $(n+1)$-connected (cf. p. 14–11).

Let $\mathbf{\Gamma} \xrightarrow{\nu_n} \mathbf{\Gamma} \times \mathbf{\Gamma}$ be the functor defined by $\mathbf{p} \to (\mathbf{n}, \mathbf{p})$ on objects and $\gamma \to (\mathrm{id}_{\mathbf{n}}, \gamma)$ on morphisms. Given a proper special $\mathbf{\Gamma}$-space X, call $\overline{X}_n$ the composite $\mathbf{\Gamma} \xrightarrow{\nu_n} \mathbf{\Gamma} \times \mathbf{\Gamma} \xrightarrow{\#} \mathbf{\Gamma} \xrightarrow{X} \mathbf{CG}_*$, $\#$ being the smash product (cf. p. 14–28). So: $\overline{X}_n(\mathbf{p}) = X_{np}$ and $\overline{X}_n$ is a proper special $\mathbf{\Gamma}$-space.

[Note: Suppose that $\gamma : \mathbf{m} \to \mathbf{n}$ is a morphism in $\mathbf{\Gamma}$. Set $\gamma_p = \gamma\#\mathrm{id}_{\mathbf{p}} : \mathbf{mp} \to \mathbf{np}$—then the γ_p induce a $\mathbf{\Gamma}$-map $\overline{X}_m \to \overline{X}_n$, thus $\overline{X}$ is a functor from $\mathbf{\Gamma}$ to $\mathbf{ps\,\Gamma\text{-}SP}$.]

The classifying space of a proper special $\mathbf{\Gamma}$-space X is the proper special $\mathbf{\Gamma}$-space BX which takes $\mathbf{n}$ to $B_nX = |\overline{X}_n|_{\mathbf{\Gamma}}$. In particular: $B_1X = |X|_{\mathbf{\Gamma}}$ is path connected, hence $B_1X \to \Omega|BX|_{\mathbf{\Gamma}}$ is a weak homotopy equivalence.

FACT Let G be an abelian cofibered monoid in $\mathbf{CG}$—then the classifying space of the $\mathbf{\Gamma}$-nerve of $\mathbf{G}$ is the $\mathbf{\Gamma}$-nerve of $\mathbf{BG}$.

Notation: Given a proper special $\mathbf{\Gamma}$-space X, write $B^{(0)}X = X$, $B^{(q+1)}X = B(B^{(q)}X)$, and put $S_0X = \Omega|X|_{\mathbf{\Gamma}}$, $S_{q+1}X = |B^{(q)}X|_{\mathbf{\Gamma}}$ $(q \geq 0)$.

EXAMPLE Let X be a proper special $\mathbf{\Gamma}$-space—then $\forall\, q > 0$, $S_qX \approx \overline{\mathrm{pow}\,\mathbf{S}^q} \otimes_{\mathbf{\Gamma}} X$.

A prespectrum $\mathbf{X}$ is a sequence of pointed Δ-separated compactly generated spaces X_q and pointed continuous functions $X_q \xrightarrow{\sigma_q} \Omega X_{q+1}$. **PRESPEC** is the category whose objects are the prespectra and whose morphisms $\mathbf{f} : \mathbf{X} \to \mathbf{Y}$ are sequences of pointed continuous functions $f_q : X_q \to Y_q$ such that the diagram

$$\begin{tikzcd} X_q \arrow[r, "f_q"] \arrow[d] & Y_q \arrow[d] \\ \Omega X_{q+1} \arrow[r, "\Omega f_{q+1}"'] & \Omega Y_{q+1} \end{tikzcd}$$

commutes $\forall\, q$. Every spectrum is a prespectrum.

[Note: The indexing begins at 0.]

EXAMPLE Let $\mathcal{O}$ be an E_∞ operad. Suppose that X is a Δ-separated proper special $\widehat{\mathcal{O}}$-space—then the assignment $q \to B(\Sigma^q L; \widehat{\mathcal{O}}^q; X)$ is a prespectrum.

Remark: **PRESPEC** is complete and cocomplete (limits and colimits are calculated levelwise).

PROPOSITION 56 Equip $\mathbf{\Delta\text{-}CG_*}$ with its singular structure—then **PRESPEC** is a model category if weak equivalences and fibrations are levelwise, a cofibration $\mathbf{f} : \mathbf{X} \to \mathbf{Y}$ being a levelwise cofibration with the additional property that $\forall\, q$, the arrow $P_{q+1} \to Y_{q+1}$ is a cofibration, where P_{q+1} is defined by the pushout square

$$\begin{tikzcd} \Sigma X_q \arrow[r] \arrow[d] & \Sigma Y_q \arrow[d] \\ X_{q+1} \arrow[r] & P_{q+1} \end{tikzcd}$$

[Note: In the presence of the condition on the $P_{q+1} \to Y_{q+1}$, to describe the cofibrations in **PRESPEC**, it suffices to require that $f_0 : X_0 \to Y_0$ be a cofibration.]

If $\mathbf{C}$ is a category and if $F, G : \mathbf{C} \rightarrow \mathbf{PRESPEC}$ are functors, then a natural transformation $\Xi : F \rightarrow G$ is a function that assigns to each $X \in \operatorname{Ob}\mathbf{C}$ an element $\Xi_X \in \operatorname{Mor}(FX, GX)$ natural in X. Using the notation $\begin{cases} FX = \{F_{X,q}\} \\ GX = \{G_{X,q}\} \end{cases}$, $\Xi_X = \{\Xi_{X,q}\}$, the fact that $\Xi_X \in \operatorname{Mor}(FX, GX)$ is expressed by the commutativity of

$$\begin{tikzcd} F_{X,q} \arrow[r, "\Xi_{X,q}"] \arrow[d, "\sigma_{F,q}"'] & G_{X,q} \arrow[d, "\sigma_{G,q}"] \\ \Omega F_{X,q+1} \arrow[r, "\Omega\Xi_{X,q+1}"'] & \Omega G_{X,q+1} \end{tikzcd}$$

$\forall\, q$. A pseudo natural transformation $\Xi : F \rightarrow G$ is a function that assigns to each $X \in \operatorname{Ob}\mathbf{C}$ a sequence of pointed continuous functions $\Xi_{X,q} : F_{X,q} \rightarrow G_{X,q}$ natural in X and a sequence of pointed homotopies $H_{X,q}$ between $\Omega\Xi_{X,q+1} \circ \sigma_{F,q}$ and $\sigma_{G,q} \circ \Xi_{X,q}$ natural in X (thus natural $\Rightarrow$ pseudo natural (constant homotopies)). A pseudo natural homotopy between pseudo natural transformations $\Xi_0, \Xi_1 : F \rightarrow G$ is a pseudo natural transformation $\Upsilon : F\#I_+ \rightarrow G$ such that $\begin{cases} \Upsilon \circ i_0 = \Xi_0 \\ \Upsilon \circ i_1 = \Xi_1 \end{cases}$, where $(F\#I_+)(X) = FX\#I_+$ $(= \{F_{X,q}\#I_+\})$ (cf. p. 3–28).

[Note: A natural (pseudo natural) transformation Ξ is called a natural (pseudo natural) weak equivalence if the $\Xi_{X,q}$ are weak homotopy equivalences.]

EXAMPLE (Cylinder Construction) There is a functor $M : \mathbf{PRESPEC} \rightarrow \mathbf{PRESPEC}$ with the property that $\forall\, \mathbf{X}$, the arrows $(M\mathbf{X})_q \rightarrow \Omega(M\mathbf{X})_{q+1}$ are closed embeddings. And:

(M$_1$) $\exists$ a natural transformation $r : M \rightarrow \mathrm{id}$ such that $\forall\, \mathbf{X}$, $r_{\mathbf{X},q} : (M\mathbf{X})_q \rightarrow X_q$ is a pointed homotopy equivalence.

(M$_2$) $\exists$ a pseudo natural transformation $j : \mathrm{id} \rightarrow M$ such that $\forall\, \mathbf{X}$, $j_{\mathbf{X},q} : X_q \rightarrow (M\mathbf{X})_q$ is a pointed homotopy equivalence.

(M$_3$) The composite $r \circ j$ is id_M and the composite $j \circ r$ is pseudo naturally homotopic to id_M.

[Construct M by repeated use of pointed mapping cylinders (this forces the definitions of r and j).]

[Note: $\forall\, \mathbf{X}$, the rule $q \rightarrow \operatorname{colim} \Omega^n(M\mathbf{X})_{n+q}$ defines a spectrum, call it $eM\mathbf{X}$.]

FACT (Conversion Principle) Let $\mathbf{C}$ be a category and let $F, G : \mathbf{C} \rightarrow \mathbf{PRESPEC}$ be functors. Suppose given a pseudo natural transformation $\Xi : F \rightarrow G$—then there exists a natural transformation $M\Xi : M \circ F \rightarrow M \circ G$ such that the diagram

$$\begin{tikzcd} MFX \arrow[r, "M\Xi"] \arrow[d, "r"'] & MGX \arrow[d, "r"] \\ FX \arrow[r, "\Xi"'] & GX \end{tikzcd}$$

is pseudo naturally homotopy commutative.

A prespectrum $\mathbf{X}$ is said to be connective if X_1 is path connected and X_q is $(q-1)$-connected $(q > 1)$.

Example: Given a Δ-separated proper special $\mathbf{\Gamma}$-space X, the assignment $q \rightarrow S_q X$ specifies a connective prespectrum $\mathbf{S}X$.

[The arrow $S_0X \to \Omega S_1X$ is the identity map $\Omega|X|_{\mathbf{\Gamma}} \to \Omega|X|_{\mathbf{\Gamma}}$. For $q > 0$, the arrow $S_qX \to \Omega S_{q+1}X$ is the weak group completion $B_1(B^{(q-1)}X)$ $(= |B^{(q-1)}X|_{\mathbf{\Gamma}}) \to \Omega|B^{(q)}X|_{\mathbf{\Gamma}}$ of p. 14–69.]

[Note: That $\mathbf{S}X$ is connective is implied by the generalities on p. 14–11.]

A prespectrum $\mathbf{X}$ is said to be an Ω-prespectrum if $\forall\, q$, the arrow $X_q \xrightarrow{\sigma_q} \Omega X_{q+1}$ is a weak homotopy equivalence.

Example: Given a Δ-separated proper special $\mathbf{\Gamma}$-space X, the assignment $q \to S_qX$ specifies an Ω-prespectrum $\mathbf{S}X$.

EXAMPLE (Algebraic K-Theory) Let A be a ring with unit—then the prescription $q \to K_0(\Sigma^q A) \times B\mathbf{GL}(\Sigma^q A)^+$ attaches to A an Ω-prespectrum $\mathbf{W}A$. Proof: $\Omega(K_0(\Sigma^{q+1}A) \times B\mathbf{GL}(\Sigma^{q+1}A)^+) \approx \Omega B\mathbf{GL}(\Sigma^{q+1}A)^+$ (trivially) $\approx K_0(\Sigma^q A) \times B\mathbf{GL}(\Sigma^q A)^+$ (cf. p. 5–75 ff.).

[Note: As it stands, a morphism $A' \to A''$ of rings does not induce a morphism $\mathbf{W}A' \to \mathbf{W}A''$ of Ω-prespectra (the relevant diagrams are only pointed homotopy commutative).]

PROPOSITION 57 Let $\left\{\begin{matrix}\mathbf{X}\\ \mathbf{Y}\end{matrix}\right.$ be connective Ω-prespectra—then a morphism $\mathbf{f}: \mathbf{X} \to \mathbf{Y}$ is a weak equivalence provided that $f_0 : X_0 \to Y_0$ is a weak homotopy equivalence.

LEMMA Let $\left\{\begin{matrix}X\\ Y\end{matrix}\right.$ be homotopy associative H spaces such that $\left\{\begin{matrix}\pi_0(X)\\ \pi_0(Y)\end{matrix}\right.$ is a group under the induced product; let $f : X \to Y$ be a pointed continuous function such that $\pi_0(f) : \pi_0(X) \to \pi_0(Y)$ is bijective—then f is a weak homotopy equivalence if f is a homology equivalence.

[Consider the commutative diagram

$$\begin{tikzcd} {|\sin X|} \arrow[r, "{|\sin f|}"] \arrow[d] & {|\sin Y|} \arrow[d] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

. Since the hypotheses on $\left\{\begin{matrix}X\\ Y\end{matrix}\right.$ and f are also satisfied by $\left\{\begin{matrix}|\sin X|\\ |\sin Y|\end{matrix}\right.$ and $|\sin f|$ and since there are homotopy equivalences $\left\{\begin{matrix}|\sin X| \to |\sin X|_0 \times \pi_0(|\sin X|)\\ |\sin Y| \to |\sin Y|_0 \times \pi_0(|\sin Y|)\end{matrix}\right.$, where $\left\{\begin{matrix}|\sin X|_0\\ |\sin Y|_0\end{matrix}\right.$ is the path component of the identity element, Dror's Whitehead theorem implies that $|\sin f|$ is a homotopy equivalence, hence f is a weak homotopy equivalence (Giever-Milnor).]

Example: Suppose that $X \to Y$ is a group completion—then $X \to Y$ is a weak homotopy equivalence if $\pi_0(X)$ is a group.

[Note: Let X be a proper special $\mathbf{\Gamma}$-space such that $\pi_0(X_1)$ is a group. Because $\pi_0(|\sin X_1|)$ is likewise a group, the group completion $|\sin X_1| \to \Omega\|\sin X\|_{\mathbf{\Gamma}}$ is a weak homotopy equivalence, thus the same is true of the weak group completion $X_1 \to \Omega|X|_{\mathbf{\Gamma}}$ (cf. p. 14–69).]

EXAMPLE Let $\mathcal{O}$ be an E_∞ operad. Suppose that X is a Δ-separated $\mathcal{O}$-space. Assume: $\pi_0(X)$ is a group—then X has the weak homotopy type of an infinite loop space.

[The group completion $X \to B_0X$ is a weak homotopy equivalence.]

PROPOSITION 58 Let $\begin{cases}\mathbf{X}\\ \mathbf{Y}\end{cases}$ be connective Ω-prespectra—then a morphism $\mathbf{f}:\mathbf{X}\to\mathbf{Y}$ is a weak equivalence whenever $f_0 : X_0 \to Y_0$ induces a bijection $\pi_0(X_0)\to\pi_0(Y_0)$ and is a homology equivalence.

[There is a commutative diagram

$$\begin{tikzcd}
X_0 \arrow[r, "f_0"] \arrow[d] & Y_0 \arrow[d] \\
\Omega X_1 \arrow[r, "\Omega f_1"'] & \Omega Y_1
\end{tikzcd}$$

and, in view of the lemma, Ωf_1 is a weak homotopy equivalence. So, f_0 is a weak homotopy equivalence and one can quote Proposition 57.]

PROPOSITION 59 Suppose given an infinite loop space machine on $\mathbf{\Gamma}$. Let $\begin{cases}X\\ Y\end{cases}$ be Δ-separated proper special $\mathbf{\Gamma}$-spaces, $f : X \to Y$ a $\mathbf{\Gamma}$-map. Assume: $f_1 : X_1 \to Y_1$ is a weak homotopy equivalence or $Kf : KX \to KY$ is a group completion—then $\mathbf{B}f : \mathbf{B}X \to \mathbf{B}Y$ is a weak equivalence.

[Work with

$$\begin{tikzcd}
X_1 \arrow[r] \arrow[d] & KX \arrow[r] \arrow[d] & B_0X \arrow[d] \\
Y_1 \arrow[r] & KY \arrow[r] & B_0Y
\end{tikzcd}$$

and apply Proposition 58.]

[Note: There is an evident analog of this result for $\mathbf{S}$.]

PROPOSITION 60 Suppose given an infinite loop space machine on $\mathbf{\Gamma}$. Let $\begin{cases}X\\ Y\end{cases}$ be Δ-separated proper special $\mathbf{\Gamma}$-spaces—then the arrow $\mathbf{B}(X\times Y)\to\mathbf{B}X\times\mathbf{B}Y$ is a weak equivalence.

[To begin with, the arrow $K(X\times Y)\to KX\times_k KY$ is a weak homotopy equivalence (examine

$$\begin{tikzcd}
K(X\times Y) \arrow[r] & KX\times_k KY \\
L(X\times Y) \arrow[u] \arrow[r, phantom, "="] & LX\times_k LY \arrow[u]
\end{tikzcd}$$

). This said, form the commutative diagram

$$\begin{tikzcd}
K(X\times Y) \arrow[r] \arrow[d] & KX\times_k KY \arrow[d] \\
B_0(X\times Y) \arrow[r] & B_0X\times_k B_0Y
\end{tikzcd}$$

. By definition, $K(X\times Y)\to B_0(X\times Y)$ is a group completion. The same is true of $KX\times_k KY\to B_0X\times_k B_0Y$. Proof: $\overline{\pi_0(KX\times_k KY)}\approx\overline{\pi_0(KX)\times\pi_0(KY)}\approx\overline{\pi_0(KX)}\times\overline{\pi_0(KY)}\approx\pi_0(B_0X)\times\pi_0(B_0Y)\approx\pi_0(B_0X\times_k B_0Y)$ (cf. p. 14–24) and, using the Künneth formula, $H_*(KX\times_k KY;\mathbf{k})[\pi_0(KX\times_k KY)^{-1}]\approx$

$H_*(B_0X \times_k B_0Y;\mathbf{k})$ for all prime fields $\mathbf{k}$ (cf. p. 14–55). It now follows that $\pi_0(B_0(X \times Y)) \approx \pi_0(B_0X \times_k B_0Y)$ and $H_*(B_0(X \times Y)) \approx H_*(B_0X \times_k B_0Y)$, from which the assertion (cf. Proposition 58).]

Let X be a Δ-separated proper special $\mathbf{\Gamma}$-space—then an infinite loop machine on $\mathbf{\Gamma}$ defines a sequence of functors $B_q\overline{X} : \mathbf{\Gamma} \rightarrow \Delta\text{-}\mathbf{CG}_*$, viz. $\mathbf{n} \rightarrow B_q\overline{X}_n$. It is not claimed that $B_q\overline{X}$ is special. However, $B_q\overline{X}_0$ is homotopically trivial and $\forall\ n \geq 1$, the arrow $B_q\overline{X}_n \rightarrow B_q\overline{X}_1 \times_k \cdots \times_k B_q\overline{X}_1$ determined by the π_i is a weak homotopy equivalence (cf. Propositions 59 and 60).

A $\mathbf{\Gamma}$-space X is said to be semispecial or semiproper if the requirement $X_0 = *$ is relaxed to X_0 homotopically trivial, the other conditions on $X|\mathbf{\Pi}$ staying the same. Example: $\forall\ q \geq 0$, $B_q\overline{X}$ is semispecial.

LEMMA Suppose that X is a Δ-separated semispecial $\mathbf{\Gamma}$-space—then there exists a Δ-separated semiproper semispecial $\mathbf{\Gamma}$-space WX and a $\mathbf{\Gamma}$-map $\pi : WX \rightarrow X$ such that $\forall\ n$, $\pi_n : W_nX \rightarrow X_n$ is a weak homotopy equivalence.

[Equip $[0,1]$ with the structure of an abelian cofibered monoid in $\mathbf{CG}$ by writing $st = \min\{s,t\}$. Put $I = \mathbf{\Gamma}\text{-ner}[0,1]$, so for $\gamma : \mathbf{m} \rightarrow \mathbf{n}$, $I\gamma : I_m \rightarrow I_n$ is the function $(s_1, \ldots, s_m) \rightarrow (t_1, \ldots, t_n)$, where $t_j = \min\limits_{\gamma(i)=j}\{s_i\}$ (a minimum over the empty set is 1). Set $W_0X = X_0$ and define a subfunctor WX of $I \times X$ and a $\mathbf{\Gamma}$-map $\pi : WX \rightarrow X$ as follows. Given an order preserving injection $\gamma : \mathbf{m} \rightarrow \mathbf{n}$, let $[0,1]^n_\gamma$ be the subspace of $[0,1]^n$ consisting of those $(t_1, \ldots, t_n)$ such that $t_j = 0$ if $j \in \operatorname{im}\gamma$, $t_j > 0$ if $j \notin \operatorname{im}\gamma$. Now form $W_nX = \bigcup\limits_{\gamma}[0,1]^n_\gamma \times (X\gamma)X_m \subset [0,1]^n \times X_n$: X_n embeds in W_nX (consider $\gamma = \mathrm{id}_\mathbf{n}$) and the homotopy $H((t_1, \ldots, t_n, x), T) = (t_1T, \ldots, t_nT, x)$ $(0 \leq T \leq 1)$ exhibits X_n as a strong deformation retract of W_nX (hence $\pi_n(t_1, \ldots, t_n, x) = (0, \ldots, 0, x)$). Therefore the Δ-separated $\mathbf{\Gamma}$-space WX is semispecial. To establish that WX is semiproper, one has to show that for each injection $\gamma : \mathbf{m} \rightarrow \mathbf{n}$, $(WX)\gamma : W_mX \rightarrow W_nX$ is a closed S_γ-cofibration. This can be done by observing that $\operatorname{im}(WX)\gamma$ admits the description $\{(t_1, \ldots, t_n, x) : t_j = 1\ \forall\ j \notin \operatorname{im}\gamma\ \&\ x \in (X\gamma)X_m\}$.]

[Note: W is functorial and π is natural: For any $\mathbf{\Gamma}$-map $f : X \rightarrow Y$ between Δ-separated semispecial $\mathbf{\Gamma}$-spaces, the diagram

$$\begin{tikzcd} WX \arrow[r, "Wf"] \arrow[d] & WY \arrow[d] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

commutes.]

Observation: The arrow $W_0X \rightarrow W_nX$ corresponding to $\mathbf{0} \rightarrow \mathbf{n}$ is a closed cofibration. Put $\overline{W}_nX = W_nX/W_0X$—then $\overline{W}X$ is a proper special $\mathbf{\Gamma}$-space, the projection $WX \rightarrow \overline{W}X$ is a levelwise weak homotopy equivalence, and the diagram $X \leftarrow WX \rightarrow \overline{W}X$ is natural in X.

Notation: If $\mathbf{X}$ is a prespectrum, then $\Omega\mathbf{X}$ is the prespectrum specified by $(\Omega\mathbf{X})_q = \Omega X_q$, where $\Omega X_q \to \Omega\Omega X_{q+1}$ is the composite $\Omega X_q \xrightarrow{\Omega\sigma_q} \Omega\Omega X_q \xrightarrow{\mathrm{T}} \Omega\Omega X_q$, T being the twist $(\mathrm{T}f)(s)(t) = f(t)(s)$.

EXAMPLE Let X be a Δ-separated proper special $\mathbf{\Gamma}$-space. Assume: X_1 is path connected—then $\forall\, q$, $|B^{(q)}X|_{\mathbf{\Gamma}}$ is $(q+1)$-connected, hence $\Omega\mathbf{S}X$ is a connective Ω-prespectrum.

LEMMA For any proper special $\mathbf{\Gamma}$-space X, ΩX is a proper special $\mathbf{\Gamma}$-space and there is a canonical arrow $|\Omega X|_{\mathbf{\Gamma}} \xrightarrow{\gamma} \Omega|X|_{\mathbf{\Gamma}}$.

[Note: Here, of course, ΩX takes $\mathbf{n}$ to ΩX_n.]

PROPOSITION 61 Let X be a Δ-separated proper special $\mathbf{\Gamma}$-space—then there is a morphism $\mathbf{s} : \mathbf{S}\Omega X \to \Omega\mathbf{S}X$ in **PRESPEC** such that the triangle

$$\begin{tikzcd} & \Omega X_1 \arrow[dl] \arrow[dr] & \\ S_0\Omega X \arrow[rr, "s_0"'] & & \Omega S_0 X \end{tikzcd}$$

commutes.

[Explicated, the oblique arrow on the left is

$$\begin{tikzcd} & \Omega X_1 \arrow[dl] \\ \Omega|\Omega X|_{\mathbf{\Gamma}} & \end{tikzcd}$$

and the composite $\Omega X_1 \to \Omega|\Omega X|_{\mathbf{\Gamma}} \xrightarrow{\Omega\gamma} \Omega\Omega|X|_{\mathbf{\Gamma}} \xrightarrow{\mathrm{T}} \Omega\Omega|X|_{\mathbf{\Gamma}}$ is Ω of $X_1 \to \Omega|X|_{\mathbf{\Gamma}}$, the oblique arrow

$$\begin{tikzcd} \Omega X_1 \arrow[dr] & \\ & \Omega\Omega|X|_{\mathbf{\Gamma}} \end{tikzcd}$$

on the right. Definition: $s_0 = \mathrm{T} \circ \Omega\gamma$. To force compatibility, take $s_1 = \gamma : S_1\Omega X \to \Omega S_1 X$, thereby ensuring that the diagram

$$\begin{tikzcd} S_0\Omega X \arrow[r, "s_0"] \arrow[d, equal] & \Omega S_0 X \arrow[d, "\mathrm{T}"] \\ \Omega S_1\Omega X \arrow[r, "\Omega s_1"'] & \Omega\Omega S_1 X \end{tikzcd}$$

commutes. The arrows $B_n\Omega X = |\Omega\overline{X}_n|_{\mathbf{\Gamma}} \xrightarrow{\gamma} \Omega|\overline{X}_n|_{\mathbf{\Gamma}} = \Omega B_n X$ yield a $\mathbf{\Gamma}$-map $b : B\Omega X \to \Omega BX$. Setting $b^{(0)} = \mathrm{id}_{\Omega X}$, let $b^{(q)}$ $(q > 0)$ be the composite $B^{(q)}\Omega X \xrightarrow{Bb^{(q-1)}} B\Omega B^{(q-1)}X \xrightarrow{b} \Omega B^{(q)}X$. Definition: $s_q = \gamma \circ |b^{(q-1)}|_{\mathbf{\Gamma}}$ $(q > 1)$. This makes sense: $S_q\Omega X = |B^{(q-1)}\Omega X|_{\mathbf{\Gamma}} \xrightarrow{|b^{(q-1)}|_{\mathbf{\Gamma}}} |\Omega B^{(q-1)}X|_{\mathbf{\Gamma}} \xrightarrow{\gamma} \Omega|B^{(q-1)}X|_{\mathbf{\Gamma}} = \Omega S_q X$ and the diagram

$$\begin{tikzcd} S_q\Omega X \arrow[r, "s_q"] \arrow[d] & \Omega S_q X \arrow[d] \\ \Omega S_{q+1}\Omega X \arrow[r, "\Omega s_{q+1}"'] & \Omega S_{q+1}\Omega X \end{tikzcd}$$

commutes.]

[Note: If X_1 is path connected, then $\Omega\mathbf{S}X$ is a connective Ω-prespectrum (cf. p. 14–75) and s_0 is a weak homotopy equivalence (cf. p. 14–72), thus $\mathbf{s}$ is a weak equivalence (cf. Proposition 57). It is also clear that $\mathbf{s}$ is natural.]

LEMMA Suppose that X is a Δ-separated semispecial $\mathbf{\Gamma}$-space—then there exists a $\mathbf{\Gamma}$-map $\omega : W\Omega X \to \Omega WX$ such that the triangle

$$\begin{tikzcd} W\Omega X \arrow[rr, "\omega"] \arrow[dr, "\pi"'] & & \Omega WX \arrow[dl, "\Omega\pi"] \\ & \Omega X & \end{tikzcd}$$

is homotopy commutative, thus $\forall\ n$, $\omega_n : W_n\Omega X \to \Omega W_n X$ is a weak homotopy equivalence.

[Represent a typical element in $W_n\Omega X$ by $(t_1, \ldots, t_n, \sigma)$ $(\sigma \in \Omega_n X = \Omega X_n)$ and let

$$\omega_n(t_1, \ldots, t_n, \sigma)(t) = \begin{cases} (u_1(t), \ldots, u_n(t), \sigma(0)) & (0 \leq t \leq 1/3) \\ (t_1, \ldots, t_n, \sigma(3t-1)) & (1/3 \leq t \leq 2/3), \\ (v_1(t), \ldots, v_n(t), \sigma(1)) & (2/3 \leq t \leq 1) \end{cases}$$

where $u_j(t) = 1 - 3t + 3tt_j$, $v_j(t) = 3t - 2 + (3 - 3t)t_j$ $(1 \leq j \leq n)$. The prescription

$$H_\omega((t_1, \ldots, t_n, \sigma), T)(t) = \begin{cases} \sigma(0) & (0 \leq t \leq (1/3)T) \\ \sigma(\frac{3t-T}{3-2T}) & ((1/3)T \leq t \leq 1 - (1/3)T) \\ \sigma(1) & (1 - (1/3)T \leq t \leq 1) \end{cases}$$

is a homotopy between π and $\Omega\pi \circ \omega$.]

[Note: ω and H_ω are natural.]

Observation: The diagram

$$\begin{tikzcd} W\Omega X \arrow[r, "\omega"] \arrow[d] & \Omega WX \arrow[d] \\ \overline{W}\Omega X \arrow[r, "\overline{\omega}"'] & \Omega\overline{W}X \end{tikzcd}$$

commutes and $\overline{\omega}$ is a levelwise weak homotopy equivalence.

A biprespectrum $\mathbf{X}$ is a sequence of prespectra $\mathbf{X}_q$ and morphisms $\mathbf{X}_q \xrightarrow{\boldsymbol{\sigma}_q} \Omega\mathbf{X}_{q+1}$ $(q \geq 0)$. Spelled out, a biprespectrum is a doubly indexed sequence of pointed Δ-separated compactly generated spaces $X_{q,p}$ and pointed continuous functions $\sigma_{q,p} : X_{q,p} \to \Omega X_{q+1,p}$, $\sigma_{q,p} : X_{q,p} \to \Omega X_{q,p+1}$ such that the diagram

$$\begin{tikzcd} X_{q,p} \arrow[rr, "\sigma_{q,p}"] \arrow[d, "\sigma_{q,p}"'] & & \Omega X_{q,p+1} \arrow[d, "\Omega\sigma_{q,p+1}"] \\ \Omega X_{q+1,p} \arrow[r, "\Omega\sigma_{q+1,p}"'] & \Omega\Omega X_{q+1,p+1} \arrow[r, "T"'] & \Omega\Omega X_{q+1,p+1} \end{tikzcd}$$

commutes $\forall\ q, p$. **BIPRESPEC** is the category whose objects are the biprespectra and whose morphisms $\mathbf{f} : \mathbf{X} \to \mathbf{Y}$ are doubly indexed sequences of pointed continuous functions $f_{q,p} : X_{q,p} \to Y_{q,p}$ such that $f_{q,*}$ & $f_{*,p}$ are morphisms of prespectra $\forall\ q, p$.

THE UP AND ACROSS THEOREM Let $\mathbf{X}$ be a biprespectrum. Assume: $\forall\ q$, $\boldsymbol{\sigma}_q$ is a weak equivalence and $\mathbf{X}_q$ is an Ω-prespectrum—then the Ω-prespectra $\begin{cases} X_{0,*} \\ X_{*,0} \end{cases}$ are naturally weakly equivalent.

[Let **C** be the full subcategory of **BIPRESPEC** whose objects $\mathbf{X}$ have the property that $\forall\, q$, $\boldsymbol{\sigma}_q$ is a weak equivalence and $\mathbf{X}_q$ is an Ω-prespectrum. Denote by $\begin{cases} E' \\ E'' \end{cases}$ the functor $\mathbf{C} \rightarrow$ **PRESPEC** that sends $\mathbf{X}$ to $\begin{cases} X_{0,*} \\ X_{*,0} \end{cases}$ —then the claim is that $\begin{cases} E'\mathbf{X} \\ E''\mathbf{X} \end{cases}$ are naturally weakly equivalent. For this, it suffices to construct functors D', D'' : $\mathbf{C} \rightarrow$ **PRESPEC** and a pseudo natural weak equivalence $\Xi_{\mathbf{X}} : D'\mathbf{X} \rightarrow D''\mathbf{X}$ together with natural weak equivalences $\begin{cases} e'_{\mathbf{X}} : E'\mathbf{X} \rightarrow D'\mathbf{X} \\ e''_{\mathbf{X}} : E''\mathbf{X} \rightarrow D''\mathbf{X} \end{cases}$. Reason: Consider the diagram

$$\begin{tikzcd}
& MD'\mathbf{X} \arrow[r, "M\Xi"] \arrow[d, "r"'] & MD''\mathbf{X} \arrow[d, "r"] & \\
E'\mathbf{X} \arrow[r] & D'\mathbf{X} \arrow[r, "\Xi"'] & D''\mathbf{X} & E''\mathbf{X} \arrow[l]
\end{tikzcd}$$

furnished by the conversion principle. Definition: $D'\mathbf{X} = \Omega^q X_{q,q} = D''\mathbf{X}$, the arrows of structure $\sigma'_q : D'_q\mathbf{X} \rightarrow \Omega D'_{q+1}\mathbf{X}$, $\sigma''_q : D''_q\mathbf{X} \rightarrow \Omega D''_{q+1}\mathbf{X}$ being the composites $\Omega^q X_{q,q} \xrightarrow{\Omega^q \sigma_{q,q}} \Omega^{q+1} X_{q,q+1} \xrightarrow{\top_q} \Omega^{q+1} X_{q,q+1} \xrightarrow{\Omega^{q+1}\sigma_{q,q+1}} \Omega^{q+2} X_{q+1,q+1}$, $\Omega^q X_{q,q} \xrightarrow{\Omega^q \sigma_{q,q}} \Omega^{q+1} X_{q+1,q} \xrightarrow{\top_q} \Omega^{q+1} X_{q+1,q} \xrightarrow{\Omega^{q+1}\sigma_{q+1,q}} \Omega^{q+2} X_{q+1,q+1}$ and $\begin{cases} D'_q\mathbf{f} \\ D''_q\mathbf{f} \end{cases} = \Omega^q f_{q,q}$, where $\mathbf{f} : \mathbf{X} \rightarrow \mathbf{Y}$ $(f_{q,q} : X_{q,q} \rightarrow Y_{q,q})$. Here $\top_q$ is given by twisting the last coordinate past the first q coordinates: $(\top_q f)(s)(t) = f(t)(s)(s \in \mathbf{S}^q, t \in \mathbf{S}^1)$. If $\Xi_{\mathbf{X},q} : \Omega^q X_{q,q} \rightarrow \Omega^q X_{q,q}$ is the identity for even q and the negative of the identity for odd q (i.e., reverse the first coordinate), then there are pointed homotopies $H_{\mathbf{X},q}$ between $\Omega\Xi_{\mathbf{X},q+1} \circ \sigma'_q$ and $\sigma''_q \circ \Xi_{\mathbf{X},q}$. Since the data is natural in $\mathbf{X}$, $\Xi_{\mathbf{X}} : D'\mathbf{X} \rightarrow D''\mathbf{X}$ is a pseudo natural weak equivalence. Introduce weak homotopy equivalences $e'_{q,p} : X_{q,p} \rightarrow \Omega^{p-q} X_{p,p}$, taking $e'_{q,q} = \mathrm{id}$ and inductively letting $e'_{q,p}$ $(q < p)$ be the composite $X_{q,p} \xrightarrow{\sigma_{q,p}} \Omega X_{q+1,p} \xrightarrow{\Omega e'_{q+1,p}} \Omega^{p-q} X_{p,p}$. Call $\omega_{q,p}$ the composite $\Omega^{p-q} X_{p,p} \xrightarrow{\Omega^{p-q}\sigma_{p,p}} \Omega^{p+1-q} X_{p,p+1} \xrightarrow{\top_{p-q}} \Omega^{p+1-q} X_{p,p+1} \xrightarrow{\Omega^{p+1-q}\sigma_{p,p+1}} \Omega^{p+2-q} X_{p+1,p+1}$— then for each q, the $e'_{q,p}$ $(q \leq p)$ specify a morphism $\{X_{q,p} \xrightarrow{\sigma_{q,p}} \Omega X_{q,p+1}\} \rightarrow \{\Omega^{p-q} X_{p,p} \xrightarrow{\omega_{q,p}} \Omega^{p+2-q} X_{p+1,p+1}\}$ of prespectra (use induction on $p - q$) (note the shift in the indexing). Put $e'_{\mathbf{X}} = e'_{0,*}$ and define $e''_{\mathbf{X}}$ analogously.]

COMPARISON THEOREM Suppose given an infinite loop space machine on $\mathbf{\Gamma}$—then $\forall$ Δ-separated proper special $\mathbf{\Gamma}$-space X, $\mathbf{B}X$ is naturally weakly equivalent to $\mathbf{S}X$.

[Note: $\mathbf{S}$ is a functor from the category of Δ-separated proper special $\mathbf{\Gamma}$-spaces to the full subcategory of **PRESPEC** whose objects are the connective Ω-prespectra while $\mathbf{B}$ is a functor from the category of Δ-separated proper special $\mathbf{\Gamma}$-spaces to the full subcategory of **PRESPEC** whose objects are the connective spectra. It is therefore of interest to observe that the proof goes through unchanged if the definition of infinite loop machine is weakened: It suffices that $\mathbf{B}$ take values in the category of connective Ω-prespectra.]

Application: Let $\mathcal{O}$ be an E_∞ operad. Suppose given an infinite loop space machine on $\widehat{\mathcal{O}}$ (e.g., the May machine)—then $\forall$ Δ-separated proper special $\mathbf{\Gamma}$-space X, $\mathbf{B}(\epsilon^* X)(= \mathbf{B}(X \circ \epsilon))$ is naturally weakly equivalent to $\mathbf{S}X$.

FACT Let $\mathcal{O}$ be an E_∞ operad. Suppose given an infinite loop space machine on $\widehat{\mathcal{O}}$—then $\forall$ Δ-separated proper special $\widehat{\mathcal{O}}$-space X, $\mathbf{B}X$ and $\mathbf{S}(\epsilon_* X)$ are naturally weakly equivalent.

[Recalling that $\epsilon_* : \mathbf{ps}\,\widehat{\mathcal{O}}\text{-}\mathbf{SP} \rightarrow \mathbf{ps}\,\mathbf{\Gamma}\text{-}\mathbf{SP}$ respects the Δ-separation condition (cf. p. 14–66), $\mathbf{B}X$ is naturally weakly equivalent to $\mathbf{B}UX$ or still, is naturally weakly equivalent to $\mathbf{B}(\epsilon^*\epsilon_* X)$ which is naturally weakly equivalent to $\mathbf{S}(\epsilon_* X)$.]

Heuristics: The proof of the comparison theorem is complicated by a technicality: The $B_q\overline{X}$ are not necessarily Δ-separated proper special $\mathbf{\Gamma}$-spaces (but are Δ-separated semispecial $\mathbf{\Gamma}$-spaces). However, let us proceed as if they were—then one can form the connective Ω-prespectra $\mathbf{S}B_q\overline{X}$ and there are morphisms $\boldsymbol{\sigma}_q : \mathbf{S}B_q\overline{X} \xrightarrow{\mathbf{S}\boldsymbol{\sigma}_q} \mathbf{S}\Omega B_{q+1}\overline{X} \xrightarrow{\approx} \Omega\mathbf{S}B_{q+1}\overline{X}$ (cf. Proposition 61). Since $\forall$ q, $\boldsymbol{\sigma}_q$ is a weak equivalence, it follows from the up and across theorem that the connective Ω-prespectra $\mathbf{S}B_0\overline{X}(= \{S_q B_0\overline{X}\})$, $S_0\mathbf{B}\overline{X}$ $(= \{S_0 B_q\overline{X}\})$ are naturally weakly equivalent. The idea now is to show that $\mathbf{S}X$ is naturally weakly equivalent to $\mathbf{S}B_0\overline{X}$ and $\mathbf{B}X$ is naturally weakly equivalent to $S_0\mathbf{B}\overline{X}$.

($\mathbf{S}B_0\overline{X}$) $\forall$ n, there are arrows $L\overline{X}_n \rightarrow K\overline{X}_n$, $K\overline{X}_n \rightarrow B_0\overline{X}_n$, i.e., there are $\mathbf{\Gamma}$-maps $L\overline{X} \rightarrow K\overline{X}$, $K\overline{X} \rightarrow B_0\overline{X}$. Because $L\overline{X}_1 \rightarrow K\overline{X}_1$ is a weak homotopy equivalence and $K\overline{X}_1 \rightarrow B_0\overline{X}_1$ is a group completion, the arrow $\mathbf{S}L\overline{X} \rightarrow \mathbf{S}K\overline{X}$ is a weak equivalence, as is the arrow $\mathbf{S}K\overline{X} \rightarrow \mathbf{S}B_0\overline{X}$ (cf. Proposition 59). But $L\overline{X} = X$.

($S_0\mathbf{B}\overline{X}$) The weak group completions $B_qX = B_q\overline{X}_1 \rightarrow \Omega|B_q\overline{X}|_{\mathbf{\Gamma}} = S_0 B_q\overline{X}$ define a morphism $\mathbf{B}X \rightarrow S_0\mathbf{B}X$ of connective Ω-prespectra (cf. Proposition 61) which we claim is a weak equivalence. In fact, $\pi_0(B_0X)$ is a group, thus $B_0X \rightarrow S_0B_0\overline{X}$ is a weak homotopy equivalence (cf. p. 14–72), so Proposition 57 is applicable.

To establish the comparison theorem in full generality, one first has to extend the basic definitions from the context of proper special $\mathbf{\Gamma}$-spaces to that of semiproper semispecial $\mathbf{\Gamma}$-spaces. Thus let X be a semiproper semispecial $\mathbf{\Gamma}$-space—then there is a closed cofibration $X_0 \rightarrow |X|_{\mathbf{\Gamma}}$ and it is best to work with the quotient $|X|_{\overline{\mathbf{\Gamma}}} \equiv |X|_{\mathbf{\Gamma}}/X_0$. Again one has a canonical arrow $\Sigma X_1 \rightarrow |X|_{\overline{\mathbf{\Gamma}}}$ whose adjoint $X_1 \rightarrow \Omega|X|_{\overline{\mathbf{\Gamma}}}$ is a weak group completion. It still makes sense to form $\overline{X}$ and the classifying space BX of X takes $\mathbf{n}$ to $B_nX = |\overline{X}_n|_{\overline{\mathbf{\Gamma}}}$. The definition of $B^{(q)}X$ is as before but $S_0X = \Omega|X|_{\overline{\mathbf{\Gamma}}}$, $S_{q+1}X = |B^{(q)}X|_{\overline{\mathbf{\Gamma}}}$ $(q \geq 0)$.

Turning to the proof of the comparison theorem, let X be a Δ-separated proper special $\mathbf{\Gamma}$-space—then $\forall\ q$, $WB_q\overline{X}$ is a Δ-separated semiproper semispecial $\mathbf{\Gamma}$-space (cf. p. 14–74), $\mathbf{S}WB_q\overline{X}$ is a connective Ω-prespectrum, and there are morphisms $\boldsymbol{\sigma}_q : \mathbf{S}WB_q\overline{X} \xrightarrow{\mathbf{S}W\sigma_q} \mathbf{S}W\Omega B_{q+1}\overline{X} \xrightarrow{\mathbf{S}\omega} \mathbf{S}\Omega WB_{q+1}\overline{X} \xrightarrow{\bullet} \Omega\mathbf{S}WB_{q+1}\overline{X}$ (cf. Proposition 61) (ω as in the lemma on p. 14–76). Since $\forall\ q$, $\boldsymbol{\sigma}_q$ is a weak equivalence, it follows from the up and across theorem that the connective Ω-prespectra $\mathbf{S}WB_0\overline{X}$ $(= \{S_qWB_0\overline{X}\})$, $S_0W\mathbf{B}\overline{X}$ $(= \{S_0WB_q\overline{X}\})$ are naturally weakly equivalent. The idea now is to show that $\mathbf{S}X$ is naturally weakly equivalent to $\mathbf{S}WB_0\overline{X}$ and $\mathbf{B}X$ is naturally weakly equivalent to $S_0W\mathbf{B}\overline{X}$.

($\mathbf{S}WB_0\overline{X}$) There is a natural weak equivalence $\mathbf{S}WX \rightarrow \mathbf{S}X$. On the other hand, there are natural weak equivalences $\mathbf{S}WL\overline{X} \rightarrow \mathbf{S}WK\overline{X}$, $\mathbf{S}WK\overline{X} \rightarrow \mathbf{S}WB_0\overline{X}$ and $L\overline{X} = X$.

($S_0W\mathbf{B}\overline{X}$) Let $\mathbf{WB}X$ be the connective Ω-prespectrum specified by $q \rightarrow W_1B_q\overline{X}$ and $W_1B_q\overline{X} \xrightarrow{W_1\sigma_q} W_1\Omega B_{q+1}\overline{X} \xrightarrow{\omega_1} \Omega W_1B_{q+1}\overline{X}$—then there is a natural weak equivalence $\mathbf{WB}X \rightarrow S_0W\mathbf{B}\overline{X}$. But there is also a pseudo natural weak equivalence $\mathbf{WB}X \rightarrow \mathbf{B}X$, hence $\mathbf{B}X$ is naturally weakly equivalent to $\mathbf{WB}X$ (conversion principle).

LEMMA Let X be a Δ-separated proper special $\mathbf{\Gamma}$-space—then ΣX_1 is homeomorphic to $(|X|_{\mathbf{\Gamma}})_1$, thus the arrow $X_1 \rightarrow \Omega|X|_{\mathbf{\Gamma}}$ is a closed embedding.

Application: Let X be a Δ-separated proper special $\mathbf{\Gamma}$-space—then $\forall\ q$, the arrow $S_qX \rightarrow \Omega S_{q+1}X$ is a closed embedding.

Consequently, if X is a Δ-separated proper special $\mathbf{\Gamma}$-space, then the rule $q \rightarrow \operatorname{colim} \Omega^n S_{n+q}X$ defines a spectrum, call it $e\mathbf{S}X$.

PROPOSITION 62 Suppose given an infinite loop space machine on $\mathbf{\Gamma}$—then $\forall\ \Delta$-separated proper special $\mathbf{\Gamma}$-space X, $\mathbf{B}X$ is naturally weakly equivalent to $e\mathbf{S}X$.

[There is an obvious natural weak equivalence $\mathbf{S}X \rightarrow e\mathbf{S}X$, so the assertion follows from the comparison theorem.]

Remark: It is a fact that **SPEC** carries a model category structure in which the weak equivalences are the levelwise weak homotopy equivalences (cf. §15, Proposition 8). One can therefore interpret Proposition 62 as saying that $\mathbf{B}X$ and $e\mathbf{S}X$ are isomorphic in **HSPEC** (a.k.a. "the" stable homotopy category).

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§15. TRIANGULATED CATEGORIES

Because the theory of triangulated categories lies outside the usual categorical experience, an exposition of the basics seems to be in order. Topologically, the rationale is that the stable homotopy category is triangulated.

Let $\mathbf{C}$ be an additive category—then an additive functor $\Sigma : \mathbf{C} \to \mathbf{C}$ is said to be a suspension functor if it is an equivalence of categories.

[Note: Thus there is also a functor $\Omega : \mathbf{C} \to \mathbf{C}$ which is simultaneously a right and left adjoint for Σ and the four arrows of adjunction $\Sigma \circ \Omega \xrightarrow{\nu} \mathrm{id}_{\mathbf{C}}$, $\mathrm{id}_{\mathbf{C}} \xrightarrow{\mu} \Omega \circ \Sigma$, $\Omega \circ \Sigma \xrightarrow{\mu^{-1}} \mathrm{id}_{\mathbf{C}}$, $\mathrm{id}_{\mathbf{C}} \xrightarrow{\nu^{-1}} \Sigma \circ \Omega$ are natural isomorphisms.]

Let $\mathbf{C}$ be an additive category, Σ a suspension functor—then a triangle in $\mathbf{C}$ consists of objects X, Y, Z and morphisms u, v, w, where $X \xrightarrow{u} Y$, $Y \xrightarrow{v} Z$, $Z \xrightarrow{w} \Sigma X$, a morphism of triangles being a triple (f, g, h) such that the diagram

$$\begin{tikzcd}
X \arrow[r, "u"] \arrow[d, "f"] & Y \arrow[r, "v"] \arrow[d, "g"] & Z \arrow[r, "w"] \arrow[d, "h"] & \Sigma X \arrow[d, "\Sigma f"] \\
X' \arrow[r, "u'"'] & Y' \arrow[r, "v'"'] & Z' \arrow[r, "w'"'] & \Sigma X'
\end{tikzcd}$$

commutes.

Let $\mathbf{C}$ be an additive category—then a triangulation of $\mathbf{C}$ is a pair (Σ, Δ), where Σ is a suspension functor and Δ is a class of triangles (the exact triangles), subject to the following assumptions.

(TR_1) Every triangle isomorphic to an exact triangle is exact.

(TR_2) For any $X \in \mathrm{Ob}\,\mathbf{C}$, the triangle $X \xrightarrow{\mathrm{id}_X} X \to 0 \to \Sigma X$ is exact.

(TR_3) Every morphism $X \xrightarrow{u} Y$ can be completed to an exact triangle $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} \Sigma X$.

(TR_4) The triangle $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} \Sigma X$ is exact iff the triangle $Y \xrightarrow{v} Z \xrightarrow{w} \Sigma X \xrightarrow{-\Sigma u} \Sigma Y$ is exact.

(TR_5) If $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} \Sigma X$, $X' \xrightarrow{u'} Y' \xrightarrow{v'} Z' \xrightarrow{w'} \Sigma X'$ are exact triangles and if in the diagram

$$\begin{tikzcd}
X \arrow[r, "u"] \arrow[d, "f"] & Y \arrow[r, "v"] \arrow[d, "g"] & Z \arrow[r, "w"] & \Sigma X \arrow[d, "\Sigma f"] \\
X' \arrow[r, "u'"'] & Y' \arrow[r, "v'"'] & Z' \arrow[r, "w'"'] & \Sigma X'
\end{tikzcd}$$

, $g \circ u = u' \circ f$, then there is a morphism $h : Z \to Z'$ such that (f, g, h) is a morphism of triangles.

EXAMPLE Suppose that $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} \Sigma X$ is exact. Let $f : X \to X'$, $g : Y \to Y'$, $h : Z \to Z'$ be isomorphisms. Put $u' = g \circ u \circ f^{-1}$, $v' = h \circ v \circ g^{-1}$, $w' = \Sigma f \circ w \circ h^{-1}$—then $X' \xrightarrow{u'} Y' \xrightarrow{v'} Z' \xrightarrow{w'} \Sigma X'$ is exact (cf. TR_1). Examples: (1) $X \xrightarrow{-u} Y \xrightarrow{-v} Z \xrightarrow{w} \Sigma X$ is exact; (2) $Y \xrightarrow{-v} Z \xrightarrow{-w} \Sigma X \xrightarrow{-\Sigma u} \Sigma Y$ is exact (cf. TR_4).

EXAMPLE $\forall X \in \operatorname{Ob}\mathbf{C}$, the triangle $0 \rightarrow X \xrightarrow{\mathrm{id}_X} X \rightarrow 0\ (=\Sigma 0)$ is in Δ (cf. TR_2 & TR_4).

EXAMPLE Suppose that $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} \Sigma X$ is exact—then there is a commutative diagram

$$\begin{tikzcd}
X \arrow[r, "u"] \arrow[d, equal] & Y \arrow[r, "\nu_Z^{-1}\circ v"] \arrow[d, equal] & \Sigma\Omega Z \arrow[r, "w\circ\nu_Z"] \arrow[d, "\nu_Z"] & \Sigma X \arrow[d, equal] \\
X \arrow[r, "u"'] & Y \arrow[r, "v"'] & Z \arrow[r, "w"'] & \Sigma X
\end{tikzcd}$$

, thus the triangle $X \xrightarrow{u} Y \xrightarrow{\nu_Z^{-1}\circ v} \Sigma\Omega Z \xrightarrow{w\circ\nu_Z} \Sigma X$ is exact (cf. TR_1) and so, by TR_4, the triangle $\Omega Z \xrightarrow{-(\mu_X^{-1}\circ\Omega w)} X \xrightarrow{u} Y \xrightarrow{\nu_Z^{-1}\circ v} \Sigma\Omega Z$ is exact.

[Note: Under the bijection of adjunction $\operatorname{Mor}(Z,\Sigma X) \approx \operatorname{Mor}(\Omega Z, X)$, w corresponds to $\mu_X^{-1}\circ\Omega w$ and $\Sigma(\mu_X^{-1}\circ\Omega w)$ equals $w\circ\nu_Z$.]

EXAMPLE Suppose that $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} \Sigma X$ is exact—then there is a commutative diagram

$$\begin{tikzcd}
\Omega Z \arrow[r, "-(\mu_X^{-1}\circ\Omega w)"] \arrow[d, equal] & X \arrow[r, "\nu_Y^{-1}\circ u"] \arrow[d, equal] & \Sigma\Omega Y \arrow[r, "\Sigma\Omega v"] \arrow[d, "\nu_Y"] & \Sigma\Omega Z \arrow[d, equal] \\
\Omega Z \arrow[r, "-(\mu_X^{-1}\circ\Omega w)"'] & X \arrow[r, "u"'] & Y \arrow[r, "\nu_Z^{-1}\circ v"'] & \Sigma\Omega Z
\end{tikzcd}$$

, thus the triangle $\Omega Z \xrightarrow{-(\mu_X^{-1}\circ\Omega w)} X \xrightarrow{\nu_Y^{-1}\circ u} \Sigma\Omega Y \xrightarrow{\Sigma\Omega v} \Sigma\Omega Z$ is exact (cf. TR_1) and so, by TR_4, the triangle $\Omega Y \xrightarrow{-\Omega v} \Omega Z \xrightarrow{-(\mu_X^{-1}\circ\Omega w)} X \xrightarrow{\nu_Y^{-1}\circ u} \Sigma\Omega Y$ is exact or still, the triangle $\Omega Y \xrightarrow{\Omega v} \Omega Z \xrightarrow{\mu_X^{-1}\circ\Omega w} X \xrightarrow{\nu_Y^{-1}\circ u} \Sigma\Omega Y$ is exact.

A triangulated category is an additive category $\mathbf{C}$ equipped with a triangulation (Σ,Δ).

[Note: The opposite of a triangulated category is triangulated. In detail: The suspension functor is Ω^{OP} and the elements of Δ^{OP} are those triangles $X \xrightarrow{u^{\mathrm{OP}}} Y \xrightarrow{v^{\mathrm{OP}}} Z \xrightarrow{w^{\mathrm{OP}}} \Omega^{\mathrm{OP}}X$ in $\mathbf{C}^{\mathrm{OP}}$ such that $\Omega X \xrightarrow{-w} Z \xrightarrow{v} Y \xrightarrow{\nu_X^{-1}\circ u} \Sigma\Omega X$ is exact.]

Example: Let $\mathbf{C}$ be a triangulated category. Call a triangle $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} \Sigma X$ antiexact if the triangle $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{-w} \Sigma X$ is exact—then $\mathbf{C}$ endowed with the class of antiexact triangles is triangulated.

EXAMPLE Let $\mathbf{A}$ be an abelian category. Write $\mathbf{CXA}$ for the abelian category of cochain complexes over $\mathbf{A}$. Let $\Sigma : \mathbf{CXA} \rightarrow \mathbf{CXA}$ be the additive functor that sends X to $X[1]$, where $\begin{cases} X[1]^n = X^{n+1} \\ d^n_{X[1]} = -d^{n+1}_X \end{cases}$—then Σ is an automorphism of $\mathbf{CXA}$, hence is a suspension functor. The quotient category $\mathbf{K(A)}$ of $\mathbf{CXA}$ per cochain homotopy is an additive category and the projection $\mathbf{CXA} \rightarrow \mathbf{K(A)}$ is an additive functor. Moreover, Σ induces a suspension functor $\mathbf{K(A)} \rightarrow \mathbf{K(A)}$. Definition: A triangle $X' \xrightarrow{u'} Y' \xrightarrow{v'} Z' \xrightarrow{w'} \Sigma X'$ in $\mathbf{K(A)}$ is exact if it is isomorphic to a triangle $X \xrightarrow{f} Y \xrightarrow{i} C_f \xrightarrow{\pi} \Sigma X$ for some f. Here

C_f is the mapping cone of $f: C_f^n = X^{n+1} \oplus Y^n$, $d^n_{C_f} = \begin{pmatrix} d^n_{\Sigma X} & 0 \\ f^{n+1} & d^n_Y \end{pmatrix}$ ($j^n = \begin{pmatrix} 0 \\ \mathrm{id}_{Y^n} \end{pmatrix}$, $\pi^n = (\mathrm{id}_{X^{n+1}}, 0)$). With these choices, one can check by direct computation that $\mathbf{K}(\mathbf{A})$ is triangulated (a detailed explanation can be found in Kashiwara-Schapira[†]).

PROPOSITION 1 Let **C** be a triangulated category. Suppose that

$$\begin{tikzcd} X \arrow[r, "u"] & Y \arrow[r, "v"] \arrow[d, "g"] & Z \arrow[r, "w"] \arrow[d, "h"] & \Sigma X \\ X' \arrow[r, "u'"'] & Y' \arrow[r, "v'"'] & Z' \arrow[r, "w'"'] & \Sigma X' \end{tikzcd}$$

is a diagram with rows in Δ. Assume: $h \circ v = v' \circ g$—then there is a morphism $f: X \to X'$ such that (f, g, h) is a morphism of triangles.

[Bearing in mind TR$_4$, pass to

$$\begin{tikzcd} Y \arrow[r, "v"] \arrow[d, "g"] & Z \arrow[r, "w"] \arrow[d, "h"] & \Sigma X \arrow[r, "-\Sigma u"] & \Sigma Y \arrow[d, "\Sigma g"] \\ Y' \arrow[r, "v'"'] & Z' \arrow[r, "w'"'] & \Sigma X' \arrow[r, "-\Sigma u'"'] & \Sigma Y' \end{tikzcd}$$

and apply TR$_5$.]

PROPOSITION 2 Let **C** be a triangulated category. Suppose that

$$\begin{tikzcd} X \arrow[r, "u"] \arrow[d, "f"] & Y \arrow[r, "v"] & Z \arrow[r, "w"] \arrow[d, "h"] & \Sigma X \arrow[d, "\Sigma f"] \\ X' \arrow[r, "u'"'] & Y' \arrow[r, "v'"'] & Z' \arrow[r, "w'"'] & \Sigma X' \end{tikzcd}$$

is a diagram with rows in Δ. Assume: $\Sigma f \circ w = w' \circ h$—then there is a morphism $g: Y \to Y'$ such that (f, g, h) is a morphism of triangles.

PROPOSITION 3 Let **C** be a triangulated category—then for any exact triangle $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} \Sigma X$, $v \circ u = 0$ and $w \circ v = 0$.

[It suffices to prove that $v \circ u = 0$. But the diagram

$$\begin{tikzcd} X \arrow[r, equal] \arrow[d, equal] & X \arrow[r] \arrow[d, "u"] & 0 \arrow[r] \arrow[d, dashed] & \Sigma X \arrow[d, equal] \\ X \arrow[r, "u"'] & Y \arrow[r, "v"'] & Z \arrow[r, "w"'] & \Sigma X \end{tikzcd}$$

must commute (cf. TR$_5$), thus $v \circ u = 0$.]

Application: Every morphism $X \xrightarrow{u} Y$ admits a weak cokernel.

[Thanks to TR$_3$, $\exists$ an exact triangle $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} \Sigma X$ and $v \circ u = 0$. On the other hand, if $g \circ u = 0$ $(g : Y \rightarrow W)$, then the diagram

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[†] *Sheaves on Manifolds*, Springer Verlag (1990), 35–38; see also Weibel, *An Introduction to Homological Algebra*, Cambridge University Press (1994), 376.

$$\begin{tikzcd}
X \arrow[r, "u"] \arrow[d] & Y \arrow[r, "v"] \arrow[d, "g"] & Z \arrow[r, "w"] & \Sigma X \arrow[d] \\
0 \arrow[r] & W \arrow[r, "\mathrm{id}_W"'] & W \arrow[r] & 0
\end{tikzcd}$$

has a filler $h : Z \rightarrow W$ such that $h \circ v = g$ (cf. TR_5).]

Suppose that a triangulated category **C** has coproducts—then **C** has weak pushouts, hence weak colimits. One can be specific. Thus let $\Delta : \mathbf{I} \rightarrow \mathbf{C}$ be a diagram. Given $\delta \in \mathrm{Mor}\,\mathbf{I}$, say $i \xrightarrow{\delta} j$, put $s\delta = i$, $t\delta = j$. Define an arrow $\coprod\limits_{\mathrm{Mor}\,\mathbf{I}} \Delta_{s\delta} \rightarrow \coprod\limits_{\mathrm{Ob}\,\mathbf{I}} \Delta_i$ by taking the coproduct of the arrows $\Delta_{s\delta} \xrightarrow{\begin{pmatrix} \mathrm{id} \\ -\Delta\delta \end{pmatrix}} \Delta_{s\delta} \amalg \Delta_{t\delta}$—then a candidate for a weak colimit of Δ is any completion L of $\coprod\limits_{\mathrm{Mor}\,\mathbf{I}} \Delta_{s\delta} \rightarrow \coprod\limits_{\mathrm{Ob}\,\mathbf{I}} \Delta_i$ to an exact triangle (cf. TR_3).

Let **C** be a triangulated category, **D** an abelian category—then an additive functor (cofunctor) $F : \mathbf{C} \rightarrow \mathbf{D}$ is said to be exact if for every exact triangle $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} \Sigma X$, the sequence $FX \rightarrow FY \rightarrow FZ$ $(FZ \rightarrow FY \rightarrow FX)$ is exact.

[Note: An exact functor (cofunctor) generates a long exact sequence involving Σ and Ω.]

PROPOSITION 4 Let **C** be a triangulated category—then $\forall\, W \in \mathrm{Ob}\,\mathbf{C}$, $\mathrm{Mor}\,(W,—)$ is an exact functor and $\mathrm{Mor}\,(—,W)$ is an exact cofunctor.

[Take any exact triangle $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} \Sigma X$ and consider $\mathrm{Mor}\,(W,X) \xrightarrow{u_*} \mathrm{Mor}\,(W,Y) \xrightarrow{v_*} \mathrm{Mor}\,(W,Z)$. In view of Proposition 3, $\mathrm{im}\, u_* \subset \ker v_*$. To go the other way, assume that $v \circ \psi = 0$ $(\psi \in \mathrm{Mor}\,(W,Y))$—then $\exists\, \phi \in \mathrm{Mor}\,(W,X) : \psi = u \circ \phi$. Proof: Examine

$$\begin{tikzcd}
W \arrow[r, "\mathrm{id}_W"] \arrow[d, dashed, "\phi"] & W \arrow[r] \arrow[d, "\psi"] & 0 \arrow[r] \arrow[d] & \Sigma W \arrow[d, dashed, "\Sigma\phi"] \\
X \arrow[r, "u"'] & Y \arrow[r, "v"'] & Z \arrow[r, "w"'] & \Sigma X
\end{tikzcd}$$

(cf. Proposition 1).]

Application: If

$$\begin{tikzcd}
X \arrow[r, "u"] \arrow[d, "f"] & Y \arrow[r, "v"] \arrow[d, "g"] & Z \arrow[r, "w"] \arrow[d, "h"] & \Sigma X \arrow[d, "\Sigma f"] \\
X' \arrow[r, "u'"'] & Y' \arrow[r, "v'"'] & Z' \arrow[r, "w'"'] & \Sigma X'
\end{tikzcd}$$

is a commutative diagram with rows in Δ and if any two of f, g, h are isomorphisms, then so is the third.

[For instance, suppose that f and g are isomorphisms—then the five lemma implies that $h_* : \mathrm{Mor}\,(Z',Z) \rightarrow \mathrm{Mor}\,(Z',Z')$, $h^* : \mathrm{Mor}\,(Z',Z) \rightarrow \mathrm{Mor}\,(Z,Z)$ are isomorphisms, so $\exists\, \phi, \psi \in \mathrm{Mor}\,(Z',Z) : h \circ \phi = \mathrm{id}_{Z'}$, $\psi \circ h = \mathrm{id}_Z$, i.e., h an isomorphism.]

EXAMPLE Let **C** be a triangulated category with finite coproducts—then $\forall\, X, Y \in \mathrm{Ob}\,\mathbf{C}$, the triangle $X \rightarrow X \amalg Y \rightarrow Y \xrightarrow{0} \Sigma X$ is in Δ.

[According to TR$_3$, the morphism $X \to X \amalg Y$ can be completed to an exact triangle $X \to X \amalg Y \to Z \to \Sigma X$. Compare it with the exact triangle $0 \to Y \xrightarrow{\mathrm{id}_Y} Y \to 0$ to get a filler $h : Z \to Y$ (cf. TR$_5$). Consideration of

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Mor}(W, \Sigma X) \arrow[r] & \mathrm{Mor}(W, \Sigma X \amalg \Sigma Y) \arrow[r] \arrow[d] & \mathrm{Mor}(W, \Sigma Z) \arrow[r] \arrow[d, "(\Sigma h)_*"] & 0 \\
 & & \mathrm{Mor}(W, \Sigma Y) \arrow[r, equal] & \mathrm{Mor}(W, \Sigma Y) &
\end{tikzcd}$$

allows one to say that $(\Sigma h)_*$ is an isomorphism $\forall$ W, hence Σh is an isomorphism or still, h is an isomorphism.]

EXAMPLE Let **C** be a triangulated category with finite coproducts—then any exact triangle of the form $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{0} \Sigma X$ is isomorphic to $X \to X \amalg Z \to Z \xrightarrow{0} \Sigma X$. Indeed, the triangle $Y \xrightarrow{v} Z \xrightarrow{0} \Sigma X \xrightarrow{-\Sigma u} \Sigma Y$ is exact (cf. TR$_4$) and there is a morphism $Y \to X \amalg Z$ rendering the diagram

$$\begin{tikzcd}
Y \arrow[r, "v"] \arrow[d, dashed] & Z \arrow[r, "0"] \arrow[d, equal] & \Sigma X \arrow[r, "-\Sigma u"] \arrow[d, equal] & \Sigma Y \arrow[d, dashed] \\
X \amalg Z \arrow[r] & Z \arrow[r, "0"'] & \Sigma X \arrow[r] & \Sigma X \amalg \Sigma Z
\end{tikzcd}$$

commutative (cf. Proposition 1).

[Note: Analogously, an exact triangle of the form $X \xrightarrow{0} Y \xrightarrow{v} Z \xrightarrow{w} \Sigma X$ is isomorphic to $X \xrightarrow{0} Y \to Y \amalg \Sigma X \to \Sigma X$.]

EXAMPLE Let **C** be a triangulated category with finite coproducts. Suppose given a morphism $i : X \to Y$ that admits a left inverse $r : Y \to X$—then there exists an isomorphism $Y \to X \amalg Z$ and a commutative triangle

$$\begin{tikzcd}
X \arrow[r, "i"] \arrow[dr] & Y \arrow[d] \\
 & X \amalg Z
\end{tikzcd}$$

.

[Complete $X \xrightarrow{i} Y$ to an exact triangle $X \xrightarrow{i} Y \xrightarrow{v} Z \xrightarrow{w} \Sigma X$ (cf. TR$_3$) and choose a filler

$$\begin{tikzcd}
X \arrow[r, "i"] \arrow[d, equal] & Y \arrow[r, "v"] \arrow[d, "r"] & Z \arrow[r, "w"] \arrow[d, dashed] & \Sigma X \arrow[d, equal] \\
X \arrow[r, "\mathrm{id}_X"'] & X \arrow[r] & 0 \arrow[r] & \Sigma X
\end{tikzcd}$$

(cf. TR$_5$) to see that $w = 0$.]

EXAMPLE Let **C** be a triangulated category with finite coproducts—then the triangles $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} \Sigma X$, $X' \xrightarrow{u'} Y' \xrightarrow{v'} Z' \xrightarrow{w'} \Sigma X'$ are exact iff the triangle $X \amalg X' \xrightarrow{\begin{pmatrix} u & 0 \\ 0 & u' \end{pmatrix}} Y \amalg Y' \xrightarrow{\begin{pmatrix} v & 0 \\ 0 & v' \end{pmatrix}} Z \amalg Z' \xrightarrow{\begin{pmatrix} w & 0 \\ 0 & w' \end{pmatrix}} \Sigma X \amalg \Sigma X'$ is exact.

EXAMPLE Let **C** be a triangulated category with finite coproducts. Suppose that $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w}$

ΣX is exact—then for any $Y' \in \mathrm{Ob}\,\mathbf{C}$ and any $g \in \mathrm{Mor}\,(Y, Y')$, the triangle $Y' \amalg Y \xrightarrow{\begin{pmatrix} \mathrm{id}_{Y'} & g \\ 0 & -v \end{pmatrix}} Y' \amalg Z \xrightarrow{(0 - w)} \Sigma X \xrightarrow{\begin{pmatrix} \Sigma(g \circ u) \\ -\Sigma u \end{pmatrix}} \Sigma Y' \amalg \Sigma Y$ is exact.

FACT Let $\mathbf{C}$ be a triangulated category—then a morphism $X \xrightarrow{u} Y$ is an isomorphism iff the triangle $X \xrightarrow{u} Y \rightarrow 0 \rightarrow \Sigma X$ is exact.

FACT Let $\mathbf{C}$ be a triangulated category. Suppose that $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w_i} \Sigma X$ $(i = 1, 2)$ are exact triangles—then $w_1 = w_2$ if $\mathrm{Mor}\,(\Sigma X, Z) = 0$.

PROPOSITION 5 Let $\mathbf{C}$ be a triangulated category. Fix a morphism $X \xrightarrow{u} Y$ in $\mathbf{C}$ and suppose that $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} \Sigma X$, $X \xrightarrow{u} Y \xrightarrow{v'} Z' \xrightarrow{w'} \Sigma X$ are exact triangles (cf. TR_3)—then $Z \approx Z'$.

[Any filler for

$$\begin{tikzcd} X \arrow[r, "u"] \arrow[d, equal] & Y \arrow[r, "v"] \arrow[d, equal] & Z \arrow[r, "w"] \arrow[d, dashed] & \Sigma X \arrow[d, equal] \\ X \arrow[r, "u"'] & Y \arrow[r, "v'"'] & Z' \arrow[r, "w'"'] & \Sigma X \end{tikzcd}$$

is an isomorphism (cf. p. 15–4).]

Let $\mathbf{C}$ be a triangulated category—then a full, isomorphism closed subcategory $\mathbf{D}$ of $\mathbf{C}$ containing 0 and stable under Σ and Ω is said to be a triangulated subcategory of $\mathbf{C}$ if $\forall\, X \xrightarrow{u} Y$ in $\mathrm{Mor}\,\mathbf{D}$, there exists an exact triangle $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} \Sigma X$ with Z in $\mathrm{Ob}\,\mathbf{D}$.

[Note: $\mathbf{D}$ is, in its own right, a triangulated category (the suspension functor is the restriction of Σ to $\mathbf{D}$ and the exact triangles $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} \Sigma X$ are those elements of Δ such that $X, Y, Z \in \mathrm{Ob}\,\mathbf{D}$).]

EXAMPLE Let $\mathbf{A}$ be an abelian category. Write $\mathbf{CXA}^+$ for the full subcategory of $\mathbf{CXA}$ consisting of those X which are bounded below $(X^n = 0\ (n << 0))$, write $\mathbf{CXA}^-$ for the full subcategory of $\mathbf{CXA}$ consisting of those X which are bounded above $(X^n = 0\ (n >> 0))$, and put $\mathbf{CXA}^b = \mathbf{CXA}^+ \cap \mathbf{CXA}^-$—then, in obvious notation, $\mathbf{K}^+(\mathbf{A})$, $\mathbf{K}^-(\mathbf{A})$, and $\mathbf{K}^b(\mathbf{A})$ are triangulated subcategories of $\mathbf{K}(\mathbf{A})$.

PROPOSITION 6 Let $\mathbf{C}$ be a triangulated category. Suppose that O is the object class of a triangulated subcategory of $\mathbf{C}$—then for any exact triangle $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} \Sigma X$, if two of X, Y, Z are in O, so is the third.

[Assuming that $X, Y \in O$, choose $Z' \in O : X \xrightarrow{u} Y \xrightarrow{v'} Z' \xrightarrow{w'} \Sigma X$ is exact. On the basis of Proposition 5, $Z \approx Z'$, hence $Z \in O$ (O is isomorphism closed). Next assume that $Y, Z \in O$ and fix an exact triangle $Y \xrightarrow{v} Z \rightarrow W \rightarrow \Sigma Y$ with $W \in O$. By TR_4,

$Y \xrightarrow{v} Z \xrightarrow{w} \Sigma X \xrightarrow{-\Sigma u} \Sigma Y$ is exact. Therefore $W \approx \Sigma X$ (cf. Proposition 5) $\Rightarrow \Omega W \approx \Omega \Sigma X$. But $\Omega W \in O \Rightarrow \Omega \Sigma X \in O \Rightarrow X \in O$. The argument that $X, Z \in O \Rightarrow Y \in O$ is similar.]

PROPOSITION 7 Let **C** be a triangulated category. Suppose given a nonempty class $O \subset \mathrm{Ob}\,\mathbf{C}$—then O is the object class of a triangulated subcategory of **C** provided that for any exact triangle $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} \Sigma X$, if two of X, Y, Z are in O, so is the third.

[(1) $0 \in O$. Proof: $\forall\, X \in O$, $X \xrightarrow{\mathrm{id}_X} X \rightarrow 0 \rightarrow \Sigma X$ is exact (cf. TR_2). (2) O is isomorphism closed. Proof: If $X \in O$ and if $X \xrightarrow{u} X'$ is an isomorphism, then the triangle $X \xrightarrow{u} X' \rightarrow 0 \rightarrow \Sigma X$ is exact (cf. p. 15–6). (3) $\Sigma O \subset O$. Proof: For any $X \in O$, $X \rightarrow 0 \rightarrow \Sigma X \xrightarrow{-\mathrm{id}_{\Sigma X}} \Sigma X$ is exact (cf. TR_4), thus $\Sigma X \in O$. (4) $\Omega O \subset O$. Proof: For any $X \in O$, $0 \rightarrow X \xrightarrow{\mathrm{id}_X} X \rightarrow 0$ is exact (cf. p. 15–2), hence $\Omega X \rightarrow 0 \rightarrow X \xrightarrow{\nu_X^{-1}} \Sigma \Omega X$ is exact (cf. p. 15–2), thus $\Omega X \in O$. The final requirement that O must satisfy is clear.]

EXAMPLE Let **C** be a triangulated category, **D** an abelian category. Suppose that $F : \mathbf{C} \rightarrow \mathbf{D}$ is an exact functor. Let S_F be the class of morphisms $X \xrightarrow{u} Y$ such that $\forall\, n \geq 0$, $\left\{\begin{array}{l} F\Sigma^n u \\ F\Omega^n u \end{array}\right.$ is an isomorphism and let O_F be the class of objects Z for which there exists an exact triangle $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} \Sigma X$ with $u \in S_F$—then O_F is the object class of a triangulated subcategory of **C**.

[Note: O_F is the class of objects Z such that $\forall\, n \geq 0$, $\left\{\begin{array}{l} F\Sigma^n Z = 0 \\ F\Omega^n Z = 0 \end{array}\right.$.]

EXAMPLE Let **A** be an abelian category with a separator. Suppose that $\mathcal{A}$ is a Serre class in **A**—then $S_{\mathcal{A}}^{-1}\mathbf{A}$ exists (cf. p. 0–39) and the composite $\mathbf{K}(\mathbf{A}) \xrightarrow{H^0} \mathbf{A} \rightarrow S_{\mathcal{A}}^{-1}\mathbf{A}$ is exact, hence determines a triangulated subcategory $\mathbf{K}_{\mathcal{A}}(\mathbf{A})$ of $\mathbf{K}(\mathbf{A})$ whose objects X are characterized by the condition that $H^n(X) \in \mathcal{A}\ \forall\, n$.

Let **C**, **D** be triangulated categories—then an additive functor $F : \mathbf{C} \rightarrow \mathbf{D}$ is said to be a triangulated functor if there exists a natural isomorphism $\Phi : F \circ \Sigma \rightarrow \Sigma \circ F$ such that $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} \Sigma X$ exact $\Rightarrow FX \xrightarrow{Fu} FY \xrightarrow{Fv} FZ \xrightarrow{\Phi_X \circ Fw} \Sigma FX$ exact.

Example: The inclusion functor determined by a triangulated subcategory of a triangulated category is a triangulated functor.

FACT Let **C**, **D** be triangulated categories, $F : \mathbf{C} \rightarrow \mathbf{D}$ a triangulated functor. Assume: $G : \mathbf{D} \rightarrow \mathbf{C}$ is a left adjoint for F—then G is triangulated.

[Note: The same conclusion obtains if G is a right adjoint for F. Proof: G^{OP} is a left adjoint for F^{OP}, hence G^{OP} is triangulated, which implies that G is triangulated.]

Let $\mathbf{C}, \mathbf{D}$ be triangulated categories—then a triangulated functor $F : \mathbf{C} \to \mathbf{D}$ is said to be a triangulated equivalence if there exists a triangulated functor $G : \mathbf{D} \to \mathbf{C}$ and natural isomorphisms $\begin{cases} \mu : \mathrm{id}_{\mathbf{C}} \to G \circ F \\ \nu : F \circ G \to \mathrm{id}_{\mathbf{D}} \end{cases}$ such that the diagrams

$$\begin{tikzcd} GF\Sigma X \arrow[r, "(\Psi F)\circ(G\Phi)"] & \Sigma GFX \\ \Sigma X \arrow[u, "\mu_{\Sigma X}"] \arrow[r, equal] & \Sigma X \arrow[u, "\Sigma\mu_X"'] \end{tikzcd},$$

$$\begin{tikzcd} FG\Sigma Y \arrow[r, "(\Phi G)\circ(F\Psi)"] \arrow[d, "\nu_{\Sigma Y}"'] & \Sigma FGY \arrow[d, "\Sigma\nu_Y"] \\ \Sigma Y \arrow[r, equal] & \Sigma Y \end{tikzcd}$$

commute.

[Note: Φ and Ψ are the natural isomorphisms implicit in the definition of F and G.]

FACT Let $\mathbf{C}, \mathbf{D}$ be triangulated categories, $F : \mathbf{C} \to \mathbf{D}$ an additive functor. Suppose that there exists a natural transformation $\Phi : F \circ \Sigma \to \Sigma \circ F$ such that $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} \Sigma X$ exact $\Rightarrow$ $FX \xrightarrow{Fu} FY \xrightarrow{Fv} FZ \xrightarrow{\Phi_X \circ Fw} \Sigma FX$ exact—then Φ is a natural isomorphism.

[For any $X \in \mathrm{Ob}\,\mathbf{C}$, the triangle $X \to 0 \to \Sigma X \xrightarrow{-\mathrm{id}_{\Sigma X}} \Sigma X$ is exact.]

FACT Let $\mathbf{C}$, $\mathbf{D}$ be triangulated categories, $F : \mathbf{C} \to \mathbf{D}$ a triangulated functor. Assume: F is an equivalence—then F is a triangulated equivalence.

[Given G and natural isomorphisms $\begin{cases} \mu : \mathrm{id}_{\mathbf{C}} \to G \circ F \\ \nu : F \circ G \to \mathrm{id}_{\mathbf{D}} \end{cases}$, consider the inverse of $(G\Sigma\nu) \circ (G\Phi G) \circ (\mu\Sigma G)$.]

Let $\mathbf{C}$ be a triangulated category—then $\mathbf{C}$ is said to be strict if its suspension functor Σ is an isomorphism (and not just an equivalence).

[Note: When $\mathbf{C}$ is strict, the role of Ω is played by Σ^{-1}.]

Example: For any abelian category $\mathbf{A}$, $\mathbf{K}(\mathbf{A})$ is a strict triangulated category.

EXAMPLE Let $\mathbf{C}$ be a strict triangulated category. Suppose that $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} \Sigma X$ is exact—then $\Sigma^{-1} Z \xrightarrow{-\Sigma^{-1} w} X \xrightarrow{u} Y \xrightarrow{v} Z$ is exact (cf. p. 15-2).

Given a triangulated category $\mathbf{C}$, let $\mathbf{ZC}$ be the additive category whose objects are the ordered pairs (n, X) $(n \in \mathbf{Z}, X \in \mathrm{Ob}\,\mathbf{C})$, the morphisms from (n, X) to (m, Y) being $\underset{q \geq n,m}{\mathrm{colim}}\, \mathrm{Mor}(\Sigma^{q-n} X, \Sigma^{q-m} Y)$. Composition in $\mathbf{ZC}$ comes from composition in $\mathbf{C}$: $\Sigma^{q-n} X \to \Sigma^{q-m} Y \to \Sigma^{q-k} Z$. To equip $\mathbf{ZC}$ with the structure of a strict triangulated category, take for the suspension functor the isomorphism $(n, X) \to (n-1, X)$ and take for the exact triangles the $(n, X) \to (m, Y) \to (k, Z) \to (n-1, X)$ associated with the $\Sigma^{q-n} X \xrightarrow{u} \Sigma^{q-m} Y \xrightarrow{v} \Sigma^{q-k} Z \xrightarrow{w} \Sigma\Sigma^{q-n} X$ such that $(u, v, (-1)^q w)$ is exact.

PROPOSITION 8 The functor $F : \mathbf{C} \to \mathbf{ZC}$ that sends X to $(0, X)$ is a triangulated equivalence of categories.

[Note: The natural isomorphism $\Phi : F\circ\Sigma \to \Sigma\circ F$ is defined by letting $\Phi_X : (0, \Sigma X) \to (-1, X)$ be the canonical image of $\mathrm{id}_{\Sigma X}$ in $\mathrm{Mor}((0, \Sigma X), (-1, X))$.]

(Octahedral Axiom) Let $\mathbf{C}$ be a triangulated category. Suppose given exact triangles $X \xrightarrow{u} Y \to Z' \to \Sigma X$, $Y \xrightarrow{v} Z \to X' \to \Sigma Y$, $X \xrightarrow{v\circ u} Z \to Y' \to \Sigma X$—then there exists an exact triangle $Z' \to Y' \to X' \to \Sigma Z'$ such that the diagram

$$\begin{tikzcd}
X \arrow[r, "u"] \arrow[d, equal] & Y \arrow[r] \arrow[d, "v"] & Z' \arrow[r] \arrow[d] & \Sigma X \arrow[d, equal] \\
X \arrow[r, "v\circ u"'] \arrow[d, "u"'] & Z \arrow[r] \arrow[d, equal] & Y' \arrow[r] \arrow[d] & \Sigma X \arrow[d] \\
Y \arrow[r, "v"'] \arrow[d] & Z \arrow[r] \arrow[d] & X' \arrow[r] \arrow[d, equal] & \Sigma Y \arrow[d] \\
Z' \arrow[r] & Y' \arrow[r] & X' \arrow[r] & \Sigma Z'
\end{tikzcd}$$

commutes.

[Note: An explanation for the term "octahedral" is the diagram

$$\begin{tikzcd}
& Y' \arrow[dr, dashed] \arrow[ddl, "\circ" marking] & \\
Z' \arrow[ur, dashed] \arrow[d, "\circ" marking] & & X' \arrow[ll, "\circ" marking] \arrow[ddl, "\circ" marking] \\
X \arrow[rr, "v\circ u"] \arrow[dr, "u"'] & & Z \arrow[u] \arrow[uul] \\
& Y \arrow[ur, "v"'] \arrow[uul] &
\end{tikzcd}$$

Here $U \overset{\circ}{\to} V$ stands for an arrow $U \to \Sigma V$.]

Example: Let $\mathbf{A}$ be an abelian category—then the triangulated category $\mathbf{K}(\mathbf{A})$ satisfies the octahedral axiom.

The stable homotopy category is a triangulated category satisfying the octahedral axiom.

EXAMPLE Let $\mathbf{C}$ be a triangulated category satisfying the octahedral axiom. Suppose that O is the object class of a triangulated subcategory of $\mathbf{C}$ and write S_O for the class of morphisms $X \xrightarrow{u} Y$ which can be completed to an exact triangle $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} \Sigma X$ with Z in O—then S_O admits a calculus of left and right fractions.

[S_O contains the identities of $\mathbf{C}$ ($\forall\, X \in \mathrm{Ob}\,\mathbf{C}$, $X \xrightarrow{\mathrm{id}_X} X \to 0 \to \Sigma X$ is exact and $0 \in O$). To check that S_O is closed under composition, let $X \xrightarrow{u} Y \to Z' \to \Sigma X$ and $Y \xrightarrow{v} Z \to X' \to \Sigma Y$ be exact triangles with $Z', X' \in O$. Choose a completion of $X \xrightarrow{v\circ u} Z$ to an exact triangle $X \xrightarrow{v\circ u} Z \to Y' \to \Sigma X$ (cf. TR$_3$)—then by the octahedral axiom, there exists an exact triangle $Z' \to Y' \to X' \to \Sigma Z'$. Since $Z', X' \in O$, it follows from Proposition 6 that $Y' \in O$. The remaining verifications do not involve the octahedral axiom.]

[Note: S_O contains the isomorphisms of $\mathbf{C}$.]

EXAMPLE Let $\mathbf{C}$ be a triangulated category satisfying the octahedral axiom. Given classes $O_1, O_2 \subset \operatorname{Ob}\mathbf{C}$, denote by $O_1 * O_2$ the class consisting of those X which occur in an exact triangle $X_1 \to X \to X_2 \to \Sigma X_1$ $(X_1 \in O_1, X_2 \in O_2)$—then the octahedral axiom implies that operation $*$ is associative.

[Note: Given a class $O \subset \operatorname{Ob}\mathbf{C}$, an extension of objects of O is an element of $\operatorname{Ext} O = \bigcup\limits_{l \geq 0} O * \cdots * O$ (l factors), the elements of $O * \cdots * O$ being the extensions of objects of O of length l.]

FACT Let $\mathbf{C}$ be a triangulated category with finite coproducts satisfying the octahedral axiom—then, in the notation of TR$_5$, $\exists$ an $h : Z \to Z'$ such that (f, g, h) is a morphism of triangles and the triangle $X' \amalg Y \xrightarrow{\begin{pmatrix} u' & g \\ 0 & -v \end{pmatrix}} Y' \amalg Z \xrightarrow{\begin{pmatrix} v' & h \\ 0 & -w \end{pmatrix}} Z' \amalg \Sigma X \xrightarrow{\begin{pmatrix} w' & \Sigma f \\ 0 & -\Sigma u \end{pmatrix}} \Sigma X' \amalg \Sigma Y$ is exact.

PROPOSITION 9 Let $\mathbf{C}$ be a triangulated category satisfying the octahedral axiom—then every commutative square

$$\begin{tikzcd}
X \arrow[r] \arrow[d] & Y \arrow[d] \\
X' \arrow[r] & Y'
\end{tikzcd}$$

can be completed to a diagram

$$\begin{tikzcd}
X \arrow[r] \arrow[d] & Y \arrow[r] \arrow[d] & Z \arrow[r] \arrow[d] & \Sigma X \arrow[d] \\
X' \arrow[r] \arrow[d] & Y' \arrow[r] \arrow[d] & Z' \arrow[r] \arrow[d] & \Sigma X' \arrow[d] \\
X'' \arrow[r] \arrow[d] & Y'' \arrow[r] \arrow[d] & Z'' \arrow[r] \arrow[d] \arrow[dr, phantom, "-"] & \Sigma X'' \arrow[d] \\
\Sigma X \arrow[r] & \Sigma Y \arrow[r] & \Sigma Z \arrow[r] & \Sigma^2 X
\end{tikzcd}$$

in which the first three rows and the first three columns are exact and all the squares commute except for the one marked with a minus sign which anticommutes.

EXAMPLE Let $\mathbf{C}$ be a triangulated category satisfying the octahedral axiom. Suppose that O is the object class of a triangulated subcategory of $\mathbf{C}$. Let $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} \Sigma X$, $X' \xrightarrow{u'} Y' \xrightarrow{v'} Z' \xrightarrow{w'} \Sigma X'$ be exact triangles. Assume: There is a diagram

$$\begin{tikzcd}
X \arrow[r, "u"] \arrow[d, "f"] & Y \arrow[r, "v"] \arrow[d, "g"] & Z \arrow[r, "w"] & \Sigma X \arrow[d, "\Sigma f"] \\
X' \arrow[r, "u'"'] & Y' \arrow[r, "v'"'] & Z' \arrow[r, "w'"'] & \Sigma X'
\end{tikzcd}$$

where $f, g \in S_O$ and $g \circ u = u' \circ f$—then $\exists$ an $h : Z \to Z'$ in S_O such that (f, g, h) is a morphism of triangles.

[Note: The metacategory $S_O^{-1}\mathbf{C}$ is triangulated and satisfies the octahedral axiom. For instance, consider $\mathbf{K}(\mathbf{A})$, where $\mathbf{A}$ is an abelian category. Let $O = \{X : H^n(X) = 0 \ \forall \ n\}$—then S_O is the class of quasiisomorphisms of $\mathbf{A}$ (i.e., the f such that $H^n(f)$ is an isomorphism $\forall \ n$ or, equivalently, the f such that $H^n(C_f) = 0 \ \forall \ n$) and the derived category $\mathbf{D}(\mathbf{A})$ of $\mathbf{A}$ is the localization $S_O^{-1}\mathbf{K}(\mathbf{A})$. But there is a problem with the terminology. Reason: A priori, $\mathbf{D}(\mathbf{A})$ is only a metacategory. However, the assumption that $\mathbf{A}$ is Grothendieck and has a separator suffices to ensure that $\mathbf{D}(\mathbf{A})$ is a category (Weibel[†]). One can also form $\mathbf{D}^+(\mathbf{A})$, $\mathbf{D}^-(\mathbf{A})$, and $\mathbf{D}^b(\mathbf{A})$. Here $\mathbf{D}^+(\mathbf{A})$ will be a category if $\mathbf{A}$ has enough injectives and $\mathbf{D}^-(\mathbf{A})$ will be a category if $\mathbf{A}$ has enough projectives.]

The derived category $\mathbf{D}(\mathbf{A})$ of Freyd's[‡] "large" abelian category $\mathbf{A}$ is not isomorphic to a category, hence exists only as a metacategory. Therefore one cannot find a model category structure on $\mathbf{A}$ whose weak equivalences are the quasiisomorphisms (cf. p. 12–32).

Let $\mathbf{C}$ be a triangulated category—then a subcategory $\mathbf{D}$ of $\mathbf{C}$ is said to be thick provided that it is triangulated and for any pair of morphisms $i : X \rightarrow Y$, $r : Y \rightarrow X$ with $r \circ i = \mathrm{id}_X$, $Y \in \operatorname{Ob}\mathbf{D} \Rightarrow X \in \operatorname{Ob}\mathbf{D}$.

PROPOSITION 10 Let $\mathbf{C}$ be a triangulated category with finite coproducts—then a triangulated subcategory $\mathbf{D}$ of $\mathbf{C}$ is thick iff every object of $\mathbf{C}$ which is a direct summand of an object of $\mathbf{D}$ is itself an object of $\mathbf{D}$, i.e., $Y \in \operatorname{Ob}\mathbf{D}$ & $Y \approx X \amalg Z \Rightarrow X \in \operatorname{Ob}\mathbf{D}$.

[Necessity: Since $\mathbf{D}$ is isomorphism closed, $X \amalg Z \in \operatorname{Ob}\mathbf{D}$, so one only has to consider $X \xrightarrow{\mathrm{in}_X} X \amalg Z \xrightarrow{\mathrm{pr}_X} X$.

Sufficiency: There exists an isomorphism $Y \rightarrow X \amalg Z$ and a commutative diagram

$$\begin{tikzcd} X \arrow[r, "i"] \arrow[dr] & Y \arrow[d] \\ & X \amalg Z \end{tikzcd}$$

(cf. p. 15–5), hence $X \in \operatorname{Ob}\mathbf{D}$.]

PROPOSITION 11 Let $\mathbf{C}$ be a triangulated category with finite coproducts satisfying the octahedral axiom—then a triangulated subcategory $\mathbf{D}$ of $\mathbf{C}$ is thick iff every morphism $X \xrightarrow{u} Y$ in $\mathbf{C}$ admitting a factorization

$$\begin{tikzcd} X \arrow[rr, "u"] \arrow[dr, "\phi"'] & & Y \\ & W \arrow[ur, "\psi"'] & \end{tikzcd}$$

through an object W of $\mathbf{D}$ and contained in an exact triangle $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} \Sigma X$, where $Z \in \operatorname{Ob}\mathbf{D}$, is a morphism in $\mathbf{D}$, i.e., $X, Y \in \operatorname{Ob}\mathbf{D}$.

[†] *An Introduction to Homological Algebra*, Cambridge University Press (1994), 386–387.

[‡] *Abelian Categories*, Harper & Row (1964), 131–132.

[Necessity: Complete $X \xrightarrow{\phi} W$ to an exact triangle $X \xrightarrow{\phi} W \xrightarrow{\omega} W' \xrightarrow{\omega'} \Sigma X$ (cf. TR$_3$)—then the triangle $Y \amalg W \xrightarrow{\begin{pmatrix} \mathrm{id}_Y & \psi \\ 0 & -\omega \end{pmatrix}} Y \amalg W' \xrightarrow{(0 - \omega')} \Sigma X \xrightarrow{\begin{pmatrix} \Sigma u \\ -\Sigma\phi \end{pmatrix}} \Sigma Y \amalg \Sigma W$ is exact (cf. p. 15–5 ff.), thus the triangle $X \xrightarrow{\begin{pmatrix} -u \\ \phi \end{pmatrix}} Y \amalg W \xrightarrow{\begin{pmatrix} \mathrm{id}_Y & \psi \\ 0 & -\omega \end{pmatrix}} Y \amalg W' \xrightarrow{(0 - \omega')} \Sigma X$ is exact (cf. TR$_4$). On the other hand, the triangle $Y \amalg W \rightarrow Y \xrightarrow{0} \Sigma W \rightarrow \Sigma Y \amalg \Sigma W$ is exact (cf. p. 15–5), as is the triangle $X \xrightarrow{-u} Y \xrightarrow{-v} Z \xrightarrow{w} \Sigma X$ (cf. p. 15–1). So, in the notation of the octahedral axiom, taking $Z' = Y \amalg W'$, $X' = \Sigma W$, and $Y' = Z$, one concludes that there is an exact triangle $Y \amalg W' \rightarrow Z \rightarrow \Sigma W \rightarrow \Sigma Y \amalg \Sigma W'$. But $Z, \Sigma W \in \mathrm{Ob}\,\mathbf{D} \Rightarrow Y \amalg W' \in \mathrm{Ob}\,\mathbf{D}$ (cf. Proposition 6) $\Rightarrow Y \in \mathrm{Ob}\,\mathbf{D}$ (cf. Proposition 10) $\Rightarrow X \in \mathrm{Ob}\,\mathbf{D}$ (cf. Proposition 6).

Sufficiency: Suppose that $Y \in \mathrm{Ob}\,\mathbf{D}$ & $Y \approx X \amalg Z$—then the triangle $X \rightarrow X \amalg Z \rightarrow Z \xrightarrow{0} \Sigma X$ is exact (cf. p. 15–5), thus the triangle $\Omega Z \xrightarrow{0} X \rightarrow X \amalg Z \rightarrow \Sigma\Omega Z$ is exact (cf. p. 15–2). But $0 \in \mathrm{Ob}\,\mathbf{D}$ and there is a factorization

$$\begin{tikzcd} \Omega Z \arrow[rr, "0"] \arrow[dr] & & X \\ & 0 \arrow[ur] & \end{tikzcd}$$

. Our assumption implies that $X \in \mathrm{Ob}\,\mathbf{D}$, so $\mathbf{D}$ is thick (cf. Proposition 10).]

FACT Let $\mathbf{C}$ be a triangulated category with finite coproducts satisfying the octahedral axiom. Suppose that O is the object class of a thick subcategory of $\mathbf{C}$—then $u \in S_O$ iff $\exists\, f, g \in \mathrm{Mor}\,\mathbf{C} : u \circ f \in S_O$, $g \circ u \in S_O$.

[Complete $X \xrightarrow{u} Y$ to an exact triangle $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} \Sigma X$ (cf. TR$_3$), the claim being that $Z \in O$. By hypothesis, there are exact triangles $X' \xrightarrow{u \circ f} Y \rightarrow Z_f \rightarrow \Sigma X'$, $X \xrightarrow{g \circ u} Y' \rightarrow Z_g \rightarrow \Sigma X$, where Z_f, $Z_g \in O$. Since $v \circ (u \circ f) = (v \circ u) \circ f = 0$ (cf. Proposition 3) and $\mathrm{Mor}\,(Z_f, Z) \rightarrow \mathrm{Mor}\,(Y, Z) \rightarrow \mathrm{Mor}\,(X', Z)$ is exact (cf. Proposition 4), $\exists$ a factorization

$$\begin{tikzcd} Y \arrow[rr, "v"] \arrow[dr] & & Z \\ & Z_f \arrow[ur] & \end{tikzcd}$$

. Complete $Y \xrightarrow{g} Y'$ to an exact triangle $Y \xrightarrow{g} Y' \rightarrow W \rightarrow \Sigma Y$ (cf. TR$_3$) and use the octahedral axiom on $X \xrightarrow{u} Y \rightarrow Z \rightarrow \Sigma X$, $Y \xrightarrow{g} Y' \rightarrow W \rightarrow \Sigma Y$, $X \xrightarrow{g \circ u} Y' \rightarrow Z_g \rightarrow \Sigma X$ to get an exact triangle $Z \rightarrow Z_g \rightarrow W \rightarrow \Sigma Z$ or still, an exact triangle $W \rightarrow \Sigma Z \rightarrow \Sigma Z_g \rightarrow \Sigma W$. From the above, the arrow $W \rightarrow \Sigma Z$ factors through $\Sigma Z_f \in O$. But also $\Sigma Z_g \in O$, thus, as O is thick, $\Sigma Z \in O$ (cf. Proposition 11), i.e., $Z \in O$.]

[Note: The condition implies that S_O is saturated: $S_O = \overline{S}_O$ (cf. p. 0–30), hence $X \in O$ iff $L_{S_O} X$ is a zero object.]

Given a triangulated category $\mathbf{C}$, call a class $S \subset \mathrm{Mor}\,\mathbf{C}$ multiplicative if (1) S admits a calculus of left and right fractions and contains the isomorphisms of $\mathbf{C}$; (2) $u \in S \Rightarrow \Sigma u$ & $\Omega u \in S$; (3) $f, g \in S \Rightarrow \exists\, h \in S$ (data as in TR$_5$); (4) $u \in S$ iff $\exists\, f, g \in \mathrm{Mor}\,\mathbf{C} : u \circ f \in S$, $g \circ u \in S$.

Example: Let **C** be a triangulated category with finite coproducts satisfying the octahedral axiom—then S_O is multiplicative provided that O is the object class of a thick subcategory of **C**. In fact, the assignment $O \to S_O$ establishes a one-to-one correspondence between the object classes of thick subcategories of **C** and the multiplicative classes of morphisms of **C**.

[Note: To place this conclusion in perspective, recall that in an abelian category there is a one-to-one correspondence between the Serre classes and the saturated morphism classes which admit a calculus of left and right fractions (Schubert[†]).]

PROPOSITION 12 Let **C** be a triangulated category. Assume: **C** has coproducts—then for any collection $\{X_i \to Y_i \to Z_i \to \Sigma X_i\}$ of exact triangles, the triangle $\coprod_i X_i \to \coprod_i Y_i \to \coprod_i Z_i \to \coprod_i \Sigma X_i$ is exact.

[Note: The suspension functor preserves coproducts, so $\Sigma \coprod_i X_i \approx \coprod_i \Sigma X_i$.]

Let **C** be a triangulated category with coproducts—then an $X \in \mathrm{Ob}\,\mathbf{C}$ is said to be compact if $\forall$ collection $\{X_i\}$ of objects in **C**, the arrow $\bigoplus_i \mathrm{Mor}(X, X_i) \to \mathrm{Mor}(X, \coprod_i X_i)$ is an isomorphism.

[Note: X compact $\Rightarrow \Sigma X$ & ΩX compact.]

EXAMPLE Let A be a commutative ring with unit—then the compact objects in $\mathbf{D}(A\text{-}\mathbf{MOD})$ are those objects which are isomorphic to bounded complexes of finitely generated projective A-modules (Bökstedt-Neeman[‡]).

FACT If **C** is a triangulated category with coproducts, then the class of compact objects in **C** is the object class of a thick subcategory of **C**.

Notation: Let **C** be a triangulated category with coproducts. Suppose given an object $(\mathbf{X}, \mathbf{f})$ in $\mathbf{FIL}(\mathbf{C})$—then $\mathrm{tel}(\mathbf{X}, \mathbf{f})$ is any completion of $\coprod_n X_n \xrightarrow{\mathrm{sf}} \coprod_n X_n$ to an exact triangle (cf. TR$_3$), the nth component of sf being the arrow $X_n \xrightarrow{\begin{pmatrix} \mathrm{id} \\ -f_n \end{pmatrix}} X_n \coprod X_{n+1}$.

PROPOSITION 13 Let **C** be a triangulated category with coproducts. Fix an $(\mathbf{X}, \mathbf{f})$ in $\mathbf{FIL}(\mathbf{C})$—then $\forall$ compact X, the arrow $\mathrm{colim}\, \mathrm{Mor}(X, X_n) \to \mathrm{Mor}(X, \mathrm{tel}(\mathbf{X}, \mathbf{f}))$ is an isomorphism.

[†] *Categories*, Springer Verlag (1972), 276.

[‡] *Compositio Math.* **86** (1993), 209–234.

[First consider the exact sequence $\operatorname{Mor}(X, \coprod_n X_n) \xrightarrow{\Phi} \operatorname{Mor}(X, \operatorname{tel}(\mathbf{X}, \mathbf{f})) \rightarrow \operatorname{Mor}(X, \coprod_n \Sigma X_n) \rightarrow \operatorname{Mor}(X, \coprod_n \Sigma X_n)$ (cf. Proposition 4). Due to the compactness of X, in the commutative diagram

$$\begin{tikzcd}
\bigoplus_n \operatorname{Mor}(X, \Sigma X_n) \arrow[r] \arrow[d] & \bigoplus_n \operatorname{Mor}(X, \Sigma X_n) \arrow[d] \\
\operatorname{Mor}(X, \coprod_n \Sigma X_n) \arrow[r] & \operatorname{Mor}(X, \coprod_n \Sigma X_n)
\end{tikzcd}$$

, the vertical arrows are isomorphisms. Because the horizontal arrow on the top is injective, the same holds for the horizontal arrow on the bottom. Therefore Φ is surjective. Now write down the commutative diagram

$$\begin{tikzcd}
\bigoplus_n \operatorname{Mor}(X, X_n) \arrow[r, "\phi"] \arrow[d] & \bigoplus_n \operatorname{Mor}(X, X_n) \arrow[d] & & \\
\operatorname{Mor}(X, \coprod_n X_n) \arrow[r] & \operatorname{Mor}(X, \coprod_n X_n) \arrow[r, "\Phi"] & \operatorname{Mor}(X, \operatorname{tel}(\mathbf{X}, \mathbf{f})) \arrow[r] & 0
\end{tikzcd}$$

and observe that $\operatorname{colim} \operatorname{Mor}(X, X_n)$ can be identified with the cokernel of ϕ.]

FACT Let $\mathbf{C}$ be a triangulated category with coproducts. Fix an $(\mathbf{X}, \mathbf{f})$ in $\mathbf{FIL}(\mathbf{C})$—then $\forall\, Y$, there is an exact sequence $0 \rightarrow \lim^1 \operatorname{Mor}(\Sigma X_n, Y) \rightarrow \operatorname{Mor}(\operatorname{tel}(\mathbf{X}, \mathbf{f}), Y) \rightarrow \lim \operatorname{Mor}(X_n, Y) \rightarrow 0$.

A triangulated category $\mathbf{C}$ is said to be compactly generated if it has coproducts and $\operatorname{Ob} \mathbf{C}$ contains a set $\mathcal{U} = \{U\}$ of compact objects such that $\operatorname{Mor}(U, X) = 0\ \forall\, U \in \mathcal{U} \Rightarrow X = 0$.

[Note: The closure $\overline{\mathcal{U}} = \{\overline{U}\}$ of $\mathcal{U}$ is the set $\bigcup_U \{\Sigma^n U : n \geq 0\} \cup \bigcup_U \{\Omega^n U : n \geq 0\}$.]

The stable homotopy category is a compactly generated triangulated category.

EXAMPLE Let X be a scheme, $\mathcal{O}_X$ its structure sheaf. Denote by $\mathcal{O}_X$-**MOD** the category of $\mathcal{O}_X$-modules and write $\mathbf{QC}/X$ for the full subcategory whose objects are quasicoherent—then $\mathcal{O}_X$-**MOD** and $\mathbf{QC}/X$ are abelian categories and the inclusion $\mathbf{QC}/X \rightarrow \mathcal{O}_X$-**MOD** is exact. In addition, $\mathcal{O}_X$-**MOD** is Grothendieck and has a separator, thus the derived category $\mathbf{D}(\mathcal{O}_X\text{-}\mathbf{MOD})$ exists. When X is quasicompact (= compact) and separated, $\mathbf{QC}/X$ is Grothendieck and has a separator, thus in this situation, the derived category $\mathbf{D}(\mathbf{QC}/X)$ also exists. Moreover, $\mathbf{D}(\mathbf{QC}/X)$ is compactly generated, the compact objects being those objects which are isomorphic to perfect complexes (Neeman[†]).

BROWN REPRESENTABILITY THEOREM Let $\mathbf{C}$ be a compactly generated triangulated category—then an exact cofunctor $F : \mathbf{C} \rightarrow \mathbf{AB}$ is representable iff it converts coproducts into products.

[†] *J. Amer. Math. Soc.* **9** (1996), 205–236.

[The condition is clearly necessary and the proof of sufficiency is a variation on the argument used in Proposition 27 of §5. Thus setting $X_0 = \coprod_{\overline{U}} F\overline{U} \cdot \overline{U}$, one has $FX_0 = \prod_{\overline{U}} (F\overline{U})^{F\overline{U}}$. Call ξ_0 that element of the product defined by $\xi_{0,\overline{U}} = \mathrm{id}_{F\overline{U}}\ \forall\ \overline{U}$ and let $\Xi_0 : \mathrm{Mor}\,(—, X_0) \to F$ be the natural transformation associated with ξ_0 via Yoneda. Note that $\Xi_{0,\overline{U}} : \mathrm{Mor}\,(\overline{U}, X_0) \to F\overline{U}$ is surjective $\forall\ \overline{U}$. Proceeding inductively, we shall construct an object $(\mathbf{X}, \mathbf{f})$ in $\mathbf{FIL(C)}$ and natural transformations $\Xi_n : \mathrm{Mor}\,(—, X_n) \to F$ such that $\forall\ n$, the triangle

$$\begin{tikzcd}
\mathrm{Mor}\,(—, X_n) \arrow[d] \arrow[dr, "\Xi_n"] & \\
\mathrm{Mor}\,(—, X_{n+1}) \arrow[r, "\Xi_{n+1}"'] & F
\end{tikzcd}$$

commutes. To this end, put $K_n = \coprod_{\overline{U}} (\ker \Xi_{n,\overline{U}}) \cdot \overline{U}$ and complete the canonical arrow $K_n \to X_n$ to an exact triangle $K_n \to X_n \xrightarrow{f_n} X_{n+1} \to \Sigma K_n$ (cf. TR$_3$). If $\xi_n \in FX_n$ corresponds to Ξ_n, then $\xi_n \in \ker(FX_n \to FK_n)$ and since the sequence $FX_{n+1} \to FX_n \to FK_n$ is exact, $\exists\ \xi_{n+1} \in FX_{n+1} : \xi_{n+1} \to \xi_n$. Definition: $\Xi_{n+1} \leftrightarrow \xi_{n+1}$, which finishes the induction. Abbreviating $\mathrm{tel}(\mathbf{X}, \mathbf{f})$ to X_ω, there is a natural transformation $\Xi_\omega : \mathrm{Mor}\,(—, X_\omega) \to F$ rendering the triangle

$$\begin{tikzcd}
\mathrm{Mor}\,(—, X_n) \arrow[d] \arrow[dr, "\Xi_n"] & \\
\mathrm{Mor}\,(—, X_\omega) \arrow[r, "\Xi_\omega"'] & F
\end{tikzcd}$$

commutative $\forall\ n$. Proof: Consider the diagram

$$\begin{tikzcd}
FX_\omega \arrow[r] & F(\coprod_n X_n) \arrow[r, "F\mathrm{sf}"] \arrow[d, equal] & F(\coprod_n X_n) \arrow[d, equal] \\
 & \prod_n FX_n \arrow[r] & \prod_n FX_n
\end{tikzcd}.$$

Because $\prod_n \xi_n$ lies in the kernel of $\prod_n FX_n \to \prod_n FX_n$, exactness gives a $\xi_\omega \in FX_\omega : \xi_\omega \to \prod_n \xi_n$, hence $\Xi_\omega \leftrightarrow \xi_\omega$ has the stated property. The final step is to establish that $\Xi_{\omega,X} : \mathrm{Mor}\,(X, X_\omega) \to FX$ is bijective $\forall\ X$. But it is certainly true that $\Xi_{\omega,\overline{U}}$ is bijective $\forall\ \overline{U}$ (injectivity follows from the construction of X_ω (cf. Proposition 13)) while $\mathrm{Mor}\,(\overline{U}, X_0) \to F\overline{U}$ surjective $\Rightarrow \mathrm{Mor}\,(\overline{U}, X_\omega) \to F\overline{U}$ surjective) and this turns out to be enough (cf. infra).]

The assumption that $\mathrm{Mor}\,(U, X) = 0\ \forall\ U \in \mathcal{U} \Rightarrow X = 0$ has yet to be employed. To do so, let $\mathbf{C}_F$ be the full, isomorphism closed subcategory of $\mathbf{C}$ whose objects are those X such that $\Xi_{\omega,\Sigma^n X} : \mathrm{Mor}\,(\Sigma^n X, X_\omega) \to F\Sigma^n X$ is bijective $\forall\ n \geq 0$ and $\Xi_{\omega,\Omega^n X} : \mathrm{Mor}\,(\Omega^n X, X_\omega) \to F\Omega^n X$ is bijective $\forall\ n \geq 0$. Obviously, $\mathbf{C}_F$ contains 0 and $\mathcal{U}$.

Claim: $\mathbf{C}_F$ is stable under Σ & Ω.

[To check stability under Σ, fix an $X \in \mathrm{Ob}\,\mathbf{C}_F$—then $\forall\ n \geq 0$, $\mathrm{Mor}\,(\Sigma^n \Sigma X, X_\omega) = \mathrm{Mor}\,(\Sigma^{n+1} X, X_\omega) \approx F\Sigma^{n+1} X = F\Sigma^n \Sigma X$. On the other hand, the arrow of adjunction $X \to \Omega\Sigma X$ is an isomorphism, thus one sees inductively from the commutative diagram

$$\begin{tikzcd}
\text{Mor}(\Omega^n\Sigma X, X_\omega) \arrow[r] \arrow[d] & \text{Mor}(\Omega^{n-1}X, X_\omega) \arrow[d] \\
F\Omega^n\Sigma X \arrow[r] & F\Omega^{n-1}X
\end{tikzcd}$$

that $\text{Mor}(\Omega^n\Sigma X, X_\omega) \approx F\Omega^n\Sigma X$. Therefore $\Sigma X \in \mathbf{C}_F$.]

Claim: If $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} \Sigma X$ is an exact triangle with $X, Y \in \text{Ob}\,\mathbf{C}_F$, then $Z \in \text{Ob}\,\mathbf{C}_F$.

[Use the five lemma.]

Claim: $\mathbf{C}_F$ is closed under the formation of coproducts in $\mathbf{C}$.

[E.g.: $\text{Mor}(\Sigma^n \coprod_i X_i, X_\omega) \approx \text{Mor}(\coprod_i X_i, \Omega^n X_\omega) \approx \prod_i \text{Mor}(X_i, \Omega^n X_\omega) \approx \prod_i \text{Mor}(\Sigma^n X_i, X_\omega) \approx \prod_i F\Sigma^n X_i \approx F(\coprod_i \Sigma^n X_i) \approx F(\Sigma^n \coprod_i X_i)$.]

In summary, $\mathbf{C}_F$ is a triangulated subcategory of $\mathbf{C}$ containing $\mathcal{U}$ and closed under the formation of coproducts in $\mathbf{C}$. To conclude that $\Xi_{\omega,X} : \text{Mor}(X, X_\omega) \to FX$ is bijective $\forall\, X$, it need only be shown that $\mathbf{C}_F = \mathbf{C}$, which is a special case of the following result.

PROPOSITION 14 Let $\mathbf{C}$ be a compactly generated triangulated category. Suppose that $\mathbf{D}$ is a triangulated subcategory of $\mathbf{C}$ containing $\mathcal{U}$ and closed under the formation of coproducts in $\mathbf{C}$—then $\mathbf{D} = \mathbf{C}$.

[Let $\overline{\mathbf{D}}$ be the smallest triangulated subcategory of $\mathbf{C}$ containing $\mathcal{U}$ and closed under the formation of coproducts in $\mathbf{C}$. Fix an X in $\mathbf{C}$—then the restriction of $\text{Mor}(—, X)$ to $\overline{\mathbf{D}}$ is an exact cofunctor. Applying what has been proved above about Brown representability to $\text{Mor}(—, X)$, one concludes that there exists an $\overline{X}_\omega$ in $\overline{\mathbf{D}}$ and a natural isomorphism $\text{Mor}(—, \overline{X}_\omega) \to \text{Mor}(—, X)$ (the minimality of $\overline{\mathbf{D}}$ enters the picture at this point). Accordingly, $\exists$ a morphism $\overline{X}_\omega \to X$ such that $\forall\, \overline{X}$ in $\overline{\mathbf{D}}$, the arrow $\text{Mor}(\overline{X}, \overline{X}_\omega) \to \text{Mor}(\overline{X}, X)$ is bijective. Complete $\overline{X}_\omega \to X$ to an exact triangle $\overline{X}_\omega \to X \to Y \to \Sigma\overline{X}_\omega$ in $\mathbf{C}$ (cf. TR_3)—then $\forall\, \overline{X}$ in $\overline{\mathbf{D}}$, $\text{Mor}(\overline{X}, Y) = 0 \Rightarrow \forall\, U \in \mathcal{U}$, $\text{Mor}(U, Y) = 0 \Rightarrow Y = 0$. Consequently, the morphism $\overline{X}_\omega \to X$ is an isomorphism (cf. p. 15–6), so $X \in \text{Ob}\,\overline{\mathbf{D}}$ ($\overline{\mathbf{D}}$ is isomorphism closed), hence $\overline{\mathbf{D}} = \mathbf{C} \Rightarrow \mathbf{D} = \mathbf{C}$.]

Application: Let $\mathbf{C}$ be a compactly generated triangulated category. Suppose that $\Xi : \text{Mor}(—, Y) \to \text{Mor}(—, Z)$ is a natural transformation such that $\forall\, \overline{U} \in \overline{\mathcal{U}}$, $\Xi_{\overline{U}}$ is bijective—then for all X in $\mathbf{C}$, $\Xi_X : \text{Mor}(X, Y) \to \text{Mor}(X, Z)$ is bijective.

[Note: If $\Xi_f : \text{Mor}(—, Y) \to \text{Mor}(—, Z)$ is the natural transformation corresponding to $f : Y \to Z$, then f is an isomorphism whenever $\Xi_{f,\overline{U}}$ is bijective $\forall\, \overline{U} \in \overline{\mathcal{U}}$.]

Example: Suppose that $\mathbf{C}$ is a compactly generated triangulated category. Let $\Delta : \mathbf{I} \to \mathbf{C}$ be a diagram—then a weak colimit L of Δ is said to be a minimal weak colimit provided that $\forall\, \overline{U} \in \overline{\mathcal{U}}$, colim $\mathrm{Mor}(\overline{U}, \Delta_i) \approx \mathrm{Mor}(\overline{U}, L)$. If L is a minimal weak colimit of Δ and if K is an arbitrary weak colimit of Δ, there are arrows $L \xrightarrow{\phi} K \xrightarrow{\psi} L$ and $\forall\, \overline{U} \in \overline{\mathcal{U}}$, $\Xi_{\psi\circ\phi,\overline{U}} : \mathrm{Mor}(\overline{U}, L) \to \mathrm{Mor}(\overline{U}, L)$ is bijective, thus by the above, $\psi \circ \phi$ is an isomorphism. Corollary: L is a direct summand of K (cf. p. 15–5).

[Note: L & K minimal $\Rightarrow L \approx K$. Example: $\forall\, (\mathbf{X}, \mathbf{f})$ in $\mathbf{FIL}(\mathbf{C})$, $\mathrm{tel}(\mathbf{X}, \mathbf{f})$ is a minimal weak colimit of $(\mathbf{X}, \mathbf{f})$ (cf. Proposition 13).]

EXAMPLE Suppose that $\mathbf{C}$ is a compactly generated triangulated category. Fix a compact object X—then for any divisible abelian group A, $\mathrm{Hom}(\mathrm{Mor}(X, —), A)$ is an exact cofunctor which converts coproducts into products, thus is representable.

EXAMPLE (Idempotents Split) Suppose that $\mathbf{C}$ is a compactly generated triangulated category. Let $e \in \mathrm{Mor}(Y, Y)$ be idempotent—then $\exists\, X, Z$ and an isomorphism $Y \to X \amalg Z$ such that the diagram

$$\begin{tikzcd}
Y \arrow[rr, "e"] \arrow[d] & & Y \arrow[d] \\
X \amalg Z \arrow[r] & X \arrow[r] & X \amalg Z
\end{tikzcd}$$

commutes.

[Using suggestive notation, write $\mathrm{Mor}(—, Y)$ as a direct sum $e\mathrm{Mor}(—, Y) \oplus (1-e)\mathrm{Mor}(—, Y)$ of two exact cofunctors which convert coproducts into products and choose $X, Z : e\mathrm{Mor}(—, Y) \approx \mathrm{Mor}(—, X)$, $(1-e)\mathrm{Mor}(—, Y) \approx \mathrm{Mor}(—, Z)$.]

[Note: Defining $r : Y \to X$ and $i : X \to Y$ in the obvious way, one has $e = i \circ r$ and $r \circ i = \mathrm{id}_X$. Moreover, $r : Y \to X$ is a split coequalizer of e, $\mathrm{id}_Y : Y \to Y$, as can be seen from the diagram

$$\begin{tikzcd}
Y \arrow[r, shift left, "e"] \arrow[r, shift right, "\mathrm{id}_Y"'] & Y \arrow[r, "r"] \arrow[l, bend right, "\mathrm{id}_Y"'] \arrow[dr, "e"'] & X \arrow[d, "i"] \arrow[l, bend right, "i"'] \\
& & Y
\end{tikzcd}$$

.]

EXAMPLE (The Eilenberg Swindle) Suppose that $\mathbf{C}$ is a compactly generated triangulated category. Let $\mathbf{D}$ be a triangulated subcategory of $\mathbf{C}$. Assume: $\mathbf{D}$ is closed under the formation of coproducts in $\mathbf{C}$—then $\mathbf{D}$ is thick.

[Fix a pair of morphisms $i : X \to Y$, $r : Y \to X$ with $r \circ i = \mathrm{id}_X$ and $Y \in \mathrm{Ob}\,\mathbf{D}$. Put $e = i \circ r$. Since e is an idempotent, by the preceding example $Y \approx X \amalg Z$ for some Z. Write $W = X \amalg (Z \amalg X) \amalg (Z \amalg X) \amalg \cdots \approx (X \amalg Z) \amalg (X \amalg Z) \amalg \cdots$ to get $W \in \mathrm{Ob}\,\mathbf{D}$. But $W \approx X \amalg W \approx W \amalg X \Rightarrow W \amalg X \in \mathrm{Ob}\,\mathbf{D}$. Because the triangle $W \to W \amalg X \to X \xrightarrow{0} \Sigma W$ is exact (cf. p. 15–4 ff.), it follows that $X \in \mathrm{Ob}\,\mathbf{D}$.]

EXAMPLE Suppose that $\mathbf{C}$ is a compactly generated triangulated category—then $\mathbf{C}$ has products. Proof: Given a set of objects X_i, apply the Brown representability theorem to the exact cofunctor $Y \to \prod\limits_i \mathrm{Mor}(Y, X_i)$.

[Note: The morphism $t : \coprod_i X_i \to \prod_i X_i$ of p. 0–34 is an isomorphism iff $\forall\, \overline{U} \in \overline{\mathcal{U}} : \#\{i : \operatorname{Mor}(\overline{U}, X_i) \neq 0\} < \omega$. To see this, consider the arrow $\operatorname{Mor}(\overline{U}, \coprod_i X_i) = \bigoplus_i \operatorname{Mor}(\overline{U}, X_i) \to \prod_i \operatorname{Mor}(\overline{U}, X_i) = \operatorname{Mor}(\overline{U}, \prod_i X_i)$.]

PROPOSITION 15 Let $\mathbf{C}$ be a compactly generated triangulated category and let $\mathbf{D}$ be an arbitrary triangulated category. Suppose that $F : \mathbf{C} \to \mathbf{D}$ is a triangulated functor which preserves coproducts—then F has a right adjoint $G : \mathbf{D} \to \mathbf{C}$.

[Given a $Y \in \operatorname{Ob}\mathbf{D}$, the cofunctor $X \to \operatorname{Mor}(FX, Y)$ is exact and converts coproducts into products, thus is representable: $\operatorname{Mor}(F—, Y) \approx \operatorname{Mor}(—, GY)$.]

FACT Let $\mathbf{C}$ be a compactly generated triangulated category and let $\mathbf{D}$ be an arbitrary triangulated category. Suppose that $F : \mathbf{C} \to \mathbf{D}$ is a triangulated functor which preserves coproducts—then its right adjoint $G : \mathbf{D} \to \mathbf{C}$ preserves coproducts iff $\forall\, U \in \mathcal{U}$, FU is compact.

[Necessity: $\bigoplus_j \operatorname{Mor}(FU, Y_j) \approx \bigoplus_j \operatorname{Mor}(U, GY_j) \approx \operatorname{Mor}(U, \coprod_j GY_j) \approx \operatorname{Mor}(U, G\coprod_j Y_j) \approx \operatorname{Mor}(FU, \coprod_j Y_j)$.

Sufficiency: The natural transformation $\Xi : \operatorname{Mor}(—, \coprod_j GY_j) \to \operatorname{Mor}(—, G\coprod_j Y_j)$ corresponding to the arrow $\coprod_j GY_j \to G\coprod_j Y_j$ has the property that $\Xi_{\overline{U}}$ is bijective $\forall\, \overline{U} \in \overline{\mathcal{U}}$, hence $\coprod_j GY_j \approx G\coprod_j Y_j$ (cf. p. 15–16).]

Notation: $\mathcal{U}^+$ is the class of objects in $\mathbf{C}$ that are coproducts of objects in $\overline{\mathcal{U}}$.

Definition: An object $(\mathbf{X}, \mathbf{f})$ in $\mathbf{FIL}(\mathbf{C})$ is completable in $\mathcal{U}^+$ if $X_0 \in \mathcal{U}^+$ and $\forall\, n \geq 0$, there is an exact triangle $X_n \xrightarrow{f_n} X_{n+1} \to Z_n \to \Sigma X_n$ with Z_n in $\mathcal{U}^+$.

PROPOSITION 16 Let $\mathbf{C}$ be a compactly generated triangulated category. Suppose that $F : \mathbf{C} \to \mathbf{AB}$ is an exact cofunctor which converts coproducts into products—then $\exists$ an object $(\mathbf{X}, \mathbf{f})$ in $\mathbf{FIL}(\mathbf{C})$, completable in $\mathcal{U}^+$, such that $\operatorname{tel}(\mathbf{X}, \mathbf{f})$ represents F.

[This is implicit in the proof of the Brown representability theorem. Thus by definition, $X_0 \in \mathcal{U}^+$. Consider the exact triangle $K_n \to X_n \xrightarrow{f_n} X_{n+1} \to \Sigma K_n$. Since $\Sigma K_n = \Sigma(\coprod_{\overline{U}} (\ker \Xi_{n,\overline{U}}) \cdot \overline{U}) \approx \coprod_{\overline{U}} (\ker \Xi_{n,\overline{U}}) \cdot \Sigma\overline{U}$, there is an exact triangle $X_n \xrightarrow{f_n} X_{n+1} \to Z_n \to \Sigma X_n$ with $Z_n \in \mathcal{U}^+$.]

[Note: If $\overline{U} = \Omega^n U$ $(n \geq 1)$, then $\Sigma\overline{U} = \Sigma\Omega^n U = \Sigma\Omega(\Omega^{n-1}U) \approx \Omega^{n-1}U \in \overline{\mathcal{U}}$.]

Application: Fix an $X \in \operatorname{Ob}\mathbf{C}$—then $\exists$ an object $(\mathbf{X}, \mathbf{f})$ in $\mathbf{FIL}(\mathbf{C})$, completable in $\mathcal{U}^+$, such that $X \approx \operatorname{tel}(\mathbf{X}, \mathbf{f})$.

[In Proposition 16, take $F = \operatorname{Mor}(—, X)$.]

Let **C** be a compactly generated triangulated category satisfying the octahedral axiom—then one may form $\operatorname{Ext}\overline{\mathcal{U}}$ and $\operatorname{Ext}\mathcal{U}^+$ (cf. p. 15–10). Example: Using the notation of Proposition 16, $\forall\, n \geq 0$, $X_n \in \operatorname{Ext}\mathcal{U}^+$.

LEMMA Let **C** be a compactly generated triangulated category satisfying the octahedral axiom. Fix a compact object X and suppose that $Z' \rightarrow Z \rightarrow Z'' \rightarrow \Sigma Z'$ is an exact triangle with $Z'' \in \operatorname{Ext}\mathcal{U}^+$—then every diagram

$$\begin{tikzcd} & X \arrow[d] \\ Z' \arrow[r] & Z \end{tikzcd}$$

can be completed to a commutative diagram

$$\begin{tikzcd} X' \arrow[r] \arrow[d] & X \arrow[d] \\ Z' \arrow[r] & Z \end{tikzcd}$$

in such a way that there is an exact triangle $X' \rightarrow X \rightarrow X'' \rightarrow \Sigma X'$ with $X'' \in \operatorname{Ext}\overline{\mathcal{U}}$.

[Argue by induction on the length l of Z''.

Case 1: $l = 1$. Here $Z'' \in \mathcal{U}^+$. Since X is compact, the composition $X \rightarrow Z \rightarrow Z''$ factors through a finite coproduct $X'' \subset Z''$ and

$$\begin{tikzcd} & X \arrow[r] \arrow[d] & X'' \arrow[d] & \\ Z' \arrow[r] & Z \arrow[r] & Z'' \arrow[r] & \Sigma Z' \end{tikzcd}$$

extends to a morphism of exact triangles

$$\begin{tikzcd} X' \arrow[r] \arrow[d] & X \arrow[r] \arrow[d] & X'' \arrow[r] \arrow[d] & \Sigma X' \arrow[d] \\ Z' \arrow[r] & Z \arrow[r] & Z'' \arrow[r] & \Sigma Z' \end{tikzcd}$$

(cf. Proposition 1).

Case 2: $l > 1$. By assumption, Z'' occurs in an exact triangle $Z_0'' \rightarrow Z'' \rightarrow Z_1'' \rightarrow \Sigma Z_0''$, where Z_0'', $Z_1'' \in \operatorname{Ext}\mathcal{U}^+$ and have length $< l$. Complete the composite $Z \rightarrow Z'' \rightarrow Z_1''$ to an exact triangle $Z \rightarrow Z_1'' \rightarrow W \rightarrow \Sigma Z$ (cf. TR$_3$). Using the octahedral axiom on $Z \rightarrow Z'' \rightarrow \Sigma Z' \rightarrow \Sigma Z$, $Z'' \rightarrow Z_1'' \rightarrow \Sigma Z_0'' \rightarrow \Sigma Z''$, construct a factorization

$$\begin{tikzcd} Z' \arrow[r] \arrow[dr] & \overline{Z} \arrow[d] \\ & Z \end{tikzcd}$$

of $Z' \rightarrow Z$ and exact triangles $Z' \rightarrow \overline{Z} \rightarrow Z_0'' \rightarrow \Sigma Z'$, $\overline{Z} \rightarrow Z \rightarrow Z_1'' \rightarrow \Sigma \overline{Z}$. Owing to the induction hypothesis, there is a commutative diagram

$$\begin{tikzcd} X' \arrow[r] \arrow[d] & \overline{X} \arrow[r] \arrow[d] & X \arrow[d] \\ Z' \arrow[r] & \overline{Z} \arrow[r] & Z \end{tikzcd}$$

and exact triangles $X' \rightarrow \overline{X} \rightarrow X_0'' \rightarrow \Sigma X'$, $\overline{X} \rightarrow X \rightarrow X_1'' \rightarrow \Sigma \overline{X}$, where X_0'', $X_1'' \in \operatorname{Ext}\overline{\mathcal{U}}$. Complete the composite $X' \rightarrow \overline{X} \rightarrow X$ to an exact triangle $X' \rightarrow X \rightarrow X'' \rightarrow \Sigma X'$ (cf. TR$_3$)—then the octahedral axiom implies that $X'' \in \operatorname{Ext}\overline{\mathcal{U}}$.]

PROPOSITION 17 Let **C** be a compactly generated triangulated category satisfying the octahedral axiom—then every compact object X in **C** is a direct summand of an object in $\operatorname{Ext}\overline{\mathcal{U}}$.

[Write $X \approx \mathrm{tel}(\mathbf{X},\mathbf{f})$ (cf. supra). Since $\operatorname{colim} \operatorname{Mor}(X, X_n) \approx \operatorname{Mor}(X, X)$ (cf. Proposition 13), id_X factors through some X_n :

$$\begin{tikzcd} X \arrow[rr, "\mathrm{id}_X"] \arrow[dr, "i"'] & & X \\ & X_n \arrow[ur, "r"'] & \end{tikzcd}$$

. On the other hand, $X_n \in \operatorname{Ext} \mathcal{U}^+$ and $0 \to X_n \xrightarrow{\mathrm{id}_{X_n}} X_n \to 0$ is exact. One may therefore apply the lemma to

$$\begin{tikzcd} & X \arrow[d] \\ 0 \arrow[r] & X_n \end{tikzcd}$$

and produce a commutative diagram

$$\begin{tikzcd} X' \arrow[r] \arrow[d] & X \arrow[d] \\ 0 \arrow[r] & X_n \end{tikzcd}$$

plus an exact triangle $X' \to X \to X'' \to \Sigma X'$ with $X'' \in \operatorname{Ext} \overline{\mathcal{U}}$. But the arrow $X' \to X$ is the zero morphism, thus $X'' \approx X \amalg \Sigma X'$ (cf. p. 15–5).]

Notation: $\operatorname{cpt} \mathbf{C}$ is the thick subcategory of $\mathbf{C}$ whose objects are compact.

THEOREM OF NEEMAN-RAVENEL Let $\mathbf{C}$ be a compactly generated triangulated category satisfying the octahedral axiom—then the thick subcategory generated by $\mathcal{U}$ is $\operatorname{cpt} \mathbf{C}$.

[This is a consequence of Proposition 10 and Proposition 17.]

[Note: The thick subcategory generated by $\mathcal{U}$ is, of course, the intersection of the conglomerate of thick subcategories of $\mathbf{C}$ containing $\mathcal{U}$.]

The proof of the Neeman-Ravenel theorem depends on the octahedral axiom (by way of Proposition 17) but its use can be eliminated. Thus let $\mathbf{\Lambda}$ be the thick subcategory generated by $\mathcal{U}$ and fix a skeleton $\overline{\mathbf{\Lambda}}$ of $\mathbf{\Lambda}$—then $\overline{\mathbf{\Lambda}}$ is small (since $\mathcal{U}$ is a set) and for any X in $\mathbf{C}$, $\overline{\mathbf{\Lambda}}/X$ is the category whose objects are the arrows $K \to X$ and whose morphisms $(K \to X) \to (L \to X)$ are the commutative triangles

$$\begin{tikzcd} K \arrow[rr] \arrow[dr] & & L \arrow[dl] \\ & X & \end{tikzcd}$$

(K, L in $\overline{\mathbf{\Lambda}}$).

LEMMA $\forall\, X$, the category $\overline{\mathbf{\Lambda}}/X$ is filtered.

[Note: The assignment $X \to \overline{\mathbf{\Lambda}}/X$ defines a functor $\mathbf{C} \to \mathbf{CAT}$.]

In what follows, $\operatorname*{colim}_X$ stands for a colimit calculated over $\overline{\mathbf{\Lambda}}/X$.

PROPOSITION 18 Let $\mathbf{C}$ be a compactly generated triangulated category. Suppose that $F : \mathbf{\Lambda} \to \mathbf{AB}$ is an exact functor. Given an $X \in \operatorname{Ob} \mathbf{C}$, put $\overline{F}X = \operatorname*{colim}_X FK$—then $\overline{F} : \mathbf{C} \to \mathbf{AB}$ is an exact functor which converts coproducts into direct sums.

[Note: $\forall\, K$ in $\mathbf{\Lambda}$, $\overline{F}K \approx FK$.]

Remark: Suppose that $F : \mathbf{C} \to \mathbf{AB}$ is an exact functor which converts coproducts into direct sums—then the natural transformation $\overline{F|\mathbf{\Lambda}} \to F$ is a natural isomorphism. Proof: The X such that the arrows $\left\{\begin{array}{lcl} \overline{F|\mathbf{\Lambda}}\Sigma^n X & \to & F\Sigma^n X \\ \overline{F|\mathbf{\Lambda}}\Omega^n X & \to & F\Omega^n X \end{array}\right.$ are isomorphisms $\forall\, n \geq 0$ constitute the object class of a triangulated subcategory of $\mathbf{C}$ containing $\mathcal{U}$ and closed under the formation of coproducts in $\mathbf{C}$, thus is all of $\mathbf{C}$ (cf. Proposition 14).

THEOREM OF NEEMAN-RAVENEL (bis) Let $\mathbf{C}$ be a compactly generated triangulated category—then the thick subcategory generated by $\mathcal{U}$ is cpt $\mathbf{C}$.

[$\forall$ compact X, the exact functor Mor $(X, —)$ converts coproducts into direct sums. Therefore, by the above remark, $\overline{\text{Mor}\,(X, —)|\mathbf{\Lambda}} \approx \text{Mor}\,(X, —)$, so id_X factors through some K in $\overline{\mathbf{\Lambda}}$:

$$\begin{tikzcd} X \arrow[rr, "\text{id}_X"] \arrow[dr, "i"'] & & X \\ & K \arrow[ur, "r"'] & \end{tikzcd}$$

.]

PROPOSITION 19 Let $\mathbf{C}$ be a compactly generated triangulated category—then cpt $\mathbf{C}$ has a small skeleton.

Let $\mathbf{C}$ be a compactly generated triangulated category—then the additive functor category $[(\text{cpt}\,\mathbf{C})^{\text{OP}}, \mathbf{AB}]^+$ is a complete and cocomplete abelian category and has enough projectives (cf. p. 0–38). Call $\mathbf{EX}[(\text{cpt}\,\mathbf{C})^{\text{OP}}, \mathbf{AB}]^+$ the full subcategory of $[(\text{cpt}\,\mathbf{C})^{\text{OP}}, \mathbf{AB}]^+$ whose objects are the exact cofunctors $F : \text{cpt}\,\mathbf{C} \to \mathbf{AB}$.

PROPOSITION 20 Let $\mathbf{C}$ be a compactly generated triangulated category—then all the projective objects of $[(\text{cpt}\,\mathbf{C})^{\text{OP}}, \mathbf{AB}]^+$ lie in $\mathbf{EX}[(\text{cpt}\,\mathbf{C})^{\text{OP}}, \mathbf{AB}]^+$.

[Every projective object of $[(\text{cpt}\,\mathbf{C})^{\text{OP}}, \mathbf{AB}]^+$ is a direct summand of a coproduct of representable cofunctors.]

PROPOSITION 21 Let $\mathbf{C}$ be a compactly generated triangulated category—then every object in $[(\text{cpt}\,\mathbf{C})^{\text{OP}}, \mathbf{AB}]^+$ of finite projective dimension belongs to $\mathbf{EX}[(\text{cpt}\,\mathbf{C})^{\text{OP}}, \mathbf{AB}]^+$.

Notation: Write h_X for the restriction Mor $(—, X)|\text{cpt}\,\mathbf{C}$ and write $h_f : h_X \to h_Y$ for the natural transformation induced by the morphism $f : X \to Y$.

FACT Let $\mathbf{C}$ be a compactly generated triangulated category—then the functor $h : \mathbf{C} \to [(\text{cpt}\,\mathbf{C})^{\text{OP}}, \mathbf{AB}]^+$ is exact, conservative, and preserves products & coproducts.

Let $\mathbf{C}$ be a compactly generated triangulated category—then $\mathbf{C}$ is said to admit Adams representability if the following conditions are satisfied.

(ADR$_1$) Every exact cofunctor $F : \text{cpt}\,\mathbf{C} \to \mathbf{AB}$ is representable in the large, i.e., $\exists$ an $X \in \text{Ob}\,\mathbf{C}$ and a natural isomorphism $h_X \to F$.

(ADR$_2$) Every natural transformation $h_X \to h_Y$ is induced by a morphism $f : X \to Y$.

FACT Suppose that $\mathbf{C}$ admits Adams representability—then $\mathbf{IND}(\text{cpt}\,\mathbf{C})$ is equivalent to $\mathbf{EX}[(\text{cpt}\,\mathbf{C})^{\text{OP}}, \mathbf{AB}]^+$.

LEMMA Let $\mathbf{C}$ be a compactly generated triangulated category. Assume: $\mathbf{C}$ admits Adams representability—then $h_X \approx h_Y \Rightarrow X \approx Y$, thus an object representing a given exact cofunctor $F : \text{cpt}\,\mathbf{C} \to \mathbf{AB}$ is unique up to isomorphism.

Suppose that $\mathbf{C}$ admits Adams representability—then $\forall\ X, Y \in \text{Ob}\,\mathbf{C}$, there is a surjection $\text{Mor}(X,Y) \to \text{Nat}(h_X, h_Y)$, viz. $f \to h_f$. Definition: f is said to be a phantom map provided that $h_f = 0$. So, if $\text{Ph}(X,Y)$ is the subgroup of $\text{Mor}(X,Y)$ consisting of the phantom maps, then the sequence $0 \to \text{Ph}(X,Y) \to \text{Mor}(X,Y) \to \text{Nat}(h_X, h_Y) \to 0$ is short exact.

[Note: Let $f \in \text{Ph}(X,Y)$—then for any $\phi : X' \to X$, $f \circ \phi \in \text{Ph}(X', Y)$, and for any $\psi : Y \to Y'$, $\psi \circ f \in \text{Ph}(X, Y')$. This has the consequence that it makes sense to form the quotient category $\mathbf{C}/\mathbf{Ph}$, where the set of morphisms from X to Y is $\text{Mor}(X,Y)/\text{Ph}(X,Y)$.]

LEMMA Let $\mathbf{C}$ be a compactly generated triangulated category. Assume: $\mathbf{C}$ admits Adams representability—then h_X is projective iff X is a direct summand of a coproduct of compact objects.

EXAMPLE Consider any exact triangle $W \xrightarrow{w} \coprod\limits_i X_i \xrightarrow{t} \prod\limits_i X_i \to \Sigma W$ (t as on p. 0–34)—then w is a phantom map.

FACT Suppose that $\mathbf{C}$ admits Adams representability—then $f : X \to Y$ is a phantom map iff $\forall$ compact K and every $\phi : K \to X$, the composite $f \circ \phi$ vanishes.

EXAMPLE Given an $X \in \text{Ob}\,\mathbf{C}$, complete $\coprod\limits_{\overline{\mathbf{\Lambda}}/X} K \to X$ to an exact triangle $\coprod\limits_{\overline{\mathbf{\Lambda}}/X} K \to X \xrightarrow{\Theta} \overline{X} \to \coprod\limits_{\overline{\mathbf{\Lambda}}/X} \Sigma K$ (cf. TR$_3$)—then Θ is a phantom map. Moreover, every $f \in \text{Ph}(X,Y)$ factors through Θ.

Corollary: All phantom maps out of X vanish iff $\Theta = 0$. And, when $\Theta = 0$, X is a direct summand of $\coprod\limits_{\overline{\Lambda}/X} K$.

[Note: Therefore Θ is a "universal" phantom map (cf. p. 5–90).]

FACT Suppose that $\mathbf{C}$ admits Adams representability—then $f : X \to Y$ is a phantom map iff $\forall$ exact functor $F : \mathbf{C} \to \mathbf{AB}$ which converts coproducts into direct sums, $Ff = 0$.

PROPOSITION 22 Let $\mathbf{C}$ be a compactly generated triangulated category. Assume: $\mathbf{C}$ admits Adams representability. Let $\Delta : \mathbf{I} \to \mathbf{C}$ be a diagram, where $\mathbf{I}$ is filtered and $\forall\, i \in \operatorname{Ob} \mathbf{I}$, Δ_i is compact—then Δ has a minimal weak colimit.

[Put $F = \operatorname{colim} h_{\Delta_i}$ (thus $\forall$ compact K, $FK = \operatorname{colim} \operatorname{Mor}(K, \Delta_i)$). Since $\mathbf{AB}$ is Grothendieck, F is exact, so by ADR_1, $\exists$ an $X \in \operatorname{Ob} \mathbf{C}$ and a natural isomorphism $h_X \to F$. Claim: X is a minimal weak colimit of Δ. Indeed, $\forall\, i$, there is a natural transformation $\Xi_i : h_{\Delta_i} \to h_X$ and, by ADR_2, $\Xi_i = h_{f_i}$ ($\exists\, f_i : \Delta_i \to X$). Moreover, f_i is determined up to an element of $\operatorname{Ph}(\Delta_i, X)$. But Δ_i compact $\Rightarrow \operatorname{Ph}(\Delta_i, X) = 0$, hence f_i is unique. Consequently, $\{\Delta_i \xrightarrow{f_i} X\}$ is a natural sink. If now $\{\Delta_i \xrightarrow{g_i} Y\}$ is another natural sink, then $\exists\, \Xi \in \operatorname{Nat}(h_X, h_Y) : \forall\, i$, $h_{g_i} = \Xi \circ h_{f_i}$. However $\Xi = h_\phi$ for some $\phi : X \to Y$ (cf. ADR_2) and this means that $g_i = \phi \circ f_i$. Therefore X is a weak colimit of Δ. Minimality is obvious.]

EXAMPLE Suppose that $\mathbf{C}$ admits Adams representability. Fix an $X \in \operatorname{Ob} \mathbf{C}$ and consider the functor $\overline{\Lambda}/X \to \mathbf{C}$ that sends $K \to X$ to K. Since $\overline{\Lambda}/X$ is filtered, this functor has a minimal weak colimit L_X (cf. Proposition 22). There is an arrow $L_X \to X$ and $\forall\, \overline{U} \in \overline{\mathcal{U}}$, $\operatorname{Mor}(\overline{U}, L_X) \approx \operatorname*{colim}\limits_{X} \operatorname{Mor}(\overline{U}, K) \approx \operatorname{Mor}(\overline{U}, X) \Rightarrow L_X \approx X$ (cf. p. 15–16).

FACT Let $\mathbf{C}$ be a compactly generated triangulated category. Assume: Every functor from a filtered category $\mathbf{I}$ to $\mathbf{C}$ with compact values has a minimal weak colimit—then $\mathbf{C}$ admits Adams representability.

LEMMA Let $\mathbf{C}$ be a compactly generated triangulated category. Assume: $\mathbf{C}$ admits Adams representability—then for any X in $\mathbf{C}$, there is an exact triangle $P \to Q \to X \to \Sigma P$ such that h_P & h_Q are projective and the sequence $0 \to h_P \to h_Q \to h_X \to 0$ is short exact.

[The functor $\overline{\Lambda}/X \to \mathbf{C}$ that sends $K \to X$ to K has a minimal weak colimit, viz. X (see the preceding example). It also has a weak colimit Y constructed via the procedure on p. 15–4: $\coprod\limits_{K \to L} K \to \coprod\limits_{\overline{\Lambda}/X} K \to Y \to \coprod\limits_{K \to L} \Sigma K$. Since X is minimal, $\exists$ arrows $\phi : X \to Y$,

$\psi : Y \to X$ such that $\psi \circ \phi$ is an isomorphism and the triangles $$\begin{tikzcd} \coprod\limits_{\overline{\Lambda}/X} K \arrow[r] \arrow[dr] & X \arrow[d, "\phi"] \\ & Y \end{tikzcd}$$, $$\begin{tikzcd} \coprod\limits_{\overline{\Lambda}/X} K \arrow[r] \arrow[dr] & Y \arrow[d, "\psi"] \\ & X \end{tikzcd}$$ commute (cf. p. 15–16 ff.). Define P by requiring that $P \to \coprod\limits_{\overline{\Lambda}/X} K \to X \to \Sigma P$ be exact. Using Proposition 1, determine arrows $f : P \to \coprod\limits_{K \to L} K$, $g : \coprod\limits_{K \to L} K \to P$ such that the diagram

$$\begin{tikzcd} P \arrow[r] \arrow[d, "f"] & \coprod\limits_{\overline{\Lambda}/X} K \arrow[r] \arrow[d, equal] & X \arrow[r] \arrow[d, "\phi"] & \Sigma P \arrow[d, "\Sigma f"] \\ \coprod\limits_{K \to L} K \arrow[r] \arrow[d, "g"] & \coprod\limits_{\overline{\Lambda}/X} K \arrow[r] \arrow[d, equal] & Y \arrow[r] \arrow[d, "\psi"] & \coprod\limits_{K \to L} \Sigma K \arrow[d, "\Sigma g"] \\ P \arrow[r] & \coprod\limits_{\overline{\Lambda}/X} K \arrow[r] & X \arrow[r] & \Sigma P \end{tikzcd}$$

commutes—then $g \circ f$ is an isomorphism (cf. p. 15–4), hence h_P is a direct summand of $\coprod\limits_{K \to L} h_K$ which implies that h_P is projective. And with $Q = \coprod\limits_{\overline{\Lambda}/X} K$, the sequence $0 \to h_P \to h_Q \to h_X \to 0$ is short exact.]

Remark: The arrow $X \to \Sigma P$ is a phantom map and if $f : X \to Y$ is a phantom map, then there is a commutative triangle $$\begin{tikzcd} X \arrow[r] \arrow[d, "f"'] & \Sigma P \arrow[dl] \\ Y \end{tikzcd}$$ (cf. p. 15–22 ff.).

Example: $f \in \mathrm{Ph}(X, Y)$ & $g \in \mathrm{Ph}(Y, Z) \Rightarrow g \circ f = 0$. Proof: h_P projective $\Rightarrow h_{\Sigma P}$ projective $\Rightarrow \mathrm{Ph}(\Sigma P, Z) = 0$.

PROPOSITION 23 Let $\mathbf{C}$ be a compactly generated triangulated category. Assume: $\mathbf{C}$ admits Adams representability—then $\mathbf{EX}[(\mathrm{cpt}\,\mathbf{C})^{\mathrm{OP}}, \mathbf{AB}]^+$ is the full subcategory of $[(\mathrm{cpt}\,\mathbf{C})^{\mathrm{OP}}, \mathbf{AB}]^+$ whose objects have projective dimension ≤ 1.

[On account of Proposition 21, it need only be shown that every F in $\mathbf{EX}[(\mathrm{cpt}\,\mathbf{C})^{\mathrm{OP}}, \mathbf{AB}]^+$ has projective dimension ≤ 1. But by ADR_1, $\exists\, X : h_X \approx F$ and the lemma implies that h_X has a projective resolution of length ≤ 1.]

FACT Suppose that $\mathbf{C}$ admits Adams representability—then $\forall\ X, Y \in \mathrm{Ob}\,\mathbf{C}$, $\mathrm{Ph}(\Omega X, Y) \approx \mathrm{Ext}(h_X, h_Y)$.

LEMMA Let $\mathbf{C}$ be a compactly generated triangulated category—then every exact cofunctor $F : \operatorname{cpt}\mathbf{C} \to \mathbf{AB}$ of projective dimension ≤ 1 has a projective resolution $0 \to H \to G \to F \to 0$, where G, H are coproducts of representable cofunctors.

[By hypothesis, there is a projective resolution $0 \to F'' \to F' \to F \to 0$. Here F' is a coproduct of representable cofunctors, while F'' is a direct summand of a coproduct of representable cofunctors, say $F'' \amalg \overline{F}'' \approx \Phi$. Noting that $\coprod\limits_1^\infty \Phi \approx F'' \amalg \coprod\limits_1^\infty \Phi$, consider $0 \to F'' \amalg \coprod\limits_1^\infty \Phi \to F' \amalg \coprod\limits_1^\infty \Phi \to F \to 0$.]

PROPOSITION 24 Let $\mathbf{C}$ be a compactly generated triangulated category. Assume: Every exact cofunctor $F : \operatorname{cpt}\mathbf{C} \to \mathbf{AB}$ has projective dimension ≤ 1—then $\mathbf{C}$ admits Adams representability.

[It is a question of checking the validity of ADR_1 and ADR_2.

Re: ADR_1. Fix an exact cofunctor $F : \operatorname{cpt}\mathbf{C} \to \mathbf{AB}$ and resolve it per the lemma: $0 \to H \to G \to F \to 0$. Write $G = \amalg \operatorname{Mor}(-, K)$, $H = \amalg \operatorname{Mor}(-, L)$—then the arrow $H \to G$ gives rise to a morphism $\amalg L \to \amalg K$ which can be completed to an exact triangle $\amalg L \to \amalg K \to X \to \amalg \Sigma L$ (cf. TR_3) and $h_X \approx F$.

Re: ADR_2. Fix a natural transformation $\Xi : h_X \to h_Y$. Choose projective resolutions $0 \to H_X \to G_X \to h_X \to 0$, $0 \to H_Y \to G_Y \to h_Y \to 0$ per the lemma and lift Ξ to a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & H_X \arrow[r] \arrow[d] & G_X \arrow[r] \arrow[d] & h_X \arrow[r] \arrow[d, "\Xi"] & 0 \\
0 \arrow[r] & H_Y \arrow[r] & G_Y \arrow[r] & h_Y \arrow[r] & 0
\end{tikzcd}$$

. Write $G_X = \amalg \operatorname{Mor}(-, K_X)$, $H_X = \amalg \operatorname{Mor}(-, L_X)$, $G_Y = \amalg \operatorname{Mor}(-, K_Y)$, $H_Y = \amalg \operatorname{Mor}(-, L_Y)$—then there is a commutative diagram

$$\begin{tikzcd}
\amalg L_X \arrow[r] \arrow[d] & \amalg K_X \arrow[d] \\
\amalg L_Y \arrow[r] & \amalg K_Y
\end{tikzcd}$$

in $\mathbf{C}$. Complete it to a morphism

$$\begin{tikzcd}
\amalg L_X \arrow[r] \arrow[d] & \amalg K_X \arrow[r] \arrow[d] & X' \arrow[r] \arrow[d, "f'"] & \amalg \Sigma L_X \arrow[d] \\
\amalg L_Y \arrow[r] & \amalg K_Y \arrow[r] & Y' \arrow[r] & \amalg \Sigma L_Y
\end{tikzcd}$$

of exact triangles (cf. TR_3 & TR_5). The rows in the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & H_X \arrow[r] \arrow[d] & G_X \arrow[r] \arrow[d] & h_{X'} \arrow[r] \arrow[d, "h_{f'}"] & 0 \\
0 \arrow[r] & H_Y \arrow[r] & G_Y \arrow[r] & h_{Y'} \arrow[r] & 0
\end{tikzcd}$$

are short exact. Working with X, the composite $\amalg L_X \to \amalg K_X \to X$ is a phantom map, hence vanishes, thus $\exists$ a commutative triangle

$$\begin{tikzcd}
\amalg K_X \arrow[r] \arrow[d] & X' \arrow[dl, "\phi"] \\
X &
\end{tikzcd}$$

and $h_\phi : h_{X'} \to h_X$ is a natural isomorphism, so $\phi : X' \to X$ is an isomorphism (h is conservative (cf. p. 15–21)).

Similar considerations apply to Y. Since

$$\begin{tikzcd} h_{X'} \arrow[r, "h_\phi"] \arrow[d, "h_{f'}"'] & h_X \arrow[d, "\Xi"] \\ h_{Y'} \arrow[r, "h_\psi"'] & h_Y \end{tikzcd}$$

commutes, it follows that $\Xi = h_{\psi \circ f' \circ \phi^{-1}}$.]

Let $\mathbf{C}$ be a compactly generated triangulated category—then Propositions 23 and 24 tell us that $\mathbf{C}$ admits Adams representability iff every object of $\mathbf{EX}[(\text{cpt}\,\mathbf{C})^{\text{OP}}, \mathbf{AB}]^+$ has projective dimension ≤ 1 in $[(\text{cpt}\,\mathbf{C})^{\text{OP}}, \mathbf{AB}]^+$. And this condition can be realized. Indeed, it suffices that cpt $\mathbf{C}$ possess a countable skeleton (cf. infra).

[Note: Recall that in any event cpt $\mathbf{C}$ has a small skeleton (cf. Proposition 19).]

NEEMAN'S COUNTABILITY CRITERION Let $\mathbf{C}$ be a triangulated category with finite coproducts and a countable skeleton—then every object of $\mathbf{EX}[\mathbf{C}^{\text{OP}}, \mathbf{AB}]^+$ has projective dimension ≤ 1 in $[\mathbf{C}^{\text{OP}}, \mathbf{AB}]^+$.

[Note: $\mathbf{EX}[\mathbf{C}^{\text{OP}}, \mathbf{AB}]^+$ is the full subcategory of $[\mathbf{C}^{\text{OP}}, \mathbf{AB}]^+$ whose objects are the exact cofunctors $F : \mathbf{C} \to \mathbf{AB}$.]

The stable homotopy category is a compactly generated triangulated category and its full subcategory of compact objects has a countable skeleton. Therefore the stable homotopy category admits Adams representability.

The proof of Neeman's countability criterion requires some preparation. Call an object of $[\mathbf{C}^{\text{OP}}, \mathbf{AB}]^+$ free if it is a coproduct of representable cofunctors. Definition: $\forall$ F in $[\mathbf{C}^{\text{OP}}, \mathbf{AB}]^+$, $\#(F)$ is the smallest infinite cardinal κ for which there is a free presentation $F'' \to F' \to F \to 0$, where F', F'' are coproducts of $\leq \kappa$ representable cofunctors.

Observation: If $0 \to F'' \to F' \to F \to 0$ is a short exact sequence in $[\mathbf{C}^{\text{OP}}, \mathbf{AB}]^+$ and if $\#(F'') \leq \kappa$, $\#(F') \leq \kappa$, then $\#(F) \leq \kappa$.

Let κ be an infinite cardinal—then $\mathbf{C}$ is said to satisfy condition κ if for any F in $\mathbf{EX}[\mathbf{C}^{\text{OP}}, \mathbf{AB}]^+$ and any morphism $\Phi \to F$, where $\#(\Phi) \leq \kappa$, there is a factorization $\Phi \to \Psi \to F$ such that $\Psi \to F$ is a monomorphism and Ψ has a free resolution $0 \to \Psi'' \to \Psi' \to \Psi \to 0$, where Ψ', Ψ'' are coproducts of $\leq \kappa$ representable cofunctors ($\Rightarrow \#(\Psi) \leq \kappa$).

Observation: Suppose that $\mathbf{C}$ satisfies condition κ—then every object F of $\mathbf{EX}[\mathbf{C}^{\text{OP}}, \mathbf{AB}]^+$ with $\#(F) \leq \kappa$ has a free resolution $0 \to F'' \to F' \to F \to 0$, where F', F'' are coproducts of $\leq \kappa$ representable cofunctors. In particular: The projective dimension of F is ≤ 1.

LEMMA Suppose that $\mathbf{C}$ satisfies condition κ. Let $F \to G$ be a monomorphism of exact cofunctors, where $\#(F) \leq \kappa$, $\#(G) \leq \kappa$—then for any free resolution $0 \to F'' \to$

$F' \to F \to 0$ of F, there exists a free resolution $0 \to G'' \to G' \to G \to 0$ of G and a commutative diagram
$$\begin{tikzcd}
0 \arrow[r] & F'' \arrow[r] \arrow[d] & F' \arrow[r] \arrow[d] & F \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & G'' \arrow[r] & G' \arrow[r] & G \arrow[r] & 0
\end{tikzcd}$$
such that $F'' \to G''$, $F' \to G'$ are split monomorphisms.

[Complete $F \to G$ to a short exact sequence $0 \to F \to G \to H \to 0$. Since F, G are exact, so is H. Moreover, $\#(F) \leq \kappa$, $\#(G) \leq \kappa \Rightarrow \#(H) \leq \kappa$ (cf. supra). Fix a free resolution $0 \to H'' \to H' \to H \to 0$, where H', H'' are coproducts of $\leq \kappa$ representable cofunctors, and extend
$$\begin{tikzcd}
 & & & 0 \arrow[d] & \\
0 \arrow[r] & F'' \arrow[r] & F' \arrow[r] & F \arrow[r] \arrow[d] & 0 \\
 & & & G \arrow[d] & \\
0 \arrow[r] & H'' \arrow[r] & H' \arrow[r] & H \arrow[r] \arrow[d] & 0 \\
 & & & 0 &
\end{tikzcd}$$
in the obvious way: $0 \to F'' \oplus H'' \to F' \oplus H' \to G \to 0$.]

[Note: Therefore if F', F'' are coproducts of $\leq \kappa$ representable cofunctors, then $G' = F' \oplus H'$, $G'' = F'' \oplus H''$ are coproducts of $\leq \kappa$ representable cofunctors.]

MAIN LEMMA Let $\mathbf{C}$ be a countable triangulated category with finite coproducts—then $\mathbf{C}$ satisfies condition κ for every κ, hence Neeman's countability criterion is valid.

[Fix an F in $\mathbf{EX}[\mathbf{C}^{\mathrm{OP}}, \mathbf{AB}]^+$ and a morphism $\Phi \to F$.

$\#(\Phi) = \omega$. There is a free presentation $\Phi'' \to \Phi' \to \Phi \to 0$, where Φ', Φ'' are countable coproducts of representable cofunctors. Accordingly, one can assume without loss of generality that Φ is a countable coproduct of representable cofunctors (replace $\Phi \to F$ by $\Phi' \to \Phi \to F$), say $\Phi = \coprod\limits_0^\infty \mathrm{Mor}(—, X_i)$, the morphism $\Phi \to F$ corresponding to a sequence of natural transformations $\mathrm{Mor}(—, X_i) \to F$. Put $X_i^0 = X_i$. Since $\mathbf{C}$ is countable, $\forall\, X \in \mathrm{Ob}\,\mathbf{C}$, $\mathrm{Mor}(X, \coprod\limits_{i=0}^k X_i^0)$ is countable, thus its subset $S_{X,k}$ consisting of the arrows for which the composite $\mathrm{Mor}(—, X) \to \coprod\limits_{i=0}^k \mathrm{Mor}(—, X_i^0) \to F$ vanishes is countable. Enumerate the elements of $\bigcup\limits_{X,k} S_{X,k}$. Supposing that $X \to \coprod\limits_{i=0}^k X_i^0$ is the lth such, define X_l^1 by the exact triangle $X \to \coprod\limits_{i=0}^k X_i^0 \to X_l^1 \to \Sigma X$ (cf. TR_3). The natural transformation $\coprod_{i=0}^{k} \mathrm{Mor}\,(—, X_i^0) \to F$ determines an element $x \in F \coprod_{i=0}^{k} X_i^0$ that, under the arrow $F \coprod_{i=0}^{k} X_i^0 \to FX$, is sent to 0. Since F is exact, $\exists$ an element of FX_l^1 mapping to x. This means that $\coprod_{i=0}^{k} \mathrm{Mor}\,(—, X_i^0) \to F$ factors as $\coprod_{i=0}^{k} \mathrm{Mor}\,(—, X_i^0) \to \mathrm{Mor}\,(—, X_l^1) \to F$. Iterate the procedure: From the set $\{X_l^1\}$ one can produce the set $\{X_l^2\}$. Continuing, the upshot is a countable filtered category $\mathbf{I}$ whose objects are the X_l^k and whose morphisms $X_l^k \to X_{l'}^{k'}$ are the identities and the composites arising from the construction. There is a functor $\mathbf{I} \to [\mathbf{C}, \mathbf{AB}]^+$ that sends X_l^k to $\mathrm{Mor}\,(—, X_l^k)$. The natural transformations $\mathrm{Mor}\,(—, X_l^k) \to F$ constitute a natural sink and the arrow $\mathrm{colim}\ \mathrm{Mor}\,(—, X_l^k) \to F$ is a monomorphism. Definition: $\Psi = \mathrm{colim}\ \mathrm{Mor}\,(—, X_l^k)$. It is clear that the $\mathrm{Mor}\,(—, X_i) \to F$ factor through Ψ. To show that Ψ has a free resolution $0 \to \Psi'' \to \Psi' \to \Psi \to 0$, where Ψ', Ψ'' are countable coproducts of representable cofunctors, fix a final functor $\nabla : [\mathrm{N}] \to \mathbf{I}$ (see below)—then $\Psi \approx \mathrm{colim}\ \mathrm{Mor}\,(—, \nabla_n)$ and there is a short exact sequence $0 \to \coprod_n \mathrm{Mor}\,(—, \nabla_n) \xrightarrow{\mathrm{sf}} \coprod_n \mathrm{Mor}\,(—, \nabla_n) \to \Psi \to 0$. Here the nth component of sf is the arrow $\nabla_n \xrightarrow{\begin{pmatrix} \mathrm{id} \\ -f_n \end{pmatrix}} \nabla_n \amalg \nabla_{n+1}$ $(f_n : \nabla_n \to \nabla_{n+1})$.

$\#(\Phi) = \kappa$ $(> \omega)$. The induction hypothesis is that $\mathbf{C}$ satisfies condition κ' for all infinite cardinals $\kappa' < \kappa$. One can assume from the start that Φ is a coproduct of $\leq \kappa$ representable cofunctors. If Φ is the coproduct of $< \kappa$ representable cofunctors, we are done. Suppose, therefore, that $\Phi = \coprod_{0 \leq \alpha < \kappa} \mathrm{Mor}\,(—, X_\alpha)$. The idea then is to define for each $\alpha \in [\omega, \kappa[$ a subobject $\Psi_\alpha \subset F$ such that $\alpha < \beta \Rightarrow \Psi_\alpha \subset \Phi_\beta$ and which has a free resolution $0 \to \Psi''_\alpha \to \Psi'_\alpha \to \Psi_\alpha \to 0$, where Ψ'_α, Ψ''_α are coproducts of $\leq \#(\alpha)$ representable cofunctors. Matters will be arranged so as to ensure that $\coprod_{i<\alpha} \mathrm{Mor}\,(—, X_i) \to F$ factors as $\coprod_{i<\alpha} \mathrm{Mor}\,(—, X_i) \to \Psi_\alpha \to F$. In addition, when $\alpha < \beta$, there will be a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \Psi''_\alpha \arrow[r] \arrow[d] & \Psi'_\alpha \arrow[r] \arrow[d] & \Psi_\alpha \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \Psi''_\beta \arrow[r] & \Psi'_\beta \arrow[r] & \Psi_\beta \arrow[r] & 0
\end{tikzcd}$$

with $\Psi'_\alpha \to \Psi'_\beta$, $\Psi''_\alpha \to \Psi''_\beta$ split monomorphisms, and when $\alpha < \beta < \gamma$, the composite

$$\begin{tikzcd}
0 \arrow[r] & \Psi''_\alpha \arrow[r] \arrow[d] & \Psi'_\alpha \arrow[r] \arrow[d] & \Psi_\alpha \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \Psi''_\beta \arrow[r] \arrow[d] & \Psi'_\beta \arrow[r] \arrow[d] & \Psi_\beta \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \Psi''_\gamma \arrow[r] & \Psi'_\gamma \arrow[r] & \Psi_\gamma \arrow[r] & 0
\end{tikzcd}$$

will equal

$$\begin{tikzcd} 0 \arrow[r] & \Psi''_\alpha \arrow[r] \arrow[d] & \Psi'_\alpha \arrow[r] \arrow[d] & \Psi_\alpha \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & \Psi''_\gamma \arrow[r] & \Psi'_\gamma \arrow[r] & \Psi_\gamma \arrow[r] & 0 \end{tikzcd}$$

. Thus determine Ψ_ω by applying the above to the arrow $\coprod\limits_{i<\omega} \text{Mor}\,(\text{—}, X_i) \to F$. Proceeding, let $\omega < \alpha$, the supposition being that the Ψ_i have been defined $\forall\ i < \alpha$. If α is a successor ordinal, say $\alpha = \beta + 1$, set $\kappa' = \#(\Psi_\beta)$ and consider the morphism $\Psi_\beta \oplus \text{Mor}\,(\text{—}, X_\beta) \to F$. Appeal to the induction hypothesis secures a factorization $\Psi_\beta \oplus \text{Mor}\,(\text{—}, X_\beta) \to \Psi_{\beta+1} \to F$. Ψ_β is obviously a subobject of $\Psi_{\beta+1}$ and since **C** satisfies condition κ', the lemma guarantees that the free resolution $0 \to \Psi''_\beta \to \Psi'_\beta \to \Psi_\beta \to 0$ can be extended to a map of free resolutions

$$\begin{tikzcd} 0 \arrow[r] & \Psi''_\beta \arrow[r] \arrow[d] & \Psi'_\beta \arrow[r] \arrow[d] & \Psi_\beta \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & \Psi''_{\beta+1} \arrow[r] & \Psi'_{\beta+1} \arrow[r] & \Psi_{\beta+1} \arrow[r] & 0 \end{tikzcd}$$

with $\Psi'_\beta \to \Psi'_{\beta+1}$, $\Psi''_\beta \to \Psi''_{\beta+1}$ split monomorphisms and $\Psi'_{\beta+1}$, $\Psi''_{\beta+1}$ (as well as $\Psi'_{\beta+1}/\Psi'_\beta$, $\Psi''_{\beta+1}/\Psi''_\beta$) a coproduct of $\leq \kappa'$ representable cofunctors. If α is a limit ordinal, put $\Psi_\alpha = \text{colim}\ \Psi_i$, $\Psi'_\alpha = \text{colim}\ \Psi'_i$, $\Psi''_\alpha = \text{colim}\ \Psi''_i$. That Ψ'_α, Ψ''_α are in fact coproducts of $\leq \#(\alpha)$ representable cofunctors follows upon observing that $\Psi'_\alpha = \Psi'_\omega \oplus \{\coprod\limits_{\omega \leq i < \alpha} \Psi'_{i+1}/\Psi'_i\}$, $\Psi''_\alpha = \Psi''_\omega \oplus \{\coprod\limits_{\omega \leq i < \alpha} \Psi''_{i+1}/\Psi''_i\}$. Conclusion: **C** satisfies condition κ.]

LEMMA Suppose that **I** is a countable filtered category—then $\exists$ a final functor $[\mathbf{N}] \to \mathbf{I}$.

[One can find a directed set $(J, \leq)$ and a final functor $\mathbf{J} \to \mathbf{I}$ (cf. p. 0–11). Since **I** is countable, so is **J** (this fact is contained in the passage from **I** to **J** (Cordier-Porter[†])). Arrange the elements of **J** in a sequence $j_0, j_1, \ldots$, and take $k_0 = j_0$, $k_n \geq k_{n-1}, j_n$ $(n \geq 1)$ to get a final functor $[\mathbf{N}] \to \mathbf{J}$.]

EXAMPLE Consider $\mathbf{D}(A\text{-}\mathbf{MOD})$, where A is commutative and noetherian—then if $\mathbf{D}(A\text{-}\mathbf{MOD})$ admits Adams representability, every flat A-module has projective dimension ≤ 1 (Neeman[‡]). Example: Take $A = \mathbf{C}[x, y]$—then the projective dimension of $\mathbf{C}(x, y)$ is 2, therefore in this case $\mathbf{D}(A\text{-}\mathbf{MOD})$ does not admit Adams representability.

[Note: Recalling the characterization of compact objects in $\mathbf{D}(A\text{-}\mathbf{MOD})$ mentioned on p. 15–13, Neeman's countability criterion implies that $\mathbf{D}(A\text{-}\mathbf{MOD})$ admits Adams representability provided that A is countable.]

Let **C** be a compactly generated triangulated category. Suppose that **D** is a reflective subcategory of **C**, R a reflector for **D**. Put $T = \iota \circ R$, where $\iota : \mathbf{D} \to \mathbf{C}$ is the inclusion, and let (S, D) be the associated orthogonal pair (cf. p. 0–22)—then T is said to be a localization functor if T is a triangulated functor.

[†] *Shape Theory*, Ellis Horwood (1989), 42–44.

[‡] *Topology* 36 (1997), 619–645.

[Note: The elements of S are the T-equivalences. The elements of D (i.e., the X such that $\epsilon_X : X \to TX$ is an isomorphism) are the T-local objects and the elements of $\ker T$ (i.e., the X such that $TX = 0$) are the T-acyclic objects.]

Observation: If X is T-acyclic and if Y is T-local, then $\begin{cases} \mathrm{Mor}\,(\Sigma^n X, Y) = 0 \\ \mathrm{Mor}\,(\Omega^n X, Y) = 0 \end{cases}$ $(n \geq 0)$.

PROPOSITION 25 Let $\mathbf{C}$ be a compactly generated triangulated category. Suppose that T is a localization functor—then $\forall\, X \in \mathrm{Ob}\,\mathbf{C}$, $\exists$ an exact triangle $X_T \to X \xrightarrow{\epsilon_X} TX \to \Sigma X_T$, where X_T is T-acyclic.

[Place $X \xrightarrow{\epsilon_X} TX$ in an exact triangle $X_T \to X \xrightarrow{\epsilon_X} TX \to \Sigma X_T$ and apply T to get an exact triangle $TX_T \to TX \xrightarrow{T\epsilon_X} T^2X \to \Sigma TX_T$. Since $T\epsilon_X$ is an isomorphism, $TX_T = 0$.]

The following lemma has been implicitly used in the proof of Proposition 25.

LEMMA Let $\mathbf{C}$ be a triangulated category. Suppose that $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} \Sigma X$ is an exact triangle, where v is an isomorphism—then $X = 0$.

[The triangle $Y \xrightarrow{v} Z \to 0 \to \Sigma Y$ is exact (cf. p. 15–6), thus the triangle $0 \to Y \xrightarrow{v} Z \to 0$ is exact (cf. p. 15–2) and $\begin{cases} v \circ u = 0 \Rightarrow u = 0 \\ w \circ v = 0 \Rightarrow w = 0 \end{cases}$ (cf. Proposition 3). Therefore the diagram

$$\begin{tikzcd}
0 \arrow[r] \arrow[d] & Y \arrow[r, "v"] \arrow[d, equal] & Z \arrow[r] \arrow[d, equal] & 0 \arrow[d] \\
X \arrow[r, "0"'] & Y \arrow[r, "v"'] & Z \arrow[r, "0"'] & \Sigma X
\end{tikzcd}$$

commutes, so $0 \to X$ is an isomorphism (cf. p. 15–4).]

PROPOSITION 26 Let $\mathbf{C}$ be a compactly generated triangulated category. Suppose that T is a localization functor—then the T-acyclic objects are the object class of a coreflective subcategory of $\mathbf{C}$, the coreflector being the functor that sends X to X_T.

[Note: There is a natural isomorphism $(\Sigma X)_T \to \Sigma X_T$ and $X \to Y \to Z \to \Sigma X$ exact $\Rightarrow X_T \to Y_T \to Z_T \to \Sigma X_T$ exact.]

PROPOSITION 27 Let $\mathbf{C}$ be a compactly generated triangulated category. Suppose that T is a localization functor—then X is T-local iff $\mathrm{Mor}\,(Y, X) = 0$ for all T-acyclic Y and X is T-acyclic iff $\mathrm{Mor}\,(X, Y) = 0$ for all T-local Y.

[To see that the condition characterizes the T-local objects, take $Y = X_T$. Thus the arrow $X_T \to X$ is the zero morphism, so the isomorphism $(X_T)_T \to X_T$ is the zero morphism, hence $X_T = 0$, which implies that $\epsilon_X : X \to TX$ is an isomorphism.]

Using the notation of p. 15–48, take for $\mathcal{T}$ the class of T-acyclic objects and take for $\mathcal{F}$ the class of T-local objects—then $\text{Ann}_{\text{L}}\,\mathcal{F} = \mathcal{T}$ and $\text{Ann}_{\text{R}}\,\mathcal{T} = \mathcal{F}$ (cf. Proposition 27), i.e., the pair $(\mathcal{T}, \mathcal{F})$ is a torsion theory on **C**.

PROPOSITION 28 Let **C** be a compactly generated triangulated category. Suppose that T is a localization functor—then the class of T-local objects is the object class of a thick subcategory of **C** which is closed under the formation of products in **C**.

[Given an exact triangle $X \to Y \to Z \to \Sigma X$, there is a commutative diagram

$$\begin{tikzcd} X \arrow[r] \arrow[d, "\epsilon_X"] & Y \arrow[r] \arrow[d, "\epsilon_Y"] & Z \arrow[r] \arrow[d, "\epsilon_Z"] & \Sigma X \arrow[d, "\Sigma\epsilon_X"] \\ TX \arrow[r] & TY \arrow[r] & TZ \arrow[r] & \Sigma TX \end{tikzcd}$$

of exact triangles, thus if two of ϵ_X, ϵ_Y, ϵ_Z are isomorphisms, so is the third (cf. p. 15–4). Therefore **D** is a triangulated subcategory of **C** (cf. Proposition 7). Next, for any pair of morphisms $i : X \to Y$, $r : Y \to X$ with $r \circ i = \text{id}_X$, there is a commutative diagram

$$\begin{tikzcd} X \arrow[r, "i"] \arrow[d, "\epsilon_X"] & Y \arrow[r, "r"] \arrow[d, "\epsilon_Y"] & X \arrow[d, "\epsilon_X"] \\ TX \arrow[r, "Ti"'] & TY \arrow[r, "Tr"'] & TX \end{tikzcd}$$

. Accordingly, ϵ_X is a retract of ϵ_Y (cf. p. 12–1) and if ϵ_Y is an isomorphism, then the same is true of ϵ_X, hence **D** is thick.]

[Note: Analogously, the class of T-acyclic objects is the object class of a thick subcategory of **C** which is closed under the formation of coproducts in **C**.]

Remark: **D** is not necessarily compactly generated. In fact, there may be no nonzero compact objects in **D** at all.

EXAMPLE Suppose that **C** is a compactly generated triangulated category. Let $\mathcal{K} = \{K\}$ be a set of compact objects. Denote by **K** the thick subcategory generated by $\mathcal{K}$ and denote by **L** the smallest triangulated subcategory of **C** containing $\mathcal{K}$ and closed under the formation of coproducts in **C**—then **K** is a subcategory of **L** (via the Eilenberg swindle) and there is a localization functor $T_{\mathcal{K}}$ whose acyclic objects are the objects of **L**. Moreover, every compact object in **C** which lies in **L** must lie in **K**.

[Write $\overline{\mathcal{K}} = \{\overline{K}\}$ for the set $\bigcup\limits_{K}\{\Sigma^n K : n \geq 0\} \cup \bigcup\limits_{K}\{\Omega^n K : n \geq 0\}$ and let $\mathcal{K}^+$ be the class of objects in **C** that are coproducts of objects in $\overline{\mathcal{K}}$—then $\forall\, X \in \text{Ob}\,\mathbf{C}$, $\exists$ an object $(\mathbf{X}, \mathbf{f})$ in $\mathbf{FIL}(\mathbf{C})$, completable in $\mathcal{K}^+$ (obvious definition), and an arrow $\text{tel}(\mathbf{X}, \mathbf{f}) \to X$ such that $\text{Mor}\,(Y, \text{tel}(\mathbf{X}, \mathbf{f})) \approx \text{Mor}\,(Y, X)$ for all Y in **L** (proceed as in the proof of the Brown representability theorem) (cf. Proposition 16)). Taking $X_{\mathcal{K}} = \text{tel}(\mathbf{X}, \mathbf{f})$, define $T_{\mathcal{K}}X$ by the exact triangle $X_{\mathcal{K}} \to X \to T_{\mathcal{K}}X \to \Sigma X_{\mathcal{K}}$.]

[Note: The $T_{\mathcal{K}}$ are the compact localization functors.]

Let **C** be a compactly generated triangulated category—then a localization functor T is said to be smashing if it preserves coproducts or, equivalently, if **D** is closed under the formation of coproducts in **C** (recall Proposition 12).

Example: A compact localization functor is smashing.

[Note: The telescope conjecture is said to hold for **C** if every smashing localization functor is compact. In the stable homotopy category, the telescope conjecture is false but in the derived category $\mathbf{D}(A\text{-}\mathbf{MOD})$, where A is commutative and noetherian, the telescope conjecture is true.]

FACT Suppose that **C** is a compactly generated triangulated category. Let T be a localization functor—then T is smashing iff K compact in $\mathbf{C} \Rightarrow RK$ compact in **D**.

Application: If T is smashing, then **D** is a compactly generated triangulated category.

FACT Suppose that **C** admits Adams representability. Let T be a localization functor—then **D** admits Adams representability provided that T is smashing.

Notation: Let **C** be a triangulated category with products. Suppose given an object $(\mathbf{X},\mathbf{f})$ in $\mathbf{TOW}(\mathbf{C})$—then $\Sigma\,\mathrm{mic}(\mathbf{X},\mathbf{f})$ is any completion of $\prod_n X_n \xrightarrow{\mathrm{sf}} \prod_n X_n$ to an exact triangle (cf. TR$_3$), where $\mathrm{pr}_n \circ \mathrm{sf} = \mathrm{pr}_n - f_n \circ \mathrm{pr}_{n+1}$.

EXAMPLE Suppose that **C** is a compactly generated triangulated category. Let T be a localization functor and let $(\mathbf{X},\mathbf{f})$ be an object in $\mathbf{TOW}(\mathbf{C})$ such that $\forall\, n$, X_n is T-local—then $\mathrm{mic}(\mathbf{X},\mathbf{f})$ is T-local.

Let **C** be a compactly generated triangulated category. Suppose that $F : \mathbf{C} \to \mathbf{AB}$ is an exact functor. Let S_F be the class of morphisms $X \xrightarrow{u} Y$ such that $\forall\, n \geq 0$, $\begin{cases} F\Sigma^n u \\ F\Omega^n u \end{cases}$ is an isomorphism—then (1) S_F admits a calculus of left and right fractions and contains the isomorphisms of **C**; (2) $u \in S_F \Rightarrow \Sigma u\ \&\ \Omega u \in S_F$; (3) $f, g \in S_F \Rightarrow \exists\, h \in S_F$ (data as in TR$_5$); (4) $u \in S_F$ iff $\exists\, f, g \in \mathrm{Mor}\,\mathbf{C} : u \circ f \in S_F$, $g \circ u \in S_F$. Therefore the metacategory $S_F^{-1}\mathbf{C}$ is triangulated and $L_{S_F} : \mathbf{C} \to S_F^{-1}\mathbf{C}$ is a triangulated functor.

[Note: In the terminology of p. 15–12, S_F is multiplicative.]

PROPOSITION 29 Let **C** be a compactly generated triangulated category. Suppose that $F : \mathbf{C} \to \mathbf{AB}$ is an exact functor which converts coproducts into direct sums. Assume: The metacategory $S_F^{-1}\mathbf{C}$ is isomorphic to a category—then $S_F^{\perp}$ is the object class of a reflective subcategory of **C**.

[Argue as in the example on p. 5–79. Thus the triangulated functor $L_{S_F} : \mathbf{C} \to S_F^{-1}\mathbf{C}$ preserves coproducts, so $\forall\, Y \in \mathrm{Ob}\, S_F^{-1}\mathbf{C}$, $\mathrm{Mor}\,(L_{S_F}—, Y)$ is an exact cofunctor $\mathbf{C} \to \mathbf{AB}$

which converts coproducts into products, hence by the Brown representability theorem, $\exists\, Y_{S_F} \in \operatorname{Ob}\mathbf{C} : \operatorname{Mor}(L_{S_F}X, Y) \approx \operatorname{Mor}(X, Y_{S_F})$.]

[Note: The procedure generates an idempotent triple $\mathbf{T}_F = (T_F, m, \epsilon)$ in $\mathbf{C}$ ($T_F : \mathbf{C} \to \mathbf{C}$ is a localization functor, S_F is the class of T_F-equivalences, and $O_F = \ker T_F$ (i.e., X is T_F-acyclic iff $\forall\, n \geq 0$, $\begin{cases} F\Sigma^n X = 0 \\ F\Omega^n X = 0 \end{cases}$ (cf. p. 15–7))).]

Maintaining the assumption that $\mathbf{C}$ is a compactly generated triangulated category, given any $X \in \operatorname{Ob}\mathbf{C}$, put $\kappa_X = \sum\limits_{U} \#(\operatorname{Mor}(\overline{U}, X))$ and for κ an infinite cardinal $\geq \kappa_{\mathcal{U}} \equiv \sum\limits_{U} \kappa_U$, let $\mathbf{C}_\kappa$ be the full subcategory of $\mathbf{C}$ whose objects are the X such that $\kappa_X \leq \kappa$—then $\mathbf{C}_\kappa$ is a thick subcategory of $\mathbf{C}$ which is closed under the formation of coproducts in $\mathbf{C}$ indexed by sets of cardinality $\leq \kappa$ and $\mathbf{C} = \bigcup\limits_{\kappa} \mathbf{C}_\kappa$.

[Note: $\mathbf{C}_\kappa$ contains $\mathcal{U}$, hence $\mathbf{C}_\kappa$ contains $\operatorname{cpt}\mathbf{C}$ (by the theorem of Neeman-Ravenel).]

Notation: $\mathcal{U}_\kappa^+$ is the class of objects in $\mathbf{C}$ that are coproducts of $\leq \kappa$ objects in $\overline{\mathcal{U}}$.

LEMMA Let $\{G_n\}$ be a sequence of abelian groups. Assume: $\forall\, n$, $\#(G_n) \leq \kappa$, where κ is an infinite cardinal—then the cardinality of $\bigoplus\limits_{n} G_n$ is bounded by κ.

[Note: Another triviality is the fact that if $G' \to G \to G''$ is an exact sequence of abelian groups and if $\#(G') \leq \kappa$, $\#(G'') \leq \kappa$, where κ is an infinite cardinal, then $\#(G) \leq \kappa^2 = \kappa$.]

PROPOSITION 30 Let $\mathbf{C}$ be a compactly generated triangulated category. Fix an infinite cardinal $\kappa \geq \kappa_{\mathcal{U}}$—then $X \in \operatorname{Ob}\mathbf{C}_\kappa$ iff $X \approx \operatorname{tel}(\mathbf{X}, \mathbf{f})$, where $(\mathbf{X}, \mathbf{f})$ is completable in $\mathcal{U}_\kappa^+$.

[The sufficiency is clear (cf. Proposition 13) and the necessity can be established by reworking the proof of Proposition 16 (with $F = \operatorname{Mor}(—, X)$).]

[Note: It is a corollary that $\mathbf{C}_\kappa$ has a small skeleton $\overline{\mathbf{C}}_\kappa$.]

LEMMA Let $\mathbf{C}$ be a compactly generated triangulated category. Suppose that $F : \mathbf{C} \to \mathbf{AB}$ is an exact functor which converts coproducts into direct sums. Put $H = \bigoplus\limits_{n \geq 0} F \circ \Sigma^n \oplus \bigoplus\limits_{n > 0} F \circ \Omega^n$—then $H : \mathbf{C} \to \mathbf{AB}$ is an exact functor which converts coproducts into direct sums and a morphism $X \xrightarrow{u} Y$ is in S_F iff $Hu : HX \to HY$ is an isomorphism.

PROPOSITION 31 Let $\mathbf{C}$ be a compactly generated triangulated category. Suppose that $F : \mathbf{C} \to \mathbf{AB}$ is an exact functor which converts coproducts into direct sums—then $\forall$ infinite cardinal $\kappa >> \kappa_{\mathcal{U}}$, $\exists$ an infinite cardinal $\delta(\kappa) \geq \kappa$ such that $\forall\, Y : \#(HY) \leq \kappa$, $\exists\, X \in \operatorname{Ob}\mathbf{C}_{\delta(\kappa)}$ & $X \xrightarrow{u} Y$ with $Hu : HX \to HY$ an isomorphism.

[Bearing in mind that cpt $\mathbf{C}$ has a small skeleton $\overline{\text{cpt}\,\mathbf{C}}$ (cf. Proposition 19), fix an infinite cardinal $\kappa_H > \sup\{\#(H\overline{K}) : \overline{K} \in \text{Ob}\,\overline{\text{cpt}\,\mathbf{C}}\}$ and take $\kappa = \delta_0(\kappa) > \max\{\kappa_H, \kappa_U\}$. Since $HY \approx \underset{Y}{\text{colim}}\, HL$ (cf. p. 15–21), $\forall\, y \in HY$, $\exists$ an object $L \to Y$ in $\overline{\Lambda}/Y : y \in \text{im}(HL \to HY)$. Therefore one can choose objects $L_i \to Y$ in $\overline{\Lambda}/Y$ indexed by a set I of cardinality $\leq \delta_0(\kappa)$ such that $Hu_0 : HX_0 \to HY$ is surjective. Here $X_0 = \coprod\limits_I L_i$ and $u_0 : X_0 \to Y$ is the coproduct of the $L_i \to Y$. Because the L_i are compact and $\#(I) \leq \delta_0(\kappa)$, $X_0 \in \text{Ob}\,\mathbf{C}_{\delta_0(\kappa)}$. Embed $X_0 \xrightarrow{u_0} Y$ in an exact triangle $Y' \xrightarrow{u'} X_0 \xrightarrow{u_0} Y \to \Sigma Y'$. Claim: $\exists$ an infinite cardinal $\delta_1(\kappa) \geq \delta_0(\kappa)$ for which $\#(HY') \leq \delta_1(\kappa)$ independently of the choices (i.e., the bound is a function only of the initial supposition that $\#(HY) \leq \kappa$). To see this, note that $\#(H\Sigma^n Y) \leq \kappa$, $\#(H\Omega^n Y) \leq \kappa$ and $\#(H\Sigma^n X_0) \leq \kappa_H^\kappa$, $\#(H\Omega^n X_0) \leq \kappa_H^\kappa$ and use the long exact sequence generated by H. Repeat the process: $u_0' : \coprod\limits_{I'} L_{i'}' \to Y'$ $(\#(I') \leq \delta_1(\kappa))$ and place $u' \circ u_0'$ in an exact triangle $Z \to \coprod\limits_{I'} L_{i'}' \xrightarrow{u' \circ u_0'} X_0 \to \Sigma Z$. Consider now the diagram

$$\begin{tikzcd}
X_0 \arrow[r] \arrow[d, equal] & \Sigma Z \arrow[r] & \coprod\limits_{I'} \Sigma L_{i'}' \arrow[r, "-\Sigma(u' \circ u_0')"] \arrow[d, "\Sigma u_0'"] & \Sigma X_0 \arrow[d, equal] \\
X_0 \arrow[r, "u_0"'] & Y \arrow[r] & \Sigma Y' \arrow[r, "-\Sigma u'"'] & \Sigma X_0
\end{tikzcd}$$

. The rows being in Δ, one can find a filler $u_1 : \Sigma Z \to Y$ (cf. Proposition 2). Put $X_1 = \Sigma Z$ (thus $X_1 \in \text{Ob}\,\mathbf{C}_{\delta_1(\kappa)}$) and let f_0 be the arrow $X_0 \to X_1$. By construction,

$$\begin{tikzcd}
X_0 \arrow[r, "f_0"] \arrow[d, "u_0"'] & X_1 \arrow[d, "u_1"] \\
Y \arrow[r, equal] & Y
\end{tikzcd}$$

commutes and $\ker Hf_0 = \ker Hu_0$. Continuing, one produces $\forall\, n$ a commutative diagram

$$\begin{tikzcd}
X_n \arrow[r, "f_n"] \arrow[d, "u_n"'] & X_{n+1} \arrow[d, "u_{n+1}"] \\
Y \arrow[r, equal] & Y
\end{tikzcd}$$

, where $\ker Hf_n = \ker Hu_n$ and $X_n \in \mathbf{C}_{\delta_n(\kappa)}$ $(\delta_n(\kappa) \leq \delta_{n+1}(\kappa))$. Definition: $X = \text{tel}(\mathbf{X}, \mathbf{f})$—then $X \in \text{Ob}\,\mathbf{C}_{\delta(\kappa)}$ $(\delta(\kappa) > (\sup\{\delta_n(\kappa)\})^\omega$ (cf. infra)), $HX \approx \text{colim}\, HX_n$ and there is an arrow $X \xrightarrow{u} Y$ with $Hu : HX \to HY$ an isomorphism (injectivity from the condition on the kernels, surjectivity from the surjectivity of Hu_0).]

Thanks to Proposition 13, $\forall\, \overline{U} \in \overline{\mathcal{U}}$, colim $\text{Mor}(\overline{U}, X_n) \approx \text{Mor}(\overline{U}, \text{tel}(\mathbf{X}, \mathbf{f}))$, hence $\#(\text{Mor}(\overline{U}, \text{tel}(\mathbf{X}, \mathbf{f}))) \leq \prod\limits_n \#(\text{Mor}(\overline{U}, X_n)) \leq \prod\limits_n \delta_n(\kappa) \leq (\sup\{\delta_n(\kappa)\})^\omega$.

BOUSFIELD-MARGOLIS LOCALIZATION THEOREM Let $\mathbf{C}$ be a compactly generated triangulated category. Suppose that $F : \mathbf{C} \to \mathbf{AB}$ is an exact functor which converts coproducts into direct sums—then there exists a localization functor T_F such that $S_F^\perp$ is the class of T_F-local objects.

[In view of Proposition 29, the point is to show that the metacategory $S_F^{-1}\mathbf{C}$ is isomorphic to a category. Thus fix $X, Y \in \operatorname{Ob} S_F^{-1}\mathbf{C}$ $(= \operatorname{Ob}\mathbf{C})$ and $\kappa >> \kappa_{\mathcal{U}} : X, Y \in \operatorname{Ob}\mathbf{C}_\kappa$ & $\#(HX) \leq \kappa$, $\#(HY) \leq \kappa$. By definition, $\operatorname{Mor}(X,Y)$ is a conglomerate of equivalence classes of pairs $(s,f) : X \xrightarrow{f} Y' \xleftarrow{s} Y$ (cf. p. 0–31). Given such a pair (s,f), consider an exact triangle $Z \rightarrow X \amalg Y \rightarrow Y' \rightarrow \Sigma Z$. Since $HY \approx HY'$, $\#(HZ) \leq \kappa$. Using Proposition 31, choose $W \in \operatorname{Ob}\mathbf{C}_{\delta(\kappa)}$ & $W \xrightarrow{u} Z$ with $Hu : HW \rightarrow HZ$ an isomorphism. There is a diagram

$$\begin{tikzcd}
W \arrow[r] \arrow[d, "u"] & X \amalg Y \arrow[r, "\pi''"] \arrow[d, equal] & Y'' \arrow[r] & \Sigma W \arrow[d, "\Sigma u"] \\
Z \arrow[r] & X \amalg Y \arrow[r, "\pi'"'] & Y' \arrow[r] & \Sigma Z
\end{tikzcd}$$

and a filler $\phi : Y'' \rightarrow Y'$ (cf. TR_5) which is necessarily in S_F. Note too that $Y'' \in \operatorname{Ob}\mathbf{C}_{\delta(\kappa)}$. Put $g = \pi'' \circ \text{in}_X$, $t = \pi'' \circ \text{in}_Y$—then $\phi \circ g = f$, $\phi \circ t = s$, and $t \in S_F$, so the pair (s,f) is equivalent to the pair (t,g). But $\mathbf{C}_{\delta(\kappa)}$ has a small skeleton $\overline{\mathbf{C}}_{\delta(\kappa)}$ (cf. Proposition 30) and there is just a set of diagrams of the form $X \xrightarrow{\overline{g}} \overline{Y}'' \xleftarrow{\overline{t}} Y$, where $\overline{Y}'' \in \operatorname{Ob}\overline{\mathbf{C}}_{\delta(\kappa)}$.]

EXAMPLE Take for $\mathbf{C}$ the stable homotopy category **HSPEC** and fix an $\mathbf{X} \in \operatorname{Ob}\mathbf{C}$—then $H_{\mathbf{X}}(\mathbf{Y}) = [\mathbf{S}^0, \mathbf{X} \wedge \mathbf{Y}]$ is an exact functor $\mathbf{C} \rightarrow \mathbf{AB}$ which converts coproducts into direct sums and by the Bousfield-Margolis localization theorem, $S_{\mathbf{X}}^{\perp}$ is the object class of a reflective subcategory of $\mathbf{C}$, where $S_{\mathbf{X}}$ is the class of morphisms $\mathbf{Y}' \rightarrow \mathbf{Y}''$ such that $\forall\, n \in \mathbf{Z}$, $[\mathbf{S}^n, \mathbf{X} \wedge \mathbf{Y}'] \approx [\mathbf{S}^n, \mathbf{X} \wedge \mathbf{Y}'']$.

Given a closed category $\mathbf{C}$, the dual DX of an object X is $\hom(X, e)$.

(DU_1) $\forall\, X, X' \in \operatorname{Ob}\mathbf{C}$, $\exists$ a natural morphism $DX \otimes DX' \rightarrow D(X \otimes X')$.

[In the pairing $\hom(X,Y) \otimes \hom(X',Y') \rightarrow \hom(X \otimes X', Y \otimes Y')$, specialize and take $Y = e$, $Y' = e$.]

(DU_2) $\forall\, X \in \operatorname{Ob}\mathbf{C}$, $\exists$ a natural morphism $X \rightarrow D^2X$.

[$\operatorname{Mor}(X, D^2X) \approx \operatorname{Mor}(X, \hom(DX, e)) \approx \operatorname{Mor}(X \otimes DX, e) \approx \operatorname{Mor}(DX, \hom(X, e)) \approx \operatorname{Mor}(DX, DX)$.]

LEMMA Suppose that $\mathbf{C}$ is a closed category—then there is an arrow $\hom(X,Y) \otimes Z \rightarrow \hom(X, Y \otimes Z)$ natural in X, Y, Z.

Given a closed category $\mathbf{C}$, an object X is said to be dualizable if $\forall\, Y \in \operatorname{Ob}\mathbf{C}$, the arrow $DX \otimes Y \rightarrow \hom(X,Y)$ is an isomorphism. Example: e is dualizable.

[Note: When X is dualizable, $DX \otimes$— is a right adjoint for —$\otimes X$, hence $DX \otimes$— $\approx$ $\hom(X,$—$)$.]

EXAMPLE Let A be a commutative ring with unit—then an object X in A-**MOD** is dualizable iff X is finitely generated and projective.

Let **C** be a closed category—then an object X in **C** is invertible if there is an object X^{-1} in **C** and an isomorphism $X \otimes X^{-1} \to e$.

FACT Every invertible element X in **C** is dualizable and $DX \approx X^{-1}$.

PROPOSITION 32 Suppose that **C** is a closed category. Assume: X is dualizable—then DX is dualizable and the morphism $X \to D^2X$ is an isomorphism.

Remark: If **C** has coproducts, then $\forall\, Y$, $\coprod_i Y \otimes X_i \approx Y \otimes \coprod_i X_i$. If **C** has products, then $\forall$ dualizable X, $X \otimes \prod_i Y_i \approx \prod_i X \otimes Y_i$. Proof: $X \otimes \prod_i Y_i \approx D^2X \otimes \prod_i Y_i \approx \text{hom}(DX, \prod_i Y_i) \approx \prod_i \text{hom}(DX, Y_i) \approx \prod_i D^2X \otimes Y_i \approx \prod_i X \otimes Y_i$.

LEMMA Suppose that **C** is a closed category—then the pairing $\text{hom}(X,Y) \otimes \text{hom}(X',Y') \to \text{hom}(X \otimes X', Y \otimes Y')$ is an isomorphism if X and X' are dualizable or if X (X') is dualizable and $Y = e$ $(Y' = e)$.

PROPOSITION 33 Suppose that **C** is a closed category—then X, X' dualizable $\Rightarrow$ $X \otimes X'$ dualizable.

[$\forall\, Y$, $D(X \otimes X') \otimes Y \approx DX \otimes DX' \otimes Y \approx DX \otimes \text{hom}(X', Y) \approx \text{hom}(X, \text{hom}(X', Y)) \approx \text{hom}(X \otimes X', Y)$.]

LEMMA Suppose that **C** is a closed category—then the arrow $\text{hom}(X,Y) \otimes Z \to \text{hom}(X, Y \otimes Z)$ is an isomorphism if either X or Z is dualizable.

PROPOSITION 34 Suppose that **C** is a closed category—then X, X' dualizable $\Rightarrow$ $\text{hom}(X, X')$ dualizable.

[$\forall\, Y$, $D\,\text{hom}(X,X') \otimes Y \approx \text{hom}(\text{hom}(X,X'),e) \otimes Y \approx \text{hom}(DX \otimes X', e) \otimes Y \approx \text{hom}(DX, \text{hom}(X', e)) \otimes Y \approx \text{hom}(DX, DX') \otimes Y \approx \text{hom}(DX, DX' \otimes Y) \approx \text{hom}(DX, \text{hom}(X', Y)) \approx \text{hom}(DX \otimes X', Y) \approx \text{hom}(\text{hom}(X, X'), Y)$.]

FACT Let **C** be a closed category. Assume: X is dualizable—then X is a retract of $X \otimes DX \otimes X$.

Let **C** be a category with finite coproducts. Assume: **C** is closed and triangulated—then **C** is said to be a closed triangulated category (CTC) if there is a natural isomorphism ζ, where $\zeta_{X,Y} : \Sigma X \otimes Y \to \Sigma(X \otimes Y)$, subject to the following assumptions.

[Note: From the existence of ζ, one derives the existence of a natural isomorphism η, where $\eta_{X,Y} : \Omega\,\text{hom}(X,Y) \to \text{hom}(\Sigma X, Y)$.]

(CTC$_1$) The diagram
$$\begin{tikzcd} \Sigma X \otimes e \arrow[r, "\zeta_{X,e}"] \arrow[dr, "R_{\Sigma X}"'] & \Sigma(X \otimes e) \arrow[d, "\Sigma R_X"] \\ & \Sigma X \end{tikzcd}$$
commutes.

(CTC$_2$) The diagram
$$\begin{tikzcd} (\Sigma X \otimes Y) \otimes Z \arrow[r, "\zeta_{X,Y} \otimes \mathrm{id}_Z"] & \Sigma(X \otimes Y) \otimes Z \arrow[r, "\zeta_{X \otimes Y, Z}"] & \Sigma((X \otimes Y) \otimes Z) \\ \Sigma X \otimes (Y \otimes Z) \arrow[u, "A"] \arrow[rr, "\zeta_{X, Y \otimes Z}"'] & & \Sigma(X \otimes (Y \otimes Z)) \arrow[u, "\Sigma A"'] \end{tikzcd}$$
commutes.

(CTC$_3$) If $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} \Sigma X$ is an exact triangle, then $\forall\ W \in \operatorname{Ob} \mathbf{C}$, the triangle $X \otimes W \xrightarrow{u \otimes \mathrm{id}_W} Y \otimes W \xrightarrow{v \otimes \mathrm{id}_W} Z \otimes W \xrightarrow{\zeta_{X,W} \circ (w \otimes \mathrm{id}_W)} \Sigma(X \otimes W)$ is exact.

(CTC$_4$) If $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} \Sigma X$ is an exact triangle, then $\forall\ W \in \operatorname{Ob} \mathbf{C}$, the triangle $\Omega \operatorname{hom}(X, W) \xrightarrow{-(w^* \circ \eta_{X,W})} \operatorname{hom}(Z, W) \xrightarrow{v^*} \operatorname{hom}(Y, W) \xrightarrow{\nu^{-1}_{\operatorname{hom}(X,W)} \circ u^*} \Sigma\Omega \operatorname{hom}(X, W)$ is exact.

(CTC$_5$) The diagram
$$\begin{tikzcd} \Sigma e \otimes \Sigma e \arrow[r, "\approx"] \arrow[d, "\mathrm{T}"'] & \Sigma^2 e \arrow[d, "-1"] \\ \Sigma e \otimes \Sigma e \arrow[r, "\approx"'] & \Sigma^2 e \end{tikzcd}$$
commutes.

Remarks: (1) If $\overline{\zeta}_{e,X}$ is the composite $\Sigma \mathrm{T}_{X,e} \circ \zeta_{X,e} \circ \mathrm{T}_{e,\Sigma X}$, then the diagram
$$\begin{tikzcd} e \otimes \Sigma X \arrow[r, "\overline{\zeta}_{e,X}"] \arrow[d, "L_{\Sigma X}"'] & \Sigma(e \otimes X) \arrow[dl, "\Sigma L_X"] \\ \Sigma X \end{tikzcd}$$
commutes; (2) The additive functor $- \otimes W : \mathbf{C} \to \mathbf{C}$ is a triangulated functor (this is the content of CTC$_3$); (3) The additive functor $\operatorname{hom}(-, W) : \mathbf{C} \to \mathbf{C}^{\mathrm{OP}}$ is a triangulated functor (this is the content of CTC$_4$); (4) If $m, n \in \mathbf{N}$, then the diagram
$$\begin{tikzcd} \Sigma^m e \otimes \Sigma^n e \arrow[r, "\approx"] \arrow[d, "\mathrm{T}"'] & \Sigma^{m+n} e \arrow[d, "(-1)^{mn}"] \\ \Sigma^n e \otimes \Sigma^m e \arrow[r, "\approx"'] & \Sigma^{m+n} e \end{tikzcd}$$
commutes.

Example: $D : \mathbf{C} \to \mathbf{C}^{\mathrm{OP}}$ is a triangulated functor.

Since the additive functor $\operatorname{hom}(W, -) : \mathbf{C} \to \mathbf{C}$ is a right adjoint for $- \otimes W$, it is necessarily triangulated (cf. p. 15–7).

Notation: $\operatorname{du} \mathbf{C}$ is the full, isomorphism closed subcategory of $\mathbf{C}$ whose objects are dualizable.

PROPOSITION 35 Let **C** be a CTC—then du **C** is a thick subcategory of **C**.

[Observe first that 0 is dualizable. This said, take any morphism $X \xrightarrow{u} Y$ in du **C** and complete it to an exact triangle $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} \Sigma X$ (cf. TR_3)—then $\forall\, W \in \text{Ob}\,\mathbf{C}$, there is a commutative diagram

$$\begin{tikzcd}
\Omega DX \otimes W \arrow[r] \arrow[d] & DZ \otimes W \arrow[r] \arrow[d] & DY \otimes W \arrow[r] \arrow[d] & \Sigma(\Omega DX \otimes W) \arrow[d] \\
\Omega\,\text{hom}(X,W) \arrow[r] & \text{hom}(Z,W) \arrow[r] & \text{hom}(Y,W) \arrow[r] & \Sigma\Omega\,\text{hom}(X,W)
\end{tikzcd},$$

where, by CTC_3 & CTC_4, the rows are exact. Specialized to the case $X = X$, $Y = X$, $Z = 0$, $u = \text{id}_X$ (cf. TR_2), it follows that the arrow $\Omega DX \otimes W \rightarrow \Omega\,\text{hom}(X,W)$ is an isomorphism (cf. p. 15–4), i.e., that the arrow $\text{hom}(\Sigma X, e) \otimes W \rightarrow \text{hom}(\Sigma X, W)$ is an isomorphism, so X dualizable $\Rightarrow$ ΣX dualizable. Next, X dualizable $\Rightarrow$ ΩX dualizable. Proof: $X \approx \text{hom}(e,X) \Rightarrow \Omega X \approx \Omega\,\text{hom}(e,X) \approx \text{hom}(\Sigma e, X)$ and e dualizable $\Rightarrow$ Σe dualizable, hence Proposition 34 is applicable. Returning to $X \xrightarrow{u} Y$, one concludes that the arrow $DZ \otimes W \rightarrow \text{hom}(Z,W)$ is an isomorphism (cf. p. 15–4), thus Z is dualizable. Therefore du **C** is a triangulated subcategory of **C**. Finally, suppose given a pair of morphisms $i : X \rightarrow Y$, $r : Y \rightarrow X$ with $r \circ i = \text{id}_X$ and Y dualizable—then $\forall\, W \in \text{Ob}\,\mathbf{C}$, there is a commutative diagram

$$\begin{tikzcd}
DX \otimes W \arrow[r, "Dr"] \arrow[d] & DY \otimes W \arrow[r, "Di"] \arrow[d] & DX \otimes W \arrow[d] \\
\text{hom}(X,W) \arrow[r, "r^*"'] & \text{hom}(Y,W) \arrow[r, "i^*"'] & \text{hom}(X,W)
\end{tikzcd}$$

, which shows that the arrow $DX \otimes W \rightarrow \text{hom}(X,W)$ is a retract of the arrow $DY \otimes W \rightarrow \text{hom}(Y,W)$. But the retract of an isomorphism is an isomorphism and this means that X is dualizable. Therefore du **C** is a thick subcategory of **C**.]

EXAMPLE Suppose that **C** is a CTC—then e dualizable $\Rightarrow$ Σe dualizable and $D\Sigma e = \text{hom}(\Sigma e, e) \approx \Omega\,\text{hom}(e,e) = \Omega e$. Therefore $\text{Mor}\,(Y, X \otimes \Omega e) \approx \text{Mor}\,(Y, D\Sigma e \otimes X) \approx \text{Mor}\,(Y, \text{hom}(\Sigma e, X)) \approx \text{Mor}\,(Y \otimes \Sigma e, X) \approx \text{Mor}\,(\Sigma Y, X) \approx \text{Mor}\,(Y, \Omega X) \Rightarrow X \otimes \Omega e \approx \Omega X$. Consequently, $\text{hom}(\Sigma X, Y) \approx \text{hom}(X, \text{hom}(\Sigma e, Y)) \approx \text{hom}(X, D\Sigma e \otimes Y) \approx \text{hom}(X, \Omega e \otimes Y) \approx \text{hom}(X, \Omega Y)$.

Suppose that **C** is a CTC—then **C** is said to be a compactly generated CTC if **C** is compactly generated and every $U \in \mathcal{U}$ is dualizable.

PROPOSITION 36 Let **C** be a compactly generated CTC—then X compact $\Rightarrow$ X dualizable.

[The thick subcategory generated by $\mathcal{U}$ is cpt **C** (theorem of Neeman-Ravenel). On the other hand, du **C** is thick (cf. Proposition 35) and contains $\mathcal{U}$.]

FACT Suppose that $\mathbf{C}$ is a compactly generated CTC—then X is dualizable iff $\forall$ collection $\{X_i\}$ of objects in $\mathbf{C}$, the arrow $\coprod_i \hom(X, X_i) \to \hom(X, \coprod_i X_i)$ is an isomorphism.

[Necessity: $\coprod_i \hom(X, X_i) \approx \coprod_i DX \otimes X_i \approx DX \otimes \coprod_i X_i \approx \hom(X, \coprod_i X_i)$.

Sufficiency: Let $\mathbf{D}$ be the full, isomorphism closed subcategory of $\mathbf{C}$ consisting of those Y for which the arrow $DX \otimes Y \to \hom(X, Y)$ is an isomorphism—then $\mathbf{D}$ is triangulated and closed under the formation of coproducts in $\mathbf{C}$. Moreover, $\mathbf{D}$ contains all the dualizable objects, so $\mathcal{U} \subset \operatorname{Ob} \mathbf{D}$. Therefore $\mathbf{D} = \mathbf{C}$ (cf. Proposition 14).]

LEMMA Let $\mathbf{C}$ be a CTC with coproducts—then X compact and Y dualizable $\Rightarrow X \otimes Y$ compact.

[$\bigoplus_i \operatorname{Mor}(X \otimes Y, Z_i) \approx \bigoplus_i \operatorname{Mor}(X, \hom(Y, Z_i)) \approx \bigoplus_i \operatorname{Mor}(X, DY \otimes Z_i) \approx \operatorname{Mor}(X, \coprod_i DY \otimes Z_i) \approx \operatorname{Mor}(X, DY \otimes \coprod_i Z_i) \approx \operatorname{Mor}(X \otimes Y, \coprod_i Z_i)$.]

Application: Let $\mathbf{C}$ be a compactly generated CTC—then X compact $\Rightarrow DX$ compact.

[X is dualizable (cf. Proposition 36), so DX is dualizable (cf. Proposition 32), hence DX is a retract of $DX \otimes D^2X \otimes DX$ (cf. p. 15–36) or still, is a retract of $DX \otimes X \otimes DX$ (cf. Proposition 32) and the lemma implies that $DX \otimes X \otimes DX$ is compact.]

Suppose that $\mathbf{C}$ is a compactly generated CTC—then $\mathbf{C}$ is said to be unital provided that e is compact.

PROPOSITION 37 Let $\mathbf{C}$ be a unital compactly generated CTC—then X dualizable $\Rightarrow X$ compact.

[By the lemma, $e \otimes X$ is compact.]

Consequently, in a unital compactly generated CTC, "compact" = "dualizable".

The stable homotopy category is a unital compactly generated CTC.

EXAMPLE Let A be a commutative ring with unit—then $\mathbf{D}(A\text{-}\mathbf{MOD})$ is a unital compactly generated CTC (Bökstedt-Neeman[†]).

Suppose that $\mathbf{C}$ is a compactly generated CTC—then a cohomology theory is an exact cofunctor $F : \mathbf{C} \to \mathbf{AB}$ which converts coproducts into products and a homology theory is an exact functor $F : \mathbf{C} \rightarrow \mathbf{AB}$ which converts coproducts into direct sums. According to the Brown representability theorem, every cohomology theory is representable. The situation for homology theories is different. Put $H_e(X) = \operatorname*{colim}_X \operatorname{Mor}(e, K)$ $(= \overline{\operatorname{Mor}(e, —)|\Lambda X})$ and $H_X(Y) = H_e(X \otimes Y)(X, Y \in \operatorname{Ob}\mathbf{C})$. Proposition 18 guarantees that H_e is a homology theory, thus H_X is also a homology theory (cf. CTC_3), and there is an arrow $H_X(Y) \rightarrow \operatorname{Mor}(e, X \otimes Y)$.

[†] *Compositio Math.* **86** (1993), 209–234.

[Note: When $\mathbf{C}$ is unital, $H_X(Y) \approx \operatorname{Mor}(e, X \otimes Y)$.]

LEMMA The arrow $H_X(Y) \rightarrow \operatorname{Mor}(e, X \otimes Y)$ is an isomorphism if X is compact.

[X compact $\Rightarrow$ X dualizable (cf. Proposition 36) $\Rightarrow \operatorname{Mor}(e, X \otimes Y) \approx \operatorname{Mor}(e, D^2X \otimes Y) \approx \operatorname{Mor}(e, D(DX) \otimes Y) \approx \operatorname{Mor}(e, \hom(DX, Y)) \approx \operatorname{Mor}(DX, Y)$. Since DX is compact (cf. p. 15–39), $\operatorname{Mor}(DX, —)$ is a homology theory. Therefore $\operatorname{Mor}(e, X \otimes —)$ is a homology theory. But Y compact $\Rightarrow X \otimes Y$ compact $\Rightarrow H_X(Y) \approx \operatorname{Mor}(e, X \otimes Y)$. In other words, the arrow $H_X \rightarrow \operatorname{Mor}(e, X \otimes —)$ is an isomorphism for compact Y, hence for all Y.]

FACT Suppose that $\mathbf{C}$ is a compactly generated CTC. Fix $X \in \operatorname{Ob}\mathbf{C}$—then $X \otimes Y = 0$ iff $\forall\, Z$, $H_X(Y \otimes Z) = 0$.

PROPOSITION 38 Let $\mathbf{C}$ be a compactly generated CTC. Assume: $\mathbf{C}$ admits Adams representability. Suppose that $F : \mathbf{C} \rightarrow \mathbf{AB}$ is a homology theory—then $\exists$ an $X \in \operatorname{Ob}\mathbf{C}$ and a natural isomorphism $H_X \rightarrow F$.

[The composite $F \circ D : \operatorname{cpt}\mathbf{C} \rightarrow \mathbf{AB}$ is an exact cofunctor, thus by ADR_1, $\exists$ an $X \in \operatorname{Ob}\mathbf{C}$ and a natural isomorphism $h_X \rightarrow F \circ D$. And: $\forall$ compact K, $H_X(K) \approx H_K(X) \approx \operatorname{Mor}(e, K \otimes X) \approx \operatorname{Mor}(DK, X) \approx h_X(DK) \approx FD^2K \approx FK$.]

[Note: It follows from ADR_2 that $\operatorname{Nat}(H_X, H_Y) \approx \operatorname{Mor}(X, Y)/\operatorname{Ph}(X, Y)$. Of course, $H_X \approx H_Y \Rightarrow X \approx Y$.]

EXAMPLE Suppose that $\mathbf{C}$ is a compactly generated CTC which admits Adams representability. Let $\Delta : \mathbf{I} \rightarrow \mathbf{C}$ be a diagram, where $\mathbf{I}$ is filtered—then a weak colimit L of Δ is a minimal weak colimit iff for every homology theory $F : \mathbf{C} \rightarrow \mathbf{AB}$, the arrow $\operatorname{colim} F\Delta_i \rightarrow FL$ is an isomorphism.

Suppose that $\mathbf{C}$ is a compactly generated CTC. Let T be a localization functor—then T is said to have the ideal property (IP) if $TX = 0 \Rightarrow T(X \otimes Y) = 0\ \forall\, Y$.

PROPOSITION 39 Let $\mathbf{C}$ be a compactly generated CTC. Suppose that T is a localization functor with the IP—then X T-acyclic and Y T-local $\Rightarrow \hom(X, Y) = 0$.

[$\forall\, Z$, $\operatorname{Mor}(Z, \hom(X, Y)) \approx \operatorname{Mor}(Z \otimes X, Y) \approx \operatorname{Mor}(X \otimes Z, Y) \approx \operatorname{Mor}(T(X \otimes Z), Y) = 0$.]

[Note: Conversely, X is T-local if $\hom(Y,X) = 0$ for all T-acyclic Y. In fact, $\mathrm{Mor}\,(Y,X) \approx \mathrm{Mor}\,(e \otimes Y, X) \approx \mathrm{Mor}\,(e, \hom(Y,X)) \approx \hom(Y,X) = 0$, so Proposition 27 is applicable. Example: X T-local $\Rightarrow \hom(Y,X)$ T-local $\forall\, Y$.]

Assuming still that T is a localization functor with the IP, consider the exact triangle $e_T \to e \xrightarrow{\epsilon} Te \to \Sigma e_T$ (cf. Proposition 25)—then by CTC$_3$, $\forall\ X \in \mathrm{Ob}\,\mathbf{C}$, the triangle $e_T \otimes X \to e \otimes X \xrightarrow{\epsilon_e \otimes \mathrm{id}_X} Te \otimes X \to \Sigma(e_T \otimes X)$ is exact. But $T(e_T \otimes X) = 0$, hence $TX \approx T(Te \otimes X)$. On the other hand, $Te \otimes X$ is T-local if X is dualizable. Proof: $Te \otimes X \approx \hom(DX, Te)$ and $\forall\ T$-acyclic Y, $\hom(Y, \hom(DX, Te)) \approx \hom(Y \otimes DX, Te) = 0$ (cf. Proposition 39).

EXAMPLE Suppose that $\mathbf{C}$ is a compactly generated CTC. Let T be a localization functor with the IP—then T is smashing iff $\forall\ X$, the composite $Te \otimes X \to T(Te \otimes X) \xrightarrow{\approx} TX$ is an isomorphism.

[By the above, $\mathcal{U}$ is contained in the class of X for which the composite in question is an isomorphism.]

FACT Suppose that $\mathbf{C}$ is a compactly generated CTC. Let T be a localization functor with the IP—then there is a canonical arrow $TX \otimes TY \to T(X \otimes Y)$.

[Working with the exact triangles $X \otimes Y_T \to X \otimes Y \to X \otimes TY \to \Sigma(X \otimes Y_T)$, $X_T \otimes TY \to X \otimes TY \to TX \otimes TY \to \Sigma(X_T \otimes TY)$, one finds that $T(\epsilon_X \otimes \epsilon_Y) : T(X \otimes Y) \to T(TX \otimes TY)$ is an isomorphism.]

FACT Suppose that $\mathbf{C}$ is a compactly generated CTC. Let T be a localization functor with the IP—then $\mathbf{D}$ is a CTC.

[Define $\otimes_T : \mathbf{D} \times \mathbf{D} \to \mathbf{D}$ by $X \otimes_T Y = R(X \otimes Y)$. Thus Re serves as the unit and the internal hom functor $\hom_T : \mathbf{D}^{\mathrm{OP}} \times \mathbf{D} \to \mathbf{D}$ sends (X,Y) to $\hom(X,Y)$ (which is automatically T-local).]

[Note: X dualizable in $\mathbf{C} \Rightarrow RX$ dualizable in $\mathbf{D}$.]

EXAMPLE Suppose that $\mathbf{C}$ is a compactly generated CTC. Let T be a localization functor with the IP. Assume: T is smashing—then $\mathbf{D}$ is a compactly generated CTC. In addition, $\mathbf{D}$ is a coreflective subcategory of $\mathbf{C}$.

[The coreflector $\mathbf{C} \to \mathbf{D}$ is the assignment $X \to \hom(Te, X)$.]

Suppose that $\mathbf{C}$ is a compactly generated CTC—then $\mathbf{C}$ is said to be monogenic if $\mathbf{C}$ is unital and $\begin{cases} \mathrm{Mor}\,(\Sigma^n e, X) = 0 \\ \mathrm{Mor}\,(\Omega^n e, X) = 0 \end{cases} \forall\, n \geq 0 \Rightarrow X = 0.$

The stable homotopy category is monogenic.

FACT Suppose that $\mathbf{C}$ is a monogenic compactly generated CTC. Let $\mathbf{D}$ be a thick subcategory of $\mathbf{C}$—then $\forall$ compact X, $X \otimes \mathrm{Ob}\,\mathbf{D} \subset \mathrm{Ob}\,\mathbf{D}$.

Notation: When **C** is monogenic, write S in place of e and Σ^{-1} in place of Ω, letting $S^{\pm n} = \Sigma^{\pm n} S (\Rightarrow S^k \otimes S^l \approx S^{k+l} \ \forall\, k, l \in \mathbf{Z})$, so $\forall\, X$, $\Sigma^{\pm 1} X \approx X \otimes S^{\pm 1}$.

[Note: The nth homotopy group $\pi_n(X)$ of X $(n \in \mathbf{Z})$ is $\operatorname{Mor}(S^n, X)$.]

LEMMA Let **C** be a monogenic compactly generated CTC—then a morphism $f : X \to Y$ in **C** is an isomorphism iff $\forall\, n$, $\pi_n(f) : \pi_n(X) \to \pi_n(Y)$ is bijective.

EXAMPLE Let A be a commutative ring with unit—then **D**(A-**MOD**) is monogenic. Here the role of S is played by A concentrated in degree 0 and $\pi_n(X) = H^{-n}(X)$.

PROPOSITION 40 Let **C** be a monogenic compactly generated CTC. Suppose that $F : \mathbf{C} \to \mathbf{AB}$ is a homology theory—then T_F has the IP (notation per the Bousfield-Margolis localization theorem).

[The class of T_F-acyclic objects coincides with O_F, the class of X such that $F\Sigma^n X = 0 \ \forall\, n \in Z$ (cf. p. 15–33). Therefore the claim is that for all such X, $F(\Sigma^n(X \otimes Y))$ $(= F(\Sigma^n X \otimes Y)) = 0 \ \forall\, n \in Z$. To see this, note that $F(\Sigma^n X \otimes \text{—}) : \mathbf{C} \to \mathbf{AB}$ is a homology theory with the property that $F(\Sigma^n X \otimes S^k) = F(\Sigma^{n+k} X) = 0 \ \forall\, k \in \mathbf{Z}$, thus, as **C** is monogenic, $F(\Sigma^n X \otimes \text{—}) = 0$.]

FACT Suppose that **C** is a monogenic compactly generated CTC. Let T be a localization functor. Assume: T is smashing—then T has the IP.

[Fix an X in $\ker T$ and consider the class of $Y : T(X \otimes \Sigma^n Y) = 0 \ \forall\, n \in \mathbf{Z}$. This class is the object class of a triangulated subcategory of **C** containing the S^n and is closed under the formation of coproducts in **C** (T being smashing), hence equals **C** (cf. Proposition 14).]

Suppose that **C** is a monogenic compactly generated CTC. Fix an $X \in \operatorname{Ob} \mathbf{C}$—then an object Y is said to be X-acyclic if $X \otimes Y = 0$ and an object Z is said to be X-local if $\hom(Y, Z) = 0$ for all X-acyclic Y. The Bousfield class $\langle X \rangle$ of X is the class of X-local objects.

Example: Let T be a localization functor. Assume: T is smashing—then $\langle TS \rangle$ is the class of T-local objects.

[Since T has the IP, $TS \otimes Y \approx TY$ (cf. p. 15–41), thus Y is TS-acyclic iff Y is T-acyclic.]

[Note: Another point is that $\forall\, X \in \operatorname{Ob} \mathbf{C}$, $\langle TX \rangle = \langle TS \rangle \cap \langle X \rangle$.]

LEMMA $\langle X \rangle$ is a thick subcategory of **C** which is closed under the formation of products in **C**. And: $\forall\, Y \in \operatorname{Ob} \mathbf{C}$ & $\forall\, Z \in \langle X \rangle$, $\hom(Y, Z) \in \langle X \rangle$.

[Note: To interpret $\langle X\rangle$, define a homology theory $H_X : \mathbf{C} \to \mathbf{AB}$ by the rule $H_X(Y) = \pi_0(X \otimes Y)$—then Y is X-acyclic iff $H_X(Y \otimes Z) = 0\ \forall\ Z$ (cf. p. 15–40). Letting T_X be the localization functor attached to H_X by the Bousfield-Margolis localization theorem and taking into account Proposition 40, it follows that Y is X-acyclic iff Y is T_X-acyclic. Therefore $\langle X\rangle$ is the class of T_X-local objects.]

Write $\langle X\rangle \leq \langle Y\rangle$ if $\langle X\rangle \subseteq \langle Y\rangle$, calling X, Y Bousfield equivalent when $\langle X\rangle = \langle Y\rangle$.

PROPOSITION 41 $\langle X\rangle \leq \langle Y\rangle$ iff $Y \otimes Z = 0 \Rightarrow X \otimes Z = 0$.

[Note: Consequently $\langle S\rangle$ is the largest Bousfield class and $\langle 0\rangle$ is the smallest.]

Notation: $\langle X\rangle \amalg \langle Y\rangle = \langle X \amalg Y\rangle$ and $\langle X\rangle \otimes \langle Y\rangle = \langle X \otimes Y\rangle$.

[Note: Both operations are welldefined. Examples: (1) $\langle X\rangle \amalg \langle 0\rangle = \langle X\rangle$, $\langle X\rangle \amalg \langle S\rangle = \langle S\rangle$; (2) $\langle X\rangle \otimes \langle 0\rangle = \langle 0\rangle$, $\langle X\rangle \otimes \langle S\rangle = \langle X\rangle$.]

FACT If $X \to Y \to Z \to \Sigma X$ is an exact triangle, then $\langle Y\rangle \leq \langle X\rangle \amalg \langle Z\rangle$.

Maintaining the assumption that $\mathbf{C}$ is monogenic, let $\langle \mathbf{C}\rangle$ be the conglomerate whose elements are the Bousfield classes. Denote by $\mathbf{DL}\langle \mathbf{C}\rangle$ the subconglomerate of $\langle \mathbf{C}\rangle$ consisting of those $\langle X\rangle$ with $\langle X\rangle \otimes \langle X\rangle = \langle X\rangle$ and denote by $\mathbf{BA}\langle \mathbf{C}\rangle$ the subconglomerate of $\langle \mathbf{C}\rangle$ consisting of those $\langle X\rangle$ that admit a complement, i.e., for which $\exists\ \langle Y\rangle : \langle X\rangle \otimes \langle Y\rangle = \langle 0\rangle$ and $\langle X\rangle \amalg \langle Y\rangle = \langle S\rangle$.

[Note: $\mathbf{DL}\langle \mathbf{C}\rangle$ is a "distributive lattice" and $\mathbf{BA}\langle \mathbf{C}\rangle$ is a "boolean algebra".]

Complements, if they exist, are unique. Thus suppose that $\langle X\rangle$ admits two complements $\langle Y'\rangle$ and $\langle Y''\rangle$—then $\langle Y'\rangle = \langle Y'\rangle \otimes \langle S\rangle = \langle Y'\rangle \otimes (\langle X\rangle \amalg \langle Y''\rangle) = (\langle Y'\rangle \otimes \langle X\rangle) \amalg (\langle Y'\rangle \otimes \langle Y''\rangle) = \langle 0\rangle \amalg (\langle Y'\rangle \otimes \langle Y''\rangle) = \langle Y'\rangle \otimes \langle Y''\rangle = \langle Y''\rangle$ (by symmetry).

Notation: Given $\langle X\rangle \in \mathbf{BA}\langle \mathbf{C}\rangle$, let $\langle X\rangle^c$ be its complement.

LEMMA $\mathbf{BA}\langle \mathbf{C}\rangle$ is contained in $\mathbf{DL}\langle \mathbf{C}\rangle$.

[$\langle X\rangle = \langle X\rangle \otimes (\langle X\rangle \amalg \langle X\rangle^c) = (\langle X\rangle \otimes \langle X\rangle) \amalg (\langle X\rangle \otimes \langle X\rangle^c) = \langle X\rangle \otimes \langle X\rangle$.]

Examples in the stable homotopy category show that the inclusions $\mathbf{BA}\langle \mathbf{C}\rangle \subset \mathbf{DL}\langle \mathbf{C}\rangle \subset \langle \mathbf{C}\rangle$ are strict (Bousfield[†]).

[†] *Comment. Math. Helv.* **54** (1979), 368–377.

EXAMPLE Let T be a localization functor—then there is an exact triangle $S_T \to S \xrightarrow{\epsilon_S} TS \to \Sigma S_T$, where S_T is T-acyclic (cf. Proposition 25), hence $\langle S\rangle = \langle S_T\rangle \amalg \langle TS\rangle$. If further T is smashing, then $\langle S_T\rangle \otimes \langle TS\rangle = \langle S_T \otimes TS\rangle = \langle TS_T\rangle = \langle 0\rangle \Rightarrow \langle S_T\rangle^c = \langle TS\rangle$.

[Note: Take for **C** the stable homotopy category—then X compact $\Rightarrow \langle X\rangle \in \mathbf{BA}\langle \mathbf{C}\rangle$ and $T_Y(\langle Y\rangle = \langle X\rangle^c)$ is smashing (Bousfield (ibid.)).]

EXAMPLE If X is dualizable, then $\langle X\rangle = \langle DX\rangle$. Indeed, X is a retract of $X \otimes DX \otimes X$ (cf. p. 15–36), thus $\langle X\rangle \leq \langle X \otimes DX \otimes X\rangle \leq \langle DX\rangle$. But DX is dualizable, so $\langle DX\rangle \leq \langle D^2X\rangle = \langle X\rangle$ (cf. Proposition 32).

Suppose that **C** is a monogenic compactly generated CTC—then a ring object in **C** is an object R equipped with a product $R \otimes R \to R$ and a unit $S \to R$ such that

$$\begin{tikzcd} R \otimes R \otimes R \arrow[r] \arrow[d] & R \otimes R \arrow[d] \\ R \otimes R \arrow[r] & R \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} S \otimes R \arrow[r] \arrow[dr] & R \otimes R \arrow[d] & R \otimes S \arrow[l] \arrow[dl] \\ & R & \end{tikzcd}$$

commute. A ring object R is commutative if

$$\begin{tikzcd} R \otimes R \arrow[rr, "T"] \arrow[dr] & & R \otimes R \arrow[dl] \\ & R & \end{tikzcd}$$

commutes.

Example: $\forall\, X \in \operatorname{Ob} \mathbf{C}$, $\hom(X, X)$ is a ring object, hence $DX \otimes X$ is a ring object if X is dualizable.

EXAMPLE If R is a ring object, then $\langle R\rangle \otimes \langle R\rangle = \langle R\rangle$ (R is a retract of $R \otimes R$).

LEMMA If R is a ring object, then $\pi_*(R)$ is a graded ring with unit which is graded commutative provided that R is commutative.

Given a ring object R, a (left) R-module is an object M equipped with an arrow $R \otimes M \to M$ such that

$$\begin{tikzcd} R \otimes R \otimes M \arrow[r] \arrow[d] & R \otimes M \arrow[d] \\ R \otimes M \arrow[r] & M \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} S \otimes M \arrow[r] \arrow[dr] & R \otimes M \arrow[d] \\ & M \end{tikzcd}$$

commute.

Example: $\forall\, X \in \operatorname{Ob} \mathbf{C}$, $R \otimes X$ and $\hom(X, R)$ are R-modules. R-**MOD** is the category whose objects are the R-modules.

[Note: If $f : M \to N$ is a morphism of R-modules and if $M \xrightarrow{f} N \to C_f \to \Sigma M$ is exact, then C_f need not admit an R-module structure.]

EXAMPLE If R is a ring object and if M is an R-module, then $\langle M\rangle \leq \langle R\rangle$ (M is a retract of $R \otimes M$).

[Note: M is necessarily T_R-local.]

EXAMPLE Let T be a localization functor with the IP—then TS is a commutative ring object (via $TS \otimes TS \rightarrow T(S \otimes S) = TS$ and $\epsilon_S : S \rightarrow TS$). Moreover, every T-local object X is a TS-module (via $TS \otimes X = TS \otimes TX \rightarrow T(S \otimes X) = TX = X$).

EXAMPLE If R is a ring object with the property that the product $R \otimes R \rightarrow R$ is an isomorphism, then T_R is smashing. Proof: $\forall\, X \in \operatorname{Ob} \mathbf{C}$, $R \otimes X$ is T_R-local and here $T_R X = R \otimes X$ (since $R \otimes R \approx R$), thus T_R preserves coproducts.

Definitions: (1) An R-module M is free if it is isomorphic to a coproduct $\coprod\limits_i \Sigma^{n_i} R$; (2) A nonzero ring object R is a skew field object if every M in R-**MOD** is free; (3) A skew field object R is a field object if R is commutative.

PROPOSITION 42 Let $\mathbf{C}$ be a monogenic compactly generated CTC. Suppose that R is a nonzero ring object in $\mathbf{C}$. Assume: The homogeneous elements of $\pi_*(R)$ are invertible—then R is a skew field object.

[Fix an M in R-**MOD**. Owing to our assumption, $\pi_*(M) \approx \bigoplus\limits_i \Sigma^{n_i} \pi_*(R)$, where $(\Sigma^{n_i} \pi_*(R))_n = \operatorname{Mor}(S^{n-n_i}, R) = \operatorname{Mor}(S^n, \Sigma^{n_i} R) = \pi_n(\Sigma^{n_i} R)$. Thus there is a morphism $\coprod\limits_i \Sigma^{n_i} R \rightarrow M$ of R-modules inducing an isomorphism $\bigoplus\limits_i \pi_{*-n_i}(R) \rightarrow \pi_*(M)$ in homotopy, hence $\coprod\limits_i \Sigma^{n_i} R \approx M$.]

In the stable homotopy category, the nth Morava K-theory spectrum $\mathbf{K}(n)$ at the prime p is a skew field object.

EXAMPLE Let R be a skew field object. Assume: $\langle R \rangle \in \mathbf{BA}\langle\mathbf{C}\rangle$—then $\langle R \rangle$ is minimal among nontrivial Bousfield classes.

[Note: In the stable homotopy category, the Eilenberg-MacLane spectrum $\mathbf{H}(\mathbf{F}_p)$ is a field object but $\langle \mathbf{H}(\mathbf{F}_p) \rangle$ is not minimal.]

Suppose that $\mathbf{C}$ is a monogenic compactly generated CTC. Given $X \in \operatorname{Ob} \mathbf{C}$ and $f \in \operatorname{Mor}(\Sigma^n X, X)$, let X/f be a completion of $\Sigma^n X \xrightarrow{f} X$ to an exact triangle (cf. TR$_3$) and write $f^{-1}X$ for tel(X,f), where (X,f) is the object in **FIL(C)** defined by $X \rightarrow \Sigma^{-n} X \rightarrow \Sigma^{-2n} X \rightarrow \cdots$.

LEMMA If $f : \Sigma^n X \rightarrow X$ is an isomorphism, then $X \approx f^{-1} X$.

PROPOSITION 43 For every $f : \Sigma^n X \rightarrow X$, $\langle X \rangle = \langle X/f \rangle \amalg \langle f^{-1} X \rangle$.

[To prove that $\langle X\rangle \leq \langle X/f\rangle \amalg \langle f^{-1}X\rangle$, one must show that $X/f\otimes Z = 0$ & $f^{-1}X\otimes Z = 0 \Rightarrow X \otimes Z = 0$. But $\Sigma^n X \xrightarrow{f} X \rightarrow X/f \rightarrow \Sigma(\Sigma^n X)$ exact $\Rightarrow \Sigma^n X \otimes Z \rightarrow X \otimes Z \rightarrow X/f \otimes Z \rightarrow \Sigma(\Sigma^n X \otimes Z)$ exact (cf. CTC$_3$) $\Rightarrow \Sigma^n X \otimes Z \approx X \otimes Z$ (cf. p. 15–6) $\Rightarrow X\otimes Z \approx (f\otimes \mathrm{id}_Z)^{-1}(X\otimes Z)$ (by the lemma). And: $(f\otimes \mathrm{id}_Z)^{-1}(X\otimes Z) = f^{-1}X\otimes Z = 0$.]

FACT Suppose that X is compact—then $f^{-1}X = 0$ iff $\exists\, k$ such that the composite $f \circ \Sigma^n f \circ \cdots \circ \Sigma^{(k-1)n} f : \Sigma^{kn} X \xrightarrow{f^k} X$ vanishes.

FACT Let R be a ring object. Fix $\alpha \in \pi_n(R)$ and let $\overline{\alpha}$ be the map $S^n \otimes R \xrightarrow{\alpha\otimes \mathrm{id}_R} R\otimes R \rightarrow R$—then α is nilpotent in $\pi_*(R)$ iff $\overline{\alpha}^{-1}R = 0$.

FACT Given $f : S \rightarrow X$, write $X_f^{(\infty)}$ for $\mathrm{tel}(\mathbf{X},\mathbf{f})$, where $(\mathbf{X},\mathbf{f})$ is the object in $\mathbf{FIL}(\mathbf{C})$ defined by $S \xrightarrow{f} X \xrightarrow{f\otimes\mathrm{id}} X \otimes X \xrightarrow{f\otimes\mathrm{id}} \cdots$, and let $f^{(\infty)}$ be the arrow $S \rightarrow X_f^{(\infty)}$—then $X_f^{(\infty)} = 0$ iff $f^{(\infty)} = 0$.

Let $\mathbf{C}$ be a triangulated category; let $\mathbf{C}^{\leq 0}$, $\mathbf{C}^{\geq 0}$ be full, isomorphism closed subcategories of $\mathbf{C}$ containing 0 and denote by $\mathbf{C}^{\leq -1}$, $\mathbf{C}^{\geq 1}$ the isomorphism closure of $\Sigma\mathbf{C}^{\leq 0}$, $\Omega\mathbf{C}^{\geq 0}$—then the pair $(\mathbf{C}^{\leq 0}, \mathbf{C}^{\geq 0})$ is said to be a t-structure on $\mathbf{C}$ if the following conditions are satisfied.

(t-st$_1$) $\mathbf{C}^{\leq -1}$ is a subcategory of $\mathbf{C}^{\leq 0}$ and $\mathbf{C}^{\geq 1}$ is a subcategory of $\mathbf{C}^{\geq 0}$.

(t-st$_2$) $\forall\, X \in \mathrm{Ob}\,\mathbf{C}^{\leq 0}, \forall\, Y \in \mathrm{Ob}\,\mathbf{C}^{\geq 1}, \mathrm{Mor}(X,Y) = 0$.

(t-st$_3$) $\forall\, X \in \mathrm{Ob}\,\mathbf{C}$, $\exists$ an exact triangle $X_0 \rightarrow X \rightarrow X_1 \rightarrow \Sigma X_0$ with $X_0 \in \mathrm{Ob}\,\mathbf{C}^{\leq 0}$, $X_1 \in \mathrm{Ob}\,\mathbf{C}^{\geq 1}$.

[Note: $\mathbf{H}(\mathbf{C}) = \mathbf{C}^{\leq 0} \cap \mathbf{C}^{\geq 0}$ is called the heart of the t-structure.]

Remark: If $(\mathbf{C}^{\leq 0}, \mathbf{C}^{\geq 0})$ is a t-structure on $\mathbf{C}$, then $((\mathbf{C}^{\geq 0})^{\mathrm{OP}}, (\mathbf{C}^{\leq 0})^{\mathrm{OP}})$ is a t-structure on $\mathbf{C}^{\mathrm{OP}}$.

EXAMPLE Let $\mathbf{A}$ be an abelian category. Given an X in $\mathbf{CXA}$, $\forall\ n \in \mathbf{Z}$, define the nth truncated cochain complexes $\tau^{\leq n}X$ & $\tau^{\geq n}X$ of X by $\cdots \rightarrow X^{n-2} \rightarrow X^{n-1} \rightarrow \ker d_X^n \rightarrow 0 \rightarrow \cdots$ & $\cdots \rightarrow 0 \rightarrow \operatorname{coker} d_X^{n-1} \rightarrow X^{n+1} \rightarrow X^{n+2} \rightarrow \cdots$. So, the cohomology of $\tau^{\leq n}X$ is trivial in degree $> n$ and the cohomology of $\tau^{\geq n}X$ is trivial in degree $< n$ and there is an arrow $\tau^{\leq n}X \rightarrow X$ which induces an isomorphism in cohomology in degree $\leq n$ and there is an arrow $X \rightarrow \tau^{\geq n}X$ which induces an isomorphism in cohomology in degree $\geq n$. The functors $\begin{cases} \tau^{\leq n} : \mathbf{CXA} \rightarrow \mathbf{CXA} \\ \tau^{\geq n} : \mathbf{CXA} \rightarrow \mathbf{CXA} \end{cases}$ pass through $\mathbf{K}(\mathbf{A})$ to the derived category $\mathbf{D}(\mathbf{A})$: $\begin{cases} \tau^{\leq n} : \mathbf{D}(\mathbf{A}) \rightarrow \mathbf{D}(\mathbf{A}) \\ \tau^{\geq n} : \mathbf{D}(\mathbf{A}) \rightarrow \mathbf{D}(\mathbf{A}) \end{cases}$ and $\forall\ X$, $\exists$ an exact triangle $\tau^{\leq n}X \rightarrow X \rightarrow \tau^{\geq n+1}X \rightarrow \Sigma\tau^{\leq n}X$. Write $\mathbf{D}^{\leq 0}(\mathbf{A})$ for the full subcategory of $\mathbf{D}(\mathbf{A})$ consisting of those X such that $H^q(X) = 0$ $(q > 0)$ and write $\mathbf{D}^{\geq 0}(\mathbf{A})$ for the full subcategory of $\mathbf{D}(\mathbf{A})$ consisting of those X such that

$H^q(X) = 0$ $(q < 0)$—then the pair $(\mathbf{D}^{\leq 0}(\mathbf{A}), \mathbf{D}^{\geq 0}(\mathbf{A}))$ is a t-structure on $\mathbf{D}(\mathbf{A})$ and its heart is equivalent to $\mathbf{A}$.

Given a t-structure $(\mathbf{C}^{\leq 0}, \mathbf{C}^{\geq 0})$ on $\mathbf{C}$, let $\begin{cases} \mathbf{C}^{\leq n} \\ \mathbf{C}^{\geq n} \end{cases}$ be the isomorphism closure of $\begin{cases} \Omega^n \mathbf{C}^{\leq 0} \\ \Omega^n \mathbf{C}^{\geq 0} \end{cases}$ $(n > 0)$ and let $\begin{cases} \mathbf{C}^{\leq n} \\ \mathbf{C}^{\geq n} \end{cases}$ be the isomorphism closure of $\begin{cases} \Sigma^{|n|} \mathbf{C}^{\leq 0} \\ \Sigma^{|n|} \mathbf{C}^{\geq 0} \end{cases}$ $(n < 0)$—then $\forall\, n \in \mathbf{Z}$, the pair $(\mathbf{C}^{\leq n}, \mathbf{C}^{\geq n})$ is a t-structure on $\mathbf{C}$.

PROPOSITION 44 Suppose that $(\mathbf{C}^{\leq 0}, \mathbf{C}^{\geq 0})$ is a t-structure on $\mathbf{C}$—then $\forall\, n \in \mathbf{Z}$, $\mathbf{C}^{\leq n}$ is a coreflective subcategory of $\mathbf{C}$ with coreflector $\tau^{\leq n}X \to X$ and $\mathbf{C}^{\geq n}$ is a reflective subcategory of $\mathbf{C}$ with reflector $X \to \tau^{\geq n}X$.

[It suffices to construct $\tau^{\leq 0}$. Thus for any $X \in \operatorname{Ob}\mathbf{C}$, $\exists$ an exact triangle $X_0 \to X \to X_1 \to \Sigma X_0$, where $X_0 \in \operatorname{Ob}\mathbf{C}^{\leq 0}$ & $X_1 \in \operatorname{Ob}\mathbf{C}^{\geq 1}$ (cf. t-st$_3$), so $\forall\, Y \in \operatorname{Ob}\mathbf{C}^{\leq 0}$, there is an exact sequence $\operatorname{Mor}(Y, \Omega X_1) \to \operatorname{Mor}(Y, X_0) \to \operatorname{Mor}(Y, X) \to \operatorname{Mor}(Y, X_1)$. Here $\operatorname{Mor}(Y, X_1) = 0$ (cf. t-st$_2$). In addition, $\operatorname{Mor}(Y, \Omega X_1) \approx \operatorname{Mor}(\Sigma Y, X_1)$ and $\Sigma\mathbf{C}^{\leq 0} \subset \mathbf{C}^{\leq -1} \subset \mathbf{C}^{\leq 0}$ (cf. t-st$_1$) $\Rightarrow \operatorname{Mor}(\Sigma Y, X_1) = 0$ (cf. t-st$_2$). Therefore, $\forall\, Y \in \operatorname{Ob}\mathbf{C}^{\leq 0}$, $\operatorname{Mor}(Y, X_0) \approx \operatorname{Mor}(Y, X)$ and we can let $\tau^{\leq 0}X = X_0$.]

[Note: Similar reasoning gives $\tau^{\geq 1}X = X_1$.]

The functors $\tau^{\leq n}$, $\tau^{\geq n}$ figuring in Proposition 44 are called the truncation functors of the t-structure.

[Note: $\forall\, X$, $\exists$ an exact triangle $\tau^{\leq n}X \to X \to \tau^{\geq n+1}X \to \Sigma\tau^{\leq n}X$ and since $\operatorname{Mor}(\Sigma\tau^{\leq n}X, \tau^{\geq n+1}X) = 0$, the arrow $\tau^{\geq n+1}X \to \Sigma\tau^{\leq n}X$ is unique (cf. p. 15–6).]

EXAMPLE Let $\mathbf{A}$ be an abelian category. Working with the t-structure on $\mathbf{D}(\mathbf{A})$ spelled out above, $\mathbf{D}^{\leq n}(\mathbf{A})$ is the coreflective subcategory of $\mathbf{D}(\mathbf{A})$ consisting of those X such that $H^q(X) = 0$ $(q > n)$ and $\mathbf{D}^{\geq n}(\mathbf{A})$ is the reflective subcategory of $\mathbf{D}(\mathbf{A})$ consisting of those X such that $H^q(X) = 0$ $(q < n)$.

Observations: Let $m, n \in \mathbf{Z}$—then (1) $m \leq n \Rightarrow \tau^{\geq n} \circ \tau^{\geq m} \approx \tau^{\geq m} \circ \tau^{\geq n} \approx \tau^{\geq n}$ and $\tau^{\leq n} \circ \tau^{\leq m} \approx \tau^{\leq m} \circ \tau^{\leq n} \approx \tau^{\leq m}$; (2) $m > n \Rightarrow \tau^{\leq n} \circ \tau^{\geq m} = 0$ and $\tau^{\geq m} \circ \tau^{\leq n} = 0$.

FACT If $m \leq n$, then $\forall\, X \in \operatorname{Ob}\mathbf{C}$, $\exists$ a unique arrow $\tau^{\geq m}\tau^{\leq n}X \to \tau^{\leq n}\tau^{\geq m}X$ such that the diagram

$$\begin{tikzcd}
\tau^{\leq n}X \arrow[r] \arrow[d] & X \arrow[r] & \tau^{\geq m}X \\
\tau^{\geq m}\tau^{\leq n}X \arrow[rr] & & \tau^{\leq n}\tau^{\geq m}X \arrow[u]
\end{tikzcd}$$

commutes.

[Note: The arrow $\tau^{\geq m}\tau^{\leq n}X \to \tau^{\leq n}\tau^{\geq m}X$ is an isomorphism provided that $\mathbf{C}$ satisfies the octahedral axiom. To see this, consider the exact triangles $\tau^{\leq m-1}X \to \tau^{\leq n}X \to \tau^{\geq m}\tau^{\leq n}X \to \Sigma\tau^{\leq m-1}X$, $\tau^{\leq n}X \to X \to \tau^{\geq n+1}X \to \Sigma\tau^{\leq n}X$, $\tau^{\leq m-1}X \to X \to \tau^{\geq m}X \to \Sigma\tau^{\leq m-1}X$.]

Notation: Write $\left\{\begin{array}{l}\mathbf{C}^{<n}\\ \mathbf{C}^{>n}\end{array}\right.$ in place of $\left\{\begin{array}{l}\mathbf{C}^{\leq n-1}\\ \mathbf{C}^{\geq n+1}\end{array}\right.$ and $\left\{\begin{array}{l}\tau^{<n}\\ \tau^{>n}\end{array}\right.$ in place of $\left\{\begin{array}{l}\tau^{\leq n-1}\\ \tau^{\geq n+1}\end{array}\right.$.

LEMMA Let $X \in \operatorname{Ob}\mathbf{C}$—then $X \in \left\{\begin{array}{l}\operatorname{Ob}\mathbf{C}^{\leq n}\\ \operatorname{Ob}\mathbf{C}^{\geq n}\end{array}\right.$ iff $\left\{\begin{array}{l}\tau^{>n}X = 0\\ \tau^{<n}X = 0\end{array}\right.$.

PROPOSITION 45 Suppose that $(\mathbf{C}^{\leq 0}, \mathbf{C}^{\geq 0})$ is a t-structure on $\mathbf{C}$. Let $X' \to X \to X'' \to \Sigma X'$ be an exact triangle—then $\left\{\begin{array}{l}X'\\ X''\end{array}\right. \in \operatorname{Ob}\mathbf{C}^{\leq 0} \Rightarrow X \in \operatorname{Ob}\mathbf{C}^{\leq 0}$ & $\left\{\begin{array}{l}X'\\ X''\end{array}\right. \in \operatorname{Ob}\mathbf{C}^{\geq 0} \Rightarrow X \in \operatorname{Ob}\mathbf{C}^{\geq 0}$.

Let $\mathbf{A}$ be an additive category. Given a class $O \subset \operatorname{Ob}\mathbf{A}$, the $\left\{\begin{array}{l}\text{left annihilator } \operatorname{Ann}_{\mathrm{L}}O\\ \text{right annihilator } \operatorname{Ann}_{\mathrm{R}}O\end{array}\right.$ of O is $\left\{\begin{array}{l}\{Y : \operatorname{Mor}(Y, X) = 0\ \forall\, X \in O\}\\ \{Y : \operatorname{Mor}(X, Y) = 0\ \forall\, X \in O\}\end{array}\right.$.

EXAMPLE Let $\mathbf{A}$ be an additive category. Suppose that $\mathcal{T}$, $\mathcal{F}$ are subclasses of $\operatorname{Ob}\mathbf{A}$—then the pair $(\mathcal{T}, \mathcal{F})$ is said to be a torsion theory on $\mathbf{A}$ if $\operatorname{Ann}_{\mathrm{L}}\mathcal{F} = \mathcal{T}$ and $\operatorname{Ann}_{\mathrm{R}}\mathcal{T} = \mathcal{F}$. Example: $\forall$ t-structure $(\mathbf{C}^{\leq 0}, \mathbf{C}^{\geq 0})$ on $\mathbf{C}$, $\operatorname{Ann}_{\mathrm{L}}\mathbf{C}^{\geq 1} = \mathbf{C}^{\leq 0}$ and $\operatorname{Ann}_{\mathrm{R}}\mathbf{C}^{\leq 0} = \mathbf{C}^{\geq 1}$, i.e., $(\mathbf{C}^{\leq 0}, \mathbf{C}^{\geq 1})$ is a torsion theory on $\mathbf{C}$.

LEMMA Let $\mathbf{C}$ be a triangulated category satisfying the octahedral axiom. Suppose that $(\mathbf{C}^{\leq 0}, \mathbf{C}^{\geq 0})$ is a t-structure on $\mathbf{C}$—then $\forall\, X \in \operatorname{Ob}\mathbf{C}$, $\tau^{\geq 0}\tau^{\leq 0}X \approx \tau^{\leq 0}\tau^{\geq 0}X$.

THEOREM OF THE HEART Let $\mathbf{C}$ be a triangulated category with finite coproducts satisfying the octahedral axiom. Suppose that $(\mathbf{C}^{\leq 0}, \mathbf{C}^{\geq 0})$ is a t-structure on $\mathbf{C}$—then its heart $\mathbf{H}(\mathbf{C})$ is an abelian category.

[$\mathbf{H}(\mathbf{C})$ is closed under the formation of finite coproducts in $\mathbf{C}$ (use the exact triangle $X \to X \amalg Y \to Y \xrightarrow{0} \Sigma X$ and quote Proposition 45). To prove that $\mathbf{H}(\mathbf{C})$ has kernels and cokernels and that parallel morphisms are isomorphisms, take an arrow $f : X \to Y$ in $\mathbf{H}(\mathbf{C})$ and place it in an exact triangle $X \xrightarrow{f} Y \to Z \to \Sigma X$ ($\Rightarrow Z \in \operatorname{Ob}\mathbf{C}^{\leq 0} \cap \operatorname{Ob}\mathbf{C}^{\geq -1}$ (cf. Proposition 45)). For any $W \in \operatorname{Ob}\mathbf{H}(\mathbf{C})$, there are exact sequences $\operatorname{Mor}(W, \Omega Y) \to \operatorname{Mor}(W, \Omega Z) \to \operatorname{Mor}(W, X) \to \operatorname{Mor}(W, Y)$, $\operatorname{Mor}(\Sigma X, W) \to \operatorname{Mor}(Z, W) \to \operatorname{Mor}(Y, W) \to \operatorname{Mor}(X, W)$. Since $\operatorname{Mor}(W, \Omega Y) = 0$, $\operatorname{Mor}(\Sigma X, W) = 0$ and $\operatorname{Mor}(W, \Omega Z) \approx \operatorname{Mor}(W, \tau^{\leq 0}\Omega Z)$, $\operatorname{Mor}(Z, W) \approx \operatorname{Mor}(\tau^{\geq 0}Z, W)$, it follows that $\ker f \approx \tau^{\leq 0}\Omega Z$, $\operatorname{coker} f \approx \tau^{\geq 0}Z$. In this connection, note that $Z \in \operatorname{Ob}\mathbf{C}^{\leq 0} \Rightarrow \tau^{\geq 0}Z \approx \tau^{\geq 0}\tau^{\leq 0}Z \approx \tau^{\leq 0}\tau^{\geq 0}Z \Rightarrow \operatorname{coker} f \in \operatorname{Ob}\mathbf{H}(\mathbf{C})$ and $Z \in \operatorname{Ob}\mathbf{C}^{\geq -1} \Rightarrow \Omega Z \in \operatorname{Ob}\mathbf{C}^{\geq 0} \Rightarrow$

$\tau^{\leq 0}\Omega Z \approx \tau^{\leq 0}\tau^{\geq 0}\Omega Z \approx \tau^{\geq 0}\tau^{\leq 0}\Omega Z \Rightarrow \ker f \in \operatorname{Ob}\mathbf{H}(\mathbf{C})$. Now fix an exact triangle $I \to Y \to \tau^{\geq 0}Z \to \Sigma I$ ($\Rightarrow I \in \operatorname{Ob}\mathbf{C}^{\geq 0}$ (cf. Proposition 45)). Applying the octahedral axiom to $Y \to Z \to \Sigma X \to \Sigma Y$, $Z \to \tau^{\geq 0}Z \to \Sigma\tau^{<0}Z \to \Sigma Z$, $Y \to \tau^{\geq 0}Z \to \Sigma I \to \Sigma Y$, one gets an exact triangle $\Sigma X \to \Sigma I \to \Sigma\tau^{<0}Z \to \Sigma^2 X$, which leads to an exact triangle $\tau^{\leq 0}\Omega Z \to X \to I \to \Sigma\tau^{\leq 0}\Omega Z$, thus $I \in \operatorname{Ob}\mathbf{C}^{\leq 0}$ (cf. Proposition 45) and so $I \in \operatorname{Ob}\mathbf{H}(\mathbf{C})$. Finally, $I \approx \operatorname{coim} f$ (consider $\ker f \to X \to I \to \Sigma\ker f$) and $I \approx \operatorname{im} f$ (consider $I \to Y \to \operatorname{coker} f \to \Sigma I$). Therefore $\mathbf{H}(\mathbf{C})$ is abelian.]

[Note: In general, there is no a priori connection between $\mathbf{C}$ and the derived category of $\mathbf{H}(\mathbf{C})$.]

EXAMPLE Take for $\mathbf{C}$ the stable homotopy category and let $\begin{cases} \mathbf{C}^{\geq 0} = \{\mathbf{X} : \pi_q(\mathbf{X}) = 0 \ (q > 0)\} \\ \mathbf{C}^{\leq 0} = \{\mathbf{X} : \pi_q(\mathbf{X}) = 0 \ (q < 0)\} \end{cases}$ — then $(\mathbf{C}^{\leq 0}, \mathbf{C}^{\geq 0})$ is a t-structure on $\mathbf{C}$. Its heart is equivalent to $\mathbf{AB}$ (cf. p. 17–2).

[Note: $\tau^{\leq 0}\mathbf{X}$ is called the connective cover of $\mathbf{X}$ (the arrow $\tau^{\leq 0}\mathbf{X} \to \mathbf{X}$ induces an isomorphism $\pi_n(\tau^{\leq 0}\mathbf{X}) \to \pi_n(\mathbf{X})$ for $n \geq 0$).]

Let $\mathbf{C}$ be a triangulated category with finite coproducts satisfying the octahedral axiom. Suppose that $(\mathbf{C}^{\leq 0}, \mathbf{C}^{\geq 0})$ is a t-structure on $\mathbf{C}$—then $H^0 : \mathbf{C} \to \mathbf{H}(\mathbf{C})$ is the functor that sends X to $\tau^{\geq 0}\tau^{\leq 0}X \approx \tau^{\leq 0}\tau^{\geq 0}X$.

FACT H^0 is an exact functor.

[Fix an exact triangle $X \to Y \to Z \to \Sigma X$ and proceed in stages.

(I) Assume that $X, Y, Z \in \operatorname{Ob}\mathbf{C}^{\geq 0}$—then $0 \to H^0(X) \to H^0(Y) \to H^0(Z)$ is exact.

(II$^{\geq 0}$) Assume that $Z \in \operatorname{Ob}\mathbf{C}^{\geq 0}$—then $0 \to H^0(X) \to H^0(Y) \to H^0(Z)$ is exact.

[For $\tau^{<0}X \approx \tau^{<0}Y$ and the octahedral axiom furnishes an exact triangle $\tau^{\geq 0}X \to \tau^{\geq 0}Y \to Z \to \Sigma\tau^{\geq 0}X$.]

(II$^{\leq 0}$) Assume that $X \in \operatorname{Ob}\mathbf{C}^{\leq 0}$—then $H^0(X) \to H^0(Y) \to H^0(Z) \to 0$ is exact.

Reduce the general case to II$^{\geq 0}$ & II$^{\leq 0}$.]

Notation: $H^q : \mathbf{C} \to \mathbf{H}(\mathbf{C})$ is the functor that sends X to $\begin{cases} H^0(\Sigma^q X) & (q > 0) \\ H^0(\Omega^q X) & (q < 0) \end{cases}$.

FACT Assume: The intersections $\bigcap\limits_n \operatorname{Ob}\mathbf{C}^{\leq n}$, $\bigcap\limits_n \operatorname{Ob}\mathbf{C}^{\geq n}$ contain only zero objects—then $H^q(X) = 0 \ \forall\, q \Rightarrow X = 0$, thus the H^q comprise a conservative system of functors (i.e., f is an isomorphism iff $H^q(f)$ is an isomorphism $\forall\, q$).

[Note: The objects of $\mathbf{C}^{\leq n}$ are characterized by the condition that $H^q(X) = 0$ $(q > n)$ and the objects of $\mathbf{C}^{\geq n}$ are characterized by the condition that $H^q(X) = 0$ $(q < n)$.]

§ 15

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§16. SPECTRA

In this §, I shall give a concise exposition of the theory of spectra, concentrating on foundational issues and using model category theoretic methods whenever possible to ease the way.

A prespectrum $\mathbf{X}$ is said to be separated if $\forall\, q$, $\sigma_q : X_q \to \Omega X_{q+1}$ is a **CG** embedding. **SEPPRESPEC** is the full subcategory of **PRESPEC** whose objects are the separated prespectra.

Notation: Given a continuous function $f : X \to Y$, where X & Y are compactly generated, write $\operatorname{im} f$ for $kf(X)$ (so $f : X \to Y$ factors as $X \to \operatorname{im} f \to Y$ and $\operatorname{im} f \to Y$ is a **CG** embedding).

PROPOSITION 1 **SEPPRESPEC** is a reflective subcategory of **PRESPEC**.

[We shall construct the reflector E^{∞} by transfinite induction.

Claim: There is a functor E : **PRESPEC** $\to$ **PRESPEC** and a natural transformation $\Xi : \mathrm{id} \to E$ such that $\forall\, \mathbf{X}$, $\Xi_{\mathbf{X}} : \mathbf{X} \to E\mathbf{X}$ is a levelwise surjection, $\mathbf{X}$ being separated iff $\Xi_{\mathbf{X}}$ is a levelwise homeomorphism. In addition, if $\mathbf{f} : \mathbf{X} \to \mathbf{Y}$ is a morphism of prespectra and if $\mathbf{Y}$ is separated, then $\mathbf{f}$ factors uniquely through $\Xi_{\mathbf{X}}$.

[Let $(E\mathbf{X})_q = \operatorname{im}(X_q \overset{\sigma_q}{\to} \Omega X_{q+1})$ and determine the arrow $(E\mathbf{X})_q \to \Omega(E\mathbf{X})_{q+1}$ from the commutative diagram

$$\begin{tikzcd}
X_q \arrow[r, "\sigma_q"] \arrow[d, "\Xi_{\mathbf{X},q}"'] & \Omega X_{q+1} \arrow[d, "\Omega\Xi_{\mathbf{X},q+1}"] \\
(E\mathbf{X})_q \arrow[r, dashed] \arrow[d, phantom, "\bigcap"] & \Omega(E\mathbf{X})_{q+1} \arrow[d, phantom, "\bigcap"] \\
\Omega X_{q+1} \arrow[r, "\Omega\sigma_{q+1}"'] & \Omega\Omega X_{q+2}
\end{tikzcd}$$

It is clear that E is functorial and Ξ is natural.]

Claim: For each ordinal α, there is a functor E^{α} : **PRESPEC** $\to$ **PRESPEC** and for each pair $\alpha \leq \beta$ of ordinals, there is a natural transformation $\Xi^{\alpha,\beta} : E^{\alpha} \to E^{\beta}$ such that $\forall\, \mathbf{X}$, $\Xi^{\alpha,\beta}_{\mathbf{X}} : E^{\alpha}\mathbf{X} \to E^{\beta}\mathbf{X}$ is a levelwise surjection, $E^{\alpha}\mathbf{X}$ being separated iff $\Xi^{\alpha,\alpha+1}_{\mathbf{X}} : E^{\alpha}\mathbf{X} \to E^{\alpha+1}\mathbf{X}$ is a levelwise homeomorphism. In addition, if $\mathbf{f} : \mathbf{X} \to \mathbf{Y}$ is a morphism of prespectra and if $\mathbf{Y}$ is separated, then $\mathbf{f}$ factors uniquely through $\Xi^{0,\alpha}_{\mathbf{X}}$.

[Here, $E^0 = \mathrm{id}$, $E^1 = E$, $\Xi^{0,1} = \Xi$, $\Xi^{\alpha,\alpha} = \mathrm{id}$, $E^{\alpha+1} = E \circ E^{\alpha}$, and $\Xi^{\alpha,\beta+1} = \Xi \circ \Xi^{\alpha,\beta}$ $(\alpha \leq \beta)$. At a limit ordinal λ, put $E^{\lambda}\mathbf{X} = \operatorname*{colim}_{\alpha<\lambda} E^{\alpha}\mathbf{X}$ and define $\Xi^{\alpha,\lambda}_{\mathbf{X}} : E^{\alpha}\mathbf{X} \to E^{\lambda}\mathbf{X}$ in the obvious manner.]

[Note: If $E^{\alpha}\mathbf{X}$ is separated, then $\forall\, \beta \geq \alpha$, $\Xi^{\alpha,\beta}_{\mathbf{X}} : E^{\alpha}\mathbf{X} \to E^{\beta}\mathbf{X}$ is a levelwise homeomorphism.]

To finish the proof, it suffices to show that $\forall\ \mathbf{X}$, $\exists$ an $\alpha_{\mathbf{X}}$ such that $E^{\alpha_{\mathbf{X}}}\mathbf{X}$ is separated. But for this, one can take $\alpha_{\mathbf{X}}$ to be any infinite cardinal greater than the cardinality of $(\coprod\limits_q X_q \times X_q) \amalg (\coprod\limits_q \tau(X_q))$ ($\tau(X_q)$ the set of open subsets of X_q).]

[Note: The arrow of reflection $\mathbf{X} \to E^{\infty}\mathbf{X}$ is a levelwise surjection. It is a levelwise homeomorphism iff $\mathbf{X}$ is separated.]

The existence of the reflector E^{∞} can be established by applying the general adjoint functor theorem: **SEPPRESPEC** is a priori complete, the inclusion **SEPPRESPEC** $\to$ **PRESPEC** preserves limits, and the solution set condition is satisfied. The drawback to this approach is that it provides no information about the behavior of E^{∞} with respect to finite limits, a situation that can be partially clarified by using the iterative definition of E^{∞} in terms of the E^{α}.

LEMMA Suppose that $(I, \leq)$ is a nonempty directed set, regarded as a filtered category $\mathbf{I}$. Let $\Delta', \Delta'' : \mathbf{I} \to \mathbf{\Delta}\text{-}\mathbf{CG}$ be diagrams—then the arrow $\operatorname{colim}_{\mathbf{I}}(\Delta' \times \Delta'') \to \operatorname{colim}_{\mathbf{I}} \Delta' \times_k \operatorname{colim}_{\mathbf{I}} \Delta''$ is a homeomorphism.

[Note: A directed colimit in $\mathbf{\Delta}\text{-}\mathbf{CG}_*$ is formed by assigning the evident base point to the corresponding directed colimit in $\mathbf{\Delta}\text{-}\mathbf{CG}$, thus the lemma is valid in $\mathbf{\Delta}\text{-}\mathbf{CG}_*$ as well.]

FACT E^{∞} preserves finite products.

[Note: E^{∞} does not preserve equalizers.]

LEMMA Suppose that $(I, \leq)$ is a nonempty directed set, regarded as a filtered category $\mathbf{I}$. Let $\Delta : \mathbf{I} \to \mathbf{\Delta}\text{-}\mathbf{CG}$ be a diagram such that $\forall\ i \xrightarrow{\delta} j$, $\Delta\delta : \Delta_i \to \Delta_j$ is an injection—then $\operatorname{colim}_{\mathbf{I}} \Delta$ in $\mathbf{\Delta}\text{-}\mathbf{CG} =$ $\operatorname{colim}_{\mathbf{I}} \Delta$ in $\mathbf{CG}$ ($= \operatorname{colim}_{\mathbf{I}} \Delta$ in **TOP**) and $\forall\ i$, the canonical arrow $\Delta_i \to \operatorname{colim}_{\mathbf{I}} \Delta$ is one-to-one.

[Note: The set underlying $\operatorname{colim}_{\mathbf{I}} \Delta$ is therefore the colimit of the underlying diagram in **SET**.]

LEMMA In $\mathbf{\Delta}\text{-}\mathbf{CG}$, directed colimits of diagrams whose arrows are injections commute with finite limits.

[Note: A finite limit in $\mathbf{\Delta}\text{-}\mathbf{CG}_*$ is formed by assigning the evident base point to the corresponding finite limit in $\mathbf{\Delta}\text{-}\mathbf{CG}$, thus the lemma is valid in $\mathbf{\Delta}\text{-}\mathbf{CG}_*$ as well.]

A prespectrum $\mathbf{X}$ is said to be injective if $\forall\ q$, $\sigma_q : X_q \to \Omega X_{q+1}$ is an injection. **INJPRESPEC** is the full subcategory of **PRESPEC** whose objects are the injective prespectra.

[Note: **SEPPRESPEC** is a full subcategory of **INJPRESPEC**.]

FACT The arrow of reflection $\mathbf{X} \to E^{\infty}\mathbf{X}$ is a levelwise injection iff $\mathbf{X}$ is injective.

[If $\mathbf{X}$ is injective, then so are the $E^{\alpha}\mathbf{X}$. Moreover, $\Xi_{\mathbf{X}}^{\alpha,\beta} : E^{\alpha}\mathbf{X} \to E^{\beta}\mathbf{X}$ $(\alpha \leq \beta)$ is one-to-one.]

[Note: It therefore follows that the arrow of reflection $\mathbf{X} \rightarrow E^{\infty}\mathbf{X}$ is a levelwise bijection iff $\mathbf{X}$ is injective.]

FACT The restriction of E^{∞} to **INJPRESPEC** preserves finite limits.

LEMMA Suppose given a sequence $\{X_n, f_n\}$, where X_n is a Δ-separated compactly generated space and $f_n : X_n \rightarrow X_{n+1}$ is a **CG** embedding—then $\forall$ compact Hausdorff space K, $\operatorname{colim} X_n^K \approx (\operatorname{colim} X_n)^K$ (exponential objects in **Δ-CG**).

[Note: There is an analogous assertion in the pointed category.]

PROPOSITION 2 **SPEC** is a reflective subcategory of **SEPPRESPEC**.

[The reflector sends $\mathbf{X}$ to $e\mathbf{X}$, the latter being defined by the rule $q \rightarrow \operatorname{colim} \Omega^n X_{n+q}$.]

LEMMA Suppose that $(I, \leq)$ is a nonempty directed set, regarded as a filtered category $\mathbf{I}$. Let $\Delta : \mathbf{I} \rightarrow \mathbf{\Delta\text{-}CG}$ be a diagram such that $\forall\, i \xrightarrow{\delta} j$, $\Delta\delta : \Delta_i \rightarrow \Delta_j$ is a **CG** embedding—then $\forall\, i$, the canonical arrow $\Delta_i \rightarrow \operatorname{colim}_{\mathbf{I}} \Delta$ is a **CG** embedding.

[Note: Changing the assumption to "closed embedding" changes the conclusion to "closed embedding".]

FACT The arrow of reflection $\mathbf{X} \rightarrow e\mathbf{X}$ is a levelwise **CG** embedding.

FACT e preserves finite limits.

PROPOSITION 3 **SPEC** is a reflective subcategory of **PRESPEC**.

[This is implied by Propositions 1 and 2.]

[Note: The composite **PRESPEC** $\xrightarrow{E^{\infty}}$ **SEPPRESPEC** $\xrightarrow{e}$ **SPEC** is the spectrification functor: $\mathbf{X} \rightarrow s\mathbf{X}$ $(s = e \circ E^{\infty})$.]

Application: **SPEC** is complete and cocomplete.

[Note: The colimit of a diagram $\Delta : \mathbf{I} \rightarrow$ **SPEC** is the spectrification of its colimit in **PRESPEC**. Example: The coproduct in **PRESPEC** or **SPEC** is denoted by a wedge. If $\{\mathbf{X}_i\}$ is a set of spectra, then its coproduct in **PRESPEC** is separated, so $e(\bigvee_i \mathbf{X}_i)$ is the coproduct $\bigvee_i \mathbf{X}_i$ of the $\mathbf{X}_i$ in **SPEC**.]

FACT Spectrification preserves finite products and its restriction to **INJPRESPEC** preserves finite limits.

EXAMPLE Let X be in $\mathbf{\Delta\text{-}CG}_*$—then the suspension prespectrum of X is the assignment $q \rightarrow \Sigma^q X$, where $\Sigma^q X \rightarrow \Omega\Sigma\Sigma^q X \approx \Omega\Sigma^{q+1} X$ (a **CG** embedding). Its spectrification is the suspension spectrum of X. Thus, in the notation of p. 14–59, the suspension spectrum of X is $\mathbf{Q}^\infty X : (\mathbf{Q}^\infty X)_q = \operatorname{colim} \Omega^n \Sigma^{n+q} X = \Omega^\infty \Sigma^\infty \Sigma^q X$.

EXAMPLE Fix $q \geq 0$. Given an X in $\mathbf{\Delta}$-$\mathbf{CG}_*$, let $\mathbf{Q}_q^\infty X$ be the spectrification of the prespectrum $p \rightarrow \begin{cases} \Sigma^{p-q} X & (p \geq q) \\ * & (p < q) \end{cases}$, where $\Sigma^{p-q} X \rightarrow \Omega \Sigma \Sigma^{p-q} X \approx \Omega \Sigma^{p+1-q} X$ $(p \geq q)$ (if $p < q$, the arrow is the inclusion of the base point). Viewed as a functor from $\mathbf{\Delta}$-$\mathbf{CG}_*$ to **SPEC**, $\mathbf{Q}_q^\infty$ is a left adjoint for the qth space functor $\mathbf{U}_q^\infty : \mathbf{SPEC} \rightarrow \mathbf{\Delta}$-$\mathbf{CG}_*$ that sends $\mathbf{X} = \{X_q\}$ to X_q. Special case: $\mathbf{Q}_0^\infty = \mathbf{Q}^\infty$, $\mathbf{U}_0^\infty = \mathbf{U}^\infty$.

[Note: $\forall\, X$, $q' \leq q'' \Rightarrow \mathbf{Q}_{q'}^\infty X \approx \mathbf{Q}_{q''}^\infty \Sigma^{q''-q'} X$.]

FACT Suppose that $\mathbf{X}$ is a prespectrum—then $s\mathbf{X} \approx \operatorname{colim} \mathbf{Q}_q^\infty X_q$.

[For any spectrum $\mathbf{Y}$, $\operatorname{Mor}(\operatorname{colim} \mathbf{Q}_q^\infty X_q, \mathbf{Y}) \approx \lim \operatorname{Mor}(\mathbf{Q}_q^\infty X_q, \mathbf{Y}) \approx \lim \operatorname{Mor}(X_q, Y_q) \approx \operatorname{Mor}(\mathbf{X}, \mathbf{Y}) \approx \operatorname{Mor}(s\mathbf{X}, \mathbf{Y})$.]

FACT Let $(\mathbf{X}, \mathbf{f})$ be an object in **FIL(SPEC)** (cf. p. 0–10). Assume: $\forall\, n$, $\mathbf{f}_n : \mathbf{X}_n \rightarrow \mathbf{X}_{n+1}$ is a levelwise CG embedding—then $\forall$ pointed compact Hausdorff space K, $\operatorname{colim} \operatorname{Mor}(\mathbf{Q}_q^\infty K, \mathbf{X}_n) \approx \operatorname{Mor}(\mathbf{Q}_q^\infty K, \operatorname{colim} \mathbf{X}_n)$.

[The assumption guarantees that the prespectrum colimit of $(\mathbf{X}, \mathbf{f})$ is a spectrum. Therefore $\operatorname{colim} \operatorname{Mor}(\mathbf{Q}_q^\infty K, \mathbf{X}_n) \approx \operatorname{colim} \operatorname{Mor}(K, \mathbf{U}_q^\infty \mathbf{X}_n) \approx \operatorname{Mor}(K, \operatorname{colim} \mathbf{U}_q^\infty \mathbf{X}_n) \approx \operatorname{Mor}(K, \mathbf{U}_q^\infty \operatorname{colim} \mathbf{X}_n) \approx \operatorname{Mor}(\mathbf{Q}_q^\infty K, \operatorname{colim} \mathbf{X}_n)$.]

FACT Let $\{\mathbf{X}_i\}$ be a set of spectra, K a pointed compact Hausdorff space—then every morphism $\mathbf{f} : \mathbf{Q}_q^\infty K \rightarrow \bigvee_i \mathbf{X}_i$ factors through a finite subwedge.

[Since $\operatorname{Mor}(\mathbf{Q}_q^\infty K, \bigvee_i \mathbf{X}_i) \approx \operatorname{Mor}(K, \mathbf{U}_q^\infty(\bigvee_i \mathbf{X}_i))$, $\mathbf{f}$ corresponds to an arrow $g : K \rightarrow \mathbf{U}_q^\infty(\bigvee_i \mathbf{X}_i)$ $(= (\bigvee_i \mathbf{X}_i)_q)$, i.e., to an arrow $g : K \rightarrow \operatorname{colim} \Omega^n(\bigvee_i (\mathbf{X}_i)_{n+q})$, which factors through $\Omega^n(\bigvee_i (\mathbf{X}_i)_{n+q})$ for some n:

$$\begin{tikzcd} K \arrow[r, "g_n"] \arrow[dr, "g"'] & \Omega^n(\bigvee_i (\mathbf{X}_i)_{n+q}) \arrow[d] \\ & (\bigvee_i \mathbf{X}_i)_q \end{tikzcd}$$

. The adjoint $\overline{g}_n : \Sigma^n K \rightarrow \bigvee_i (\mathbf{X}_i)_{n+q}$ factors through a finite subwedge $\bigvee_k (\mathbf{X}_{i_k})_{n+q}$, so $\mathbf{f}$ factors through $\bigvee_k \mathbf{X}_{i_k}$.]

Notation: Given $\mathbf{X}, \mathbf{Y}$ in **PRESPEC**, write $\operatorname{HOM}(\mathbf{X}, \mathbf{Y})$ for $\operatorname{Mor}(\mathbf{X}, \mathbf{Y})$ topologized via the equalizer diagram $\operatorname{Mor}(\mathbf{X}, \mathbf{Y}) \rightarrow \prod_q Y_q^{X_q} \rightrightarrows \prod_q (\Omega Y_{q+1})^{X_q}$.

PROPOSITION 4 Spectrification is a continuous functor in the sense that $\forall\, \mathbf{X}, \mathbf{Y}$ in **PRESPEC**, the arrow $\operatorname{HOM}(\mathbf{X}, \mathbf{Y}) \rightarrow \operatorname{HOM}(s\mathbf{X}, s\mathbf{Y})$ is a continuous function.

($\square$ and $\wedge$) Fix a K in $\mathbf{\Delta\text{-}CG}_*$. Given an $\mathbf{X}$ in **PRESPEC**, let $\mathbf{X}\square K$ be the prespectrum $q \to X_q\#_k K$, where $X_q\#_k K \to \Omega(X_{q+1}\#_k K)$ is $X_q\#_k K \to \Omega X_{q+1}\#_k K \to \Omega(X_{q+1}\#_k K)$, and given an $\mathbf{X}$ in **SPEC**, let $\mathbf{X}\wedge K$ be the spectrification of $\mathbf{X}\square K$.

Examples: (1) $\Gamma\mathbf{X} = \mathbf{X}\square[0,1]$ or $\mathbf{X}\wedge[0,1]$, the cone of $\mathbf{X}$; (2) $\Sigma\mathbf{X} = \mathbf{X}\square\mathbf{S}^1$ or $\mathbf{X}\wedge\mathbf{S}^1$, the suspension of $\mathbf{X}$.

($\textsc{hom}$) Fix a K in $\mathbf{\Delta\text{-}CG}_*$. Given an $\mathbf{X}$ in **PRESPEC**, let $\textsc{hom}(K,\mathbf{X})$ be the prespectrum $q \to X_q^K$, where $X_q^K \to \Omega X_{q+1}^K$ is $X_q^K \to (\Omega X_{q+1})^K \approx \Omega X_{q+1}^K$.

[Note: If $\mathbf{X}$ is a spectrum, then $\textsc{hom}(K,\mathbf{X})$ is a spectrum.]

Example: $\forall\,\mathbf{X}$, $\Omega\mathbf{X} = \textsc{hom}(\mathbf{S}^1,\mathbf{X})$ (cf. p. 14–75).

PROPOSITION 5 For $\mathbf{X},\mathbf{Y}$ in **PRESPEC** and K in $\mathbf{\Delta\text{-}CG}_*$, there are natural homeomorphisms $\mathrm{HOM}(\mathbf{X}\square K,\mathbf{Y}) \approx \mathrm{HOM}(\mathbf{X},\mathbf{Y})^K \approx \mathrm{HOM}(\mathbf{X},\textsc{hom}(K,\mathbf{Y}))$.

[Note: Consequently, the functor $\mathbf{X}\square\text{—} : \mathbf{\Delta\text{-}CG}_* \to \mathbf{PRESPEC}$ has a right adjoint, viz. $\mathrm{HOM}(\mathbf{X},\text{—})$, and the functor $\text{—}\square K : \mathbf{PRESPEC} \to \mathbf{PRESPEC}$ has a right adjoint, viz. $\textsc{hom}(K,\text{—})$.]

PROPOSITION 6 For $\mathbf{X},\mathbf{Y}$ in **SPEC** and K in $\mathbf{\Delta\text{-}CG}_*$, there are natural homeomorphisms $\mathrm{HOM}(\mathbf{X}\wedge K,\mathbf{Y}) \approx \mathrm{HOM}(\mathbf{X},\mathbf{Y})^K \approx \mathrm{HOM}(\mathbf{X},\textsc{hom}(K,\mathbf{Y}))$.

[Note: Consequently, the functor $\mathbf{X}\wedge\text{—} : \mathbf{\Delta\text{-}CG}_* \to \mathbf{SPEC}$ has a right adjoint, viz. $\mathrm{HOM}(\mathbf{X},\text{—})$, and the functor $\text{—}\wedge K : \mathbf{SPEC} \to \mathbf{SPEC}$ has a right adjoint, viz. $\textsc{hom}(K,\text{—})$.]

Examples: (1) $\mathbf{Q}_q^\infty(K\#_k L) \approx (\mathbf{Q}_q^\infty K)\wedge L$ and $\mathbf{U}_q^\infty\textsc{hom}(K,\mathbf{X}) \approx (\mathbf{U}_q^\infty\mathbf{X})^K$; (2) $s(\mathbf{X}\square K) \approx s\mathbf{X}\wedge K$.

Example: (Σ,Ω) is an adjoint pair.

EXAMPLE (1) $\mathbf{X}\wedge\mathbf{S}^0 \approx \mathbf{X}$; (2) $\textsc{hom}(\mathbf{S}^0,\mathbf{X}) \approx \mathbf{X}$; (3) $(\mathbf{X}\wedge K)\wedge L \approx \mathbf{X}\wedge(K\#_k L)$; (4) $\textsc{hom}(K\#_k L,\mathbf{X}) \approx \textsc{hom}(K,\textsc{hom}(L,\mathbf{X}))$.

FACT Suppose that $\mathbf{X}$ is an injective prespectrum—then $\forall\,K$, $\mathbf{X}\square K$ is an injective prespectrum.

FACT Suppose that $\mathbf{X}$ is a separated prespectrum—then $\forall$ nonempty compact Hausdorff space K, $\mathbf{X}\square K_+$ is a separated prespectrum.

LEMMA Suppose that

$$\begin{tikzcd} P \arrow[r, "\eta"] \arrow[d, "\xi"'] & Y \arrow[d, "g"] \\ X \arrow[r, "f"'] & Z \end{tikzcd}$$

is a pullback square in $\mathbf{\Delta\text{-}CG}$. Assume: g is a closed embedding—then ξ is a closed embedding.

EXAMPLE Let $\mathbf{f} : \mathbf{X} \to \mathbf{Y}$ be a morphism of prespectra—then the mapping cylinder $\mathbf{M_f}$ of $\mathbf{f}$ is defined by the pushout square

$$\begin{tikzcd}
\mathbf{X} \square \{0\}_+ \arrow[r] \arrow[d] & \mathbf{Y} \square \{0\}_+ \arrow[d] \\
\mathbf{X} \square I_+ \arrow[r] & \mathbf{M_f}
\end{tikzcd}$$

($I_+ = [0,1] \amalg *$ (cf. p. 3–28)). There is a natural arrow $\mathbf{M_f} \to \mathbf{Y} \square I_+$ and the commutative diagram

$$\begin{tikzcd}
\mathbf{X} \arrow[r] \arrow[d] & \mathbf{M_f} \arrow[d] \\
\mathbf{Y} \arrow[r] & \mathbf{Y} \square I_+
\end{tikzcd}$$

is a pullback square. Definition: $\mathbf{f}$ is a prespectral cofibration if $\mathbf{M_f} \to \mathbf{Y} \square I_+$ has a left inverse. Every prespectral cofibration is a levelwise closed embedding.

FACT Let $\mathbf{f} : \mathbf{X} \to \mathbf{Y}$ be a morphism of prespectra. Assume: $\begin{cases} \mathbf{X} \\ \mathbf{Y} \end{cases}$ are injective—then $\mathbf{M_f}$ is injective.

EXAMPLE Let $\mathbf{f} : \mathbf{X} \to \mathbf{Y}$ be a morphism of spectra—then the mapping cylinder $\mathbf{M_f}$ of $\mathbf{f}$ is defined by the pushout square

$$\begin{tikzcd}
\mathbf{X} \wedge \{0\}_+ \arrow[r] \arrow[d] & \mathbf{Y} \wedge \{0\}_+ \arrow[d] \\
\mathbf{X} \wedge I_+ \arrow[r] & \mathbf{M_f}
\end{tikzcd}$$

($I_+ = [0,1] \amalg *$ (cf. p. 3–28)). There is a natural arrow $\mathbf{M_f} \to \mathbf{Y} \wedge I_+$ and the commutative diagram

$$\begin{tikzcd}
\mathbf{X} \arrow[r] \arrow[d] & \mathbf{M_f} \arrow[d] \\
\mathbf{Y} \arrow[r] & \mathbf{Y} \wedge I_+
\end{tikzcd}$$

is a pullback square. Indeed, the mapping cylinder of $\mathbf{f}$ in **SPEC** is the spectrification of the mapping cylinder of $\mathbf{f}$ in **PRESPEC**. And: All data is injective, so

$$s\left(\begin{tikzcd}
\mathbf{X} \arrow[r] \arrow[d] & \mathbf{M_f} \arrow[d] \\
\mathbf{Y} \arrow[r] & \mathbf{Y} \square I_+
\end{tikzcd}\right)$$

is a pullback square in **SPEC** (cf. p. 16–3). Definition: $\mathbf{f}$ is a spectral cofibration if $\mathbf{M_f} \to \mathbf{Y} \wedge I_+$ has a left inverse. Every spectral cofibration is a levelwise closed embedding.

[Note: The arrow $\mathbf{f} : \mathbf{X} \to \mathbf{Y}$ is a spectral cofibration iff the commutative diagram

$$\begin{tikzcd}
\mathbf{X} \wedge \{0\}_+ \arrow[r] \arrow[d] & \mathbf{Y} \wedge \{0\}_+ \arrow[d] \\
\mathbf{X} \wedge I_+ \arrow[r] & \mathbf{Y} \wedge I_+
\end{tikzcd}$$

is a weak pushout square or, equivalently, iff $\forall$ $\mathbf{Z}$, $\mathbf{f}$ has the LLP w.r.t. $\mathrm{HOM}(I_+, \mathbf{Z}) \xrightarrow{p_0} \mathbf{Z}$. Example: Suppose that $L \to K$ is a pointed cofibration—then $\forall$ $\mathbf{X}$, $\mathbf{X} \wedge L \to \mathbf{X} \wedge K$ is a spectral cofibration.]

Notation: For $n \geq 0$, put $\mathbf{S}^n = \mathbf{Q}^\infty \mathbf{S}^n$ and for $n > 0$, put $\mathbf{S}^{-n} = \mathbf{Q}_n^\infty \mathbf{S}^0$.

[Note: $\forall$ n & $\forall$ $m \geq 0$, $\Sigma^m \mathbf{S}^n (= \mathbf{S}^n \wedge \mathbf{S}^m) \approx \mathbf{S}^{m+n}$ and $\forall$ $n \geq 0$ & $\forall$ $m \geq 0$, $\mathbf{S}^{-m} \wedge \mathbf{S}^n \approx (\mathbf{Q}_m^\infty \mathbf{S}^0) \wedge \mathbf{S}^n \approx \mathbf{Q}_m^\infty (\mathbf{S}^0 \#_k \mathbf{S}^n) \approx \mathbf{Q}_m^\infty \mathbf{S}^n \approx \mathbf{S}^{n-m}$.]

EXAMPLE $\forall\, X$, $\mathbf{Q}_q^\infty X \approx \mathbf{S}^{-q} \wedge X$. So, the arrow of adjunction id $\rightarrow \mathbf{U}_q^\infty \circ \mathbf{Q}_q^\infty$ is given by $X \rightarrow (\mathbf{S}^{-q} \wedge X)_q$ and the arrow of adjunction $\mathbf{Q}_q^\infty \circ \mathbf{U}_q^\infty \rightarrow$ id is given by $\mathbf{S}^{-q} \wedge X_q \rightarrow \mathbf{X}$.

PROPOSITION 7 The qth space functor $\mathbf{U}_q^\infty : \mathbf{SPEC} \rightarrow \mathbf{\Delta}\text{-}\mathbf{CG}_*$ is represented by $\mathbf{S}^{-q}$.

[$\forall\, \mathbf{X}$, $\operatorname{Mor}(\mathbf{S}^{-q}, \mathbf{X}) = \operatorname{Mor}(\mathbf{Q}_q^\infty \mathbf{S}^0, \mathbf{X}) \approx \operatorname{Mor}(\mathbf{S}^0, \mathbf{U}_q^\infty \mathbf{X}) = \mathbf{U}_q^\infty \mathbf{X}$.]

A homotopy in **SPEC** is an arrow $\mathbf{X} \wedge I_+ \rightarrow \mathbf{Y}$. Homotopy is an equivalence relation which respects composition, so there is an associated quotient category $\mathbf{SPEC}/\simeq$: $[\mathbf{X}, \mathbf{Y}]_0 = \operatorname{Mor}(\mathbf{X}, \mathbf{Y})/\simeq$, i.e., $[\mathbf{X}, \mathbf{Y}]_0 = \pi_0(\operatorname{HOM}(\mathbf{X}, \mathbf{Y}))$.

EXAMPLE (Homotopy Groups of Spectra) Let **X** be a spectrum—then the nth homotopy group $\pi_n(\mathbf{X})$ of **X** ($n \in \mathbf{Z}$) is $[\mathbf{S}^n, \mathbf{X}]_0$. The $\pi_n(\mathbf{X})$ are necessarily abelian. And: $\forall\, n \geq 0$, $\pi_n(\mathbf{X}) = \pi_n(X_0)$, while $\pi_{-n}(\mathbf{X}) = \pi_0(X_n)$. Therefore **X** is connective iff $\pi_n(\mathbf{X}) = 0$ for $n \leq -1$. Example: $\forall\, X$ in $\mathbf{\Delta}\text{-}\mathbf{CG}_{*c}$, the suspension spectrum $\mathbf{Q}^\infty X$ of X is connective. Proof: ΣX is path connected and wellpointed ($\Rightarrow \Sigma^2 X$ simply connected), thus $\forall\, n \geq 1$, $\pi_q(\Sigma^{q+n} X) = *$ (by the suspension isomorphism and Hurewicz), so $\pi_{-n}(\mathbf{Q}^\infty X) = \pi_0(\Omega^\infty \Sigma^\infty \Sigma^n X) = \operatorname{colim} \pi_q(\Sigma^{q+n} X) = *$.

[Note: The stable homotopy groups $\pi_n^s(X)$ ($n \geq 0$) of X are the $\pi_n(\mathbf{Q}^\infty X)$ ($= \pi_n(\Omega^\infty \Sigma^\infty X)$). Example: $\pi_0^s(X) \approx \widetilde{H}_0(X)$.]

FACT Let $(\mathbf{X}, \mathbf{f})$ be an object in **FIL(SPEC)** (cf. p. 0–10). Assume: $\forall\, n$, $\mathbf{f}_n : \mathbf{X}_n \rightarrow \mathbf{X}_{n+1}$ is a levelwise CG embedding—then $\forall$ pointed compact Hausdorff space K, $\operatorname{colim}[\mathbf{Q}_q^\infty K, \mathbf{X}_n]_0 \approx [\mathbf{Q}_q^\infty K, \operatorname{colim} \mathbf{X}_n]_0$ (cf. p. 16–4).

EXAMPLE Imitating the construction in pointed spaces, one can attach to each object $(\mathbf{X}, \mathbf{f})$ in **FIL(SPEC)** a spectrum $\operatorname{tel}(\mathbf{X}, \mathbf{f})$, its mapping telescope. Thus $\operatorname{tel}(\mathbf{X}, \mathbf{f}) = \operatorname{colim} \operatorname{tel}_n(\mathbf{X}, \mathbf{f})$ and the arrow $\operatorname{tel}_n(\mathbf{X}, \mathbf{f}) \rightarrow \operatorname{tel}_{n+1}(\mathbf{X}, \mathbf{f})$ is a spectral cofibration (hence is a levelwise closed embedding (cf. p. 16–6)). Since there are canonical homotopy equivalences $\operatorname{tel}_n(\mathbf{X}, \mathbf{f}) \rightarrow \mathbf{X}_n$, it follows that $\forall$ pointed compact Hausdorff space K, $\operatorname{colim}[\mathbf{Q}_q^\infty K, \mathbf{X}_n]_0 \approx [\mathbf{Q}_q^\infty K, \operatorname{tel}(\mathbf{X}, \mathbf{f})]_0$.

LEMMA Suppose that $\mathbf{f} : \mathbf{X} \rightarrow \mathbf{Y}$ is a homotopy equivalence—then $\forall\, q$, $f_q : X_q \rightarrow Y_q$ is a homotopy equivalence.

[The qth space functor $\mathbf{U}_q^\infty : \mathbf{SPEC} \rightarrow \mathbf{\Delta}\text{-}\mathbf{CG}_*$ is a **V**-functor ($\mathbf{V} = \mathbf{\Delta}\text{-}\mathbf{CG}_*$), hence preserves homotopies.]

FACT **SPEC** is a cofibration category if weak equivalence=homotopy equivalence, cofibration= spectral cofibration. All objects are cofibrant and fibrant.

[Note: One way to proceed is to show that **SPEC** is an I-category in the sense of Baues[†].]

A prespectrum $\mathbf{X}$ is said to satisfy the cofibration condition if $\forall\, q$, the arrow $\Sigma X_q \rightarrow X_{q+1}$ adjoint to σ_q is a pointed cofibration. An $\mathbf{X}$ which satisfies the cofibration condition is necessarily separated (for then σ_q is a closed embedding). Example: $\forall\, \mathbf{X}$, $M\mathbf{X}$ satisfies the cofibration condition (cf. p. 14–71).

EXAMPLE Equip **PRESPEC** with the model category structure supplied by Proposition 56 in §14—then every cofibrant $\mathbf{X}$ satisfies the cofibration condition.

[Note: The converse is false. To see this, take any X in $\mathbf{\Delta}$-$\mathbf{CG}_*$ and consider the prespectrum whose spectrification is $\mathbf{Q}_q^\infty X$, bearing in mind that the inclusion of a point is always a pointed cofibration.]

A spectrum $\mathbf{X}$ is said to be tame if it is homotopy equivalent to a spectrum of the form $s\mathbf{Y}$, where $\mathbf{Y}$ is a prespectrum satisfying the cofibration condition ($\Rightarrow s\mathbf{Y} \approx e\mathbf{Y}$).

LEMMA Let $\mathbf{f} : \mathbf{X} \rightarrow \mathbf{Y}$ be a morphism of spectra. Assume: $\mathbf{f}$ is a levelwise pointed homotopy equivalence—then $\forall$ tame spectrum $\mathbf{Z}$, $\mathbf{f}_* : [\mathbf{Z}, \mathbf{X}]_0 \rightarrow [\mathbf{Z}, \mathbf{Y}]_0$ is bijective.

Application: A levelwise pointed homotopy equivalence between tame spectra is a homotopy equivalence of spectra.

FACT Let $\mathbf{f} : \mathbf{X} \rightarrow \mathbf{Y}$ be a morphism of prespectra. Assume: $\begin{cases} \mathbf{X} \\ \mathbf{Y} \end{cases}$ satisfy the cofibration condition and $\mathbf{f}$ is a levelwise pointed homotopy equivalence—then $s\mathbf{f} : s\mathbf{X} \rightarrow s\mathbf{Y}$ is a homotopy equivalence of spectra.

Equip $\mathbf{\Delta}$-$\mathbf{CG}_*$ with its singular structure.

LEMMA Let $\mathbf{f} : \mathbf{X} \rightarrow \mathbf{Y}$ be a morphism of spectra—then $\mathbf{f}$ is a levelwise fibration iff $\mathbf{f}$ has the RLP w.r.t. the spectral cofibrations $\mathbf{S}^{-q} \wedge [0,1]_+^n \rightarrow \mathbf{S}^{-q} \wedge I[0,1]_+^n$ $(n \geq 0, q \geq 0)$.

LEMMA Let $\mathbf{f} : \mathbf{X} \rightarrow \mathbf{Y}$ be a morphism of spectra—then $\mathbf{f}$ is a levelwise acyclic fibration iff $\mathbf{f}$ has the RLP w.r.t. the spectral cofibrations $\mathbf{S}^{-q} \wedge \mathbf{S}_+^{n-1} \rightarrow \mathbf{S}^{-q} \wedge \mathbf{D}_+^n$ $(n \geq 0, q \geq 0)$.

[†] *Algebraic Homotopy*, Cambridge University Press (1989), 18–27.

Since $(\mathbf{Q}_q^\infty, \mathbf{U}_q^\infty)$ is an adjoint pair, the lifting problem
$$\begin{tikzcd} \mathbf{Q}_q^\infty L \arrow[r] \arrow[d, "\mathbf{Q}_q^\infty f"'] & \mathbf{X} \arrow[d, "\mathbf{f}"] \\ \mathbf{Q}_q^\infty K \arrow[r] \arrow[ur, dashed] & \mathbf{Y} \end{tikzcd}$$
is equivalent to the lifting problem
$$\begin{tikzcd} L \arrow[r] \arrow[d, "f"'] & \mathbf{U}_q^\infty \mathbf{X} \arrow[d, "\mathbf{U}_q^\infty \mathbf{f}"] \\ K \arrow[r] \arrow[ur, dashed] & \mathbf{U}_q^\infty \mathbf{Y} \end{tikzcd}$$
.

PROPOSITION 8 Equip $\mathbf{\Delta}\text{-}\mathbf{CG}_*$ with its singular structure—then **SPEC** is a model category if weak equivalences and fibrations are levelwise, the cofibrations being those morphisms which have the LLP w.r.t. the levelwise acyclic fibrations.

[The proof is basically the same as that for the singular structure on **TOP** (cf. p. 12–10 ff.). Thus there are two claims.

Claim: Every morphism $\mathbf{f} : \mathbf{X} \to \mathbf{Y}$ can be written as a composite $\mathbf{f}_\omega \circ \mathbf{i}_\omega$, where $\mathbf{i}_\omega : \mathbf{X} \to \mathbf{X}_\omega$ is a weak equivalence and has the LLP w.r.t. all fibrations and $\mathbf{f}_\omega : \mathbf{X}_\omega \to \mathbf{Y}$ is a fibration.

[In the small object argument, take $S_0 = \{\mathbf{S}^{-q} \wedge [0,1]^n_+ \to \mathbf{S}^{-q} \wedge I[0,1]^n_+ \ (n \geq 0, q \geq 0)$—then $\forall\ k$, the arrow $\mathbf{X}_k \to \mathbf{X}_{k+1}$ is a spectral cofibration, hence is a levelwise closed embedding (cf. p. 16–6). Since $\mathbf{Q}_q^\infty [0,1]^n_+ \approx \mathbf{S}^{-q} \wedge [0,1]^n_+$, it follows that colim $\text{Mor}(\mathbf{S}^{-q} \wedge [0,1]^n_+, \mathbf{X}_k) \approx \text{Mor}(\mathbf{S}^{-q} \wedge [0,1]^n_+, \mathbf{X}_\omega)\ \forall\ n$ (cf. p. 16–4), so $\mathbf{f}_\omega$ has the RLP w.r.t. the $\mathbf{S}^{-q} \wedge [0,1]^n_+ \to \mathbf{S}^{-q} \wedge I[0,1]^n_+$, i.e., is a fibration. The assertions regarding $\mathbf{i}_\omega$ are implicit in its construction.]

Claim: Every morphism $\mathbf{f} : \mathbf{X} \to \mathbf{Y}$ can be written as a composite $\mathbf{f}_\omega \circ \mathbf{i}_\omega$, where $\mathbf{i}_\omega : \mathbf{X} \to \mathbf{X}_\omega$ has the LLP w.r.t. levelwise acyclic fibrations and $\mathbf{f}_\omega$ is both a weak equivalence and a fibration.

[Run the small object argument once again, taking $S_0 = \{\mathbf{S}^{-q} \wedge \mathbf{S}^{n-1}_+ \to \mathbf{S}^{-q} \wedge \mathbf{D}^n_+ \ (n \geq 0, q \geq 0).]$

Combining the claims gives MC–5 and the nontrivial half of MC–4 can be established in the usual way.]

[Note: All objects are fibrant and every cofibration is a spectral cofibration.]

True or false: The model category structure on **SPEC** is proper.

HSPEC is the homotopy category of **SPEC** (cf. p. 12–24 ff.). In this situation, $I\mathbf{X} = \mathbf{X} \wedge I_+$ is a cylinder object when $\mathbf{X}$ is cofibrant while $P\mathbf{X} = \text{HOM}(I_+, \mathbf{X})$ serves as a path object. And: It can be assumed that the "cofibrant replacement" $\mathcal{L}\mathbf{X}$ is functorial in $\mathbf{X}$, so $\mathcal{L} : \mathbf{SPEC} \to \mathbf{SPEC}_c$.

[Note: Recall too that the inclusion $\mathbf{HSPEC_c} \rightarrow \mathbf{HSPEC}$ is an equivalence of categories (cf. §12, Proposition 13).]

Remark: Suppose that $\mathbf{X}$ is cofibrant—then for any $\mathbf{Y}$, $[\mathbf{X},\mathbf{Y}]_0 \approx [\mathbf{X},\mathbf{Y}]$ (cf. p. 12–25) (all objects are fibrant), thus if $\mathbf{Y} \rightarrow \mathbf{Z}$ is a weak equivalence, then $[\mathbf{X},\mathbf{Y}]_0 \approx [\mathbf{X},\mathbf{Z}]_0$.

Example: Let (K, k_0) be a pointed CW complex—then $\mathbf{Q}_q^\infty K$ is cofibrant.

FACT Let $\mathbf{f} : \mathbf{X} \rightarrow \mathbf{Y}$ be a morphism of spectra—then $\mathbf{f}$ is a weak equivalence iff $\forall\ n$, $\pi_n(\mathbf{f}) : \pi_n(\mathbf{X}) \rightarrow \pi_n(\mathbf{Y})$ is an isomorphism.

LEMMA $\mathbf{HSPEC_c}$ has coproducts and weak pushouts.

[Note: The wedge $\bigvee\limits_i \mathbf{X}_i$ is the coproduct of the $\mathbf{X}_i$ in $\mathbf{HSPEC_c}$. Proof: $\bigvee\limits_i \mathbf{X}_i$ is cofibrant and for any cofibrant $\mathbf{Y}$, $[\bigvee\limits_i \mathbf{X}_i, \mathbf{Y}] \approx [\bigvee\limits_i \mathbf{X}_i, \mathbf{Y}]_0 \approx \pi_0(\mathrm{HOM}(\bigvee\limits_i \mathbf{X}_i, \mathbf{Y})) \approx \pi_0(\prod\limits_i \mathrm{HOM}(\mathbf{X}_i, \mathbf{Y})) \approx \prod\limits_i \pi_0(\mathrm{HOM}(\mathbf{X}_i, \mathbf{Y})) \approx \prod\limits_i [\mathbf{X}_i, \mathbf{Y}]_0 \approx \prod\limits_i [\mathbf{X}_i, \mathbf{Y}]$.]

BROWN REPRESENTABILITY THEOREM A cofunctor $F : \mathbf{HSPEC_c} \rightarrow \mathbf{SET}$ is representable iff it converts coproducts into products and weak pushouts into weak pullbacks.

[In the notation of p. 5–79, let $\mathcal{U} = \{\mathbf{S}^n : n \in \mathbf{Z}\}$. If $\mathbf{f} : \mathbf{X} \rightarrow \mathbf{Y}$ is a morphism such that $\forall\ n$, the arrow $[\mathbf{S}^n, \mathbf{X}] \rightarrow [\mathbf{S}^n, \mathbf{Y}]$ is bijective, then $\mathbf{f}$ is a weak equivalence (cf. supra), thus is a homotopy equivalence (cf. §12, Proposition 10). Therefore $\mathcal{U}_1$ holds. As for $\mathcal{U}_2$, given an object $(\mathbf{X}, \mathbf{f})$ in $\mathbf{FIL}(\mathbf{HSPEC_c})$, $\mathrm{tel}(\mathbf{X}, \mathbf{f})$ is a weak colimit and $\forall\ n$, the arrow $\mathrm{colim}[\mathbf{S}^n, \mathbf{X}_k] \rightarrow [\mathbf{S}^n, \mathrm{tel}(\mathbf{X}, \mathbf{f})]$ is bijective (cf. p. 16–7).]

EXAMPLE $\mathbf{HSPEC_c}$ has products. For if $\{\mathbf{X}_i\}$ is a set of cofibrant spectra, then the cofunctor $\mathbf{Y} \rightarrow \prod\limits_i [\mathbf{Y}, \mathbf{X}_i]$ satisfies the hypotheses of the Brown representability theorem.

PROPOSITION 9 Suppose that $\mathbf{A} \rightarrow \mathbf{Y}$ is a cofibration and $\mathbf{X} \rightarrow \mathbf{B}$ is a fibration—then the arrow $\mathrm{HOM}(\mathbf{Y}, \mathbf{X}) \rightarrow \mathrm{HOM}(\mathbf{A}, \mathbf{X}) \times_{\mathrm{HOM}(\mathbf{A},\mathbf{B})} \mathrm{HOM}(\mathbf{Y}, \mathbf{B})$ is a Serre fibration which is a weak homotopy equivalence if $\mathbf{A} \rightarrow \mathbf{Y}$ or $\mathbf{X} \rightarrow \mathbf{B}$ is acyclic.

Proposition 9 implies (and is implied by) the following equivalent statements (cf. §13, Propositions 31 and 32).

FACT If $\mathbf{A} \rightarrow \mathbf{Y}$ is a cofibration in $\mathbf{SPEC}$ and if $L \rightarrow K$ is a cofibration in $\mathbf{\Delta}$-$\mathbf{CG}_*$, then the arrow $\mathbf{A} \wedge K \underset{\mathbf{A} \wedge L}{\sqcup} \mathbf{Y} \wedge L \rightarrow \mathbf{Y} \wedge K$ is a cofibration in $\mathbf{SPEC}$ which is acyclic if $\mathbf{A} \rightarrow \mathbf{Y}$ or $L \rightarrow K$ is acyclic.

FACT If $L \rightarrow K$ is a cofibration in $\mathbf{\Delta}$-$\mathbf{CG}_*$ and if $\mathbf{X} \rightarrow \mathbf{B}$ is a fibration in $\mathbf{SPEC}$, then the arrow $\mathrm{HOM}(K, \mathbf{X}) \rightarrow \mathrm{HOM}(L, \mathbf{X}) \times_{\mathrm{HOM}(L,\mathbf{B})} \mathrm{HOM}(K, \mathbf{B})$ is a fibration in $\mathbf{SPEC}$ which is acyclic if $L \rightarrow K$ or $\mathbf{X} \rightarrow \mathbf{B}$ is acyclic.

The shift suspension is the functor $\Lambda : \mathbf{SPEC} \to \mathbf{SPEC}$ defined by $(\Lambda \mathbf{X})_q = X_{q+1}$ $(q \geq 0)$ and the shift desuspension is the functor $\Lambda^{-1} : \mathbf{SPEC} \to \mathbf{SPEC}$ defined by $(\Lambda^{-1}\mathbf{X})_q = \begin{cases} X_{q-1} & (q > 0) \\ \Omega X_0 & (q = 0) \end{cases}$.

PROPOSITION 10 The pair (Λ, Λ^{-1}) is an adjoint equivalence of categories.

EXAMPLE Λ^q is a left adjoint for Λ^{-q} and, by Proposition 10, Λ^{-q} is a left adjoint for Λ^q. On the other hand, $\mathbf{Q}^\infty$ is a left adjoint for $\mathbf{U}^\infty$. Therefore $\Lambda^{-q} \circ \mathbf{Q}^\infty$ is a left adjoint for $\mathbf{U}^\infty \circ \Lambda^q$. But $\mathbf{U}^\infty \circ \Lambda^q = \mathbf{U}_q^\infty$, thus $\forall\, q \geq 0$, $\Lambda^{-q} \circ \mathbf{Q}^\infty \approx \mathbf{Q}_q^\infty$.

Remarks: (1) Λ preserves weak equivalences, so $Q \circ \Lambda : \mathbf{SPEC} \to \mathbf{HSPEC}$ sends weak equivalences to isomorphisms and there is a commutative triangle

$$\begin{tikzcd}
\mathbf{SPEC} \arrow[r, "Q \circ \Lambda"] \arrow[d, "Q"'] & \mathbf{HSPEC} \\
\mathbf{HSPEC} \arrow[ur, "\mathbf{L}\Lambda"'] &
\end{tikzcd}$$

, $\mathbf{L}\Lambda$ the total left derived functor for Λ; (2) Λ^{-1} preserves weak equivalences, so $Q \circ \Lambda^{-1} : \mathbf{SPEC} \to \mathbf{HSPEC}$ sends weak equivalences to isomorphisms and there is a commutative triangle

$$\begin{tikzcd}
\mathbf{SPEC} \arrow[r, "Q \circ \Lambda^{-1}"] \arrow[d, "Q"'] & \mathbf{HSPEC} \\
\mathbf{HSPEC} \arrow[ur, "\mathbf{R}\Lambda^{-1}"'] &
\end{tikzcd}$$

, $\mathbf{R}\Lambda^{-1}$ the total right derived functor for Λ^{-1}.

PROPOSITION 11 The pair $(\mathbf{L}\Lambda, \mathbf{R}\Lambda^{-1})$ is an adjoint equivalence of categories.

[Λ^{-1} preserves fibrations and acyclic fibrations (the data is levelwise). Therefore Λ preserves cofibrations and the TDF theorem implies that $(\mathbf{L}\Lambda, \mathbf{R}\Lambda^{-1})$ is an adjoint pair. Consider now the bijection of adjunction $\Xi_{\mathbf{X},\mathbf{Y}} : \mathrm{Mor}\,(\Lambda\mathbf{X}, \mathbf{Y}) \to \mathrm{Mor}\,(\mathbf{X}, \Lambda^{-1}\mathbf{Y})$, so $\Xi_{\mathbf{X},\mathbf{Y}}\mathbf{f}$ is the composition $\mathbf{X} \to \Lambda^{-1}\Lambda\mathbf{X} \xrightarrow{\Lambda^{-1}\mathbf{f}} \Lambda^{-1}\mathbf{Y}$. Since the arrow $\mathbf{X} \to \Lambda^{-1}\Lambda\mathbf{X}$ is an isomorphism, $\Xi_{\mathbf{X},\mathbf{Y}}\mathbf{f}$ is a weak equivalence iff $\Lambda^{-1}\mathbf{f}$ is a weak equivalence, i.e., iff $\mathbf{f}$ is a weak equivalence. Therefore the pair $(\mathbf{L}\Lambda, \mathbf{R}\Lambda^{-1})$ is an adjoint equivalence of categories (cf. p. 12–29).]

Λ^{-1} is naturally isomorphic to $\overline{\Omega}$. Here $(\overline{\Omega}\mathbf{X})_q = \Omega X_q$, the arrow of structure $\Omega X_q \to \Omega\Omega X_{q+1}$ being $\Omega\sigma_q$. Therefore the difference between $\overline{\Omega}$ and Ω is the twist T (cf. p. 14–75). Define a pseudo natural weak equivalence $\Xi_{\mathbf{X}} : \Omega\mathbf{X} \to \overline{\Omega}\mathbf{X}$ by letting $\Xi_{\mathbf{X},q} : \Omega X_q \to \Omega X_q$ be the identity for even q and the negative of the identity for odd q (i.e., coordinate reversal).

LEMMA Let $\mathbf{C}$ be a category and let $F, G : \mathbf{C} \to \mathbf{PRESPEC}$ be functors. Suppose given a pseudo natural weak equivalence $\Xi : F \to G$—then in the notation of the conversion principle, there are natural transformations $sFX \xleftarrow{sr} sMFX \xrightarrow{sm\Xi} sMGX \xrightarrow{sr} sGX$.

[Note: $sM\Xi$ is a weak equivalence. Moreover, the sr are weak equivalences if F, G factor through **SEPPRESPEC**.]

Application: $\forall$ $\mathbf{X}$ in **SPEC**, $\Omega\mathbf{X}$ is naturally weakly equivalent to $\overline{\Omega}\mathbf{X}$ or still, is naturally weakly equivalent to $\Lambda^{-1}\mathbf{X}$.

Example: In **HSPEC**, $\mathbf{S}^{-n} \approx \Omega^n\mathbf{S}^0$ $(n \geq 0)$.

PROPOSITION 12 The total left derived functor $\mathbf{L}\Sigma$ for Σ exists and the total right derived functor $\mathbf{R}\Omega$ for Ω exists. And: $(\mathbf{L}\Sigma, \mathbf{R}\Omega)$ is an adjoint pair.

[Σ preserves cofibrations and Ω preserves fibrations. Now quote the TDF theorem.]

[Note: Since Ω, $\overline{\Omega}$ preserve weak equivalences, there are commutative triangles

$$\begin{tikzcd} \mathbf{SPEC} \arrow[r, "Q\circ\Omega"] \arrow[d, "Q"'] & \mathbf{HSPEC} \\ \mathbf{HSPEC} \arrow[ur, "\mathbf{R}\Omega"'] & \end{tikzcd}, \quad \begin{tikzcd} \mathbf{SPEC} \arrow[r, "Q\circ\overline{\Omega}"] \arrow[d, "Q"'] & \mathbf{HSPEC} \\ \mathbf{HSPEC} \arrow[ur, "\mathbf{R}\overline{\Omega}"'] & \end{tikzcd}$$

and, by the above, natural isomorphisms, $\mathbf{R}\Omega \rightarrow \mathbf{R}\overline{\Omega}$, $\mathbf{R}\overline{\Omega} \rightarrow \mathbf{R}\Lambda^{-1}$.]

Σ preserves weak equivalences between cofibrant objects. So, unraveling the definitions, one finds that $\mathbf{L}\Sigma(= L(Q\circ\Sigma))$ "is" $\mathbf{L}(\Sigma\circ\iota\circ\mathcal{L})$ $(\mathbf{L}(\Sigma\circ\iota\circ\mathcal{L})\circ Q = Q\circ\Sigma\circ\iota\circ\mathcal{L})$, $\iota : \mathbf{SPEC}_c \rightarrow \mathbf{SPEC}$ the inclusion. In particular: $\forall\,\mathbf{X}$, $\mathbf{L}\Sigma\mathbf{X} = \Sigma\mathcal{L}\mathbf{X}$.

PROPOSITION 13 The pair $(\mathbf{L}\Sigma, \mathbf{R}\Omega)$ is an adjoint equivalence of categories.

[According to Proposition 11, the arrows of adjunction $\mathrm{id} \xrightarrow{\mu} \mathbf{R}\Lambda^{-1}\circ\mathbf{L}\Lambda$, $\mathbf{L}\Lambda\circ\mathbf{R}\Lambda^{-1} \xrightarrow{\nu}$ id are natural isomorphisms and the claim is that the arrows of adjunction $\mathrm{id} \xrightarrow{\mu} \mathbf{R}\Omega\circ\mathbf{L}\Sigma$, $\mathbf{L}\Sigma\circ\mathbf{R}\Omega \xrightarrow{\nu}$ id are natural isomorphisms. Thus fix a natural isomorphism $\mathbf{R}\Omega \rightarrow \mathbf{R}\Lambda^{-1}$— then there exists a unique natural isomorphism $\mathbf{L}\Lambda \rightarrow \mathbf{L}\Sigma$ characterized by the commutativity of

$$\begin{tikzcd} {[\mathbf{L}\Sigma\mathbf{X}, \mathbf{Y}]} \arrow[r] \arrow[d] & {[\mathbf{X}, \mathbf{R}\Omega\mathbf{Y}]} \arrow[d] \\ {[\mathbf{L}\Lambda\mathbf{X}, \mathbf{Y}]} \arrow[r] & {[\mathbf{X}, \mathbf{R}\Lambda^{-1}\mathbf{Y}]} \end{tikzcd}$$

$\forall\,\mathbf{X}, \mathbf{Y}$. It remains only to note that the diagrams

$$\begin{tikzcd} \mathrm{id} \arrow[r, "\mu"] \arrow[d, "\mu"'] & \mathbf{R}\Lambda^{-1}\circ\mathbf{L}\Lambda \arrow[d] \\ \mathbf{R}\Omega\circ\mathbf{L}\Sigma \arrow[r] & \mathbf{R}\Lambda^{-1}\circ\mathbf{L}\Sigma \end{tikzcd}, \quad \begin{tikzcd} \mathbf{L}\Lambda\circ\mathbf{R}\Omega \arrow[r] \arrow[d] & \mathbf{L}\Lambda\circ\mathbf{R}\Lambda^{-1} \arrow[d, "\nu"] \\ \mathbf{L}\Sigma\circ\mathbf{R}\Omega \arrow[r, "\nu"'] & \mathrm{id} \end{tikzcd}$$

of natural transformations commute.]

Application: **HSPEC** is an additive category and $\mathbf{L}\Sigma$ is an additive functor.

[Note: **HSPEC** has coproducts and products (since $\mathbf{HSPEC}_c$ does (cf. p. 16–10)). Standard categorical generalities then imply that the arrow $\mathbf{X}\vee\mathbf{Y} \rightarrow \mathbf{X}\times\mathbf{Y}$ is an isomorphism for all $\mathbf{X}, \mathbf{Y}$ in **HSPEC** (cf. p. 0–36).]

Notation: Write $\left\{\begin{matrix}\Sigma\\ \Omega\end{matrix}\right.$ in place of $\left\{\begin{matrix}\mathbf{L}\Sigma\\ \mathbf{R}\Omega\end{matrix}\right.$ and $\left\{\begin{matrix}\Lambda\\ \Lambda^{-1}\end{matrix}\right.$ in place of $\left\{\begin{matrix}\mathbf{L}\Lambda\\ \mathbf{R}\Lambda^{-1}\end{matrix}\right.$.

PROPOSITION 14 **HSPEC** is a triangulated category satisfying the octahedral axiom.

[Working in $\mathbf{HSPEC_c}$, stipulate that a triangle $\mathbf{X}' \xrightarrow{\mathbf{u}'} \mathbf{Y}' \xrightarrow{\mathbf{v}'} \mathbf{Z}' \xrightarrow{\mathbf{w}'} \Sigma\mathbf{X}'$ is exact if it is isomorphic to a triangle $\mathbf{X} \xrightarrow{\mathbf{f}} \mathbf{Y} \xrightarrow{\mathbf{j}} \mathbf{C_f} \xrightarrow{\boldsymbol{\pi}} \Sigma\mathbf{X}$ for some $\mathbf{f}$ ($\mathbf{C_f}$ = mapping cone of $\mathbf{f}$) (obvious definition). Since TR_1–TR_5 are immediate, it will be enough to deal just with the octahedral axiom. Suppose given exact triangles $\mathbf{X} \xrightarrow{\mathbf{u}} \mathbf{Y} \rightarrow \mathbf{Z}' \rightarrow \Sigma\mathbf{X}$, $\mathbf{Y} \xrightarrow{\mathbf{v}} \mathbf{Z} \rightarrow \mathbf{X}' \rightarrow \Sigma\mathbf{Y}$, $\mathbf{X} \xrightarrow{\mathbf{v}\circ\mathbf{u}} \mathbf{Z} \rightarrow \mathbf{Y}' \rightarrow \Sigma\mathbf{X}$, where without loss of generality, $\mathbf{Z}' = \mathbf{C_u}$, $\mathbf{X}' = \mathbf{C_v}$, $\mathbf{Y}' = \mathbf{C_{v\circ u}}$. Starting at the prespectrum level, define a pointed continuous function $f_n : C_{u_n} \rightarrow C_{v_n \circ u_n}$ by letting f_n be the identity on ΓX_n and v_n on Y_n and define a pointed continuous function $g_n : C_{v_n \circ u_n} \rightarrow C_{v_n}$ by letting g_n be Γu_n on ΓX_n and the identity on Z_n—then the f_n and the g_n combine to give morphisms of prespectra, so applying s, $\exists$ morphisms $\mathbf{f} : \mathbf{Z}' \rightarrow \mathbf{Y}'$ and $\mathbf{g} : \mathbf{Y}' \rightarrow \mathbf{X}'$ of spectra. By construction, the composite $\mathbf{Z}' \xrightarrow{\mathbf{f}} \mathbf{Y}' \rightarrow \Sigma\mathbf{X}$ is the arrow $\mathbf{Z}' \rightarrow \Sigma\mathbf{X}$ and the composite $\mathbf{Z} \rightarrow \mathbf{Y}' \xrightarrow{\mathbf{g}} \mathbf{X}'$ is the arrow $\mathbf{Z} \rightarrow \mathbf{X}'$. Letting $\mathbf{h} : \mathbf{X}' \rightarrow \Sigma\mathbf{Z}'$ be the composite $\mathbf{X} \rightarrow \Sigma\mathbf{Y} \rightarrow \Sigma\mathbf{Z}'$, one sees that all the commutativity required of the octahedral axiom is present, thus the final task is to establish that the triangle $\mathbf{Z}' \xrightarrow{\mathbf{f}} \mathbf{Y}' \xrightarrow{\mathbf{g}} \mathbf{X}' \xrightarrow{\mathbf{h}} \Sigma\mathbf{Z}'$ is exact. But there is a canonical commutative diagram

$$\begin{tikzcd}
\mathbf{Z}' \arrow[r, "\mathbf{f}"] \arrow[d, equal] & \mathbf{Y}' \arrow[r, "\mathbf{g}"] \arrow[d, equal] & \mathbf{X}' \arrow[r, "\mathbf{h}"] \arrow[d, "\phi"] & \Sigma\mathbf{Z}' \arrow[d, equal] \\
\mathbf{Z}' \arrow[r, "\mathbf{f}"'] & \mathbf{Y}' \arrow[r, "\mathbf{j}"'] & \mathbf{C_f} \arrow[r, "\boldsymbol{\pi}"'] & \Sigma\mathbf{Z}'
\end{tikzcd}$$

. And: ϕ is a homotopy equivalence.]

Application: An exact triangle $\mathbf{X} \xrightarrow{\mathbf{u}} \mathbf{Y} \xrightarrow{\mathbf{v}} \mathbf{Z} \xrightarrow{\mathbf{w}} \Sigma\mathbf{X}$ in **HSPEC** gives rise to a long exact sequence in homotopy $\cdots \rightarrow \pi_{n+1}(\mathbf{Z}) \rightarrow \pi_n(\mathbf{X}) \rightarrow \pi_n(\mathbf{Y}) \rightarrow \pi_n(\mathbf{Z}) \rightarrow \pi_{n-1}(\mathbf{X}) \rightarrow \cdots$.

EXAMPLE If $\mathbf{f} : \mathbf{X} \rightarrow \mathbf{Y}$, $\mathbf{g} : \mathbf{Y} \rightarrow \mathbf{Z}$ are morphisms in **HSPEC**, then there is an exact triangle $\mathbf{C_f} \rightarrow \mathbf{C_{g\circ f}} \rightarrow \mathbf{C_g} \rightarrow \Sigma\mathbf{C_f}$.

Remark: **HSPEC** is compactly generated (take $\mathcal{U} = \{\mathrm{S}^n : n \in \mathbf{Z}\}$) and admits Adams representability (by Neeman's countability criterion).

EXAMPLE The homotopy groups of a compact spectrum are finitely generated.

[The thick subcategory of **HSPEC** whose objects are those $\mathbf{X}$ such that $\pi_q(\mathbf{X})$ is finitely generated $\forall\, q$ contains the $\mathbf{S}^n$.]

It is also true that **HSPEC** is a closed category (indeed, a CTC) but the proof requires some preliminary work which is best carried out in a more general context.

The main difficulty lies in equipping **HSPEC** with the structure of a closed category (cf. p. 16–30). Granted this, the fact that **HSPEC** is a CTC can be seen as follows.

Recall that if $f : X \to Y$ is a map in the pointed category, then there is a homotopy commutative diagram

$$\begin{tikzcd}
 & & \Sigma\Omega X \arrow[r] \arrow[d] & \Sigma\Omega Y \arrow[r] \arrow[d] & \Sigma E_f \arrow[r] \arrow[d] & \Sigma X \arrow[d, equal] \\
\Omega Y \arrow[r] \arrow[d, equal] & E_f \arrow[r] \arrow[d] & X \arrow[r] \arrow[d] & Y \arrow[r] \arrow[d] & C_f \arrow[r] & \Sigma X, \\
\Omega Y \arrow[r] & \Omega C_f \arrow[r] & \Omega\Sigma X \arrow[r] & \Omega\Sigma Y & &
\end{tikzcd}$$

a formalism which also holds in the category of prespectra or spectra. Of course, when viewed in **HSPEC**, the arrows $\mathbf{E_f} \to \Omega\mathbf{C_f}$, $\Sigma\mathbf{E_f} \to \mathbf{C_f}$ are isomorphisms (cf. Proposition 13). Turning to the axioms for a CTC, the only one that is potentially troublesome is CTC$_4$. In order not to obscure the issue, we shall proceed informally, omitting all mention of $\mathcal{L}$ and the underlying total derived functors. Thus given $\mathbf{X} \xrightarrow{\mathbf{f}} \mathbf{Y} \xrightarrow{\mathbf{j}} \mathbf{C_f} \xrightarrow{\pi} \Sigma\mathbf{X}$, one has to show that $\forall$ $\mathbf{Z}$, the triangle $\Omega \operatorname{hom}(\mathbf{X},\mathbf{Z}) \xrightarrow{-(\pi^* \circ \eta_{\mathbf{X},\mathbf{Z}})} \operatorname{hom}(\mathbf{C_f},\mathbf{Z}) \xrightarrow{\mathbf{j}^*} \operatorname{hom}(\mathbf{Y},\mathbf{Z}) \xrightarrow{\nu^{-1}_{\operatorname{hom}(\mathbf{X},\mathbf{Z})} \circ \mathbf{f}^*} \Sigma\Omega \operatorname{hom}(\mathbf{X},\mathbf{Z})$ is exact. Consider the commutative diagram

$$\begin{tikzcd}
\Omega\mathbf{Y} \arrow[r, "\Omega\mathbf{j}"] \arrow[d, equal] & \Omega\mathbf{C_f} \arrow[r, "\Omega\pi"] \arrow[d, equal] & \Omega\Sigma\mathbf{X} \arrow[r, "\nu_{\mathbf{Y}}^{-1} \circ \mu_{\mathbf{Y}}^{-1} \circ \Omega\Sigma\mathbf{f}"] \arrow[d, "\mu_{\mathbf{X}}^{-1}"] & \Sigma\Omega\mathbf{Y} \arrow[d, equal] \\
\Omega\mathbf{Y} \arrow[r, "\Omega\mathbf{j}"'] & \Omega\mathbf{C_f} \arrow[r, "\mu_{\mathbf{X}}^{-1} \circ \Omega\pi"'] & \mathbf{X} \arrow[r, "\nu_{\mathbf{Y}}^{-1} \circ \mathbf{f}"'] & \Sigma\Omega\mathbf{Y}
\end{tikzcd}.$$

Since the triangle on the bottom is exact (cf. p. 15–2), so is the triangle on the top. But then, on the basis of the commutative diagram

$$\begin{tikzcd}
\Omega\mathbf{Y} \arrow[r] \arrow[d, equal] & \mathbf{E_f} \arrow[r] \arrow[d] & \mathbf{X} \arrow[r, "\mathbf{f}"] \arrow[d, "\mu_{\mathbf{X}}"] & \mathbf{Y} \arrow[r, "\nu_{\mathbf{Y}}^{-1}"] \arrow[d, "\mu_{\mathbf{Y}}"] & \Sigma\Omega\mathbf{Y} \arrow[d, equal] \\
\Omega\mathbf{Y} \arrow[r] & \Omega\mathbf{C_f} \arrow[r] & \Omega\Sigma\mathbf{X} \arrow[r, "\Omega\Sigma\mathbf{f}"'] & \Omega\Sigma\mathbf{Y} \arrow[r, "\nu_{\mathbf{Y}}^{-1} \circ \mu_{\mathbf{Y}}^{-1}"'] & \Sigma\Omega\mathbf{Y}
\end{tikzcd},$$

the triangle $\Omega\mathbf{Y} \to \mathbf{E_f} \to \mathbf{X} \xrightarrow{\nu_{\mathbf{Y}}^{-1} \circ \mathbf{f}} \Sigma\Omega\mathbf{Y}$ is exact. In particular: The triangle $\Omega \operatorname{hom}(\mathbf{X},\mathbf{Z}) \to \mathbf{E}_{\mathbf{f}^*} \to \operatorname{hom}(\mathbf{Y},\mathbf{Z}) \xrightarrow{\nu^{-1}_{\operatorname{hom}(\mathbf{X},\mathbf{Z})} \circ \mathbf{f}^*} \Sigma\Omega \operatorname{hom}(\mathbf{X},\mathbf{Z})$ is exact. However, there is an isomorphism $\mathbf{E}_{\mathbf{f}^*} \to \operatorname{hom}(\mathbf{C_f},\mathbf{Z})$ and a commutative diagram

$$\begin{tikzcd}
\Omega \operatorname{hom}(\mathbf{X},\mathbf{Z}) \arrow[r] \arrow[d, equal] & \mathbf{E}_{\mathbf{f}^*} \arrow[r] \arrow[d] & \operatorname{hom}(\mathbf{Y},\mathbf{Z}) \arrow[r, "\nu^{-1}_{\operatorname{hom}(\mathbf{X},\mathbf{Z})} \circ \mathbf{f}^*"] \arrow[d, equal] & \Sigma\Omega \operatorname{hom}(\mathbf{X},\mathbf{Z}) \arrow[d, equal] \\
\Omega \operatorname{hom}(\mathbf{X},\mathbf{Z}) \arrow[r, "-(\pi^* \circ \eta_{\mathbf{X},\mathbf{Z}})"'] & \operatorname{hom}(\mathbf{C_f},\mathbf{Z}) \arrow[r, "\mathbf{j}^*"'] & \operatorname{hom}(\mathbf{Y},\mathbf{Z}) \arrow[r, "\nu^{-1}_{\operatorname{hom}(\mathbf{X},\mathbf{Z})} \circ \mathbf{f}^*"'] & \Sigma\Omega \operatorname{hom}(\mathbf{X},\mathbf{Z})
\end{tikzcd},$$

hence the triangle on the bottom is exact, this being the case of the triangle on the top.

LEMMA HSPEC is a compactly generated CTC.

[In general, X dualizable $\Rightarrow \begin{cases} \Sigma\mathbf{X} \\ \Omega\mathbf{X} \end{cases}$ **dualizable (cf. §15, Proposition 35). But trivially the unit** $\mathbf{S}^0$ **is dualizable, thus** $\forall\ n > 0$, $\mathbf{S}^n \approx \Sigma^n\mathbf{S}^0$ **&** $\mathbf{S}^{-n} \approx \Omega^n\mathbf{S}^0$ **are dualizable, i.e., all the elements of** $\mathcal{U} = \{\mathbf{S}^n : n \in \mathbf{Z}\}$ **are dualizable.]**

[Note: Observe too that $\forall\, n$, $D\mathbf{S}^n \approx \mathbf{S}^{-n}$.**]**

Remark: HSPEC is a unital compactly generated CTC (since $\mathbf{S}^0$ **is compact). Accordingly, du HSPEC = cpt HSPEC (cf. p. 15–39), the thick subcategory generated by the** $\mathbf{S}^n$ **(theorem of Neeman-Ravenel).**

[Note: It is clear that HSPEC is actually monogenic.]

EXAMPLE The compact objects in HSPEC are those objects which are isomorphic to a $\mathbf{Q}_q^\infty K$, **where** K **is a pointed finite CW complex.**

Notation: Given a real finite dimensional inner product space V, let $\mathbf{S}^V$ denote its one point compactification (base point at ∞) and for any X in **Δ-CG**$_*$, put $\Sigma^V X = X\#_k\mathbf{S}^V$, $\Omega^V X = X^{\mathbf{S}^V}$.

[Note: If V and W are two real finite dimensional inner product spaces such that $V \subset W$, write $W-V$ for the orthogonal complement of V in W—then $\forall\ X$, $\Sigma^{W-V}\Sigma^V X \approx \Sigma^W X$ and $\Omega^V\Omega^{W-V}X \approx \Omega^W X$.]

A universe is a real inner product space $\mathcal{U}$ with $\dim\mathcal{U} = \omega$ equipped with the finite topology. **UN** is the category whose objects are the universes and whose morphisms are the linear isometries. An indexing set in a universe $\mathcal{U}$ is a set $\mathcal{A}$ of finite dimensional subspaces of $\mathcal{U}$ such that each finite dimensional subspace V of $\mathcal{U}$ is contained in some $U \in \mathcal{A}$. The standard indexing set is the set of all finite dimensional subspaces of $\mathcal{U}$. Example: Take $\mathcal{U} = \mathbf{R}^\infty$—then $\{\mathbf{R}^q : q \geq 0\}$ is an indexing set in $\mathbf{R}^\infty$.

Let $\mathcal{A}$ be an indexing set in a universe $\mathcal{U}$—then a $(\mathcal{U}, \mathcal{A})$-prespectrum $\mathbf{X}$ is a collection of pointed Δ-separated compactly generated spaces X_U $(U \in \mathcal{A})$ and a collection of pointed continuous functions $X_V \xrightarrow{\sigma_{V,W}} \Omega^{W-V}X_W$ $(V, W \in \mathcal{A}$ & $V \subset W)$ such that $X_V \xrightarrow{\sigma_{V,V}} X_V$ is the identity and for $U \subset V \subset W$ in $\mathcal{A}$, the diagram

$$\begin{tikzcd} X_U \arrow[r, "\sigma_{U,V}"] \arrow[d, "\sigma_{U,W}"'] & \Omega^{V-U}X_V \arrow[d, "\Omega^{V-U}\sigma_{V,W}"] \\ \Omega^{W-U}X_W \arrow[r, equal] & \Omega^{V-U}\Omega^{W-V}X_W \end{tikzcd}$$

commutes. $\mathbf{PRESPEC}_{\mathcal{U},\mathcal{A}}$ is the category whose objects are the $(\mathcal{U}, \mathcal{A})$-prespectra and whose morphisms $\mathbf{f} : \mathbf{X} \to \mathbf{Y}$ are collections of pointed continuous functions $f_U : X_U \to Y_U$

such that the diagram

$$\begin{tikzcd} X_V \arrow[r, "f_V"] \arrow[d] & Y_V \arrow[d] \\ \Omega^{W-V}X_W \arrow[r, "\Omega^{W-V}f_W"'] & \Omega^{W-V}Y_W \end{tikzcd}$$

commutes for $V \subset W$ in $\mathcal{A}$. A $(\mathcal{U},\mathcal{A})$-prespectrum $\mathbf{X}$ is a $\underline{(\mathcal{U},\mathcal{A})\text{-spectrum}}$ if the $\sigma_{V,W}$ are homeomorphisms. $\mathbf{SPEC}_{\mathcal{U},\mathcal{A}}$ is the full subcategory of $\mathbf{PRESPEC}_{\mathcal{U},\mathcal{A}}$ with object class the $(\mathcal{U},\mathcal{A})$-spectra. Example: Take $\mathcal{U} = \mathbf{R}^\infty$, $\mathcal{A} = \{\mathbf{R}^q : q \geq 0\}$—then $\mathbf{PRESPEC}_{\mathcal{U},\mathcal{A}} = \mathbf{PRESPEC}$, $\mathbf{SPEC}_{\mathcal{U},\mathcal{A}} = \mathbf{SPEC}$.

[Note: When $\mathcal{A}$ is the standard indexing set, write $\mathbf{PRESPEC}_{\mathcal{U}}$, $\mathbf{SPEC}_{\mathcal{U}}$ in place of $\mathbf{PRESPEC}_{\mathcal{U},\mathcal{A}}$, $\mathbf{SPEC}_{\mathcal{U},\mathcal{A}}$.]

What has been said earlier can now be said again. Thus introduce the notion of a separated $(\mathcal{U},\mathcal{A})$-prespectrum by requiring that the $\sigma_{V,W} : X_V \rightarrow \Omega^{W-V}X_W$ be **CG** embeddings. This done, repeat the proof of Proposition 1 to see that $\mathbf{SEPPRESPEC}_{\mathcal{U},\mathcal{A}}$ is a reflective subcategory of $\mathbf{PRESPEC}_{\mathcal{U},\mathcal{A}}$ with reflector E^∞. Next, as in Proposition 2, $\mathbf{SPEC}_{\mathcal{U},\mathcal{A}}$ is a reflective subcategory of $\mathbf{SEPPRESPEC}_{\mathcal{U},\mathcal{A}}$ (the reflector sends $\mathbf{X}$ to $e\mathbf{X}$, where $(e\mathbf{X})_V = \operatorname*{colim}_{W \supset V} \Omega^{W-V}X_W$). Conclusion: $\mathbf{SPEC}_{\mathcal{U},\mathcal{A}}$ is a reflective subcategory of $\mathbf{PRESPEC}_{\mathcal{U},\mathcal{A}}$ (cf. Proposition 3), hence is complete and cocomplete.

[Note: The composite $\mathbf{PRESPEC}_{\mathcal{U},\mathcal{A}} \xrightarrow{E^\infty} \mathbf{SEPPRESPEC}_{\mathcal{U},\mathcal{A}} \xrightarrow{e} \mathbf{SPEC}_{\mathcal{U},\mathcal{A}}$ is the $\underline{\text{spectrification functor:}}$ $\mathbf{X} \rightarrow s\mathbf{X}$ $(s = e \circ E^\infty)$.]

EXAMPLE Fix $U \in \mathcal{A}$. Given an X in $\mathbf{\Delta}\text{-}\mathbf{CG}_*$, let $\mathbf{Q}_U^\infty X$ be the spectrification of the prespectrum $V \rightarrow \begin{cases} \Sigma^{V-U}X & (V \supset U) \\ * & (V \not\supset U) \end{cases}$, where $\Sigma^{V-U}X \rightarrow \Omega^{W-V}\Sigma^{W-V}\Sigma^{V-U}X \approx \Omega^{W-V}\Sigma^{W-U}X$ $(V, W \in \mathcal{A}$ & $U \subset V \subset W)$ (otherwise, the arrow is the inclusion of the base point). Viewed as a functor from $\mathbf{\Delta}\text{-}\mathbf{CG}_*$ to $\mathbf{SPEC}_{\mathcal{U},\mathcal{A}}$, $\mathbf{Q}_U^\infty$ is a left adjoint for the Uth space functor $\mathbf{U}_U^\infty : \mathbf{SPEC}_{\mathcal{U},\mathcal{A}} \rightarrow \mathbf{\Delta}\text{-}\mathbf{CG}_*$ that sends $\mathbf{X} = \{X_U\}$ to X_U.

FACT If $\mathbf{X}$ is a $(\mathcal{U},\mathcal{A})$-spectrum and if $\dim V_1 = \dim V_2$ $(V_1, V_2 \in \mathcal{A})$, then $X_{V_1} \approx X_{V_2}$.

[Embed V_1 and V_2 in a common finite dimensional $W \in \mathcal{A}$ and observe that $X_{V_1} \approx \Omega^{W-V_1}X_W \approx \Omega^{W-V_2}X_W \approx X_{V_2}$.]

Notation: Given $\mathbf{X}, \mathbf{Y}$ in $\mathbf{PRESPEC}_{\mathcal{U},\mathcal{A}}$, write $\text{HOM}(\mathbf{X},\mathbf{Y})$ for $\text{Mor}(\mathbf{X},\mathbf{Y})$ topologized via the equalizer diagram $\text{Mor}(\mathbf{X},\mathbf{Y}) \rightarrow \prod\limits_{V \in \mathcal{A}} Y_V^{X_V} \rightrightarrows \prod\limits_{\substack{V,W \in \mathcal{A} \\ V \subset W}} (\Omega^{W-V}Y_W)^{X_V}$.

So, just as before, spectrification is a continuous functor (cf. Proposition 4) and there are analogs of Propositions 5 and 6 ($\square$ $(\wedge)$ and HOM being defined in the obvious way).

Remark: $\mathbf{PRESPEC}_{\mathcal{U},\mathcal{A}}$ and $\mathbf{SPEC}_{\mathcal{U},\mathcal{A}}$ are $\mathbf{V}$-categories, where $\mathbf{V} = \mathbf{\Delta}\text{-}\mathbf{CG}_*$. Accordingly, to say that s is continuous simply means that s is a $\mathbf{V}$-functor.

[Note: The interpretation of $\square$ ($\wedge$) and HOM is that $\mathbf{PRESPEC}_{\mathcal{U},\mathcal{A}}$ and $\mathbf{SPEC}_{\mathcal{U},\mathcal{A}}$ admit a closed $\mathbf{\Delta}$-$\mathbf{CG}_*$ action (the topological parallel of closed simplicial action).]

LEMMA Let $\mathcal{A}$ and $\mathcal{B}$ be indexing sets in a universe $\mathcal{U}$ with $\mathcal{A} \subset \mathcal{B}$—then the arrow of restriction $i^* : \mathbf{PRESPEC}_{\mathcal{U},\mathcal{B}} \rightarrow \mathbf{PRESPEC}_{\mathcal{U},\mathcal{A}}$ has a left adjoint i_* and a right adjoint $i_!$.

[For $\mathbf{X}$ in $\mathbf{PRESPEC}_{\mathcal{U},\mathcal{A}}$ and W an element of $\mathcal{B}$, $(i_*\mathbf{X})_W$ is the coequalizer of $\coprod\limits_{\substack{V'' \subset V' \\ V' \subset W}} \Sigma^{W-V'}\Sigma^{V'-V''}X_{V''} \rightrightarrows \coprod\limits_{\substack{V \\ V \subset W}} \Sigma^{W-V}X_V$ and $(i_!\mathbf{X})_W$ is the equalizer of $\prod\limits_{\substack{V \\ W \subset V}} \Omega^{V-W}X_V$ $\rightrightarrows \prod\limits_{\substack{V' \subset V'' \\ W \subset V'}} \Omega^{V'-W}\Omega^{V''-V'}X_{V''}$ $(V, V', V'' \in \mathcal{A})$.]

The formulas figuring in the lemma can be understood in terms of "enriched" Kan extensions. Thus let $\mathbf{I}_{\mathcal{A}}$ be the category whose objects are the elements of $\mathcal{A}$, with $\operatorname{Mor}(V', V'') = \begin{cases} \mathbf{S}^{V''-V'} & (V'' \supset V') \\ * & (V'' \not\supset V') \end{cases}$ (composition comes from the identification $\mathbf{S}^{V-U} \#_k \mathbf{S}^{W-V} \approx \mathbf{S}^{W-U}$)—then $\mathbf{I}_{\mathcal{A}}$ is a small $\mathbf{V}$-category and $\mathbf{PRESPEC}_{\mathcal{U},\mathcal{A}}$ "is" $\mathbf{V}[\mathbf{I}_{\mathcal{A}}, \mathbf{\Delta}\text{-}\mathbf{CG}_*]$ (cf. p. 0–42) ($\mathbf{V} = \mathbf{\Delta}\text{-}\mathbf{CG}_*$). So, if $\mathcal{A} \subset \mathcal{B}$ and $i : \mathbf{I}_{\mathcal{A}} \rightarrow \mathbf{I}_{\mathcal{B}}$ is the inclusion, $i_* = \operatorname{lan}$ & $i_! = \operatorname{ran}$, i.e., $i_*\mathbf{X} = \operatorname{lan}\mathbf{X}$ (the left Kan extension of $\mathbf{X}$ along i) & $i_!\mathbf{X} = \operatorname{ran}\mathbf{X}$ (the right Kan extension of $\mathbf{X}$ along i).

PROPOSITION 15 Let $\mathcal{A}$ and $\mathcal{B}$ be indexing sets in a universe $\mathcal{U}$ with $\mathcal{A} \subset \mathcal{B}$—then the arrow of restriction $i^* : \mathbf{SPEC}_{\mathcal{U},\mathcal{B}} \rightarrow \mathbf{SPEC}_{\mathcal{U},\mathcal{A}}$ is an equivalence of categories.

[The functor $s \circ i_*$ is a left adjoint for i^* and the arrows of adjunction $\operatorname{id} \xrightarrow{\mu} i^* \circ (s \circ i_*)$, $(s \circ i_*) \circ i^* \xrightarrow{\nu} \operatorname{id}$ are natural isomorphisms.]

Application: Let $\mathcal{U}$ be a universe—then $\forall$ indexing set $\mathcal{A}$ in $\mathcal{U}$, $\mathbf{SPEC}_{\mathcal{U},\mathcal{A}}$ is equivalent to $\mathbf{SPEC}_{\mathcal{U}}$.

EXAMPLE (Thom Spectra) If $\mathcal{U}$ is a universe and if $\mathbf{G}_n(\mathcal{U})$ is the grassmannian of n-dimensional subspaces of $\mathcal{U}$, then $\mathbf{G}_n(\mathcal{U})$ is topologized as the colimit of the $\mathbf{G}_n(U)$ ($U \subset \mathcal{U}$ & $\dim U < \omega$), so every compact subspace of $\mathbf{G}_n(\mathcal{U})$ is contained in some $\mathbf{G}_n(U)$. Let K be a compact Hausdorff space and suppose that $f : K \rightarrow \mathbf{G}_n(\mathcal{U})$ is a continuous function. Write $\mathcal{A}_f$ for the set of $U : f(K) \subset \mathbf{G}_n(U)$—then $\mathcal{A}_f$ is an indexing set in $\mathcal{U}$. Given $U \in \mathcal{A}_f$, call K^{U-f} the Thom space of the vector bundle defined by the pullback square

$$\begin{tikzcd} f^*\gamma_n^\perp \arrow[r] \arrow[d] & \gamma_n^\perp \arrow[d] \\ K \arrow[r, "f"'] & \mathbf{G}_n(U) \end{tikzcd}$$

(γ_n the canonical n-plane bundle over $\mathbf{G}_n(U)$). The assignment $U \rightarrow K^{U-f}$ defines an object in $\mathbf{PRESPEC}_{\mathcal{U},\mathcal{A}_f}$. Pass to its spectrification in $\mathbf{SPEC}_{\mathcal{U},\mathcal{A}_f}$, thence by the above to an object in $\mathbf{SPEC}_{\mathcal{U}}$, say K^{-f}. In general, an arbitrary X in $\mathbf{\Delta}$-$\mathbf{CG}$ can be represented as the colimit of its compact subspaces $K : X \approx \operatorname{colim} K$. Accordingly, for $f : X \to \mathbf{G}_n(\mathcal{U})$ a continuous function, put $X^{-f} = \operatorname{colim} K^{-f|K}$, the Thom spectrum of the virtual vector bundle $-f$. Example: An n-dimensional U determines a map $* \xrightarrow{U} \mathbf{G}_n(\mathcal{U})$ and $*^{-U} \approx \mathbf{S}^{-U}$.

The Uth space functor $\mathbf{U}_U^\infty : \mathbf{SPEC}_{\mathcal{U}} \to \mathbf{\Delta\text{-}CG}_*$ is represented by $\mathbf{S}^{-U}$, where $\mathbf{S}^{-U} = \mathbf{Q}_U^\infty \mathbf{S}^0$ (cf. Proposition 7). Equipping $\mathbf{\Delta\text{-}CG}_*$ with its singular structure, if $\mathbf{f} : \mathbf{X} \to \mathbf{Y}$ is a morphism of $\mathcal{U}$-spectra, then $\mathbf{f}$ is a levelwise fibration iff $\mathbf{f}$ has the RLP w.r.t. the spectral cofibrations $\mathbf{S}^{-U} \wedge [0,1]_+^n \to \mathbf{S}^{-U} \wedge I[0,1]_+^n$ and $\mathbf{f}$ is a levelwise acyclic fibration iff $\mathbf{f}$ has the RLP w.r.t. the spectral cofibrations $\mathbf{S}^{-U} \wedge \mathbf{S}_+^{n-1} \to \mathbf{S}^{-U} \wedge \mathbf{D}_+^n$ $(n \geq 0, U \subset \mathcal{U}\ \&\ \dim U < \omega)$ (cf. p. 16–8). Using this, it follows that $\mathbf{SPEC}_{\mathcal{U}}$ is a model category if weak equivalences and fibrations are levelwise, the cofibrations being those morphisms which have the LLP w.r.t. the levelwise acyclic fibrations (cf. Proposition 8) (bear in mind that a spectral cofibration is necessarily a levelwise closed embedding (cf. p. 16–6)). Proposition 9 and its variants go through without change.

[Note: $\mathbf{HSPEC}_{\mathcal{U}}$ is the homotopy category of $\mathbf{SPEC}_{\mathcal{U}}$ (cf. p. 12–24 ff.).]

Remark: The functor $\mathbf{U}_U^\infty$ preserves fibrations and acyclic fibrations, thus the TDF theorem implies that $\mathbf{LQ}_U^\infty$ and $\mathbf{RU}_U^\infty$ exist and $(\mathbf{LQ}_U^\infty, \mathbf{RU}_U^\infty)$ is an adjoint pair (the requisite assumptions are validated by the generalities on p. 12–3 ff.).

EXAMPLE Take $\mathcal{U} = \mathbf{R}^\infty$—then $i^* : \mathbf{SPEC}_{\mathcal{U}} \to \mathbf{SPEC}$ preserves fibrations and acyclic fibrations, so the hypotheses of the TDF theorem are satisfied (cf. p. 12–3 ff.). Therefore $\mathbf{L}i_*$ and $\mathbf{R}i^*$ exist and $(\mathbf{L}i_*, \mathbf{R}i^*)$ is an adjoint pair. Dissecting the bijection of adjunction $\Xi_{\mathbf{X},\mathbf{Y}} : \operatorname{Mor}(i_*\mathbf{X}, \mathbf{Y}) \to \operatorname{Mor}(\mathbf{X}, i^*\mathbf{Y})$, it follows that $\Xi_{\mathbf{X},\mathbf{Y}}\mathbf{f}$ is a weak equivalence iff $\mathbf{f}$ is a weak equivalence, thus the pair $(\mathbf{L}i_*, \mathbf{R}i^*)$ is an adjoint equivalence of categories (cf. p. 12–29).

Let $\mathcal{U}$, $\mathcal{U}'$ be universes, $f : \mathcal{U} \to \mathcal{U}'$ a linear isometry—then there is a functor $f^* : \mathbf{PRESPEC}_{\mathcal{U}'} \to \mathbf{PRESPEC}_{\mathcal{U}}$ which assigns to each $\mathbf{X}'$ in $\mathbf{PRESPEC}_{\mathcal{U}'}$ the $\mathcal{U}$-prespectrum $f^*\mathbf{X}'$ specified by $(f^*\mathbf{X}')_U = X'_{f(U)}$, where $(f^*\mathbf{X}')_V \to \Omega^{W-V}(f^*\mathbf{X}')_W$ is the composite $X'_{f(V)} \to \Omega^{f(W)-f(V)} X'_{f(W)} \to \Omega^{W-V} X'_{f(W)}$. It has a left adjoint $f_* : \mathbf{PRESPEC}_{\mathcal{U}} \to \mathbf{PRESPEC}_{\mathcal{U}'}$, viz. $(f_*\mathbf{X})_{U'} = \Sigma^{U'-f(U)} X_U$ $(U = f^{-1}(U'))$, where $(f_*\mathbf{X})_{V'} \to \Omega^{W'-V'}(f_*\mathbf{X})_{W'}$ is the composite $\Sigma^{V'-f(V)} X_V \to \Omega^{W'-V'} \Sigma^{W'-V'} \Sigma^{V'-f(V)} X_V$ $\to \Omega^{W'-V'} \Sigma^{W'-f(W)} \Sigma^{f(W)-f(V)} X_V \to \Omega^{W'-V'} \Sigma^{W'-f(W)} \Sigma^{W-V} X_V \to \Omega^{W'-V'} \Sigma^{W'-f(W)}$ X_W $(V = f^{-1}(V'), W = f^{-1}(W'))$. Since f^* sends $\mathcal{U}'$-spectra to $\mathcal{U}$-spectra, there is an induced functor $f^* : \mathbf{SPEC}_{\mathcal{U}'} \to \mathbf{SPEC}_{\mathcal{U}}$ and a left adjoint for it is $s \circ f_*$, denoted still by f_*.

Let $\mathbf{I}_{\mathcal{U}}$, $\mathbf{I}_{\mathcal{U}'}$ be the small **V**-categories associated with the standard indexing sets in $\mathcal{U}$, $\mathcal{U}'$—then the linear isometry $f : \mathcal{U} \rightarrow \mathcal{U}'$ determines a continuous functor $F_f : \mathbf{I}_{\mathcal{U}} \rightarrow \mathbf{I}_{\mathcal{U}'}$. Viewing $\mathbf{PRESPEC}_{\mathcal{U}}$ as $\mathbf{V}[\mathbf{I}_{\mathcal{U}}, \mathbf{\Delta\text{-}CG}_*]$ and $\mathbf{PRESPEC}_{\mathcal{U}'}$ as $\mathbf{V}[\mathbf{I}_{\mathcal{U}'}, \mathbf{\Delta\text{-}CG}_*]$, f^* becomes precomposition with F_f and $f_* = \text{lan}$.

EXAMPLE $f_*(\mathbf{X} \wedge K) \approx (f_*\mathbf{X}) \wedge K$ and $f_*(\mathbf{Q}_U^\infty X) \approx \mathbf{Q}_{f(U)}^\infty X$.

FACT Let $\mathcal{U}$, $\mathcal{U}'$ be universes, $f : \mathcal{U} \rightarrow \mathcal{U}'$ a linear isometric isomorphism—then the pair (f_*, f^*) is an adjoint isomorphism of categories.

[Note: Here, of course, it is a question of spectra, not prespectra.]

Let $\mathcal{U}$, $\mathcal{U}'$ be universes—then a $(\mathcal{U}',\mathcal{U})$-spectrum $\mathbf{X}'$ is a collection of $\mathcal{U}'$-spectra $\mathbf{X}'_U$ indexed by the finite dimensional subspaces U of $\mathcal{U}$ and a collection of isomorphisms $\Sigma^{W-V}\mathbf{X}'_W \xrightarrow{\rho_{W,V}} \mathbf{X}'_V$ $(V \subset W)$ such that $\mathbf{X}'_V \xrightarrow{\rho_{V,V}} \mathbf{X}'_V$ is the identity and for $U \subset V \subset W$, the diagram

$$\begin{tikzcd}
\Sigma^{V-U}\Sigma^{W-V}\mathbf{X}'_W \arrow[r, equal] \arrow[d, "\Sigma^{V-U}\rho_{W,V}"'] & \Sigma^{W-U}\mathbf{X}'_W \arrow[d, "\rho_{W,U}"] \\
\Sigma^{V-U}\mathbf{X}'_V \arrow[r, "\rho_{V,U}"'] & \mathbf{X}'_U
\end{tikzcd}$$

commutes. $\mathbf{SPEC}(\mathcal{U}',\mathcal{U})$ is the category whose objects are the $(\mathcal{U}',\mathcal{U})$-spectra and whose morphisms $\mathbf{f} : \mathbf{X}' \rightarrow \mathbf{Y}'$ are collections of morphisms of $\mathcal{U}'$-spectra $\mathbf{f}'_U : \mathbf{X}'_U \rightarrow \mathbf{Y}'_U$ such that the diagram

$$\begin{tikzcd}
\Sigma^{W-V}\mathbf{X}'_W \arrow[r, "\Sigma^{W-V}\mathbf{f}'_W"] \arrow[d] & \Sigma^{W-V}\mathbf{Y}'_W \arrow[d] \\
\mathbf{X}'_V \arrow[r, "\mathbf{f}'_V"'] & \mathbf{Y}'_V
\end{tikzcd}$$

commutes for $V \subset W$.

[Note: It makes sense to suspend a $\mathcal{U}'$-spectrum by a finite dimensional subspace of $\mathcal{U}$ (this being an instance of smashing with an object in $\mathbf{\Delta\text{-}CG}_*$).]

EXAMPLE Let $\mathcal{U}$, $\mathcal{U}'$ be universes, $\mathbf{f} : \mathcal{U} \rightarrow \mathcal{U}'$ a linear isometry. Given an X in $\mathbf{\Delta\text{-}CG}_*$, let $\mathbf{Q}'_f X$ be the object in $\mathbf{SPEC}(\mathcal{U}',\mathcal{U})$ defined by $(\mathbf{Q}'_f X)_U = \mathbf{Q}_{f(U)}^\infty X$, where $\Sigma^{W-V}(\mathbf{Q}'_f X)_W \rightarrow (\mathbf{Q}'_f X)_V$ is the identification $\Sigma^{W-V}\mathbf{Q}_{f(W)}^\infty X \approx \Sigma^{f(W)-f(V)}\mathbf{Q}_{f(W)}^\infty X \approx \mathbf{Q}_{f(V)}^\infty X$.

Notation: Given $\mathbf{X}'$, $\mathbf{Y}'$ in $\mathbf{SPEC}(\mathcal{U}',\mathcal{U})$, write $\text{HOM}(\mathbf{X}', \mathbf{Y}')$ for $\text{Mor}\,(\mathbf{X}', \mathbf{Y}')$ topologized via the equalizer diagram $\text{Mor}\,(\mathbf{X}', \mathbf{Y}') \rightarrow \prod\limits_{V} \text{HOM}(\mathbf{X}'_V, \mathbf{Y}'_V) \rightrightarrows \prod\limits_{\substack{V,W \\ V\subset W}} \text{HOM}(\Sigma^{W-V}\mathbf{X}'_W, \mathbf{Y}'_V)$.

($\wedge$) Fix a K in $\mathbf{\Delta\text{-}CG}_*$. Given an $\mathbf{X}'$ in $\mathbf{SPEC}(\mathcal{U}',\mathcal{U})$, let $\mathbf{X}' \wedge K$ be the $(\mathcal{U}',\mathcal{U})$-spectrum $U \rightarrow \mathbf{X}'_U \wedge K$, where $\Sigma^{W-V}(\mathbf{X}'_W \wedge K) \approx (\mathbf{X}'_W \wedge K) \wedge \mathbf{S}^{W-V} \approx (\mathbf{X}'_W \wedge \mathbf{S}^{W-V}) \wedge K \approx \mathbf{X}'_V \wedge K$.

PROPOSITION 16 For $\mathbf{X}', \mathbf{Y}'$ in $\mathbf{SPEC}(\mathcal{U}',\mathcal{U})$ and K in $\mathbf{\Delta\text{-}CG}_*$, there is a natural homeomorphism $\text{HOM}(\mathbf{X}' \wedge K, \mathbf{Y}') \approx \text{HOM}(\mathbf{X}', \mathbf{Y}')^K$.

(HOM) Fix an $\mathbf{X}'$ in $\mathbf{SPEC}(\mathcal{U}',\mathcal{U})$. Given a $\mathbf{Y}'$ in $\mathbf{SPEC}_{\mathcal{U}'}$, let $\text{HOM}(\mathbf{X}',\mathbf{Y}')$ be the $\mathcal{U}$-spectrum $U \rightarrow \text{HOM}(\mathbf{X}'_U,\mathbf{Y}')$, where $\text{HOM}(\mathbf{X}'_V,\mathbf{Y}') \approx \text{HOM}(\Sigma^{W-V}\mathbf{X}'_W,\mathbf{Y}') \approx \text{HOM}(\mathbf{X}'_W,\Omega^{W-V}\mathbf{Y}') \approx \Omega^{W-V}\text{HOM}(\mathbf{X}'_W,\mathbf{Y}')$.

Observation: $\forall$ $\mathbf{X}$ in $\mathbf{SPEC}_{\mathcal{U}}$, $\text{Mor}(\mathbf{X},\text{HOM}(\mathbf{X}',\mathbf{Y}')) \approx \lim \text{Mor}(X_U,\text{HOM}(\mathbf{X}'_U,\mathbf{Y}')) \approx \lim \text{Mor}(\mathbf{X}'_U \wedge X_U,\mathbf{Y}') \approx \text{Mor}(\text{colim}\ \mathbf{X}'_U \wedge X_U,\mathbf{Y}')$, the colimit being taken over the arrows $\mathbf{X}'_V \wedge X_V \approx \Sigma^{W-V}\mathbf{X}'_W \wedge X_V \approx \mathbf{X}'_W \wedge \Sigma^{W-V}X_V \rightarrow \mathbf{X}'_W \wedge X_W$.

Definition: $\mathbf{X}' \wedge \mathbf{X}$ is the $\mathcal{U}'$-spectrum $\text{colim}\ \mathbf{X}'_U \wedge X_U$.

PROPOSITION 17 For $\mathbf{X}$ in $\mathbf{SPEC}_{\mathcal{U}}$, $\mathbf{Y}'$ in $\mathbf{SPEC}_{\mathcal{U}'}$, and $\mathbf{X}'$ in $\mathbf{SPEC}(\mathcal{U}',\mathcal{U})$, there is a natural homeomorphism $\text{HOM}(\mathbf{X}' \wedge \mathbf{X},\mathbf{Y}') \approx \text{HOM}(\mathbf{X},\text{HOM}(\mathbf{X}',\mathbf{Y}'))$.

EXAMPLE (1) $\mathbf{X}' \wedge \mathbf{Q}^\infty_U X \approx \mathbf{X}'_U \wedge X$; (2) $(\mathbf{X}' \wedge \mathbf{X}) \wedge K \approx \mathbf{X}' \wedge (\mathbf{X} \wedge K) \approx (\mathbf{X}' \wedge K) \wedge \mathbf{X}$.

Notation: Given a vector bundle $\xi : E \rightarrow B$, $T(\xi)$ is its Thom space.

[Note: If S^ξ is the sphere bundle obtained from ξ by fiberwise one point compactification, then $T(\xi) = S^\xi/S_\infty$, where S_∞ is the section at infinity. Example: If $\underline{V}$ is the trivial vector bundle $B \times V \rightarrow B$, then $T(\xi \oplus \underline{V}) \approx \Sigma^V T(\xi)$.]

Let $\mathcal{U}$, $\mathcal{U}'$ be universes. Fix an object $A \xrightarrow{\alpha} \mathcal{I}(\mathcal{U},\mathcal{U}')$ in $\mathbf{\Delta}\text{-}\mathbf{CG}/\mathcal{I}(\mathcal{U},\mathcal{U}')$ ($\mathcal{I}(\mathcal{U},\mathcal{U}')$ topologized as on p. 14–52). Given finite dimensional $U \subset \mathcal{U}$, $U' \subset \mathcal{U}'$, define $A_{U,U'}$ by the pullback square

$$\begin{tikzcd}
A_{U,U'} \arrow[rr] \arrow[d] & & \mathcal{I}(U,U') \arrow[d] \\
A \arrow[r, "\alpha"] & \mathcal{I}(\mathcal{U},\mathcal{U}') \arrow[r] & \mathcal{I}(U,\mathcal{U}')
\end{tikzcd}$$

(so $A_{U,U'} = \{a \in A : \alpha(a)U \subset U'\}$, which can be empty). Write $\xi(\alpha)_{U,U'}$ for the vector bundle over $A_{U,U'}$ with total space $\{(a,u') \in A_{U,U'} \times U' : u' \perp \alpha(a)U\}$ and let $T\alpha_{U,U'}$ be the associated Thom space (if $A_{U,U'}$ is empty, then the Thom space is a singleton). For each U, the assignment $U' \rightarrow T\alpha_{U,U'}$ specifies a $\mathcal{U}'$-prespectrum, call it $\mathbf{T}'\alpha_U$ (the arrow $T\alpha_{U,V'} \rightarrow \Omega^{W'-V'}T\alpha_{U,W'}$ is the adjoint of the arrow $\Sigma^{W'-V'}T\alpha_{U,V'} \rightarrow T\alpha_{U,W'}$ induced by the morphism $\xi(\alpha)_{U,V'} \oplus (\underline{W}' - \underline{V}') \rightarrow \xi(\alpha)_{U,W'}$ of vector bundles). Let $\mathbf{M}'\alpha_U$ be the spectrification of $\mathbf{T}'\alpha_U$—then there are morphisms $\Sigma^{W-V}\mathbf{M}'\alpha_W \rightarrow \mathbf{M}'\alpha_V$ of $\mathcal{U}'$-spectra arising from the morphisms $\xi(\alpha)_{W,V'} \oplus (\underline{W} - \underline{V}) \rightarrow \xi(\alpha)_{V,V'}$ of vector bundles.

PROPOSITION 18 The morphisms $\Sigma^{W-V}\mathbf{M}'\alpha_W \rightarrow \mathbf{M}'\alpha_V$ are isomorphisms, thus the collection $\mathbf{M}'\alpha = \{\mathbf{M}'\alpha_U\}$ is an object in $\mathbf{SPEC}(\mathcal{U}',\mathcal{U})$.

[Since all constructions are natural in $\mathbf{\Delta}\text{-}\mathbf{CG}/\mathcal{I}(\mathcal{U},\mathcal{U}')$ and commute with colimits, one can assume that A is compact. But then, for $V \subset W$, $\exists$ $V' : A_{V,V'} = A_{W,V'} = A$, hence $\Sigma^{W-V}\mathbf{M}'\alpha_W \approx \Sigma^{W-V}\mathbf{Q}^\infty_{V'}T\alpha_{W,V'} \approx \mathbf{Q}^\infty_{V'}\Sigma^{W-V}T\alpha_{W,V'} \approx \mathbf{Q}^\infty_{V'}T\alpha_{V,V'} \approx \mathbf{M}'\alpha_V$.]

Example: There is an isomorphism $\mathbf{M}'\alpha_{\{0\}} \approx \mathbf{Q}^\infty_{\{0\}}A_+$ natural in α.

[In fact, $\xi(\alpha)_{\{0\},U'}$ is the trivial vector bundle $A \times U' \to A$.]

EXAMPLE Suppose that α is the constant map at $f \in \mathcal{I}(\mathcal{U},\mathcal{U}')$—then $\mathbf{M}'\alpha \approx \mathbf{Q}'_f A_+$.

Let $\mathcal{U},\mathcal{U}'$ be universes. Fix an $A \xrightarrow{\alpha} \mathcal{I}(\mathcal{U},\mathcal{U}')$ in $\mathbf{\Delta}$-$\mathbf{CG}/\mathcal{I}(\mathcal{U},\mathcal{U}')$.

($\ltimes$) Given an $\mathbf{X}$ in $\mathbf{SPEC}_{\mathcal{U}}$, let $\alpha \ltimes \mathbf{X}$ be the $\mathcal{U}'$-spectrum $\mathbf{M}'\alpha \wedge \mathbf{X}$.

($\mathcal{HOM}$) Given a $\mathbf{Y}'$ in $\mathbf{SPEC}_{\mathcal{U}'}$, let $\mathcal{HOM}[\alpha, \mathbf{Y}')$ be the $\mathcal{U}$-spectrum $\textsc{hom}(\mathbf{M}'\alpha, \mathbf{Y}')$.

Remark: $\alpha \ltimes \mathbf{X} \approx \operatorname{colim} \alpha|K \ltimes \mathbf{X}$ and $\mathcal{HOM}[\alpha, \mathbf{Y}') \approx \lim \mathcal{HOM}[\alpha|K, \mathbf{Y}')$, where K runs over the compact subspaces of A.

[Note: $\ltimes : \mathbf{\Delta}\text{-}\mathbf{CG}/\mathcal{I}(\mathcal{U},\mathcal{U}') \times \mathbf{SPEC}_{\mathcal{U}} \to \mathbf{SPEC}_{\mathcal{U}'}$ and $\mathcal{HOM} : (\mathbf{\Delta}\text{-}\mathbf{CG}/\mathcal{I}(\mathcal{U},\mathcal{U}'))^{\mathrm{OP}} \times \mathbf{SPEC}_{\mathcal{U}'} \to \mathbf{SPEC}_{\mathcal{U}}$ are continuous functors of their respective arguments. Moreover, $\alpha \ltimes \mathbf{X}$ preserves colimits in α and $\mathbf{X}$, while $\mathcal{HOM}[\alpha, \mathbf{Y}')$ converts colimits in α to limits and preserves limits in $\mathbf{Y}'$.]

PROPOSITION 19 For $\mathbf{X}$ in $\mathbf{SPEC}_{\mathcal{U}}$, $\mathbf{Y}'$ in $\mathbf{SPEC}_{\mathcal{U}'}$, and α in $\mathbf{\Delta}$-$\mathbf{CG}/\mathcal{I}(\mathcal{U},\mathcal{U}')$, there is a natural homeomorphism $\mathrm{HOM}(\alpha \ltimes \mathbf{X}, \mathbf{Y}') \approx \mathrm{HOM}(\mathbf{X}, \mathcal{HOM}[\alpha, \mathbf{Y}'))$ (cf. Proposition 17).

Example: Fix a linear isometry $f : \mathcal{U} \to \mathcal{U}'$, viewed as an object in $* \to \mathcal{I}(\mathcal{U},\mathcal{U}')$—then $f \ltimes \mathbf{X} \approx f_* \mathbf{X}$ and $\mathcal{HOM}[f, \mathbf{Y}') \approx f^* \mathbf{Y}'$ (cf. p. 16–18).

[E.g.: $\mathbf{M}' f_U \approx \mathbf{Q}^{\infty}_{f(U)} \mathbf{S}^0 \Rightarrow \mathcal{HOM}[f, \mathbf{Y}') \approx \mathrm{HOM}(\mathbf{Q}^{\infty}_{f(U)} \mathbf{S}^0, \mathbf{Y}') \approx Y'_{f(U)}$.]

Examples: (1) $(\alpha \ltimes \mathbf{X}) \wedge K \approx \alpha \ltimes (\mathbf{X} \wedge K)$; (2) $\textsc{hom}(K, \mathcal{HOM}[\alpha, \mathbf{Y}')) \approx \mathcal{HOM}[\alpha, \textsc{hom}(K, \mathbf{Y}'))$.

Addendum: Let $\mathbf{HOM}(\mathbf{X}, \mathbf{Y}')$ be the set of ordered pairs $(f, \mathbf{f})$, where $f \in \mathcal{I}(\mathcal{U},\mathcal{U}')$ and $\mathbf{f} : \mathbf{X} \to f^* \mathbf{Y}'$ is a morphism of $\mathcal{U}$-spectra, and let $\epsilon : \mathbf{HOM}(\mathbf{X}, \mathbf{Y}') \to \mathcal{I}(\mathcal{U},\mathcal{U}')$ be the projection $(f, \mathbf{f}) \to f$—then Elmendorf† has shown that one may equip $\mathbf{HOM}(\mathbf{X}, \mathbf{Y}')$ with the structure of a Δ-separated compactly generated space in such a way that ϵ is continuous (and $\epsilon^{-1}(f) \approx \mathrm{HOM}(\mathbf{X}, f^* \mathbf{Y}')\ \forall\ f$). Moreover, there are natural homeomorphisms $\mathrm{HOM}(\alpha \ltimes \mathbf{X}, \mathbf{Y}') \approx \mathrm{HOM}(\alpha, \epsilon) \approx \mathrm{HOM}(\mathbf{X}, \mathcal{HOM}[\alpha, \mathbf{Y}'))$.

[Note: $\mathrm{HOM}(\alpha, \epsilon)$ is the set of continuous functions

$$\begin{tikzcd} A \arrow[rr] \arrow[dr, "\alpha"'] & & \mathbf{HOM}(\mathbf{X}, \mathbf{Y}') \arrow[dl, "\epsilon"] \\ & \mathcal{I}(\mathcal{U},\mathcal{U}') & \end{tikzcd}$$

regarded as a closed subspace of $\mathbf{HOM}(\mathbf{X}, \mathbf{Y}')^A$ (viz., the fiber of $\mathbf{HOM}(\mathbf{X}, \mathbf{Y}')^A \xrightarrow{\epsilon_*} \mathcal{I}(\mathcal{U},\mathcal{U}')^A$ over α).]

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† *J. Pure Appl. Algebra* **54** (1988), 37–94.

FACT Suppose given $\alpha : A \to \mathcal{I}(\mathcal{U},\mathcal{U}')$. Let B be in **Δ-CG** and call π the projection $A \times_k B \to A$—then $(\alpha \circ \pi) \ltimes \mathbf{X} \approx (\alpha \ltimes \mathbf{X}) \wedge B_+$ and $\mathcal{HOM}[\alpha \circ \pi, \mathbf{Y}') \approx \text{HOM}(B_+, \mathcal{HOM}[\alpha, \mathbf{Y}'))$.

FACT Suppose given $\alpha : A \to \mathcal{I}(\mathcal{U},\mathcal{U}')$ and $\beta : B \to \mathcal{I}(\mathcal{U}',\mathcal{U}'')$. Let $\beta \times_c \alpha$ be the composite $B \times_k A \xrightarrow{\beta \times_k \alpha} \mathcal{I}(\mathcal{U}',\mathcal{U}'') \times_k \mathcal{I}(\mathcal{U},\mathcal{U}') \xrightarrow{c} \mathcal{I}(\mathcal{U},\mathcal{U}'')$—then $(\beta \times_c \alpha) \ltimes \mathbf{X} \approx \beta \ltimes (\alpha \ltimes \mathbf{X})$ and $\mathcal{HOM}[\beta \times_c \alpha, \mathbf{Y}'') \approx \mathcal{HOM}[\alpha, \mathcal{HOM}[\beta, \mathbf{Y}''))$.

PROPOSITION 20 Fix an α in $\mathbf{\Delta\text{-}CG}/\mathcal{I}(\mathcal{U},\mathcal{U}')$—then for $\mathbf{X}$ in $\mathbf{SPEC}_{\mathcal{U}}$ and $\mathbf{Y}'$ in $\mathbf{SPEC}_{\mathcal{U}'}$, a morphism $\phi : \alpha \ltimes \mathbf{X} \to \mathbf{Y}'$ determines and is determined by morphisms $\phi(a) : \mathbf{X} \to \alpha(a)^* \mathbf{Y}'$ $(a \in A)$ such that the functions $T\alpha_{U,U'} \#_k X_U \to \Sigma^{U' - \alpha(a)U} Y'_{\alpha(a)U} \to Y'_{U'}$ are continuous, the first arrow being the assignment $(a, u') \#_k x \to \phi(a)_U(x) \#_k u'$ $(a \in A_{U,U'}, u' \in U' - \alpha(a)U, x \in X_U)$.

[Write $\mathbf{M}'\alpha_U = \operatorname{colim}_{U'} \mathbf{Q}^{\infty}_{U'} T\alpha_{U,U'}$ to get $\operatorname{Mor}(\alpha \ltimes \mathbf{X}, \mathbf{Y}') \approx \operatorname{Mor}(\operatorname{colim}_U \mathbf{M}'\alpha_U \wedge X_U, \mathbf{Y}') \approx \lim_U \operatorname{Mor}(\mathbf{M}'\alpha_U \wedge X_U, \mathbf{Y}') \approx \lim_U \operatorname{Mor}(\operatorname{colim}_{U'} \mathbf{Q}^{\infty}_{U'}(T\alpha_{U,U'} \#_k X_U), \mathbf{Y}') \approx \lim_U \lim_{U'} \operatorname{Mor}(\mathbf{Q}^{\infty}_{U'}(T\alpha_{U,U'} \#_k X_U), \mathbf{Y}') \approx \lim_U \lim_{U'} \operatorname{Mor}(T\alpha_{U,U'} \#_k X_U, Y'_{U'})$. Take now a $\phi : \alpha \ltimes \mathbf{X} \to \mathbf{Y}'$ and let $\phi(a)$ be the adjoint of the composite $\alpha(a)_* \mathbf{X} \to \alpha \ltimes \mathbf{X} \xrightarrow{\phi} \mathbf{Y}'$. Projecting from the double limit thus gives rise to continuous functions $T\alpha_{U,U'} \#_k X_U \to Y'_{U'}$ as stated. Conversely, a collection of morphisms $\phi(a) : \mathbf{X} \to \alpha(a)^* \mathbf{Y}'$ $(a \in A)$ satisfying the hypotheses define continuous functions compatible with the maps in the double limit, hence specify a morphism $\phi : \alpha \ltimes \mathbf{X} \to \mathbf{Y}'$.]

Given a universe $\mathcal{U}$, $\mathbf{O}(\mathcal{U})$ is its orthogonal group, so topologically, $\mathbf{O}(\mathcal{U}) = \operatorname{colim} \mathbf{O}(U)$, where $\mathbf{O}(U)$ is the orthogonal group of the ambient finite dimensional subspace U of $\mathcal{U}$.

LEMMA Let $\mathcal{U}$ be a universe—then $\forall$ finite dimensional $U \subset \mathcal{U}$, the arrow of restriction $\mathbf{O}(\mathcal{U}) \to \mathcal{I}(U,\mathcal{U})$ is a Serre fibration.

Application: $\sec_{\mathcal{I}(U,\mathcal{U})}(\mathbf{O}(\mathcal{U}))$ is not empty.

[$\mathcal{I}(U,\mathcal{U})$ is a CW complex and, being contractible (cf. p. 14–52), the identity map $\mathcal{I}(U,\mathcal{U}) \to \mathcal{I}(U,\mathcal{U})$ admits a lifting $\mathcal{I}(U,\mathcal{U}) \to \mathbf{O}(\mathcal{U})$ (cf. p. 4–7).]

UNTWISTING LEMMA Let $\mathcal{U}$, $\mathcal{U}'$ be universes. Fix $U \subset \mathcal{U}$, $U' \subset \mathcal{U}'$ such that $U \approx U'$—then there is an isomorphism $\mathbf{M}'\alpha_U \approx \mathbf{Q}^{\infty}_{U'} A_+$ natural in α.

[Choose a linear isometric isomorphism $f : U \to U'$ and a section $s' : \mathcal{I}(U',\mathcal{U}') \to \mathbf{O}(\mathcal{U}')$. Put $s = s' \circ (f^*)^{-1}$. Define $A_{[U,V']}$ by the pullback square

$$\begin{tikzcd}
A_{[U,V']} \arrow[rr] \arrow[d] & & \mathbf{O}(V') \arrow[d] \\
A \arrow[r, "\alpha|U"] & \mathcal{I}(U,\mathcal{U}') \arrow[r, "s"] & \mathbf{O}(\mathcal{U}')
\end{tikzcd}$$

if $U' \subset V'$ and let $A_{[U,V']} = \emptyset$ otherwise (thus $A_{[U,V']} \subset A_{U,V'}$). Write $\xi(\alpha)_{[U,V']}$ for the trivial vector bundle $A_{[U,V']} \times (V' - U')$ and, passing to Thom spaces, let $\mathbf{T}'\alpha_{[U]}$ be the $\mathcal{U}'$-prespectrum $V' \to T(\xi(\alpha)_{[U,V']}) \approx \Sigma^{V'-U'} A_{[U,V']+}$. Call $\mathbf{M}'\alpha_{[U]}$ the spectrification of $\mathbf{T}'\alpha_{[U]}$—then there are two claims: (1) $\mathbf{M}'\alpha_{[U]} \approx \mathbf{Q}^{\infty}_{U'} A_+$; (2) $\mathbf{M}'\alpha_{[U]} \approx \mathbf{M}'\alpha_U$. For the first, one can assume that A is compact, in which case $A_{[U,V']} = A$ for V' large enough and the claim follows. Turning to the second, define a morphism $\xi(\alpha)_{[U,V']} \to \xi(\alpha)_{U,V'}$ of vector bundles by sending (a, v') to $(a, s(\alpha(a)|U)(v'))$. These morphisms lead to a morphism $\mathbf{T}'\alpha_{[U]} \to \mathbf{T}'\alpha_U$ of $\mathcal{U}'$-prespectra or still, to a morphism $\mathbf{M}'\alpha_{[U]} \to \mathbf{M}'\alpha_U$ of $\mathcal{U}'$-spectra. But when A is compact and $A_{[U,V']} = A$, the bundle map is an isomorphism.]

PROPOSITION 21 Let $\mathcal{U}, \mathcal{U}'$ be universes. Fix $U \subset \mathcal{U}$, $U' \subset \mathcal{U}'$ such that $U \approx U'$—then there is an isomorphism $\alpha \ltimes \mathbf{Q}^{\infty}_{U} X \approx \mathbf{Q}^{\infty}_{U'}(A_+ \#_k X)$ natural in α and X.

[For $\alpha \ltimes \mathbf{Q}^{\infty}_{U} X = \mathbf{M}'\alpha \wedge \mathbf{Q}^{\infty}_{U} X \approx \mathbf{M}'\alpha_U \wedge X$ and, by the untwisting lemma, $\mathbf{M}'\alpha_U \wedge X \approx \mathbf{Q}^{\infty}_{U'} A_+ \wedge X$.]

EXAMPLE Fix $U \subset \mathcal{U}$, $U' \subset \mathcal{U}'$ such that $U \approx U'$—then the functor $\mathbf{M}'\text{—}_U : \mathbf{\Delta\text{-}CG}/\mathcal{I}(\mathcal{U},\mathcal{U}') \to \mathbf{SPEC}_{\mathcal{U}'}$ has for a right adjoint the functor $\mathbf{M}\text{—}_{U'} : \mathbf{SPEC}_{\mathcal{U}'} \to \mathbf{\Delta\text{-}CG}/\mathcal{I}(\mathcal{U},\mathcal{U}')$ that sends $\mathbf{Y}'$ to $\mathcal{I}(\mathcal{U},\mathcal{U}') \times_k Y'_{U'} \to \mathcal{I}(\mathcal{U},\mathcal{U}')$.

[$\text{Mor}(\mathbf{M}'\alpha_U, \mathbf{Y}') \approx \text{Mor}(\mathbf{Q}^{\infty}_{U'} A_+, \mathbf{Y}') \approx \text{Mor}(A_+, Y'_{U'}) \approx \text{Mor}(A, Y'_{U'}) \approx \text{Mor}(\alpha, \mathbf{M}\mathbf{Y}'_{U'})$.]

FACT Suppose that A is a CW complex—then the functor $\mathcal{HOM}[\alpha, \text{—})$ preserves weak equivalences.

[Let $\mathbf{f}' : \mathbf{X}' \to \mathbf{Y}'$ be a weak equivalence of $\mathcal{U}'$-spectra and consider the induced morphism $\mathcal{HOM}[\alpha, \mathbf{X}') \to \mathcal{HOM}[\alpha, \mathbf{Y}')$ of $\mathcal{U}$-spectra. Given $U \subset \mathcal{U}$, $\exists\, U' \subset \mathcal{U}' : U \approx U' \Rightarrow \mathcal{HOM}[\alpha, \mathbf{X}')_U \approx (X'_{U'})^{A+}$, $\mathcal{HOM}[\alpha, \mathbf{Y}')_U \approx (Y'_{U'})^{A+}$ (cf. Proposition 21). Since A_+ is a CW complex and $X'_{U'} \to Y'_{U'}$ is a weak homotopy equivalence, $(X'_{U'})^{A+} \to (Y'_{U'})^{A+}$ is also a weak homotopy equivalence (cf. p. 9–39).]

Rappel: $\mathbf{\Delta\text{-}CG}/\mathcal{I}(\mathcal{U},\mathcal{U}')$ is a model category (singular structure) (cf. p. 12–3).

PROPOSITION 22 If $\mathbf{X} \to \mathbf{Y}$ is a cofibration in $\mathbf{SPEC}_{\mathcal{U}}$ and if

$$\begin{tikzcd} A \arrow[rr] \arrow[dr, "\alpha"'] & & B \arrow[dl, "\beta"] \\ & \mathcal{I}(\mathcal{U},\mathcal{U}') & \end{tikzcd}$$

is a cofibration in $\mathbf{\Delta\text{-}CG}/\mathcal{I}(\mathcal{U},\mathcal{U}')$, then the arrow $\beta \ltimes \mathbf{X} \sqcup_{\alpha \ltimes \mathbf{X}} \alpha \ltimes \mathbf{Y} \to \beta \ltimes \mathbf{Y}$ is a cofibration in $\mathbf{SPEC}_{\mathcal{U}'}$ which is acyclic if $\mathbf{X} \to \mathbf{Y}$ or

$$\begin{tikzcd} A \arrow[rr] \arrow[dr, "\alpha"'] & & B \arrow[dl, "\beta"] \\ & \mathcal{I}(\mathcal{U},\mathcal{U}') & \end{tikzcd}$$

is acyclic (cf. §13, Proposition 31).

PROPOSITION 23 If

$$\begin{tikzcd} A \arrow[rr] \arrow[dr, "\alpha"'] & & B \arrow[dl, "\beta"] \\ & \mathcal{I}(\mathcal{U},\mathcal{U}') & \end{tikzcd}$$

is a cofibration in **$\mathbf{\Delta}$-CG**$/\mathcal{I}(\mathcal{U},\mathcal{U}')$ and if $\mathbf{Y}' \rightarrow \mathbf{X}'$ is a fibration in $\mathbf{SPEC}_{\mathcal{U}'}$, then the arrow $\mathcal{HOM}[\beta,\mathbf{Y}'] \rightarrow \mathcal{HOM}[\alpha,\mathbf{Y}'] \times_{\mathcal{HOM}[\alpha,\mathbf{X}']} \mathcal{HOM}[\beta,\mathbf{X}']$ is a fibration in $\mathbf{SPEC}_{\mathcal{U}}$ which is acyclic if

$$\begin{tikzcd} A \arrow[rr] \arrow[dr, "\alpha"'] & & B \arrow[dl, "\beta"] \\ & \mathcal{I}(\mathcal{U},\mathcal{U}') & \end{tikzcd}$$

or $\mathbf{Y}' \rightarrow \mathbf{X}'$ is acyclic (cf. §13, Proposition 32).

Propositions 22 and 23 are formally equivalent. To establish the fibration contention in Proposition 23, use Proposition 21 and convert the lifting problem

$$\begin{tikzcd} \mathbf{S}^{-U} \wedge [0,1]^n_+ \arrow[r] \arrow[d] & \mathcal{HOM}[\beta,\mathbf{Y}'] \arrow[d] \\ \mathbf{S}^{-U} \wedge I[0,1]^n_+ \arrow[r] \arrow[ur, dashed] & \mathcal{HOM}[\alpha,\mathbf{Y}'] \times_{\mathcal{HOM}[\alpha,\mathbf{X}']} \mathcal{HOM}[\beta,\mathbf{X}'] \end{tikzcd}$$

in $\mathbf{SPEC}_{\mathcal{U}}$ to the lifting problem

$$\begin{tikzcd} {[0,1]^n} \arrow[r] \arrow[d] & (Y'_{U'})^B \arrow[d] \\ I[0,1]^n \arrow[r] \arrow[ur, dashed] & (Y'_{U'})^A \times_{(X'_{U'})^A} (X'_{U'})^B \end{tikzcd}$$

in **$\mathbf{\Delta}$-CG**.

LEMMA Let A, B be cofibrant objects in **$\mathbf{\Delta}$-CG** and suppose that

$$\begin{tikzcd} A \arrow[rr] \arrow[dr, "\alpha"'] & & B \arrow[dl, "\beta"] \\ & \mathcal{I}(\mathcal{U},\mathcal{U}') & \end{tikzcd}$$

is an acyclic cofibration in **$\mathbf{\Delta}$-CG**$/\mathcal{I}(\mathcal{U},\mathcal{U}')$. Fix a cofibrant $\mathbf{X}$ in $\mathbf{SPEC}_{\mathcal{U}}$ and consider the commutative diagram

$$\begin{tikzcd} \mathbf{X} \arrow[r] \arrow[d] & \mathcal{HOM}[\alpha, \alpha \ltimes \mathbf{X}] \arrow[d] \\ \mathcal{HOM}[\beta, \beta \ltimes \mathbf{X}] \arrow[r] & \mathcal{HOM}[\alpha, \beta \ltimes \mathbf{X}] \end{tikzcd}$$

—then the arrow of adjunction $\mathbf{X} \rightarrow \mathcal{HOM}[\alpha, \alpha \ltimes \mathbf{X}]$ is a weak equivalence iff the arrow of adjunction $\mathbf{X} \rightarrow \mathcal{HOM}[\beta, \beta \ltimes \mathbf{X}]$ is a weak equivalence.

[Since the arrow $\beta \ltimes \mathbf{X} \rightarrow *$ is a fibration, it follows from Proposition 23 that $\mathcal{HOM}[\beta, \beta \ltimes \mathbf{X}] \rightarrow \mathcal{HOM}[\alpha, \beta \ltimes \mathbf{X}]$ is an acyclic fibration. On the other hand, since the arrow $* \rightarrow \mathbf{X}$ is a cofibration, it follows from Proposition 22 that the arrow $\alpha \ltimes \mathbf{X} \rightarrow \beta \ltimes \mathbf{X}$ is an acyclic cofibration. But from the assumptions, $\alpha \ltimes \mathbf{X}$ and $\beta \ltimes \mathbf{X}$ are cofibrant, thus as fibrancy is automatic, the arrow $\alpha \ltimes \mathbf{X} \rightarrow \beta \ltimes \mathbf{X}$ is a homotopy equivalence (cf. §12, Proposition 10). Therefore $\mathcal{HOM}[\alpha, \alpha \ltimes \mathbf{X}] \rightarrow \mathcal{HOM}[\alpha, \beta \ltimes \mathbf{X}]$ is a homotopy equivalence.]

EXAMPLE Let $\mathcal{U}, \mathcal{U}'$ be universes, $f : \mathcal{U} \rightarrow \mathcal{U}'$ a linear isometry—then $f^* : \mathbf{SPEC}_{\mathcal{U}'} \rightarrow \mathbf{SPEC}_{\mathcal{U}}$ preserves fibrations and acyclic fibrations, so the hypotheses of the TDF theorem are satisfied (cf. p. 12–3 ff.). Therefore $\mathbf{L}f_*$ and $\mathbf{R}f^*$ exist and $(\mathbf{L}f_*, \mathbf{R}f^*)$ is an adjoint pair. Claim: $\forall$ cofibrant $\mathbf{X}$ in $\mathbf{SPEC}_{\mathcal{U}}$, the arrow of adjunction $\mathbf{X} \rightarrow f^* f_* \mathbf{X}$ is a weak equivalence. To see this, choose a linear isometric isomorphism $\phi \in \mathcal{I}(\mathcal{U}, \mathcal{U}')$ and a path $H : [0,1] \rightarrow \mathcal{I}(\mathcal{U}, \mathcal{U}')$ such that $H \circ i_0 = \phi$ and $H \circ i_1 = f$. Because

$$\begin{tikzcd} * \arrow[rr, "i_0"] \arrow[dr, "\phi"'] & & {[0,1]} \arrow[dl, "H"] \\ & \mathcal{I}(\mathcal{U},\mathcal{U}') & \end{tikzcd}$$

is an acyclic cofibration in $\mathbf{\Delta}\text{-}\mathbf{CG}/\mathcal{I}(\mathcal{U}, \mathcal{U}')$ with $*$, $[0,1]$ cofibrant and because the arrow of adjunction $\mathbf{X} \rightarrow \phi^* \phi_* \mathbf{X}$ is an isomorphism, the lemma implies that the arrow of adjunction $\mathbf{X} \rightarrow \mathcal{HOM}[H, H \ltimes \mathbf{X})$ is a weak equivalence. Another application of the lemma to

$$\begin{tikzcd} * \arrow[rr, "i_1"] \arrow[dr, "f"'] & & {[0,1]} \arrow[dl, "H"] \\ & \mathcal{I}(\mathcal{U},\mathcal{U}') & \end{tikzcd}$$

then leads to the conclusion that the arrow of adjunction $\mathbf{X} \rightarrow f^* f_* \mathbf{X}$ is indeed a weak equivalence. Since $\mathbf{X}' \rightarrow \mathbf{Y}'$ is a weak equivalence iff $f^* \mathbf{X}' \rightarrow f^* \mathbf{Y}'$ is a weak equivalence, the pair $(\mathbf{L}f_*, \mathbf{R}f^*)$ is an adjoint equivalence of categories (see the note on p. 12–29 to the TDF theorem). Example: $\forall$ universe $\mathcal{U}$, $\mathbf{HSPEC}_{\mathcal{U}}$ "is" $\mathbf{HSPEC}$. Proof: $\mathbf{HSPEC}_{\mathcal{U}}$ "is" $\mathbf{HSPEC}_{\mathbf{R}^\infty}$ which "is" $\mathbf{HSPEC}$ (cf. p. 16–18).

[Note: The functors $\mathbf{L}f_* : \mathbf{HSPEC}_{\mathcal{U}} \rightarrow \mathbf{HSPEC}_{\mathcal{U}'}$ obtained from the $f \in \mathcal{I}(\mathcal{U}, \mathcal{U}')$ are naturally isomorphic. Thus let $g \in \mathcal{I}(\mathcal{U}, \mathcal{U}')$ and choose a path $H : [0,1] \rightarrow \mathcal{I}(\mathcal{U}, \mathcal{U}')$ such that $H \circ i_0 = f$ and $H \circ i_1 = g$—then for cofibrant $\mathbf{X}$, there are natural homotopy equivalences $f_* \mathbf{X} \rightarrow H \ltimes \mathbf{X} \leftarrow g_* \mathbf{X}$ and the natural isomorphism $\mathbf{L}f_* \approx \mathbf{L}g_*$ is independent of the choice of H. In effect, if $\sigma, \tau : [0,1] \rightarrow \mathcal{I}(\mathcal{U}, \mathcal{U}')$ are paths in $\mathcal{I}(\mathcal{U}, \mathcal{U}')$ such that $\begin{cases} \sigma(0) = f \\ \sigma(1) = g \end{cases}$, $\begin{cases} \tau(0) = f \\ \tau(1) = g \end{cases}$ and if $\Phi : [0,1]^2 \rightarrow \mathcal{I}(\mathcal{U}, \mathcal{U}')$ is a homotopy between σ, τ through paths from f to g, then there is a commutative diagram

$$\begin{tikzcd} f_* \mathbf{X} \arrow[r] \arrow[d] & f_* \mathbf{X} \wedge I_+ \arrow[d] \arrow[r, dashed, shift right] & f_* \mathbf{X} \arrow[d] \arrow[l, shift right] \\ \sigma \ltimes \mathbf{X} \arrow[r] & \Phi \ltimes \mathbf{X} & \tau \ltimes \mathbf{X} \arrow[l] \\ g_* \mathbf{X} \arrow[r] \arrow[u] & g_* \mathbf{X} \wedge I_+ \arrow[u] \arrow[r, dashed, shift left] & g_* \mathbf{X} \arrow[u] \arrow[l, shift left] \end{tikzcd}$$

of natural homotopy equivalences, where $\dashrightarrow \circ \longrightarrow = \text{id}$. Similar remarks apply to the $\mathbf{R}f^* : \mathbf{HSPEC}_{\mathcal{U}'} \rightarrow \mathbf{HSPEC}_{\mathcal{U}}$.]

FACT If $\mathbf{X} \rightarrow \mathbf{Y}$ is a cofibration in $\mathbf{SPEC}_{\mathcal{U}}$ and if $\mathbf{Y}' \rightarrow \mathbf{X}'$ is a fibration in $\mathbf{SPEC}_{\mathcal{U}'}$, then the arrow $\mathbf{HOM}(\mathbf{Y}, \mathbf{Y}') \rightarrow \mathbf{HOM}(\mathbf{X}, \mathbf{Y}') \times_{\mathbf{HOM}(\mathbf{X}, \mathbf{X}')} \mathbf{HOM}(\mathbf{Y}, \mathbf{X}')$ is a fibration in $\mathbf{\Delta}\text{-}\mathbf{CG}/\mathcal{I}(\mathcal{U}, \mathcal{U}')$ which is a weak equivalence if $\mathbf{X} \rightarrow \mathbf{Y}$ or $\mathbf{Y}' \rightarrow \mathbf{X}'$ is acyclic (the notation is that of the addendum on p. 16–21).

PROPOSITION 24 Suppose that A is a cofibrant object in $\mathbf{\Delta}\text{-}\mathbf{CG}$—then the functor $\mathcal{HOM}[\alpha, —)$ preserves fibrations and acyclic fibrations (cf. Proposition 23). Therefore the assumptions of the TDF theorem are met (cf. p. 12–3 ff.), so $\mathbf{L}\alpha \ltimes —$ and $\mathbf{R}\mathcal{HOM}[\alpha, —)$ exist and $(\mathbf{L}\alpha \ltimes —, \mathbf{R}\mathcal{HOM}[\alpha, —))$ is an adjoint pair.

FACT Fix a cofibrant object in A in **Δ-CG** and let $H : IA \rightarrow \mathcal{I}(\mathcal{U},\mathcal{U}')$ be a homotopy—then $\forall$ cofibrant $\mathbf{X}$ in $\mathbf{SPEC}_{\mathcal{U}}$, the arrow $H \circ i_t \ltimes \mathbf{X} \rightarrow H \ltimes \mathbf{X}$ is a homotopy equivalence ($t \in \{0,1\}$).

[Note: Consequently the functors $\mathbf{L}\alpha \ltimes —: \mathbf{HSPEC}_{\mathcal{U}} \rightarrow \mathbf{HSPEC}_{\mathcal{U}'}$ corresponding to the $\alpha : A \rightarrow \mathcal{I}(\mathcal{U},\mathcal{U}')$ are naturally isomorphic, as are the functors $\mathbf{R}\mathcal{HOM}[\alpha,—) : \mathbf{HSPEC}_{\mathcal{U}'} \rightarrow \mathbf{HSPEC}_{\mathcal{U}}$.]

FACT Let A, B be cofibrant objects in **Δ-CG** and suppose that $\phi : A \rightarrow B$ is a homotopy equivalence—then $\forall\, \beta : B \rightarrow \mathcal{I}(\mathcal{U},\mathcal{U}')$, the arrow $\beta \circ \phi \ltimes \mathbf{X} \rightarrow \beta \ltimes \mathbf{X}$ is a homotopy equivalence provided that $\mathbf{X}$ is cofibrant.

[Fix a homotopy inverse $\psi : B \rightarrow A$ for ϕ, choose $H : IA \rightarrow A$ such that $\begin{cases} H \circ i_0 = \text{id}_A \\ H \circ i_1 = \psi \circ \phi \end{cases}$ & $G : IB \rightarrow B$ such that $\begin{cases} G \circ i_0 = \text{id}_B \\ G \circ i_1 = \phi \circ \psi \end{cases}$, and, keeping in mind the preceding result, use the commutative diagrams

$$\begin{tikzcd}
\beta \circ \phi \circ \psi \circ \phi \ltimes \mathbf{X} \arrow[r, "i_1 \ltimes \text{id}"] \arrow[d] & \beta \circ \phi \circ H \ltimes \mathbf{X} \arrow[ddl, "H \ltimes \text{id}"] \\
\beta \circ \phi \circ \psi \ltimes \mathbf{X} \arrow[d] & \\
\beta \circ \phi \ltimes \mathbf{X} \arrow[r, equal] & \beta \circ \phi \ltimes \mathbf{X} \arrow[uu, "i_0 \ltimes \text{id}"']
\end{tikzcd}
\qquad
\begin{tikzcd}
\beta \circ \phi \circ \psi \ltimes \mathbf{X} \arrow[r, "i_1 \ltimes \text{id}"] \arrow[d] & \beta \circ G \ltimes \mathbf{X} \arrow[ddl, "G \ltimes \text{id}"] \\
\beta \circ \phi \ltimes \mathbf{X} \arrow[d] & \\
\beta \ltimes \mathbf{X} \arrow[r, equal] & \beta \ltimes \mathbf{X} \arrow[uu, "i_0 \ltimes \text{id}"']
\end{tikzcd}$$

to deduce that the arrow $\beta \circ \phi \ltimes \mathbf{X} \rightarrow \beta \ltimes \mathbf{X}$ is a weak equivalence, hence a homotopy equivalence.]

[Note: The cofibrancy assumption on A, B can be dropped. Thus let $\mathbf{Y}'$ be any $\mathcal{U}'$-spectrum. Given $U \subset \mathcal{U}$, $\exists\, U' \subset \mathcal{U}' : U \approx U' \Rightarrow \mathcal{HOM}[\beta \circ \phi, \mathbf{Y}')_U \approx (Y'_{U'})^{A}{+}$, $\mathcal{HOM}[\beta, \mathbf{Y}')_U \approx (Y'_{U'})^{B}{+}$ (cf. Proposition 21). Because $\phi : A \rightarrow B$ is a homotopy equivalence, it follows that $\mathcal{HOM}[\beta, \mathbf{Y}')_U \rightarrow \mathcal{HOM}[\beta \circ \phi, \mathbf{Y}')_U$ is a homotopy equivalence $\forall\, U$. But $\mathbf{X}$ is cofibrant, so $[\mathbf{X},—]_0 \approx [\mathbf{X},—]$ (cf. p. 12–25) (all objects are fibrant). Therefore $[\mathbf{X}, \mathcal{HOM}[\beta, \mathbf{Y}')]_0 \approx [\mathbf{X}, \mathcal{HOM}[\beta \circ \phi, \mathbf{Y}')]_0 \Rightarrow [\beta \ltimes \mathbf{X}, \mathbf{Y}']_0 \approx [\beta \circ \phi \ltimes \mathbf{X}, \mathbf{Y}']_0$ (cf. Proposition 19). And this means that the arrow $\beta \circ \phi \ltimes \mathbf{X} \rightarrow \beta \ltimes \mathbf{X}$ is a homotopy equivalence ($\mathbf{Y}'$ being arbitrary). Variant: The same conclusion obtains if $\mathbf{X}$ is tame.]

EXAMPLE Take $\mathcal{U} = \mathcal{U}'$—then $\forall\, f \in \mathcal{I}(\mathcal{U},\mathcal{U})$, there is a commutative diagram

$$\begin{tikzcd}
* \arrow[rr, "f"] \arrow[dr, "f"'] & & \mathcal{I}(\mathcal{U},\mathcal{U}) \arrow[dl, "\text{id}"] \\
& \mathcal{I}(\mathcal{U},\mathcal{U}) &
\end{tikzcd}$$

, thus $\forall$ cofibrant $\mathbf{X}$, the arrow $f_*\mathbf{X} \rightarrow \text{id} \ltimes \mathbf{X}$ is a homotopy equivalence.

[Note: The point here is this: $\mathcal{I}(\mathcal{U},\mathcal{U})$ is contractible but it is unknown whether it is a cofibrant object in **Δ-CG**.]

FACT Let A, B be objects in **Δ-CG** and suppose that $\phi : A \rightarrow B$ is a closed cofibration—then $\forall\, \beta : B \rightarrow \mathcal{I}(\mathcal{U},\mathcal{U}')$, the arrow $\beta \circ \phi \ltimes \mathbf{X} \rightarrow \beta \ltimes \mathbf{X}$ is a spectral cofibration provided that $\mathbf{X}$ is cofibrant.

[Given a $\mathcal{U}'$-spectrum $\mathbf{Y}'$, finding a filler for the diagram

$$\begin{tikzcd}
\beta\circ\phi\ltimes\mathbf{X} \arrow[rr] \arrow[dd] & & \beta\ltimes\mathbf{X} \arrow[dl] \arrow[dd] \\
& \mathbf{Y}' & \\
(\beta\circ\phi\ltimes\mathbf{X})\wedge I_+ \arrow[rr] \arrow[ur] & & (\beta\ltimes\mathbf{X})\wedge I_+ \arrow[ul, dashed]
\end{tikzcd}$$

amounts to finding a filler for the diagram

$$\begin{tikzcd}
\mathbf{X} \arrow[r] \arrow[d] & \mathcal{HOM}[\beta,\mathbf{Y}') \arrow[d] \\
\mathbf{X}\wedge I_+ \arrow[r] \arrow[ur, dashed] & \mathcal{HOM}[\beta\circ\phi,\mathbf{Y}')
\end{tikzcd}$$

. However, the arrow $\mathcal{HOM}[\beta,\mathbf{Y}') \rightarrow \mathcal{HOM}[\beta\circ\phi,\mathbf{Y}')$ is a levelwise CG fibration, therefore is a levelwise Serre fibration, and, as $\mathbf{X}$ is cofibrant, the arrow $\mathbf{X}\rightarrow\mathbf{X}\wedge I_+$ is an acyclic cofibration in our model category structure on $\mathbf{SPEC}_{\mathcal{U}}$ (cf. p. 12–16 ff.).]

EXAMPLE Take $\mathcal{U}=\mathcal{U}'$—then $\forall\ f\in\mathcal{I}(\mathcal{U},\mathcal{U})$, there is a commutative diagram

$$\begin{tikzcd}
* \arrow[r, "f"] \arrow[d, "f"'] & \mathcal{I}(\mathcal{U},\mathcal{U}) \arrow[dl, "\mathrm{id}"] \\
\mathcal{I}(\mathcal{U},\mathcal{U}) &
\end{tikzcd}$$

, thus $\forall$ cofibrant $\mathbf{X}$, the arrow $f_*\mathbf{X}\rightarrow\mathrm{id}\ltimes\mathbf{X}$ is a spectral cofibration.

[In fact, $\mathcal{I}(\mathcal{U},\mathcal{U})$ is Δ-cofibered (cf. p. 14–52), so $\forall\ f\in\mathcal{I}(\mathcal{U},\mathcal{U})$, $\{f\}\rightarrow\mathcal{I}(\mathcal{U},\mathcal{U})$ is a closed cofibration (cf. p. 3–15).]

Let $\mathcal{U}$, $\mathcal{V}$ be universes. Put $\mathcal{A}\oplus\mathcal{B}=\{U\oplus V: U\subset\mathcal{U}\ \&\ \dim U<\omega,\ V\subset\mathcal{V}\ \&\ \dim V<\omega\}$ (which is not the standard indexing set in $\mathcal{U}\oplus\mathcal{V}$).

($\underline{\wedge}$) Given $\mathbf{X}$ in $\mathbf{SPEC}_{\mathcal{U}}$ and $\mathbf{Y}$ in $\mathbf{SPEC}_{\mathcal{V}}$, the data $\{X_U\#_k Y_V\}$ defines a $(\mathcal{U}\oplus\mathcal{V},\ \mathcal{A}\oplus\mathcal{B})$-prespectrum. Spectrify and let $\mathbf{X}\underline{\wedge}\mathbf{Y}$ be its image in $\mathbf{SPEC}_{\mathcal{U}\oplus\mathcal{V}}$ under the canonical equivalence $\mathbf{SPEC}_{\mathcal{U}\oplus\mathcal{V},\mathcal{A}\oplus\mathcal{B}}\rightarrow\mathbf{SPEC}_{\mathcal{U}\oplus\mathcal{V}}$ provided by Proposition 15.

Examples: (1) $\mathbf{Q}_U^\infty X\underline{\wedge}\mathbf{Q}_V^\infty Y\approx\mathbf{Q}_{U\oplus V}^\infty(X\#_k Y)$; (2) $(\mathbf{X}\wedge K)\underline{\wedge}\mathbf{Y}\approx(\mathbf{X}\underline{\wedge}\mathbf{Y})\wedge K\approx\mathbf{X}\underline{\wedge}(\mathbf{Y}\wedge K)$.

[Note: Take $X=Y=\mathbf{S}^0$ in (1) to get $\mathbf{S}^{-U}\underline{\wedge}\mathbf{S}^{-V}\approx\mathbf{S}^{-(U\oplus V)}$.]

Remark: It is not literally true that $\underline{\wedge}$ is an associative, commutative operation. Consider, e.g., commutativity. If $\mathrm{T}:\mathcal{U}\oplus\mathcal{V}\rightarrow\mathcal{V}\oplus\mathcal{U}$ is the switching map, then $\mathrm{T}_*(\mathbf{X}\underline{\wedge}\mathbf{Y})$ is naturally isomorphic to $\mathbf{Y}\underline{\wedge}\mathbf{X}$. The situation for associativity is analogous (consider the isomorphism $\mathcal{U}\oplus(\mathcal{V}\oplus\mathcal{W})\approx(\mathcal{U}\oplus\mathcal{V})\oplus\mathcal{W}$ of universes).

Another way to proceed is this. Write $\mathbf{X}\underline{\square}\mathbf{Y}$ for the composite $\mathbf{I}_{\mathcal{U}}\times\mathbf{I}_{\mathcal{V}}\xrightarrow{\mathbf{X}\times\mathbf{Y}}\mathbf{\Delta}\text{-}\mathbf{CG}_*\times\mathbf{\Delta}\text{-}\mathbf{CG}_*\xrightarrow{\#_k}\mathbf{\Delta}\text{-}\mathbf{CG}_*$—then, relative to the arrow $\mathbf{I}_{\mathcal{U}}\times\mathbf{I}_{\mathcal{V}}\rightarrow\mathbf{I}_{\mathcal{U}\oplus\mathcal{V}}((U,V)\rightarrow U\oplus V)$, $\operatorname{lan}\mathbf{X}\underline{\square}\mathbf{Y}$ is a $\mathcal{U}\oplus\mathcal{V}$-prespectrum, i.e., an object of $\mathbf{V}[\mathbf{I}_{\mathcal{U}\oplus\mathcal{V}},\ \mathbf{\Delta}\text{-}\mathbf{CG}_*]$, and its spectrification can be identified with $\mathbf{X}\underline{\wedge}\mathbf{Y}$. Therefore $\underline{\wedge}:\mathbf{SPEC}_{\mathcal{U}}\times\mathbf{SPEC}_{\mathcal{V}}\rightarrow\mathbf{SPEC}_{\mathcal{U}\oplus\mathcal{V}}$ is a continuous functor.

FACT Suppose given $\alpha: A\rightarrow\mathcal{I}(\mathcal{U},\mathcal{U}')$ and $\beta: B\rightarrow\mathcal{I}(\mathcal{V},\mathcal{V}')$. Let $\alpha\times_{\oplus}\beta$ be the composite $A\times_k B\xrightarrow{\alpha\times_k\beta}\mathcal{I}(\mathcal{U},\mathcal{U}')\times_k\mathcal{I}(\mathcal{V},\mathcal{V}')\xrightarrow{\oplus}\mathcal{I}(\mathcal{U}\oplus\mathcal{V},\mathcal{U}'\oplus\mathcal{V}')$—then $(\alpha\times_{\oplus}\beta)\ltimes(\mathbf{X}\underline{\wedge}\mathbf{Y})\approx(\alpha\ltimes\mathbf{X})\underline{\wedge}(\beta\ltimes\mathbf{Y})$.

Given $\mathbf{Y}$ in $\mathbf{SPEC}_{\mathcal{V}}$ and $\mathbf{Z}$ in $\mathbf{SPEC}_{\mathcal{U}\oplus\mathcal{V}}$, let $\mathbf{Z}^{\mathbf{Y}}$ be the $\mathcal{U}$-spectrum $U \to \mathrm{HOM}(\mathbf{S}^{-U}\underline{\wedge}\mathbf{Y}, \mathbf{Z})$—then there is a natural homeomorphism $\mathrm{HOM}(\mathbf{X}\underline{\wedge}\mathbf{Y}, \mathbf{Z}) \approx \mathrm{HOM}(\mathbf{X}, \mathbf{Z}^{\mathbf{Y}})$.

Example: $(\mathbf{Z}^{\mathbf{S}^{-V}})_U = \mathrm{HOM}(\mathbf{S}^{-U}\underline{\wedge}\mathbf{S}^{-V}, \mathbf{Z}) \approx \mathrm{HOM}(\mathbf{S}^{-(U\oplus V)}, \mathbf{Z}) \approx Z_{U\oplus V}$.

PROPOSITION 25 If $\mathbf{A} \to \mathbf{X}$ is a cofibration in $\mathbf{SPEC}_{\mathcal{U}}$ and if $\mathbf{B} \to \mathbf{Y}$ is a cofibration in $\mathbf{SPEC}_{\mathcal{V}}$, then the arrow $\mathbf{A}\underline{\wedge}\mathbf{Y} \underset{\mathbf{A}\underline{\wedge}\mathbf{B}}{\sqcup} \mathbf{X}\underline{\wedge}\mathbf{B} \to \mathbf{X}\underline{\wedge}\mathbf{Y}$ is a cofibration in $\mathbf{SPEC}_{\mathcal{U}\oplus\mathcal{V}}$ which is acyclic if $\mathbf{A} \to \mathbf{X}$ or $\mathbf{B} \to \mathbf{Y}$ is acyclic.

PROPOSITION 26 If $\mathbf{B} \to \mathbf{Y}$ is a cofibration in $\mathbf{SPEC}_{\mathcal{V}}$ and if $\mathbf{Z} \to \mathbf{C}$ is a fibration in $\mathbf{SPEC}_{\mathcal{U}\oplus\mathcal{V}}$, then the arrow $\mathbf{Z}^{\mathbf{Y}} \to \mathbf{Z}^{\mathbf{B}} \times_{\mathbf{C}^{\mathbf{B}}} \mathbf{C}^{\mathbf{Y}}$ is a fibration in $\mathbf{SPEC}_{\mathcal{U}}$ which is acyclic if $\mathbf{B} \to \mathbf{Y}$ or $\mathbf{Z} \to \mathbf{C}$ is acyclic.

Propositions 25 and 26 are formally equivalent. To establish the fibration contention in Proposition 26, one can assume that $\mathbf{B} \to \mathbf{Y}$ has the form $\mathbf{S}^{-V} \wedge L \to \mathbf{S}^{-V} \wedge K$, where $L \to K$ is a cofibration in $\mathbf{\Delta}\text{-}\mathbf{CG}_*$. The fact that $\mathbf{Z} \to \mathbf{C}$ is a fibration in $\mathbf{SPEC}_{\mathcal{U}\oplus\mathcal{V}}$ implies that the arrow $\mathrm{HOM}(K, \mathbf{Z}) \to \mathrm{HOM}(L, \mathbf{Z}) \times_{\mathrm{HOM}(L,\mathbf{C})} \mathrm{HOM}(K, \mathbf{C})$ is a fibration in $\mathbf{SPEC}_{\mathcal{U}\oplus\mathcal{V}}$ which is acyclic if $L \to K$ or $\mathbf{Z} \to \mathbf{C}$ is acyclic (cf. p. 16–10). But the functor $(—)^{\mathbf{S}^{-V}}$ preserves fibrations and acyclic fibrations and $\forall\ X$, $\mathrm{HOM}(X, \mathbf{Z})^{\mathbf{S}^{-V}} \approx \mathbf{Z}^{\mathbf{S}^{-V}\wedge X}$, thus the arrow $\mathbf{Z}^{\mathbf{S}^{-V}\wedge K} \to \cdots$ is a fibration in $\mathbf{SPEC}_{\mathcal{U}}$ which is acyclic if $L \to K$ or $\mathbf{Z} \to \mathbf{C}$ is acyclic.

[Note: The functor $\mathbf{Q}_V^\infty = \mathbf{S}^{-V} \wedge —$ preserves cofibrations and acyclic cofibrations.]

Example: $\left\{\begin{matrix}\mathbf{X}\\ \mathbf{Y}\end{matrix}\right.$ cofibrant $\Rightarrow$ $\mathbf{X}\underline{\wedge}\mathbf{Y}$ cofibrant (cf. Proposition 25).

PROPOSITION 27 Suppose that $\mathbf{Y}$ is a cofibrant object in $\mathbf{SPEC}_{\mathcal{V}}$—then the functor $(—)^{\mathbf{Y}}$ preserves fibrations and acyclic fibrations (cf. Proposition 26). Therefore the assumptions of the TDF theorem are met (cf. p. 12–3 ff.), so $\mathbf{L}(—\underline{\wedge}\mathbf{Y})$ and $\mathbf{R}(—)^{\mathbf{Y}}$ exist and $(\mathbf{L}(—\underline{\wedge}\mathbf{Y}), \mathbf{R}(—)^{\mathbf{Y}})$ is an adjoint pair.

[Note: Since all objects are fibrant, $(—)^{\mathbf{Y}}$ necessarily preserves weak equivalences (cf. p. 12–28).]

If $\mathbf{C}$ and $\mathbf{D}$ are model categories, then $\mathbf{C} \times \mathbf{D}$ becomes a model category upon imposing the obvious slotwise structure. In particular: $\mathbf{SPEC}_{\mathcal{U}} \times \mathbf{SPEC}_{\mathcal{V}}$ is a model category.

PROPOSITION 28 The functor $\underline{\wedge} : \mathbf{SPEC}_{\mathcal{U}} \times \mathbf{SPEC}_{\mathcal{V}} \to \mathbf{SPEC}_{\mathcal{U}\oplus\mathcal{V}}$ sends weak equivalences between cofibrant objects to weak equivalences, thus the total left derived functor $\mathbf{L}\underline{\wedge} : \mathbf{HSPEC}_{\mathcal{U}} \times \mathbf{HSPEC}_{\mathcal{V}} \to \mathbf{HSPEC}_{\mathcal{U}\oplus\mathcal{V}}$ exists (cf. §12, Proposition 14).

[Suppose that $\mathbf{A} \to \mathbf{X}$ is an acyclic cofibration in $\mathbf{SPEC}_{\mathcal{U}}$ and $\mathbf{B} \to \mathbf{Y}$ is an acyclic cofibration in $\mathbf{SPEC}_{\mathcal{V}}$, where $\mathbf{A}$ & $\mathbf{B}$ (hence $\mathbf{X}$ & $\mathbf{Y}$) are cofibrant. Factor the arrow

$\mathbf{A}\underline{\wedge}\mathbf{B} \to \mathbf{X}\underline{\wedge}\mathbf{Y}$ as the composite $\mathbf{A}\underline{\wedge}\mathbf{B} \to \mathbf{X}\underline{\wedge}\mathbf{B} \to \mathbf{X}\underline{\wedge}\mathbf{Y}$. Owing to Proposition 25, $\mathbf{A}\underline{\wedge}\mathbf{B} \to \mathbf{X}\underline{\wedge}\mathbf{B}$ and $\mathbf{X}\underline{\wedge}\mathbf{B} \to \mathbf{X}\underline{\wedge}\mathbf{Y}$ are acyclic cofibrations. Therefore $\mathbf{A}\underline{\wedge}\mathbf{B} \to \mathbf{X}\underline{\wedge}\mathbf{Y}$ is an acyclic cofibration. The lemma on p. 12–28 then implies that $\underline{\wedge}$ preserves weak equivalences between cofibrant objects.]

[Note: $\mathbf{L}\underline{\wedge}(\mathbf{X},\mathbf{Y}) = \mathcal{L}\mathbf{X}\underline{\wedge}\mathcal{L}\mathbf{Y}$, the value of the total left derived functor of $-\underline{\wedge}\mathcal{L}\mathbf{Y}$ at $\mathbf{X}$ (cf. Proposition 27).]

Take in the above $\mathcal{U} = \mathcal{V}$ and choose any $f \in \mathcal{I}(\mathcal{U}^2,\mathcal{U})$ $(\mathcal{U}^2 = \mathcal{U}\oplus\mathcal{U})$. Definition: $\mathbf{X}\wedge_f \mathbf{Y} = f_*(\mathbf{X}\underline{\wedge}\mathbf{Y})$, $\hom_f(\mathbf{Y},\mathbf{Z}) = (f^*\mathbf{Z})^{\mathbf{Y}}$. So: $\mathrm{HOM}(\mathbf{X}\wedge_f \mathbf{Y},\mathbf{Z}) = \mathrm{HOM}(f_*(\mathbf{X}\underline{\wedge}\mathbf{Y}),\mathbf{Z}) \approx \mathrm{HOM}(\mathbf{X}\underline{\wedge}\mathbf{Y}, f^*\mathbf{Z}) \approx \mathrm{HOM}(\mathbf{X},(f^*\mathbf{Z})^{\mathbf{Y}}) = \mathrm{HOM}(\mathbf{X},\hom_f(\mathbf{Y},\mathbf{Z}))$.

[Note: While each of the functors $-\wedge_f \mathbf{Y}$ has a right adjoint $\mathbf{Z} \to \hom_f(\mathbf{Y},\mathbf{Z})$, $\mathbf{SPEC}_{\mathcal{U}}$ is definitely not a symmetric monoidal category under $\otimes = \wedge_f$.]

EXAMPLE Write $\mathbf{Q}^\infty$ in place of $\mathbf{Q}^\infty_{\{0\}}$ and put $\mathbf{S} = \mathbf{Q}^\infty\mathbf{S}^0$. Letting $i : \mathcal{U} \to \mathcal{U}\oplus\mathcal{U}$ be the inclusion of $\mathcal{U}$ onto the first summand, one has $i_*(\mathbf{X}\wedge\mathbf{S}^0) \approx \mathbf{X}\underline{\wedge}\mathbf{S}$, thus $(f\circ i)_*(\mathbf{X}\wedge\mathbf{S}^0) \approx f_*\circ i_*(\mathbf{X}\wedge\mathbf{S}^0) \approx f_*(\mathbf{X}\underline{\wedge}\mathbf{S}) = \mathbf{X}\wedge_f \mathbf{S}$. And, when $\mathbf{X}$ is cofibrant, $\mathbf{X}\wedge\mathbf{S}^0 \approx (f\circ i)_*(\mathbf{X}\wedge\mathbf{S}^0)$ in $\mathbf{HSPEC}_{\mathcal{U}}$, i.e., $\mathbf{X} \approx \mathbf{X}\wedge_f \mathbf{S}$ in $\mathbf{HSPEC}_{\mathcal{U}}$.

Definition: $\mathbf{X}\wedge\mathbf{Y} = \mathbf{L}f_*(\mathbf{L}\underline{\wedge}(\mathbf{X},\mathbf{Y}))$, $\hom(\mathbf{Y},\mathbf{Z}) = \mathbf{R}(\mathbf{R}f^*\mathbf{Z})^{\mathcal{L}\mathbf{Y}}(= (f^*\mathbf{Z})^{\mathcal{L}\mathbf{Y}}$, all objects being fibrant).

[Note: This apparent abuse of notation is justified on the grounds that, up to natural isomorphism, these functors are independent of the choice of f (cf. p. 16–25). Terminology: Call $\wedge$ the smash product.]

Observation: Since f_* sends cofibrant objects to cofibrant objects and $\mathcal{L}\mathbf{X}\underline{\wedge}\mathcal{L}\mathbf{Y}$ is cofibrant (cf. p. 16–28), $[\mathbf{X}\wedge\mathbf{Y},\mathbf{Z}] = [\mathbf{L}f_*(\mathbf{L}\underline{\wedge}(\mathbf{X},\mathbf{Y})),\mathbf{Z}] \approx [\mathbf{L}f_*(\mathcal{L}\mathbf{X}\underline{\wedge}\mathcal{L}\mathbf{Y}),\mathbf{Z}] \approx [f_*(\mathcal{L}\mathbf{X}\underline{\wedge}\mathcal{L}\mathbf{Y}),\mathbf{Z}] \approx \pi_0(\mathrm{HOM}(f_*(\mathcal{L}\mathbf{X}\underline{\wedge}\mathcal{L}\mathbf{Y}),\mathbf{Z})) \approx \pi_0(\mathrm{HOM}(\mathcal{L}\mathbf{X}\underline{\wedge}\mathcal{L}\mathbf{Y},f^*\mathbf{Z})) \approx \pi_0(\mathrm{HOM}(\mathcal{L}\mathbf{X},(f^*\mathbf{Z})^{\mathcal{L}\mathbf{Y}})) \approx [\mathcal{L}\mathbf{X},(f^*\mathbf{Z})^{\mathcal{L}\mathbf{Y}}] \approx [\mathbf{X},(f^*\mathbf{Z})^{\mathcal{L}\mathbf{Y}}] \approx [\mathbf{X},\mathbf{R}(\mathbf{R}f^*\mathbf{Z})^{\mathcal{L}\mathbf{Y}}] = [\mathbf{X},\hom(\mathbf{Y},\mathbf{Z})]$.

FACT In $\mathbf{HSPEC}_{\mathcal{U}}$, $\mathbf{X}\wedge Y \approx \mathbf{X}\wedge\mathbf{Q}^\infty Y$, hence $\mathbf{Q}^\infty(K\#_k L) \approx (\mathbf{Q}^\infty K)\wedge L \approx \mathbf{Q}^\infty K\wedge\mathbf{Q}^\infty L$ and $\mathrm{HOM}(K,\mathbf{X}) \approx \hom(\mathbf{Q}^\infty K,\mathbf{X})$.

PROPOSITION 29 $\mathbf{HSPEC}_{\mathcal{U}}$ is a monoidal category.

[Taking $\otimes = \wedge$ and $e = \mathbf{S}$ $(= \mathbf{Q}^\infty\mathbf{S}^0)$, one has to define natural isomorphisms $\begin{cases} R_{\mathbf{X}} : \mathbf{X}\wedge\mathbf{S} \to \mathbf{X} \\ L_{\mathbf{X}} : \mathbf{S}\wedge\mathbf{X} \to \mathbf{X} \end{cases}$ and $A_{\mathbf{X},\mathbf{Y},\mathbf{Z}} : \mathbf{X}\wedge(\mathbf{Y}\wedge\mathbf{Z}) \to (\mathbf{X}\wedge\mathbf{Y})\wedge\mathbf{Z}$ satisfying MC_1 and MC_2 on p. 0–24. The definitions of $R_{\mathbf{X}}$ and $L_{\mathbf{X}}$ are clear (cf. supra). Letting Φ be the isomorphism $(\mathcal{U}\oplus\mathcal{U})\oplus\mathcal{U} \to \mathcal{U}\oplus(\mathcal{U}\oplus\mathcal{U})$, define $A_{\mathbf{X},\mathbf{Y},\mathbf{Z}}$ for cofibrant $\mathbf{X},\mathbf{Y},\mathbf{Z}$ via the following string of natural isomorphisms in $\mathbf{HSPEC}_{\mathcal{U}}$: $\mathbf{X}\wedge(\mathbf{Y}\wedge\mathbf{Z}) = \mathbf{L}f_*(\mathbf{L}\underline{\wedge}(\mathbf{X},\mathbf{Y}\wedge$

$\mathbf{Z})) \approx \mathbf{L}f_*(\mathbf{X}\underline{\wedge}\mathcal{L}(\mathbf{Y} \wedge \mathbf{Z})) \approx \mathbf{L}f_*(\mathbf{X}\underline{\wedge}\mathcal{L}(\mathbf{L}f_*(\mathbf{L}\underline{\wedge}(\mathbf{Y},\mathbf{Z})))) \approx \mathbf{L}f_*(\mathbf{X}\underline{\wedge}\mathcal{L}(\mathbf{L}f_*(\mathbf{Y}\underline{\wedge}\mathbf{Z}))) \approx \mathbf{L}f_*(\mathbf{X}\underline{\wedge}f_*(\mathbf{Y}\underline{\wedge}\mathbf{Z})) \approx f_*(\mathbf{X}\underline{\wedge}f_*(\mathbf{Y}\underline{\wedge}\mathbf{Z})) \approx f_* \circ (\mathrm{id}_{\mathcal{U}} \oplus f)_* \circ \Phi_*((\mathbf{X}\underline{\wedge}\mathbf{Y})\underline{\wedge}\mathbf{Z}) \approx f_* \circ (f \oplus \mathrm{id}_{\mathcal{U}})_*((\mathbf{X}\underline{\wedge}\mathbf{Y})\underline{\wedge}\mathbf{Z}) \approx f_*(f_*(\mathbf{X}\underline{\wedge}\mathbf{Y})\underline{\wedge}\mathbf{Z}) \approx (\mathbf{X} \wedge \mathbf{Y}) \wedge \mathbf{Z}$ (reverse the steps). That MC_1 and MC_2 obtain can then be established by using the contractibility of $\mathcal{I}(\mathcal{U}^n,\mathcal{U})$.]

[Note: $\mathbf{HSPEC}_{\mathcal{U}}$ admits an evident compatible symmetry, thus is a symmetric monoidal category (cf. p. 0–25). Since each of the functors $— \wedge \mathbf{Y} : \mathbf{HSPEC}_{\mathcal{U}} \to \mathbf{HSPEC}_{\mathcal{U}}$ has a right adjoint $\mathbf{Z} \to \hom(\mathbf{Y},\mathbf{Z})$, it follows that $\mathbf{HSPEC}_{\mathcal{U}}$ is a closed category.]

Therefore **HSPEC** is a closed category.

EXAMPLE If $\mathbf{f} : \mathbf{X} \to \mathbf{Y}$, $\mathbf{g} : \mathbf{Z} \to \mathbf{W}$ are morphisms in **HSPEC**, then there is an exact triangle $\mathbf{X} \wedge \mathbf{C_g} \to \mathbf{C_{f\wedge g}} \to \mathbf{C_f} \wedge \mathbf{W} \to \Sigma(\mathbf{X} \wedge \mathbf{C_g})$.]

[Consider the factorization $\mathbf{f} \wedge \mathbf{g} = \mathbf{f} \wedge \mathrm{id}_{\mathbf{W}} \circ \mathrm{id}_{\mathbf{X}} \wedge \mathbf{g}$ and use the result on p. 16–13.]

FACT $\mathbf{X} \wedge \mathbf{Y}$ is connective if $\mathbf{X}$ & $\mathbf{Y}$ are connective.

Given a finite dimensional subspace U of $\mathcal{U}$, put $\Sigma^U\mathbf{X} = \mathbf{X}\wedge\mathbf{S}^U$, $\Omega^U\mathbf{X} = \mathrm{HOM}(\mathbf{S}^U,\mathbf{X})$—then (Σ^U,Ω^U) is an adjoint pair.

PROPOSITION 30 The total left derived functor $\mathbf{L}\Sigma^U$ for Σ^U exists and the total right derived functor $\mathbf{R}\Omega^U$ for Ω^U exists. And: $(\mathbf{L}\Sigma^U,\mathbf{R}\Omega^U)$ is an adjoint pair (cf. Proposition 12).

PROPOSITION 31 The pair $(\mathbf{L}\Sigma^U,\mathbf{R}\Omega^U)$ is an adjoint equivalence of categories (cf. Proposition 13).

[Suppose that $\mathbf{X}$ is cofibrant—then in $\mathbf{HSPEC}_{\mathcal{U}}$ there are, on the one hand, natural isomorphisms $\Sigma^U(\mathbf{X} \wedge \mathbf{S}^{-U}) \approx f_*(\mathbf{X}\underline{\wedge}\mathbf{S}^{-U}) \wedge \mathbf{S}^U \approx f_*((\mathbf{X}\underline{\wedge}\mathbf{S}^{-U}) \wedge \mathbf{S}^U) \approx f_*(\mathbf{X}\underline{\wedge}(\mathbf{S}^{-U} \wedge \mathbf{S}^U)) \approx f_*(\mathbf{X}\underline{\wedge}\mathbf{Q}^\infty\mathbf{S}^0) \approx \mathbf{X}$ and, on the other, natural isomorphisms $\Sigma^U\mathbf{X} \wedge \mathbf{S}^{-U} \approx f_*(\Sigma^U\mathbf{X}\underline{\wedge}\mathbf{S}^{-U}) \approx f_*((\mathbf{X} \wedge \mathbf{S}^U)\underline{\wedge}\mathbf{S}^{-U}) \approx f_*(\mathbf{X}\underline{\wedge}(\mathbf{S}^{-U} \wedge \mathbf{S}^U)) \approx f_*(\mathbf{X}\underline{\wedge}\mathbf{Q}^\infty\mathbf{S}^0) \approx \mathbf{X}$. Therefore $\mathbf{L}\Sigma^U$ is an equivalence of categories and $— \wedge \mathbf{S}^{-U} \approx \mathbf{R}\Omega^U$.]

Fix a universe $\mathcal{U}$—then S_n operates to the left on $\mathcal{U}^n$ by permutations, hence $\forall\, \sigma \in S_n$ there are functors $\sigma_* : \mathbf{SPEC}_{\mathcal{U}^n} \to \mathbf{SPEC}_{\mathcal{U}^n}$. Agreeing to write $S_n \ltimes —$ for the functor corresponding to the arrow $\chi_n : S_n \to \mathcal{I}(\mathcal{U}^n,\mathcal{U}^n)$, one has $S_n \ltimes \mathbf{X} \approx \bigvee\limits_{\sigma \in S_n} \sigma_*\mathbf{X}$. The multiplication and unit of S_n induce natural transformations $m_n : S_n \ltimes S_n \ltimes — \to S_n \ltimes —$ & $\epsilon_n : \mathrm{id} \to S_n \ltimes —$, so $(S_n \ltimes —, m_n, \epsilon_n)$ is a triple in $\mathbf{SPEC}_{\mathcal{U}^n}$. Its associated category of algebras is called the category of $\underline{S_n\text{-spectra}}$ (relative to $\mathcal{U}$):S_n-$\mathbf{SPEC}_{\mathcal{U}^n}$. An S_n-spectrum is therefore a $\mathcal{U}^n$-spectrum $\mathbf{X}$ equipped with a morphism $\xi : S_n \ltimes \mathbf{X} \to \mathbf{X}$ satisfying TA_1 and TA_2 (cf. p. 0–27 ff.), i.e., equipped with morphisms $\xi_\sigma : \sigma_* \mathbf{X} \to \mathbf{X}$ such that $\xi_e = \text{id}_{\mathbf{X}}$ and $\xi_\sigma \circ \sigma_*(\xi_\tau) = \xi_{\sigma\tau}$.

[Note: Given $(\mathbf{X}, \xi), (\mathbf{Y}, \eta)$ in S_n-$\mathbf{SPEC}_{\mathcal{U}^n}$, write S_n-HOM($\mathbf{X}, \mathbf{Y}$) for Mor $((\mathbf{X}, \xi), (\mathbf{Y}, \eta))$ topologized via the equalizer diagram Mor $((\mathbf{X}, \xi), (\mathbf{Y}, \eta)) \to \text{HOM}(\mathbf{X}, \mathbf{Y}) \rightrightarrows \text{HOM}(S_n \ltimes \mathbf{X}, \mathbf{Y})$.]

Example: $\forall$ $\mathbf{X}$ in $\mathbf{SPEC}_{\mathcal{U}}$, $\mathbf{X}^{(n)} \equiv \mathbf{X} \underline{\wedge} \cdots \underline{\wedge} \mathbf{X}$ (n factors) is an S_n-spectrum.

[Note: $\forall$ X in $\mathbf{\Delta}$-$\mathbf{CG}_*$, $X^{(n)} \equiv X \#_k \cdots \#_k X$ (n factors) and $(\mathbf{Q}^\infty X)^{(n)} \approx \mathbf{Q}^\infty(X^{(n)})$.]

The functor $S_n \ltimes$ — is a left adjoint, hence preserves colimits. Since $\mathbf{SPEC}_{\mathcal{U}^n}$ is complete and cocomplete, specialization of the following generality allows one to conclude that S_n-$\mathbf{SPEC}_{\mathcal{U}^n}$ is complete and cocomplete.

LEMMA Suppose that $\mathbf{C}$ is a complete and cocomplete category. Let $\mathbf{T} = (T, m, \epsilon)$ be a triple in $\mathbf{C}$. Assume: T preserves filtered colimits—then $\mathbf{T}$-$\mathbf{ALG}$ is complete and cocomplete.

[A proof can be found in Borceux[†].]

LEMMA Suppose that A is a right S_n-space in $\mathbf{\Delta}$-$\mathbf{CG}$. Let $\alpha : A \to \mathcal{I}(\mathcal{U}^n, \mathcal{U})$ be S_n-equivariant—then for every S_n-spectrum $\mathbf{X}$, there is a coequalizer diagram $\alpha \ltimes S_n \ltimes \mathbf{X} \rightrightarrows \alpha \ltimes \mathbf{X} \to \alpha \ltimes_{S_n} \mathbf{X}$.

[One of the arrows is $\text{id}_\alpha \ltimes \xi$. As for the other, $\alpha \ltimes S_n \ltimes \mathbf{X} \approx (\alpha \times_c \chi_n) \ltimes \mathbf{X}$ (cf. p. 16–22) and the diagram

$$\begin{tikzcd}
A \times S_n \arrow[rr, "\pi"] \arrow[dr, "\alpha \times_c \chi_n"'] & & A \arrow[dl, "\alpha"] \\
& \mathcal{I}(\mathcal{U}^n, \mathcal{U}) &
\end{tikzcd}$$

commutes ($\pi(a, \sigma) = a \cdot \sigma$). Proof: $\alpha \times_c \chi_n(a, \sigma) = \alpha(a) \circ \chi_n(\sigma)$, $\alpha \circ \pi(a, \sigma) = \alpha(a \cdot \sigma) = \alpha(a) \cdot \sigma$ and $\forall$ $u \in \mathcal{U}^n$, $(\alpha(a) \circ \chi_n(\sigma))(u) = \alpha(a)(\sigma \cdot u) = (\alpha(a) \cdot \sigma)(u)$ (by the very definition of the right action of S_n on $\mathcal{I}(\mathcal{U}^n, \mathcal{U})$).]

Remark: $\alpha \ltimes_{S_n}$ — is a functor from S_n-$\mathbf{SPEC}_{\mathcal{U}^n}$ to $\mathbf{SPEC}_{\mathcal{U}}$. On the other hand, $\mathcal{HOM}[\alpha, —)$ is a functor from $\mathbf{SPEC}_{\mathcal{U}}$ to S_n-$\mathbf{SPEC}_{\mathcal{U}^n}$. And: $\text{HOM}(\alpha \ltimes_{S_n} \mathbf{X}, \mathbf{Y}) \approx S_n$-HOM($\mathbf{X}, \mathcal{HOM}[\alpha, \mathbf{Y})$).

It is sometimes necessary to consider G-spectra, where G is a subgroup of S_n (the objects of G-$\mathbf{SPEC}_{\mathcal{U}^n}$ are thus the algebras per $G \ltimes$ —). Given a subgroup K of G, there is a forgetful functor G-$\mathbf{SPEC}_{\mathcal{U}^n} \to K$-$\mathbf{SPEC}_{\mathcal{U}^n}$ and, in obvious notation, it has a left adjoint $G \ltimes_K$ —, so that G-HOM($G \ltimes_K \mathbf{X}, \mathbf{Y}) \approx K$-HOM($\mathbf{X}, \mathbf{Y}$).

[†] *Handbook of Categorical Algebra* 2, Cambridge University Press (1994), 206–211.

FACT Let $U : G\text{-}\mathbf{SPEC}_{\mathcal{U}^n} \to \mathbf{SPEC}_{\mathcal{U}^n}$ be the forgetful functor. Call a morphism $\mathbf{f} : \mathbf{X} \to \mathbf{Y}$ of G-spectra a weak equivalence if $U\mathbf{f}$ is a weak equivalence, a fibration if $U\mathbf{f}$ is a fibration, and a cofibration if $\mathbf{f}$ has the LLP w.r.t. acyclic fibrations—then with these choices, $G\text{-}\mathbf{SPEC}_{\mathcal{U}^n}$ is a model category.

[Note: This is the external structure. To define the internal structure, stipulate that $\mathbf{f} : \mathbf{X} \to \mathbf{Y}$ is a weak equivalence or a fibration if for each finite dimensional G-stable $U \subset \mathcal{U}^n$, and each subgroup $K \subset G$, the induced map of fixed point spaces $X_U^K \to Y_U^K$ is a weak equivalence or a fibration and let the cofibrations be the $\mathbf{f}$ which have the LLP w.r.t. acyclic fibrations. Example: Take $G = S_n$—then $\forall$ cofibrant $\mathbf{X}$ in $\mathbf{SPEC}_{\mathcal{U}}$, $\mathbf{X}^{(n)}$ is cofibrant in the internal structure on $S_n\text{-}\mathbf{SPEC}_{\mathcal{U}^n}$.]

The preceding formalities are the spectral counterpart of a standard topological setup. Thus given a right S_n-space A in $\mathbf{\Delta}\text{-}\mathbf{CG}$ and a left S_n-space X in $\mathbf{\Delta}\text{-}\mathbf{CG}_*$, define $A \ltimes_{S_n} X$ by the coequalizer diagram $(A \times S_n)_+ \#_k X \rightrightarrows A_+ \#_k X \to A \ltimes_{S_n} X((A \times S_n)+ \approx A_+ \#_k S_{n+})$—then $A \ltimes_{S_n}$ — is a functor from the category of pointed Δ-separated compactly generated left S_n-spaces to the category of pointed Δ-separated compactly generated spaces. It has a right adjoint, viz. the functor that sends Y to $Y^{A_+}((\sigma \cdot f)(a) = f(a \cdot \sigma)$, with trivial action on the disjoint base point).

Example: Let $\mathcal{C}$ be a Δ-separated creation operator, i.e., a functor $\mathcal{C} : \mathbf{\Gamma}_{\text{in}}^{\text{OP}} \to \mathbf{\Delta}\text{-}\mathbf{CG}$ such that $\mathcal{C}_0 = *$—then in the notation of §14, Proposition 27, the filtration quotient $\mathcal{C}_n[X]/\mathcal{C}_{n-1}[X]$ is homeomorphic to $\mathcal{C}_n \ltimes_{S_n} X^{(n)}$.

FACT $\forall X$ in $\mathbf{\Delta}\text{-}\mathbf{CG}_*$, $\alpha \ltimes_{S_n} (\mathbf{Q}^\infty X)^{(n)} \approx \mathbf{Q}^\infty(A \ltimes_{S_n} X^{(n)})$.

EXAMPLE (Extended Powers) Take $A = XS_n$ (which is S_n-universal) and fix an equivariant arrow $XS_n \to \mathcal{I}(\mathcal{U}^n, \mathcal{U})$. Using suggestive notation, the assignment $\mathbf{X} \to XS_n \ltimes_{S_n} \mathbf{X}^{(n)}$ specifies a functor $D_n : \mathbf{SPEC}_{\mathcal{U}} \to \mathbf{SPEC}_{\mathcal{U}}$ (conventionally, $D_0\mathbf{X} = \mathbf{S}$), the nth extended power. Defining $D_n : \mathbf{\Delta}\text{-}\mathbf{CG}_* \to \mathbf{\Delta}\text{-}\mathbf{CG}_*$ in exactly the same way, one has $D_n\mathbf{Q}^\infty X = XS_n \ltimes_{S_n} (\mathbf{Q}^\infty X)^{(n)} \approx \mathbf{Q}^\infty(XS_n \ltimes_{S_n} X^{(n)}) = \mathbf{Q}^\infty(D_n X)$. Example: $D_n\mathbf{S}^0 = BS_{n+}$ ($\Rightarrow \bigvee_{n \geq 0} D_n\mathbf{S}^0 = B\mathbf{M}_{\infty+}$, $\mathbf{M}_\infty$ the permutative category of p. 14–28).]

[Note: Extended powers have many applications in homotopy theory. For an account, see Bruner[†] et al..]

Let $\mathcal{C}$ be a Δ-separated creation operator—then $\forall X$ in $\mathbf{\Delta}\text{-}\mathbf{CG}_*$, the realization $\mathcal{C}[X]$ of $\mathcal{C}$ at X is $\int^{\mathbf{n}} \mathcal{C}_n \times_k X^n$ (cf. p. 14–38 (the assumption there that (X, x_0) be wellpointed has been omitted here)), so $\mathcal{C}[X]$ can be described by the coequalizer diagram $\coprod_{\gamma:\mathbf{m}\to\mathbf{n}} \mathcal{C}_n \times_k$

[†] *SLN* **1176** (1986).

$X^m \overset{u}{\underset{v}{\rightrightarrows}} \coprod_{m\geq 0} \mathcal{C}_m \times_k X^m \to \mathcal{C}[X]$ (on the term indexed by $\gamma : \mathbf{m} \to \mathbf{n}$, u is the arrow $\mathcal{C}_n \times_k X^m \to \mathcal{C}_n \times_k X^n$ and v is the arrow $\mathcal{C}_n \times_k X^m \to \mathcal{C}_m \times_k X^m$). It is this interpretation of $\mathcal{C}[X]$ that carries over to spectra provided they are unital.

Definition: A unital $\mathcal{U}$-spectrum is a pair $(\mathbf{X}, \mathbf{e})$, where $\mathbf{e} : \mathbf{S} \to \mathbf{X}$ is a morphism of $\mathcal{U}$-spectra. Therefore the unital $\mathcal{U}$-spectra are simply the objects of the category $\mathbf{S} \backslash \mathbf{SPEC}_{\mathcal{U}}$. Example: $\forall\, X$ in $\mathbf{\Delta}$-$\mathbf{CG}_*$, map $\mathbf{S}^0$ to X_+ by $\begin{cases} 0 \to x_0 \\ 1 \to * \end{cases}$ —then $\mathbf{Q}^\infty X_+$ is unital.

[Note: Morphisms in $\mathbf{S} \backslash \mathbf{SPEC}_{\mathcal{U}}$ are termed unital.]

Let $\mathbf{X}$ be a unital $\mathcal{U}$-spectrum. Viewing $\mathcal{U}$ as an object in $\mathbf{\Delta}$-$\mathbf{CG}_*$ with base point 0, each $\gamma : \mathbf{m} \to \mathbf{n}$ in $\mathbf{\Gamma}_{\text{in}}$ induces a linear isometry $\gamma : \mathcal{U}^m \to \mathcal{U}^n$ and $\gamma_* \mathbf{X}^{(m)}$ can be identified with $\mathbf{X}_1 \underline{\wedge} \cdots \underline{\wedge} X_n, \mathbf{X}_j$ being $\mathbf{X}$ if $\gamma^{-1}(j) \neq \emptyset$ and $\mathbf{S}$ if $\gamma^{-1}(j) = \emptyset$. There is an arrow $\gamma_* \mathbf{X}^{(m)} \approx \mathbf{X}_1 \underline{\wedge} \cdots \underline{\wedge} \mathbf{X}_n \to \mathbf{X}^{(n)}$ which is $\mathrm{id}_{\mathbf{X}}$ or $\mathbf{e}$ according to whether $\mathbf{X}_j = \mathbf{X}$ or $\mathbf{S}$.

Suppose now that $\phi : \mathcal{C} \to \mathcal{L}$ is a morphism of creation operators, where $\mathcal{L}$ is the linear isometries operad attached to our universe (recall that $\mathcal{L}$ extends to a Δ-separated creation operator (cf. §14, Proposition 35))—then $\forall\, n$, $\phi_n : \mathcal{C}_n \to \mathcal{L}_n$ $(= \mathcal{I}(\mathcal{U}^n, \mathcal{U}))$ is S_n-equivariant. Given a morphism $\gamma : \mathbf{m} \to \mathbf{n}$ in $\mathbf{\Gamma}_{\text{in}}$, let $\phi_\gamma : \mathcal{C}_n \to \mathcal{L}_m$ be either composite in the commutative diagram

$$\begin{tikzcd} \mathcal{C}_n \arrow[r] \arrow[d] \arrow[dr, dashed] & \mathcal{C}_m \arrow[d] \\ \mathcal{L}_n \arrow[r] & \mathcal{L}_m \end{tikzcd}$$

and for $\mathbf{X}$ in $\mathbf{S} \backslash \mathbf{SPEC}_{\mathcal{U}}$, put $\mathcal{C}_\gamma \ltimes \mathbf{X}^{(m)} = \phi_\gamma \ltimes \mathbf{X}^{(m)}$, $\mathcal{C}_m \ltimes \mathbf{X}^{(m)} = \phi_m \ltimes \mathbf{X}^{(m)}$ to get an arrow $\mathcal{C}_\gamma \ltimes \mathbf{X}^{(m)} \to \mathcal{C}_m \ltimes \mathbf{X}^{(m)}$. The realization $\mathcal{C}[\mathbf{X}]$ of $\mathcal{C}$ at $\mathbf{X}$ is then defined by the coequalizer diagram $\bigvee_{\gamma:\mathbf{m}\to\mathbf{n}} \mathcal{C}_\gamma \ltimes \mathbf{X}^{(m)} \overset{\mathbf{u}}{\underset{\mathbf{v}}{\rightrightarrows}} \bigvee_{m \geq 0} \mathcal{C}_m \ltimes \mathbf{X}^{(m)} \to \mathcal{C}[\mathbf{X}]$ (on the term indexed by $\gamma : \mathbf{m} \to \mathbf{n}$, $\mathbf{u}$ is the arrow $\mathcal{C}_\gamma \ltimes \mathbf{X}^{(m)} \approx \mathcal{C}_n \ltimes \gamma_* \mathbf{X}^{(m)} \to \mathcal{C}_n \ltimes \mathbf{X}^{(n)}$ and $\mathbf{v}$ is the arrow $\mathcal{C}_\gamma \ltimes \mathbf{X}^{(m)} \to \mathcal{C}_m \ltimes \mathbf{X}^{(m)}$).

[Note: The isomorphism $\mathcal{C}_\gamma \ltimes \mathbf{X}^{(m)} \approx \mathcal{C}_n \ltimes \gamma_* \mathbf{X}^{(m)}$ is an instance of the "composition rule" on p. 16–22. To see this, consider $* \xrightarrow{\gamma} \mathcal{I}(\mathcal{U}^m, \mathcal{U}^n)$ and $\mathcal{C}_n \xrightarrow{\phi_n} \mathcal{I}(\mathcal{U}^n, \mathcal{U}) : \phi_n \times_c \gamma = \phi_\gamma \Rightarrow \phi_\gamma \ltimes \mathbf{X}^{(m)} \approx \phi_n \ltimes \gamma_* \mathbf{X}^{(m)}$.]

Remark: $\mathcal{C}[\mathbf{X}]$ is unital (since $\mathbf{S} = \mathcal{C}_0 \ltimes \mathbf{X}^{(0)}$) and $\mathcal{C}[?]$ is functorial.

PROPOSITION 32 Let $\mathcal{C}$ be a Δ-separated creation operator, augmented over $\mathcal{L}$ via $\phi : \mathcal{C} \to \mathcal{L}$—then $\forall\, X$ in $\mathbf{\Delta}$-$\mathbf{CG}_*$, $\mathcal{C}[\mathbf{Q}^\infty X_+] \approx \mathbf{Q}^\infty \mathcal{C}[X]_+$.

[Apply $\mathbf{Q}^\infty$ to the coequalizer diagram $\bigvee_{\gamma:\mathbf{m}\to\mathbf{n}} \mathcal{C}_{n+} \#_k (X_+)^{(m)} \rightrightarrows \bigvee_{m\geq 0} \mathcal{C}_{m+} \#_k (X_+)^{(m)} \to \mathcal{C}[X]_+$.]

[Note: The isomorphism is natural in X.]

The coequalizer diagram describing $\mathcal{C}[X]$ can be reduced to $\coprod\limits_{n\geq 0}\coprod\limits_{0\leq i\leq n} \mathcal{C}_{n+1}\times_k X^n \overset{u}{\underset{v}{\rightrightarrows}} \coprod\limits_{n\geq 0} \mathcal{C}_n \times_{S_n} X^n \rightarrow \mathcal{C}[X]$ and the coequalizer diagram describing $\mathcal{C}[\mathbf{X}]$ can be reduced to $\bigvee\limits_{n\geq 0}\bigvee\limits_{0\leq i\leq n} \mathcal{C}_{\sigma_i} \ltimes \mathbf{X}^{(n)} \overset{u}{\underset{v}{\rightrightarrows}} \bigvee\limits_{n\geq 0} \mathcal{C}_n \ltimes_{S_n} \mathbf{X}^{(n)} \rightarrow \mathcal{C}[\mathbf{X}]$, the $(n,i)^{\text{th}}$ term being indexed on $\sigma_i : \mathbf{n} \rightarrow \mathbf{n+1}$ $(0 \leq i \leq n)$ (notation as in the proof of Proposition 35 in §14). There is also a coequalizer diagram $\coprod\limits_{m\leq n-1}\coprod\limits_{0\leq j\leq m} \mathcal{C}_{m+1} \times_k X^m \overset{u}{\underset{v}{\rightrightarrows}} \coprod\limits_{m\leq n} \mathcal{C}_m \times_{S_m} X^m \rightarrow \mathcal{C}_n[X]$ (cf. §14, Proposition 27). Here $\mathcal{C}_0[X] = *$, $\mathcal{C}[X] = \operatorname{colim} \mathcal{C}_n[X]$, and the arrows $\mathcal{C}_n[X] \rightarrow \mathcal{C}_{n+1}[X]$ are closed embeddings. Proceeding by analogy, define $\mathcal{C}_n[\mathbf{X}]$ by the coequalizer diagram $\bigvee\limits_{m\leq n-1}\bigvee\limits_{0\leq j\leq m} \mathcal{C}_{\sigma_j} \ltimes \mathbf{X}^{(m)} \overset{u}{\underset{v}{\rightrightarrows}} \bigvee\limits_{m\leq n} \mathcal{C}_m \ltimes_{S_m} \mathbf{X}^{(m)} \rightarrow \mathcal{C}_n[\mathbf{X}]$—then $\mathcal{C}_0[\mathbf{X}] = \mathbf{S}$, $\mathcal{C}[\mathbf{X}] = \operatorname{colim} \mathcal{C}_n[\mathbf{X}]$, and the arrows $\mathcal{C}_n[\mathbf{X}] \rightarrow \mathcal{C}_{n+1}[\mathbf{X}]$ are levelwise closed embeddings if $\mathbf{e} : \mathbf{S} \rightarrow \mathbf{X}$ is a levelwise closed embedding.

Recalling that X_*^{n+1} is the subspace of X^{n+1} consisting of those points having at least one coordinate the base point x_0, the commutative diagram

$$\begin{tikzcd}
\mathcal{C}_{n+1} \times_{S_{n+1}} X_*^{n+1} \arrow[r] \arrow[d] & \mathcal{C}_n[X] \arrow[d] \\
\mathcal{C}_{n+1} \times_{S_{n+1}} X^{n+1} \arrow[r] & \mathcal{C}_{n+1}[X]
\end{tikzcd}$$

is a pushout square. To formulate its spectral analog, one first has to define $\mathbf{X}_*^{(n+1)}$. The arrow $\mathbf{X}^{(n)}\underline{\Delta}\mathbf{S} \rightarrow \mathbf{X}^{(n+1)}$ is a morphism of S_n-spectra $(S_n \subset S_{n+1})$, hence determines by adjointness a morphism $\boldsymbol{\theta} : S_{n+1} \ltimes_{S_n} (\mathbf{X}^{(n)}\underline{\Delta}\mathbf{S}) \rightarrow \mathbf{X}^{(n+1)}$ of S_{n+1}-spectra. Noting that $S_{n+1} \ltimes_{S_n} (\mathbf{X}^{(n)}\underline{\Delta}\mathbf{S}) \approx \bigvee\limits_{0\leq i\leq n} \mathbf{X}^{(i)}\underline{\Delta}\mathbf{S}\underline{\Delta}\mathbf{X}^{(n-i)}$, the arrows $\mathbf{X}^{(n-1)}\underline{\Delta}\mathbf{S}\underline{\Delta}\mathbf{S} \rightarrow \mathbf{X}^{(n)}\underline{\Delta}\mathbf{S} \subset S_{n+1} \ltimes_{S_n} (\mathbf{X}^{(n)}\underline{\Delta}\mathbf{S})$, $\mathbf{X}^{(n-1)}\underline{\Delta}\mathbf{S}\underline{\Delta}\mathbf{S} \rightarrow \mathbf{X}^{(n-1)}\underline{\Delta}\mathbf{S}\underline{\Delta}\mathbf{X} \subset S_{n+1} \ltimes_{S_n} (\mathbf{X}^{(n)}\underline{\Delta}\mathbf{S})$ are morphisms of S_{n-1}-spectra $(S_{n-1} \subset S_n \subset S_{n+1})$, hence determine by adjointness morphisms $\mathbf{f}, \mathbf{g} : S_{n+1} \ltimes_{S_{n-1}} (\mathbf{X}^{(n-1)}\underline{\Delta}\mathbf{S}\underline{\Delta}\mathbf{S}) \rightarrow S_{n+1} \ltimes_{S_n} (\mathbf{X}^{(n)}\underline{\Delta}\mathbf{S})$ of S_{n+1}-spectra. One then defines $\mathbf{X}_*^{(n+1)}$ by the coequalizer diagram $S_{n+1} \ltimes_{S_{n-1}} (\mathbf{X}^{(n-1)}\underline{\Delta}\mathbf{S}\underline{\Delta}\mathbf{S}) \overset{\mathbf{f}}{\underset{\mathbf{g}}{\rightrightarrows}} S_{n+1} \ltimes_{S_n} (\mathbf{X}^{(n)}\underline{\Delta}\mathbf{S}) \rightarrow \mathbf{X}_*^{(n+1)}$ (calculated in S_{n+1}-$\mathbf{SPEC}_{\mathcal{U}^{n+1}}$ (cf. p. 16–31)). Since $\boldsymbol{\theta}$ coequalizes $(\mathbf{f}, \mathbf{g})$, there is a morphism $\mathbf{X}_*^{(n+1)} \rightarrow \mathbf{X}^{(n+1)}$ of S_{n+1}-spectra (which is a levelwise closed embedding if this is the case of $\mathbf{e} : \mathbf{S} \rightarrow \mathbf{X}$). Finally, the composites $\mathcal{C}_{n+1} \ltimes (\mathbf{X}^{(i)}\underline{\Delta}\mathbf{S}\underline{\Delta}\mathbf{X}^{(n-i)}) \approx \mathcal{C}_{\sigma_i} \ltimes \mathbf{X}^{(n)} \rightarrow \mathcal{C}_n \ltimes \mathbf{X}^{(n)} \rightarrow \mathcal{C}_n[\mathbf{X}]$ give rise to an arrow $\mathcal{C}_{n+1} \ltimes_{S_{n+1}} \mathbf{X}_*^{(n+1)} \rightarrow \mathcal{C}_n[\mathbf{X}]$ and the commutative diagram

$$\begin{tikzcd}
\mathcal{C}_{n+1} \ltimes_{S_{n+1}} \mathbf{X}_*^{(n+1)} \arrow[r] \arrow[d] & \mathcal{C}_n[\mathbf{X}] \arrow[d] \\
\mathcal{C}_{n+1} \ltimes_{S_{n+1}} \mathbf{X}^{(n+1)} \arrow[r] & \mathcal{C}_{n+1}[\mathbf{X}]
\end{tikzcd}$$

is a pushout square.

Observation: The forgetful functor $\mathbf{S} \backslash \mathbf{SPEC}_{\mathcal{U}} \rightarrow \mathbf{SPEC}_{\mathcal{U}}$ has a left adjoint $\mathbf{X} \rightarrow \mathbf{S} \vee \mathbf{X}$ ($\mathbf{e} : \mathbf{S} \rightarrow \mathbf{S} \vee \mathbf{X}$ is the inclusion of the wedge summand $\mathbf{S}$).

PROPOSITION 33 Let $\mathcal{C}$ be a Δ-separated creation operator, augmented over $\mathcal{L}$ via $\phi : \mathcal{C} \rightarrow \mathcal{L}$—then there is an isomorphism $\mathcal{C}[\mathbf{S} \vee \mathbf{X}] \approx \bigvee\limits_{n\geq 0} \mathcal{C}_n \ltimes_{S_n} \mathbf{X}^{(n)}$ natural in $\mathbf{X}$.

[In fact, $(\mathbf{S} \vee \mathbf{X})^{(n+1)} \approx (\mathbf{S} \vee \mathbf{X})_*^{(n+1)} \vee \mathbf{X}^{(n+1)}$ as S_{n+1}-spectra, thus by induction, $\mathcal{C}_n[\mathbf{S} \vee \mathbf{X}] \approx \bigvee_{m \leq n} \mathcal{C}_m \ltimes_{S_m} \mathbf{X}^{(m)}$ $(n \geq 0)$.]

The spacewise version of Proposition 33 is the relation $\mathcal{C}[X_+] \approx \coprod_{n \geq 0} \mathcal{C}_n \times_{S_n} X^n$.

LEMMA Suppose that (X, x_0) is Δ-separated and wellpointed—then there are unital morphisms $\mathbf{Q}^\infty X_+ \to \mathbf{S} \vee \mathbf{Q}^\infty X$ and $\mathbf{S} \vee \mathbf{Q}^\infty X \to \mathbf{Q}^\infty X_+$ which are unital homotopy equivalences.

[Note: A homotopy $\mathbf{H}$ is unital if $\forall\, t$, $\mathbf{H}_t$ is unital.]

PROPOSITION 34 Let $\mathcal{C}$ be a Δ-separated creation operator, augmented over $\mathcal{L}$ via $\phi : \mathcal{C} \to \mathcal{L}$—then $\forall$ Δ-separated, wellpointed X, there is a natural weak equivalence $\mathbf{Q}^\infty \mathcal{C}[X] \to \bigvee_{n \geq 1} \mathbf{Q}^\infty(\mathcal{C}_n \ltimes_{S_n} X^{(n)})$ of $\mathcal{U}$-spectra.

[$\mathcal{C}[X]$ is Δ-separated and wellpointed (cf. §14, Proposition 27). The lemma thus provides a weak equivalence $\mathbf{S} \vee \mathbf{Q}^\infty \mathcal{C}[X] \to \mathbf{Q}^\infty \mathcal{C}[X]_+ \approx \mathcal{C}[\mathbf{Q}^\infty X_+]$ (cf. Proposition 32). But $\mathcal{C}[?] : \mathbf{S} \backslash \mathbf{SPEC}_\mathcal{U} \to \mathbf{S} \backslash \mathbf{SPEC}_\mathcal{U}$ is a continuous functor, so it's homotopy preserving. Accordingly, there is a weak equivalence $\mathcal{C}[\mathbf{Q}^\infty X_+] \to \mathcal{C}[\mathbf{S} \vee \mathbf{Q}^\infty X] \approx \bigvee_{n \geq 0} \mathcal{C}_n \ltimes_{S_n} (\mathbf{Q}^\infty X)^{(n)}$ (cf. Proposition 33). And: $\bigvee_{n \geq 0} \mathcal{C}_n \ltimes_{S_n} (\mathbf{Q}^\infty X)^{(n)} \approx \mathbf{S} \vee \bigvee_{n \geq 1} \mathcal{C}_n \ltimes_{S_n} (\mathbf{Q}^\infty X)^{(n)} \approx \mathbf{S} \vee \bigvee_{n \geq 1} \mathbf{Q}^\infty(\mathcal{C}_n \ltimes_{S_n} X^{(n)})$ (cf. p. 16–32). The weak equivalence in question now follows upon quotienting out by $\mathbf{S}$.]

Application: $\mathbf{Q}^\infty \mathcal{C}[X]$ and $\bigvee_{n \geq 1} \mathbf{Q}^\infty(\mathcal{C}_n[X]/\mathcal{C}_{n-1}[X])$ are isomorphic in $\mathbf{HSPEC}_\mathcal{U}$.

LEMMA Let X, Y be in $\mathbf{\Delta}\text{-}\mathbf{CG}_{*c}$ and let $f : X \to Y$ be a pointed continuous function. Assume: f is a weak homotopy equivalence—then $\mathbf{Q}^\infty f : \mathbf{Q}^\infty X \to \mathbf{Q}^\infty Y$ is a weak equivalence.

[Since it suffices to work in $\mathbf{SPEC}$, one has only to show that the $\pi_n^s(f) : \pi_n^s(X) \to \pi_n^s(Y)$ $(n \geq 0)$ are bijective ($\mathbf{Q}^\infty X, \mathbf{Q}^\infty Y$ being connective (cf. p. 16–7)). But $\pi_n^s(X) = \operatorname{colim} \pi_{n+q}(\Sigma^q X)$, $\pi_n^s(Y) = \operatorname{colim} \pi_{n+q}(\Sigma^q Y)$ and $\Sigma^q f : \Sigma^q X \to \Sigma^q Y$ is a weak homotopy equivalence (cf. p. 14–35).]

PROPOSITION 35 Let $\begin{cases} \mathcal{C} \\ \mathcal{D} \end{cases}$ be creation operators, where $\forall\, n$, $\begin{cases} \mathcal{C}_n \\ \mathcal{D}_n \end{cases}$ is a compactly generated Hausdorff space and the action of S_n is free. Suppose given an arrow $\phi : \mathcal{C} \to \mathcal{D}$ such that $\forall$ n, $\phi_n : \mathcal{C}_n \to \mathcal{D}_n$ is a weak homotopy equivalence—then $\forall$ Δ-separated, wellpointed X, there is a weak equivalence $\mathbf{Q}^\infty \mathcal{C}[X] \to \mathbf{Q}^\infty \mathcal{D}[X]$.

[$\mathcal{C}[X]$ and $\mathcal{D}[X]$ are Δ-separated and wellpointed (cf. §14, Proposition 27). But the hypotheses imply that ϕ induces a weak homotopy equivalence $\mathcal{C}[X] \to \mathcal{D}[X]$ (cf. p. 14–54).]

Application: Let $\mathcal{C}$ be a creation operator, where $\forall\ n$, $\mathcal{C}_n$ is a compactly generated Hausdorff space and the action of S_n is free—then $\forall\ \Delta$-separated, wellpointed X, there is a natural weak equivalence $\mathbf{Q}^\infty\mathcal{C}[X] \to \bigvee_{n\geq 1} \mathbf{Q}^\infty(\mathcal{C}_n \ltimes_{S_n} X^{(n)})$ of $\mathcal{U}$-spectra.

[The projection $\mathcal{C} \times \mathcal{L} \to \mathcal{L}$ augments $\mathcal{C} \times \mathcal{L}$ over $\mathcal{L}$. On the other hand, $\forall\ n$, the projection $\mathcal{C}_n \times_k \mathcal{L}_n \to \mathcal{C}_n$ is a weak homotopy equivalence. Quote Propositions 34 and 35.]

[Note: To justify the tacit use of the lemma, it is necessary to observe that $(\mathcal{C}_n \times_k \mathcal{L}_n) \ltimes_{S_n} X^{(n)}$, $\mathcal{C}_n \ltimes_{S_n} X^{(n)}$ are wellpointed and the arrow $(\mathcal{C}_n \times_k \mathcal{L}_n) \ltimes_{S_n} X^{(n)} \to \mathcal{C}_n \ltimes_{S_n} X^{(n)}$ is a weak homotopy equivalence.]

Example: In $\mathbf{HSPEC}_{\mathcal{U}}$, $\mathbf{Q}^\infty\mathrm{BV}^q[X] \approx \bigvee_{n\geq 1} \mathbf{Q}^\infty(\mathrm{BV}(R(q), n) \ltimes_{S_n} X^{(n)})$.

[Note: $\mathrm{BV}^q[X]$ can be replaced by $\Omega^q\Sigma^q X$ if X is path connected (May's approximation theorem).]

Example: In $\mathbf{HSPEC}_{\mathcal{U}}$, $\mathbf{Q}^\infty\mathrm{BV}^\infty[X] \approx \bigvee_{n\geq 1} \mathbf{Q}^\infty(\mathrm{BV}(R(\infty), n) \ltimes_{S_n} X^{(n)})$.

[Note: $\mathrm{BV}^\infty[X]$ can be replaced by $\Omega^\infty\Sigma^\infty X$ if X is path connected and Δ-cofibered (cf. §14, Proposition 33) (X Δ-cofibered $\Rightarrow \Omega^\infty\Sigma^\infty X$ wellpointed (cf. p. 14–44)).]

EXAMPLE Take $\mathcal{C} = \mathrm{PER}$—then in $\mathbf{HSPEC}_{\mathcal{U}}$, $\mathbf{Q}^\infty\mathrm{PER}[X] \approx \bigvee_{n\geq 1} \mathbf{Q}^\infty(XS_n \ltimes_{S_n} X^{(n)}) \approx \bigvee_{n\geq 1} D_n\mathbf{Q}^\infty X$ (cf. p. 16–32).

LEMMA Let **S** be a triple in a category **C** and let **T** be a triple in the category **S-ALG** of **S**-algebras—then the category **T-(S-ALG)** of **T**-algebras in **S-ALG** is isomorphic to the category **T ∘ S-ALG** of **T ∘ S** algebras in **C**.

Let $\mathcal{O}$ be a reduced operad in **Δ-CG**, augmented over $\mathcal{L}$ via $\phi : \mathcal{O} \to \mathcal{L}$—then $\mathcal{O}$ determines a triple $\mathbf{T}_{\mathcal{O}} = (T_{\mathcal{O}}, m, \epsilon)$ in $\mathbf{S}\backslash\mathbf{SPEC}_{\mathcal{U}}$ (cf. §14, Proposition 36) ($T_{\mathcal{O}}\mathbf{X} = \mathcal{O}[\mathbf{X}]$, the realization of $\mathcal{O}$ at $\mathbf{X}$). But $\mathcal{O}$ also determines a triple $\overline{\mathbf{T}}_{\mathcal{O}} = (\overline{T}_{\mathcal{O}}, \overline{m}, \overline{\epsilon})$ in $\mathbf{SPEC}_{\mathcal{U}}$, where $\overline{T}_{\mathcal{O}}\mathbf{X} = \bigvee_{n\geq 0} \mathcal{O}_n \ltimes_{S_n} \mathbf{X}^{(n)}$. To explain the connection between the two, note that $\mathbf{S}\backslash\mathbf{SPEC}_{\mathcal{U}} = \mathbf{S}\text{-}\mathbf{ALG}$, S the functor that sends $\mathbf{X}$ to $\mathbf{S}\vee\mathbf{X}$. And, according to Proposition 33, $T_{\mathcal{O}} \circ S$ "is" $\overline{T}_{\mathcal{O}}$, so by the lemma, the categories $\mathbf{T}_{\mathcal{O}}$-**ALG**, $\overline{\mathbf{T}}_{\mathcal{O}}$-**ALG** are isomorphic.

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§17. STABLE HOMOTOPY THEORY

A complete treatment of stable homotopy theory would require a book of many pages. Therefore, to avoid getting bogged down in a welter of detail, I shall admit some of the results without proof and keep the calculations to a minimum. Despite working within these limitations, it is nevertheless still possible to gain a reasonable understanding of the subject in the "large".

Recapitulation: The stable homotopy category **HSPEC** is a triangulated category satisfying the octahedral axiom (cf. §16, Proposition 14). Furthermore, **HSPEC** is a monogenic compactly generated CTC (cf. p. 16–15) and admits Adams representability (by Neeman's countability criterion).

[Note: **S** is the unit in **HSPEC** and Σ^{-1} stands for Ω (cf. p. 15–42), so $\Lambda^{\pm 1} \approx \Sigma^{\pm 1}$ (recall the convention on p. 16–13).]

EXAMPLE (Complex K-Theory) Let $\mathbf{U} = \operatorname{colim} \mathbf{U}(n)$ be the infinite unitary group—then $\mathbf{U}$ is a pointed CW complex and there is a pointed homotopy equivalence $\mathbf{U} \to \Omega^2\mathbf{U}$ (Bott periodicity). Therefore the prescription $X_q = \Omega^k\mathbf{U}$ ($q \equiv 1 - k \mod 2$ $(0 \leq k \leq 1)$) defines an Ω-prespectrum $\mathbf{X}$ and by definition, $\mathbf{KU} = eM\mathbf{X}$ (cf. p. 14–71) is the spectrum of complex K-theory.

EXAMPLE (Real K-Theory) Let $\mathbf{O} = \operatorname{colim} \mathbf{O}(n)$ be the infinite orthogonal group—then $\mathbf{O}$ is a pointed CW complex and there is a pointed homotopy equivalence $\mathbf{O} \to \Omega^8\mathbf{O}$ (Bott periodicity). Therefore the prescription $X_q = \Omega^k\mathbf{O}$ ($q \equiv 7 - k \mod 8$ $(0 \leq k \leq 7)$) defines an Ω-prespectrum $\mathbf{X}$ and by definition, $\mathbf{KO} = eM\mathbf{X}$ (cf. p. 14–71) is the spectrum of real K-theory.

A Z-graded cohomology theory E^* on **SPEC** is a sequence of exact cofunctors E^n : **HSPEC** $\to$ **AB** and a sequence of natural isomorphisms $\sigma^n : E^{n+1} \circ \Sigma \to E^n$ such that the E^n convert coproducts into products. $\mathbf{CT_Z(SPEC)}$ is the category whose objects are the **Z**-graded cohomology theories on **SPEC** and whose morphisms $\Xi^* : E^* \to F^*$ are sequences of natural transformations $\Xi^n : E^n \to F^n$ such that the diagram

$$\begin{tikzcd}
E^{n+1} \circ \Sigma \arrow[r, "\Xi^{n+1}\Sigma"] \arrow[d, "\sigma^n"'] & F^{n+1} \circ \Sigma \arrow[d, "\sigma^n"] \\
E^n \arrow[r, "\Xi^n"'] & F^n
\end{tikzcd}$$

commutes $\forall\, n$.

Definition: The **Z**-graded cohomology theory $\mathbf{E}^*$ on **SPEC** attached to a spectrum **E** is given by $\mathbf{E}^n(\mathbf{X}) = [\mathbf{X}, \Sigma^n\mathbf{E}]$ $(= \pi_{-n}(\operatorname{hom}(\mathbf{X}, \mathbf{E})))$.

[Note: The coefficient groups of $\mathbf{E}^*$ are the $\mathbf{E}^n(\mathbf{S})$ $(= \pi_{-n}(\mathbf{E}))$, i.e., $\mathbf{E}^*(\mathbf{S}) = \pi_{-*}(\mathbf{E})$ $(= \pi_*(\mathbf{E})^{\mathrm{OP}})$.]

Remark: Owing to the Brown representability theorem (cf. p. 15–14), every $\mathbf{Z}$-graded cohomology theory on **SPEC** is naturally isomorphic to some $\mathbf{E}^*$, thus **HSPEC** is the represented equivalent of $\mathbf{CT_Z}(\mathbf{SPEC})$.

[Note: Needless to say, $\text{Mor}(\mathbf{E}^*, \mathbf{F}^*) \approx [\mathbf{E}, \mathbf{F}]$.]

EXAMPLE Take $\mathbf{E} = \mathbf{S}$—then the corresponding $\mathbf{Z}$-graded cohomology theory on **SPEC** is called stable cohomotopy, the coefficient groups being $\begin{cases} 0 & (n > 0) \\ \mathbf{Z} & (n = 0) \\ \pi_{-n}^s & (n < 0) \end{cases}$.

[Note: As on p. 14–61, the π_*^s are the stable homotopy groups of spheres.]

LEMMA If $\begin{cases} \pi_n(\mathbf{X}) = 0 & (n < 0) \\ \pi_n(\mathbf{Y}) = 0 & (n > 0) \end{cases}$, then $\pi_0 : [\mathbf{X}, \mathbf{Y}] \to \text{Hom}(\pi_0(\mathbf{X}), \pi_0(\mathbf{Y}))$ is an isomorphism.

EXAMPLE **HSPEC** carries a t-structure (cf. p. 15–49) and the elements of its heart are the Eilenberg-MacLane spectra. An explanation for the terminology is that $\pi_0 : \mathbf{H}(\mathbf{HSPEC}) \to \mathbf{AB}$ is an equivalence of categories. To see this, consider the functor $\mathbf{H} : \mathbf{AB} \to \mathbf{H}(\mathbf{HSPEC})$ that sends $\mathbf{Z}$ to $\tau^{\geq 0}\tau^{\leq 0}\mathbf{S} \approx \tau^{\leq 0}\tau^{\geq 0}\mathbf{S}$, defining $\mathbf{H}(\pi)$ for an arbitrary abelian group π by the exact triangle $\bigvee\limits_j \mathbf{H}(\mathbf{Z}) \to \bigvee\limits_i \mathbf{H}(\mathbf{Z}) \to \mathbf{H}(\pi) \to \bigvee\limits_j \Sigma\mathbf{H}(\mathbf{Z})$, where $0 \to \bigoplus\limits_j \mathbf{Z} \to \bigoplus\limits_i \mathbf{Z} \to \pi \to 0$ is a presentation of π (the lemma implies that $\pi_0 : [\bigvee\limits_j \mathbf{H}(\mathbf{Z}), \bigvee\limits_i \mathbf{H}(\mathbf{Z})] \to \text{Hom}(\bigoplus\limits_j \mathbf{Z}, \bigoplus\limits_i \mathbf{Z})$ is an isomorphism). Therefore $\pi_0(\mathbf{H}(\pi)) = \pi$, $\pi_n(\mathbf{H}(\pi)) = 0$ $(n \neq 0)$ and $[\mathbf{H}(\pi'), \mathbf{H}(\pi'')] = \text{Hom}(\pi', \pi'')$. Example: $[\Sigma^{-1}\mathbf{H}(\pi'), \mathbf{H}(\pi'')] = \text{Ext}(\pi', \pi'')$ but $\text{Ph}(\Sigma^{-1}\mathbf{H}(\pi'), \mathbf{H}(\pi'')) = \text{PurExt}(\pi', \pi'')$ (Christensen-Strickland[†]).

[Note: Given π, $\exists$ an Ω-prespectrum $\mathbf{K}(\pi)$ such that $K(\pi)_q = K(\pi, q)$ (realized as a pointed CW complex with $K(\pi, 0) = \pi$ (discrete topology)). Since $\pi_n(eM\mathbf{K}(\pi)) = \text{colim}\,\pi_{n+q}(K(\pi)_q) = \begin{cases} \pi & (n = 0) \\ 0 & (n > 0) \end{cases}$, $eM\mathbf{K}(\pi)$ "is" $\mathbf{H}(\pi)$ (M the cylinder functor of p. 14–71).]

EXAMPLE Lin[‡] has shown that $\mathbf{S}^*(\mathbf{H}(\mathbf{F}_p)) = 0$, hence $D\mathbf{H}(\mathbf{F}_p)$ is trivial and $[\mathbf{H}(\mathbf{F}_p), \mathbf{K}] = 0$ for all compact $\mathbf{K}$. Therefore the stable cohomotopy $\mathbf{S}^*(\mathbf{H}(\pi))$ of $\mathbf{H}(\pi)$ vanishes if π is torsion (but not in general (consider $\pi = \mathbf{Z}$)).

[Note: $\text{Ph}(\mathbf{H}(\mathbf{F}_p), \mathbf{Y})$ is a vector space over $\mathbf{F}_p$ which is nonzero for some $\mathbf{Y}$. Reason: If the contrary held, then $h_{\mathbf{H}(\mathbf{F}_p)}$ would be projective and since $[\mathbf{H}(\mathbf{F}_p), \mathbf{K}] = 0$ for all compact $\mathbf{K}$, it would follow that $\mathbf{H}(\mathbf{F}_p) = 0$.]

PROPOSITION 1 The graded abelian group $\mathbf{E}^*(\mathbf{E})$ is a graded ring with unit.

[†]

[‡] *Proc. Amer. Math. Soc.* 56 (1976), 291–299.

[Given $\mathbf{f} \in \mathbf{E}^n(\mathbf{E})$, $\mathbf{g} \in \mathbf{E}^m(\mathbf{E})$, let $\mathbf{f} \cdot \mathbf{g} \in \mathbf{E}^{n+m}(\mathbf{E})$ be the composite $\mathbf{E} \xrightarrow{\mathbf{g}} \Sigma^m\mathbf{E} \xrightarrow{\Sigma^m\mathbf{f}} \Sigma^{n+m}\mathbf{E}$ ($\mathrm{id}_{\mathbf{E}} \in \mathbf{E}^0(\mathbf{E})$ thus serves as the unit).]

[Note: $\forall$ **X**, $\mathbf{E}^*(\mathbf{X})$ is a graded left $\mathbf{E}^*(\mathbf{E})$-module.]

EXAMPLE The $\mathbf{F}_p$-algebra $\mathbf{H}(\mathbf{F}_p)^*(\mathbf{H}(\mathbf{F}_p))$ is isomorphic to $\mathcal{A}_p$, the mod p Steenrod algebra.

PROPOSITION 2 Fix a spectrum **E**—then $\forall$ n and $\forall$ **X**, there is a short exact sequence $0 \to \lim^1 \mathbf{E}^{n+q-1}(\mathbf{Q}^\infty X_q) \to \mathbf{E}^n(\mathbf{X}) \to \lim \mathbf{E}^{n+q}(\mathbf{Q}^\infty X_q) \to 0$.

Specialized to the case $n = 0$, the conclusion is that the homomorphism $[\mathbf{X}, \mathbf{E}] \to \lim[X_q, E_q]$ is surjective with kernel $\lim^1[\Sigma X_q, E_q]$.

[Note: This is a recipe for the calculation of morphisms in **HSPEC** by means of morphisms in **HΔ-CG**$_*$.]

A Z-graded cohomology theory E^* on $\mathbf{CW}_*$ is a sequence of cofunctors $E^n : \mathbf{CW}_* \to \mathbf{AB}$ and a sequence of natural isomorphisms $\sigma^n : E^{n+1} \circ \Sigma \to E^n$ such that the E^n convert coproducts into products and satisfy the following conditions.

(Homotopy) If $f, g : X \to Y$ are homotopic, then $E^n(f) = E^n(g) : E^n(Y) \to E^n(X)$ $\forall$ n.

(Exactness) If (X, A, x_0) is a pointed CW pair, then the sequence $E^n(X/A) \to E^n(X) \to E^n(A)$ is exact $\forall$ n.

(Isotropy) If $f : X \to Y$ is a homotopy equivalence, then $E^n(f) : E^n(Y) \to E^n(X)$ is an isomorphism $\forall$ n.

[Note: The homotopy axiom implies that a **Z**-graded cohomology theory on $\mathbf{CW}_*$ passes to $\mathbf{HCW}_*$, thus the isotropy axiom is redundant.]

Example: Given a spectrum **E**, the assignment $X \to \mathbf{E}^n(\mathbf{Q}^\infty X)$ defines a **Z**-graded cohomology theory on $\mathbf{CW}_*$.

$\mathbf{CT_Z}(\mathbf{CW}_*)$ is the category whose objects are the **Z**-graded cohomology theories on $\mathbf{CW}_*$ and whose morphisms $\Xi^* : E^* \to F^*$ are sequences of natural transformations $\Xi^n : E^n \to F^n$ such that the diagram

$$\begin{tikzcd} E^{n+1} \circ \Sigma \arrow[r, "\Xi^{n+1}\Sigma"] \arrow[d, "\sigma^n"'] & F^{n+1} \circ \Sigma \arrow[d, "\sigma^n"] \\ E^n \arrow[r, "\Xi^n"'] & F^n \end{tikzcd}$$

commutes $\forall$ n.

Let E^* be a **Z**-graded cohomology theory on $\mathbf{CW}_*$—then the coefficient groups of E^* are the $E^n(\mathbf{S}^0)$. Example: Reduced singular cohomology with coefficients in an abelian group π is a **Z**-graded cohomology theory on $\mathbf{CW}_*$ whose only nontrivial coefficient group is π itself.

[Note: $E^n(*) = 0 \;\forall\, n$. Proof: $* \approx */*$, so the composite $E^n(*) \to E^n(*) \to E^n(*)$ is both the identity map and the zero map.]

FACT Let π be an abelian group. Suppose that E_1^*, E_2^* are $\mathbf{Z}$-graded cohomology theories on $\mathbf{CW}_*$ such that $E_1^0(\mathbf{S}^0) = \pi$, $E_2^0(\mathbf{S}^0) = \pi$ and $E_1^n(\mathbf{S}^0) = 0$, $E_2^n(\mathbf{S}^0) = 0$ $(n \neq 0)$—then E_1^*, E_2^* are naturally isomorphic.

EXAMPLE The $\mathbf{Z}$-graded cohomology theory on $\mathbf{CW}_*$ determined by $\mathbf{H}(\pi)$ is naturally isomorphic to reduced singular cohomology $\widetilde{H}^*(—;\pi)$.

Notation: Let $T : \mathbf{CW}^2 \to \mathbf{CW}^2$ be the functor that sends (X,A) to $(A,\emptyset)$.
[Note: The lattice of (X,A) is the diagram

$$\begin{tikzcd}
 & & (X,\emptyset) \arrow[dr] & & \\
(\emptyset,\emptyset) \arrow[r] & (A,\emptyset) \arrow[ur] \arrow[dr] & & (X,A) \arrow[r] & (X,X).] \\
 & & (A,A) \arrow[ur] & &
\end{tikzcd}$$

A $\mathbf{Z}$-graded cohomology theory H^* on $\mathbf{CW}^2$ is a sequence of cofunctors $H^n : \mathbf{CW}^2 \to \mathbf{AB}$ and a sequence of natural transformations $d^n : H^{n-1} \circ T \to H^n$ such that the H^n convert coproducts into products and satisfy the following conditions.

(Homotopy) If $f, g : (X,A) \to (Y,B)$ are homotopic, then $H^n(f) = H^n(g) : H^n(Y,B) \to H^n(X,A) \;\forall\, n$.

(Exactness) If (X,A) is a $\mathbf{CW}$ pair, then the sequence $\cdots \to H^{n-1}(A,\emptyset) \xrightarrow{d^n} H^n(X,A) \to H^n(X,\emptyset) \to H^n(A,\emptyset) \xrightarrow{d^{n+1}} H^{n+1}(X,A) \to \cdots$ is exact.

(Excision) If A, B are subcomplexes of X, then the arrow $H^n(A \cup B, B) \to H^n(A, A \cap B)$ is an isomorphism $\forall\, n$.

(Isotropy) If $f : (X,A) \to (Y,B)$ is a homotopy equivalence, then $E^n(f) : H^n(Y,B) \to H^n(X,A)$ is an isomorphism $\forall\, n$.

[Note: The homotopy axiom implies that a $\mathbf{Z}$-graded cohomology theory on $\mathbf{CW}^2$ passes to $\mathbf{HCW}^2$, thus the isotropy axiom is redundant.]

$\mathbf{CT_Z}(\mathbf{CW}^2)$ is the category whose objects are the $\mathbf{Z}$-graded cohomology theories on $\mathbf{CW}^2$ and whose morphisms $\Xi^* : H^* \to G^*$ are sequences of natural transformations $\Xi^n : H^n \to G^n$ such that the diagram

$$\begin{tikzcd}
H^{n-1} \circ T \arrow[r, "\Xi^{n-1}T"] \arrow[d, "d^n"'] & G^{n-1} \circ T \arrow[d, "d^n"] \\
H^n \arrow[r, "\Xi^n"'] & G^n
\end{tikzcd}$$

commutes $\forall\, n$.

PROPOSITION 3 $\mathbf{CT_Z(CW_*)}$ and $\mathbf{CT_Z(CW^2)}$ are equivalent categories.

[On objects, consider the functor $\mathbf{CT_Z(CW_*)} \to \mathbf{CT_Z(CW^2)}$ that sends E^* to H^*, where $H^n(X, A) = E^n(X_+/A_+)$, and the functor $\mathbf{CT_Z(CW^2)} \to \mathbf{CT_Z(CW_*)}$ that sends H^* to E^*, where $E^n(X) = H^n(X, \{x_0\})$.]

[Note: Consult Whitehead[†] for a verification down to the last detail.]

The definition of a $\mathbf{Z}$-graded homology theory E_* on $\mathbf{CW_*}$, $\mathbf{CW^2}$ is dual and, in obvious notation, the categories $\mathbf{HT_Z(CW_*)}$, $\mathbf{HT_Z(CW^2)}$ are equivalent (cf. Proposition 3).

FACT Fix a $\mathbf{Z}$-graded cohomology theory H^* on $\mathbf{CW^2}$. Let (X, A) be a CW pair. Suppose given a sequence $\{X_q\}$ of subcomplexes of X such that $A \subset X_0$, $X_q \subset X_{q+1}$, and $X = \operatorname{colim} X_q$—then $\forall\, n$, there is a short exact sequence $0 \to \lim^1 H^{n-1}(X_q, A) \to H^n(X, A) \to \lim H^n(X_q, A) \to 0$.

[Note: Modulo some additional assumptions on H^*, one can establish a variant involving the finite subcomplexes of X which contain A (Huber-Meier[‡]).]

PROPOSITION 4 Let $\mathbf{E}$ be an Ω-prespectrum—then the prescription $E^n(X) = \begin{cases} [X, E_n] & (n \geq 0) \\ [X, \Omega^{-n} E_0] & (n < 0) \end{cases}$ specifies a $\mathbf{Z}$-graded cohomology theory on $\mathbf{CW_*}$.

[Note: When $\mathbf{E}$ is a spectrum, $E^n(X) = \mathbf{E}^n(\mathbf{Q}^\infty X)$ (cf. p. 17–3).]

PROPOSITION 5 Every $\mathbf{Z}$-graded cohomology theory E^* on $\mathbf{CW_*}$ is represented by an Ω-prespectrum $\mathbf{E}$.

[Let $U : \mathbf{AB} \to \mathbf{SET}$ be the forgetful functor—then $\forall\, n$, $U \circ E^n$ is representable (cf. p. 5–81 ff.): $U \circ E^n(X) \approx [X, E_n]$. And: The E_n $(n \geq 0)$ assemble into an Ω-prespectrum.]

The precise connection between Ω-prespectra, spectra, and $\mathbf{Z}$-graded cohomology theories on $\mathbf{CW_*}$ can be pinned down. Thus let **WPRESPEC** be the category whose objects are the prespectra and whose morphisms $\mathbf{f} : \mathbf{X} \to \mathbf{Y}$ are sequences of pointed continuous functions $f_q : X_q \to Y_q$ such that the diagram

$$\begin{tikzcd} X_q \arrow[r, "f_q"] \arrow[d] & Y_q \arrow[d] \\ \Omega X_{q+1} \arrow[r, "\Omega f_{q+1}"'] & \Omega Y_{q+1} \end{tikzcd}$$

is pointed homotopy commutative $\forall\, q$. Denote by **HWPRESPEC** the localization of **WPRESPEC** at the class of levelwise weak homotopy equivalences (there is no difficulty

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[†] *Elements of Homotopy Theory*, Springer Verlag (1978), 571–600.

[‡] *Comment. Math. Helv.* **53** (1978), 239–257; see also Yosimura, *Osaka J. Math.* **25** (1988), 881–890, and Ohkawa, *Hiroshima Math. J.* **23** (1993), 1–14.

in seeing that this procedure leads to a category). Write HWΩ-PRESPEC for the full subcategory of HWPRESPEC whose objects are the Ω-prespectra—then Mor$(\mathbf{X},\mathbf{Y}) = \lim[X_q, Y_q]$, where the limit is taken with respect to the composites $[X_{q+1}, Y_{q+1}] \to [\Omega X_{q+1}, \Omega Y_{q+1}] \to [X_q, Y_q]$.

FACT HWΩ-PRESPEC is the represented equivalent of $\mathbf{CT_Z}(\mathbf{CW_*})$.

Let HWSPEC be the full subcategory of HWΩ-PRESPEC whose objects are the spectra.

FACT The inclusion HWSPEC$\to$ HWΩ-PRESPEC is an equivalence of categories.

[Consider the functor that on objects sends an Ω-prespectrum $\mathbf{X}$ to $eM\mathbf{X}$ (M as on p. 14–71).]

[Note: If E^* is a Z-graded cohomology theory on $\mathbf{CW_*}$ which is represented by an Ω-prespectrum $\mathbf{E}$, then $eM\mathbf{E}$ is a spectrum which also represents E^*.]

Summary: HSPEC $\leftrightarrow$ $\mathbf{CT_Z}$(SPEC), HWSPEC $\leftrightarrow$ $\mathbf{CT_Z}(\mathbf{CW_*})$ and there is a functor HSPEC $\to$ HWSPEC that on morphisms is the arrow $[\mathbf{X},\mathbf{Y}] \to \lim[X_q, Y_q]$. Accordingly, every Z-graded cohomology theory on $\mathbf{CW_*}$ lifts to a Z-graded cohomology theory on SPEC and every morphism of Z-graded cohomology theories on $\mathbf{CW_*}$ lifts to a morphism of Z-graded cohomology theories on SPEC (but not uniquely due to the potential nonvanishing of $\lim^1[\Sigma X_q, Y_q]$ (cf. Proposition 2)).

A Z-graded homology theory E_* on **SPEC** is a sequence of exact functors E_n : **HSPEC** $\to$ **AB** and a sequence of natural isomorphisms $\sigma_n : E_n \to E_{n+1} \circ \Sigma$ such that the E_n convert coproducts into direct sums. $\mathbf{HT_Z}$(**SPEC**) is the category whose objects are the **Z**-graded homology theories on **SPEC** and whose morphisms $\Xi_* : E_* \to F_*$ are sequences of natural transformations $\Xi_n : E_n \to F_n$ such that the diagram

$$\begin{tikzcd}
E_n \arrow[r, "\Xi_n"] \arrow[d, "\sigma_n"'] & F_n \arrow[d, "\sigma_n"] \\
E_{n+1} \circ \Sigma \arrow[r, "\Xi_{n+1}\Sigma"'] & F_{n+1} \circ \Sigma
\end{tikzcd}$$

commutes $\forall\, n$.

Definition: The **Z**-graded homology theory $\mathbf{E_*}$ on **SPEC** attached to a spectrum **E** is given by $\mathbf{E}_n(\mathbf{X}) = \pi_n(\mathbf{E} \wedge \mathbf{X})$.

[Note: The coefficient groups of $\mathbf{E_*}$ are the $\mathbf{E}_n(\mathbf{S})$ $(= \pi_n(\mathbf{E}))$, i.e., $\mathbf{E_*}(\mathbf{S}) = \pi_*(\mathbf{E})$.]

Remark: Because **HSPEC** admits Adams representability, every **Z**-graded homology theory on **SPEC** is naturally isomorphic to some $\mathbf{E_*}$ (cf. §15, Proposition 38), thus **HSPEC/Ph** (cf. p. 15–22) is the represented equivalent of $\mathbf{HT_Z}$(**SPEC**).

[Note: Here Mor$(\mathbf{E_*}, \mathbf{F_*}) \approx [\mathbf{E}, \mathbf{F}]/\mathrm{Ph}(\mathbf{E}, \mathbf{F})$.]

EXAMPLE Take $\mathbf{E} = \mathbf{S}$—then the corresponding $\mathbf{Z}$-graded homology theory on **SPEC** is called stable homotopy, the coefficient groups being $\begin{cases} \pi_n^s & (n > 0) \\ \mathbf{Z} & (n = 0) \\ 0 & (n < 0) \end{cases}$.

EXAMPLE For any two spectra $\mathbf{E}, \mathbf{F}$, the arrow $\pi_*(\mathbf{E}) \otimes \pi_*(\mathbf{F}) \otimes \mathbf{Q} \rightarrow \pi_*(\mathbf{E} \wedge \mathbf{F}) \otimes \mathbf{Q}$ is an isomorphism.

[Fix $\mathbf{E}$ and let $\mathbf{F}$ vary—then the arrow $\pi_*(\mathbf{E}) \otimes \pi_*(—) \otimes \mathbf{Q} \rightarrow \pi_*(\mathbf{E} \wedge —) \otimes \mathbf{Q}$ is a morphism of $\mathbf{Z}$-graded homology theories on **SPEC**. But $\pi_0^s(\mathbf{S}) = \mathbf{Z}$ and $\pi_n^s(\mathbf{S})$ is finite if $n > 0$ (cf. p. 5–44), hence $\pi_*(\mathbf{E}) \otimes \pi_*(\mathbf{S}) \otimes \mathbf{Q} \approx \pi_*(\mathbf{E} \wedge \mathbf{S}) \otimes \mathbf{Q}$.]

PROPOSITION 6 Let $\begin{cases} \mathbf{E} \\ \mathbf{F} \end{cases}, \begin{cases} \mathbf{X} \\ \mathbf{Y} \end{cases}$ be spectra—then there is an external product $\mathbf{E}^*(\mathbf{X}) \otimes \mathbf{F}^*(\mathbf{Y}) \rightarrow (\mathbf{E} \wedge \mathbf{F})^*(\mathbf{X} \wedge \mathbf{Y})$ in cohomology.

[Work with the arrow $\hom(\mathbf{X}, \mathbf{E}) \wedge \hom(\mathbf{Y}, \mathbf{F}) \rightarrow \hom(\mathbf{X} \wedge \mathbf{Y}, \mathbf{E} \wedge \mathbf{F})$.]

PROPOSITION 7 Let $\begin{cases} \mathbf{E} \\ \mathbf{F} \end{cases}, \begin{cases} \mathbf{X} \\ \mathbf{Y} \end{cases}$ be spectra—then there is an external product $\mathbf{E}_*(\mathbf{X}) \otimes \mathbf{F}_*(\mathbf{Y}) \rightarrow (\mathbf{E} \wedge \mathbf{F})_*(\mathbf{X} \wedge \mathbf{Y})$ in homology.

[Work with the arrow $\mathbf{E} \wedge \mathbf{X} \wedge \mathbf{F} \wedge \mathbf{Y} \rightarrow \mathbf{E} \wedge \mathbf{F} \wedge \mathbf{X} \wedge \mathbf{Y}$.]

PROPOSITION 8 Let $\begin{cases} \mathbf{E} \\ \mathbf{F} \end{cases}, \begin{cases} \mathbf{X} \\ \mathbf{Y} \end{cases}$ be spectra—then there is an external slant product $\mathbf{E}^*(\mathbf{X} \wedge \mathbf{Y}) \otimes \mathbf{F}_*(\mathbf{X}) \xrightarrow{/} (\mathbf{E} \wedge \mathbf{F})^*(\mathbf{Y})$.

[Use the commutative diagram

$$\begin{tikzcd}
\hom(\mathbf{X} \wedge \mathbf{Y}, \mathbf{E}) \wedge \mathbf{F} \wedge \mathbf{X} \arrow[r, "/"] \arrow[d] & \hom(\mathbf{Y}, \mathbf{E} \wedge \mathbf{F}) \\
\hom(\mathbf{X}, \hom(\mathbf{Y}, \mathbf{E})) \wedge \mathbf{X} \wedge \mathbf{F} \arrow[r] & \hom(\mathbf{Y}, \mathbf{E}) \wedge \mathbf{F} \arrow[u]
\end{tikzcd}$$

.]

PROPOSITION 9 Let $\begin{cases} \mathbf{E} \\ \mathbf{F} \end{cases}, \begin{cases} \mathbf{X} \\ \mathbf{Y} \end{cases}$ be spectra—then there is an external slant product $\mathbf{E}_*(\mathbf{X} \wedge \mathbf{Y}) \otimes \mathbf{F}^*(\mathbf{X}) \xrightarrow{\backslash} (\mathbf{E} \wedge \mathbf{F})_*(\mathbf{Y})$.

[Use the commutative diagram

$$\begin{tikzcd}
\mathbf{E} \wedge \mathbf{X} \wedge \mathbf{Y} \wedge \hom(\mathbf{X}, \mathbf{F}) \arrow[r, "\backslash"] \arrow[d] & \mathbf{E} \wedge \mathbf{F} \wedge \mathbf{Y} \\
\mathbf{E} \wedge \hom(\mathbf{X}, \mathbf{F}) \wedge \mathbf{X} \wedge \mathbf{Y} \arrow[ur] &
\end{tikzcd}$$

.]

The external products are morphisms of graded abelian groups but this is not the case of the slant products. Explicated: $\mathbf{E}^n(\mathbf{X}\wedge\mathbf{Y})\otimes\mathbf{F}_m(\mathbf{X})\xrightarrow{/}(\mathbf{E}\wedge\mathbf{F})^{n-m}(\mathbf{Y})$ and $E_n(\mathbf{X}\wedge\mathbf{Y})\otimes\mathbf{F}^m(\mathbf{X})\xrightarrow{\backslash}(\mathbf{E}\wedge\mathbf{F})_{n-m}(\mathbf{Y})$, thus to get a morphism of graded abelian groups one must give $\mathbf{F}_*(\mathbf{X})$ and $\mathbf{F}^*(\mathbf{X})$ the opposite gradings.

A ring spectrum is a ring object in **HSPEC**. Example: $\mathbf{S}$ is a commutative ring spectrum and every spectrum is an $\mathbf{S}$-module.

EXAMPLE Let $\mathbf{k}$ be a commutative ring with unit—then $\mathbf{H}(\mathbf{k})$ is a commutative ring spectrum and for any $\mathbf{k}$-module M, $\mathbf{H}(M)$ is an $\mathbf{H}(\mathbf{k})$-module.

EXAMPLE McClure[†] has shown that $\mathbf{KU}$ is a commutative ring spectrum. The homotopy $\pi_*(\mathbf{KU})$ of $\mathbf{KU}$ has period 2 and $\pi_0(\mathbf{KU})=\mathbf{Z}$, $\pi_1(\mathbf{KU})=0$. In addition, there exists a multiplicatively invertible generator $\mathbf{b_U}\in\pi_2(\mathbf{KU})\approx\mathbf{Z}$ inducing the homotopy periodicity and as a graded ring, $\pi_*(\mathbf{KU})\approx\mathbf{Z}[\mathbf{b_U},\mathbf{b_U^{-1}}]$.

[Note: $\mathbf{KO}$ is also a commutative ring spectrum.]

EXAMPLE For any X in $\mathbf{\Delta}$-$\mathbf{CG}_*$, $(\Omega X)_+$ $(=\Omega X\amalg *)$ is wellpointed, $\mathbf{Q}^\infty((\Omega X)_+)$ is a ring spectrum, and $\pi_0(\Omega^\infty\Sigma^\infty(\Omega X)_+)\approx\mathbf{Z}[\pi_1(X)]$ (as rings).

[To define the product, note that $\mathbf{Q}^\infty(\Omega X)_+\wedge\mathbf{Q}^\infty(\Omega X)_+\approx\mathbf{Q}^\infty((\Omega X)_+\#_k(\Omega X)_+)$ (cf. p. 16–29), which is isomorphic to $\mathbf{Q}^\infty((\Omega X\times_k\Omega X)_+)$.]

FACT If $\mathbf{E}$ is a connective ring spectrum, then $\mathrm{Hom}(\pi_0(\mathbf{E}),\pi_0(\mathbf{E}))\approx[\mathbf{E},\mathbf{H}(\pi_0(\mathbf{E}))]$ and the arrow $\mathbf{E}\to\mathbf{H}(\pi_0(\mathbf{E}))$ realizing the identity $\pi_0(\mathbf{E})\to\pi_0(\mathbf{E})$ is a morphism of ring spectra.

FACT If $\mathbf{E}$ is a ring spectrum and $\mathbf{e}(=\tau^{\leq 0}\mathbf{E})$ is its connective cover, then $\mathbf{e}$ admits a unique ring spectrum structure such that the arrow $\mathbf{e}\to\mathbf{E}$ is a morphism of ring spectra.

If $\mathbf{E}$ is a ring spectrum and $\mathbf{F}$ is an $\mathbf{E}$-module, then the products figuring in the preceding propositions can be made "internal" through $\mathbf{E}\wedge\mathbf{F}\to\mathbf{F}$.

Example: Take $\mathbf{E}=\mathbf{F}$ and fix an $\mathbf{X}$—then Proposition 8 furnishes an arrow $\mathbf{E}^*(\mathbf{X})\otimes\mathbf{E}_*(\mathbf{X})\xrightarrow{/}(\mathbf{E}\wedge\mathbf{E})^*(\mathbf{S})\to\mathbf{E}^*(\mathbf{S})=\pi_{-*}(\mathbf{E})$ and Proposition 9 furnishes an arrow $\mathbf{E}_*(\mathbf{X})\otimes\mathbf{E}^*(\mathbf{X})\xrightarrow{\backslash}(\mathbf{E}\wedge\mathbf{E})_*(\mathbf{S})\to\mathbf{E}_*(\mathbf{S})=\pi_*(\mathbf{E})$.

EXAMPLE Let $\mathbf{E}$ be a ring spectrum—then for spectra $\mathbf{F}$ & $\mathbf{X}$, the Hurewicz homomorphism $\mathbf{F}_*(\mathbf{X})\to(\mathbf{E}\wedge\mathbf{F})_*(\mathbf{X})$ is defined by the arrow $\mathbf{F}_n(\mathbf{X})=\pi_n(\mathbf{F}\wedge\mathbf{X})\approx\pi_n(\mathbf{S}\wedge\mathbf{F}\wedge\mathbf{X})\to\pi_n(\mathbf{E}\wedge\mathbf{F}\wedge$

[†] *SLN* **1176** (1986), 241–242.

X) = $(\mathbf{E}\wedge\mathbf{F})_n(\mathbf{X})$ and the Boardman homomorphism $\mathbf{F}^*(\mathbf{X})\to(\mathbf{E}\wedge\mathbf{F})^*(\mathbf{X})$ is defined by the arrow $\mathbf{F}^n(\mathbf{X})=[\mathbf{X},\Sigma^n\mathbf{F}]\approx[\mathbf{X},\Sigma^n(\mathbf{S}\wedge\mathbf{F})]\to[\mathbf{X},\Sigma^n(\mathbf{E}\wedge\mathbf{F})]=(\mathbf{E}\wedge\mathbf{F})^n(\mathbf{X})$. Assuming that both **E** and **F** are ring spectra, the commutative diagram

$$\begin{tikzcd}
\mathbf{F}_n(\mathbf{X})\otimes\mathbf{F}^m(\mathbf{X}) \arrow[r] \arrow[d] & \pi_{n-m}(\mathbf{F}) \arrow[d] \\
(\mathbf{E}\wedge\mathbf{F})_n(\mathbf{X})\otimes(\mathbf{E}\wedge\mathbf{F})^m(\mathbf{X}) \arrow[r] & \pi_{n-m}(\mathbf{E}\wedge\mathbf{F})
\end{tikzcd}$$

serves to relate the two.

[Note: In particular, there are arrows $\begin{cases}\mathbf{S}_*(\mathbf{X})\to\mathbf{E}_*(\mathbf{X})\\ \mathbf{S}^*(\mathbf{X})\to\mathbf{E}^*(\mathbf{X})\end{cases}$.]

If **E** is a ring spectrum and **F** is an **E**-module, then $\forall$ **X**, $\begin{cases}\mathbf{F}^*(\mathbf{X})\\ \mathbf{F}_*(\mathbf{X})\end{cases}$ is a graded $\begin{cases}\mathbf{E}^*(\mathbf{S})\text{-module}\\ \mathbf{E}_*(\mathbf{S})\text{-module}\end{cases}$ (cf. Propositions 6 and 7).

[Note: The structure is on the left. Observe, however, that $\begin{cases}\mathbf{E}^*(\mathbf{X})\\ \mathbf{E}_*(\mathbf{X})\end{cases}$ is a graded left and right $\begin{cases}\mathbf{E}^*(\mathbf{S})\text{-module}\\ \mathbf{E}_*(\mathbf{S})\text{-module}\end{cases}$, in fact, $\begin{cases}\mathbf{E}^*(\mathbf{X})\\ \mathbf{E}_*(\mathbf{X})\end{cases}$ is a graded $\begin{cases}\mathbf{E}^*(\mathbf{S})\text{-bimodule}\\ \mathbf{E}_*(\mathbf{S})\text{-bimodule}\end{cases}$.]

In view of the associativity of the operations, the arrows $\begin{cases}\mathbf{E}^*(\mathbf{X})\otimes\mathbf{E}^*(\mathbf{Y})\to\mathbf{E}^*(\mathbf{X}\wedge\mathbf{Y})\\ \mathbf{E}_*(\mathbf{X})\otimes\mathbf{E}_*(\mathbf{Y})\to\mathbf{E}_*(\mathbf{X}\wedge\mathbf{Y})\end{cases}$ pass to the quotient, thereby giving arrows $\begin{cases}\mathbf{E}^*(\mathbf{X})\otimes_{\mathbf{E}^*(\mathbf{S})}\mathbf{E}^*(\mathbf{Y})\to\mathbf{E}^*(\mathbf{X}\wedge\mathbf{Y})\\ \mathbf{E}_*(\mathbf{X})\otimes_{\mathbf{E}_*(\mathbf{S})}\mathbf{E}_*(\mathbf{Y})\to\mathbf{E}_*(\mathbf{X}\wedge\mathbf{Y})\end{cases}$.

PROPOSITION 10 Suppose that **E** is a ring spectrum. Let $\begin{cases}\mathbf{X}\\ \mathbf{Y}\end{cases}$ be spectra. Assume: Either $\mathbf{E}_*(\mathbf{X})$, as a graded right $\mathbf{E}_*(\mathbf{S})$-module, is flat or $\mathbf{E}_*(\mathbf{Y})$, as a graded left $\mathbf{E}_*(\mathbf{S})$-module, is flat—then the arrow $\mathbf{E}_*(\mathbf{X})\otimes_{\mathbf{E}_*(\mathbf{S})}\mathbf{E}_*(\mathbf{Y})\to\mathbf{E}_*(\mathbf{X}\wedge\mathbf{Y})$ is an isomorphism.

[The situation being symmetric, take **Y** fixed and $\mathbf{E}_*(\mathbf{Y})$ flat—then the arrow $\mathbf{E}_*(—)\otimes_{\mathbf{E}_*(\mathbf{S})}\mathbf{E}_*(\mathbf{Y})\to\mathbf{E}_*(—\wedge\mathbf{Y})$ is a morphism of **Z**-graded homology theories on **SPEC**. But $\mathbf{E}_*(\mathbf{S})\otimes_{\mathbf{E}_*(\mathbf{S})}\mathbf{E}_*(\mathbf{Y})\approx\mathbf{E}_*(\mathbf{S}\wedge\mathbf{Y})$.]

FACT Let **E** be a ring spectrum, **F** an **E**-module. Assume: $\pi_*(\mathbf{F})$, as a graded left $\pi_*(\mathbf{E})$-module, is flat—then $\forall$ **X**, the arrow $\mathbf{E}_*(\mathbf{X})\otimes_{\pi_*(\mathbf{E})}\pi_*(\mathbf{F})\to\mathbf{F}_*(\mathbf{X})$ is an isomorphism.

Notation: Given an abelian group π, put $\mathbf{H}_*(\mathbf{X};\pi)=\mathbf{H}(\pi)_*(\mathbf{X})$ and $\mathbf{H}^*(\mathbf{X};\pi)=\mathbf{H}(\pi)^*(\mathbf{X})$.

EXAMPLE Let A be a PID, M an A-module—then $\forall$ **X**, there is an exact sequence $0\to\mathbf{H}_n(\mathbf{X};A)\otimes_A M\to\mathbf{H}_n(\mathbf{X};M)\to\operatorname{Tor}^A(\mathbf{H}_{n-1}(X;A),M)\to 0$.

[Since A is a PID, the projective dimension of M is ≤ 1, so $\exists$ an exact sequence $0\to Q\to P\to M\to 0$, where P and Q are projective, hence flat. Applying the above result then gives $\mathbf{H}_*(\mathbf{X};A)\otimes_A P\approx\mathbf{H}_*(\mathbf{X};P)$

and $\mathbf{H}_*(\mathbf{X};A)\otimes_A Q \approx \mathbf{H}_*(\mathbf{X};Q)$. On the other hand, the exact triangle $\mathbf{H}(Q)\to\mathbf{H}(P)\to\mathbf{H}(M)\to\Sigma\mathbf{H}(Q)$ leads to an exact sequence $\mathbf{H}_n(\mathbf{X};Q)\to\mathbf{H}_n(\mathbf{X};P)\to\mathbf{H}_n(\mathbf{X};M)\to\mathbf{H}_{n-1}(\mathbf{X};Q)\to\mathbf{H}_{n-1}(\mathbf{X};P)$.]

[Note: Under the same hypotheses, there is an exact sequence $0\to\mathrm{Ext}_A(\mathbf{H}_{n-1}(\mathbf{X};A),M)\to\mathbf{H}^n(\mathbf{X};M)\to\mathrm{Hom}_A(\mathbf{H}_n(\mathbf{X};A),M)\to 0$.]

FACT Suppose that A is a PID—then $\forall\,\mathbf{X}$, $\mathbf{X}\wedge\mathbf{H(A)}\approx\bigvee\limits_n\Sigma^n\mathbf{H}(G_n)$, where $G_n=\mathbf{H}_n(\mathbf{X};A)$.

[Here $\bigvee\limits_n\Sigma^n\mathbf{H}(G_n)\approx\prod\limits_n\Sigma^n\mathbf{H}(G_n)$ (cf. p. 15–17 ff.), thus it suffices to specify arrows $\mathbf{f}_n:\mathbf{X}\wedge\mathbf{H}(A)\to\Sigma^n\mathbf{H}(G_n)$ such that $\pi_n(\mathbf{f}_n)$ is an isomorphism $\forall\, n$.]

EXAMPLE Let A be a PID—then $\forall\,\mathbf{X},\mathbf{Y}$ & $\forall\, i,j$, there is an exact sequence $0\to\mathbf{H}_i(\mathbf{X};A)\otimes_A\mathbf{H}_j(\mathbf{Y};A)\to\mathbf{H}_i(\mathbf{X};\mathbf{H}_j(\mathbf{Y};A))\to\mathrm{Tor}^A(\mathbf{H}_{i-1}(\mathbf{X};A),\,\mathbf{H}_j(\mathbf{Y};A))\to 0$. Now sum over all $(i,j):i+j=k$. Setting aside the flanking terms and putting $G_j=\mathbf{H}_j(\mathbf{Y};A)$, the middle term assumes the form $\bigoplus\limits_{i+j=k}\mathbf{H}_i(\mathbf{X};\mathbf{H}_j(\mathbf{Y};A))=\bigoplus\limits_j\pi_k(\mathbf{X}\wedge\Sigma^j\mathbf{H}(G_j))=\pi_k(\mathbf{X}\wedge\bigvee\limits_j\Sigma^j\mathbf{H}(G_j))=\pi_k(\mathbf{X}\wedge\mathbf{Y}\wedge\mathbf{H}(A))=\mathbf{H}_k(\mathbf{X}\wedge\mathbf{Y};A)$.

In a category $\mathbf{C}$ with pushouts, one has the notion of an internal cocategory (or a cocategory object) (cf. p. 0–42), which can be specialized to the notion of an internal cogroupoid (or a cogroupoid object). Definition: Let $\mathbf{k}$ be a commutative ring with unit—then a graded Hopf algebroid over $\mathbf{k}$ is a cogroupoid object in the category of graded commutative $\mathbf{k}$-algebras with unit. So, a graded Hopf algebroid over $\mathbf{k}$ consists of a pair (A,Γ) of graded commutative $\mathbf{k}$-algebras with unit and morphisms $\eta_R:A\to\Gamma$ (right unit="cosource"), $\eta_L:A\to\Gamma$ (left unit="cotarget"), $\epsilon:\Gamma\to A$ (augmentation="coidentity"), $\Delta:\Gamma\to\Gamma\otimes_A\Gamma$ (diagonal="cocomposition"), $c:\Gamma\to\Gamma$ (conjugation="coinversion") satisfying the dual of the usual category theoretic relations (cf. infra). Therefore (A,Γ) attaches to a graded commutative $\mathbf{k}$-algebra T with unit a groupoid $\mathbf{G}_T$, where $\mathrm{Ob}\,\mathbf{G}_T=\mathrm{Hom}(A,T)$ and $\mathrm{Mor}\,\mathbf{G}_T=\mathrm{Hom}(\Gamma,T)$. Example: $(\mathbf{k},\mathbf{k})$ is a graded Hopf algebroid over $\mathbf{k}$ (trivial grading).

[Note: When $A=\mathbf{k}$ and $\eta_L=\eta_R$, Γ is a graded commutative Hopf algebra over $\mathbf{k}$ or still, a cogroup object in the category of graded commutative $\mathbf{k}$-algebras with unit.]

Remark: Graded Hopf algebroids over $\mathbf{k}$ can be organized into a (large) double category (Borceux[†]).

There is a coequalizer diagram $\Gamma\otimes_{\mathbf{k}}A\otimes_{\mathbf{k}}\Gamma\overset{\eta_R\otimes\mathrm{id}_\Gamma}{\underset{\mathrm{id}_\Gamma\otimes\eta_L}{\rightrightarrows}}\Gamma\otimes_{\mathbf{k}}\Gamma\to\Gamma\otimes_A\Gamma$ and

$$\begin{tikzcd} A \arrow[r, "\eta_R"] \arrow[d, "\eta_L"'] & \Gamma \arrow[d, "\mathrm{in}_R"] \\ \Gamma \arrow[r, "\mathrm{in}_L"'] & \Gamma\otimes_A\Gamma \end{tikzcd}$$

is a pushout square.

[†] *Handbook of Categorical Algebra* 1, Cambridge University Press (1994), 327–328.

[Note: Tacitly, one uses η_R to equip Γ with the structure of a graded right A-module and η_L to equip Γ with the structure of a graded left A-module.]

As for η_R, η_L, ϵ, Δ, and c, they must have the following properties: $\epsilon \circ \eta_R = \mathrm{id}_A = \epsilon \circ \eta_L$, $\Delta \circ \eta_R = \mathrm{in}_L \circ \eta_R$, $\Delta \circ \eta_L = \mathrm{in}_R \circ \eta_L$, $(\mathrm{id}_\Gamma \otimes \epsilon) \circ \Delta = \mathrm{id}_\Gamma$, $(\epsilon \otimes \mathrm{id}_\Gamma) \circ \Delta = \mathrm{id}_\Gamma$, $(\mathrm{id}_\Gamma \otimes \Delta) \circ \Delta = (\Delta \otimes \mathrm{id}_\Gamma) \circ \Delta$, $c \circ \eta_R = \eta_L$, $c \circ \eta_L = \eta_R$, $(c \otimes \mathrm{id}_\Gamma) \circ \Delta = \eta_R \circ \epsilon$, and $(\mathrm{id}_\Gamma \otimes c) \circ \Delta = \eta_L \circ \epsilon$.

[Note: The formulas relating c to the other arrows are the duals of those on p. 13–36 (the role of χ in the groupoid object situation is played here by c). Corollaries: (1) $c \circ c = \mathrm{id}_\Gamma$; (2) $\epsilon \circ c = \epsilon$.]

EXAMPLE The dual of the mod p Steenrod algebra is isomorphic to $\mathbf{H}(\mathbf{F}_p)_*(\mathbf{H}(\mathbf{F}_p))$, a graded commutative Hopf algebra over $\mathbf{F}_p$. One has $\mathbf{H}(\mathbf{F}_2)_*(\mathbf{H}(\mathbf{F}_2)) \approx \mathbf{F}_2[\xi_1, \xi_2, \ldots]$, where $|\xi_k| = 2^k - 1$ and $\Delta(\xi_k) = \sum_{i=0}^{k} \xi_{k-i}^{2^i} \otimes \xi_i$, and for $p > 2$, $\mathbf{H}(\mathbf{F}_p)_*(\mathbf{H}(\mathbf{F}_p)) \approx \mathbf{F}_p[\xi_1, \xi_2, \ldots] \otimes_{\mathbf{F}_p} \bigwedge(\tau_0, \tau_1, \ldots)$, where $|\xi_k| = 2(p^k - 1)$, $|\tau_k| = 2p^k - 1$, and $\Delta(\xi_k) = \sum_{i=0}^{k} \xi_{k-i}^{p^i} \otimes \xi_i$, $\Delta(\tau_k) = \tau_k \otimes 1 + \sum_{i=0}^{k} \xi_{k-i}^{p^i} \otimes \tau_i$. The unit and augmentation are isomorphisms in degree 0 and the conjugation c is given recursively by $\sum_{i=0}^{k} \xi_{k-i}^{p^i} c(\xi_i) = 0$ $(k > 0)$ and $\tau_k + \sum_{i=0}^{k} \xi_{k-i}^{p^i} c(\tau_i) = 0$ $(k \geq 0)$.

[Note: In the above, it is understood that $\xi_0 = 1$.]

PROPOSITION 11 Suppose that $\mathbf{E}$ is a ring spectrum. Assume: $\mathbf{E}$ is commutative and $\mathbf{E}_*(\mathbf{E})$, as a graded right $\mathbf{E}_*(\mathbf{S})$-module, is flat—then the pair $(\mathbf{E}_*(\mathbf{S}), \mathbf{E}_*(\mathbf{E}))$ is a graded Hopf algebroid over $\mathbf{Z}$.

[$\mathbf{E}_*(\mathbf{E})$ is a graded commutative $\mathbf{Z}$-algebra with unit. Proof: The product is defined by $\mathbf{E}_*(\mathbf{E}) \otimes \mathbf{E}_*(\mathbf{E}) \to (\mathbf{E} \wedge \mathbf{E})_*(\mathbf{E} \wedge \mathbf{E}) \to \mathbf{E}_*(\mathbf{E} \wedge \mathbf{E}) \to \mathbf{E}_*(\mathbf{E})$ and the unit $\mathbf{Z} \to \mathbf{E}_0(\mathbf{E})$ is defined by sending 1 to the arrow $\mathbf{S} = \mathbf{S} \wedge \mathbf{S} \to \mathbf{E} \wedge \mathbf{E}$. This said, let $\begin{cases} \eta_R : \mathbf{E}_*(\mathbf{S}) \approx \pi_*(\mathbf{S} \wedge \mathbf{E}) \to \pi_*(\mathbf{E} \wedge \mathbf{E}) = \mathbf{E}_*(\mathbf{E}) \\ \eta_L : \mathbf{E}_*(\mathbf{S}) \approx \pi_*(\mathbf{E} \wedge \mathbf{S}) \to \pi_*(\mathbf{E} \wedge \mathbf{E}) = \mathbf{E}_*(\mathbf{E}) \end{cases}$ and $\epsilon : \mathbf{E}_*(\mathbf{E}) = \pi_*(\mathbf{E} \wedge \mathbf{E}) \to \pi_*(\mathbf{E}) = \mathbf{E}_*(\mathbf{S})$. Next, take for Δ the composite $\mathbf{E}_*(\mathbf{E}) = \pi_*(\mathbf{E} \wedge \mathbf{E}) \approx \pi_*(\mathbf{E} \wedge \mathbf{S} \wedge \mathbf{E}) \to \pi_*(\mathbf{E} \wedge \mathbf{E} \wedge \mathbf{E}) = \mathbf{E}_*(\mathbf{E} \wedge \mathbf{E}) \approx \mathbf{E}_*(\mathbf{E}) \otimes_{\mathbf{E}_*(\mathbf{S})} \mathbf{E}_*(\mathbf{E})$ (cf. Proposition 10). Finally, $c : \mathbf{E}_*(\mathbf{E}) = \pi_*(\mathbf{E} \wedge \mathbf{E}) \to \pi_*(\mathbf{E} \wedge \mathbf{E}) = \mathbf{E}_*(\mathbf{E})$ is induced by the interchange $\mathrm{T} : \mathbf{E} \wedge \mathbf{E} \to \mathbf{E} \wedge \mathbf{E}$.]

[Note: Due to the presence of c and the relations $\begin{cases} c \circ \eta_R = \eta_L \\ c \circ \eta_L = \eta_R \end{cases}$, $\mathbf{E}_*(\mathbf{E})$, as a graded right $\mathbf{E}_*(\mathbf{S})$-module, is flat iff $\mathbf{E}_*(\mathbf{E})$, as a graded left $\mathbf{E}_*(\mathbf{S})$-module, is flat (the $\mathbf{E}_*(\mathbf{S})$-module structures on $\mathbf{E}_*(\mathbf{E})$ per η_R and η_L are the same as those introduced on p. 17–9). Example: The flatness assumption is met if $\mathbf{E} \wedge \mathbf{E} \approx \bigvee_i \Sigma^{n_i} \mathbf{E}$ (isomorphism of $\mathbf{E}$-modules) (for then $\pi_*(\mathbf{E} \wedge \mathbf{E}) \approx \bigoplus_i \pi_{*-n_i}(\mathbf{E})$, thus is a graded free $\pi_*(\mathbf{E})$-module).]

Tied to the definitions are various diagrams and a complete proof of Proposition 11 entails checking that these diagrams commute, which is straightforward if tedious (a discussion can be found in Adams[†]).

EXAMPLE $\mathbf{KU}_*(\mathbf{KU})$ is a graded free $\mathbf{KU}_*(\mathbf{S})$-module (Adams-Clarke[‡]), thus the hypotheses of Proposition 11 are met in this case.

[Note: The structure of $\mathbf{KU}_*(\mathbf{KU})$ has been worked out by Adams-Harris-Switzer[‖].]

Given a graded Hopf algebroid (A,Γ) over $\mathbf{k}$, a (left) (A,Γ)-comodule is a graded left A-module M equipped with a morphism $M \to \Gamma \otimes_A M$ of graded left A-modules such that

$$\begin{tikzcd} M \arrow[r] \arrow[d] & \Gamma \otimes_A M \arrow[d] \\ \Gamma \otimes_A M \arrow[r] & \Gamma \otimes_A \Gamma \otimes_A M \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} M \arrow[r] \arrow[dr] & \Gamma \otimes_A M \arrow[d] \\ & A \otimes_A M \end{tikzcd} \quad \text{commute.}$$

PROPOSITION 12 Suppose that $\mathbf{E}$ is a ring spectrum. Assume: $\mathbf{E}$ is commutative and $\mathbf{E}_*(\mathbf{E})$, as a graded right $\mathbf{E}_*(\mathbf{S})$-module, is flat—then $\forall\, \mathbf{X}$, $\mathbf{E}_*(\mathbf{X})$ is an $(\mathbf{E}_*(\mathbf{S}), \mathbf{E}_*(\mathbf{E}))$-comodule.

[The arrow $\mathbf{E}_*(\mathbf{X}) \to \mathbf{E}_*(\mathbf{E}) \otimes_{\mathbf{E}_*(\mathbf{S})} \mathbf{E}_*(\mathbf{X})$ is the composite $\mathbf{E}_*(\mathbf{X}) = \pi_*(\mathbf{E} \wedge \mathbf{X}) \approx \pi_*(\mathbf{E} \wedge \mathbf{S} \wedge \mathbf{X}) \to \pi_*(\mathbf{E} \wedge \mathbf{E} \wedge \mathbf{X}) = \mathbf{E}_*(\mathbf{E} \wedge \mathbf{X}) \approx \mathbf{E}_*(\mathbf{E}) \otimes_{\mathbf{E}_*(\mathbf{S})} \mathbf{E}_*(\mathbf{X})$ (cf. Proposition 10).]

Rappel: A spectrum $\mathbf{E}$ defines a $\mathbf{Z}$-graded cohomology theory E^* on $\mathbf{CW}_*$ (cf. Proposition 4) and $\forall\, X$ in $\mathbf{CW}_*$, $E^n(X_+) \approx E^n(X) \oplus E^n(\mathbf{S}^0)$.

[Note: When $\mathbf{E}$ is a ring spectrum, there is a cup product $\cup$, viz. the composite $E^*(X) \otimes E^*(X) \to E^*(X \#_k X) \to E^*(X)$, where $X \to X \#_k X$ is the reduced diagonal. Therefore $E^*(X)$ is a graded ring and $E^*(X_+)$ is a graded ring with unit (both are graded commutative if $\mathbf{E}$ is commutative).]

Let $\mathbf{E}$ be a commutative ring spectrum—then $\mathbf{E}$ is said to be complex orientable if $\exists$ an element $x_{\mathbf{E}} \in E^2(\mathbf{P}^\infty(\mathbf{C}))$ with the property that the arrow of restriction $E^2(\mathbf{P}^\infty(\mathbf{C})) \to E^2(\mathbf{P}^1(\mathbf{C})) \approx \pi_0(\mathbf{E})$ sends $x_{\mathbf{E}}$ to the unit $\mathbf{S} \to \mathbf{E}$ of $\mathbf{E}$. One calls $x_{\mathbf{E}}$ a complex orientation of $\mathbf{E}$.

[Note: $\pi_0(\mathbf{E}) = [\mathbf{S}, \mathbf{E}] \approx [\mathbf{S}^0, E_0] \approx [\mathbf{S}^0, \Omega^2 E_2] \approx [\Sigma^2 \mathbf{S}^0, E_2] = E^2(\mathbf{S}^2)$ and $\mathbf{S}^2 \approx \mathbf{P}^1(\mathbf{C})$.]

Remark: Identify $\pi_0(\mathbf{E}) = [\mathbf{S}, \mathbf{E}]$ with $[\mathbf{Q}_{2n}^\infty \mathbf{S}^{2n}, \mathbf{E}] \approx [\mathbf{S}^{2n}, E_{2n}]$ and let top : $\mathbf{P}^n(\mathbf{C}) \to \mathbf{S}^{2n} (= \mathbf{P}^n(\mathbf{C})/\mathbf{P}^{n-1}(\mathbf{C}))$ be the top cell map—then the arrow of restriction $E^{2n}(\mathbf{P}^\infty(\mathbf{C}))$

[†] *SLN* **99** (1969), 56–71.

[‡] *Illinois J. Math.* **21** (1977), 826–829.

[‖] *Proc. London Math. Soc.* **23** (1971), 385–408.

Remark: Identify $\pi_0(\mathbf{E}) = [\mathbf{S}, \mathbf{E}]$ with $[\mathbf{Q}_{2n}^{\infty}\mathbf{S}^{2n}, \mathbf{E}] \approx [\mathbf{S}^{2n}, E_{2n}]$ and let top : $\mathbf{P}^n(\mathbf{C}) \rightarrow \mathbf{S}^{2n}(= \mathbf{P}^n(\mathbf{C})/\mathbf{P}^{n-1}(\mathbf{C}))$ be the top cell map—then the arrow of restriction $E^{2n}(\mathbf{P}^{\infty}(\mathbf{C})) \rightarrow E^{2n}(\mathbf{P}^n(\mathbf{C}))$ sends $x_{\mathbf{E}}^n$ to the image of the unit of $\mathbf{E}$ under the precomposition arrow $[\mathbf{S}^{2n}, E_{2n}] \xrightarrow{\text{top}^*} [\mathbf{P}^n(\mathbf{C}), E_{2n}]$.

[The diagram

$$\begin{tikzcd}
\mathbf{P}^n(\mathbf{C}) \arrow[r] \arrow[d, "\text{top}"'] & \mathbf{P}^n(\mathbf{C})\#_k \cdots \#_k \mathbf{P}^n(\mathbf{C}) \\
\mathbf{S}^{2n} \arrow[r] & \mathbf{S}^2\#_k \cdots \#_k \mathbf{S}^2 \arrow[u]
\end{tikzcd}$$

is pointed homotopy commutative.]

Example: Let A be a commutative ring with unit—then $\mathbf{H}(A)$ is complex orientable. [Recall that $H^*(\mathbf{P}^{\infty}(\mathbf{C}); A) \approx A[x]$, $|x| = 2$.]

PROPOSITION 13 Suppose that $\mathbf{E}$ is a commutative ring spectrum. Assume: $\mathbf{E}$ is complex orientable with complex orientation $x_{\mathbf{E}}$—then $E^*(\mathbf{P}^{\infty}(\mathbf{C})_+) \approx \mathbf{E}^*(\mathbf{S})[[x_{\mathbf{E}}]]$.

[Note: $\mathbf{E}^*(\mathbf{S})[[x_{\mathbf{E}}]]$ is the graded $\mathbf{E}^*(\mathbf{S})$-algebra of formal power series in $x_{\mathbf{E}}$ ($|x_{\mathbf{E}}| = 2$). So: A typical element in $\mathbf{E}^q(\mathbf{S})[[x_{\mathbf{E}}]]$ has the form $\sum_{i=0}^{\infty} \lambda_i x_{\mathbf{E}}^i$, where $\lambda_i \in \mathbf{E}^{q-2i}(\mathbf{S})$.]

PROPOSITION 14 Suppose that $\mathbf{E}$ is a commutative ring spectrum. Assume: $\mathbf{E}$ is complex orientable with complex orientation $x_{\mathbf{E}}$—then $E^*((\mathbf{P}^{\infty}(\mathbf{C}) \times_k \mathbf{P}^{\infty}(\mathbf{C}))_+) \approx \mathbf{E}^*(\mathbf{S})[[x_{\mathbf{E}} \otimes 1, 1 \otimes x_{\mathbf{E}}]]$.

Cole[†] has given a proof of these propositions which does not involve the Atiyah-Hirzebruch spectral sequence.

[Note: The method is to show from first principles that there are splittings $\mathbf{E} \wedge \mathbf{P}^n(\mathbf{C}) = \bigvee_{i=1}^{n} \Sigma^{2i}\mathbf{E}$, $\text{HOM}(\mathbf{P}^n(\mathbf{C}), \mathbf{E}) \approx \prod_{i=1}^{n} \Omega^{2i}\mathbf{E}$ in $\mathbf{E}$-$\mathbf{MOD}$.]

EXAMPLE If $\mathbf{E}$ is complex orientable, then $E_*(\mathbf{P}^{\infty}(\mathbf{C})_+)$ is a graded free $\mathbf{E}_*(\mathbf{S})$-module and $E_*(\mathbf{P}^{\infty}(\mathbf{C})_+) \otimes_{\mathbf{E}_*(\mathbf{S})} E_*(\mathbf{P}^{\infty}(\mathbf{C})_+) \approx E_*(\mathbf{P}^{\infty}(\mathbf{C})_+ \#_k \mathbf{P}^{\infty}(\mathbf{C})_+)$ (cf. Proposition 10).

The standard reference for the theory of formal groups is Hazewinkel[‡]. There the reader can look up the proofs but to establish notation, I shall review some of the definitions.

Let A be a graded commutative ring with unit. Consider $A[[x, y]]$, where $\begin{cases} |x| = 2 \\ |y| = 2 \end{cases}$—then a formal group law (FGL) over A is an element $F(x, y) \in A[[x, y]]$ of the form $x + y +$

[†] Ph.D. Thesis, University of Chicago, Chicago (1996).

[‡] *Formal Groups and Applications*, Academic Press (1978); see also Ravenel, *Complex Cobordism and Stable Homotopy Groups of Spheres*, Academic Press (1986), 354–379.

$\sum\limits_{i,j\geq 1} a_{ij}x^iy^j$, where $a_{ij} \in A_{2-2i-2j}$, such that $F(x,F(y,z)) = F(F(x,y),z)$ (associativity) and $F(x,y) = F(y,x)$ (commutativity).

[Note: In algebra, one does not usually work in the graded setting, the standing assumption being that A is a commutative ring with unit (as, e.g., in Hazewinkel). Of course, if A is a graded commutative ring with unit, then A_{even} $(= \bigoplus\limits_n A_{2n})$ is a commutative ring with unit and every FGL over A is a FGL over A_{even}. Example: $F(x,y) = x+y+uxy$ $(u \in A_{-2})$ is a FGL over A, hence over A_{even}, while $F(x,y) = x+y+xy$ is not a FGL over A (but is a FGL over A_{even}).]

Notation: Write $F(x,y) = x +_F y$, so $\begin{cases} x +_F 0 = x \\ 0 +_F y = y \end{cases}$, $x +_F (y +_F z) = (x +_F y) +_F z$, and $x +_F y = y +_F x$.

Definition: An element $\phi(x) = \sum\limits_{i\geq 1} \phi_i x^i \in A[[x]]$ $(|x| = 2)$ is said to be homogeneous if $\phi_i \in A_{2-2i}$ $\forall\, i$.

FACT If $F(x,y)$ is a FGL over A, then there is a unique homogeneous element $\iota(x) \in A[[x]]$ such that $x +_F \iota(x) = 0 = \iota(x) +_F x$.

[There exist unique homogeneous elements $\begin{cases} \iota_L(x) \\ \iota_R(x) \end{cases} \in A[[x]]$ such that $\begin{cases} \iota_L(x) +_F x = 0 \\ x +_F \iota_R(x) = 0 \end{cases}$, thus $\iota_L(x) = \iota_L(x) +_F 0 = \iota_L(x) +_F (x +_F \iota_R(x)) = (\iota_L(x) +_F x) +_F \iota_R(x) = 0 +_F \iota_R(x) = \iota_R(x)$ and one can take $\iota(x) = \iota_L(x) = \iota_R(x)$.]

PROPOSITION 15 Let $m : (\mathbf{P}^\infty(\mathbf{C}) \times_k \mathbf{P}^\infty(\mathbf{C}))_+ \to \mathbf{P}^\infty(\mathbf{C})_+$ be the multiplication classifying the tensor product of complex line bundles—then $\forall$ complex orientable $\mathbf{E}$, $F_{\mathbf{E}} = m^*(x_{\mathbf{E}})$ is a FGL over $\mathbf{E}^*(\mathbf{S})$.

Example: The FGL attached to $\mathbf{H}(\mathbf{k})$ by Proposition 15, where $\mathbf{k}$ is a commutative ring with unit, is the "additive" FGL, viz. $x + y$.

EXAMPLE: KU is complex orientable and the associated FGL is $x + y + \mathbf{b_U}xy$ (cf. p. 17–8).

Let A be a graded commutative ring with unit. Suppose that F, G are formal group laws over A—then a homomorphism $\phi : F \to G$ is a homogeneous element $\phi \in A[[x]]$ such that $\phi(x +_F y) = \phi(x) +_G \phi(y)$, i.e., $\phi(F(x,y)) = G(\phi(x),\phi(y))$. A homomorphism $\phi : F \to G$ is an isomorphism if $\phi'(0)$ (the coefficient of x) belongs to $A_0^\times$. An isomorphism $\phi : F \to G$ is a strict isomorphism if $\phi'(0) = 1$.

[Note: A homomorphism $\phi : F \to G$ is an isomorphism iff $\exists$ a homomorphism $\psi : G \to F$ such that $\phi(\psi(x)) = x = \psi(\phi(x))$.]

FGL_A is the set of formal group laws over A and $\mathbf{FGL}_A$ is the category whose objects are the elements of FGL_A and whose morphisms are the homomorphisms.

[Note: If $f : A \to A'$ is a homomorphism of graded commutative rings with unit, then f induces a functor $f_* : \mathbf{FGL}_A \to \mathbf{FGL}_{A'}$ (on objects, $f_*F(x,y) = x + y + \sum\limits_{i,j\geq 1} f(a_{ij})x^iy^j$, and on morphisms, $f_*\phi(x) = \sum\limits_{i\geq 1} f(\phi_i)x^i$.]

FACT If $\mathbf{E}$ is complex orientable and if $x'_{\mathbf{E}}$, $x''_{\mathbf{E}}$ are two complex orientations of $\mathbf{E}$, then the associated formal group laws $F'_{\mathbf{E}}$, $F''_{\mathbf{E}}$ over $\mathbf{E}^*(\mathbf{S})$ are strictly isomorphic.

Let A be a graded commutative ring with unit. Write IPS_A for the set of homogeneous elements ϕ in $A[[x]]$ such that $\phi'(0) = 1$—then IPS_A is a group under composition, functorially in A.

Notation: $B = \mathbf{Z}[b_1, b_2, \ldots]$, where $|b_i| = -2i$.

PROPOSITION 16 B is a graded Hopf algebra over $\mathbf{Z}$.

[In fact, $\text{Hom}(B, A) \approx \text{IPS}_A$, so B is a cogroup object in the category of graded commutative rings with unit.]

Remark: IPS_A operates to the left on FGL_A, viz. $(\phi, F) \to \phi \cdot F = F_\phi$, where $F_\phi(x,y) = \phi(F(\phi^{-1}(x), \phi^{-1}(y)))$.

Let A be a graded commutative ring with unit—then A is said to be graded coherent if each finitely generated graded ideal of A is finitely presented. Example: A graded noetherian $\Rightarrow$ A graded coherent.

[Note: $\pi_*(\mathbf{S})$ is not graded coherent (Cohen†).]

Remark: Suppose that A is graded coherent—then a finitely generated graded A-module M is finitely presented iff it and its finitely generated graded submodules are finitely presented.

EXAMPLE Let $\mathbf{k}$ be a commutative ring with unit. Consider $\mathbf{k}[x_1, x_2, \ldots]$, where $|x_i| = -2i$—then $\mathbf{k}[x_1, x_2, \ldots]$ is not graded noetherian but is graded coherent provided that $\mathbf{k}$ is noetherian.

LAZARD'S THEOREM The functor from the category of graded commutative rings with unit to the category of sets which sends A to FGL_A is representable. Accordingly, there is a graded commutative ring L with unit and a FGL F_L over L such that $\forall$ A and $\forall$ $F \in \text{FGL}_A$, $\exists! f \in \text{Hom}(L, A) : f_*F_L = F$.

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† *Comment. Math. Helv.* **44** (1969), 217–228.

[Note: The structure of L can be determined, viz. $L = \mathbf{Z}[x_1, x_2, \ldots]$, where $|x_i| = -2i$, hence L is graded coherent (cf. supra).]

The mere existence of L is a formality. Thus fix indeterminates t_{ij} of degree $2 - 2i - 2j$ and put $\mu(x, y) = x + y + \sum_{i,j \geq 1} t_{ij} x^i y^j$. Define homogeneous polynomials p_{ijk} in the t_{ij} by writing $\mu(x, \mu(y, z)) - \mu(\mu(x, y), z) = \sum_{i,j,k \geq 1} p_{ijk} x^i y^j z^k$—then $L = \mathbf{Z}[t_{ij} : i, j \geq 1]/I$, where I is the graded ideal generated by the $t_{ij} - t_{ji}$ and the p_{ijk}, and μ induces a FGL F_L over L having the universal property in question.

Determining the structure of L is more difficult and depends in part on the following construction. Fix indeterminates b_i of degree $-2i$ and consider, as above, $B = \mathbf{Z}[b_1, b_2, \ldots]$. Let $\exp x = x + \sum_{i \geq 1} b_i x^{i+1} \in B[[x]]$ ($|x| = 2$) and let $\log x$ be its inverse (so $\exp(\log x) = x = \log(\exp x)$)—then $F_B(x, y) = \exp(\log x + \log y)$ is a FGL over B and the homomorphism $L \to B$ classifying F_B is injective.

FACT If $A \to A'$ is a surjective map of graded commutative rings with unit, then any FGL over A' lifts to a FGL over A.

Put $LB = L[b_1, b_2, \ldots]$, where b_i is an indeterminate of degree $-2i$ ($\Rightarrow LB = L \otimes_{\mathbf{Z}} \mathbf{Z}[b_1, b_2, \ldots] = L \otimes_{\mathbf{Z}} B$).

PROPOSITION 17 The pair (L, LB) is a graded Hopf algebroid over $\mathbf{Z}$.

[Let A be a graded commutative ring with unit. Denoting by $\mathbf{G}_A$ the groupoid whose objects are the formal group laws over A and whose morphisms are the strict isomorphisms, the functor from the category of graded commutative rings with unit to the category of groupoids which sends A to $\mathbf{G}_A^{\mathrm{OP}}$ is represented by (L, LB). For Lazard gives $\mathrm{Hom}(L, A) \leftrightarrow \mathrm{FGL}_A = \mathrm{Ob}\, \mathbf{G}_A$ ($= \mathrm{Ob}\, \mathbf{G}_A^{\mathrm{OP}}$) and this identifies the objects. Turning to the morphisms, suppose that $f \in \mathrm{Hom}(LB, A)$. Put $F = (f|L)_* F_L$ and $\phi(x) = x + \sum_{i \geq 1} f(b_i) x^{i+1}$—then $\phi^{\mathrm{OP}} : G \to F$ is a strict isomorphism, where $G(x, y) = \phi(F(\phi^{-1}(x), \phi^{-1}(y)))$.]

[Note: η_L is the inclusion $L \to LB$ but there is no simple explicit formula for η_R. However, using definitions only, one can write down explicit formulas for ϵ, Δ, and c.]

A groupoid $\mathbf{G}$ is said to be split if there exists a group G and a left G-set Y such that $\mathbf{G}$ is isomorphic to tran$Y$, the translation category of Y (cf. p. 0–45).

Example: Take $G = \mathrm{IPS}_A$, $Y = \mathrm{FGL}_A$—then the translation category of FGL_A is isomorphic to $\mathbf{G}_A$, i.e., $\mathbf{G}_A$ is split.

I shall now review the theory of MU, referring the reader to Adams[†] for the details and further information.

[†] *Stable Homotopy and Generalized Homology*, University of Chicago (1974), 32–93.

Let $\mathbf{G}_n(\mathbf{C}^\infty)$ be the grassmannian of complex n-dimensional subspaces of $\mathbf{C}^\infty$, γ_n the canonical complex n-plane bundle over $\mathbf{G}_n(\mathbf{C}^\infty)$. Put $MU(n) = T(\gamma_n)$, the Thom space of γ_n—then $i^*(\gamma_{n+1}) = \gamma_n \oplus \underline{\mathbf{C}}\,(\mathbf{G}_n(\mathbf{C}^\infty) \xrightarrow{i} \mathbf{G}_{n+1}(\mathbf{C}^\infty))$ and $T(\gamma_n \oplus \underline{\mathbf{C}}) \approx \Sigma^2 T(\gamma_n) = \Sigma^2 MU(n)$, so there is an arrow $\Sigma^2 MU(n) \to MU(n+1)$. The prescription $X_{2n} = MU(n)$, $X_{2n+1} = \Sigma MU(n)$ thus defines a separated prespectrum $\mathbf{X}$ and by definition, $\mathbf{MU} = e\mathbf{X}$.

EXAMPLE $\mathbf{MU}$ and $\mathbf{KU}$ are connected by the fact that the arrow $\mathbf{MU}_*(\mathbf{X}) \otimes_{\mathbf{MU}_*(\mathbf{S})} \mathbf{KU}_*(\mathbf{S}) \to \mathbf{KU}_*(\mathbf{X})$ induced by the Todd genus is an isomorphism of graded $\mathbf{KU}_*(\mathbf{S})$-modules for all $\mathbf{X}$ (Conner-Floyd[†]).

MU THEOREM $\mathbf{MU}$ is a commutative ring spectrum with complex orientation $x_{\mathbf{MU}}$. And: The map $L \to \mathbf{MU}^*(\mathbf{S})$ classifying $F_{\mathbf{MU}}$ is an isomorphism of graded commutative rings with unit.

[Note: The pair $(\mathbf{MU}_*(\mathbf{S}), \mathbf{MU}_*(\mathbf{MU}))$ satisfies the hypotheses of Proposition 11 ($\mathbf{MU}_*(\mathbf{MU})$ is a graded free $\mathbf{MU}_*(\mathbf{S})$-module), hence is a graded Hopf algebroid over $\mathbf{Z}$. As such, it is isomorphic to $(L, LB)^{\mathrm{OP}}$ (reversal of gradings).]

An arrow $\mathbf{f} : \Sigma^n \mathbf{X} \to \mathbf{X}$ is said to be composition nilpotent if $\exists\ k$ such that the composite $\mathbf{f} \circ \Sigma^n \mathbf{f} \circ \cdots \circ \Sigma^{(k-1)n}\mathbf{f} : \Sigma^{kn}\mathbf{X} \xrightarrow{\mathbf{f}^k} \mathbf{X}$ vanishes. Example: Take $\mathbf{X}$ compact—then $\mathbf{f}$ is composition nilpotent iff $\mathbf{f}^{-1}\mathbf{X} = 0$ (cf. p. 15–46).

[Note: The same terminology is used in the category of graded abelian groups. Example: Take $\mathbf{X}$ compact and let $\mathbf{E}$ be a ring spectrum—then $\mathbf{E}_*(\mathbf{f})$ is composition nilpotent iff $\mathbf{E} \wedge \mathbf{f}^{-1}\mathbf{X} = 0$.]

An arrow $\mathbf{f} : \mathbf{X} \to \mathbf{Y}$ is said to be smash nilpotent if $\exists\ k$ such that the k-fold smash product $\mathbf{f}^{(k)} : \mathbf{X}^{(k)} \to \mathbf{Y}^{(k)}$ vanishes. Example: $\mathbf{f} : \mathbf{S} \to \mathbf{Y}$ is smash nilpotent iff $\mathbf{Y}_{\mathbf{f}}^{(\infty)} = 0$ (cf. p. 15–46).

FACT (MU Nilpotence Technology) Let $\mathbf{E}$ be a ring spectrum and consider the Hurewicz homomorphism $\mathbf{S}_*(\mathbf{E}) \to \mathbf{MU}_*(\mathbf{E})$ (cf. p. 17–8 ff.)—then the homogeneous elements of its kernel are nilpotent (Devinatz-Hopkins-Smith[‡]).

Application: If $\mathbf{X}$ is compact and if $\mathbf{f} : \Sigma^n \mathbf{X} \to \mathbf{X}$ is an arrow such that $\mathbf{MU}_*(\mathbf{f}) = 0$, then $\mathbf{f}$ is composition nilpotent.

[$\mathbf{MU}_*(\mathbf{f}) = 0 \Rightarrow \mathbf{MU} \wedge \mathbf{f}^{-1}\mathbf{X} = 0 \Rightarrow\ \exists\ k : \Sigma^{kn}\mathbf{X} \xrightarrow{\mathbf{f}^k} \mathbf{X} \to \mathbf{MU} \wedge \mathbf{X}$ vanishes. Calling $\bar{\mathbf{f}}^k \in \pi_{kn}(D\mathbf{X} \wedge \mathbf{X})$ the adjoint of $\mathbf{f}^k$ and noting that $D\mathbf{X} \wedge \mathbf{X}$ is a ring spectrum (cf. p. 15–44) ($\mathbf{X}$ compact

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[†] *The Relation of Cobordism to K-Theories*, Springer Verlag (1966); see also Hopkins-Hovey, *Math Zeit.* **210** (1992), 181–196.

[‡] *Ann. of Math.* **128** (1988), 207–241.

$\Rightarrow$ $\mathbf{X}$ dualizable), $\mathbf{MU}$ nilpotent technology secures a d such that $(\mathbf{S}^{kn})^{(d)} \xrightarrow{\bar{\mathbf{f}}^k \wedge \cdots \wedge \bar{\mathbf{f}}^k} (D\mathbf{X} \wedge \mathbf{X})^{(d)} \rightarrow D\mathbf{X} \wedge \mathbf{X}$ is trivial, so $\Sigma^{dkn}\mathbf{X} \xrightarrow{\mathbf{f}^{dk}} \mathbf{X}$ is trivial.]

[Note: The compactness assumption on $\mathbf{X}$ cannot be dropped (Ravenel[†]).]

A corollary to the foregoing is that every element of positive degree in $\pi_*(\mathbf{S})$ is nilpotent. Proof: The elements of $\pi_n(\mathbf{S})$ $(n > 0)$ are torsion and $\mathbf{MU}_*(\mathbf{S})$ has no torsion.

Application: If $\mathbf{X}$ is compact and if $\mathbf{f} : \mathbf{X} \rightarrow \mathbf{Y}$ is an arrow such that $\mathrm{id}_{\mathbf{MU}} \wedge \mathbf{f} = 0$, then $\mathbf{f}$ is smash nilpotent.

[Suppose that $\bar{\mathbf{f}} : \mathbf{S} \rightarrow D\mathbf{X} \wedge \mathbf{Y}$ corresponds to $\mathbf{f}$ under the identifications $[\mathbf{X}, \mathbf{Y}] \approx [\mathbf{S} \wedge \mathbf{X}, \mathbf{Y}] \approx [\mathbf{S}, \hom(\mathbf{X}, \mathbf{Y})] \approx [\mathbf{S}, D\mathbf{X} \wedge \mathbf{Y}]$ ($\mathbf{X}$ compact $\Rightarrow$ $\mathbf{X}$ dualizable)—then $\mathbf{f}$ is smash nilpotent iff $\bar{\mathbf{f}}$ is smash nilpotent and $\mathrm{id}_{\mathbf{MU}} \wedge \mathbf{f} = 0$ iff $\mathrm{id}_{\mathbf{MU}} \wedge \bar{\mathbf{f}} = 0$. This allows one to reduce to the case when $\mathbf{X} = \mathbf{S}$, the assumption becoming that the composite $\mathbf{S} \xrightarrow{\mathbf{f}} \mathbf{Y} \rightarrow \mathbf{MU} \wedge \mathbf{Y}$ vanishes. Put $\mathbf{EY} = \bigvee_{i \geq 0} \mathbf{Y}^{(i)}$ $(\mathbf{Y}^{(0)} = \mathbf{S})$ and view $\mathbf{EY}$ as a ring spectrum with multiplication given by concatenation. $\mathbf{MU}$ nilpotence technology now implies that the element of $\pi_*(\mathbf{EY})$ determined by $\mathbf{f}$ is nilpotent.]

FACT Suppose that $\mathbf{E}$ is complex orientable—then the set of complex orientations of $\mathbf{E}$ is in a one-to-one correspondence with the set of morphisms $\mathbf{MU} \rightarrow \mathbf{E}$ of ring spectra.

[Note: If $\mathbf{f} : \mathbf{MU} \rightarrow \mathbf{E}$ corresponds to $x_{\mathbf{E}}$, then $\mathbf{f}_* F_{\mathbf{MU}} = F_{\mathbf{E}}$.]

Notation: Given $F \in \mathrm{FGL}_A$, define homogeneous elements $[n]_F(x) \in A[[x]]$ by $[1]_F(x) = x$, $[n]_F(x) = x +_F [n-1]_F(x)$ $(n > 1)$, and for each prime p, write $[p]_F(x) = v_0 x + \cdots + v_1 x^p + \cdots + v_n x^{p^n} + \cdots (\Rightarrow v_0 = p, v_n \in A_{2(1-p^n)})$.

Specialized to $A = \mathbf{MU}^*(\mathbf{S})$, $F = F_{\mathbf{MU}}$, the v_n can and will be construed as elements of $\mathbf{MU}_*(\mathbf{S})$.

EXACT FUNCTOR THEOREM Let M be a graded left $\mathbf{MU}_*(\mathbf{S})$-module—then $\mathbf{MU}_*(—) \otimes_{\mathbf{MU}_*(\mathbf{S})} M$ is a $\mathbf{Z}$-graded homology theory on $\mathbf{SPEC}$ if $\forall\, p \in \mathbf{\Pi}$, the sequence $\{v_n\}$ is M-regular, i.e., multiplication by $v_0 = p$ on M and by v_n on $M/(v_0 M + \cdots + v_{n-1} M)$ for $n \geq 1$ is injective.

[Note: This result is due to Landweber[‡].]

Remark: Since $\mathbf{HSPEC}/\mathbf{Ph}$ is the represented equivalent of $\mathbf{HT}_{\mathbf{Z}}(\mathbf{SPEC})$ (cf. p. 17–6), the exact functor theorem implies that $\exists$ a spectrum $\mathbf{EM}$ such that $\mathbf{EM}_*(\mathbf{X}) \approx \mathbf{MU}_*(\mathbf{X}) \otimes_{\mathbf{MU}_*(\mathbf{S})} M$ $\forall\, \mathbf{X} (\Rightarrow \mathbf{EM}_*(\mathbf{S}) \approx M)$.

[†] *Amer. J. Math.* **106** (1984), 351–414 (cf. 400–401).

[‡] *Amer. J. Math.* **98** (1976), 591–610; see also Rudyak, *Math. Notes* **40** (1986), 562–569.

[Note: **EM** is unique up to isomorphism (but is not necessarily unique up to unique isomorphism). To force the latter, it suffices that M be countable and concentrated in even degrees (Franke[†]).]

Remark: Franke (ibid.) has shown that if R is a countable graded $\mathbf{MU}_*(\mathbf{S})$-algebra with unit which, when viewed as a graded left $\mathbf{MU}_*(\mathbf{S})$-module, satisfies the hypotheses of the exact functor theorem, then $\mathbf{E}R$ is a ring spectrum (commutative if R is graded commutative).

Suppose given an $F \in \mathrm{FGL}_A$—then the homomorphism $f : \mathbf{MU}^*(\mathbf{S}) \to A$ classifying F serves to equip A^{OP} with the structure of a graded left $\mathbf{MU}_*(\mathbf{S})$-module and the $f(v_n) \in A$ are the $v_n \in A$ per F.

EXAMPLE Take $A = \mathbf{Q}$ (trivial grading) and let $f : \mathbf{MU}^*(\mathbf{S}) \to \mathbf{Q}$ classify the FGL $x + y$—then $\forall\, p \in \Pi$, $f(v_0) = p$ is a unit and $f(v_n) = 0$ $(n \geq 1)$. Therefore the sequence $\{f(v_n)\}$ is $\mathbf{Q}$-regular and the spectrum produced by the exact functor theorem is $\mathbf{H}(\mathbf{Q})$.

[Note: This would not work if $\mathbf{Q}$ were replaced by $\mathbf{Z}$.]

EXAMPLE Take $A = \mathbf{Z}[u, u^{-1}]$ $(|u| = -2)$ and let $f : \mathbf{MU}^*(\mathbf{S}) \to \mathbf{Z}[u, u^{-1}]$ classify the FGL $x + y + uxy$. Here $f(v_0) = p$, $f(v_1) = u^{p-1}$, $f(v_n) = 0$ $(n > 1)$, thus the conditions of the exact functor theorem are met and the representing spectrum is $\mathbf{KU}$ (cf. p. 17–14).

Let A be a divisible abelian group—then $\mathrm{Hom}([\mathbf{S}, —, A)$ is an exact cofunctor which converts coproducts into products, thus is representable (cf. p. 15–17) ($\mathbf{S}$ is compact). So: $\exists$ a spectrum $\mathbf{S}[A]$ such that $\forall\, \mathbf{X}$, $[\mathbf{X}, \mathbf{S}[A]] \approx \mathrm{Hom}(\pi_0(\mathbf{X}), A)$.. Definition: The A-dual $\nabla_A \mathbf{X}$ of $\mathbf{X}$ is $\mathrm{hom}(\mathbf{X}, \mathbf{S}[A])$.

Observation: There is a canonical arrow $\mathbf{X} \to \nabla_A^2 \mathbf{X}$, and $\forall\, n$, $\mathbf{S}[A]^n(\mathbf{X}) \approx \mathrm{Hom}(\pi_n(\mathbf{X}), A)$.

PROPOSITION 18 There are no nonzero phantom maps to $\nabla_A \mathbf{X}$.

[Written out, the claim is that $\mathrm{Ph}(\mathbf{Y}, \nabla_A \mathbf{X}) = 0\ \forall\, \mathbf{Y}$, i.e., that the kernel of the arrow $[\mathbf{Y}, \nabla_A \mathbf{X}] \to \mathrm{Nat}(h_{\mathbf{Y}}, h_{\nabla_A \mathbf{X}})$ is trivial. But $h_{\mathbf{Y}} = \mathrm{colim}_{\mathbf{Y}}\, h_{\mathbf{L}} \Rightarrow \mathrm{Nat}(h_{\mathbf{Y}}, h_{\nabla_A \mathbf{X}}) \approx \lim_{\mathbf{Y}} \mathrm{Nat}(h_{\mathbf{L}}, h_{\nabla_A \mathbf{X}}) \approx \lim_{\mathbf{Y}} [\mathbf{L}, \nabla_A \mathbf{X}]$. On the other hand, there is an arrow $\mathrm{Hom}(\pi_0(\mathbf{Y} \wedge \mathbf{X}), A) \to \lim_{\mathbf{Y}} \mathrm{Hom}(\pi_0(\mathbf{L} \wedge \mathbf{X}), A)$ and a commutative diagram

$$\begin{tikzcd} {[\mathbf{Y}, \nabla_A \mathbf{X}]} \arrow[r] \arrow[d] & {} \\ {\lim_{\mathbf{Y}} [\mathbf{L}, \nabla_A \mathbf{X}]} \arrow[r] & {} \end{tikzcd}$$

[†] *Math. Nachr.* **158** (1992), 43–65.

$$\begin{tikzcd}\text{Hom}(\pi_0(\mathbf{Y} \wedge \mathbf{X}), A) \arrow[d] \\ \lim_{\mathbf{Y}} \text{Hom}(\pi_0(\mathbf{L} \wedge \mathbf{X}), A)\end{tikzcd}$$

. The horizontal arrows are isomorphisms, as is the vertical arrow on the right (cf. §15, Proposition 18 and subsequent remark). Therefore the vertical arrow on the left is an isomorphism, hence $\text{Ph}(\mathbf{Y}, \nabla_A \mathbf{X}) = 0$.]

EXAMPLE Take $A = \mathbf{Q}/\mathbf{Z}$—then $\nabla_{\mathbf{Q}/\mathbf{Z}}\mathbf{X}$ is the Brown-Comenetz[†] dual of $\mathbf{X}$ and, thanks to the Pontryagin duality theorem, the canonical arrow $\mathbf{X} \to \nabla^2_{\mathbf{Q}/\mathbf{Z}}\mathbf{X}$ is an isomorphism if the homotopy groups of $\mathbf{X}$ are finite. Example: $\nabla_{\mathbf{Q}/\mathbf{Z}}\mathbf{H}(\mathbf{Z}/p\mathbf{Z}) \approx \mathbf{H}(\mathbf{Z}/p\mathbf{Z})$.

[Note: In homotopy, the canonical arrow $\pi_n(\mathbf{X}) \to \pi_n(\nabla^2_{\mathbf{Q}/\mathbf{Z}}\mathbf{X})$ is the inclusion of $\pi_n(\mathbf{X})$ into its double dual per $\mathbf{Q}/\mathbf{Z}$ and if $\pi_n(\mathbf{X})$ is finitely generated, then $\pi_n(\nabla^2_{\mathbf{Q}/\mathbf{Z}}\mathbf{X}) = \text{pro}\,\pi_n(\mathbf{X})$, the profinite completion of $\pi_n(\mathbf{X})$.]

FACT Take $\mathbf{C} = \mathbf{HSPEC}$—then $\forall\, \mathbf{X}$, $h_{\nabla_{\mathbf{Q}/\mathbf{Z}}\mathbf{X}}$ is an injective object of $[(\text{cpt}\,\mathbf{C})^{\text{OP}}, \mathbf{AB}]^+$.

[It follows from the definitions (and Yoneda) that this is true if $\mathbf{X}$ is compact. In general, there are compact objects $\mathbf{K}_i$ and an arrow $\nabla_{\mathbf{Q}/\mathbf{Z}}\mathbf{X} \xrightarrow{\mathbf{f}} \prod_i \nabla_{\mathbf{Q}/\mathbf{Z}}\mathbf{K}_i$ such that $h_{\mathbf{f}}$ is a monomorphism ($\mathbf{Q}/\mathbf{Z}$ is an injective coseparator in $\mathbf{AB}$). Consider now the exact triangle $\mathbf{Y} \xrightarrow{\boldsymbol{\phi}} \nabla_{\mathbf{Q}/\mathbf{Z}}\mathbf{X} \xrightarrow{\mathbf{f}} \prod_i \nabla_{\mathbf{Q}/\mathbf{Z}}\mathbf{K}_i \to \Sigma\mathbf{Y}$. Since $\mathbf{f} \circ \boldsymbol{\phi} = 0$ (cf. §15, Proposition 3), $h_{\mathbf{f}} \circ h_{\boldsymbol{\phi}} = 0 \Rightarrow h_{\boldsymbol{\phi}} = 0 \Rightarrow \boldsymbol{\phi} \in \text{Ph}(\mathbf{Y}, \nabla_{\mathbf{Q}/\mathbf{Z}}\mathbf{X}) \Rightarrow \boldsymbol{\phi} = 0$ (cf. Proposition 18), so $\nabla_{\mathbf{Q}/\mathbf{Z}}\mathbf{X}$ is a retract of $\prod_i \nabla_{\mathbf{Q}/\mathbf{Z}}\mathbf{K}_i$.]

EXAMPLE Define $\mathbf{S}[\mathbf{Z}]$ by the exact triangle $\mathbf{S}[\mathbf{Z}] \xrightarrow{\mathbf{u}} \mathbf{S}[\mathbf{Q}] \xrightarrow{\mathbf{v}} \mathbf{S}[\mathbf{Q}/\mathbf{Z}] \xrightarrow{\mathbf{w}} \Sigma\mathbf{S}[\mathbf{Z}]$, where $\mathbf{v}_* : \pi_0(\mathbf{S}[\mathbf{Q}]) \to \pi_0(\mathbf{S}[\mathbf{Q}/\mathbf{Z}])$ corresponds to the projection $\mathbf{Q} \to \mathbf{Q}/\mathbf{Z}$—then $\pi_0(\mathbf{S}[\mathbf{Z}]) \approx \mathbf{Z}$ and $\mathbf{u}_* : \pi_0(\mathbf{S}[\mathbf{Z}]) \to \pi_0(\mathbf{S}[\mathbf{Q}])$ corresponds to the inclusion $\mathbf{Z} \to \mathbf{Q}$. Definition: The Anderson dual $\nabla_{\mathbf{Z}}\mathbf{X}$ of $\mathbf{X}$ is $\text{hom}(\mathbf{X}, \mathbf{S}[\mathbf{Z}])$. There is a canonical arrow $\mathbf{X} \to \nabla^2_{\mathbf{Z}}\mathbf{X}$ which is an isomorphism if the homotopy groups of $\mathbf{X}$ are finitely generated. Examples: (1) $\nabla_{\mathbf{Z}}\mathbf{H}(\mathbf{Z}) \approx \mathbf{H}(\mathbf{Z})$; (2) $\nabla_{\mathbf{Z}}\mathbf{KU} \approx \mathbf{KU}$.

FACT Suppose that the homotopy groups of $\mathbf{X}$ are finite—then $\Sigma\nabla_{\mathbf{Z}}\mathbf{X} \approx \nabla_{\mathbf{Q}/\mathbf{Z}}\mathbf{X}$.

Given an abelian group G, define the Moore spectrum of type G by the exact triangle $\bigvee_j \mathbf{S} \to \bigvee_i \mathbf{S} \to \mathbf{S}(G) \to \bigvee_j \Sigma\mathbf{S}$, where $0 \to \bigoplus_j \mathbf{Z} \to \bigoplus_i \mathbf{Z} \to G \to 0$ is a presentation of G—then $\mathbf{S}(G)$ is connective and $\pi_0(\mathbf{S}(G)) = G$. Example: $\mathbf{S}(\mathbf{Z}) = \mathbf{S}$.

PROPOSITION 19 Given a spectrum $\mathbf{X}$ and an abelian group G, there are short exact sequences

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[†] *Amer. J. Math.* **98** (1976), 1–27.

$$\begin{cases} 0 \longrightarrow \quad \pi_n(\mathbf{X}) \otimes G \quad \longrightarrow \pi_n(\mathbf{X} \wedge \mathbf{S}(G)) \longrightarrow \operatorname{Tor}(\pi_{n-1}(\mathbf{X}), G) \longrightarrow 0 \\ \\ 0 \longrightarrow \operatorname{Ext}(G, \pi_{n+1}(\mathbf{X})) \longrightarrow [\Sigma^n \mathbf{S}(G), \mathbf{X}] \longrightarrow \operatorname{Hom}(G, \pi_n(\mathbf{X})) \longrightarrow 0 \end{cases}$$

Application: $\mathbf{H}(\mathbf{Z}) \wedge \mathbf{S}(G) \approx \mathbf{H}(G)$, the Eilenberg-MacLane spectrum attached to G (cf. p. 17–2).

EXAMPLE Take $G = \mathbf{Z}_P$—then $\mathbf{S}(\mathbf{Z}_P)$ is a commutative ring spectrum.

[Note: $\mathbf{S}(\mathbf{Q}) \approx \mathbf{H}(\mathbf{Q})$ (since $\pi_n(\mathbf{S}) \otimes \mathbf{Q} = 0$ for $n \neq 0$).]

EXAMPLE Take $G = \mathbf{Z}/p\mathbf{Z}$, where p is odd—then $\mathbf{S}(\mathbf{Z}/p\mathbf{Z}) \wedge \mathbf{S}(\mathbf{Z}/p\mathbf{Z}) \approx \mathbf{S}(\mathbf{Z}/p\mathbf{Z}) \vee \Sigma\mathbf{S}(\mathbf{Z}/p\mathbf{Z})$ and $\mathbf{S}(\mathbf{Z}/p\mathbf{Z})$ is a commutative ring spectrum if $p > 3$.

[Note: When $p = 3$, $\mathbf{S}(\mathbf{Z}/3\mathbf{Z})$ admits a commutative multiplication with unit but associativity breaks down.]

EXAMPLE Take $G = \mathbf{Z}/2\mathbf{Z}$—then $\mathbf{S}(\mathbf{Z}/2\mathbf{Z})$ has no multiplication with unit ($\mathbf{S}(\mathbf{Z}/2\mathbf{Z})$ is not a retract of $\mathbf{S}(\mathbf{Z}/2\mathbf{Z}) \wedge \mathbf{S}(\mathbf{Z}/2\mathbf{Z})$).

[Note: $\operatorname{Hom}(\mathbf{Z}/2\mathbf{Z}, \mathbf{Z}/2\mathbf{Z}) = \mathbf{Z}/2\mathbf{Z}$ whereas $[\mathbf{S}(\mathbf{Z}/2\mathbf{Z}), \mathbf{S}(\mathbf{Z}/2\mathbf{Z})] = \mathbf{Z}/4\mathbf{Z}$. Because of this, one cannot construct an additive functor $\mathbf{AB} \xrightarrow{F} \mathbf{HSPEC}$ such that $FG = \mathbf{S}(G)$ (there is no ring homomorphism $\mathbf{Z}/2\mathbf{Z} \rightarrow \mathbf{Z}/4\mathbf{Z}$).]

EXAMPLE Fix $p \in \mathbf{\Pi}$—then $\mathbf{S}(\mathbf{Z}/p^\infty\mathbf{Z}) \approx \operatorname{tel}(\mathbf{S}(\mathbf{Z}/p\mathbf{Z}) \rightarrow \mathbf{S}(\mathbf{Z}/p^2\mathbf{Z}) \rightarrow \cdots) \Rightarrow \Sigma^{-1}\mathbf{S}(\mathbf{Z}/p^\infty\mathbf{Z}) \approx \operatorname{tel}(\Sigma^{-1}\mathbf{S}(\mathbf{Z}/p\mathbf{Z}) \rightarrow \Sigma^{-1}\mathbf{S}(\mathbf{Z}/p^2\mathbf{Z}) \rightarrow \cdots)$. But since $\mathbf{S} \rightarrow \mathbf{S} \rightarrow \mathbf{S}(\mathbf{Z}/p^n\mathbf{Z}) \rightarrow \Sigma\mathbf{S}$ is exact, $\mathbf{S}(\mathbf{Z}/p^n\mathbf{Z}) \approx \Sigma D\mathbf{S}(\mathbf{Z}/p^n\mathbf{Z})$, so $\Sigma^{-1}\mathbf{S}(\mathbf{Z}/p^\infty\mathbf{Z}) \approx \operatorname{tel}(D\mathbf{S}(\mathbf{Z}/p\mathbf{Z}) \rightarrow D\mathbf{S}(\mathbf{Z}/p^2\mathbf{Z}) \rightarrow \cdots)$. Accordingly, $\forall$ $\mathbf{X}$, $\hom(\Sigma^{-1}\mathbf{S}(\mathbf{Z}/p^\infty\mathbf{Z}), \mathbf{X}) \approx \operatorname{mic}(\hom(D\mathbf{S}(\mathbf{Z}/p\mathbf{Z}), \mathbf{X}) \leftarrow \hom(D\mathbf{S}(\mathbf{Z}/p^2\mathbf{Z}), \mathbf{X}) \leftarrow \cdots)$. However, $\forall$ n, $\mathbf{S}(\mathbf{Z}/p^n\mathbf{Z})$ is compact, hence dualizable $\Rightarrow$ $D\mathbf{S}(\mathbf{Z}/p^n\mathbf{Z})$ dualizable (cf. §15, Proposition 32) $\Rightarrow$ $\hom(D\mathbf{S}(\mathbf{Z}/p^n\mathbf{Z}), \mathbf{X}) \approx \mathbf{S}(\mathbf{Z}/p^n\mathbf{Z}) \wedge \mathbf{X}$. Thus, $\forall$ $\mathbf{X}$, $\hom(\Sigma^{-1}\mathbf{S}(\mathbf{Z}/p^\infty\mathbf{Z}), \mathbf{X}) \approx \operatorname{mic}(\mathbf{S}(\mathbf{Z}/p\mathbf{Z}) \wedge \mathbf{X} \leftarrow \mathbf{S}(\mathbf{Z}/p^2\mathbf{Z}) \wedge \mathbf{X} \leftarrow \cdots)$. Example: $\operatorname{mic}(\mathbf{S}(\mathbf{Z}/p\mathbf{Z}) \leftarrow \mathbf{S}(\mathbf{Z}/p^2\mathbf{Z}) \leftarrow \cdots) \approx \mathbf{S}(\widehat{\mathbf{Z}}_p) \Rightarrow \Sigma D\mathbf{S}(\mathbf{Z}/p^\infty\mathbf{Z}) \approx \mathbf{S}(\widehat{\mathbf{Z}}_p)$.

Fix a spectrum $\mathbf{E}$—then a morphism $\mathbf{f} : \mathbf{X} \rightarrow \mathbf{Y}$ in **HSPEC** is said to be an $\mathbf{E}_*$-equivalence if $\mathbf{f}_* : \mathbf{E}_*(\mathbf{X}) \rightarrow \mathbf{E}_*(\mathbf{Y})$ is an isomorphism. Denoting by $S_\mathbf{E}$ the class of $\mathbf{E}_*$-equivalences, the Bousfield-Margolis localization theorem guarantees the existence of a localization functor $T_\mathbf{E}$ such that $S_\mathbf{E}^\perp$ is the class of $\mathbf{E}_*$-local ($= T_\mathbf{E}$-local) spectra. In this connection, recall that $\mathbf{X}$ is $\mathbf{E}_*$-local iff $[\mathbf{Y}, \mathbf{X}] = 0$ for all $\mathbf{E}_*$-acyclic ($= T_\mathbf{E}$-acyclic) $\mathbf{Y}$ (cf. §15, Proposition 27) and the class of $\mathbf{E}_*$-local spectra is the object class of a thick subcategory of **HSPEC** which is closed under the formation of products in **HSPEC** (cf. §15, Proposition 28). Let us also bear in mind that $T_{\mathbf{E}}$ has the IP (cf. §15, Proposition 40).

Notation: $\mathbf{HSPEC}_{\mathbf{E}}$ is the full subcategory of **HSPEC** whose objects are the $\mathbf{E}_*$-local spectra, $L_{\mathbf{E}} : \mathbf{HSPEC} \to \mathbf{HSPEC}_{\mathbf{E}}$ is the associated reflector, and $l_{\mathbf{E}} : \mathbf{X} \to L_{\mathbf{E}}\mathbf{X}$ is the arrow of localization.

[Note: The objects of $\mathbf{HSPEC}_{\mathbf{E}}$ are the objects of $\langle \mathbf{E} \rangle$, the Bousfield class of **E**, and $L_{\mathbf{E}} \approx L_{\mathbf{F}}$ iff $\langle \mathbf{E} \rangle = \langle \mathbf{F} \rangle$. $\mathbf{HSPEC}_{\mathbf{E}}$ is a CTC (cf. p. 15–41) but need not be compactly generated (Strickland[†]).]

Remark: Ohkawa[‡] has shown that the conglomerate $\langle \mathbf{HSPEC} \rangle$ whose elements are the Bousfield classes is codable by a set.

LEMMA Given spectra **E** and **F**, suppose that $\langle \mathbf{E} \rangle \leq \langle \mathbf{F} \rangle$—then $\forall$ **X**, $T_{\mathbf{E}}T_{\mathbf{F}}\mathbf{X} \approx T_{\mathbf{E}}\mathbf{X} \approx T_{\mathbf{F}}T_{\mathbf{E}}\mathbf{X}$.

EXAMPLE Suppose that **X** is connective—then $\mathbf{X} = 0$ iff **X** is $\mathbf{H}(\mathbf{Z})_*$-acyclic.

[Note: $\nabla_{\mathbf{Q}/\mathbf{Z}}\mathbf{S}$ $(= \mathbf{S}[\mathbf{Q}/\mathbf{Z}])$ is $\mathbf{H}(\mathbf{Z})_*$-acyclic and nonzero (although $\nabla_{\mathbf{Q}/\mathbf{Z}}\mathbf{S} \wedge \nabla_{\mathbf{Q}/\mathbf{Z}}\mathbf{S} = 0$).]

Instead of working with $\mathbf{E}_*$-equivalences, one could work instead with $\mathbf{E}^*$-equivalences and then define the $\mathbf{E}^*$-local spectra in the obvious way. Problem: Do the $\mathbf{E}^*$-local spectra constitute the object class of a reflective subcategory of **HSPEC**? While the answer is unknown in general, one does have the following partial result due to Bousfield[‖].

COHOMOLOGICAL LOCALIZATION THEOREM Suppose that **E** has the following property: $\forall\, n$, $\mathbf{Z}/p\mathbf{Z} \otimes \pi_n(\mathbf{E})$ and $\mathrm{Tor}(\mathbf{Z}/p\mathbf{Z}, \pi_n(\mathbf{E}))$ are finite $\forall\, p \in \mathbf{\Pi}$—then there exists an **F** such that the $\mathbf{E}^*$-equivalences are the same as the $\mathbf{F}_*$-equivalences, so cohomological localization with respect to **E** exists and is given by homological localization with respect to **F**.

[Note: When the $\pi_n(\mathbf{E})$ are finitely generated, one can take $\mathbf{F} = \nabla_{\mathbf{Z}}\mathbf{E}$.]

Given an abelian group G, call $\mathcal{S}(G)$ the class of abelian groups A such that $A \otimes G = 0 = \mathrm{Tor}(A, G)$ (cf. p. 9–30).

PROPOSITION 20 $\mathcal{S}(G') = \mathcal{S}(G'')$ iff $\langle \mathbf{S}(G') \rangle = \langle \mathbf{S}(G'') \rangle$.

[†] No Small Objects, Preprint.

[‡] *Hiroshima Math. J.* **19** (1989), 631–639.

[‖] Cohomological Localizations of Spaces and Spectra, Preprint.

This result reduces the problem of inventoring the $L_{\mathbf{S}(G)}$ to when $G = \mathbf{Z}_P$ or $G = \bigoplus\limits_{p\in P} \mathbf{Z}/p\mathbf{Z}$.

EXAMPLE $\langle \mathbf{S}(\mathbf{Z}_P)\rangle = \langle \mathbf{S}(\mathbf{Q})\rangle \vee \bigvee\limits_{p\in P} \langle \mathbf{S}(\mathbf{Z}/p\mathbf{Z})\rangle (\Rightarrow \langle \mathbf{S}\rangle = \langle \mathbf{S}(\mathbf{Q})\rangle \vee \bigvee\limits_{p} \langle \mathbf{S}(\mathbf{Z}/p\mathbf{Z})\rangle)$. And: $\langle \mathbf{S}(\mathbf{Q})\rangle \wedge \langle \mathbf{S}(\mathbf{Z}/p\mathbf{Z})\rangle = \langle 0\rangle$ & $\langle \mathbf{S}(\mathbf{Z}/p\mathbf{Z})\rangle \wedge \langle \mathbf{S}(\mathbf{Z}/q\mathbf{Z})\rangle = \langle 0\rangle$ $(p \neq q)$.

PROPOSITION 21 Let $G = \mathbf{Z}_P$—then $L_{\mathbf{S}(\mathbf{Z}_P)}\mathbf{X} = \mathbf{S}(\mathbf{Z}_P) \wedge \mathbf{X}$ and $\pi_*(L_{\mathbf{S}(\mathbf{Z}_P)}\mathbf{X}) = \mathbf{Z}_P \otimes \pi_*(\mathbf{X})$.

[$\mathbf{S}(\mathbf{Z}_P)$ is a commutative ring spectrum with the property that the product $\mathbf{S}(\mathbf{Z}_P) \wedge \mathbf{S}(\mathbf{Z}_P) \to \mathbf{S}(\mathbf{Z}_P)$ is an isomorphism, thus $T_{\mathbf{S}(\mathbf{Z}_P)}$ is smashing (cf. p. 15–45) and $\mathbf{X} \approx \mathbf{S} \wedge \mathbf{X} \to \mathbf{S}(\mathbf{Z}_P) \wedge \mathbf{X}$ is the arrow of localization.]

FACT Suppose that $\mathbf{X}$ is connective—then $L_{\mathbf{S}(\mathbf{Z}_P)}\mathbf{X} \approx L_{\mathbf{H}(\mathbf{Z}_P)}\mathbf{X}$.

[Note: Take $P = \mathbf{\Pi}$ to see that $L_{\mathbf{S}(\mathbf{Z})}\mathbf{X} \approx L_{\mathbf{H}(\mathbf{Z})}\mathbf{X}$, i.e., $\mathbf{X} \approx L_{\mathbf{H}(\mathbf{Z})}\mathbf{X}$.]

Write $\mathbf{HSPEC}_P$ for the full subcategory of $\mathbf{HSPEC}$ whose objects are P-local ($= \mathbf{S}(\mathbf{Z}_P)_*$-local) (use the symbol $\mathbf{HSPEC}_{\mathbf{Q}}$ if $P = \emptyset$)—then the objects of $\mathbf{HSPEC}_P$ are those $\mathbf{X}$ which are P-local in homotopy, i.e., $\forall\, n$, $\pi_n(\mathbf{X})$ is P-local and $\mathbf{HSPEC}_P$ is a monogenic compactly generated CTC.

FACT The category $\mathbf{HSPEC}_{\mathbf{Q}}$ is equivalent to the category of graded vector spaces over $\mathbf{Q}$.

[Note: The objects of $\mathbf{HSPEC}_{\mathbf{Q}}$ are the rational spectra.]

PROPOSITION 22 Let $G = \mathbf{Z}/p\mathbf{Z}$—then $L_{\mathbf{S}(\mathbf{Z}/p\mathbf{Z})}\mathbf{X} = \hom(\Sigma^{-1}\mathbf{S}(\mathbf{Z}/p^\infty\mathbf{Z}), \mathbf{X})$ and there is a split short exact sequence $0 \to \mathrm{Ext}(\mathbf{Z}/p^\infty\mathbf{Z}, \pi_*(\mathbf{X})) \to \pi_*(L_{\mathbf{S}(\mathbf{Z}/p\mathbf{Z})}\mathbf{X}) \to \mathrm{Hom}(\mathbf{Z}/p^\infty\mathbf{Z}, \pi_{*-1}(\mathbf{X})) \to 0$.

[Consider the exact triangle $\hom(\mathbf{S}(\mathbf{Z}\left[\frac{1}{p}\right]), \mathbf{X}) \to \hom(\mathbf{S}, \mathbf{X}) \to \hom(\Sigma^{-1}\mathbf{S}(\mathbf{Z}/p^\infty\mathbf{Z}), \mathbf{X}) \to \Sigma\hom(\mathbf{S}(\mathbf{Z}\left[\frac{1}{p}\right]), \mathbf{X})$. On the one hand, $\hom(\Sigma^{-1}\mathbf{S}(\mathbf{Z}/p^\infty\mathbf{Z}), \mathbf{X})$ is $\mathbf{S}(\mathbf{Z}/p\mathbf{Z})_*$-local (for $\mathcal{S}(\mathbf{Z}/p\mathbf{Z}) = \mathcal{S}(\mathbf{Z}/p^\infty\mathbf{Z})$) and, on the other, $\hom(\mathbf{S}(\mathbf{Z}\left[\frac{1}{p}\right], \mathbf{X})$ is $\mathbf{S}(\mathbf{Z}/p\mathbf{Z})_*$-acyclic (its homotopy groups are uniquely p-divisible). Therefore $\mathbf{X} \approx \hom(\mathbf{S}, \mathbf{X}) \to \hom(\Sigma^{-1}\mathbf{S}(\mathbf{Z}/p^\infty\mathbf{Z}), \mathbf{X})$ is the arrow of localization.]

[Note: The $\mathbf{S}(\mathbf{Z}/p\mathbf{Z})_*$-local spectra are those $\mathbf{X}$ such that $\forall\, n$, $\pi_n(\mathbf{X})$ is p-cotorsion. Proof: $\hom(\mathbf{S}(\mathbf{Z}\left[\frac{1}{p}\right]), \mathbf{X}) = 0$ iff $\forall\, n$, $\mathrm{Hom}(\mathbf{Z}\left[\frac{1}{p}\right], \pi_n(\mathbf{X})) = 0$ & $\mathrm{Ext}(\mathbf{Z}\left[\frac{1}{p}\right], \pi_n(\mathbf{X})) = 0$.]

If the homotopy groups of $\mathbf{X}$ are finitely generated, put $\widehat{\mathbf{X}}_p = L_{\mathbf{S}(\mathbf{Z}/p\mathbf{Z})}\mathbf{X}$ and call $\widehat{\mathbf{X}}_p$ the p-adic completion of $\mathbf{X}$. Justification: $\forall\, n$, $\pi_n(\widehat{\mathbf{X}}_p) \approx \pi_n(\mathbf{X})^{\wedge}_p$ (cf. p. 10–2). Example:

$\widehat{\mathbf{S}}_p = L_{\mathbf{S}(\mathbf{Z}/p\mathbf{Z})}\mathbf{S} = \hom(\Sigma^{-1}\mathbf{S}(\mathbf{Z}/p^\infty\mathbf{Z}), \mathbf{S}) = D\Sigma^{-1}\mathbf{S}(\mathbf{Z}/p^\infty\mathbf{Z}) = \Sigma D\mathbf{S}(\mathbf{Z}/p^\infty\mathbf{Z}) = \mathbf{S}(\widehat{\mathbf{Z}}_p)$ (cf. p. 17–21).

PROPOSITION 23 The arrow of localization per $\bigoplus_{p\in P} \mathbf{Z}/p\mathbf{Z}$ is $\mathbf{X} \to \prod_{p\in P} L_{\mathbf{S}(\mathbf{Z}/p\mathbf{Z})}\mathbf{X}$ (cf. §9, Proposition 22).

FACT $\forall$ $\mathbf{X}$, there is an exact triangle $\hom(\mathbf{S}(\mathbf{Q}), \mathbf{X}) \to \mathbf{X} \to \prod_p \hom(\Sigma^{-1}\mathbf{S}\mathbf{Z}/p^\infty\mathbf{Z}), \mathbf{X}) \to \Sigma\hom(\mathbf{S}(\mathbf{Q}), \mathbf{X})$.

FACT $\forall$ $\mathbf{X}$, there is an exact triangle $\bigvee_p \mathbf{X} \wedge \Sigma^{-1}\mathbf{S}\mathbf{Z}/p^\infty\mathbf{Z}) \to \mathbf{X} \to \mathbf{X} \wedge \mathbf{S}(\mathbf{Q}) \to \bigvee_p \Sigma(\mathbf{X} \wedge \Sigma^{-1}\mathbf{S}(\mathbf{Z}/p^\infty\mathbf{Z}))$.

PROPOSITION 24 Let G, K be abelian groups such that $\mathcal{S}(G) = \mathcal{S}(K)$—then $\forall\, \mathbf{X}, \langle \mathbf{X} \wedge \mathbf{S}(G)\rangle = \langle \mathbf{X} \wedge \mathbf{S}(K)\rangle$.

EXAMPLE Let G, K be abelian groups such that $\mathcal{S}(G) = \mathcal{S}(K)$—then $\langle \mathbf{H}(G)\rangle = \langle \mathbf{H}(K)\rangle$. In fact, $\begin{cases} \mathbf{H}(G) \approx \mathbf{H}(\mathbf{Z}) \wedge \mathbf{S}(G) \\ \mathbf{H}(K) \approx (\mathbf{Z} \wedge \mathbf{S}(K) \end{cases}$ (cf. p. 17–21).

FACT Suppose that $\mathbf{E} \wedge \mathbf{S}(\mathbf{Q}) \neq 0$—then $\forall\, \mathbf{X}$, $L_{\mathbf{E}\wedge\mathbf{S}(\mathbf{Q})}\mathbf{X} \approx L_{\mathbf{S}(\mathbf{Q})}\mathbf{X}$.

LEMMA Given a connective spectrum $\mathbf{E}$, put $\pi\mathbf{E} = \bigoplus_n \pi_n(\mathbf{E})$—then $\langle \mathbf{H}(\pi\mathbf{E})\rangle \leq \langle \mathbf{E}\rangle \leq \langle \mathbf{S}(\pi\mathbf{E})\rangle$.

[$\langle \mathbf{H}(\pi\mathbf{E})\rangle \leq \langle \mathbf{E}\rangle$: Since $\mathbf{E}$ is connective, $\mathcal{S}(\pi\mathbf{E}) = \mathcal{S}(\bigoplus_n \mathbf{H}_n(\mathbf{E};\mathbf{Z}))$, so $\langle \mathbf{H}(\pi\mathbf{E})\rangle = \langle \mathbf{H}(\bigoplus_n \mathbf{H}_n(\mathbf{E};\mathbf{Z}))\rangle = \langle \bigvee_n \Sigma^n \mathbf{H}(\mathbf{H}_n(\mathbf{E};\mathbf{Z}))\rangle = \langle \mathbf{E} \wedge \mathbf{H}(\mathbf{Z})\rangle$ (cf. p. 17–20), which is $\leq \langle \mathbf{E}\rangle$.

$\langle \mathbf{E}\rangle \leq \langle \mathbf{S}(\pi\mathbf{E})\rangle$: Let G_1 be the direct sum of the groups in the set $\{\mathbf{Q}, \mathbf{Z}/p\mathbf{Z}(p \in \mathbf{\Pi}\}$ with $\mathcal{S}(G_1) = \mathcal{S}(\pi\mathbf{E})$ and let G_2 be the direct sum of what remains—then $\langle \mathbf{S}(G_1)\rangle \wedge \langle \mathbf{S}(G_2)\rangle = \langle 0\rangle$ & $\langle \mathbf{S}(G_1)\rangle \vee \langle \mathbf{S}(G_2)\rangle = \langle \mathbf{S}\rangle$. And: $\mathbf{E} \wedge \mathbf{S}(G_2) = 0$, hence $\langle \mathbf{E}\rangle = \langle \mathbf{E}\rangle \wedge \langle \mathbf{S}\rangle = \langle \mathbf{E}\rangle \wedge (\langle \mathbf{S}(G_1)\rangle \vee \langle \mathbf{S}(G_2)\rangle) = (\langle \mathbf{E}\rangle \wedge \mathbf{S}(G_1)\rangle) \vee (\langle \mathbf{E}\rangle \wedge \langle \mathbf{S}(G_2)\rangle) = \langle \mathbf{E}\rangle \wedge \langle \mathbf{S}(G_1)\rangle = \langle \mathbf{E}\rangle \wedge \langle \mathbf{S}(\pi\mathbf{E})\rangle \leq \langle \mathbf{S}(\pi\mathbf{E})\rangle$.]

PROPOSITION 25 Let $\mathbf{E}, \mathbf{X}$ be connective—then $L_{\mathbf{E}}\mathbf{X} \approx L_{\mathbf{S}(\pi\mathbf{E})}\mathbf{X}$, where $\pi\mathbf{E} = \bigoplus_n \pi_n(\mathbf{E})$.

[The lemma implies that the arrow of localization $\mathbf{X} \to L_{\mathbf{S}(\pi\mathbf{E})}\mathbf{X}$ is an $\mathbf{E}_*$-equivalence. But $L_{\mathbf{S}(\pi\mathbf{E})}\mathbf{X} = L_{\mathbf{H}(\pi\mathbf{E})}\mathbf{X}$ (cf. infra) and $L_{\mathbf{H}(\pi\mathbf{E})}\mathbf{X}$ is $\mathbf{E}_*$-local (by the lemma).]

LEMMA Let $\mathbf{E}, \mathbf{X}$ be spectra and let G be an abelian group—then the arrow $L_{\mathbf{S}(G)}L_{\mathbf{E}}\mathbf{X} \to L_{\mathbf{E}\wedge\mathbf{S}(G)}\mathbf{X}$ is an isomorphism if G is torsion or if $\mathbf{E} \wedge \mathbf{S}(\mathbf{Q}) \neq 0$.

[Suppose first that G is torsion, say $G = \bigoplus_{p \in P} \mathbf{Z}/p\mathbf{Z}$ (this entails no loss of generality). Since $L_{\mathbf{S}(G)} L_{\mathbf{E}} \mathbf{X} \to L_{\mathbf{E} \wedge \mathbf{S}(G)} \mathbf{X}$ is an $(\mathbf{E} \wedge \mathbf{S}(G))_*$-equivalence, it suffices to prove that $L_{\mathbf{S}(G)} L_{\mathbf{E}} \mathbf{X}$ is $(\mathbf{E} \wedge \mathbf{S}(G))_*$-local or still, that $[\mathbf{Y}, L_{\mathbf{S}(G)} L_{\mathbf{E}} \mathbf{X}] = 0$ for all $(\mathbf{E} \wedge \mathbf{S}(G))_*$-acyclic $\mathbf{Y}$. But $[\mathbf{Y}, L_{\mathbf{S}(G)} L_{\mathbf{E}} \mathbf{X}] = [\mathbf{Y}, \operatorname{hom}(\bigvee_{p \in P} \Sigma^{-1} \mathbf{S}(\mathbf{Z}/p^\infty \mathbf{Z}), L_{\mathbf{E}} \mathbf{X})] = [\mathbf{Y} \wedge \bigvee_{p \in P} \Sigma^{-1} \mathbf{S}(\mathbf{Z}/p^\infty \mathbf{Z}), L_{\mathbf{E}} \mathbf{X}]$ and $\mathbf{Y} \wedge \bigvee_{p \in P} \Sigma^{-1} \mathbf{S}(\mathbf{Z}/p^\infty \mathbf{Z})$ is $\mathbf{E}_*$-acyclic ($\mathcal{S}(\bigoplus_{p \in P} \mathbf{Z}/p^\infty \mathbf{Z}) = \mathcal{S}(G)$). To discuss the other case, viz. when $\mathbf{E} \wedge \mathbf{S}(\mathbf{Q}) \neq 0$, one can take $G = \mathbf{Z}_P$. Because $L_{\mathbf{E} \wedge \mathbf{S}(G)} \mathbf{X}$ is $\mathbf{S}(G)_*$-local, it need only be shown that $L_{\mathbf{E}} \mathbf{X} \to L_{\mathbf{E} \wedge \mathbf{S}(G)} \mathbf{X}$ is an $\mathbf{S}(G)_*$-equivalence. However $\langle \mathbf{S}(G) \rangle = \langle \mathbf{S}(\mathbf{Q}) \rangle \vee \bigvee_{p \in P} \langle \mathbf{S}(\mathbf{Z}/p\mathbf{Z}) \rangle$, which reduces the problem to showing that $L_{\mathbf{E}} \mathbf{X} \to L_{\mathbf{E} \wedge \mathbf{S}(G)} \mathbf{X}$ is an $\mathbf{S}(\mathbf{Q})_*$-equivalence and an $\mathbf{S}(\mathbf{Z}/p\mathbf{Z})_*$-equivalence for each $p \in P$. Due to our assumption that $\mathbf{E} \wedge \mathbf{S}(\mathbf{Q}) \neq 0$, just the second possibility is at issue. For this, consider the commutative triangle

$$\begin{tikzcd}
L_{\mathbf{E}} \mathbf{X} \arrow[r] \arrow[dr] & L_{\mathbf{E} \wedge \mathbf{S}(G)} \arrow[d] \\
 & L_{\mathbf{E} \wedge \mathbf{S}(\mathbf{Z}/p\mathbf{Z})}
\end{tikzcd}$$

. Here, the arrow $L_{\mathbf{E} \wedge \mathbf{S}(G)} \to L_{\mathbf{E} \wedge \mathbf{S}(\mathbf{Z}/p\mathbf{Z})}$ is an $\mathbf{S}(\mathbf{Z}/p\mathbf{Z})_*$-equivalence ($L_{\mathbf{S}(\mathbf{Z}/p\mathbf{Z})} L_{\mathbf{E} \wedge \mathbf{S}(G)} \mathbf{X} \approx L_{\mathbf{E} \wedge \mathbf{S}(G) \wedge \mathbf{S}(\mathbf{Z}/p\mathbf{Z})} \approx L_{\mathbf{E} \wedge \mathbf{S}(\mathbf{Z}/p\mathbf{Z})}$), as is the arrow $L_{\mathbf{E}} \mathbf{X} \to L_{\mathbf{E} \wedge \mathbf{S}(\mathbf{Z}/p\mathbf{Z})}$ ($L_{\mathbf{S}(\mathbf{Z}/p\mathbf{Z})} L_{\mathbf{E}} \mathbf{X} \approx L_{\mathbf{E} \wedge \mathbf{S}(\mathbf{Z}/p\mathbf{Z})}$). Therefore the arrow $L_{\mathbf{E}} \mathbf{X} \to L_{\mathbf{E} \wedge \mathbf{S}(G)} \mathbf{X}$ is an $\mathbf{S}(\mathbf{Z}/p\mathbf{Z})_*$-equivalence.]

[Note: The assumption that $\mathbf{E} \wedge \mathbf{S}(\mathbf{Q}) \neq 0$ cannot be dropped. Example: $L_{\mathbf{S}(\mathbf{Q})} L_{\mathbf{S}(\mathbf{Z}/p\mathbf{Z})} \mathbf{H}(\mathbf{Z}) \neq 0$, yet $L_{\mathbf{S}(\mathbf{Z}/p\mathbf{Z}) \wedge \mathbf{S}(\mathbf{Q})} \mathbf{H}(\mathbf{Z}) = 0$.]

To tie up the loose end in the proof of Proposition 25, observe that $\mathbf{H}(\mathbf{Z}) \wedge \mathbf{S}(\mathbf{Q}) \approx \mathbf{H}(\mathbf{Q}) \neq 0$ (cf. p. 17–21). In addition, since $\mathbf{X}$ is connective, $\mathbf{X} \approx L_{\mathbf{H}(\mathbf{Z})} \mathbf{X}$ (cf. p. 17–23), hence $L_{\mathbf{S}(\pi \mathbf{E})} \mathbf{X} \approx L_{\mathbf{S}(\pi \mathbf{E})} L_{\mathbf{H}(\mathbf{Z})} \mathbf{X} \approx L_{\mathbf{H}(\mathbf{Z}) \wedge \mathbf{S}(\pi \mathbf{E})} \mathbf{X} \approx L_{\mathbf{H}(\pi \mathbf{E})} \mathbf{X}$ (cf. p. 17–21).

LEMMA Let A be a ring with unit, M a left A-module—then $\mathcal{S}(A) = \mathcal{S}(A \oplus M)$.

Application: Suppose that $\mathbf{E}$ is a ring spectrum—then $\mathcal{S}(\pi_0(\mathbf{E})) = \mathcal{S}(\bigoplus_n \pi_n(\mathbf{E}))$.

Example: Take $\mathbf{E} = \mathbf{MU}$—then $\mathcal{S}(\mathbf{Z}) = \mathcal{S}(\bigoplus_n \pi_n(\mathbf{MU}))$, thus for any connective $\mathbf{X}$, $L_{\mathbf{MU}} \mathbf{X} \approx L_{\mathbf{S}(\mathbf{Z})} \mathbf{X} \approx L_{\mathbf{S}} \mathbf{X} \approx \mathbf{X}$.

[Note: It follows that all compact spectra are $\mathbf{MU}_*$-local. Indeed, a compact object in **HSPEC** is isomorphic to a $\mathbf{Q}_q^\infty K$, where K is a pointed finite CW complex (cf. p. 16–15). And: $\mathbf{Q}_q^\infty K \approx \mathbf{S}^{-q} \wedge K \approx \mathbf{S}^{-q} \wedge \mathbf{Q}^\infty K$ (cf. p. 16–29). But $\mathbf{Q}^\infty K$ is connective (cf. p. 16–7) (K is wellpointed). Therefore $\mathbf{Q}^\infty K$ is $\mathbf{MU}_*$-local, hence $\mathbf{S}^{-q} \wedge \mathbf{Q}^\infty K$ is too (cf. p. 15–41) ($\mathbf{S}^{-q}$ is compact and **HSPEC** is a monogenic compactly generated CTC).]

FACT Let $\mathbf{X}, \mathbf{Y}$ be spectra with $\mathbf{Y}^*(\mathbf{X}) = 0$. Assume: The homotopy groups of $\mathbf{Y}$ are finite—then $\pi_*(\mathbf{X} \wedge \nabla_{\mathbf{Q}/\mathbf{Z}} \mathbf{Y}) = 0$.

One also has a good understanding of homological localization with respect to **KU**. Here though I shall merely provide a summary (proofs can be found in Bousfield[†]).

[Note: There is no need to distinguish between $L_{\mathbf{KU}}$ and $L_{\mathbf{KO}}$ since $\langle \mathbf{KU} \rangle = \langle \mathbf{KO} \rangle$ (Meier[‡]).]

Put $\mathbf{M}(p) = \mathbf{S}(\mathbf{Z}/p\mathbf{Z})$—then there is a $\mathbf{KU}_*$-equivalence $\mathbf{A}_p : \Sigma^d \mathbf{M}(p) \to \mathbf{M}(p)$, where $d = 8$ if $p = 2$ & $d = 2p - 2$ if $p > 2$. Using the notation of p. 15–45, the arrow $\mathbf{M}(p) \to \mathbf{A}_p^{-1}\mathbf{M}(p)$ is a $\mathbf{KU}_*$-equivalence and $\mathbf{A}_p^{-1}\mathbf{M}(p)$ is $\mathbf{KU}_*$-local ($\Rightarrow L_{\mathbf{KU}}\mathbf{M}(p) = \mathbf{A}_p^{-1}\mathbf{M}(p)$).

[Note: Define $\mathbf{coA}_p$ by the exact triangle $\Sigma^d \mathbf{M}(p) \xrightarrow{\mathbf{A}_p} \mathbf{M}(p) \to \mathbf{coA}_p \to \Sigma^{d+1}\mathbf{M}(p)$—then $\langle \mathbf{KU} \rangle = \langle \bigvee_p \mathbf{coA}_p \rangle^c$.]

Remark: $T_{\mathbf{KU}}$ is smashing and the $\pi_n(L_{\mathbf{KU}}\mathbf{S})$ can be calculated in closed form ($L_{\mathbf{KU}}\mathbf{S}$ is not connective, e.g., $\pi_{-2}(L_{\mathbf{KU}}\mathbf{S}) = \mathbf{Q}/\mathbf{Z}$).

Examples: (1) $L_{\mathbf{KU}}(\mathbf{X} \wedge \mathbf{M}(p)) \approx L_{\mathbf{KU}}\mathbf{S} \wedge \mathbf{X} \wedge \mathbf{M}(p) \approx \mathbf{X} \wedge L_{\mathbf{KU}}\mathbf{S} \wedge \mathbf{M}(p) \approx \mathbf{X} \wedge L_{\mathbf{KU}}\mathbf{M}(p) \approx \mathbf{X} \wedge \mathbf{A}_p^{-1}\mathbf{M}(p)($; (2) $L_{\mathbf{E}\wedge\mathbf{M}(p)}\mathbf{X} \approx L_{\mathbf{M}(p)}L_{\mathbf{E}}\mathbf{X}$ (cf. p. 17–?).

BOUSFIELD'S FIRST KU THEOREM Fix an $\mathbf{X}$—then $\mathbf{X}$ is $\mathbf{KU}_*$-local iff $\forall\, p$ & $\forall\, n$, the arrow $[\Sigma^n \mathbf{M}(p), \mathbf{X}] \to [\Sigma^{n+d}\mathbf{M}(p), \mathbf{X}]$ induced by $\mathbf{A}_p$ is bijective or, equivalently, iff $\forall\, p$ & $\forall\, n$, the arrow $\pi_n(\mathbf{M}(p) \wedge \mathbf{X}) \to \pi_{n+d}(\mathbf{M}(p) \wedge \mathbf{X})$ induced by $\mathbf{A}_p$ is bijective.

[Note: Therefore $\mathbf{X}$ is $\mathbf{KU}_*$-local iff $\pi_*(\mathbf{M}(p) \wedge \mathbf{X}) \approx \pi_*(\mathbf{A}_p^{-1}\mathbf{M}(p) \wedge \mathbf{X})$ under the $\mathbf{KU}_*$-equivalence $\mathbf{M}(p) \to \mathbf{A}_p^{-1}\mathbf{M}(p)$.

BOUSFIELD'S SECOND KU THEOREM Fix an $\mathbf{f} : \mathbf{X} \to \mathbf{Y}$—then $\mathbf{f}$ is a $\mathbf{KU}_*$-equivalence iff $\mathbf{f}_* : \pi_*(\mathbf{X}) \otimes \mathbf{Q} \to \pi_*(\mathbf{Y}) \otimes \mathbf{Q}$ is bijective and $\forall\, p$, $\mathbf{f}_* : \pi_*(\mathbf{A}_p^{-1}\mathbf{M}(p) \wedge \mathbf{X}) \to \pi_*(\mathbf{A}_p^{-1}\mathbf{M}(p) \wedge \mathbf{Y})$ is bijective.

FACT Let **ku** be the connective cover of **KU**—then **ku** is a ring spectrum (cf. p. 17–8) and $\mathbf{K} \approx \overline{\mathbf{b}}_{\mathbf{U}}^{-1}\mathbf{ku}$ (cf. p. 15–46).

Fix a prime p—then the objects of $\mathbf{HSPEC}_p$ ($= \mathbf{HSPEC}_{\{p\}}$) are the p-local spectra and one writes $\mathbf{X}_p$ in place of $L_{\mathbf{S}(\mathbf{Z}_p)}\mathbf{X}$, $\mathbf{X}_p$ being the p-localization of $\mathbf{X}$. Example: $\mathbf{M}(p)$ is p-local.

[Note: In $\mathbf{HSPEC}_p$, $\mathbf{X} \wedge -p\mathbf{Y} = (\mathbf{X} \wedge \mathbf{Y})_p$ (cf. p. 15–41), i.e., $\mathbf{X} \wedge_p \mathbf{Y} = \mathbf{X} \wedge \mathbf{Y}$ ($T_{\mathbf{S}(\mathbf{Z}_p)}$ is smashing), and $\mathbf{S}_p$ is the unit. Example: $\langle \mathbf{S}_p \rangle = \langle \mathbf{M}(p) \rangle \vee \langle \mathbf{S}(\mathbf{Q}) \rangle$.]

[†] *Topology* **18** (1979), 257–281; see also *J. Pure Appl. Algebra* **66** (1990), 121–163.

[‡] *J. Pure Appl. Algebra* **14** (1979), 59–71.

EXAMPLE Consider $\mathbf{KU}_p$—then Adams[†] has shown that there is a splitting $\mathbf{KU}_p \approx \mathbf{KU}_p(1) \vee \Sigma^2\mathbf{KU}_p(1) \vee \cdots \vee \Sigma^{2(p-2)}\mathbf{KU}_p(1)$, where $\mathbf{KU}_p(1)$ is a p-local spectrum with $\pi_*(\mathbf{KU}_p(1)) \approx \mathbf{Z}_p[v_1, v_1^{-1}]$ ($|v_1| = 2(p-1)$).

PROPOSITION 26 Suppose that $\mathbf{X}_p = 0 \ \forall\, p$—then $\mathbf{X} = 0$.
[$\mathbf{X}_p = 0 \ \forall\, p \Rightarrow \mathbf{Z}_p \otimes \pi_*(\mathbf{X}) = 0 \ \forall\, p \Rightarrow \pi_*(\mathbf{X}) = 0$ (cf. p. 8–3) $\Rightarrow \mathbf{X} = 0$.]
[Note: The converse is trivial.]

The objects of cpt $\mathbf{HSPEC}_p$ are the p-compact spectra.

FACT A p-local spectrum is p-compact iff it is isomorphic to the p-localization of a compact spectrum.

EXAMPLE Take $\mathbf{X}$ compact—then $\mathbf{f} : \Sigma^n\mathbf{X} \to \mathbf{X}$ is composition nilpotent iff $\forall\, p$, $\mathbf{f}_p : \Sigma^n\mathbf{X}_p \to \mathbf{X}_p$ is composition nilpotent.

[$\mathbf{f}$ is composition nilpotent iff $\mathbf{f}^{-1}\mathbf{X} = 0$ (cf. p. 15–46). But $\mathbf{f}^{-1}\mathbf{X} = 0$ iff $\forall\, p$, $(\mathbf{f}^{-1}\mathbf{X})_p = 0$ (cf. Proposition 26). And: $(\mathbf{f}^{-1}\mathbf{X})_p = \mathbf{f}_p^{-1}\mathbf{X}_p$.]

BP THEOREM Formal group law theory furnishes a canonical idempotent $\mathbf{e}_p \in [\mathbf{MU}_p, \mathbf{MU}_p]$ (the Quillen idempotent) which is a morphism of ring spectra. Thus, since idempotents split (cf. p. 15–17), $\exists$ a commutative ring spectrum $\mathbf{BP}$ (called the Brown-Peterson spectrum at the prime p) and morphisms $\mathbf{i} : \mathbf{BP} \to \mathbf{MU}_p$, $\mathbf{r} : \mathbf{MU}_p \to \mathbf{BP}$ of ring spectra such that $\mathbf{r} \circ \mathbf{i} = \mathrm{id}_{\mathbf{BP}}$ and $\mathbf{E}_p = \mathbf{i} \circ \mathbf{r}$. $\mathbf{BP}$ is complex orientable and $\mathbf{BP}^*(\mathbf{S}) = \mathbf{Z}_p[v_1, v_2, \ldots]$, where $|v_i| = -2(p^i - 1)$. And: $\mathbf{MU}_p$ is isomorphic to a wedge of suspensions of $\mathbf{BP}$, hence $\langle \mathbf{MU}_p \rangle = \langle \mathbf{BP} \rangle$.

[Note: The construction is spelled out in Adams[‡] (a sketch of the underlying ideas is given below).]

Notation: A is a commutative $\mathbf{Z}_p$-algebra with unit, FGL_A is the set of formal group laws over A, and $\mathrm{FGL}_{A,p}$ is the set of p-typical formal group laws over A.

[Note: Initially, it is best to keep the graded picture in the background.]

CARTIER'S THEOREM There is an idempotent $\epsilon_A : \mathbf{FGL}_A \to \mathbf{FGL}_{A,p}$, functorial in A, such that $\epsilon_A(\mathrm{FGL}_A) = \mathrm{FGL}_{A,p}$. Furthermore, there is a natural strict isomorphism $F \to \epsilon_A F$ such that if F is p-typical, then $\epsilon_A F = F$ and $F \to \epsilon_A F$ is the identity.

[†] *SLN* **99** (1969), 77–98; see also Bousfield, *Amer. J. Math.* **107** (1985), 895–932.

[‡] *Stable Homotopy and Generalized Homology*, University of Chicago (1974), 104–116; see also Wilson, *CBMS Regional Conference* **48** (1982), 1–86.

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Using this result, one can establish a p-typical variant of Lazard's theorem: The functor from the category of commutative $\mathbf{Z}_p$-algebras with unit to the category of sets which sends A to $\mathrm{FGL}_{A,p}$ is representable. Proof: Let $\epsilon_p : L \otimes \mathbf{Z}_p \to L \otimes \mathbf{Z}_p$ be the homomorphism classifying $\epsilon_{L\otimes\mathbf{Z}_p} F_L$—then ϵ_p is idempotent, $F_V = \epsilon_{L\otimes\mathbf{Z}_p} F_L$ is defined over $V = \operatorname{im}\epsilon_p$, and F_V is the universal p-typical FGL.

[Note: Structurally, $V = \mathbf{Z}_p[v_1, v_2, \ldots]$, a polynomial algebra on generators v_i of degree $-2(p^i-1)$.]

Remark: To explain the origin of the Quillen idempotent, identify $L \otimes \mathbf{Z}_p$ with $\mathbf{MU}^*(\mathbf{S}) \otimes \mathbf{Z}_p$, so $F_L \leftrightarrow F_{\mathbf{MU}}$. Let $\phi_p : F_{\mathbf{MU}} \to F_V$ be the natural strict isomorphism provided by Cartier, put $x_{\mathbf{MU}_p} = \phi_p(x_{\mathbf{MU}}) \in \mathbf{MU}_p^2(\mathbf{P}^\infty(\mathbf{C}))$ (a complex orientation of $\mathbf{MU}_p$), and let $\mathbf{e}_p : \mathbf{MU}_p \to \mathbf{MU}_p$ be the unique morphism of ring spectra such that $\mathbf{e}_p \circ x_{\mathbf{MU}} = x_{\mathbf{MU}_p}$—then from the definitions, $\mathbf{e}_p \circ \mathbf{e}_p \circ x_{\mathbf{MU}} = \mathbf{e}_p \circ x_{\mathbf{MU}}$, hence $\mathbf{e}_p$ is idempotent: $\mathbf{e}_p \circ \mathbf{e}_p = \mathbf{e}_p$.

[Note: $\mathbf{BP}$ is a commutative ring spectrum with complex orientation $x_{\mathbf{BP}}$. The associated FGL $F_{\mathbf{BP}}$ is p-typical and the map $V \to \mathbf{BP}^*(\mathbf{S})$ classifying $F_{\mathbf{BP}}$ is an isomorphism of graded commutative $\mathbf{Z}_p$-algebras with unit. Therefore $\pi_*(\mathbf{MU}_p) = \pi_*(\mathbf{BP}) \otimes_{\mathbf{Z}_p} \mathbf{Z}_p[x_1, \ldots, \widehat{x}_{p-1}, x_p, \ldots, \widehat{x}_{p^2-1}, x_{p^2}, \ldots]$. Now let S be the set of monomials drawn from $\{x_k : k \neq p^i - 1 \ \forall\, i\}$. Given an $x_I \in S$, write d_I for its degree and call $\mathbf{f}_I$ the composite $\mathbf{S}^{d_I} \wedge \mathbf{BP} \to \mathbf{MU}_p \wedge \mathbf{MU}_p \to \mathbf{MU}_p$—then the wedge of the $\mathbf{f}_I$ defines a morphism $\bigvee_{x_I \in S} \Sigma^{d_I} \mathbf{BP} \to \mathbf{MU}_p$ which induces an isomorphism in homotopy.]

Rappel: If $F \in \mathrm{FGL}_{A,p}$ and if $\phi(x) = \sum_{i\geq 1} \phi_i x^i \in A[[x]]$ with $\phi'(0) = 1$, then the formal group law $G(x,y) = \phi(F(\phi^{-1}(x), \phi^{-1}(y)))$ is p-typical iff $\phi^{-1}(x)$ has the form $x +_F a_1 x^p +_F a_2 x^{p^2} +_F \cdots (a_i \in A)$.

Set $VT = V[t_1, t_2, \ldots]$, a polynomial algebra on indeterminates t_i ($|t_i| = -2(p^i - 1)$)—then the pair (V, VT) is a Hopf algebroid over $\mathbf{Z}_p$, i.e., is a cogroupoid object in the category of commutative $\mathbf{Z}_p$-algebras with unit (cf. Proposition 17). Thus let A be a commutative $\mathbf{Z}_p$-algebra with unit. Denoting by $\mathbf{G}_{A,p}$ the groupoid whose objects are the p-typical formal group laws over A and whose morphisms are the strict isomorphisms, the functor from the category of commutative $\mathbf{Z}_p$-algebras with unit to the category of groupoids which sends A to $\mathbf{G}_{A,p}^{\mathrm{OP}}$ is represented by (V, VT). Indeed, $\mathrm{Hom}(V, A) \leftrightarrow \mathrm{FGL}_{A,p} = \mathrm{Ob}\,\mathbf{G}_{A,p}$ $(= \mathrm{Ob}\,\mathbf{G}_{A,p}^{\mathrm{OP}})$ and this identifies the objects. Turning to the morphisms, suppose that $f \in \mathrm{Hom}(VT, A)$. Put $F = (f|V)_* F_V$ and let $\phi : F \to G$ be the morphism with $\phi^{-1}(x) = x +_F f(t_1)x^p +_F f(t_2)x^{p^2} +_F \cdots$, so $\phi^{\mathrm{OP}} : G \to F$ is a strict isomorphism, where $G(x,y) = \phi(F(\phi^{-1}(x), \phi^{-1}(y)))$ is again p-typical.

[Note: η_L is the inclusion $V \to VT$ but there is no simple explicit formula for η_R. Incidentally, the groupoid $\mathbf{G}_{A,p}$ is not split.]

To understand the grading on V and VT, define an action $A^\times \times \text{Ob}\,\mathbf{G}_{A,p}^{\text{OP}} \to \text{Ob}\,\mathbf{G}_{A,p}^{\text{OP}}$ by $(u, F) \to F^u$, where $F^u(x,y) = uF(u^{-1}x, u^{-1}y)$, and define an action $A^\times \times \text{Mor}\,\mathbf{G}_{A,p}^{\text{OP}} \to \text{Mor}\,\mathbf{G}_{A,p}^{\text{OP}}$ by $(u, \phi^{\text{OP}}) \to (\phi^u)^{\text{OP}}$, where $\phi^u(x) = u\phi(u^{-1}X)$—then this action grades V and VT and one can check that $|v_i| = -2(p^i - 1) = |t_i|$. Because the five arrows of structure η_R, η_L, ϵ, Δ, c are gradation preserving, it follows that (V, VT) is a graded Hopf algebroid over $\mathbf{Z}_p$.

[Note: Therefore $(V, VT)^{\text{OP}}$ is but another name for $(\mathbf{BP}_*(\mathbf{S}), \mathbf{BP}_*(\mathbf{BP}))$ and $\mathbf{BP}_*(\mathbf{BP})$ is a graded free $\mathbf{BP}_*(\mathbf{S})$-module.]

FACT (BP Nilpotence Technology) Let $\mathbf{E}$ be a p-local ring spectrum and consider the Hurewicz homomorphism $\mathbf{S}_*(\mathbf{E}) \to \mathbf{BP}_*(\mathbf{E})$ (cf. p. 17–8 ff.)—then the homogeneous elements of its kernel are nilpotent (Devinatz-Hopkins-Smith[†]).

Application: If $\mathbf{X}$ is p-compact and if $\mathbf{f} : \Sigma^n\mathbf{X} \to \mathbf{X}$ is an arrow such that $\mathbf{BP}_*(\mathbf{f}) = 0$, then $\mathbf{f}$ is composition nilpotent (cf. p. 17–17 ff.).

Application: If $\mathbf{X}$ is p-compact and $\mathbf{Y}$ is p-local and if $\mathbf{f} : \mathbf{X} \to \mathbf{Y}$ is an arrow such that $\text{id}_{\mathbf{BP}} \wedge \mathbf{f} = 0$, then $\mathbf{f}$ is smash nilpotent (cf. p. 17–18).

[Note: Write $\mathbf{X} = \overline{\mathbf{X}}_p$, where $\overline{\mathbf{X}}$ is compact (cf. p. 17–27)—then $\hom(\mathbf{X}, \mathbf{Y}) \approx \hom(\overline{\mathbf{X}}, \mathbf{Y}) \approx D\overline{\mathbf{X}} \wedge \mathbf{Y} \approx D\overline{\mathbf{X}} \wedge \mathbf{S}_p \wedge \mathbf{Y} \approx \hom(\overline{\mathbf{X}}, \mathbf{S}_p) \wedge \mathbf{Y} \approx \hom(\overline{\mathbf{X}}_p, \mathbf{S}_p) \wedge \mathbf{Y} \approx \hom(\mathbf{X}, \mathbf{S}_p) \wedge \mathbf{Y}$ and $\hom(\mathbf{X}, \mathbf{S}_p)$ is the dual of $\mathbf{X}$ in $\mathbf{HSPEC}_p$.]

There are two particularly important classes of spectra attached to $\mathbf{BP}$, viz. the $\mathbf{K}(n)$ and the $\mathbf{P}(n)$ $(0 < n < \infty)$ with $\pi_*(\mathbf{K}(n)) = \mathbf{F}_p[v_n, v_n^{-1}]$ and $\pi_*(\mathbf{P}(n)) = \mathbf{F}_p[v_n, v_{n+1}, \ldots]$. Both are p-local ring spectra (commutative if $p > 2$) and $\mathbf{BP}$-module spectra but the exact details of their construction need not detain us since all that really counts are the properties possessed by them, which will be listed below. Example: $\mathbf{P}(1) \approx \mathbf{BP} \wedge \mathbf{M}(p)$.

[Note: The theory has been surveyed by Würgler[‡].]

The role of the $\mathbf{P}(n)$ is basically technical. Since $v_n \in \pi_{2(p^n-1)}(\mathbf{P}(n))$, one can form $\overline{v}_n : \Sigma^{2(p^n-1)}\mathbf{P}(n) \to \mathbf{P}(n)$ (cf. p. 15–46)—then there is an exact triangle $\Sigma^{2(p^n-1)}\mathbf{P}(n) \xrightarrow{\overline{v}_n} \mathbf{P}(n) \to \mathbf{P}(n+1) \to \Sigma^{2p^n-1}\mathbf{P}(n)$. Moreover, $\langle \mathbf{K}(n) \rangle = \langle \overline{v}_n^{-1}\mathbf{P}(n) \rangle$ and $\mathbf{H}(\mathbf{F}_p) \approx \text{tel}(\mathbf{P}(1) \to \mathbf{P}(2) \to \cdots)$. On the other hand, $\langle \mathbf{BP} \rangle = \langle \mathbf{H}(\mathbf{Q}) \rangle \vee \langle \mathbf{P}(1) \rangle$ and $\langle \mathbf{P}(n) \rangle = \langle \mathbf{K}(n) \rangle \vee \langle \mathbf{P}(n+1) \rangle$ (cf. §15, Proposition 43), hence $\langle \mathbf{BP} \rangle = \langle \mathbf{H}(\mathbf{Q}) \rangle \vee \langle \mathbf{K}(1) \rangle \vee \cdots \vee$

[†] *Ann. of Math.* **128** (1988), 207–241.

[‡] *SLN* **1474** (1991), 111–138.

$\langle \mathbf{K}(n)\rangle \vee \langle \mathbf{P}(n+1)\rangle$. In addition, $\langle \mathbf{H}(\mathbf{Q})\rangle \wedge \langle \mathbf{P}(1)\rangle = \langle 0\rangle$, $\langle \mathbf{K}(i)\rangle \wedge \langle \mathbf{P}(n+1)\rangle = \langle 0\rangle$ $(i = 1, \ldots, n)$.

By contrast, $\mathbf{K}(n)$ (called the nth Morava K-theory at the prime p) is a major player.

(Mo$_1$) $\mathbf{K}(n)$ is a skew field object in **HSPEC**.

[This is because the homogeneous elements of $\pi_*(\mathbf{K}(n))$ are invertible (cf. §15, Proposition 42).]

(Mo$_2$) $\forall\, \mathbf{X}$, $\mathbf{K}(n) \wedge \mathbf{X}$ is isomorphic to a wedge of suspensions of $\mathbf{K}(n)$.

[$\mathbf{K}(n) \wedge \mathbf{X}$ is a $\mathbf{K}(n)$-module, thus the assertion follows from the definition of a skew field object (to accommodate $\mathbf{K}(n) \wedge \mathbf{X} = 0$, use the empty wedge).]

(Mo$_3$) $\forall\, \mathbf{X}$ & $\forall\, \mathbf{Y}$, $\mathbf{K}(n)_*(\mathbf{X}) \otimes_{\mathbf{K}(n)_*(\mathbf{S})} \mathbf{K}(n)_*(\mathbf{Y}) \approx \mathbf{K}(n)_*(\mathbf{X} \wedge \mathbf{Y})$.

[This is a special case of Proposition 10.]

(Mo$_4$) $\langle \mathbf{K}(n)\rangle \wedge \langle \mathbf{K}(m)\rangle = \langle 0\rangle$ $(m \neq n)$.

[Suppose that $n < m$—then $\langle \mathbf{K}(m)\rangle \leq \langle \mathbf{P}(m)\rangle \leq \langle \mathbf{P}(n+1)\rangle$ and $\langle \mathbf{K}(n)\rangle \wedge \langle \mathbf{P}(n+1)\rangle = \langle 0\rangle$.]

(Mo$_5$) $\langle \mathbf{H}(\mathbf{Q})\rangle \wedge \langle \mathbf{K}(n)\rangle = \langle 0\rangle$ & $\langle \mathbf{H}(\mathbf{F}_p)\rangle \wedge \langle \mathbf{K}(n)\rangle = \langle 0\rangle$.

[$\langle \mathbf{H}(\mathbf{Q})\rangle \wedge \langle \mathbf{P}(1)\rangle = \langle 0\rangle$ and $\langle \mathbf{K}(n)\rangle \leq \langle \mathbf{P}(n)\rangle \Rightarrow \langle \mathbf{H}(\mathbf{Q})\rangle \wedge \langle \mathbf{K}(n)\rangle = \langle 0\rangle$. And: $\mathbf{H}(\mathbf{F}_p) \approx \operatorname{tel}(\mathbf{P}(1) \to \mathbf{P}(2) \to \cdots) \Rightarrow \langle \mathbf{H}(\mathbf{F}_p)\rangle \leq \langle \mathbf{P}(n+1)\rangle \Rightarrow \langle \mathbf{H}(\mathbf{F}_p)\rangle \wedge \langle \mathbf{K}(n)\rangle = \langle 0\rangle$.]

(Mo$_6$) $\forall$ compact $\mathbf{X}$, $\mathbf{K}(n)_*(\mathbf{X}) \approx \mathbf{K}(n)_*(\mathbf{S}) \otimes_{\mathbf{F}_p} \mathbf{H}_*(\mathbf{X}; \mathbf{F}_p)$ $\forall\, n >> 0$.

[Apply the Atiyah-Hirzebruch spectral sequence.]

Remarks: (1) $\mathbf{K}(n)$ is complex orientable if p is odd; (2) $\mathbf{K}(1)$ can be identified with $\mathbf{KU}_p(1) \wedge \mathbf{M}(p)$ (cf. p. 17–27).

EXAMPLE (Algebraic K-Theory) Suppose that A is a ring with unit and let $\mathbf{W}A$ be the Ω-prespectrum attached to A by algebraic K-theory (cf. p. 14–72). Consider $\mathbf{K}A = eM\mathbf{W}A$—then Mitchell[†] has shown that $\forall\, p$ & $\forall\, n \geq 2$, the connective cover of $\mathbf{K}A$ is $\mathbf{K}(n)_*$-acyclic.

FACT Let $\mathbf{k}(n)$ be the connective cover of $\mathbf{K}(n)$—then $\mathbf{k}(n)$ is a ring spectrum (cf. p. 17–8) and $\mathbf{K}(n) \approx \overline{v}_n^{-1}\mathbf{k}(n)$ (cf. p. 15–46).

[Note: There is an exact triangle $\Sigma^{2(p^n-1)}\mathbf{k}(n) \xrightarrow{\overline{v}_n} \mathbf{k}(n) \to \mathbf{H}(\mathbf{F}_p) \to \Sigma^{2p^n-1}\mathbf{k}(n)$, so by §15, Proposition 43, $\langle \mathbf{k}(n)\rangle = \langle \mathbf{H}(\mathbf{F}_p)\rangle \vee \langle \mathbf{K}(n)\rangle$.]

LEMMA Any retract of a $\mathbf{K}(n)$-module is a $\mathbf{K}(n)$-module.

EXAMPLE A spectrum $\mathbf{Y}$ is indecomposable if it has no nontrivial direct summands, i.e., $\mathbf{Y} \approx \mathbf{X} \vee \mathbf{Z} \Rightarrow \mathbf{X} = 0$ or $\mathbf{Z} = 0$. Since idempotents split (cf. p. 15–17), $\mathbf{Y}$ is indecomposable iff $[\mathbf{Y}, \mathbf{Y}]$ has no nontrivial idempotents. Example: $\mathbf{K}(n)$ is indecomposable.

[†] *K-Theory* **3** (1990), 607–626.

[Note: One can also prove that **BP** is indecomposable.]

Notation: For uniformity of statement, it is convenient to put $\mathbf{K}(0) = \mathbf{H}(\mathbf{Q})$, $\mathbf{K}(\infty) = \mathbf{H}(\mathbf{F}_p)$.

Hovey[†] has shown that $\langle \mathbf{K}(n) \rangle$ is minimal if $n < \infty$ (but this is false if $n = \infty$).

LEMMA Given $\mathbf{f} : \mathbf{X} \to \mathbf{Y}$, suppose that $\mathbf{K}(n)_*(\mathbf{f}) = 0$, where $n \in [0, \infty]$—then the composite $\mathbf{X} \xrightarrow{\mathbf{f}} \mathbf{Y} \approx \mathbf{S} \wedge \mathbf{Y} \to \mathbf{K}(n) \wedge \mathbf{Y}$ vanishes.

[For any $\mathbf{K}(n)$-module $\mathbf{E}$, $\begin{cases} \mathbf{E}^*(\mathbf{X}) \approx \mathrm{Hom}_{\pi_*(\mathbf{K}(n))}(\mathbf{K}(n)_*(\mathbf{X}), \pi_*(\mathbf{E})) \\ \mathbf{E}^*(\mathbf{Y}) \approx \mathrm{Hom}_{\pi_*(\mathbf{K}(n))}(\mathbf{K}(n)_*(\mathbf{Y}), \pi_*(\mathbf{E})) \end{cases}$, hence the induced map $\mathbf{E}^*(\mathbf{Y}) \to \mathbf{E}^*(\mathbf{X})$ is the zero map. Now specialize to $\mathbf{E} = \mathbf{K}(n) \wedge \mathbf{Y}$.]

PROPOSITION 27 If $\mathbf{X}$ is p-compact and $\mathbf{Y}$ is p-local and if $\mathbf{f} : \mathbf{X} \to \mathbf{Y}$ is an arrow such that $\mathbf{K}(n)_*(\mathbf{f}) = 0 \ \forall \ n \in [0, \infty]$, then $\mathbf{f}$ is smash nilpotent.

[It is enough to prove that $\mathrm{id}_{\mathbf{BP}} \wedge \mathbf{f}^{(k)} = 0$ ($\exists \ k >> 0$) (cf. p. 17–29) and for this, one can take $\mathbf{X} = \mathbf{S}_p$. So, passing to $\mathbf{Y}_{\mathbf{f}}^{(\infty)}$ (defined by $\mathbf{S}_p$ instead of $\mathbf{S}$ (cf. p. 15–46)), it suffices to show that $\mathbf{BP} \wedge \mathbf{Y}_{\mathbf{f}}^{(\infty)} = 0$. But $\langle \mathbf{BP} \rangle = \langle \mathbf{K}(0) \rangle \vee \cdots \vee \langle \mathbf{K}(n) \rangle \vee \langle \mathbf{P}(n+1) \rangle$ and from our hypotheses and the lemma, $\mathbf{K}(m) \wedge \mathbf{Y}_{\mathbf{f}}^{(\infty)} = 0$ ($m \leq n$), thus we are left with proving that $\mathbf{P}(n) \wedge \mathbf{Y}_{\mathbf{f}}^{(\infty)} = 0$ ($n >> 0$), which however is clear since $\mathbf{H}(\mathbf{F}_p) \wedge \mathbf{Y}_{\mathbf{f}}^{(\infty)} = 0$ and $\mathbf{H}(\mathbf{F}_p) \approx \mathrm{tel}(\mathbf{P}(1) \to \mathbf{P}(2) \to \cdots)$.]

Application: If $\mathbf{E} \neq 0$ is a p-local ring spectrum, then for some $n \in [0, \infty]$, $\mathbf{K}(n)_*(\mathbf{E}) \neq 0$.

[Consider the unit $\mathbf{S}_p \to \mathbf{E}$.]

Let **R** be a ring spectrum—then **R** is said to detect nilpotence if for any ring spectrum **E**, the homogeneous elements of the kernel of the Hurewicz homomorphism $\mathbf{S}_*(\mathbf{E}) \to \mathbf{R}_*(\mathbf{E})$ are nilpotent. Example: **MU** detects nilpotence (cf. p. 17–17).

LEMMA **R** detects nilpotence iff for all compact **X** and any $\mathbf{f} : \mathbf{X} \to \mathbf{Y}$ such that $\mathrm{id}_{\mathbf{R}} \wedge \mathbf{f} = 0$, $\mathbf{f}$ is smash nilpotent.

[Necessity: Argue as on p. 17–18, with **MU** replaced by **R**.

Sufficiency: Given a ring spectrum **E**, fix a homogeneous element $\mathbf{f} : \mathbf{S}^n \to \mathbf{E}$ in the kernel of the Hurewicz homomorphism $\mathbf{S}_*(\mathbf{E}) \to \mathbf{R}_*(\mathbf{E})$—then $\mathrm{id}_{\mathbf{R}} \wedge \mathbf{f} = 0$, so $\mathbf{f}$ is smash nilpotent, thus nilpotent.]

[†] *Contemp. Math.* **181** (1995), 230.

Remark: For a compact $\mathbf{X}$, $\mathbf{f} : \mathbf{X} \to \mathbf{Y}$ is smash nilpotent iff $\bar{\mathbf{f}} : \mathbf{S} \to D\mathbf{X} \wedge \mathbf{Y}$ is smash nilpotent (cf. p. 17–18). This said, the problem of determining the smash nilpotency of $\mathbf{f} : \mathbf{S} \to \mathbf{Y}$ is local, i.e., one has only to check that $\mathbf{f}_p : \mathbf{S}_p \to \mathbf{Y}_p$ is smash nilpotent $\forall\, p$. Proof: $\mathbf{f} : \mathbf{S} \to \mathbf{Y}$ is smash nilpotent iff $\mathbf{Y}_{\mathbf{f}}^{(\infty)} = 0$ (cf. p. 15–46). But $\mathbf{Y}_{\mathbf{f}}^{(\infty)} = 0$ iff $(\mathbf{Y}_{\mathbf{f}}^{(\infty)})_p = 0 \;\forall\, p$ (cf. Proposition 26). And: $(\mathbf{Y}_{\mathbf{f}}^{(\infty)})_p = \mathbf{Y}_{\mathbf{f}_p}^{(\infty)}$.

EXAMPLE A ring spectrum $\mathbf{R}$ detects nilpotence iff $\forall\, p$ & $\forall\, n \in [0, \infty]$, $\mathbf{K}(n)_*(\mathbf{R}) \neq 0$.

[Consider an $\mathbf{f} : \mathbf{S} \to \mathbf{Y}$ such that $\mathrm{id}_{\mathbf{R}} \wedge \mathbf{f} = 0$. Fixing p, one has $\mathbf{K}(n)_*(\mathbf{f}_p) = 0 \;\forall\, n \in [0, \infty]$ ($\mathbf{K}(n) \wedge \mathbf{R}$ is isomorphic to a wedge of suspensions of $\mathbf{K}(n)$), thus by Proposition 27, $\mathbf{f}_p$ is smash nilpotent. Therefore $\mathbf{R}$ detects nilpotence.]

FACT Suppose that $\mathbf{E}$ is a skew field object in **HSPEC**—then $\mathbf{E}$ is isomorphic to a wedge of suspensions of some $\mathbf{K}(n)$ ($\exists\, n \in [0, \infty]$).

[$\exists\, p : \mathbf{E}_p \neq 0$ (cf. Proposition 26) $\Rightarrow \mathbf{K}(n)_*(\mathbf{E}) \neq 0$ ($\exists\, n \in [0, \infty]$) (cf. p. 17–31). Since $\mathbf{K}(n)$ and $\mathbf{E}$ are both skew field objects, $\mathbf{K}(n) \wedge \mathbf{E} \neq 0$ is simultaneously a wedge of suspensions of $\mathbf{K}(n)$ and a wedge of suspensions of $\mathbf{E}$. Deduce that $\mathbf{E}$ is a retract of a wedge of suspensions of $\mathbf{K}(n)$, hence is a $\mathbf{K}(n)$-module (cf. p. 17–30).]

A skew field object in **HSPEC** is said to be prime if it is indecomposable. The $\mathbf{K}(n)$ ($n \in [0, \infty]$) for $p \in \mathbf{\Pi}$ are prime and the preceding result implies that, up to isomorphism, they are the only primes in **HSPEC**.

EXAMPLE Suppose that p is odd—then $\mathbf{KU}_p(1) \wedge \mathbf{M}(p)$ is a field object (being isomorphic to $\mathbf{K}(1) \vee \Sigma^2\mathbf{K}(1) \vee \cdots \vee \Sigma^{2(p-2)}\mathbf{K}(1)$ (cf. p. 17–27)) but it is not prime.

PROPOSITION 28 Fix a prime p—then $\mathbf{H}(\mathbf{F}_p)$ is $\mathbf{K}(n)_*$-acyclic ($n \in [0, \infty[$).

[Trivially, $\mathbf{H}(\mathbf{Q}) \wedge \mathbf{H}(\mathbf{F}_p) = 0$. Proceeding by contradiction, assume that $\mathbf{K}(n) \wedge \mathbf{H}(\mathbf{F}_p) \neq 0$ for some $n \in [1, \infty[$. Since $\mathbf{H}(\mathbf{F}_p)$ is a field object, $\mathbf{H}(\mathbf{F}_p)$ is isomorphic to a wedge of suspensions of $\mathbf{K}(n)$ (cf. supra), an impossibility.]

FACT Let $\mathbf{X}$ be a spectrum with the property that $\exists\, N : \pi_n(\mathbf{X}) = 0$ ($n > N$)—then $\mathbf{X}$ is $\mathbf{K}(n)_*$-acyclic ($n \in [1, \infty[$).

[Using Proposition 28, prove it first under the assumption that $\pi_*(\mathbf{X})$ is torsion. To handle the general case, smash $\mathbf{S} \xrightarrow{p} \mathbf{S} \to \mathbf{M}(p) \to \Sigma\mathbf{S}$ with $\mathbf{K}(n) \wedge \mathbf{X}$ to see that $\pi_*(\mathbf{K}(n) \wedge \mathbf{X})$ injects into $\pi_*(\mathbf{K}(n) \wedge \mathbf{X} \wedge \mathbf{M}(p))$. But Proposition 19 implies that $\mathbf{X} \wedge \mathbf{M}(p)$, like $\mathbf{X}$, is "bounded above", and $\pi_*(\mathbf{X} \wedge \mathbf{M}(p))$ is torsion.]

[Note: In particular, $\mathbf{K}(n) \wedge \mathbf{H}(\pi) = 0$ ($n \in [1, \infty[$, π any abelian group.]

Application: If $\mathbf{X}$ is a spectrum and $\mathbf{x}$ ($= \tau^{\leq 0}\mathbf{X}$) is its connective cover, then the arrow $\mathbf{x} \to \mathbf{X}$ is a $\mathbf{K}(n)_*$-equivalence ($n \in [1, \infty[$).

[For $\mathbf{K}(n) \wedge \mathbf{F} = 0$, where $\mathbf{F}$ is defined by the exact triangle $\mathbf{F} \rightarrow \mathbf{x} \rightarrow \mathbf{X} \rightarrow \Sigma\mathbf{F}$.]

[Note: Let A be a ring with unit—then $\forall\, p$ & $\forall\, n \geq 2$, the connective cover of $\mathbf{K}A$ is $\mathbf{K}(n)_*$-acyclic (cf. p. 17–30), hence so is $\mathbf{K}A$ itself.]

PROPOSITION 29 If $\mathbf{X}$ is p-compact and if $\mathbf{f} : \Sigma^d\mathbf{X} \rightarrow \mathbf{X}$ is an arrow such that $\mathbf{K}(n)_*(\mathbf{f}) = 0 \;\forall\, n \in [0, \infty[$, then $\mathbf{f}$ is composition nilpotent.

[This is a consequence of Proposition 27 (one doesn't need the $n = \infty$ case).]

EXAMPLE If $\mathbf{X}$ is p-compact and if $\mathbf{K}(n)_*(\mathbf{X}) = 0$ ($\forall\, n \in [0, \infty[$), then $\mathbf{X} = 0$ (in Proposition 29, take $\mathbf{f} = \mathrm{id}_{\mathbf{X}}$).

[Note: Accordingly, if $\mathbf{X}$ is compact and if $\forall\, p$ & $\forall\, n \in [0, \infty[$, $\mathbf{K}(n)_*(\mathbf{X}) = 0$, then $\mathbf{X} = 0$. In fact, $\mathbf{K}(n)_*(\mathbf{X}) = \pi_*(\mathbf{K}(n) \wedge \mathbf{X}) = \pi_*(\mathbf{K}(n) \wedge \mathbf{X}_p) = \mathbf{K}(n)_*(\mathbf{X}_p) \Rightarrow \mathbf{X}_p = 0 \;\forall\, p \Rightarrow \mathbf{X} = 0$ (cf. Proposition 26).]

Given a prime p, write $\mathbf{C}(0)$ for cpt $\mathbf{HSPEC}_p$ and let $\mathbf{C}(n)$ be the thick subcategory of $\mathbf{C}(0)$ whose objects are those $\mathbf{X}$ such that $\mathbf{K}(n-1)_*(\mathbf{X}) = 0$ ($n \in [1, \infty[$) (conventionally, the objects of $\mathbf{C}(\infty)$ are the zero objects)—then $\mathbf{C}(n+1) \subset \mathbf{C}(n)$, i.e., $\mathbf{K}(n)_*(\mathbf{X}) = 0 \Rightarrow \mathbf{K}(n-1)_*(\mathbf{X}) = 0$ (Ravenel[†]) and the containment is strict (Mitchell[‡]).

[Note: A p-compact $\mathbf{X}$ is said to have type n if $n = \min\{m : \mathbf{K}(m)_*(\mathbf{X}) \neq 0\}$ ($\mathbf{X} = 0$ has type ∞). The objects of type n are the objects in $\mathbf{C}(n)$ which are not in $\mathbf{C}(n+1)$. Examples: (1) $\mathbf{S}_p$ has type 0; (2) $\mathbf{M}(p)$ has type 1; (3) $\mathbf{coA}_p$ has type 2.]

LEMMA Let $\mathbf{X}$ be a p-compact spectrum, $\mathbf{E}$ a p-local ring spectrum. Suppose given a p-local spectrum $\mathbf{Z}$ and a morphism $\mathbf{f} : \mathbf{X} \rightarrow \mathbf{E} \wedge \mathbf{Z}$ in $\mathbf{HSPEC}_p$ such that $\mathbf{K}(n)_*(\mathbf{f}) = 0 \;\forall\, n \in [0, \infty]$—then the composite $\mathbf{X}^{(N)} \xrightarrow{\mathbf{f}^{(N)}} (\mathbf{E} \wedge \mathbf{Z})^{(N)} \approx \mathbf{E}^{(N)} \wedge \mathbf{Z}^{(N)} \rightarrow \mathbf{E} \wedge \mathbf{Z}^{(N)}$ vanishes if $N >> 0$ (cf. Proposition 27).

Application: Let $\mathbf{X}, \mathbf{Y}$ be p-compact spectra. Suppose given a p-local spectrum $\mathbf{Z}$ and a morphism $\mathbf{f} : \mathbf{X} \rightarrow \mathbf{Z}$ in $\mathbf{HSPEC}_p$ such that $\mathbf{K}(n)_*(\mathbf{f} \wedge \mathrm{id}_{\mathbf{Y}}) = 0 \;\forall\, n \in [0, \infty]$—then $\mathbf{f}^{(N)} \wedge \mathrm{id}_{\mathbf{Y}} : \mathbf{X}^{(N)} \wedge \mathbf{Y} \rightarrow \mathbf{Z}^{(N)} \wedge \mathbf{Y}$ vanishes if $N >> 0$.

[One has $[\mathbf{X} \wedge \mathbf{Y}, \mathbf{Z} \wedge \mathbf{Y}] \approx [\mathbf{X}, \hom(\mathbf{Y}, \mathbf{Z} \wedge \mathbf{Y})]$. But $\mathbf{Y}$ is p-compact, so $\hom(\mathbf{Y}, \mathbf{Z} \wedge \mathbf{Y}) \approx \hom(\mathbf{Y}, \mathbf{S}_p) \wedge \mathbf{Y} \wedge \mathbf{Z} \approx \hom(\mathbf{Y}, \mathbf{Y}) \wedge \mathbf{Z}$. Now specialize the lemma to $\mathbf{E} = \hom(\mathbf{Y}, \mathbf{Y})$.]

THICK SUBCATEGORY THEOREM The thick subcategories of $\mathbf{C}(0)$ are the $\mathbf{C}(n)$.

[†] *Amer. J. Math.* **106** (1984), 351–414 (cf. 366-367).

[‡] *Topology* **24** (1985), 227–246; see also Palmieri-Sadofsky, *Math. Zeit.* **215** (1994), 477–490.

[Fix a thick subcategory $\mathbf{C}$ of $\mathbf{C}(0)$ and let $n_{\mathbf{C}} = \min\{n : \mathbf{C}(n) \subset \mathbf{C}\}$. Claim: If $X \in \operatorname{Ob}\mathbf{C}$ has type n, then $\mathbf{C}(n) \subset \mathbf{C}$ ($\Rightarrow \mathbf{C} = \mathbf{C}(n_{\mathbf{C}})$). Define $\mathbf{F}, \mathbf{f}$ by the exact triangle $\mathbf{F} \xrightarrow{\mathbf{f}} \mathbf{S}_p \to \hom(\mathbf{X},\mathbf{X}) \to \Sigma\mathbf{F}$. Because $\mathbf{HSPEC}_p$ is monogenic ($\Rightarrow$ unital), $\hom(\mathbf{X},\mathbf{S}_p)$ is p-compact, so $\hom(\mathbf{X},\mathbf{X}) \approx \hom(\mathbf{X},\mathbf{S}_p) \wedge \mathbf{X} \in \operatorname{Ob}\mathbf{C}$ ($\mathbf{C}$ being thick (cf. p. 15–41)). Putting $\mathbf{C}_\mathbf{f} = \hom(\mathbf{X},\mathbf{X})$, one thus concludes that $\mathbf{F} \wedge \mathbf{C}_\mathbf{f} \in \operatorname{Ob}\mathbf{C}$ (here again the assumption that $\mathbf{C}$ is thick comes in). But there is an exact triangle $\mathbf{F} \wedge \mathbf{C}_{\mathbf{f}^{(N-1)}} \to \mathbf{C}_{\mathbf{f}^{(N)}} \to \mathbf{C}_\mathbf{f} \wedge \mathbf{S}_p^{(N-1)} \to \Sigma(\mathbf{F} \wedge \mathbf{C}_{\mathbf{f}^{(N-1)}})$ (cf. p. 16–30), from which inductively, $\mathbf{C}_{\mathbf{f}^{(N)}} \in \operatorname{Ob}\mathbf{C}\ \forall\ N \geq 1$. Take a $\mathbf{Y}$ in $\mathbf{C}(n)$. Since $\mathbf{K}(m)_*(\mathbf{f} \wedge \mathrm{id}_\mathbf{Y}) = 0\ \forall\ m \in [0,\infty](\mathbf{K}(m)_*(\mathbf{X}) \neq 0\ \forall\ m \geq n)$, $\forall\ N >> 0$, $\mathbf{f}^{(N)} \wedge \mathrm{id}_\mathbf{Y} = 0$ (cf. supra). Working with the exact triangle $\mathbf{F}^{(N)} \wedge \mathbf{Y} \xrightarrow{\mathbf{f}^{(N)} \wedge \mathrm{id}_\mathbf{Y}} \mathbf{S}_p^{(N)} \wedge \mathbf{Y} \to \mathbf{C}_{\mathbf{f}^{(N)}} \wedge \mathbf{Y} \to \Sigma(\mathbf{F}^{(N)} \wedge \mathbf{Y})$, it then follows that $\mathbf{C}_{\mathbf{f}^{(N)}} \wedge \mathbf{Y} \approx (\mathbf{S}_p^{(N)} \wedge \mathbf{Y}) \vee \Sigma(\mathbf{F}^{(N)} \wedge \mathbf{Y})$ (cf. p. 15–5). And: $\mathbf{C}_{\mathbf{f}^{(N)}} \wedge \mathbf{Y} \in \operatorname{Ob}\mathbf{C} \Rightarrow \mathbf{S}_p^{(N)} \wedge \mathbf{Y} \in \operatorname{Ob}\mathbf{C} \Rightarrow \mathbf{Y} \in \operatorname{Ob}\mathbf{C}$.]

EXAMPLE Fix a spectrum $\mathbf{E}$ and write $\mathbf{ACY}_p(\mathbf{E})$ for the class of p-compact $\mathbf{X}$ such that $\mathbf{E} \wedge \mathbf{X} = 0$—then $\mathbf{ACY}_p(\mathbf{E})$ is the object class of a thick subcategory of $\mathbf{C}(0)$, hence $\mathbf{ACY}_p(\mathbf{E}) = \operatorname{Ob}\mathbf{C}(n)$ for some n.

FACT (Class Invariance Principle) Let $\mathbf{X}, \mathbf{Y}$ be p-compact. Suppose that $\mathbf{X}$ has type n and $\mathbf{Y}$ has type m—then $\langle \mathbf{X} \rangle = \langle \mathbf{Y} \rangle$ iff $n = m$.

[The necessity is obvious. To establish the sufficiency, note that the full, isomorphism closed subcategory of cpt $\mathbf{HSEC}_p$ whose objects are the $\mathbf{Z}$ with $\langle \mathbf{Z} \rangle \leq \langle \mathbf{X} \rangle$ is thick.]

Given a prime p and a p-compact $\mathbf{X}$, an arrow $\mathbf{f} : \Sigma^d\mathbf{X} \to \mathbf{X}$ is said to be a v_n-map ($n \in [0,\infty[$) if $\mathbf{K}(n)_*(\mathbf{f})$ is an isomorphism and $\mathbf{K}(m)_*(\mathbf{f}) = 0\ \forall\ m \neq n$ ($m \in [0,\infty[$) (cf. Proposition 29). Example: $\mathbf{X} \xrightarrow{p} \mathbf{X}$ is a v_0-map.

[Note: For $m >> 0$, $\mathbf{K}(m)_*(\mathbf{f}) = \mathbf{H}(\mathbf{F}_p)_*(\mathbf{f}) \otimes_{\mathbf{F}_p} \mathrm{id}_{\mathbf{K}(m)_*} \Rightarrow \mathbf{H}(\mathbf{F}_p)_*(\mathbf{f}) = 0$.]

Example: $\mathbf{A}_p : \Sigma^d\mathbf{M}(p) \to \mathbf{M}(p)$ is a v_1-map ($d = 8$ if $p = 2$ & $d = 2p-2$ if $p > 2$ (cf. p. 17–26)).

PROPOSITION 30 Let $\mathbf{X}$ be p-compact and fix $n \geq 1$. Suppose that $\mathbf{X}$ admits a v_n-map—then $\mathbf{X}$ belongs to $\mathbf{C}(n)$, i.e., $\mathbf{K}(n-1)_*(\mathbf{X}) = 0$.

[Defining $\mathbf{Y}$ by the exact triangle $\Sigma^d\mathbf{X} \xrightarrow{\mathbf{f}} \mathbf{X} \to \mathbf{Y} \to \Sigma^{d+1}\mathbf{X}$, one has $\mathbf{K}(n)_*(\mathbf{Y}) = 0$, thus $0 = \mathbf{K}(n-1)_*(\mathbf{Y}) = \mathbf{K}(n-1)_*(\mathbf{X}) \oplus \mathbf{K}(n-1)_*(\Sigma^{d+1}\mathbf{X}) \Rightarrow \mathbf{K}(n-1)_*(\mathbf{X}) = 0$.]

I shall omit the proof of the following result as it is quite involved.

HOPKINS-SMITH[†] EXISTENCE THEOREM Given $n \geq 1$, $\exists$ a p-compact $\mathbf{X}$ of type n which admits a v_n-map.

[Note: In fact, $\mathbf{X}$ admits a v_n-map $\mathbf{f} : \Sigma^{p^N 2(p^n-1)}\mathbf{X} \to \mathbf{X}$ such that $\mathbf{K}(n)_*(\mathbf{f}) = v_n^{p^N}$ $(N >> 0)$.]

Remark: A p-compact $\mathbf{X}$ admits a v_n-map iff $\mathbf{X}$ is in $\mathbf{C}(n)$. To see this, call $\mathbf{V}_n$ the full, isomorphism closed subcategory of $\mathbf{C}(0)$ (= cpt $\mathbf{HSPEC}_p$) whose objects are those $\mathbf{X}$ which admit a v_n-map. Owing to Proposition 30, $\mathbf{C}(n) \supset \mathbf{V}_n$. On the other hand, $\mathbf{X} \xrightarrow{0} \mathbf{X}$ is a v_n-map if $\mathbf{K}(n)_*(\mathbf{X}) = 0$, so $\mathbf{V}_n \supset \mathbf{C}(n+1)$. However $\mathbf{V}_n$ is thick (cf. p. 17–36), hence by the thick subcategory theorem, either $\mathbf{V}_n = \mathbf{C}(n)$ or $\mathbf{V}_n = \mathbf{C}(n+1)$. Since the containment $\mathbf{C}(n+1) \subset \mathbf{C}(n)$ is proper, the Hopkins-Smith existence theorem eliminates the second possibility.

Notation: Write $[\mathbf{X}, \mathbf{X}]_*$ for the graded ring with unit defined by $[\mathbf{X}, \mathbf{X}]_n = [\Sigma^n \mathbf{X}, \mathbf{X}]$ (cf. Proposition 1).

[Note: An arrow $\mathbf{f} : \Sigma^n \mathbf{X} \to \mathbf{X}$ is composition nilpotent iff $\mathbf{f}^k = 0$ for some k or still, is nilpotent when viewed as an element of $[\mathbf{X}, \mathbf{X}]_*$.]

PROPOSITION 31 Let $\mathbf{X}$ be p-compact and fix $n \geq 1$. Suppose that $\mathbf{f} : \Sigma^d \mathbf{X} \to \mathbf{X}$, $\mathbf{g} : \Sigma^e \mathbf{X} \to \mathbf{X}$ are v_n-maps—then $\exists\, i, j : \mathbf{f}^i = \mathbf{g}^j$.

The proof of Proposition 31 rests on the following considerations.

Given a p-compact $\mathbf{X}$ in $\mathbf{C}(n)$ $(n \geq 1)$, put $\mathbf{RX} = \hom(\mathbf{X}, \mathbf{S}_p) \wedge \mathbf{X}$ $(\approx \hom(\mathbf{X}, \mathbf{X}))$—then $\mathbf{RX}$ is a p-compact ring spectrum, $\mathbf{H}(\mathbf{Q}) \wedge \mathbf{RX} = 0$, and $[\mathbf{X}, \mathbf{X}]_* \approx \pi_*(\mathbf{RX})$.

Definition: An element $\alpha \in \pi_d(\mathbf{RX})$ is a $\underline{v_n\text{-element}}$ provided that its image $\mathbf{K}(m)_*(\alpha)$ under the Hurewicz homomorphism $\mathbf{S}_*(\mathbf{RX}) \to \mathbf{K}(m)_*(\mathbf{RX})$ is a unit if $m = n$ and vanishes otherwise $(m \in [1, \infty[)$.

[Note: By contrast, if $\mathbf{K}(m)_*(\alpha) = 0$ $\forall\, m \in [0, \infty[$, then α is nilpotent.]

Example: The adjoint $\bar{\mathbf{f}} \in \pi_d(\mathbf{RX})$ of a v_n-map $\mathbf{f} \in [\mathbf{X}, \mathbf{X}]_d$ is a v_n-element (and conversely).

Claim: Fix a v_n-element α—then $\exists\, i$ such that $\mathbf{K}(n)_*(\alpha^i) = v_n^N$ for some N.

[The ungraded quotient $\mathbf{K}(n)_*(\mathbf{RX})/(v_n - 1)$ is a finite dimensional $\mathbf{F}_p$-algebra, thus its group of units is finite.]

Claim: Fix a v_n-element α—then $\exists\, i$ such that α^i is in the center of $\pi_*(\mathbf{RX})$.

[There is no loss of generality in supposing that $\mathbf{K}(m)_*(\alpha)$ is in the center of $\mathbf{K}(m)_*(\mathbf{RX})$ $\forall\, m \in [0, \infty[$. Letting $\operatorname{ad}(\alpha) : \Sigma^d \mathbf{RX} \to \mathbf{RX}$ be the composite $\mathbf{S}^d \wedge \mathbf{RX} \xrightarrow{\alpha \wedge \mathrm{id}} \mathbf{RX} \wedge \mathbf{RX} \xrightarrow{\mathrm{id} - \mathrm{T}} \mathbf{RX} \wedge \mathbf{RX} \to$

[†] *Ann. of Math.* ; see also Ravenel, *Nilpotence and Periodicity in Stable Homotopy Theory*, Princeton University Press (1992), 53–68.

$\mathbf{RX}$, $\mathrm{ad}(\alpha)_*(\beta) = \alpha\beta - \beta\alpha$ and $\forall\ i$, $\mathrm{ad}(\alpha^i)_*(\beta) = \sum\limits_j \binom{i}{j}\mathrm{ad}^j(\alpha)_*(\beta)\alpha^{i-j}$. Since $p^k\alpha = 0$ for some k and $\mathrm{ad}(\alpha) \in [\mathbf{RX},\mathbf{RX}]_*$ is nilpotent (cf. Proposition 29), one can take $i = p^N$ $(N >> 0)$ to get that $\alpha^i\beta - \beta\alpha^i = 0 \ \forall\ \beta \in \pi_*(\mathbf{RX})$.]

Claim: Fix v_n-elements α, β—then $\exists\ i, j$ such that $\alpha^i = \beta^j$.

[Assuming, as is permissible, that $\alpha\beta = \beta\alpha$ and $\mathbf{K}(m)_*(\alpha - \beta) = 0\ \forall\ m \in [0,\infty[$, use the binomial theorem on $\alpha^{p^N} = (\beta + (\alpha - \beta))^{p^N}$ $(N >> 0)$, observing that $\alpha - \beta$ is both torsion and nilpotent.]

The last claim serves to complete the proof of Proposition 31.

PROPOSITION 32 Let $\mathbf{X}, \mathbf{Y}$ be p-compact and fix $n \geq 1$. Suppose that $\mathbf{f}: \Sigma^d\mathbf{X} \to \mathbf{X}$, $\mathbf{g} : \Sigma^e\mathbf{Y} \to \mathbf{Y}$ are v_n-maps—then $\exists\ i,j$ such that $\forall\ \mathbf{h} \in [\mathbf{X},\mathbf{Y}]$ the diagram

$$\begin{tikzcd}
\Sigma^{id}\mathbf{X} \arrow[r, "\Sigma^{id}\mathbf{h} = \Sigma^{je}\mathbf{h}"] \arrow[d, "\mathbf{f}^i"'] & \Sigma^{je}\mathbf{Y} \arrow[d, "\mathbf{g}^j"] \\
\mathbf{X} \arrow[r, "\mathbf{h}"'] & \mathbf{Y}
\end{tikzcd}$$

commutes.

[Pass to $\hom(\mathbf{X}, \mathbf{S}_p) \wedge \mathbf{Y}$ and apply Proposition 31.]

To round out the discussion on p. 17–35, we shall now verify that $\mathbf{V}_n$ is thick. Obviously, $\mathbf{V}_n$ contains 0 and is stable under $\Sigma^{\pm 1}$. Next, let $\mathbf{X}, \mathbf{Y}$ be objects of $\mathbf{V}_n$ with v_n-maps $\mathbf{f}: \Sigma^d\mathbf{X} \to \mathbf{X}$, $\mathbf{g} : \Sigma^e\mathbf{Y} \to \mathbf{Y}$. Choose i, j per Proposition 32 and put $k = id$ $(= je)$. Take $\mathbf{X} \xrightarrow{\mathbf{u}} \mathbf{Y}$ and complete it to an exact triangle $\mathbf{X} \xrightarrow{\mathbf{u}} \mathbf{Y} \xrightarrow{\mathbf{v}} \mathbf{Z} \xrightarrow{\mathbf{w}} \Sigma\mathbf{X}$—then the claim is that $\mathbf{Z}$ admits a v_n-map. For consider the diagram

$$\begin{tikzcd}
\Sigma^{id}\mathbf{X} \arrow[r, "\Sigma^k\mathbf{u}"] \arrow[d, "\mathbf{f}^i"'] & \Sigma^{je}\mathbf{Y} \arrow[r, "\Sigma^k\mathbf{v}"] \arrow[d, "\mathbf{g}^j"] & \Sigma^k\mathbf{Z} \\
\mathbf{X} \arrow[r, "\mathbf{u}"'] & \mathbf{Y} \arrow[r, "\mathbf{v}"'] & \mathbf{Z}
\end{tikzcd}$$

. Since $\Sigma^{je}\mathbf{Y} \xrightarrow{\Sigma^k\mathbf{v}} \Sigma^k\mathbf{Z}$ is a weak cokernel of $\Sigma^{id}\mathbf{X} \xrightarrow{\Sigma^k\mathbf{u}} \Sigma^{je}\mathbf{Y}$ and since $\mathbf{v} \circ \mathbf{g}^j \circ \Sigma^k\mathbf{u} = \mathbf{v} \circ \mathbf{u} \circ \mathbf{f}^i = 0$, $\exists$ an arrow $\mathbf{h} : \Sigma^k\mathbf{Z} \to \mathbf{Z}$ such that $\mathbf{h} \circ \Sigma^k\mathbf{v} = \mathbf{v} \circ \mathbf{g}^j$ (cf. p. 15–3 ff.). The five lemma gives that $\mathbf{K}(n)_*(\mathbf{h})$ is an isomorphism. And: $\forall\ m \neq n$ $(m \in [0,\infty[)$, $\mathbf{K}(m)_*(\mathbf{h}^2) = 0$. Therefore $\mathbf{h}^2$ is a v_n-map, so $\mathbf{Z}$ is in $\mathbf{V}_n$, which means that $\mathbf{V}_n$ is triangulated. Finally, if $\mathbf{Y} \in \mathrm{Ob}\,\mathbf{V}_n$ and $\mathbf{Y} \approx \mathbf{X} \vee \mathbf{Z}$ with $\mathbf{i} : \mathbf{X} \to \mathbf{Y}$, $\mathbf{r} : \mathbf{Y} \to \mathbf{X}$ and $\mathbf{r} \circ \mathbf{i} = \mathrm{id}_{\mathbf{X}}$, then $\mathbf{X} \in \mathrm{Ob}\,\mathbf{V}_n$. Thus fix a v_n-map $\mathbf{g} : \Sigma^e\mathbf{Y} \to \mathbf{Y}$. By raising $\mathbf{g}$ to a sufficiently high power, it can be arranged that the diagram

$$\begin{tikzcd}
\Sigma^e\mathbf{Y} \arrow[r, "\Sigma^e(\mathbf{i} \circ \mathbf{r})"] \arrow[d, "\mathbf{g}"'] & \Sigma^e\mathbf{Y} \arrow[d, "\mathbf{g}"] \\
\mathbf{Y} \arrow[r, "\mathbf{i} \circ \mathbf{r}"'] & \mathbf{Y}
\end{tikzcd}$$

commutes (cf. Proposition 32). Applying $\mathbf{K}(n)_*$ to

$$\begin{tikzcd}
\Sigma^e\mathbf{X} \arrow[r, "\Sigma^e\mathbf{i}"] \arrow[d, "\mathbf{f}"'] & \Sigma^e\mathbf{Y} \arrow[r, "\Sigma^e\mathbf{r}"] \arrow[d, "\mathbf{g}"] & \Sigma^e\mathbf{X} \arrow[d, "\mathbf{f}"] \\
\mathbf{X} \arrow[r, "\mathbf{i}"'] & \mathbf{Y} \arrow[r, "\mathbf{r}"'] & \mathbf{X}
\end{tikzcd}$$

, where $\mathbf{f} = \mathbf{r} \circ \mathbf{g} \circ \Sigma^e\,\mathbf{i}$, and using the fact that the retract of an isomorphism is an isomorphism, one concludes that $\mathbf{f}$ is a v_n-map. Accordingly, $\mathbf{V}_n$ is thick.

PROPOSITION 33 If $\mathbf{E}$ is p-local, then $\forall\, \mathbf{X}$, $L_{\mathbf{E}}\mathbf{X}_p \approx L_{\mathbf{E}}\mathbf{X} \approx (L_{\mathbf{E}}\mathbf{X})_p$.

[Since $\mathbf{E}$ is p-local, $\mathbf{E} \approx \mathbf{E} \wedge \mathbf{S}(\mathbf{Z}_p)$, hence $\langle \mathbf{E} \rangle \leq \langle \mathbf{S}(\mathbf{Z}_p) \rangle$, and the lemma on p. 17–22 can be quoted.]

[Note: In order that $\mathbf{X}$ be $\mathbf{E}_*$-local, it is therefore necessary that $\mathbf{X}$ be p-local.]

Application: If $\mathbf{E}$ is p-local and if $L_{\mathbf{E}}\mathbf{X} \approx \mathbf{X} \wedge L_{\mathbf{E}}\mathbf{S}_p$ $\forall$ p-local $\mathbf{X}$, then $T_{\mathbf{E}}$ is smashing.

[Given an arbitrary $\mathbf{X}$, $L_{\mathbf{E}}\mathbf{X} \approx L_{\mathbf{E}}\mathbf{X}_p \approx \mathbf{X}_p \wedge L_{\mathbf{E}}\mathbf{S}_p \approx \mathbf{X} \wedge \mathbf{S}(\mathbf{Z}_p) \wedge L_{\mathbf{E}}\mathbf{S}_p \approx \mathbf{X} \wedge (L_{\mathbf{E}}\mathbf{S}_p)_p \approx \mathbf{X} \wedge L_{\mathbf{E}}(\mathbf{S}_p)_p \approx \mathbf{X} \wedge L_{\mathbf{E}}\mathbf{S}_p \approx \mathbf{X} \wedge L_{\mathbf{E}}\mathbf{S}$.]

Recall that for any $\mathbf{E}$ and any compact $\mathbf{X}$, $L_{\mathbf{E}}\mathbf{X} \approx \mathbf{X} \wedge L_{\mathbf{E}}\mathbf{S}$ (cf. p. 15–41). Corollary: For any p-local $\mathbf{E}$ and any p-compact $\mathbf{X}$, $L_{\mathbf{E}}\mathbf{X} \approx \mathbf{X} \wedge L_{\mathbf{E}}\mathbf{S}_p$. Proof: Write $\mathbf{X} = \overline{\mathbf{X}}_p$, where $\overline{\mathbf{X}}$ is compact (cf. p. 17–27)—then $L_{\mathbf{E}}\mathbf{X} \approx L_{\mathbf{E}}\overline{\mathbf{X}}_p \approx L_{\mathbf{E}}\overline{\mathbf{X}} \approx \overline{\mathbf{X}} \wedge L_{\mathbf{E}}\mathbf{S} \approx \overline{\mathbf{X}} \wedge L_{\mathbf{E}}\mathbf{S}_p \approx \overline{\mathbf{X}} \wedge \mathbf{S}(\mathbf{Z}_p) \wedge L_{\mathbf{E}}\mathbf{S}_p \approx \overline{\mathbf{X}}_p \wedge L_{\mathbf{E}}\mathbf{S}_p \approx \mathbf{X} \wedge L_{\mathbf{E}}\mathbf{S}_p$. Example: Taking $\mathbf{E} = \mathbf{S}(\mathbf{Z}/p\mathbf{Z})$ ($= \mathbf{M}(p)$), $L_{\mathbf{S}(\mathbf{Z}/p\mathbf{Z})}\mathbf{X} \approx \mathbf{X} \wedge \widehat{\mathbf{S}}_p$ if $\mathbf{X}$ is p-compact.

EXAMPLE Let $\mathbf{E} \neq 0$ be p-local and suppose that there exists an $\mathbf{E}_*$-local object in $\mathbf{C}(n)$ for some $n < \infty$. Case 1: $\mathbf{H}(\mathbf{Q}) \wedge \mathbf{E} \neq 0$—then $L_{\mathbf{E}}\mathbf{X} \approx \mathbf{X}$ $\forall$ p-compact $\mathbf{X}$. Case 2: $\mathbf{H}(\mathbf{Q}) \wedge \mathbf{E} = 0$—then $L_{\mathbf{E}}\mathbf{X} \approx \mathbf{X} \wedge \widehat{\mathbf{S}}_p$ $\forall$ p-compact $\mathbf{X}$.

[The class of all p-local $\mathbf{X}$ which are $\mathbf{E}_*$-local must contain $\operatorname{Ob}\mathbf{C}(1)$. In addition, $\widehat{\mathbf{S}}_p$ is $\mathbf{E}_*$-local (consider the exact triangle $\widehat{\mathbf{S}}_p \xrightarrow{p} \widehat{\mathbf{S}}_p \rightarrow \mathbf{M}(p) \rightarrow \Sigma\widehat{\mathbf{S}}_p$) and if $\mathbf{F}$ is defined by the exact triangle $\mathbf{F} \rightarrow \mathbf{S}_p \rightarrow \widehat{\mathbf{S}}_p \rightarrow \Sigma\mathbf{F}$, then $\mathbf{F}$ is $\mathbf{E}_*$-local or $\mathbf{E}_*$-acyclic depending on whether $\mathbf{H}(\mathbf{Q}) \wedge \mathbf{E} \neq 0$ or $\mathbf{H}(\mathbf{Q}) \wedge \mathbf{E} = 0$ ($\mathbf{F}$ is rational). Working now with the commutative diagram

$$\begin{tikzcd}
\mathbf{F} \arrow[r] \arrow[d] & \mathbf{S}_p \arrow[r] \arrow[d] & \widehat{\mathbf{S}}_p \arrow[r] \arrow[d] & \Sigma\mathbf{F} \arrow[d] \\
T_{\mathbf{E}}\mathbf{F} \arrow[r] & T_{\mathbf{E}}\mathbf{S}_p \arrow[r] & T_{\mathbf{E}}\widehat{\mathbf{S}}_p \arrow[r] & \Sigma T_{\mathbf{E}}\mathbf{F}
\end{tikzcd},$$

one thus sees that in case 1, $\mathbf{S}_p$ is $\mathbf{E}_*$-local ($\Rightarrow L_{\mathbf{E}}\mathbf{X} \approx \mathbf{X} \wedge L_{\mathbf{E}}\mathbf{S}_p \approx \mathbf{X} \wedge \mathbf{S}_p \approx \mathbf{X}$) while in case 2, $L_{\mathbf{E}}\mathbf{S}_p \approx \widehat{\mathbf{S}}_p$ ($\Rightarrow L_{\mathbf{E}}\mathbf{X} \approx \mathbf{X} \wedge L_{\mathbf{E}}\mathbf{S}_p \approx \mathbf{X} \wedge \widehat{\mathbf{S}}_p$).]

EXAMPLE Let $\mathbf{E} \neq 0$ be a p-local ring spectrum with the property that $\mathbf{ACY}_p(\mathbf{E}) = 0$. Case 1: $\mathbf{H}(\mathbf{Q}) \wedge \mathbf{E} \neq 0$—then $L_{\mathbf{E}}\mathbf{X} \approx \mathbf{X}$ $\forall$ p-compact $\mathbf{X}$. Case 2: $\mathbf{H}(\mathbf{Q}) \wedge \mathbf{E} = 0$—then $L_{\mathbf{E}}\mathbf{X} \approx \mathbf{X} \wedge \widehat{\mathbf{S}}_p$ $\forall$ p-compact $\mathbf{X}$.

[In view of the preceding example, one has only to exhibit an $\mathbf{E}_*$-local object in $\mathbf{C}(1)$. Choose $n \in [0, \infty] : \mathbf{K}(n)_*(\mathbf{E}) \neq 0$ (cf. p. 17–31). If $\mathbf{K}(\infty)_*(\mathbf{E}) = \mathbf{H}(\mathbf{F}_p)_*(\mathbf{E}) \neq 0$, then $\langle \mathbf{H}(\mathbf{F}_p) \rangle \leq \langle \mathbf{E} \rangle$ and $\mathbf{M}(p)$ is $\mathbf{H}(\mathbf{F}_p)_*$-local, hence is $\mathbf{E}_*$-local. So suppose that $\mathbf{H}(\mathbf{F}_p) \wedge \mathbf{E} = 0$. Claim: $\exists$ a sequence $k_1 < k_2 < \cdots$ such that $\mathbf{E} \wedge \mathbf{K}(k_i) \neq 0$ ($i = 1, 2, \ldots$). Proof: $\forall$ $n < \infty$, $\exists$ a p-compact ring spectrum $\mathbf{X}_n$ of type n

and $\mathbf{E}\wedge\mathbf{X}_n \neq 0$ (by hypothesis) $\Rightarrow \mathbf{K}(m)_*(\mathbf{E}\wedge\mathbf{X}_n) = 0$ ($m < n$ or $m = \infty$) $\Rightarrow \mathbf{K}(m)_*(\mathbf{E}\wedge\mathbf{X}_n) \neq 0$ ($\exists\, m \in [n,\infty[$). But $\langle\mathbf{K}\rangle \leq \langle\mathbf{E}\rangle$ and $\mathbf{M}(p)$ is $\mathbf{K}_*$-local, where $\mathbf{K} = \bigvee_i \mathbf{K}(k_i)$.]

FACT Let $\mathbf{E} \neq 0$ be p-local. Assume: $\mathbf{ACY}_p(\mathbf{E}) = 0$ and $T_{\mathbf{E}}$ is smashing—then $\langle\mathbf{E}\rangle = \langle\mathbf{S}_p\rangle$.

[Since $T_{\mathbf{E}}$ is smashing, $\langle\mathbf{E}\rangle = \langle L_{\mathbf{E}}\mathbf{S}\rangle = \langle L_{\mathbf{E}}\mathbf{S}_p\rangle$. However $L_{\mathbf{E}}\mathbf{S}_p \neq 0$ is a p-local ring spectrum with the property that $\mathbf{ACY}_p(L_{\mathbf{E}}\mathbf{S}_p) = 0$. Therefore $L_{L_{\mathbf{E}}\mathbf{S}_p}\mathbf{S}_p \approx L_{\mathbf{E}}\mathbf{S}_p \approx \mathbf{S}_p$ or $\widehat{\mathbf{S}}_p$. And: $\langle\mathbf{S}_p\rangle = \langle\widehat{\mathbf{S}}_p\rangle \Rightarrow \langle\mathbf{E}\rangle = \langle\mathbf{S}_p\rangle$.]

Let $\mathbf{X}(n)$ be a p-compact spectrum of type n—then by the class invariance principle, $\langle\mathbf{X}(n)\rangle$ depends only on n. Write $\mathbf{T}(n)$ for $\mathbf{f}^{-1}\mathbf{X}(n)$, where $\mathbf{f} : \Sigma^d\mathbf{X}(n) \rightarrow \mathbf{X}(n)$ is a v_n-map. Thanks to Proposition 31, $\mathbf{T}(n)$ is independent of the choice of $\mathbf{f}$. Moreover, its Bousfield class $\langle\mathbf{T}(n)\rangle$ is independent of the choice of $\mathbf{X}(n)$ and applying Proposition 43 in §15 repeatedly, one obtains a decomposition $\langle\mathbf{S}_p\rangle = \langle\mathbf{T}(0)\rangle \vee \langle\mathbf{T}(1)\rangle \vee \cdots \vee \langle\mathbf{T}(n)\rangle \vee \langle\mathbf{X}(n+1)\rangle$ with $\langle\mathbf{T}(i)\rangle \wedge \langle\mathbf{X}(n+1)\rangle = \langle 0\rangle$ ($i = 0, 1, \ldots, n$), $\langle\mathbf{T}(n)\rangle \wedge \langle\mathbf{T}(m)\rangle = \langle 0\rangle$ ($m \neq n$) (here, $\mathbf{T}(0) = \mathbf{H}(\mathbf{Q})$). Examples: (1) $\langle\mathbf{BP}\rangle \wedge \langle\mathbf{X}(n)\rangle = \langle\mathbf{P}(n)\rangle$; (2) $\langle\mathbf{BP}\rangle \wedge \langle\mathbf{T}(n)\rangle = \langle\mathbf{K}(n)\rangle$.

Notation: Put $\mathbf{T}(\leq n) = \mathbf{T}(0) \vee \mathbf{T}(1) \vee \cdots \vee \mathbf{T}(n)$, call T_n^f the corresponding localization functor, and let L_n^f be the associated reflector.

PROPOSITION 34 T_n^f is smashing, so $\forall\, \mathbf{X}$, $L_n^f\mathbf{X} \approx \mathbf{X} \wedge L_n^f\mathbf{S}$.

[The Bousfield classes of $L_n^f\mathbf{S}_p$ ($= L_n^f\mathbf{S}$) and $\mathbf{T}(\leq n)$ are one and the same.]

FACT Suppose that $\mathbf{X}$ is p-compact and has type n—then $L_n^f\mathbf{X} \approx \mathbf{f}^{-1}\mathbf{X}$, $\mathbf{f} : \Sigma^d\mathbf{X} \rightarrow \mathbf{X}$ a v_n-map.

Notation: Put $\mathbf{K}(\leq n) = \mathbf{K}(0) \vee \mathbf{K}(1) \vee \cdots \vee \mathbf{K}(n)$, call T_n the corresponding localization functor, and let L_n be the associated reflector.

There are similarities between the "L_n^f-theory" and the "L_n-theory" (but the proofs for the latter are much more difficult). Thus, e.g., it turns out that T_n is smashing (cf. Proposition 34). Moreover, one can attach to any $\mathbf{X}$ a tower $L_0\mathbf{X} \leftarrow L_1\mathbf{X} \leftarrow \cdots$ and $\mathbf{X} \approx \operatorname{mic}(L_0\mathbf{X} \leftarrow L_1\mathbf{X} \leftarrow \cdots)$ if $\mathbf{X}$ is p-compact (it is unknown whether the analog of this with L_n replaced by L_n^f is true or not). On the other hand, L_n^f and L_n are connected by a natural transformation $L_n^f \rightarrow L_n$ and $\forall\, \mathbf{X}$, $L_n^f\mathbf{X} \rightarrow L_n\mathbf{X}$ is a $\mathbf{BP}_*$-equivalence.

[Note: These assertions are detailed in Ravenel†. They represent the point of departure for the study of the "chromatic" aspects of **HSPEC**.]

FACT Suppose that $\mathbf{X}$ is p-compact and has type n—then $L_n^f\mathbf{X} \approx L_{\mathbf{T}(n)}\mathbf{X}$ and $L_n\mathbf{X} \approx L_{\mathbf{K}(n)}\mathbf{X}$.

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§18. ALGEBRAIC K-THEORY

My objective in this § is to provide an introduction to algebraic K-theory, placing the emphasis on its homotopical underpinnings.

Consider a skeletally small category **C** equipped with two composition closed classes of morphisms termed weak equivalences (denoted $\xrightarrow{\sim}$) and cofibrations (denoted $\rightarrowtail$), each containing the isomorphisms of **C**—then **C** is said to be a Waldhausen category provided that the following axioms are satisfied.

(WC–1) **C** has a zero object 0.

(WC–2) All the objects of **C** are cofibrant, i.e., $\forall\, X \in \operatorname{Ob}\mathbf{C}$, the arrow $0 \rightarrow X$ is a cofibration.

(WC–3) Every 2-source $X \xleftarrow{f} Z \xrightarrow{g} Y$, where f is a cofibration, admits a pushout $X \xrightarrow{\xi} P \xleftarrow{\eta} Y$, where η is a cofibration.

(WC–4) If

$$\begin{tikzcd}
X \arrow[d] & Z \arrow[l, "f"'] \arrow[r, "g"] \arrow[d] & Y \arrow[d] \\
X' & Z' \arrow[l, "f'"] & Y' \arrow[l, "g'"]
\end{tikzcd}$$

is a commutative diagram, where $\left\{\begin{matrix} f \\ f' \end{matrix}\right.$ are cofibrations and the vertical arrows are weak equivalences, then the induced morphism $P \rightarrow P'$ of pushouts is a weak equivalence.

[Note: The opposite of a Waldhausen category need not be Waldhausen.]

Remark: **C** has finite coproducts (define $X \amalg Y$ by the pushout square

$$\begin{tikzcd}
0 \arrow[r] \arrow[d] & Y \arrow[d] \\
X \arrow[r] & X \amalg Y
\end{tikzcd}$$

($\Rightarrow$ in_X & in_Y are cofibrations)).

[Note: Every cofibration $X \rightarrowtail Y$ has a cokernel Y/X, viz. $Y \underset{X}{\sqcup} 0$.]

Example: A finitely cocomplete pointed skeletally small category is Waldhausen if the weak equivalences are the isomorphisms and the cofibrations are the morphisms.

EXAMPLE Take for **C** the category whose objects are the pointed finite sets—then **C** is a Waldhausen category if weak equivalence = isomorphism, cofibration = pointed injection.

EXAMPLE Take for **C** the category whose objects are the pointed finite simplicial sets—then **C** is a Waldhausen category if weak equivalence = weak homotopy equivalence, cofibration = pointed injective simplicial map.

EXAMPLE Let A be a ring with unit. Denote by $\mathbf{P}(A)$ the full subcategory of A-**MOD** whose objects are finitely generated and projective—then $\mathbf{P}(A)$ is a Waldhausen category if weak equivalence = isomorphism, cofibration = split injection.

EXAMPLE Let A be a ring with unit. Denote by $\mathbf{F}(A)$ the full subcategory of A-**MOD** whose objects are finitely generated and free—then $\mathbf{F}(A)$ is a Waldhausen category if weak equivalence = isomorphism, cofibration = split injection with free quotient.

FACT The cofibrant objects in a pointed skeletally small cofibration category are the objects of a Waldhausen category (cf. §12, Proposition 3 and p. 12–32).

PROPOSITION 1 Any skeleton of a Waldhausen category is a small Waldhausen category.

There are two other conditions which are sometimes imposed on a Waldhausen category.

(Saturation Axiom) Given composable morphisms f, g, if any two of $f, g, g \circ f$ are weak equivalences, so is the third.

(Extension Axiom) Given a commutative diagram

$$\begin{tikzcd} X \arrow[r, tail] \arrow[d] & Y \arrow[r] \arrow[d] & Y/X \arrow[d] \\ X' \arrow[r, tail] & Y' \arrow[r] & Y'/X' \end{tikzcd}$$

, if $X \to X'$ & $Y/X \to Y'/X'$ are weak equivalences, then $Y \to Y'$ is a weak equivalence.

Neither the saturation axiom nor the extension axiom is a consequence of the other axioms.

Observation: If $\mathbf{C}$ is a Waldhausen category, then its arrow category $\mathbf{C}(\to)$ is a Waldhausen category.

[The weak equivalences and cofibrations are levelwise.]

Let $\mathbf{C}$ be a Waldhausen category—then a mapping cylinder is a functor $M : \mathbf{C}(\to) \to \mathbf{C}$ together with natural transformations $i : S \to M$, $j : T \to M$, $r : M \to T$, where $S : \mathbf{C}(\to) \to \mathbf{C}$ is the source functor and $T : \mathbf{C}(\to) \to \mathbf{C}$ is the target functor, all subject to the following assumptions.

[Note: Spelled out, M assigns to each object $X \xrightarrow{f} Y$ in $\mathbf{C}(\to)$ an object M_f in $\mathbf{C}$ and to each morphism $(\phi, \psi) : f \to f'$ in $\mathbf{C}(\to)$ a morphism $M_{\phi,\psi} : M_f \to M_{f'}$ in $\mathbf{C}$.]

(MCy$_1$) For every object $X \xrightarrow{f} Y$ in $\mathbf{C}(\to)$, the diagrams

$$\begin{tikzcd} X \arrow[r, "i"] \arrow[dr, "f"'] & M_f \arrow[d, "r"] \\ & Y \end{tikzcd} , \quad \begin{tikzcd} M_f \arrow[d, "r"'] & Y \arrow[l, "j"'] \arrow[dl, equal] \\ Y & \end{tikzcd}$$

commute and $i \amalg j : X \amalg Y \to M_f$ is a cofibration (hence i & j are cofibrations).

(MCy$_2$) For every object Y in $\mathbf{C}$, $M_{0 \to Y} = Y$ with $r = \mathrm{id}_Y$ and $j = \mathrm{id}_Y$.

(MCy$_3$) For every morphism $(\phi, \psi) : f \to f'$ in $\mathbf{C}(\to)$, $M_{\phi,\psi} : M_f \to M_{f'}$ is a weak equivalence if ϕ, ψ are weak equivalences.

(MCy$_4$) For every morphism $(\phi,\psi) : f \rightarrow f'$ in $\mathbf{C}(\rightarrow)$, $M_{\phi,\psi} : M_f \rightarrow M_{f'}$ is a cofibration if ϕ, ψ are cofibrations.

(MCy$_5$) For every morphism $(\phi,\psi) : f \rightarrow f'$ in $\mathbf{C}(\rightarrow)$, the diagram

$$\begin{tikzcd} X \amalg Y \arrow[r, "i\amalg j"] \arrow[d] & M_f \arrow[r, "r"] \arrow[d] & Y \arrow[d] \\ X' \amalg Y' \arrow[r, "i\amalg j"'] & M_{f'} \arrow[r, "r"'] & Y' \end{tikzcd}$$

commutes and if ϕ, ψ are cofibrations, then the arrow $(X' \amalg Y') \underset{X \amalg Y}{\sqcup} M_f \rightarrow M_{f'}$ is a cofibration.

Example: The cone functor $\Gamma : \mathbf{C} \rightarrow \mathbf{C}$ sends X to ΓX, where $\Gamma X = M_{X\rightarrow 0}$ and the suspension functor $\Sigma : \mathbf{C} \rightarrow \mathbf{C}$ sends X to $\Sigma X = \Gamma X / X$ (per $X \xrightarrow{i} \Gamma X$).

EXAMPLE The category of pointed finite simplicial sets, where weak equivalence = weak homotopy equivalence and cofibration = pointed injective simplicial map, has a mapping cylinder.

(Mapping Cylinder Axiom) Assume that $\mathbf{C}$ admits a mapping cylinder—then $\forall\, X \xrightarrow{f} Y \in \operatorname{Ob} \mathbf{C}(\rightarrow)$, $r : M_f \rightarrow Y$ is a weak equivalence.

EXAMPLE The category of pointed finite simplicial sets, where weak equivalence = isomorphism and cofibration = pointed injective simplicial map, has a mapping cylinder which does not satisfy the mapping cylinder axiom.

In a Waldhausen category, an acyclic cofibration is a morphism which is both a weak equivalence and a cofibration.

PROPOSITION 2 If $X \xleftarrow{f} Z \xrightarrow{g} Y$ is a 2-source, where f is an acyclic cofibration, then $Y \xrightarrow{\eta} P$ is an acyclic cofibration.

[Bearing in mind WC–3, consider the commutative diagram

$$\begin{tikzcd} Z \arrow[d, "f"'] & Z \arrow[l, "\mathrm{id}_Z"'] \arrow[r, "g"] \arrow[d, equal] & Y \arrow[d, equal] \\ X & Z \arrow[l, "f"] \arrow[r, "g"'] & Y \end{tikzcd}$$

and apply WC–4.]

[Note: Therefore $0 \rightarrow Y/X$ is an acyclic cofibration if $X \rightarrow Y$ is an acyclic cofibration.]

Remark: If $\mathbf{C}$ satisfies the saturation axiom and the mapping cylinder axiom, then j is an acyclic cofibration and i is an acyclic cofibration provided that f is a weak equivalence.

Notation: Given a Waldhausen category $\mathbf{C}, \mathbf{wC}$ is the subcategory of $\mathbf{C}$ having morphisms the weak equivalences, $\mathbf{coC}$ is the subcategory of $\mathbf{C}$ having morphisms the cofibrations, and $\mathbf{wcoC}$ is the subcategory of $\mathbf{C}$ having morphisms the acyclic cofibrations.

PROPOSITION 3 Suppose that $\mathbf{C}$ is a small Waldhausen category satisfying the saturation axiom and the mapping cylinder axiom—then the inclusion $\iota : \mathbf{wcoC} \to \mathbf{wC}$ induces a pointed homotopy equivalence $B\iota : B\mathbf{wcoC} \to B\mathbf{wC}$.

[Owing to Quillen's theorem A, it suffices to show that ι is a strictly initial functor, i.e., that $\forall\, Y \in \operatorname{Ob}\mathbf{wC}$, the comma category ι/Y is contractible. An object of ι/Y is a pair (X, f), where $f : X \to Y$ is a weak equivalence. Specify a functor $m : \iota/Y \to \iota/Y$ by sending (X, f) to (M_f, r)—then i defines a natural transformation $\mathrm{id}_{\iota/Y} \to m$ and j defines a natural transformation $K_{(Y,\mathrm{id}_Y)} \to m$. Therefore $B\iota/Y$ is contractible (cf. p. 3–15).]

[Note: The base point is the 0-cell corresponding to 0.]

Let $\mathbf{C}$ be an additive category—then a pair of composable morphisms $X \xrightarrow{i} Y \xrightarrow{p} Z$ is exact if i is a kernel of p and p is a cokernel of i, a morphism of exact pairs being a triple (f, g, h) such that the diagram

$$\begin{tikzcd}
X \arrow[r, "i"] \arrow[d, "f"] & Y \arrow[r, "p"] \arrow[d, "g"] & Z \arrow[d, "h"] \\
X' \arrow[r, "i'"'] & Y' \arrow[r, "p'"'] & Z'
\end{tikzcd}$$

commutes.

[Note: The first component of an exact pair is called an inflation (denoted $\rightarrowtail$), the second component a deflation (denoted $\twoheadrightarrow$) (terminology as in Gabriel-Roiter[†]).]

Let $\mathbf{C}$ be a skeletally small additive category—then $\mathbf{C}$ is said to be a category with exact sequences (category WES) if there is given an isomorphism closed class $\mathcal{E}$ of exact pairs satisfying the following conditions.

(ES–1) The pair $0 \xrightarrow{\mathrm{id}_0} 0 \xrightarrow{\mathrm{id}_0} 0$ is in $\mathcal{E}$.

(ES–2) The composition of two inflations is an inflation and the composition of two deflations is a deflation.

(ES–3) Every 2-source $X \xleftarrow{f} Z \xrightarrow{g} Y$, where f is an inflation, admits a pushout $X \xrightarrow{\xi} P \xleftarrow{\eta} Y$, where η is an inflation, and every two sink $X \xrightarrow{f} Z \xleftarrow{g} Y$, where g is a deflation, admits a pullback $X \xleftarrow{\xi} P \xrightarrow{\eta} Y$, where ξ is a deflation.

[Note: The opposite of a category WES is again a category WES.]

A full, additive subcategory $\mathbf{C}$ of an abelian category $\mathbf{D}$ is closed under extensions if for every short exact sequence $0 \to X \to Y \to Z \to 0$ in $\mathbf{D}$, where $X, Z \in \operatorname{Ob}\mathbf{C}$, $\exists$ an object in **C** which is isomorphic to Y.

[†] *Representations of Finite Dimensional Algebras*, Springer Verlag (1992).

[Note: Such a **C** necessarily has finite coproducts.]

Example: Let **C** be a full, skeletally small additive subcategory of an abelian category **D**. Assume: **C** is closed under extensions. Declare a sequence $X \xrightarrow{i} Y \xrightarrow{p} Z$ in **C** to be exact iff $0 \rightarrow X \xrightarrow{i} Y \xrightarrow{p} Z \rightarrow 0$ is short exact in **D**—then **C** is a category WES.

[Note: This example is prototypical. Thus suppose that **C** is a category WES—then $\exists$ an abelian category **G-Q** and an additive functor $\iota : \mathbf{C} \rightarrow \mathbf{G\text{-}Q}$ which is full and faithful such that $X \xrightarrow{i} Y \xrightarrow{p} Z$ is exact iff $0 \rightarrow \iota X \xrightarrow{\iota i} \iota Y \xrightarrow{\iota p} \iota Z \rightarrow 0$ is short exact. And: $\iota\mathbf{C}$ is closed under extensions. Specifically: **G-Q** is the full subcategory of $[\mathbf{C}^{\mathrm{OP}}, \mathbf{AB}]^+$ whose objects are those F such that $X \xrightarrow{i} Y \xrightarrow{p} Z$ exact $\Rightarrow 0 \rightarrow FZ \rightarrow FY \rightarrow FX$ exact and ι is the Yoneda embedding. For a proof, consult Thomason-Trobaugh[†] (**G-Q** = Gabriel-Quillen).]

LEMMA Let **C** be a category WES—then $\forall\, X \in \mathrm{Ob}\,\mathbf{C}$, id_X is both an inflation and a deflation.

[Consider the pushout square
$$\begin{tikzcd} 0 \arrow[r] \arrow[d] & X \arrow[d, "\mathrm{id}_X"] \\ 0 \arrow[r] & X \end{tikzcd}$$
to see that id_X is an inflation and consider the pullback square
$$\begin{tikzcd} X \arrow[r] \arrow[d, "\mathrm{id}_X"'] & 0 \arrow[d] \\ X \arrow[r] & 0 \end{tikzcd}$$
to see that id_X is a deflation.]

[Note: Similarly, $0 \rightarrow X$ is an inflation and $X \rightarrow 0$ is a deflation. Therefore $0 \rightarrow X \xrightarrow{\mathrm{id}_X} X$ and $X \xrightarrow{\mathrm{id}_X} X \rightarrow 0$ are exact.]

Application: Every isomorphism $\phi : X \rightarrow Y$ is both an inflation and a deflation.

[By assumption, $\mathcal{E}$ is isomorphism closed and there are commutative diagrams
$$\begin{tikzcd} X \arrow[r, "\phi"] \arrow[d, equal] & {} \\ X \arrow[r, "\mathrm{id}_X"'] & {} \end{tikzcd}$$

$$\begin{tikzcd} Y \arrow[r] \arrow[d, "\phi^{-1}"] & 0 \arrow[d, equal] \\ X \arrow[r] & 0 \end{tikzcd}, \quad \begin{tikzcd} 0 \arrow[r] \arrow[d, equal] & X \arrow[r, "\phi"] \arrow[d, equal] & Y \arrow[d, "\phi^{-1}"] \\ 0 \arrow[r] & X \arrow[r, "\mathrm{id}_X"'] & X \end{tikzcd}.]$$

PROPOSITION 4 A category WES is a Waldhausen category.

[Take for the weak equivalences the isomorphisms and take for the cofibrations the inflations.]

[†] *The Grothendieck Festschrift*, vol. III, Birkhäuser (1990), 247–435 (cf. 399–406); see also Keller, *Manuscripta Math.* **67** (1990), 379–417 (cf. 408–409).

[Note: This interpretation entails a loss of structure.]

Remark: Any skeleton of a category WES is a small category WES (cf. Proposition 1).

Let **C** be a category WES.

FACT Consider a pushout square

$$\begin{tikzcd} Z \arrow[r, "g"] \arrow[d, "f"'] & Y \arrow[d, "\eta"] \\ X \arrow[r, "\xi"'] & P \end{tikzcd}$$

, where f is an inflation—then $Z \xrightarrow{\begin{pmatrix} -f \\ g \end{pmatrix}} X \oplus Y \xrightarrow{(\xi,\eta)} P$ is exact.

FACT Consider a pullback square

$$\begin{tikzcd} P \arrow[r, "\eta"] \arrow[d, "\xi"'] & Y \arrow[d, "g"] \\ X \arrow[r, "f"'] & Z \end{tikzcd}$$

, where g is a deflation—then $P \xrightarrow{\begin{pmatrix} -\eta \\ \xi \end{pmatrix}} X \oplus Y \xrightarrow{(f,g)} Z$ is exact.

FACT If $f : X \to Y$ has a cokernel and if $g \circ f$ is an inflation for some morphism g, then f is an inflation.

FACT If $f : X \to Y$ has a kernel and if $f \circ g$ is a deflation for some morphism g, then f is a deflation.

FACT $\forall\, X, Y \in \operatorname{Ob} \mathbf{C}$, $X \xrightarrow{\text{in}_X} X \oplus Y \xrightarrow{\text{pr}_Y} Y$ is exact.

EXAMPLE Let A be a ring with unit—then $\mathbf{P}(A)$ and $\mathbf{F}(A)$ are categories WES.

EXAMPLE Let X be a scheme, $\mathcal{O}_X$ its structure sheaf—then the category of locally free $\mathcal{O}_X$-modules of finite rank is a category WES.

EXAMPLE Let X be a topological space—then the category of real or complex vector bundles over X is a category WES.

Let **C** be a category WES—then a pair $(\mathbf{A}, \iota)$, where **A** is an abelian category and $\iota : \mathbf{C} \to \mathbf{A}$ is an additive functor which is full and faithful, satisfies the embedding condition provided that $X \xrightarrow{i} Y \xrightarrow{p} Z$ is exact iff $0 \to \iota X \xrightarrow{\iota i} \iota Y \xrightarrow{\iota p} \iota Z \to 0$ is short exact. And: $\iota\mathbf{C}$ is closed under extensions. Example: The pair $(\mathbf{G\text{-}Q}, \iota)$ satisfies the embedding condition.

(E $\Rightarrow$ D Axiom) Under the assumption that the pair $(\mathbf{A}, \iota)$ satisfies the embedding condition, an $f \in \operatorname{Mor} \mathbf{C}$ is a deflation whenever $\iota f \in \operatorname{Mor} \mathbf{A}$ is an epimorphism.

EXAMPLE Let X be a scheme, $\mathcal{O}_X$ its structure sheaf. With $\mathbf{C}$ the category of locally free $\mathcal{O}_X$-modules of finite rank, let $\mathbf{A}$ be either the abelian category of $\mathcal{O}_X$-modules or the abelian category of quasicoherent $\mathcal{O}_X$-modules—then in either case, the pair $(\mathbf{A}, \iota)$ satisfies the embedding condition and the E $\Rightarrow$ D axiom.

A pseudoabelian category is an additive category $\mathbf{C}$ with finite coproducts such that every idempotent has a kernel. Example: Let A be a ring with unit—then $\mathbf{P}(A)$ is pseudoabelian (but this need not be the case of $\mathbf{F}(A)$).

[Note: If $\mathbf{C}$ is pseudoabelian and if $e : X \to X$ is an idempotent, then $X \approx \ker e \oplus \ker(1 - e)$ and $e \leftrightarrow 0 \oplus 1$.]

LEMMA Let $\mathbf{C}$ be a category WES. Assume: $\mathbf{C}$ is pseudoabelian—then $f \in \operatorname{Mor} \mathbf{C}$ is a deflation if f has a right inverse.

Remark: Let $\mathbf{C}$ be a category WES—then, while the pair ($\mathbf{G}$-$\mathbf{Q}$, ι) satisfies the embedding condition, it is not automatic that the E $\Rightarrow$ D axiom holds. To ensure this, it suffices that retracts be deflations (Thomason-Trobaugh (ibid.)) which, by the lemma, will be true if $\mathbf{C}$ is pseudoabelian.

EXAMPLE Let X be a topological space—then the category of real or complex vector bundles over X is pseudoabelian.

Rappel: Let $\mathbf{C}$ be an additive category with finite coproducts—then there exists a pseudoabelian category $\mathbf{C}_{\mathrm{pa}}$ and an additive functor $\Phi : \mathbf{C} \to \mathbf{C}_{\mathrm{pa}}$ which is full and faithful such that for any pseudoabelian category $\mathbf{D}$ and any additive functor $F : \mathbf{C} \to \mathbf{D}$, there exists an additive functor $F_{\mathrm{pa}} : \mathbf{C}_{\mathrm{pa}} \to \mathbf{D}$ such that $F \approx F_{\mathrm{pa}} \circ \Phi$. And: $\mathbf{C}_{\mathrm{pa}}$ is unique up to equivalence.

[One model for $\mathbf{C}_{\mathrm{pa}}$ is the category whose objects are the pairs (X, e), where $X \in \operatorname{Ob} \mathbf{C}$ and $e \in \operatorname{Mor}(X, X)$ is idempotent, and whose morphisms $(X, e) \to (X', e')$ are the $f \in \operatorname{Mor}(X, X')$ such that $f = e' \circ f \circ e$. Here $\mathrm{id}_{(X,e)} = e$ and $(X, e) \oplus (X', e') = (X \oplus X', e \oplus e')$. As for $\Phi : \mathbf{C} \to \mathbf{C}_{\mathrm{pa}}$, it is defined by $\Phi X = (X, \mathrm{id}_X)$ & $\Phi f = f$.

[Note: Every object in $\mathbf{C}_{\mathrm{pa}}$ is a direct summand of an object in $\Phi\mathbf{C}$. Indeed, $(X, e) \oplus (X, 1 - e) = (X \oplus X, e \oplus (1 - e)) \approx (X, \mathrm{id}_X) = \Phi X$.]

FACT If $\mathbf{D}$ is a pseudoabelian category and $F : \mathbf{C} \to \mathbf{D}$ is an additive functor which is full and faithful such that every object in $\mathbf{D}$ is a direct summand of an object in $F\mathbf{C}$, then $F_{\text{pa}} : \mathbf{C}_{\text{pa}} \to \mathbf{D}$ is an equivalence of categories.

EXAMPLE Suppose that X is a compact Hausdorff space. Let $\mathbf{C}$ be the category of real or complex trivial vector bundles over X—then $\mathbf{C}_{\text{pa}}$ is equivalent to the category of real or complex vector bundles over X.

[Since X is compact Hausdorff, $\forall\, E \to X \ \exists\, E' \to X$ such that $E \oplus E'$ is trivial.]

Let $\mathbf{C}, \mathbf{D}$ be categories WES. Assume: $\mathbf{C}$ is a full, additive subcategory of $\mathbf{D}$ with the property that a pair $X \xrightarrow{i} Y \xrightarrow{p} Z$ is exact in $\mathbf{C}$ iff it is exact in $\mathbf{D}$—then $\mathbf{C}$ is said to be cofinal in $\mathbf{D}$ if for every exact pair $X \xrightarrow{i} Y \xrightarrow{p} Z$ in $\mathbf{D}$, where $X, Z \in \operatorname{Ob}\mathbf{C}$, $\exists$ an object in $\mathbf{C}$ which is isomorphic to Y, and $\forall\, X \in \operatorname{Ob}\mathbf{D}$, $\exists\, Z \in \operatorname{Ob}\mathbf{D}$ such that $X \oplus Z$ is isomorphic to an object in $\mathbf{C}$. Example: Given a ring A with unit, $\mathbf{F}(A)$ is cofinal in $\mathbf{P}(A)$.

EXAMPLE Let $\mathbf{C}$ be a category WES. Viewing $\mathbf{C}$ as a full, additive subcategory of $\mathbf{C}_{\text{pa}}$, stipulate that the elements of $\mathcal{E}_{\text{pa}}$ are those pairs which are direct summands of elements of $\mathcal{E}$—then $\mathbf{C}_{\text{pa}}$ is a category WES and $\mathbf{C}$ is cofinal in $\mathbf{C}_{\text{pa}}$.

If $\left\{\begin{matrix}\mathbf{C}\\ \mathbf{D}\end{matrix}\right.$ are Waldhausen categories and if $F : \mathbf{C} \to \mathbf{D}$ is a functor, then F is said to be a model functor provided that $F0 = 0$, F sends weak equivalences to weak equivalences and cofibrations to cofibrations, and F preserves pushouts along a cofibration, i.e., for any 2-source $X \xleftarrow{f} Z \xrightarrow{g} Y$, where f is a cofibration, the arrow $FX \underset{FZ}{\sqcup} FY \to F(X \underset{Z}{\sqcup} Y)$ is an isomorphism.

FACT Let $\left\{\begin{matrix}\mathbf{C}\\ \mathbf{D}\end{matrix}\right.$ be categories WES viewed as Waldhausen categories (cf. Proposition 4)—then an additive functor $F : \mathbf{C} \to \mathbf{D}$ is a model functor iff $X \xrightarrow{i} Y \xrightarrow{p} Z$ exact $\Rightarrow FX \xrightarrow{Fi} FY \xrightarrow{Fp} FZ$ exact.

[Note: In this context, a model functor is called an exact functor.]

$\mathbf{WALD}$ is the category whose objects are the small Waldhausen categories and whose morphisms are the model functors between them.

EXAMPLE Let $\mathbf{C}$ be a small Waldhausen category—then the functor category $[[n], \mathbf{C}]$ is again in $\mathbf{WALD}$ (the weak equivalences and cofibrations are levelwise) and $\operatorname{Ob}[[n], \mathbf{C}] = \operatorname{ner}_n \mathbf{C}$. Write $\mathbf{wC}(n)$ for the full subcategory of $[[n], \mathbf{C}]$ consisting of those functors that take values in $\mathbf{wC}$, i.e., the diagrams of the form $X_0 \xrightarrow{f_0} X_1 \to \cdots \to X_{n-1} \xrightarrow{f_{n-1}} X_n$, where the f_i are weak equivalences (thus $\operatorname{Ob}\mathbf{wC}(n) = \operatorname{ner}_n \mathbf{wC}$

and $[[n],\mathbf{wC}]$ is a subcategory of $\mathbf{wC}(n)$). Since pushouts are levelwise, $\mathbf{wC}(n)$ inherits the structure of a Waldhausen category from $[[n],\mathbf{C}]$.

[If $X \xleftarrow{f} Z \xrightarrow{g} Y$ is a 2-source in $\mathbf{wC}(n)$, where f is a cofibration, then there are commutative diagrams

$$\begin{tikzcd} Z_i \arrow[r, "g_i"] \arrow[d, tail, "f_i"'] & Y_i \arrow[d, "\eta_i"] \\ X_i \arrow[r, "\xi_i"'] & P_i \end{tikzcd}$$

and the claim is that $P_0 \to P_1 \to \cdots \to P_{n-1} \to P_n \in \operatorname{Ob}\mathbf{wC}(n)$. But this is implied by WC–4.]

Let $\mathbf{C}$ be a small Waldhausen category. Recalling that $[n](\to)$ is the arrow category of $[n]$ (cf. p. 0–3), denote by $\mathbf{S}_n\mathbf{C}$ the full subcategory of $[[n](\to),\mathbf{C}]$ consisting of those functors $F : [n](\to) \to \mathbf{C}$ such that $F(i \to i) = 0$ $(0 \leq i \leq n)$ and for every triple $i \leq j \leq k$ in $[n]$, $F(i \to j) \to F(i \to k)$ is a cofibration and the commutative diagram

$$\begin{tikzcd} F(i \to j) \arrow[r] \arrow[d] & F(j \to j) \arrow[d] \\ F(i \to k) \arrow[r] & F(j \to k) \end{tikzcd}$$

is a pushout square—then the assignment $[n] \to \mathbf{S}_n\mathbf{C}$ defines an internal category in **SISET**, call it **SC**.

[Note: Each $\alpha : [m] \to [n]$ in $\operatorname{Mor}\mathbf{\Delta}$ determines a functor $\alpha(\to) : [m](\to) \to [n](\to)$ from which a functor $\mathbf{S}_n\mathbf{C} \to \mathbf{S}_m\mathbf{C}$, viz. $F \to F \circ \alpha(\to)$.]

LEMMA $\mathbf{S}_n\mathbf{C}$ is a small Waldhausen category.

[The weak equivalences are those natural transformations $\Xi : F \to G$ such that $\Xi_{i \to j} : F(i \to j) \to G(i \to j)$ is a weak equivalence and the cofibrations are those natural transformations $\Xi : F \to G$ such that $\Xi_{i \to j} : F(i \to j) \to G(i \to j)$ is a cofibration and for every triple $i \leq j \leq k$ in $[n]$, the arrow $F(i \to k) \underset{F(i \to j)}{\sqcup} G(i \to j) \to G(i \to k)$ is a cofibration.]

[Note: $\mathbf{S}_0\mathbf{C} \approx \mathbf{1}$ and $\mathbf{S}_1\mathbf{C} \approx \mathbf{C}$.]

Given a $\mathbf{C}$ in **WALD**, define a simplicial set $W\mathbf{C}$ by putting $W_n\mathbf{C} = \operatorname{Ob}\mathbf{S}_n\mathbf{C}$.

FACT Suppose that $\begin{cases} \mathbf{C} \\ \mathbf{D} \end{cases}$ are small Waldhausen categories. Let $F : \mathbf{C} \to \mathbf{D}$ be a model functor—then F induces a simplicial map $WF : W\mathbf{C} \to W\mathbf{D}$.

FACT Suppose that $\begin{cases} \mathbf{C} \\ \mathbf{D} \end{cases}$ are small Waldhausen categories. Let $F, G : \mathbf{C} \to \mathbf{D}$ be model functors, $\Xi : F \to G$ a natural isomorphism—then Ξ induces a simplicial homotopy between WF and WG.

EXAMPLE Let $\mathbf{C}$ be a small Waldhausen category. Denote by $i\mathbf{C}(\to)$ the full subcategory of $\mathbf{C}(\to)$ whose objects are the $X \xrightarrow{f} Y$ such that f is an isomorphism—then there is a model functor $F : \mathbf{C} \to$

$i\mathbf{C}(\rightarrow)$, viz. $FX = X \xrightarrow{\mathrm{id}_X} X$, and a model functor $G : i\mathbf{C}(\rightarrow) \rightarrow \mathbf{C}$, viz. $G(X \xrightarrow{f} Y) = X$. Obviously, $G \circ F = \mathrm{id}_{\mathbf{C}}$ and $F \circ G \approx \mathrm{id}_{i\mathbf{C}(\rightarrow)}$, so $|W\mathbf{C}|$ and $|Wi\mathbf{C}(\rightarrow)|$ have the same pointed homotopy type.

PROPOSITION 5 Let $\mathbf{C}$ be a small Waldhausen category—then $\mathbf{SC}$ is a simplicial object in **WALD**.

[The d_i and the s_i are model functors.]

[Note: A model functor $\mathbf{C} \rightarrow \mathbf{D}$ induces a model functor $\mathbf{SC} \rightarrow \mathbf{SD}$. Therefore $\mathbf{S}$ is a functor from **WALD** to **SIWALD** ($= [\Delta^{\mathrm{OP}}, \mathbf{WALD}]$).]

Given a small Waldhausen category $\mathbf{C}$, let $B\mathbf{wSC} = |[n] \rightarrow B\mathbf{wS}_n\mathbf{C}|$—then $B\mathbf{wSC}$ is path connected and there is a closed embedding $\Sigma B\mathbf{wC} \rightarrow B\mathbf{wSC}$. Now iterate the process, i.e., form $\mathbf{S}^{(2)}\mathbf{C} = \mathbf{SSC}$, a bisimplicial object in **WALD**, and in general, $\mathbf{S}^{(q)}\mathbf{C} = \mathbf{S}\cdots\mathbf{SC}$, a multisimplicial object in **WALD**. Write $\mathbf{wS}^{(q)}\mathbf{C}$ for the weak equivalences in $\mathbf{S}^{(q)}\mathbf{C}$. If $B\mathbf{wS}^{(q)}\mathbf{C}$ is its classifying space (see below), then $B\mathbf{wS}^{(q)}\mathbf{C}$ is $(q-1)$-connected ($q > 1$) and there is a closed embedding $\Sigma B\mathbf{wS}^{(q)}\mathbf{C} \rightarrow B\mathbf{wS}^{(q+1)}\mathbf{C}$ whose adjoint $B\mathbf{wS}^{(q)}\mathbf{C} \rightarrow \Omega B\mathbf{wS}^{(q+1)}\mathbf{C}$ is a pointed homotopy equivalence (cf. p. 18–17). The data can be assembled into a separated prespectrum $\mathbf{WC}$, where $(\mathbf{WC})_0 = B\mathbf{wC}$ and $(\mathbf{WC})_q = B\mathbf{wS}^{(q)}\mathbf{C}$ ($q \geq 1$). Definition: The spectrum $\mathbf{KC} = e\mathbf{WC}$ is the algebraic K-theory of $\mathbf{C}$, its homotopy groups $\pi_n(\mathbf{KC})$ ($\approx \pi_n(\Omega B\mathbf{wSC})$) being the algebraic K-groups $K_n(\mathbf{C})$ of $\mathbf{C}$.

[Note: $\mathbf{KC}$ is connective. In addition, $\mathbf{KC}$ is tame (since $\mathbf{WC}$ satisfies the cofibration condition).]

Remark: A model functor $F : \mathbf{C} \rightarrow \mathbf{D}$ determines a morphism $\mathbf{WC} \rightarrow \mathbf{WD}$ of prespectra, hence a morphism $\mathbf{KC} \rightarrow \mathbf{KD}$ of spectra. Therefore $\mathbf{K} : \mathbf{WALD} \rightarrow \mathbf{SPEC}$ is a functor.

[Note: If $B\mathbf{wSC} \rightarrow B\mathbf{wSD}$ is a weak homotopy equivalence, then $\forall\, q$, $B\mathbf{wS}^{(q)}\mathbf{C} \rightarrow B\mathbf{wS}^{(q)}\mathbf{D}$ is a weak homotopy equivalence or still, a pointed homotopy equivalence, so $\mathbf{KC} \rightarrow \mathbf{KD}$ is a homotopy equivalence of spectra (cf. p. 16–8).]

Convention: If $\mathbf{C}$ is an arbitrary Waldhausen category, then $\mathbf{C}$ is not necessarily small. However $\mathbf{C}$ is skeletally small (by definition) and all of the above is applicable to a skeleton $\overline{\mathbf{C}}$, thus $\mathbf{KC} \equiv \mathbf{K}\overline{\mathbf{C}}$ and $K_n(\mathbf{C}) \equiv K_n(\overline{\mathbf{C}})$.

[Note: If $\mathbf{C}$ is small to begin with, then $B\mathbf{wSC}$ and $B\mathbf{wS}\overline{\mathbf{C}}$ have the same pointed homotopy type, so this is a consistent agreement.]

If $X : (\mathbf{\Delta} \times \cdots \times \mathbf{\Delta})^{\mathrm{OP}} \rightarrow \mathbf{CG}$ is a compactly generated multisimplicial space, then its geometric realization is the coend $X \otimes_{\mathbf{\Delta} \times \cdots \times \mathbf{\Delta}} (\Delta^? \times_k \cdots \times_k \Delta^?)$, which is homeomorphic to $|\mathrm{di}X|$, the geometric realization of $\mathrm{di}X$ (the diagonal of X (cf. p. 14–14)).

EXAMPLE If $\mathbf{C}$ is an internal category in **SISET**, i.e., a simplicial object in **CAT**, then $\operatorname{ner}\mathbf{C}$ is a bisimplicial set or still, a functor $(\mathbf{\Delta} \times \mathbf{\Delta})^{\mathrm{OP}} \rightarrow \mathbf{SET}(\subset \mathbf{CG})$ and its geometric realization is the classifying space $B\mathbf{C}$ of $\mathbf{C}$ (thus $B\mathbf{C} \approx |[n] \rightarrow B\mathbf{C}_n|$).

[Note: Analogous considerations apply to multisimplicial objects in **CAT**.]

EXAMPLE If $\mathbf{C}$ is an internal category in **CAT**, i.e., a double category, then the classifying space $B\mathbf{C}$ of $\mathbf{C}$ is the geometric realization of the bisimplicial set $\operatorname{ner}(\operatorname{ner}\mathbf{C})$ (cf. p. 13–67). Example: Let $\mathbf{A}$ be a subcategory of $\mathbf{B}$, where $\mathbf{B}$ is small. Call $\mathbf{A} \cdot \mathbf{B}$ the double category whose objects are those of $\mathbf{B}$, with horizontal morphisms$=\operatorname{Mor}\mathbf{B}$ and vertical morphisms$=\operatorname{Mor}\mathbf{A}$, and whose bimorphisms are the commutative squares with horizontal arrows in $\mathbf{B}$ and vertical arrows in $\mathbf{A}$. View $\mathbf{B}$ as the double category

$$\begin{tikzcd} \bullet \arrow[r] \arrow[d, equal] & \bullet \arrow[d, equal] \\ \bullet \arrow[r] & \bullet \end{tikzcd}$$

—then the inclusion $\mathbf{B} \rightarrow \mathbf{A} \cdot \mathbf{B}$ induces a homotopy equivalence $B\mathbf{B} \rightarrow B\mathbf{A} \cdot \mathbf{B}$.

FACT If $\mathbf{C}$ is a small Waldhausen category, then there is a pointed homotopy equivalence $|W\mathbf{C}| \rightarrow B\mathrm{iso}\,\mathbf{SC}$.

EXAMPLE Let $\mathbf{C}$ be the Waldhausen category whose objects are the pointed finite sets, where weak equivalence=isomorphism and cofibration=pointed injection—then $\mathbf{\Gamma}$ is a skeleton of $\mathbf{C}$, hence is a small Waldhausen category (cf. Proposition 1), and a model for $|W\mathbf{\Gamma}|$ in the pointed homotopy category is $\Omega^\infty\Sigma^\infty\mathbf{S}^1$. Proof: Thanks to the homotopy colimit theorem, $\Omega^\infty\Sigma^\infty\mathbf{S}^1$ can be identified with $\overline{\operatorname{hocolim}}\,\operatorname{pow}\mathbf{S}^1$. But, in the notation of p. 14–68, $\overline{\operatorname{hocolim}}\,\operatorname{pow}\mathbf{S}^1 \approx \overline{\operatorname{pow}}\,\mathbf{S}^1 \otimes_{\mathbf{\Gamma}} \gamma^\infty \approx |\gamma^\infty|_{\mathbf{\Gamma}} \approx B|M_\infty| \approx |W\mathbf{\Gamma}|$, where $|M_\infty| = \coprod_{n \geq 0} BS_n$. Therefore the loop space of $B\mathrm{iso}\,\mathbf{S\Gamma}$ is pointed homotopy equivalent to $\Omega\Omega^\infty\Sigma^\infty\mathbf{S}^1 \approx \Omega^\infty\Sigma^\infty\mathbf{S}^0$, so the algebraic K-groups $K_*(\mathbf{\Gamma})$ of $\mathbf{\Gamma}$ "are" the π_*^s, the stable homotopy groups of spheres.

[Note: More is true, namely $\mathbf{K\Gamma}$ and $\mathbf{S}$, when viewed as objects in **HSPEC**, are isomorphic (Rognes[†]).]

EXAMPLE Let $\mathbf{C}$ be a small category WES, $\mathbf{CXC}^{\mathrm{b}}$ the category of bounded cochain complexes over $\mathbf{C}$. Suppose that $(\mathbf{A}, \iota)$ is a pair satisfying the embedding condition and the E $\Rightarrow$ D axiom. Equip $\mathbf{CXC}^{\mathrm{b}}$ with the structure of a small Waldhausen category by stipulating that the weak equivalences are the arrows in $\mathbf{CXC}^{\mathrm{b}}$ which are quasiisomorphisms in $\mathbf{A}$ and the cofibrations are the levelwise inflations—then the exact functor $\mathbf{C} \rightarrow \mathbf{CXC}^{\mathrm{b}}$ sending X to X concentrated in degree 0 induces a homotopy equivalence $\mathbf{KC} \rightarrow \mathbf{KCXC}^{\mathrm{b}}$ of spectra (Thomason-Trobaugh[‡]).

[†] *Topology* **31** (1992), 813–845.

[‡] *The Grothendieck Festschrift*, vol. III, Birkhäuser (1990), 247–435 (cf. 278–283).

[Note: The definition of weak equivalence is independent of the choice of $(\mathbf{A}, \iota)$. Recall that when $\mathbf{C}$ is pseudoabelian one can take for $(\mathbf{A}, \iota)$ the pair $(\mathbf{G}\text{-}\mathbf{Q}, \iota)$ (cf. p. 18–7).]

PROPOSITION 6 Let $\mathbf{C}$ be a small Waldhausen category—then $K_0(\mathbf{C})$ is the free abelian group on generators $[X]$ ($X \in \operatorname{Ob}\mathbf{C}$) subject to the relations (i) $[X] = [Y]$ if $\exists$ a weak equivalence $X \to Y$ and (ii) $[Y] = [X] + [Y/X]$ for every sequence $X \rightarrowtail Y \to Y/X$.

[Since $K_0(\mathbf{C}) \approx \pi_1(B\mathrm{w}S\mathbf{C})$, $K_0(\mathbf{C})$ is the free group on generators $[X]$ ($X \in \operatorname{Ob}\mathbf{C}$) subject to the relations (i) $[X] = [Y]$ if $\exists$ a weak equivalence $X \to Y$ and (ii) $[Y] = [X]\cdot[Y/X]$ for every sequence $X \rightarrowtail Y \to Y/X$. Applying the second relation to $X \xrightarrow{\mathrm{in}_X} X \amalg Y \xrightarrow{\mathrm{pr}_Y} Y$ & $Y \xrightarrow{\mathrm{in}_Y} X \amalg Y \xrightarrow{\mathrm{pr}_X} X$ gives $[X \amalg Y] = [X]\cdot[Y]$ & $[X \amalg Y] = [Y]\cdot[X]$, thus $K_0(\mathbf{C})$ is abelian and one uses additive notation ($[0] = 0$).]

[Note: If $X \xleftarrow{f} Z \xrightarrow{g} Y$ is a 2-source, where f is a cofibration, then $[P] = [Y] + [P/Y] = [Y] + [X/Z] = [X] + [Y] - [Z]$.]

Example: Suppose that $\mathbf{C}$ satisfies the mapping cylinder axiom—then $\forall\, X \in \operatorname{Ob}\mathbf{C}$, there is a weak equivalence $\Gamma X \to 0$, hence $[X] = -[\Sigma X]$.

[Note: Under these circumstances, every element of $K_0(\mathbf{C})$ is a $[X]$ for some $X \in \operatorname{Ob}\mathbf{C}$. Proof: $[Y] - [Z] = [Y \amalg \Sigma Z]$.]

EXAMPLE Let $\mathbf{C}$ be the category whose objects are the pointed finite CW complexes and whose morphisms are the pointed skeletal maps—then $\mathbf{C}$ is a Waldhausen category if the weak equivalences are the weak homotopy equivalences and the cofibrations are the closed cofibrations which are isomorphic to the inclusion of a subcomplex. Put $A(*) = \Omega B\mathrm{w}S\overline{\mathbf{C}}$ (the algebraic K-theory of a point)—then the reduced Euler characteristic $\widetilde{\chi}$ defined by $K \to \chi(K) - 1$ is an isomorphism from $\pi_0(A(*))$ onto $\mathbf{Z}$.

[Note: Dwyer[†] has shown that the homotopy groups of $A(*)$ are finitely generated. Structurally, in the pointed homotopy category there exists a splitting $A(*) \approx \Omega^\infty\Sigma^\infty\mathbf{S}^0 \times Wh^{\mathrm{DIFF}}(*)$ (Waldhausen[‡]), so $\pi_q(A(*)) \approx \pi_q^s \oplus \pi_q(Wh^{\mathrm{DIFF}}(*))$. Here $Wh^{\mathrm{DIFF}}(*)$ is the Whitehead space of a point. It has the property that there is a pointed homotopy equivalence $\Omega^2 Wh^{\mathrm{DIFF}}(*) \to P(*)$, the stable smooth pseudoisotopy space of $*$. Rationally, it is known that $\pi_q(Wh^{\mathrm{DIFF}}(*)) \otimes \mathbf{Q} = \mathbf{Q}$ if $q \equiv 5 \mod 4$ and is zero otherwise, but the explicit determination of the torsion is difficult and unresolved.]

EXAMPLE Let $\mathbf{C}$ be a small category WES—then $\mathbf{C}$ has finite coproducts (=finite products), thus $\mathbf{C}$ can be viewed as a symmetric monoidal category. Therefore the isomorphism classes of $\mathbf{C}$ constitute an abelian monoid, call it M. Definition: $K^{\oplus}(\mathbf{C}) = \overline{M}$, the group completion of M. So: $K_0(\mathbf{C})$ is a quotient of $K_0^{\oplus}(\mathbf{C})$, the two being the same if every exact pair $X \xrightarrow{i} Y \xrightarrow{p} Z$ splits (i.e., is isomorphic to $X \xrightarrow{\mathrm{in}_X} X \oplus Z \xrightarrow{\mathrm{pr}_Z} Z$).

[†] *Ann. of Math.* 111 (1980), 239–251.

[‡] *Ann. of Math. Studies* 113 (1987), 392–417.

FACT Let $\mathbf{C}$, $\mathbf{D}$ be small categories WES. Assume: $\mathbf{C}$ is cofinal in $\mathbf{D}$—then $K_0(\mathbf{C})$ is a subgroup of $K_0(\mathbf{D})$.

[Observe first that $K^{\oplus}(\mathbf{C})$ is a subgroup of $K^{\oplus}(\mathbf{D})$. This said, suppose in addition that $\mathbf{C}$ is isomorphism closed in $\mathbf{D}$. Given an exact pair $X \xrightarrow{i} Y \xrightarrow{p} Z$ in $\mathbf{D}$, choose X', Z' in $\mathbf{D}$ such that $X \oplus X'$, $Z \oplus Z'$ are in $\mathbf{C}$—then $X \oplus X' \rightarrow Z' \oplus Y \oplus X' \rightarrow Z' \oplus Z$ is exact in $\mathbf{D}$, hence $Z' \oplus Y \oplus X' \in \mathrm{Ob}\,\mathbf{C}$. Consequently, in $K^{\oplus}(\mathbf{D})$, $[Z' \oplus Y \oplus X'] - [X \oplus X'] - [Z' \oplus Z] = [Z'] + [Y] + [X'] - [X] - [X'] - [Z'] - [Z] = [Y] - [X] - [Z]$, thus the kernel of $K_0^{\oplus}(\mathbf{C}) \rightarrow K_0(\mathbf{C})$ equals the kernel of $K_0^{\oplus}(\mathbf{D}) \rightarrow K_0(\mathbf{D})$, which implies that the arrow $K_0(\mathbf{C}) \rightarrow K_0(\mathbf{D})$ is one-to-one.]

EXAMPLE Let $\mathbf{C}$ be a small category WES—then $\mathbf{C}$ is cofinal in $\mathbf{C}_{\mathrm{pa}}$ (cf. p. 18–8), so $K_0(\mathbf{C})$ is a subgroup of $K_0(\mathbf{C}_{\mathrm{pa}})$.

[Note: Let A be a ring with unit—then $K_0(\mathbf{P}(A)) = K_0(A)$ and $\mathbf{F}(A)$ is cofinal in $\mathbf{P}(A)$. The arrow $\mathbf{Z}_{\geq 0} \rightarrow \mathbf{P}(A)$ that sends n to A^n induces a homomorphism $\mathbf{Z} \rightarrow K_0(A)$ of groups (injective iff A has the invariant basis property (i.e., $m \neq n \Rightarrow A^m \not\approx A^n$)). Since $\mathbf{F}(A)_{\mathrm{pa}} = \mathbf{P}(A)$, it follows that the cyclic group $K_0(\mathbf{F}(A))$ is a subgroup of $K_0(A)$.]

PROPOSITION 7 Suppose that $\begin{cases} \mathbf{C} \\ \mathbf{D} \end{cases}$ are small Waldhausen categories. Let $F, G : \mathbf{C} \rightarrow \mathbf{D}$ be model functors, $\Xi : F \rightarrow G$ a natural transformation such that $\forall\, X \in \mathrm{Ob}\,\mathbf{C}$, $\Xi_X : FX \rightarrow GX$ is a weak equivalence in $\mathbf{D}$—then Ξ induces a spectral homotopy between $\mathbf{K}F$ and $\mathbf{K}G$ (cf. p. 13–15 and §14, Proposition 12).

[Note: One starts from the pointed homotopy $B\mathbf{w}\mathbf{S}F \simeq B\mathbf{w}\mathbf{S}G$.]

EXAMPLE Suppose that $\mathbf{C}$ satisfies the mapping cylinder axiom—then $\forall\, X \in \mathrm{Ob}\,\mathbf{C}$, there is a weak equivalence $\Gamma X \rightarrow 0$. But $\Gamma : \mathbf{C} \rightarrow \mathbf{C}$ is a model functor, hence the induced map $B\mathbf{w}\mathbf{S}\mathbf{C} \rightarrow B\mathbf{w}\mathbf{S}\mathbf{C}$ is nullhomotopic.

Let $\mathbf{C}$, $\mathbf{C}'$, $\mathbf{C}''$ be small Waldhausen categories. Assume: $\mathbf{C}'$ and $\mathbf{C}''$ are subcategories of $\mathbf{C}$ with the property that the inclusions $\mathbf{C}' \rightarrow \mathbf{C}$, $\mathbf{C}'' \rightarrow \mathbf{C}$ are model functors. Denote by $\mathbf{E}(\mathbf{C}', \mathbf{C}, \mathbf{C}'')$ the category whose objects are the pushout squares

$$\begin{tikzcd} X' \arrow[r] \arrow[d, tail] & 0 \arrow[d] \\ X \arrow[r] & X'' \end{tikzcd}$$

in $\mathbf{C}$, where $X' \in \mathrm{Ob}\,\mathbf{C}'$, $X \in \mathrm{Ob}\,\mathbf{C}$, $X'' \in \mathrm{Ob}\,\mathbf{C}''$, and whose morphisms are the commutative

diagrams
$$\begin{tikzcd} X' \arrow[r, tail] \arrow[d] & X \arrow[r] \arrow[d] & X'' \arrow[d] \\ Y' \arrow[r, tail] & Y \arrow[r] & Y'' \end{tikzcd}$$
in $\mathbf{C}$, where $X' \to Y' \in \operatorname{Mor}\mathbf{C}'$, $X \to Y \in \operatorname{Mor}\mathbf{C}$, $X'' \to Y'' \in \operatorname{Mor}\mathbf{C}''$.

[Note: When $\mathbf{C}' = \mathbf{C}$ and $\mathbf{C}'' = \mathbf{C}$, put $\mathbf{EC} = \mathbf{E}(\mathbf{C},\mathbf{C},\mathbf{C})$.]

LEMMA $\mathbf{E}(\mathbf{C}',\mathbf{C},\mathbf{C}'')$ is a small Waldhausen category.

[A morphism in $\mathbf{E}(\mathbf{C}',\mathbf{C},\mathbf{C}'')$ is a weak equivalence if $X' \to Y'$ is a weak equivalence in $\mathbf{C}'$, $X \to Y$ is a weak equivalence in $\mathbf{C}$, $X'' \to Y''$ is a weak equivalence in $\mathbf{C}''$ and a morphism in $\mathbf{E}(\mathbf{C}',\mathbf{C},\mathbf{C}'')$ is a cofibration if $X' \to Y'$ is a cofibration in $\mathbf{C}'$, $Y' \underset{X'}{\sqcup} X \to Y$ is a cofibration in $\mathbf{C}$, $X'' \to Y''$ is a cofibration in $\mathbf{C}''$.

[Note: $X \to Y$ is then a cofibration in $\mathbf{C}$ (being the composite $X' \underset{X'}{\sqcup} X \rightarrowtail Y' \underset{X'}{\sqcup} X \rightarrowtail Y$ (cf. §12, Proposition 4)).]

There are model functors $s : \mathbf{E}(\mathbf{C}',\mathbf{C},\mathbf{C}'') \to \mathbf{C}'$, $t : \mathbf{E}(\mathbf{C}',\mathbf{C},\mathbf{C}'') \to \mathbf{C}$, $Q : \mathbf{E}(\mathbf{C}',\mathbf{C},\mathbf{C}'') \to \mathbf{C}''$, viz. $s(X' \rightarrowtail X \to X'') = X'$, $t(X' \rightarrowtail X \to X'') = X$, $Q(X' \rightarrowtail X \to X'') = X''$. In the other direction, there is a model functor $I : \mathbf{C}' \times \mathbf{C}'' \to \mathbf{E}(\mathbf{C}',\mathbf{C},\mathbf{C}'')$ which sends (X', X'') to $X' \rightarrowtail X' \amalg X'' \to X''$. Agreeing to write (s, Q) for the model functor $\mathbf{E}(\mathbf{C}',\mathbf{C},\mathbf{C}'') \to \mathbf{C}' \times \mathbf{C}''$ defined by s and Q, viz. $(s,Q)(X' \rightarrowtail X \to X'') = (X', X'')$, one has $(s,Q) \circ I = \mathrm{id}_{\mathbf{C}' \times \mathbf{C}''}$.

RELATIVE ADDITIVITY THEOREM The model functor (s,Q) induces a homotopy equivalence $\mathbf{K}(s,Q) : \mathbf{KE}(\mathbf{C}',\mathbf{C},\mathbf{C}'') \to \mathbf{KC}' \times \mathbf{KC}''$ of spectra.

ABSOLUTE ADDITIVITY THEOREM The model functor (s,Q) induces a homotopy equivalence $\mathbf{K}(s,Q) : \mathbf{KEC} \to \mathbf{KC} \times \mathbf{KC}$ of spectra.

It is a question of proving that (s,Q) induces a weak homotopy equivalence $B\mathbf{wSE}(\mathbf{C}',\mathbf{C},\mathbf{C}'') \to B\mathbf{wSC}' \times_k B\mathbf{wSC}''$ of classifying spaces. To this end, we shall proceed via a series of lemmas.

HOMOTOPY LEMMA Grant the truth of the absolute additivity theorem—then $B\mathbf{wS}t : B\mathbf{wSEC} \to B\mathbf{wSC}$ is pointed homotopic to $B\mathbf{wS}(s \amalg Q) : B\mathbf{wSEC} \to B\mathbf{wSC}$.

[Note: Here $(s \amalg Q)(X' \rightarrowtail X \to X'') = X' \amalg X''$.]

TRIAD LEMMA Grant the truth of the absolute additivity theorem. Suppose given a small Waldhausen category $\mathbf{D}$, model functors $G, G', G'' : \mathbf{D} \to \mathbf{C}$, and natural transformations $G' \to G$, $G \to G''$. Assume: (i) For every object X in $\mathbf{D}$, the arrow $G'X \to GX$

is a cofibration and the commutative diagram

$$\begin{tikzcd} G'X \arrow[r] \arrow[d, tail] & 0 \arrow[d] \\ GX \arrow[r] & G''X \end{tikzcd}$$

is a pushout square; (ii) For every cofibration $X \rightarrowtail Y$ in $\mathbf{D}$, the arrow $G'Y \underset{G'X}{\sqcup} GX \to GY$ is a cofibration—then $B\mathbf{w}\mathbf{S}G$ is pointed homotopic to $B\mathbf{w}\mathbf{S}(G' \amalg G'')$.

[There exists a model functor $\Phi : \mathbf{D} \to \mathbf{EC}$ with $G' = s \circ \Phi$, $G = t \circ \Phi$, $G'' = Q \circ \Phi$. The assertion thus follows from the homotopy lemma by naturality.]

EXAMPLE Let $\mathbf{C}$ be the Waldhausen category whose objects are the pointed finite CW complexes and whose morphisms are the pointed skeletal maps—then the arrow $B\mathbf{wS}\overline{\mathbf{C}} \to B\mathbf{wS}\overline{\mathbf{C}}$ induced by Σ is a pointed homotopy equivalence.

[In the triad lemma, take $\overline{\mathbf{C}} = \overline{\mathbf{D}}$ and let $G' = \mathrm{id}_{\overline{\mathbf{C}}}, G = \Gamma$, $G'' = \Sigma$.]

[Note: The full subcategory $\mathbf{C}_0$ of $\overline{\mathbf{C}}$ whose objects are path connected is Waldhausen (WC–3 is a consequence of AD_1 (cf. p. 3–1)). Since there is a commutative diagram

$$\begin{tikzcd} B\mathbf{wS}\overline{\mathbf{C}} \arrow[r, "B\mathbf{wS}\Sigma"] \arrow[rd, dashed] & B\mathbf{wS}\overline{\mathbf{C}} \\ B\mathbf{wS}\mathbf{C}_0 \arrow[u, "B\iota"] \arrow[r, "B\mathbf{wS}\Sigma"'] & B\mathbf{wS}\mathbf{C}_0 \arrow[u, "B\iota"'] \end{tikzcd}$$

, it follows that $B\iota$ is a pointed homotopy equivalence. Therefore the algebraic K-theory of a point can be defined using path connected objects. If now $\mathbf{C}_1$ is the full subcategory of $\mathbf{C}_0$ whose objects are simply connected, then $\mathbf{C}_1$ is Waldhausen (WC–3 is implied by the Van Kampen theorem). Repeating the argument, one concludes that the algebraic K-theory of a point can be defined using simply connected objects. As an aside, observe that $\mathbf{C}_1$ satisfies the extension axiom (via the Whitehead theorem) but $\overline{\mathbf{C}}$ does not.]

LEMMA OF REDUCTION The absolute additivity theorem implies the relative additivity theorem.

[Since $(s, Q) \circ I = \mathrm{id}_{\mathbf{C}' \times \mathbf{C}''}$, it suffices to show that $B\mathbf{wS}(I \circ (s, Q))$ is pointed homotopic to the identity. Accordingly, to apply the triad lemma, define model functors G', G, G'' : $\mathbf{E}(\mathbf{C}', \mathbf{C}, \mathbf{C}'') \to \mathbf{E}(\mathbf{C}', \mathbf{C}, \mathbf{C}'')$ by $G'(X' \rightarrowtail X \to X'') = X' \overset{\mathrm{id}_{X'}}{\rightarrowtail} X' \to 0$, $G(X' \rightarrowtail X \to X'') = X' \rightarrowtail X \to X''$, $G''(X' \rightarrowtail X \to X'') = 0 \rightarrowtail X'' \overset{\mathrm{id}_{X''}}{\rightarrowtail} X''$ and note that $B\mathbf{wS}(I \circ (s, Q)) = B\mathbf{wS}(G' \amalg G'')$.]

ADDITIVITY LEMMA The simplicial map $W(s, Q) : W\mathbf{EC} \to W\mathbf{C} \times W\mathbf{C}$ induced by (s, Q) is a weak homotopy equivalence (notation as on p. 18–9).

The additivity lemma implies the absolute additivity theorem. To see this, introduce $\mathbf{wC}(n)$ (cf. p. 18–8 ff.)—then $\forall$ n, the arrow $W\mathbf{EwC}(n) \to W\mathbf{wC}(n) \times W\mathbf{wC}(n)$ is a weak homotopy equivalence. Therefore the diagonal of the bisimplicial map $([n] \rightarrow W\mathbf{E}\mathbf{w}\mathbf{C}(n)) \rightarrow ([n] \rightarrow W\mathbf{w}\mathbf{C}(n)) \times ([n] \rightarrow W\mathbf{w}\mathbf{C}(n))$ is a weak homotopy equivalence (cf. §13, Proposition 51) or still, the induced map of geometric realizations is a weak homotopy equivalence. It remains only to observe that $\operatorname{Ob}\mathbf{S}_m\mathbf{w}\mathbf{C}(n) \approx \operatorname{ner}_n \mathbf{w}\mathbf{S}_m\mathbf{C}$.

LEMMA The projection $W\mathbf{EC} \xrightarrow{p} W\mathbf{C}$ induced by s is a homotopy fibration (cf. infra).

This result leads to the additivity lemma. In fact, $\forall\ n$ & $\forall\ x \in W_n\mathbf{C}$, the pullback square

$$\begin{tikzcd} F_x \arrow[r] \arrow[d] & W\mathbf{EC} \arrow[d, "p"] \\ \Delta[n] \arrow[r, "\Delta_x"'] & W\mathbf{C} \end{tikzcd}$$

($F_x = W\mathbf{EC}_x$) is a homotopy pullback (cf. p. 12–16). Now take $n = 0$ and recall that $W_0\mathbf{C} = *$—then $F_0 \rightarrow W\mathbf{EC} \xrightarrow{p} W\mathbf{C}$ is a homotopy pullback and F_0 can be identified with $W\mathbf{F}_0\mathbf{C}$, $\mathbf{F}_0\mathbf{C}$ being the full subcategory of $\mathbf{EC}$ whose objects are the $0 \rightarrowtail X \rightarrow X''$ ($\Rightarrow X \approx X''$). But the model functor $\mathbf{F}_0\mathbf{C} \rightarrow \mathbf{C}$ defined by

$$\begin{tikzcd} 0 \arrow[r, tail] & X \arrow[r] \arrow[d] & X'' \\ & X & \end{tikzcd}$$

gives rise to a homotopy equivalence $W\mathbf{F}_0\mathbf{C} \rightarrow W\mathbf{C}$ of simplicial sets. Therefore the sequence $W\mathbf{C} \rightarrow W\mathbf{EC} \xrightarrow{p} W\mathbf{C}$ is a homotopy pullback (the arrow $W\mathbf{C} \rightarrow W\mathbf{EC}$ corresponds to the insertion $\mathbf{C} \rightarrow \mathbf{EC}$ which sends X to $0 \rightarrowtail X \xrightarrow{\mathrm{id}_X} X$). Consider the diagram

$$\begin{tikzcd} W\mathbf{C} \arrow[r] \arrow[d, equal] & W\mathbf{C} \times W\mathbf{C} \arrow[r] \arrow[d] & W\mathbf{C} \arrow[d, equal] \\ W\mathbf{C} \arrow[r] & W\mathbf{EC} \arrow[r] & W\mathbf{C} \end{tikzcd}$$

, where the vertical arrow is determined by I. Passing to geometric realizations, the top and bottom rows become fibrations up to homotopy (per **CGH** (singular structure) (cf. p. 13–75)), thus $|WI| : |W\mathbf{C}| \times_k |W\mathbf{C}| \rightarrow |W\mathbf{EC}|$ is a pointed homotopy equivalence. Since $|W(s, Q)| \circ |WI| = \mathrm{id}_{|W\mathbf{C}| \times_k |W\mathbf{C}|}$, it follows that $|W(s, Q)|$ is also a pointed homotopy equivalence, the assertion of the additivity lemma.

Put $X = W\mathbf{EC}$, $B = W\mathbf{C}$—then to prove the lemma, one must show that for every commutative diagram

$$\begin{tikzcd} X_{b'} \arrow[r] \arrow[d] & X_b \arrow[r] \arrow[d] & X \arrow[d, "p"] \\ \Delta[n'] \arrow[r] & \Delta[n] \arrow[r, "\Delta_b"'] & B \end{tikzcd}$$

, the arrow $X_{b'} \rightarrow X_b$ is a weak homotopy equivalence (cf. p. 13–64). Since any map $[n'] \rightarrow [n]$ can be placed in a commutative triangle

$$\begin{tikzcd} & {[0]} \arrow[dl] \arrow[dr] & \\ {[n']} \arrow[rr] & & {[n]} \end{tikzcd}$$

, there is no loss of generality in supposing that $n' = 0$, thus our objective may be recast.

LEMMA Fix an element $b \in B_n$ and let $v_i : X_{b'} \to X_b$ be the simplicial map attached to the ith vertex operator $\epsilon_i : [0] \to [n]$ $(0 \leq i \leq n)$—then v_i is a homotopy equivalence.

[From the definitions, $x \in X_m$ $(= W_m\mathbf{EC}) \leftrightarrow F' \rightarrowtail F \to F'' \in \operatorname{Ob}\mathbf{ES}_m\mathbf{C}$. And: An element of $(X_b)_m$ consists of an element of X_m plus a map $\alpha : [m] \to [n]$ such that F' is equal to the composite $[m](\to) \xrightarrow{\alpha_*} [n](\to) \xrightarrow{b} \mathbf{C}$. There is an evident homotopy equivalence $W\mathbf{C} \xrightarrow{f} X_{b'}$ and $\forall\ i$, $q \circ v_i \circ f = \mathrm{id}_{W\mathbf{C}}$, where $q : X_b \to W\mathbf{C}$ is induced by the functor that takes $F' \rightarrowtail F \to F''$ to F''. It will be enough to show that q is a homotopy equivalence and for this it will be enough to show that $\mathrm{id}_{X_b} \simeq v_n \circ f \circ q$. Let X_b^* be the composite $(\mathbf{\Delta}/[1])^{\mathrm{OP}} \to \mathbf{\Delta}^{\mathrm{OP}} \xrightarrow{X_b} \mathbf{SET}$ and define a natural transformation $H : X_b^* \to X_b^*$ by assigning to $\beta : [m] \to [1]$ the function $H_\beta \in \operatorname{Mor}((X_b)_m, (X_b)_m)$ which sends $(F' \rightarrowtail F \to F'', \alpha : [m] \to [n])$ to $(\overline{F}' \rightarrowtail \overline{F} \to \overline{F}'', \overline{\alpha} : [m] \to [n])$. Here $\overline{\alpha}$ is the composite $[m] \xrightarrow{(\alpha,\beta)} [n] \times [1] \xrightarrow{\gamma} [n]$ $(\gamma(j,0) = j, \gamma(j,1) = n)$ and $\overline{F}' = b \circ \overline{\alpha}_*$. Because $\alpha \leq \overline{\alpha}$, $\exists$ a natural transformation $\alpha_* \to \overline{\alpha}_*$, hence $\exists$ a natural transformation $F' \to \overline{F}'$ and $\overline{F}$ is given by the pushout square

$$\begin{tikzcd} F' \arrow[r] \arrow[d, tail] & \overline{F}' \arrow[d] \\ F \arrow[r] & \overline{F} \end{tikzcd}$$

in $\mathbf{S}_m\mathbf{C}$ with $\overline{F}'' = \overline{F}/\overline{F}'$. Needless to say, this procedure involves certain choices and it is necessary to check that they can be made in such a way that H really is natural. Leaving this as an exercise, let us note only that matters can be arranged so that the homotopy starts at the identity (viz., if $F' \to \overline{F}'$ is the identity, choose $F \to \overline{F}$ to be the identity) and that the image of $v_n \circ f$ is fixed under the homotopy (viz., if $\overline{F}' = 0$, choose $\overline{F} \to \overline{F}''$ to be the identity).]

Rappel: Given a simplicial set X, TX is its translate (cf. p. 14–12).

[Note: $T_0X = X_1$, so there is a simplicial map $\mathrm{si}X_1 \to TX$. On the other hand, the $d_0 : X_{n+1} \to X_n$ define a simplicial map $TX \to X$.]

Example: If $\mathbf{C}$ is a simplicial object in $\mathbf{CAT}$, then $T\mathbf{C} \leftrightarrow (TM, TO)$, where $\mathbf{C} \leftrightarrow (M, O)$ (an internal category in $\mathbf{SISET}$) and there is a sequence $\mathrm{si}\mathbf{C}_1 \to T\mathbf{C} \to \mathbf{C}$.

[Note: This applies to $\mathbf{wSC}$, where $\mathbf{C}$ is a small Waldhausen category. Since $\mathbf{wS}_1\mathbf{C}$ is isomorphic to $\mathbf{wC}$, there is a sequence $\mathrm{si}\mathbf{wC} \to T\mathbf{wSC} \to \mathbf{wSC}$ and since $B\mathbf{wS}_0\mathbf{C} = *$, $BT\mathbf{wSC}$ is contractible (cf. p. 14–12). Thus one is lead again to the arrow $B\mathbf{wC} \to \Omega B\mathbf{wSC}$ whose adjoint $\Sigma B\mathbf{wC} \to B\mathbf{wSC}$ is the closed embedding on p. 18–10. By naturality, $\mathbf{C}$ can be replaced by $\mathbf{SC}$, which produces another sequence $\mathrm{si}\mathbf{wSC} \to T\mathbf{wS}^{(2)}\mathbf{C} \to \mathbf{wS}^{(2)}\mathbf{C}$. It follows from Proposition 8 below that the sequence $B\mathbf{wSC} \to BT\mathbf{wS}^{(2)}\mathbf{C} \to B\mathbf{wS}^{(2)}\mathbf{C}$ of classifying spaces is a fibration up to homotopy (per $\mathbf{CGH}$ (singular structure)). Therefore the arrow $B\mathbf{wSC} \to \Omega B\mathbf{wS}^{(2)}\mathbf{C}$ is a weak homotopy equivalence or still, a pointed homotopy equivalence. Continuing, one sees that $B\mathbf{wS}^{(q)}\mathbf{C} \to \Omega B\mathbf{wS}^{(q+1)}\mathbf{C}$ is a pointed homotopy equivalence $\forall\ q$ (cf. p. 18–10).]

Let $\begin{cases} \mathbf{C} \\ \mathbf{D} \end{cases}$ be small Waldhausen categories, $F : \mathbf{C} \to \mathbf{D}$ a model functor. Define

$\mathbf{S}(\mathbf{C}\xrightarrow{F}\mathbf{D})$ by the pullback square

$$\begin{tikzcd} \mathbf{S}(\mathbf{C}\xrightarrow{F}\mathbf{D}) \arrow[r] \arrow[d] & T\mathbf{SD} \arrow[d] \\ \mathbf{SC} \arrow[r] & \mathbf{SD} \end{tikzcd}$$

in $[\boldsymbol{\Delta}^{\mathrm{OP}},\mathbf{CAT}]$, so $\forall\ n$,

$$\begin{tikzcd} \mathbf{S}_n(\mathbf{C}\xrightarrow{F}\mathbf{D}) \arrow[r] \arrow[d] & \mathbf{S}_{n+1}\mathbf{D} \arrow[d] \\ \mathbf{S}_n\mathbf{C} \arrow[r] & \mathbf{S}_n\mathbf{D} \end{tikzcd}$$

is a pullback square in **CAT**.

[Note: There is a sequence $\mathbf{siD}\rightarrow\mathbf{S}(\mathbf{C}\xrightarrow{F}\mathbf{D})\rightarrow\mathbf{SC}$.]

LEMMA $\mathbf{S}_n(\mathbf{C}\xrightarrow{F}\mathbf{D})$ is a small Waldhausen category.

[The weak equivalences are given by the pullback square

$$\begin{tikzcd} \mathbf{wS}_n(\mathbf{C}\xrightarrow{F}\mathbf{D}) \arrow[r] \arrow[d] & \mathbf{wS}_{n+1}\mathbf{D} \arrow[d] \\ \mathbf{wS}_n\mathbf{C} \arrow[r] & \mathbf{wS}_n\mathbf{D} \end{tikzcd}$$

and the cofibrations are given by the pullback square

$$\begin{tikzcd} \mathbf{coS}_n(\mathbf{C}\xrightarrow{F}\mathbf{D}) \arrow[r] \arrow[d] & \mathbf{coS}_{n+1}\mathbf{D} \arrow[d] \\ \mathbf{coS}_n\mathbf{C} \arrow[r] & \mathbf{coS}_n\mathbf{D} \end{tikzcd}$$

.]

[Note: $\mathbf{S}(\mathbf{C}\xrightarrow{F}\mathbf{D})$ is a simplicial object in **WALD**.]

EXAMPLE Taking $\mathbf{C} = \mathbf{D}$ and $F = \mathrm{id}_{\mathbf{C}}$ gives nothing new $(\mathbf{S}(\mathbf{C}\xrightarrow{\mathrm{id}_{\mathbf{C}}}\mathbf{C}) = T\mathbf{SC})$ but there is a variant which is of some interest. Thus define $\mathbf{GC}$ by the pullback square

$$\begin{tikzcd} \mathbf{GC} \arrow[r] \arrow[d] & T\mathbf{SC} \arrow[d] \\ T\mathbf{SC} \arrow[r] & \mathbf{SC} \end{tikzcd}$$

—then $\mathbf{G}_n\mathbf{C}$ is a small Waldhausen category and $\mathbf{GC}$ is a simplicial object in **WALD**. The significance of $\mathbf{GC}$ lies in the fact that the arrow $B\mathbf{wGC}\rightarrow\Omega B\mathbf{wSC}$ is a weak homotopy equivalence if $\mathbf{C}$ is a category WES (Gillet-Grayson[†]).

PROPOSITION 8 Let $\left\{\begin{matrix}\mathbf{C}\\ \mathbf{D}\end{matrix}\right.$ be small Waldhausen categories, $F:\mathbf{C}\rightarrow\mathbf{D}$ a model functor—then the sequence $B\mathbf{wSD}\rightarrow B\mathbf{wS}^{(2)}(\mathbf{C}\xrightarrow{F}\mathbf{D})\rightarrow B\mathbf{wS}^{(2)}\mathbf{C}$ of classifying spaces is a fibration up to homotopy (per **CGH** (singular structure)).

[It suffices to verify that $\forall\ n$, the sequence $B\mathbf{wSD}\rightarrow B\mathbf{wSS}_n(\mathbf{C}\xrightarrow{F}\mathbf{D})\rightarrow B\mathbf{wSS}_n\mathbf{C}$ is a fibration up to homotopy (per **CGH** (singular structure)) (cf. p. 14–9) $(\pi_0(B\mathbf{wSS}_n\mathbf{C}) = *\ \forall\ n)$. Do this by comparing it with the sequence $B\mathbf{wSD}\rightarrow B\mathbf{wSD}\times_k B\mathbf{wSS}_n\mathbf{C}\rightarrow B\mathbf{wSS}_n\mathbf{C}$, using the triad lemma to establish that the arrow $B\mathbf{wSD}\times_k B\mathbf{wSS}_n\mathbf{C}\rightarrow B\mathbf{wSS}_n(\mathbf{C}\xrightarrow{F}\mathbf{D})$ is a "retraction up to homotopy".]

[†] *Illinois J. Math.* **31** (1987), 574–597; see also Gunnarsson et al., *J. Pure Appl. Algebra* **79** (1992), 255–270.

LEMMA Equip $\mathbf{CGH}$ with its singular structure. Suppose given a commutative diagram

$$\begin{tikzcd} A \arrow[r, "g"] \arrow[d] & X \arrow[r, "f"] \arrow[d, "\phi"] & Y \arrow[d, equal] \\ A' \arrow[r, "g'"'] & X' \arrow[r, "f'"'] & Y \end{tikzcd}$$

of pointed compactly generated Hausdorff spaces. Assume: The rows are fibrations up to homotopy—then the square

$$\begin{tikzcd} A \arrow[r, "g"] \arrow[d] & X \arrow[d, "\phi"] \\ A' \arrow[r, "g'"'] & X' \end{tikzcd}$$

is a homotopy pullback.

[The claim is that the arrow $A \to W_{g',\phi}$ is a weak homotopy equivalence. Consider the commutative diagram

$$\begin{tikzcd} A' \arrow[r, "g'"] \arrow[d] & X' \arrow[d, equal] & X \arrow[l, "\phi"'] \arrow[d, equal] \\ E_{f'} \arrow[r, "\pi'"'] & X' & X \arrow[l, "\phi"] \end{tikzcd}$$

. By hypothesis, the arrow $A' \to E_{f'}$ is a weak homotopy equivalence, so the induced map $W_{g',\phi} \to W_{\pi',\phi}$ is a weak homotopy equivalence (cf. p. 4–48). On the other hand, the projection $\pi' : E_{f'} \to X'$ is a pointed $\mathbf{CG}$ fibration (cf. p. 4–32), hence is a $\mathbf{CG}$ fibration (cf. p. 4–7). Therefore the arrow $E_{f'} \times_{X'} X \to W_{\pi',\phi}$ is a homotopy equivalence (cf. §4, Proposition 18). But $E_{f'} \times_{X'} X = \{y_0\} \times_Y W_{f'} \times_{X'} X = \{y_0\} \times_Y W_f = E_f$ and by hypothesis, the arrow $A \to E_f$ is a weak homotopy equivalence.]

PROPOSITION 9 Let $\mathbf{C}', \mathbf{C}, \mathbf{C}''$ be small Waldhausen categories. Suppose given model functors $\mathbf{C}' \to \mathbf{C}, \mathbf{C} \to \mathbf{C}''$—then the square

$$\begin{tikzcd} B\mathbf{wSC} \arrow[r] \arrow[d] & B\mathbf{wS}^{(2)}(\mathbf{C}' \to \mathbf{C}) \arrow[d] \\ B\mathbf{wSC}'' \arrow[r] & B\mathbf{wS}^{(2)}(\mathbf{C}' \to \mathbf{C}'') \end{tikzcd}$$

is a homotopy pullback (per $\mathbf{CGH}$ (singular structure)).

[Bearing in mind Proposition 8, apply the lemma to the commutative diagram

$$\begin{tikzcd} B\mathbf{wSC} \arrow[r] \arrow[d] & B\mathbf{wS}^{(2)}(\mathbf{C}' \to \mathbf{C}) \arrow[r] \arrow[d] & B\mathbf{wS}^{(2)}\mathbf{C}' \arrow[d, equal] \\ B\mathbf{wSC}'' \arrow[r] & B\mathbf{wS}^{(2)}(\mathbf{C}' \to \mathbf{C}'') \arrow[r] & B\mathbf{wS}^{(2)}\mathbf{C}' \end{tikzcd}$$

.]

Suppose given a small category $\mathbf{C}$ carrying the structure of two Waldhausen categories, both having the same subcategory of cofibrations but potentially distinct subcategories of weak equivalences, say $\mathbf{vC}$ and $\mathbf{wC}$, with $\mathbf{vC} \subset \mathbf{wC}$ (e.g., $\mathbf{vC}$ might be iso $\mathbf{C}$). Let $\mathbf{C}^{\mathbf{w}}$ be the full subcategory of $\mathbf{C}$ whose objects are the X such that $0 \to X$ is in $\mathbf{wC}$, put $\mathbf{vC}^{\mathbf{w}} = \mathbf{vC} \cap \mathbf{C}^{\mathbf{w}}$ & $\mathbf{wC}^{\mathbf{w}} = \mathbf{wC} \cap \mathbf{C}^{\mathbf{w}}$, and $\mathbf{coC}^{\mathbf{w}} = \mathbf{coC} \cap \mathbf{C}^{\mathbf{w}}$—then $\mathbf{C}^{\mathbf{w}}$ is Waldhausen relative to either notion of weak equivalence.

LOCALIZATION THEOREM Assume that $\mathbf{C}$ admits a functor $M : \mathbf{C}(\to) \to \mathbf{C}$ that is a mapping cylinder in the v-structure and the w-structure. Suppose further that in the w-structure, the saturation axiom, the extension axiom, and the mapping cylinder axiom all hold—then the square

$$\begin{tikzcd} B\mathbf{vSC}^{\mathbf{w}} \arrow[r] \arrow[d] & B\mathbf{wSC}^{\mathbf{w}} \arrow[d] \\ B\mathbf{vSC} \arrow[r] & B\mathbf{wSC} \end{tikzcd}$$

is a homotopy pullback (per **CGH** (singular structure)).

[The proof, which depends on Proposition 9, is detailed in Waldhausen[†].]

[Note: $\forall\, n, \mathbf{wS}_n\mathbf{C}^{\mathbf{w}}$ has an initial object, thus $B\mathbf{wSC}^{\mathbf{w}}$ is contractible.]

Remark: Proposition 3 enters into the proof through the assumption that the w-structure on $\mathbf{C}$ satisfies the saturation axiom and the mapping cylinder axiom. As for the role of the extension axiom, recall that if $X \to Y$ is an acyclic cofibration, then $0 \to Y/X$ is an acyclic cofibration (cf. Proposition 2), i.e., $Y/X \in \operatorname{Ob}\mathbf{C}^{\mathbf{w}}$. Conversely, if $X \to Y$ is a cofibration for which $Y/X \in \operatorname{Ob}\mathbf{C}^{\mathbf{w}}$, then the extension axiom implies that $X \to Y$ is a weak equivalence (consider the commutative diagram

$$\begin{tikzcd} X \arrow[r, equal] \arrow[d, equal] & X \arrow[r] \arrow[d] & 0 \arrow[d] \\ X \arrow[r] & Y \arrow[r] & Y/X \end{tikzcd}$$

).

[Note: For an interesting application of the localization theorem to the algebraic K-theory of a ring with unit, see Weibel-Yao[‡].]

PROPOSITION 10 Let $\left\{\begin{matrix}\mathbf{C}\\ \mathbf{D}\end{matrix}\right.$ be small Waldhausen categories, $F : \mathbf{C} \to \mathbf{D}$ a model functor—then there exists a long exact sequence $\cdots \to \pi_{n+1}(B\mathbf{wS}^{(2)}(\mathbf{C} \xrightarrow{F} \mathbf{D})) \to \pi_n(B\mathbf{wSC}) \to \pi_n(B\mathbf{wSD}) \to \pi_n(B\mathbf{wS}^{(2)}(\mathbf{C} \xrightarrow{F} \mathbf{D})) \to \cdots \to \pi_2(B\mathbf{wS}^{(2)}(\mathbf{C} \xrightarrow{F} \mathbf{D})) \to \pi_1(B\mathbf{wSC}) \to \pi_1(B\mathbf{wSD}) \to \pi_1(B\mathbf{wS}^{(2)}(\mathbf{C} \xrightarrow{F} \mathbf{D})) \to \pi_0(B\mathbf{wSC}) \to \pi_0(B\mathbf{wSD})$ in homotopy.

[Proposition 9 implies that the square

$$\begin{tikzcd} B\mathbf{wSC} \arrow[r] \arrow[d] & B\mathbf{wS}^{(2)}(\mathbf{C} \xrightarrow{\mathrm{id}_{\mathbf{C}}} \mathbf{C}) \arrow[d] \\ B\mathbf{wSD} \arrow[r] & B\mathbf{wS}^{(2)}(\mathbf{C} \xrightarrow{F} \mathbf{D}) \end{tikzcd}$$

is a homotopy pullback (per **CGH** (singular structure)), thus the Mayer-Vietoris sequence is applicable (cf. p. 4–37). And: $B\mathbf{wS}^{(2)}(\mathbf{C} \xrightarrow{\mathrm{id}_{\mathbf{C}}} \mathbf{C})$ is contractible.]

COFINALITY PRINCIPLE Let $\mathbf{C}, \mathbf{D}$ be small categories WES. Assume: $\mathbf{C}$ is cofinal in $\mathbf{D}$—then $K_0(\mathbf{C})$ is a subgroup of $K_0(\mathbf{D})$ (cf. p. 18–13) and $\forall\, n \geq 1$, $K_n(\mathbf{C}) \approx K_n(\mathbf{D})$.

[Since by definition, $K_n(\mathbf{C}) \approx \pi_{n+1}(B\mathbf{wSC})$ & $K_n(\mathbf{D}) \approx \pi_{n+1}(B\mathbf{wSD})$, one can invoke Proposition 10 if the higher homotopy groups of $B\mathbf{wS}^{(2)}(\mathbf{C} \xrightarrow{\iota} \mathbf{D})$ are trivial. This is established by showing that $B\mathbf{wS}^{(2)}(\mathbf{C}\xrightarrow{i}\mathbf{D})$ has the same pointed homotopy type as $B(K_0(\mathbf{D})/K_0(\mathbf{C}))$, the classifying space of $K_0(\mathbf{D})/K_0(\mathbf{C})$.]

[†] *SLN* 1126 (1985), 350–352.

[‡] *Contemp. Math.* 126 (1992), 219–230.

[Note: All the particulars can be found in Staffeldt[†].]

EXAMPLE Let $\mathbf{C}$ be a small category WES—then $\mathbf{C}$ is cofinal in $\mathbf{C}_{\text{pa}}$ (cf. p. 18–8), hence $\forall\, n \geq 1$, $K_n(\mathbf{C}) \approx K_n(\mathbf{C}_{\text{pa}})$.

[Note: Let A be a ring with unit—then $\mathbf{F}(A)$ is cofinal in $\mathbf{P}(A)$, so the higher algebraic K-groups of $\mathbf{F}(A)$ can be identified with the higher algebraic K-groups of $\mathbf{P}(A)$.]

Let $\mathbf{C}, \mathbf{D}$ be small Waldhausen categories, $F : \mathbf{C} \to \mathbf{D}$ a model functor—then F is said to have the approximation property provided that the following conditions are satisfied.

(App$_1$) A morphism f in $\mathbf{C}$ is in $\mathbf{wC}$ if Ff is in $\mathbf{wD}$.

(App$_2$) Given $X \in \text{Ob}\,\mathbf{C}$ and $f \in \text{Mor}(FX, Y)$, there is a $g \in \text{Mor}(X, X')$ and a weak equivalence $h : FX' \to Y$ such that $f = h \circ Fg$:

$$\begin{tikzcd} FX \arrow[r, "f"] \arrow[d, "Fg"'] & Y \\ FX' \arrow[ur, "h"'] & \end{tikzcd}$$

.

Remarks: (1) Since F is a model functor, Ff is in $\mathbf{wD}$ if f is in $\mathbf{wC}$; (2) When $\mathbf{C}$ satisfies the mapping cylinder axiom, $\exists$ a commutative triangle

$$\begin{tikzcd} X \arrow[r, "i"] \arrow[dr, "g"'] & M_g \arrow[d, "r"] \\ & X' \end{tikzcd}$$

, where r is a weak equivalence, hence in this case one can assume that the "g" is a cofibration.

APPROXIMATION THEOREM Let $\mathbf{C}, \mathbf{D}$ be small Waldhausen categories satisfying the saturation axiom, $F : \mathbf{C} \to \mathbf{D}$ a model functor. Suppose that $\mathbf{C}$ satisfies the mapping cylinder axiom and F has the approximation property—then $B\mathbf{wS}F : B\mathbf{wSC} \to B\mathbf{wSD}$ is a pointed homotopy equivalence.

[This result is due to Waldhausen[‡]. I shall omit the proof (which is long and technical) but by way of simplification, it suffices that $B\mathbf{w}F : B\mathbf{wC} \to B\mathbf{wD}$ be a pointed homotopy equivalence. Reason: $\mathbf{S}_n\mathbf{C}$ and $\mathbf{S}_n\mathbf{D}$ inherit the assumptions made on $\mathbf{C}$ and $\mathbf{D}$, thus $\forall\, n$, $B\mathbf{wS}_nF : B\mathbf{wS}_n\mathbf{C} \to B\mathbf{wS}_n\mathbf{D}$ is a pointed homotopy equivalence and so $B\mathbf{wS}F : B\mathbf{wSC} \to B\mathbf{wSD}$ is a pointed homotopy equivalence (cf. p. 14–8). One then proceeds to the crux, viz. the verification that $\mathbf{w}F : \mathbf{wC} \to \mathbf{wD}$ is a strictly initial functor, and concludes by appealing to Quillen's theorem A.]

[†] *K-Theory* **1** (1989), 511–532; see also Grayson, *Illinois J. Math.* **31** (1987), 598–617.

[‡] *SLN* **1126** (1985), 354–358.

EXAMPLE Let **C** be the Waldhausen category whose objects are the pointed finite CW complexes and whose morphisms are the pointed skeletal maps. Let **D** be the category whose objects are the wellpointed spaces with closed base point which have the pointed homotopy type of a pointed finite CW complex and whose morphisms are the pointed continuous functions—then **D** satisfies the axioms for a Waldhausen category if weak equivalence = weak homotopy equivalence, cofibration = closed cofibration. However, while **C** is skeletally small, **D** is definitely not. Still, it will be convenient to ignore this detail since the situation can be rectified by the insertion of some additional language. We claim that the inclusion $\iota : \mathbf{C} \to \mathbf{D}$ has the approximation property. App_1 is, of course, trivial. To check the validity of App_2, fix a K in **C** and suppose given a pointed continuous function $f : K \to X$, where X is in **D**. By definition, $\exists$ an L in **C** and pointed continuous functions $\phi : X \to L$, $\psi : L \to X$ such that $\psi \circ \phi \simeq \text{id}_X$, $\phi \circ \psi \simeq \text{id}_L$. Using the skeletal approximation theorem, choose a pointed skeletal $g : K \to L$ for which $g \simeq \phi \circ f$. Display the data in a commutative diagram

$$\begin{tikzcd}
K \arrow[r, "i"] \arrow[dr, "g"'] & M_g \arrow[d, "r"] & L \arrow[l, "j"'] \arrow[dl, equal] \\
& L &
\end{tikzcd}$$

and consider the composite $M_g \xrightarrow{r} L \xrightarrow{\psi} X$. Since $\psi \circ r \circ i = \psi \circ g$, $\psi \circ r \circ j = \psi$, the restriction of $\psi \circ r$ to $K \vee L$ equals $\psi \circ g \vee \psi$ (identify K & $i(K)$, L & $j(L)$). But $g \simeq \phi \circ f \Rightarrow \psi \circ g \simeq \psi \circ \phi \circ f \simeq f \Rightarrow \psi \circ g \vee \psi \simeq f \vee \psi$. Because $K \vee L \to M_g$ is a closed cofibration, it follows that $f \vee \psi$ admits an extension to M_g, call it h:

$$\begin{tikzcd}
K \arrow[r, "i"] \arrow[dr, "f"'] & M_g \arrow[d, "h"] & L \arrow[l, "j"'] \arrow[dl, "\psi"] \\
& X &
\end{tikzcd}$$

. From the triangle on the right, one sees that h is a weak homotopy equivalence. On the other hand, $f = h \circ i$ and i is skeletal.

EXAMPLE Let **C** be the Waldhausen category whose objects are the pointed finite simplicial sets with weak equivalence = weak homotopy equivalence, cofibration = pointed injective simplicial map and let **D** be as in the preceding example. We claim that the geometric realization $|?| : \mathbf{C} \to \mathbf{D}$ has the approximation property. App_1 is true by definition. Turning to App_2, fix an X in **C** and suppose given a pointed continuous function $f : |X| \to Y$, where Y is in **D**. Let us assume for the moment that it is possible to fulfill App_2 up to homotopy, i.e., that $\exists$ a pointed finite simplicial set X', a simplicial map $g : X \to X'$, and a weak homotopy equivalence $h : |X'| \to Y$ such that $f \simeq h \circ |g|$—then App_2 holds on the nose. Indeed, $|M_g| \approx M_{|g|}$ and there is a commutative diagram

$$\begin{tikzcd}
{|X|} \arrow[r, "|i|"] \arrow[dr, "|g|"'] & M_{|g|} \arrow[d, "|r|"] & {|X'|} \arrow[l, "|j|"'] \arrow[dl, equal] \\
& {|X'|} &
\end{tikzcd}$$

. Obviously, $h \circ |r| \circ |i| = h \circ |g|$, $h \circ |r| \circ |j| = h$, and $h \circ |g| \vee h \simeq f \vee h$, hence $f \vee h$ can be extended to $M_{|g|}$, call it H:

$$\begin{tikzcd}
{|X|} \arrow[r, "|i|"] \arrow[dr, "f"'] & M_{|g|} \arrow[d, "H"] & {|X'|} \arrow[l, "|j|"'] \arrow[dl, "h"] \\
& Y &
\end{tikzcd}$$

. But H is a weak homotopy equivalence and $f = H \circ |i|$, as desired. Proceeding, there exists a pointed finite CW complex having the pointed homotopy type of Y and without loss of generality, one can assume that it is the geometric realization of a pointed finite simplicial set K (cf. §5, Proposition 3 and use the barycentric subdivision of the relevant vertex scheme), thus Y may be replaced by $|K|$. Because X is finite, the argument employed in the proof of the simplicial approximation theorem produces a simplicial map $g : X \to \mathrm{Ex}^n K (\exists\, n)$ for which $|g| \simeq |e^n_K| \circ f$. And: $|e^n_K| : |K| \to |\mathrm{Ex}^n K|$ is a pointed homotopy equivalence (cf. p. 13–12).

Remark: The above considerations therefore imply that the algebraic K-theory of a point can also be defined in terms of pointed finite simplicial sets.

Let A be a ring with unit—then it is clear that $K_0(\mathbf{P}(A)) = K_0(A)$.

CONSISTENCY PRINCIPLE There is a pointed homotopy equivalence $\Omega B\mathbf{w}\mathbf{S}\overline{\mathbf{P}(A)} \to K_0(A) \times B\mathbf{GL}(A)^+$, hence $\forall\, n \geq 1$, $K_n(\mathbf{P}(A)) \approx K_n(A)$.

[Note: Recall that $K_n(A) = \pi_n(B\mathbf{GL}(A)^+)$ (cf. p. 5–73 ff.).]

This is not obvious and the existing proofs are quite roundabout in that they do not directly involve $B\mathbf{w}\mathbf{S}\overline{\mathbf{P}(A)}$. Instead, one replaces it with $B\mathbf{Q}\overline{\mathbf{P}(A)}$, where $\mathbf{Q}\overline{\mathbf{P}(A)}$ is the "**Q** construction" on $\overline{\mathbf{P}(A)}$ (cf. infra), and then introduces yet another artifice, namely the "$\mathbf{S}^{-1}\mathbf{S}$ construction" which, in effect, is a bridge between these two very different ways of defining the higher algebraic K-groups of A. For the "classical" approach to these matters, consult the seventh chapter of Srinivas[†] (a sophisticated variant has been given by Jardine[‡]).

Example: Form the monoid $\coprod\limits_P B\mathrm{Aut}P$, where P runs through the objects in $\overline{\mathbf{P}(A)}$—then in the pointed homotopy category, $\Omega B \coprod\limits_P B\mathrm{Aut}P \approx K_0(A) \times B\mathbf{GL}(A)^+$ (cf. p. 14–22 ff.).

Let $\mathbf{C}$ be a small category WES—then $\mathbf{QC}$ is the category with the same objects as $\mathbf{C}$, a morphism from X to Y in $\mathbf{QC}$ being an equivalence class of diagrams of the form $X \twoheadleftarrow A \rightarrowtail Y$, where $X \twoheadleftarrow A' \rightarrowtail Y$ & $X \twoheadleftarrow A'' \rightarrowtail Y$ are equivalent if $\exists$ an isomorphism $A' \to A''$ rendering

$$\begin{tikzcd} X \arrow[d, equal] & A' \arrow[l, two heads] \arrow[r, tail] \arrow[d] & Y \arrow[d, equal] \\ X & A'' \arrow[l, two heads] \arrow[r, tail] & Y \end{tikzcd}$$

commutative. To compose $X \twoheadleftarrow A \rightarrowtail Y$ and $Y \twoheadleftarrow B \rightarrowtail Z$, form the pullback $A \times_Y B$ and project to X and

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[†] *Algebraic K-Theory*, Birkhäuser (1991); see also Gillet-Grayson, *Illinois J. Math.* **31** (1987), 574–597 (cf. 591–593).

[‡] *J. Pure Appl. Algebra* **75** (1991), 103–194; see also Thomason, *Comm. Algebra* **10** (1982), 1589–1668.

Z, i.e.,
$$\begin{tikzcd}
A \times_Y B \arrow[r, tail] \arrow[d, two heads] & B \arrow[r, tail] \arrow[d, two heads] & Z \\
A \arrow[r, tail] \arrow[d, two heads] & Y \\
X
\end{tikzcd}$$
.

Observation: If $\mathbf{C}, \mathbf{D}$ are small categories WES and if $F : \mathbf{C} \to \mathbf{D}$ is an exact functor, then there is an induced functor $\mathbf{Q}F : \mathbf{QC} \to \mathbf{QD}$.

PROPOSITION W Let $\mathbf{C}$ be a small category WES—then $B\mathbf{wSC}$ and $B\mathbf{QC}$ have the same pointed homotopy type.

The proof of Proposition W depends on an auxiliary device.

Let $\text{sd} : \mathbf{\Delta} \to \mathbf{\Delta}$ be the functor that sends $[n]$ to $[2n+1]$ and $\alpha : [m] \to [n]$ to the arrow $[2m+1] \to [2n+1]$ defined by the prescription $0 \to \alpha(0), \ldots, m \to \alpha(m)$, $m+1 \to 2n+1-\alpha(m), \ldots, 2m+1 \to 2n+1-\alpha(0)$.

Given a simplicial space X, put $\text{sd}X = X \circ \text{sd}^{\text{OP}}$, the edgewise subdivision of X. So, $(\text{sd}X)_n = X_{2n+1}$ and the $\begin{cases} d_i \\ s_i \end{cases}$ per $\text{sd}X$ are the $\begin{cases} d_i \circ d_{2n+1-i} & (0 \leq i \leq n, n > 0) \\ s_i \circ s_{2n+1-i} & (0 \leq i \leq n, n \geq 0) \end{cases}$ per X.

LEMMA Specify a continuous function $\theta_n : (\text{sd}X)_n \times \Delta^n \to X_{2n+1} \times \Delta^{2n+1}$ via the formula $\theta_n(x, t_0, \ldots, t_n) = (x, \frac{1}{2}t_0, \ldots, \frac{1}{2}t_n, \frac{1}{2}t_n, \ldots, \frac{1}{2}t_0)$—then the θ_n induce a homeomorphism $|\text{sd}X| \to |X|$.

Let $\mathbf{C}$ be a small category WES—then the weak equivalences are isomorphisms (cf. Proposition 4), hence $B\mathbf{wSC} = B\text{iso}\,\mathbf{SC}$ and there is a pointed homotopy equivalence $|W\mathbf{C}| \to B\text{iso}\,\mathbf{SC}$ (cf. p. 18–11). On the other hand, from the lemma, $|\text{sd}W\mathbf{C}| \approx |W\mathbf{C}|$, thus to prove Proposition W, it suffices to construct a pointed homotopy equivalence $|\text{sd}W\mathbf{C}| \to B\mathbf{QC}$. An element F of $(\text{sd}W\mathbf{C})_n$ is an element of $W_{2n+1}\mathbf{C} = \text{Ob}\,\mathbf{S}_{2n+1}\mathbf{C}$. Writing $F_{i,j}$ for $F(i \to j)$, send F to that element of $\text{ner}_n\mathbf{QC}$ represented by the diagram
$$\begin{tikzcd}[column sep=small]
& F_{n-1,n+1} \arrow[dl, two heads] \arrow[dr, tail] & & \cdots & & F_{0,2n} \arrow[dl, two heads] \arrow[dr, tail] & \\
F_{n,n+1} & & F_{n-1,n+2} & & F_{1,2n} & & F_{0,2n+1}
\end{tikzcd}$$
, i.e., to the string $F_{n,n+1} \to F_{n-1,n+2} \to \cdots \to F_{1,2n} \to F_{0,2n+1}$ in $\text{ner}_n\mathbf{QC}$. This assignment defines a simplicial map $\text{sd}W\mathbf{C} \to \text{ner}\,\mathbf{QC}$ and the claim is that its geometric realization is a pointed homotopy equivalence.

Introduce the double category $i\mathbf{QC} \equiv \text{iso}\,\mathbf{QC} \cdot \mathbf{QC}$ and recall that there is a pointed homotopy equivalence $B\mathbf{QC} \to Bi\mathbf{QC}$ (cf. p. 18–11). Call $i\mathbf{Q}_n\mathbf{C}$ the category whose objects are the functors $[n] \to \mathbf{QC}$ and whose morphisms are the natural isomorphisms ($\Rightarrow i\mathbf{Q}_n\mathbf{C} = \text{iso}[[n], \mathbf{QC}]$)—then $\forall\ n$, the functor $\text{iso}\,\text{sd}\mathbf{S}_n\mathbf{C} \to i\mathbf{Q}_n\mathbf{C}$ is an equivalence of categories. Contemplation of the diagram
$$\begin{tikzcd}
{|\text{sd}W\mathbf{C}|} \arrow[r] \arrow[d] & B\mathbf{QC} \arrow[d] \\
B\text{iso}\,\text{sd}\mathbf{SC} \arrow[r] & Bi\mathbf{QC}
\end{tikzcd}$$
finishes the argument.

Let A be a ring with unit—then by definition, $\mathbf{W}A$ is the Ω-prespectrum with q^{th} space $K_0(\Sigma^q A) \times B\mathbf{GL}(\Sigma^q A)^+$ (cf. p. 14–72) and $\mathbf{K}A = eM\mathbf{W}A$ (cf. p. 17–30), thus $\pi_n(\mathbf{K}A) = K_n(A)$ $(n \geq 0)$. And: $\pi_{-n}(\mathbf{K}A) = K_0(\Sigma^n A) = (L^n K_0)(A)$ $(n \geq 0)$, the negative algebraic K-groups of A in the sense of Bass (compare, e.g., Karoubi[†]).

[Note: The $\pi_{-n}(\mathbf{K}A)$ vanish if A is left noetherian and every finitely generated left A-module has finite projective dimension.]

The consistency principle can be generalized: $\exists$ a morphism of spectra $\mathbf{KP}(A) \rightarrow \mathbf{K}A$ such that the induced map $\pi_n(\mathbf{KP}(A)) \rightarrow \pi_n(\mathbf{K}A)$ is an isomorphism $\forall\, n \geq 0$.

To conclude this §, I shall say a few words about topological K-theory.

[Note: A reference is the book of Karoubi[‡].]

Let A be a Banach algebra with unit over $\mathbf{k}$, where $\mathbf{k} = \mathbf{R}$ or $\mathbf{C}$. Write $\mathbf{GL}(A)^{\text{top}}$ for $\mathbf{GL}(A)$ in its canonical topology—then $\mathbf{GL}(A)^{\text{top}}$ is a topological group and $\pi_0(\mathbf{GL}(A)^{\text{top}})$ is abelian. Definition: $\forall\, n > 0$, $K_n^{\text{top}}(A) = \pi_n(B\mathbf{GL}(A)^{\text{top}})$, the nth topological K-group of A (put $K_0^{\text{top}}(A) = K_0(A)$).

BOTT PERIODICITY THEOREM Let A be a Banach algebra with unit over $\mathbf{k}$.

$(\mathbf{k} = \mathbf{C})$ $\forall\, n \geq 0, K_n^{\text{top}}(A) \approx K_{n+2}^{\text{top}}(A)$.

$(\mathbf{k} = \mathbf{R})$ $\forall\, n \geq 0, K_n^{\text{top}}(A) \approx K_{n+8}^{\text{top}}(A)$.

For instance, one can take for A the Banach algebra with unit whose elements are the real or complex valued continuous functions on a compact Hausdorff space X.

The identity $\mathbf{GL}(A) \rightarrow \mathbf{GL}(A)^{\text{top}}$ induces a map $B\mathbf{GL}(A) \rightarrow B\mathbf{GL}(A)^{\text{top}}$, from which an arrow $B\mathbf{GL}(A)^+ \rightarrow B\mathbf{GL}(A)^{\text{top}}$. Passing to homotopy, this gives a homomorphism $K_n(A) \rightarrow K_n^{\text{top}}(A)$ that connects the algebraic K-groups of A to the topological K-groups of A.

[Note: The fundamental group of $B\mathbf{GL}(A)^{\text{top}}$ is abelian $(\pi_1(B\mathbf{GL}(A)^{\text{top}}) \approx \pi_0(\Omega B\mathbf{GL}(A)^{\text{top}}) \approx \pi_0(\mathbf{GL}(A)^{\text{top}}))$, thus $B\mathbf{GL}(A)^{\text{top}}$ is insensitive to the plus construction.]

[†] *Ann. Sci. École Norm. Sup.* **4** (1971), 63–95.

[‡] *K-Theory: An Introduction*, Springer Verlag (1978); see also N. Wegge-Olsen, *K-Theory and C*-Algebras*, Oxford University Press (1993).

THEOREM OF FISCHER[†]-PRASOLOV[‡] Let A be a commutative Banach algebra over $\mathbf{k}$ with unit—then $\forall\, n \geq 1$, the arrow $\pi_n(B\mathbf{GL}(A)^+; \mathbf{Z}/k\mathbf{Z}) \rightarrow \pi_n(B\mathbf{GL}(A)^{\text{top}}; \mathbf{Z}/k\mathbf{Z})$ is an isomorphism.

[Note: The notation is that of p. 9–2 ($B\mathbf{GL}(A)^+$ and $B\mathbf{GL}(A)^{\text{top}}$ are H spaces).]

Therefore, in the commutative case, the algebraic and topological K-groups of A are indistinguishable if one sticks to finite coefficients.

[†] *J. Pure Appl. Algebra* **69** (1990), 33–50.

[‡] *Amer. Math. Soc. Transl.* **154** (1992), 133–137.

§ 18

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§19. DIMENSION THEORY

Dimension theory enables one to associate with each nonempty normal Hausdorff space X a topological invariant $\dim X \in \{0, 1, \ldots\} \cup \{\infty\}$ called its topological dimension. Classically, there are two central theorems, namely:

(1) The topological dimension of $\mathbf{R}^n$ is exactly n, hence as a corollary, $\mathbf{R}^n$ and $\mathbf{R}^m$ are homeomorphic iff $n = m$.

(2) Every second countable normal Hausdorff space of topological dimension n can be embedded in $\mathbf{R}^{2n+1}$.

Although I shall limit the general discussion to what is needed to prove these results, some important applications will be given, e.g., to the "invariance of domain" and the "superposition question". On the theoretical side, Čech cohomology makes an initial appearance but it does not really come to the fore until §20.

Let X be a nonempty normal Hausdorff space. Consider the following statement.

($\dim X \leq n$) There exists an integer $n = 0, 1, \ldots$ such that every finite open covering of X has a finite open refinement of order $\leq n + 1$.

If $\dim X \leq n$ is true for some n, then the topological dimension of X, denoted by $\dim X$, is the smallest value of n for which $\dim X \leq n$.

[Note: By convention, $\dim X = -1$ when $X = \emptyset$. If the statement $\dim X \leq n$ is false for every n, then we put $\dim X = \infty$.]

Our primary emphasis will be on spaces of finite topological dimension. A simple example of a compact metrizable space of infinite topological dimension is the Hilbert cube $[0, 1]^\omega$.

Why work with finite open coverings? Answer: The concept of dimension would be very different otherwise. Example: Take $X = [0, \Omega[$—then $\dim [0, \Omega[= 0$ (cf. p. 19–4). But the open covering $\{[0, \alpha[: 0 < \alpha < \Omega\}$ has no point finite open refinement, so $[0, \Omega[$ would be "infinite dimensional" if arbitrary open coverings were allowed.

Why work with normal X? A priori, this is not necessary since the definition evidently makes sense for any CRH space X. But observe: If $\dim X = 0$, then X must be normal. So, no new spaces of "dimension zero" are produced by just formally extending the definition to nonnormal X. Such an agreement would also introduce a degree of pathology. Example: The topological dimension of $X = [0, \Omega] \times [0, \omega]$ is zero (cf. p. 19–4) but the "topological dimension" of $X - \{(\Omega, \omega)\}$, the Tychonoff plank (which is not normal), is one. The escape from this predicament is to reformulate the definition of dim in such a way that it is naturally applicable to the class of all nonempty CRH spaces. The topological dimension of the Tychonoff plank then turns out to be zero, as might be expected (cf. p. 19–4).

Let X be a nonempty CRH space. Consider the following statement.

($\dim X \leq n$) There exists an integer $n = 0, 1, \ldots$ such that every finite numerable open covering of X has a finite numerable open refinement of order $\leq n+1$.

If $\dim X \leq n$ is true for some n, then the topological dimension of X, denoted by $\dim X$, is the smallest value of n for which $\dim X \leq n$.

[Note: By convention, $\dim X = -1$ iff $X = \emptyset$. If the statement $\dim X \leq n$ is false for every n, then we put $\dim X = \infty$.]

Since a nonempty CRH space X is normal iff every finite open covering of X is numerable, this agreement is a consistent extension of dim. On the other hand, the price to pay for increasing the generality is that more things can go wrong (e.g., every subspace of X now has a topological dimension). Because of this, my policy will be to concentrate on the normal case and simply indicate as we go along what changes, if any, must be made to accommodate the completely regular situation. The omitted details are invariably straightforward.

[Note: By repeating what has been said above verbatim, an arbitrary nonempty topological space X acquires a "topological dimension" $\dim X$. One can then show that $\dim X = \dim \mathrm{cr} X$, where $\mathrm{cr}X$ is the complete regularization of X (cf. p. 1–26). Example: $\dim[0,1]/[0,1[= 0$.]

PROPOSITION 1 The topological dimension of X is equal to the topological dimension of βX.

[$\dim \beta X \leq n \Rightarrow \dim X \leq n$: Let $\mathcal{U} = \{U\}$ be a finite open covering of X. Since $\mathcal{U}$ is numerable, one can assume that the U are cozero sets. The collection $\{\beta X - \mathrm{cl}_{\beta X}(X - U)\}$ is then a finite open covering of βX, thus admits a precise open refinement of order $\leq n+1$ which, when restricted to X, is a precise open refinement of $\mathcal{U}$ of order $\leq n+1$.

$\dim X \leq n \Rightarrow \dim \beta X \leq n$: Let $\mathcal{U} = \{U\}$ be a finite open covering of βX. Choose a partition of unity $\{\kappa_U\}$ on βX subordinate to $\mathcal{U}$. The collection $\{X \cap \kappa_U^{-1}(]0,1])\}$ is a finite open covering of X, hence has a precise open refinement $\mathcal{V} = \{V\}$ of order $\leq n+1$. Let $\{\kappa_V\}$ be a partition of unity on X subordinate to $\mathcal{V}$—then the collection $\{\beta X - \mathrm{cl}_{\beta X}(X - \kappa_V^{-1}(]0,1]))\}$ is a precise open refinement of $\mathcal{U}$ of order $\leq n+1$.]

The argument used in Proposition 1 carries over directly to the completely regular situation, so the result holds in that setting too.

A nonempty Hausdorff space is said to be zero dimensional if it has a basis consisting of clopen sets. Every zero dimensional space is necessarily completely regular. The class of zero dimensional spaces is closed under the formation of nonempty products and coproducts.

[Note: Recall that for nonempty LCH spaces, the notions of zero dimensional and totally disconnected are equivalent.]

A nonempty subspace of the real line is zero dimensional iff it contains no open interval.

The Isbell-Mrówka space, the van Douwen line, the van Douwen space, and the Kunen line are all zero dimensional. But of these, only the Kunen line is normal.

FACT Let X be a zero dimensional normal LCH space. Suppose that X is metacompact—then X is subparacompact.

Any metric space (X, d) for which $d(x,z) \leq \max(d(x,y), d(y,z))$ is zero dimensional. Such a metric is said to be nonarchimedean. They are common fare in algebraic number theory and p-adic analysis. Example: Suppose that X is zero dimensional and second countable—then X admits a compatible nonarchimedean metric. Indeed, let $\mathcal{U} = \{U_n\}$ be a clopen basis for X and put $d(x,y) = \max\limits_n \left\{\frac{|\chi_n(x) - \chi_n(y)|}{n}\right\}$, χ_n the characteristic function of U_n.

[Note: Suppose that X is metrizable—then de Groot[†] has shown that $\dim X = 0$ iff X admits a compatible nonarchimedean metric.]

EXAMPLE Let κ be an infinite cardinal—then the Cantor cube C_κ is the space $\{0,1\}^\kappa$, where $\{0,1\}$ has the discrete topology. It is a compact Hausdorff space of weight κ and is zero dimensional. Of course, the Cantor cube associated with $\kappa = \omega$ is homeomorphic to the usual Cantor set. Every zero dimensional space X of weight κ can be embedded in C_κ, hence has a zero dimensional compactification ζX of weight κ.

[Let $\mathcal{U} = \{U_i : i \in I\}$ be a clopen basis for X such that $\#(I) = \kappa$. Agreeing to denote by χ_i the characteristic function of U_i, call χ the diagonal of the χ_i—then $\chi : X \rightarrow C_\kappa$ is an embedding. The closure ζX of the image of X in C_κ is a zero dimensional compactification of X of weight κ. Viewing X as a subspace of ζX, to within topological equivalence ζX is the only zero dimensional compactification of X with the property: For every zero dimensional compact Hausdorff space Y and for every continuous function $f : X \rightarrow Y$ there exists a continuous function $\zeta f : \zeta X \rightarrow Y$ such that $\zeta f | X = f$.]

[Note: Consider the Cantor cube C_ω. Since $C_\omega \hookrightarrow \mathbf{R}$, it follows that if X is zero dimensional and second countable, then there is an embedding $X \rightarrow \mathbf{R}$.]

Suppose that $\dim X = 0$—then it is clear that X is zero dimensional. To what extent is the converse true?

[†] *Proc. Amer. Math. Soc.* 7 (1956), 948–953; see also Nagata, *Fund. Math.* 55 (1964), 181–194.

LEMMA If for every pair (A,B) of disjoint closed subsets of X there exists a clopen set $U \subset X$ such that $A \subset U \subset X - B$, then $\dim X = 0$.

[Let $\mathcal{U} = \{U_i : i \in I\}$ be a finite open covering of X of cardinality $\#(I) = k$. To establish the existence of a finite refinement of $\mathcal{U}$ by pairwise disjoint clopen sets, we shall argue by induction on k. For $k = 1$, the assertion is trivial. Assume that $k > 1$ and that the assertion is true for all open coverings of cardinality $k - 1$. Enumerate the elements of $\mathcal{U}$: $U_1, \ldots, U_k$ and pass to $\{U_1, \ldots, U_{k-1} \cup U_k\}$, which thus has a precise refinement $\{V_1, \ldots, V_{k-1}\}$ by pairwise disjoint clopen sets. Noting that $\left\{\begin{array}{l} V_{k-1} - U_{k-1} \\ V_{k-1} - U_k \end{array}\right.$ are disjoint closed subsets of X, choose a clopen set $U \subset X$: $\left\{\begin{array}{l} V_{k-1} - U_{k-1} \subset U \\ U \subset (X - V_{k-1}) \cup U_k \end{array}\right.$. Consideration of the covering $\{V_1, \ldots, V_{k-1} - U, V_{k-1} \cap U\}$ then finishes the induction.]

PROPOSITION 2 Suppose that X is zero dimensional and Lindelöf—then $\dim X = 0$.

[Let (A,B) be a pair of disjoint closed subsets of X. Given $x \in X$, choose a clopen neighborhood $U_x \subset X$ of x such that either $A \cap U_x = \emptyset$ or $B \cap U_x = \emptyset$. Let $\{U_{x_i}\}$ be a countable subcover of $\{U_x\}$—then the $U_i = U_{x_i} - \bigcup\limits_{j<i} U_{x_j}$ are pairwise disjoint clopen subsets of X and $\bigcup\limits_i U_i = X$. Put $U = \operatorname{st}(A, \{U_i\})$: U is clopen and $A \subset U \subset X - B$. The lemma therefore implies that $\dim X = 0$.]

Take $X = [0, \Omega]$—then X is zero dimensional and compact, thus in view of Proposition 2, $\dim[0,\Omega] = 0$. Take next $X = [0,\Omega[$—then $\beta X = [0,\Omega]$, so $\dim[0,\Omega[= 0$ too (cf. Proposition 1).

LEMMA Let X be a nonempty CRH space—then $\dim X = 0$ iff for every pair of disjoint zero sets in X there exists a clopen set in X containing the one and not the other.

Consequently, Proposition 2 is valid as it stands in the completely regular situation. Example: Consider $[0,\Omega] \times [0,\omega]$ and conclude that the topological dimension of the Tychonoff plank is zero.

LEMMA Let X be a nonempty CRH space—then $\dim X = 0$ iff every zero set in X is a countable intersection of clopen sets.

EXAMPLE Let κ be a cardinal—then $\mathbf{N}^\kappa$ is paracompact if κ is countable but is neither normal nor submetacompact if κ is uncountable. Claim: $\forall\ \kappa$, $\dim \mathbf{N}^\kappa = 0$. For this, it can be assumed that κ is uncountable. Let $Z(f)$ be a zero set in $\mathbf{N}^\kappa$—then there exists a countable subproduct through which f factors, i.e., there exists a continuous $g : \mathbf{N}^\omega \rightarrow \mathbf{R}$ such that $f = g \circ p$, $p : \mathbf{N}^\kappa \rightarrow \mathbf{N}^\omega$ the projection. Obviously, $Z(g) = p(Z(f))$. Choose a sequence $\{V_n\}$ of clopen sets in $\mathbf{N}^\omega$: $Z(g) = \bigcap_n V_n$. Put $U_n = p^{-1}(V_n)$—then U_n is clopen in $\mathbf{N}^\kappa$ and $Z(f) = \bigcap_n U_n$.

[Note: Suppose that κ is uncountable—then every open subspace of $\mathbf{N}^\kappa$ has topological dimension zero but this need not be the case of closed subspaces (cf. p. 19–10).]

FACT Let X be a nonempty CRH space—then $\dim X = 0$ iff the real valued continuous functions on X with finite range are uniformly dense in $BC(X)$.

[There is no loss of generality in assuming that X is compact. If X is totally disconnected, use Stone-Weierstrass; if X is not totally disconnected, consider the functions constant on some connected subset of X that has more than one point.]

It is false that unconditionally: X zero dimensional $\Rightarrow \dim X = 0$, even if X is a metric space (Roy[†]).

[Note: The topological dimension of Roy's metric space is equal to 1. Does there exist for each $n > 1$ a zero dimensional metric space X such that $\dim X = n$? The answer is unknown.]

EXAMPLE (Dowker's Example "M") In $[0,1]$, write $x \sim y$ iff $x - y \in \mathbf{Q}$, so $[0,1]/\!\sim\, = \coprod_\alpha Q_\alpha$. There are 2^ω equivalence classes Q_α. Each is a countable dense subset of $[0,1]$. Take a subcollection $\{Q_\alpha : \alpha < \Omega\}$, where $\forall\, \alpha < \Omega : Q_\alpha \neq \mathbf{Q} \cap [0,1]$. Put $S_\alpha = [0,1] - \bigcup\{Q_\beta : \alpha \leq \beta < \Omega\}$ and consider the subspace $X = \{(\alpha, s) : \alpha < \Omega, s \in S_\alpha\}$ of $[0,\Omega[\times[0,1]$—then X is zero dimensional and the claim is that X is normal, yet $\dim X > 0$. To see this, form $X^* = X \cup (\{\Omega\} \times [0,1])$, a subspace of $[0,\Omega] \times [0,1]$ which is normal. In addition, if A and B are disjoint closed subsets of X, then their closures A^* and B^* in X^* are also disjoint. It follows that X is normal. If $\dim X = 0$, then there exists a clopen set $U \subset X$ such that $[0,\Omega[\times\{0\} \subset U$ and $[0,\Omega[\times\{1\} \subset X - U$. But $U^* \cap (X-U)^* = \emptyset$ & $\begin{cases} (\Omega,0) \in U^* \\ (\Omega,1) \in (X-U)^* \end{cases}$, and this contradicts the connectedness of $\{\Omega\} \times [0,1]$. Therefore $\dim X > 0$. One can be precise: $\dim X = 1$. For if $\{U\}$ is a finite open covering of X, then $\forall\, t \in [0,1]$, there exists a neighborhood O of t and an α such that $X \cap (]\alpha,\Omega[\times O)$ is contained in some U, which implies that there exists a finite open covering $\{O\}$ of $[0,1]$ of order ≤ 2 and an α such that each $X \cap (]\alpha,\Omega[\times O)$ is contained in some U. Therefore $\dim X \leq 1$.

[Note: X has a zero dimensional compactification ζX and the latter has topological dimension zero (cf. Proposition 2). So: A compact Hausdorff space of zero topological dimension can have a normal subspace of positive topological dimension. Another aspect is that while X is zero dimensional, βX is not. In fact, $\dim X = \dim \beta X$ (cf. Proposition 1), which is > 0, thus Proposition 2 is applicable. Here is a final remark: By appropriately adjoining to X a single point, one can destroy its zero dimensionality or reduce its topological dimension to zero without, in either case, losing normality.]

[†] *Trans. Amer. Math. Soc.* **134** (1968), 117–132; see also Kulesza, *Topology Appl.* **35** (1990), 109–120.

Modify the preceding construction, replacing $\begin{cases} [0,1] & \text{by } [0,1]^\omega \\ S_\alpha & \text{by } S_\alpha^\omega \end{cases}$ and conclude that there exists a compact Hausdorff space of zero topological dimension with a normal subspace of infinite topological dimension.

FACT Suppose that $\dim X = 0$ and X is paracompact. Let A be a closed subset of X; let Y be a complete metric space—then every (bounded) continuous function $f : A \to Y$ has a (bounded) continuous extension $F : X \to Y$.

[For $n = 1, 2, \ldots$, let $\mathcal{V}_n$ be the covering of Y by open $1/n$ balls. Let $\mathcal{A}_n = \{A_{i,n} : i \in I_n\}$ be an open partition of A that refines $f^{-1}(\mathcal{V}_n)$. Inductively determine an open partition $\mathcal{U}_n = \{U_{i,n} : i \in I_n\}$ of X that refines $\mathcal{U}_{n-1}$ and $\forall\, i \in I_n : A \cap U_{i,n} = A_{i,n}$. Assign to a given $x \in X$ an index $i(x,n) \in I_n : x \in U_{i(x,n),n}$. Choose points $y_{i,n} \in f(A_{i,n})$. Observe that $\{y_{i(x,n),n}\}$ is Cauchy. Put $F(x) = \lim y_{i(x,n),n}$.]

Provided that Y is a separable complete metric space, the preceding result retains its validity if only $\dim X = 0$ and X is normal.

PROPOSITION 3 Suppose that X is a nonempty paracompact LCH space—then X is zero dimensional iff $\dim X = 0$.

[Since X is paracompact, X admits a representation $X = \coprod_i X_i$, where the X_i are nonempty pairwise disjoint open σ-compact (=Lindelöf) subspaces of X (cf. p. 1–2). But obviously, X is zero dimensional iff each of the X_i is zero dimensional. Now use Proposition 2.]

Proposition 3 can fail for an arbitrary normal LCH space. Consider the space X of Dowker's Example "M". It is not locally compact. To get around this, let $p : X \to [0,\Omega[$ be the projection, form $\beta p : \beta X \to \beta[0,\Omega[= [0,\Omega]$ and put $A = (\beta p)^{-1}([0,\Omega[)$. One can check that A is normal and zero dimensional. And: $X \subset A \subset \beta X \Rightarrow \beta A = \beta X \Rightarrow \dim A = \dim X > 0$ (cf. Proposition 1). But A, being open in βA, is a LCH space.

[Note: A zero dimensional $\Rightarrow A_\infty$ zero dimensional $\Rightarrow \dim(A_\infty) = 0$ (cf. Proposition 2). So: A compact Hausdorff space of zero topological dimension can have an open subspace of positive topological dimension.]

Let X be a CRH space. Suppose that $\mathcal{A}$ is a collection of subsets of X closed under the formation of finite unions and finite intersections. A subcollection $\mathcal{F} \subset \mathcal{A}$ is said to be an $\mathcal{A}$-filter if (i) $\emptyset \notin \mathcal{F}$, (ii) $A \in \mathcal{F}$ & $A \subset B \in \mathcal{A} \Rightarrow B \in \mathcal{F}$, and (iii) $\forall\, A, B \in \mathcal{F} : A \cap B \in \mathcal{F}$. Example: $\mathcal{A}$ = all zero sets in X or

$\mathcal{A}$ = all clopen sets in X, the associated $\mathcal{A}$-filters then being the zero set filters and the clopen set filters, respectively.

(Fil$_1$) An $\mathcal{A}$-filter $\mathcal{F}$ is said to be an $\mathcal{A}$-ultrafilter if $\mathcal{F}$ is a maximal $\mathcal{A}$-filter. The maximality of $\mathcal{F}$ is equivalent to the condition: If $B \in \mathcal{A}$ and if $A \cap B \neq \emptyset \ \forall \ A \in \mathcal{F}$, then $B \in \mathcal{F}$. An $\mathcal{A}$-ultrafilter $\mathcal{F}$ is prime, i.e., if A and B belong to $\mathcal{A}$ and if $A \cup B \in \mathcal{F}$, then $A \in \mathcal{F}$ or $B \in \mathcal{F}$. Every $\mathcal{A}$-filter is contained in an $\mathcal{A}$-ultrafilter.

(Fil$_2$) An $\mathcal{A}$-filter $\mathcal{F}$ is said to be fixed if $\cap \mathcal{F}$ is nonempty.

(Fil$_3$) An $\mathcal{A}$-filter $\mathcal{F}$ is said to have the countable intersection property if for every sequence $\{A_n\} \subset \mathcal{F}$, $\bigcap\limits_n A_n \neq \emptyset$.

[Note: The zero sets in X are closed under the formation of countable intersections. Therefore every zero set ultrafilter on X with the countable intersection property is closed under the formation of countable intersections.]

The following standard characterizations illustrate the terminology.

(**R**) Let X be a CRH space—then X is **R**-compact iff every zero set ultrafilter on X with the countable intersection property is fixed.

(N) Let X be a CRH space. Suppose that X is zero dimensional—then X is N-compact iff every clopen set ultrafilter on X with the countable intersection property is fixed.

LEMMA Let X be a nonempty CRH space. Suppose that $\dim X = 0$ and X is **R**-compact—then X is N-compact.

[Let $\mathcal{U}$ be a clopen set ultrafilter on X with the countable intersection property—then the claim is that $\mathcal{U}$ is fixed. Choose a zero set ultrafilter $\mathcal{Z}$ on $X : \mathcal{Z} \supset \mathcal{U}$. Take any sequence $\{Z_n\} \subset \mathcal{Z}$ and write $Z_n = \bigcap\limits_m U_{mn}$, U_{mn} clopen. Each U_{mn} meets every element of $\mathcal{U}$, thus each U_{mn} is in $\mathcal{U}$. But $\mathcal{U}$ has the countable intersection property, so $\bigcap\limits_n Z_n = \bigcap\limits_n \bigcap\limits_m U_{mn} \neq \emptyset$. Therefore $\mathcal{Z}$ has the countable intersection property, hence is fixed, and this implies that $\mathcal{U}$ is fixed as well.]

The converse to this lemma is false: There exist N-compact spaces of positive topological dimension.

EXAMPLE (Mysior Space) Let X be the subspace of ℓ^2 consisting of all sequences $\{x_n\}$, with x_n rational—then X is the textbook example of a totally disconnected space that is not zero dimensional (Erdös). Fix a countable dense subset D of X. For each $S \subset D$ with $\#(\overline{S} \cap \overline{D - S}) = 2^\omega$, choose a point $x_S \in \overline{S} \cap \overline{D - S}$ subject to: $S' \neq S'' \Rightarrow x_{S'} \neq x_{S''}$. In addition, given $x \in X - D$, let $\{s_k(x)\}$ be a sequence in D having limit x such that if $x = x_S$ for some $S \subset D$, then both S and $D - S$ contain infinitely many terms of $\{s_k(x)\}$. Topologize X as follows: Isolate the points of D and take for the basic neighborhoods of $x \in X - D$ the sets $K_k(x) = \{x\} \cup \{s_l(x) : l \geq k\}$ $(k = 1, 2, \ldots)$. The resulting topology τ on X is finer than the metric topology. And the space X_τ thereby produced is a nonnormal zero dimensional LCH space possessing a basis comprised of countable clopen compact sets. To see that X_τ is N-compact, let $\mathcal{U}$ be a clopen set ultrafilter on X_τ with the countable intersection property. The collection $\{U \in \mathcal{U} : U$ clopen in $X\}$ is a clopen set ultrafilter on X with the countable intersection property, hence there exists a point x_0 in its intersection (X is Lindelöf). This x_0 is then the intersection of countably many elements of $\mathcal{U}$, thus $\mathcal{U}$ is fixed and so X_τ is N-compact. Still, $\dim X_\tau > 0$. Observe first that since D is dense in X_τ, the frontier in X of any clopen subset of X_τ has cardinality $< 2^\omega$. Consider the disjoint zero sets $\begin{cases} Z_1 = \{x : \|x\| \leq 1\} \\ Z_2 = \{x : \|x\| \geq 2\} \end{cases}$. Let U be a clopen subset of $X_\tau : Z_1 \subset U \subset X - Z_2$—then its frontier in X necessarily has cardinality 2^ω.

FACT Let X be a nonempty CRH space—then X is N-compact iff X is zero dimensional and there exists a closed embedding $X \to \prod(\mathbf{N} \times [0,1])$.

There exist zero dimensional **R**-compact normal LCH spaces that are not N-compact. Owing to the lemma, such a space must have positive topological dimension (cf. Proposition 3).

EXAMPLE [Assume CH] (The Kunen Plane) The construction of the Kunen line starting from $X = \mathbf{R}$ can be carried out with no change whatsoever starting instead with $X = \mathbf{R}^2$, the upshot being the Kunen plane X_Ω, a space with the same general topological properties as the Kunen line. So: X_Ω is a zero dimensional perfectly normal LCH space that is not paracompact but is first countable, hereditarily separable, and collectionwise normal. The topology τ_Ω on X_Ω is finer than the usual topology on $\mathbf{R}^2$. And, $\forall\, S \subset \mathbf{R}^2 : \#(\mathrm{cl}_{\mathbf{R}^2}(S) - \mathrm{cl}_\Omega(S)) \leq \omega$. It follows from this that if A and B are disjoint closed subsets of X_Ω, then $\#(\overline{A} \cap \overline{B}) \leq \omega$, the bar denoting closure in $\mathbf{R}^2$.

Claim: X_Ω is **R**-compact.

[Let $\mathcal{Z}_\Omega$ be a zero set ultrafilter on X_Ω with the countable intersection property. Let $\mathcal{Z} \subset \mathcal{Z}_\Omega$ be the subcollection consisting of the $\mathbf{R}^2$-closed elements of $\mathcal{Z}_\Omega$. Fix a point $x_0 \in \cap\mathcal{Z}$ and choose a continuous function $\phi : \mathbf{R}^2 \to [0,1]$ such that $\phi^{-1}(0) = \{x_0\}$. The sets $\begin{cases} \phi^{-1}([0,1/n]) \\ \phi^{-1}([1/n,1]) \end{cases}$ are zero sets in $\mathbf{R}^2$, hence are zero sets in X_Ω. Of course, $X_\Omega = \phi^{-1}([0,1/n]) \cup \phi^{-1}([1/n,1])$. But obviously, $\phi^{-1}([1/n,1]) \notin \mathcal{Z}$, thus $\phi^{-1}([1/n,1]) \notin \mathcal{Z}_\Omega$ and so $\phi^{-1}([0,1/n]) \in \mathcal{Z}_\Omega$, $\mathcal{Z}_\Omega$ being prime. Consequently, $\{x_0\} = \bigcap\limits_n \phi^{-1}([0,1/n]) \in \mathcal{Z}_\Omega$, which means that $\mathcal{Z}_\Omega$ is fixed.]

Claim: X_Ω is not N-compact.

[Let $U \subset X_\Omega$ be clopen—then $\#(\overline{U} \cap \overline{X_\Omega - U}) \leq \omega$. Therefore the plane $\mathbf{R}^2$ is not disconnected by $\overline{U} \cap \overline{X_\Omega - U}$, so either $\#(U) \leq \omega$ or $\#(X_\Omega - U) \leq \omega$. Consider the collection $\mathcal{U}$ of all clopen $U \subset X_\Omega$ for which $\#(X_\Omega - U) \leq \omega$—then $\mathcal{U}$ is a clopen set ultrafilter on X_Ω with the countable intersection property such that $\cap\mathcal{U} = \emptyset$ (every $x \in X_\Omega$ has a countable clopen neighborhood).]

[Note: The Kunen line X_Ω is **R**-compact (same argument as above) but, in contrast to the Kunen plane, it is also N-compact. For this, it need only be shown that $\dim X_\Omega = 0$.

Claim: Let $A \subset X_\Omega$ be countable and closed—then there exists a countable open $U \subset X_\Omega : A \subset U$ & $U = \overline{U}$, the bar denoting closure in $\mathbf{R}$.

[One can assume that A is closed in $\mathbf{R}$. Write $A = \bigcap\limits_n O_n = \bigcap\limits_n \overline{O}_n$, where the O_n are $\mathbf{R}$-open and $\forall\, n : O_n \supset O_{n+1}$. Enumerate $A : \{a_n\}$, and for each n choose a compact countable open $U_n \subset X_\Omega : a_n \in U_n$ and $U_n \subset O_n$. Consider $U = \bigcup\limits_n U_n$.]

To prove that $\dim X_\Omega = 0$, it suffices to take an arbitrary pair (A, B) of disjoint closed subsets of X_Ω and construct a pair (U_A, U_B) of disjoint clopen subsets of $X_\Omega : \begin{cases} A \subset U_A \\ B \subset U_B \end{cases}$. Since $\#(\overline{A} \cap \overline{B}) \leq \omega$, by the claim there exists a countable open $O \subset X_\Omega$: $\overline{A} \cap \overline{B} \subset O$ & $O = \overline{O}$. Pick disjoint $\mathbf{R}$-open sets O_A and O_B: $\begin{cases} A - O \subset \overline{A} - O \subset O_A \subset \mathbf{R} - O \\ B - O \subset \overline{B} - O \subset O_B \subset \mathbf{R} - O \end{cases}$, with $\#((\mathbf{R} - O) - (O_A \cup O_B)) \leq \omega$ (possible because it is a question of $\mathbf{R}$ as opposed to $\mathbf{R}^2$). Pass to $\mathbf{R} - (O_A \cup O_B)$ and use the claim once again to choose a countable open $P \subset X_\Omega : \mathbf{R} - (O_A \cup O_B) \subset P \subset \mathbf{R} - ((\overline{A} - O) \cup (\overline{B} - O))$ & $P = \overline{P}$—then $\begin{cases} (O_A \cup P) \cap (\mathbf{R} - O) \\ (O_B - P) \cap (\mathbf{R} - O) \end{cases}$ are disjoint clopen subsets of X_Ω containing $\begin{cases} A - O \\ B - O \end{cases}$, respectively. On the other hand, O is a normal subspace of X_Ω of zero topological dimension (cf. Proposition 2), so we can find disjoint clopen sets P_A and P_B in X_Ω: $\begin{cases} A \cap O \subset P_A \subset O \\ B \cap O \subset P_B \subset O \end{cases}$. Now put $\begin{cases} U_A = ((O_A \cup P) \cap (\mathbf{R} - O)) \cup P_A \\ U_B = ((O_B - P) \cap (\mathbf{R} - O)) \cup P_B \end{cases}$.]

EXAMPLE (The van Douwen Plane) The object is to equip $X = \mathbf{R}^2$ with a first countable, separable topology that is finer than the usual topology (hence Hausdorff) and under which $X = \mathbf{R}^2$ is locally compact and normal and zero dimensional and $\mathbf{R}$-compact but not $\mathbf{N}$-compact. Let $\{U_n\}$ be a countable basis for $\mathbf{R}^2$ with $U_0 = \mathbf{R}^2$. Assign to each $x \in \mathbf{R}^2$ the sets $O_k(x) = \bigcap\limits_n \{U_n : n \leq k \,\&\, x \in U_n\}$—then the collection $\{O_k(x)\}$ is a neighborhood basis at x in $\mathbf{R}^2$. Obviously, $x \in O_l(y) \Rightarrow O_k(x) \subset O_l(y)$ $(\forall\, k \geq l)$. Let $\{x_\alpha : \alpha < 2^\omega\}$ be an enumeration of $\mathbf{R}^2$ and put $X_\alpha = \{x_\beta : \beta < \alpha\}$—then $X_c = \mathbf{R}^2$ $(c = 2^\omega)$. We shall assume that $X_\omega = \mathbf{Q}^2$. Fix an enumeration $\{(A_\alpha, B_\alpha) : \alpha < 2^\omega\}$ of the set of all pairs (A, B), where A and B are countable subsets of $\mathbf{R}^2$ with $\#(\overline{A} \cap \overline{B}) = 2^\omega$, arranging matters in such a way that each pair is listed 2^ω times. Here (and below) the bar stands for closure in $\mathbf{R}^2$, while cl_c will denote the closure operator relative to the upcoming topology τ_c on X_c. Define an injection $\Gamma : 2^\omega \rightarrow 2^\omega - \omega$ by the prescription

$$\Gamma(\gamma) = \min(\{\alpha \in 2^\omega - \omega : A_\gamma \cup B_\gamma \subset X_\alpha, x_\alpha \in \overline{A}_\gamma \cap \overline{B}_\gamma\} - \{\Gamma(\beta) : \beta < \gamma\}).$$

Given $\alpha \in 2^\omega - \omega$, choose a sequence $\{s_k(\alpha)\} \subset X_\alpha : \forall\, k, s_k(\alpha) \in O_k(x_\alpha)$, having the property that if $\alpha = \Gamma(\gamma)$, then $\{s_k(\alpha)\} \subset \mathbf{Q}^2 \cup A_\gamma \cup B_\gamma$ and each of $\mathbf{Q}^2$, A_γ, and B_γ contains infinitely many terms of $\{s_k(\alpha)\}$, otherwise $\{s_k(\alpha)\} \subset \mathbf{Q}^2$. Topologize $X = \mathbf{R}^2$ as follows: Inductively take for the basic neighborhoods of x_α the sets $K_k(x_\alpha)$, $K_k(x_\alpha)$ being $\{x_\alpha\}$ if $\alpha \in \omega$ and $\{x_\alpha\} \cup \bigcup\limits_{l \geq k} K_l(x_{s_l(\alpha)})$ if $\alpha \in 2^\omega - \omega$ $(k = 1, 2, \ldots)$. Needless to say, $\forall\, \alpha : K_k(x_\alpha) \subset O_k(x_\alpha)$, and $\forall\, \alpha, \beta$: $x_\alpha \in K_l(x_\beta) \Rightarrow K_k(x_\alpha) \subset K_l(x_\beta)$ $(\exists\, k)$. Observe too that the $K_k(x_\alpha)$ are compact and countable. Therefore X_c is a zero dimensional LCH space that is in addition first countable and separable.

Claim: Let $S, T \subset X_c$. Suppose that $\overline{S} \cap \overline{T}$ is uncountable—then $\text{cl}_c(S) \cap \text{cl}_c(T)$ is uncountable.

[There are countable $A, B \subset \mathbf{R}^2 : \begin{cases} A \subset S \subset \overline{A} \\ B \subset T \subset \overline{B} \end{cases}$. From the definitions, $(A, B) = (A_\alpha, B_\alpha)$ for 2^ω ordinals α and, by construction, $x_{\Gamma(\alpha)} \in \text{cl}_c(A_\alpha) \cap \text{cl}_c(B_\alpha)$. But Γ is one-to-one.]

To establish that X_c is normal, it suffices to show that if A and B are two disjoint closed subsets of X_c, then there exists a countable open covering $\mathcal{O} = \{O\}$ of X_c such that $\forall\, O \in \mathcal{O} : \text{cl}_c(O) \cap A = \emptyset$ or $\text{cl}_c(O) \cap B = \emptyset$. In view of the claim, $\overline{A} \cap \overline{B}$ is countable. Let $x \in \overline{A} \cap \overline{B}$—then $x \notin A \cup B$, so by regularity there exists an open set $O_x \subset X_c$ containing $x : \text{cl}_c(O_x) \cap (A \cup B) = \emptyset$. It is equally plain that for any $x \in \mathbf{R}^2 - \overline{A} \cap \overline{B}$ there exists an $\mathbf{R}^2$-open set O_x containing $x : \overline{O}_x \cap \overline{A} = \emptyset$ or $\overline{O}_x \cap \overline{B} = \emptyset$. Select a countable subcollection of $\{O_x : x \in \mathbf{R}^2 - \overline{A} \cap \overline{B}\}$ that covers $\mathbf{R}^2 - \overline{A} \cap \overline{B}$ and combine it with $\{O_x : x \in \overline{A} \cap \overline{B}\}$.

Arguing as before, one proves that X_c is $\mathbf{R}$-compact but not $\mathbf{N}$-compact.

[Note: The van Douwen plane exists in ZFC. But unlike the Kunen plane, it is not perfect. Reason: $\mathbf{Q}^2 \cup \{x_{\Gamma(\alpha)} : A_\alpha \cup B_\alpha \subset \mathbf{Q}^2\}$ is not a normal subspace of X_c. However, every closed discrete subspace of X_c is countable, so X_c, like the Kunen plane, is collectionwise normal. Of course, X_c is not Lindelöf, thus is not paracompact (being separable), although X_c is countably paracompact. By the way, if the same procedure is applied to $X = \mathbf{R}$, then the endproduct is a space very different from what was termed the van Douwen line in §1.]

Is it true that for every normal subspace $Y \subset X$, $\dim Y \leq \dim X$? In other words, is dim monotonic? On closed subspaces, this is certainly the case but, as has been seen above, this is not the case in general.

It is false that dim is monotonic on closed subspaces of a nonnormal X. For example, the topological dimension of the Mysior space is positive but it embeds as a closed subspace of some $\mathbf{N}^\kappa$ and $\dim \mathbf{N}^\kappa = 0$.

LEMMA Let X be a nonempty CRH space. Suppose that A is a subspace of X which has the EP w.r.t. $[0,1]$—then $\dim A \leq \dim X$.

PROPOSITION 4 Suppose that X is hereditarily normal—then dim is monotonic iff for every open $U \subset X : \dim U \leq \dim X$.

One might conjecture that dim is monotonic if X is hereditarily normal. This is false: Pol-Pol[†] have given an example of a hereditarily normal X that has topological dimension zero but which contains for every $n = 1, 2, \ldots$ a subspace $X_n : \dim X_n = n$. Since βX

[†] *Fund. Math.* **102** (1979), 137–142.

also has topological dimension zero (cf. Proposition 1), dim is dramatically nonmonotonic even for compact Hausdorff spaces.

Consider the Kunen plane X_Ω—then its one point compactification is hereditarily normal and has topological dimension zero, although X_Ω appears as an open subspace of positive topological dimension.

EXAMPLE The Isbell-Mrówka space $\Psi(\mathbf{N})$ is a nonnormal LCH space. While zero dimensional, its "finer" topological properties definitely depend on the choice of $\mathcal{S}$. Claim: $\exists\ \mathcal{S}$ for which $\dim \Psi(\mathbf{N}) > 0$. To this end, replace $\mathbf{N}$ by $\mathbf{Q}_{[0,1]} \equiv \mathbf{Q} \cap [0,1]$. Attach to each $r, 0 < r < 1$, a bijection $\iota_r : \{q \in \mathbf{Q}_{[0,1]} : q < r\} \rightarrow \{q \in \mathbf{Q}_{[0,1]} : q > r\}$ such that $q' < q''$ iff $\iota_r(q') > \iota_r(q'')$. Let SEQ be the collection of all sequences s of distinct elements of $\mathbf{Q}_{[0,1]}$ satisfying one of the following two conditions: (i) $\lim s = 0$ or $\lim s = 1$; (ii) $s = t \cup \iota_r(t) (0 < r < 1)$, where t converges to r from the left. Because $[0,1]$ is compact, there is a maximal infinite collection $\mathcal{S} \subset$ SEQ of almost disjoint infinite subsets of $\mathbf{Q}_{[0,1]}$. Consider the corresponding Isbell-Mrówka space $X = \Psi(\mathbf{Q}_{[0,1]})$, i.e., $X = \mathcal{S} \cup \mathbf{Q}_{[0,1]}$—then $\dim \mathcal{S} = 0$ and $\dim \mathbf{Q}_{[0,1]} = 0$, yet $\dim X > 0$. To see this, define a continuous function $f : X \rightarrow [0,1]$ by $\begin{cases} f(q) = q \ (q \in \mathbf{Q}_{[0,1]}) \\ f(s) = \lim s \ (s \in \mathcal{S}) \end{cases}$. Verify that there is no clopen subset of X containing $f^{-1}(0)$ and missing $f^{-1}(1)$.

[Note: Mrówka[†] has shown that for certain choices of $\mathcal{S}$, $\beta(\Psi(\mathbf{N})) = \Psi(\mathbf{N})_\infty$, hence $\dim \Psi(\mathbf{N}) = 0$. At the opposite extreme, Terasawa[‡] proved that for any $n = 1, 2, \ldots$ or ∞, it is possible to find an $\mathcal{S}$ such that the associated $\Psi(\mathbf{N})$ has topological dimension n but at the same time is expressible as the union of two zero sets, each having topological dimension zero.]

LEMMA Let $\mathcal{U}$ be a finite open covering of X—then $\mathcal{U}$ has a finite open refinement of order $\leq n+1$ iff $\mathcal{U}$ has a finite closed refinement of order $\leq n+1$.

[Suppose that $\mathcal{U} = \{U_1, \ldots, U_k\}$. Let $\mathcal{V} = \{V_1, \ldots, V_k\}$ be a precise open refinement of $\mathcal{U}$ of order $\leq n+1$—then $\mathcal{V}$ has a precise open refinement $\mathcal{W} = \{W_1, \ldots, W_k\}$ such that $\forall\ i : \overline{W}_i \subset V_i$. And the order of $\overline{\mathcal{W}}$ is $\leq n+1$. To go the other way, let $\mathcal{A} = \{A_1, \ldots, A_k\}$ be a precise closed refinement of $\mathcal{U}$ of order $\leq n+1$—then it will be enough to produce a precise open refinement $\mathcal{V} = \{V_1, \ldots, V_k\}$ of $\mathcal{U}$ such that $\forall\ i : A_i \subset V_i \subset U_i$ and $A_{i_1} \cap \cdots \cap A_{i_m} \neq \emptyset$ iff $V_{i_1} \cap \cdots \cap V_{i_m} \neq \emptyset$. Here $i_1, \ldots, i_m$ are natural numbers, each $\leq k$. This can be done by a simple iterative procedure. Denote by B_1 the union of all intersections of members of the collection $\{A_1, \ldots, A_k\}$ which are disjoint from A_1 and choose an open set $V_1 : \begin{cases} A_1 \subset V_1 \\ \overline{V}_1 \subset U_1 \end{cases}$ & $B_1 \cap \overline{V}_1 = \emptyset$. Denote by B_2 the union of all intersections of members of the collection $\{\overline{U}_1, A_2, \ldots, A_k\}$ which are disjoint from A_2 and choose an open set $V_2 : \begin{cases} A_2 \subset V_2 \\ \overline{V}_2 \subset U_2 \end{cases}$ & $B_2 \cap \overline{V}_2 = \emptyset$. ETC.]

[†] *Fund. Math.* **94** (1977), 83–92.

[‡] *Topology Appl.* **11** (1980), 93–102.

COUNTABLE UNION LEMMA Suppose that $X = \bigcup\limits_1^\infty A_j$, where the A_j are closed subspaces of X such that $\forall\ j, \dim A_j \leq n$—then $\dim X \leq n$, hence $\dim X = \sup \dim A_j$.

[Let $\mathcal{U} = \{U_i\}$ be a finite open covering of X. Put $A_0 = \emptyset$. Claim: There exists a sequence $\mathcal{U}_0, \mathcal{U}_1, \ldots$ of finite open coverings $\mathcal{U}_j = \{U_{i,j}\}$ of X such that $U_{i,0} \subset U_i$ but

$$\overline{U}_{i,j} \subset U_{i,j-1} \ \& \ \text{ord}(\{A_j \cap \overline{U}_{i,j}\}) \leq n+1$$

if $j \geq 1$. To prove this, we shall proceed by induction on j, setting $\mathcal{U}_0 = \mathcal{U}$ and then assuming that the $\mathcal{U}_j$ have been defined for all $j < j_0$, where $j_0 \geq 1$. Since $\{A_{j_0} \cap U_{i,j_0-1}\}$ is a finite open covering of A_{j_0} and since $\dim A_{j_0} \leq n$, there exist open subsets $V_i \subset A_{j_0} \cap U_{i,j_0-1}$ of A_{j_0} such that $A_{j_0} = \bigcup\limits_i V_i$ and $\text{ord}(\{V_i\}) \leq n+1$. Let $W_i = (U_{i,j_0-1} - A_{j_0}) \cup V_i$—then $\{W_i\}$ is a finite open covering of X and $\text{ord}(\{A_{j_0} \cap W_i\}) \leq n+1$. The induction is completed by choosing the elements U_{i,j_0} of $\mathcal{U}_{j_0}$ subject to $\overline{U}_{i,j_0} \subset W_i$. By construction, the collection $\{\bigcap\limits_{j\geq 1} \overline{U}_{i,j}\}$ is a precise closed refinement of $\mathcal{U} = \{U_i\}$ of order $\leq n+1$, so from the lemma $\dim X \leq n$.]

Example: $\dim[0,1] = 1 \Rightarrow \dim \mathbf{R} = 1$.

FACT Suppose that X is normal of topological dimension $n \geq 1$—then there exists a sequence of pairwise disjoint closed subspaces A_j of X such that $\forall\ j, \dim A_j = n$.

A CRH space X is said to be strongly paracompact if every open covering of X has a star finite open refinement. Any paracompact LCH space X is strongly paracompact (cf. §1, Proposition 2). Also: X Lindelöf $\Rightarrow$ X strongly paracompact and X connected + strongly paracompact $\Rightarrow$ X Lindelöf. Not every metric space is strongly paracompact (consider the star space $\mathbf{S}(\kappa)$, $\kappa > \omega$).

FACT Suppose that X is normal and Y is a strongly paracompact subspace of X—then $\dim Y \leq \dim X$.

[The assertion is trivial if $\dim X = \infty$, so assume that $\dim X = n$ is finite. Let $\{U_i\}$ be a finite open covering of Y; let O_i be an open subset of X such that $U_i = Y \cap O_i$ and put $O = \bigcup\limits_i O_i$. Assign to each $y \in Y$ a neighborhood O_y of y in $X : \overline{O}_y \subset O$—then $\{Y \cap O_y\}$ is an open covering of Y, thus has a star finite open refinement $\mathcal{P}$. Write $\mathcal{P} = \coprod\limits_j \mathcal{P}_j$, the equivalence relation corresponding to this partition being $P' \sim P''$ iff there exists a finite collection of sets $P_1, \ldots, P_r$ in $\mathcal{P}$ with $P_1 = P'$, $P_r = P''$ and $P_1 \cap P_2 \neq \emptyset, \ldots, P_{r-1} \cap P_r \neq \emptyset$. Since $\mathcal{P}$ is star finite, each of the $\mathcal{P}_j$ is countable. Let $Y_j = \bigcup\{\overline{P} : P \in \mathcal{P}_j\}$, where $\overline{P}$ is the closure of P in X. Being an F_σ, Y_j is normal and therefore, by the countable union lemma, $\dim Y_j \leq n$. But Y_j is contained in $O = \bigcup\limits_i O_i$, so there exists an open covering $\{O_{i,j}\}$ of Y_j such that $\forall\ i : O_{i,j} \subset O_i$ & $\operatorname{ord}(\{O_{i,j}\}) \leq n+1$. Let $V_i = Y \cap \bigcup\limits_j (O_{i,j} \bigcap \cup \mathcal{P}_j)$—then $\{V_i\}$ is a precise open refinement of $\{U_i\}$ of order $\leq n+1$.]

The preceding result is false if "paracompact" is substituted for "strongly paracompact". Example: Consider Roy's metric space X sitting inside its zero dimensional compactification ζX.

The countable union lemma retains its validity in the completely regular situation provided the A_j are subspaces of X which have the EP w.r.t. $[0,1]$. Proof: The closure of A_j in βX is βA_j, so if $Y = \bigcup\limits_1^\infty \beta A_j$, then Y is normal and therefore, by the countable union lemma, $\dim Y \leq n$, from which $\dim X = \dim \beta X = \dim \beta Y = \dim Y \leq n$.

[Note: According to Terasawa (cf. p. 19–11), there exists a completely regular X of topological dimension n such that $X = X_1 \cup X_2$, where X_1 and X_2 are zero sets with $\begin{cases} \dim X_1 = 0 \\ \dim X_2 = 0 \end{cases}$. Therefore the countable union lemma can fail even when the hypothesis "closed set" is strengthened to "zero set".]

LEMMA Let X be a nonempty CRH space. Suppose that A is a $\mathcal{Z}$-embedded subspace of X—then $\dim A \leq \dim X$.

[Assume that $\dim X \leq n$. Let $\{U_i\}$ be a finite cozero set covering of A; let O_i be a cozero set in βX such that $U_i = A \cap O_i$ and put $O = \bigcup\limits_i O_i$—then O is a cozero set in βX, so by the countable union lemma, $\dim O \leq \dim \beta X = \dim X \leq n$. Therefore there exists a cozero set covering $\{P_i\}$ of O of order $\leq n+1$ such that $\forall\ i : P_i \subset O_i$. Consider the collection $\{A \cap P_i\}$.]

Recall: Every subspace of a perfectly normal space is perfectly normal. So: X perfectly normal $\Rightarrow$ X hereditarily normal. The conjunction perfectly normal + paracompact is hereditary to all subspaces. Reason: Every open set is an F_σ and an F_σ in a paracompact space is paracompact. For example, the class of stratifiable spaces or the class of CW complexes realize this conjunction.

[Note: The ordinal space $[0, \Omega]$ is hereditarily normal but not perfectly normal and its product with $[0,1]$ is normal but not hereditarily normal.]

PROPOSITION 5 Suppose that X is perfectly normal—then dim is monotonic.

[Apply the countable union lemma to an open subset of X and then quote Proposition 4.]

Working under CH, the procedure for manufacturing the Kunen line or the Kunen plane is just a specialization to $\mathbf{R}$ or $\mathbf{R}^2$ of a general "machine" for refining topologies. Thus suppose that X is a set of cardinality Ω equipped with a Hausdorff topology τ which is first countable, hereditarily separable and perfectly normal—then a Kunen modification of τ is a topology $K\tau$ on X finer than τ which is zero dimensional, locally compact, first countable, hereditarily separable and perfectly normal (but not Lindelöf) such that each $x \in X$ has a countable clopen neighborhood and $\forall\, S \subset X : \#(\mathrm{cl}_{\tau}(S) - \mathrm{cl}_{K\tau}(S)) \leq \omega$.

[Note: Any τ having the stated properties admits a Kunen modification $K\tau$ (cf. p. 1–16).]

FACT [Assume CH] If $\dim(X, \tau) \geq n$, then $\dim(X, K\tau) \geq n - 1$ and if $\dim(X, \tau) \leq n$, then $\dim(X, K\tau) \leq n$.

PROPOSITION 6 The statement $\dim X \leq n$ is true iff every neighborhood finite open covering of X has a numerable open refinement of order $\leq n + 1$.

[Let $\mathcal{U}$ be a neighborhood finite open covering of X—then $\mathcal{U}$ is numerable, hence has a numerable open refinement that is both neighborhood finite and σ-discrete, say $\mathcal{V} = \bigcup\limits_n \mathcal{V}_n$ (cf. §1, Proposition 12). Choose a partition of unity $\{\kappa_V\}$ on X subordinate to $\mathcal{V}$. Put $f_n = \sum\limits_{V \in \mathcal{V}_n} \kappa_V$: The collection $\{f_n^{-1}(]0,1])\}$ is a countable cozero set covering of X, thus has a countable star finite cozero set refinement $\{O_k\}$ (cf. p. 1–25). Fix a sequence of integers $1 = n_1 < n_2 \cdots$: $O_k \cap O_l = \emptyset$ if $k \leq n_i$ and $l \geq n_{i+2}$ $(i = 1, 2, \ldots)$. The subspace $\bigcup\limits_{k \leq n_2} O_k$ is a cozero set and so by the countable union lemma its topological dimension is $\leq n$. Accordingly, there exists a covering $\mathcal{W}_1 = \{W_1, \ldots, W_{n_1}, W'_{n_1+1}, \ldots, W'_{n_2}\}$ of $\bigcup\limits_{k \leq n_2} O_k$ by cozero sets of order $\leq n + 1$ such that $\begin{cases} W_k \subset O_k & (k \leq n_1) \\ W'_k \subset O_k & (n_1 < k \leq n_2) \end{cases}$. Next, there exists a covering $\mathcal{W}_2 = \{W_{n_1+1}, \ldots, W_{n_2}, W'_{n_2+1}, \ldots, W'_{n_3}\}$ of $W'_{n_1+1} \cup \cdots \cup W'_{n_2} \cup O_{n_2+1} \cup \cdots \cup O_{n_3}$ by cozero sets of order $\leq n + 1$ such that $\begin{cases} W_k \subset W'_k & (n_1 < k \leq n_2) \\ W'_k \subset O_k & (n_2 < k \leq n_3) \end{cases}$. Iterate to get a covering $\mathcal{W} = \{W_k\}$ of X by cozero sets of order $\leq n + 1$ such that $\forall\, k : W_k \subset O_k$. The collection $\bigcup\limits_k \mathcal{U} \cap W_k$ is a numerable open refinement of $\mathcal{U}$ of order $\leq n + 1$.]

Suppose that X is paracompact—then it follows from Proposition 6 that $\dim X \leq n$ iff every open covering of X has an open refinement of order $\leq n + 1$.

Since cozero sets are $\mathcal{Z}$-embedded and since dim is monotonic on $\mathcal{Z}$-embedded subspaces, Proposition 6 goes through without change in the completely regular situation provided one works with numerable open coverings and numerable open refinements.

SUBLEMMA The statement $\dim X \leq n$ is true iff every open covering $\{U_1, \ldots, U_{n+2}\}$ of X has a precise open refinement $\{V_1, \ldots, V_{n+2}\}$ such that $\bigcap_1^{n+2} V_i = \emptyset$.

[When turned around, the nontrivial assertion is that if $\dim X > n$, then there exists an open covering $\{U_1, \ldots, U_{n+2}\}$ of X, every precise open refinement $\{V_1, \ldots, V_{n+2}\}$ of which satisfies the condition $\bigcap_1^{n+2} V_i \neq \emptyset$. But $\dim X > n$ means that there exists an open covering $\{O_1, \ldots, O_k\}$ of X that has no precise open refinement of order $\leq n+1$. By making at most a finite number of replacements, matters can be arranged so as to ensure that if $\{P_1, \ldots, P_k\}$ is a precise open refinement of $\{O_1, \ldots, O_k\}$, then $P_{i_1} \cap \cdots \cap P_{i_m} \neq \emptyset$ whenever $O_{i_1} \cap \cdots \cap O_{i_m} \neq \emptyset$. Here $i_1, \ldots, i_m$ are natural numbers, each $\leq k$. We can and will assume that $\bigcap_1^{n+2} O_i \neq \emptyset$. Put $U_i = O_i$ $(i \leq n+1)$, $U_{n+2} = \bigcup_{n+2}^{k} O_i$—then $\{U_1, \ldots, U_{n+2}\}$ is an open covering of X with the property in question. In fact, let $\{V_1, \ldots, V_{n+2}\}$ be an open covering of X such that $\forall\, i : V_i \subset U_i$. The covering $\{V_1, \ldots, V_{n+1}, V_{n+2} \cap O_{n+2}, \ldots, V_{n+2} \cap O_k\}$ is a precise open refinement of $\{O_1, \ldots, O_k\}$ and $\bigcap_1^{n+2} V_i \supset (\bigcap_1^{n+1} V_i) \cap (V_{n+2} \cap O_{n+2}) \neq \emptyset$.]

LEMMA The statement $\dim X \leq n$ is true iff for every collection $\{(A_i, B_i) : i = 1, \ldots, n+1\}$ of $n+1$ pairs of disjoint closed subsets of X there exists a collection $\{\phi_i : i = 1, \ldots, n+1\}$ of $n+1$ continuous functions $\phi_i : X \to [0,1]$ such that $\begin{cases} \phi_i | A_i = 0 \\ \phi_i | B_i = 1 \end{cases}$ and $\bigcap_1^{n+1} \phi_i^{-1}(1/2) = \emptyset$.

[Necessity: Put $B_{n+2} = \bigcup_1^{n+1} A_i$—then $\bigcap_1^{n+2} B_i = \emptyset$, so there exists an open covering $\{U_1, \ldots, U_{n+2}\}$ of X such that $B_i \subset U_i$ and $\bigcap_1^{n+2} U_i = \emptyset$. Since $A_i \subset U_{n+2}$, we can replace U_i by $U_i - A_i$ and force $A_i \subset X - U_i$. Fix a precise closed refinement $\{C_1, \ldots, C_{n+2}\}$ of $\{U_1, \ldots, U_{n+2}\}$ with $B_i \subset C_i$. Let $\phi_i : X \to [0,1]$ be a continuous function such that $\phi_i | X - U_i = 0$ and $\phi_i | C_i = 1$. Obviously, $\begin{cases} \phi_i | A_i = 0 \\ \phi_i | B_i = 1 \end{cases}$. And finally, $\bigcap_1^{n+1} \phi_i^{-1}(1/2) \subset \bigcap_1^{n+1} (U_i - C_i) \subset \bigcap_1^{n+2} U_i = \emptyset$.

Sufficiency: Let $\{U_1, \ldots, U_{n+2}\}$ be an open covering of X. Fix a precise closed refinement $\{C_1, \ldots, C_{n+2}\}$ for it and let $\begin{cases} A_i = X - U_i \\ B_i = C_i \end{cases}$ $(i = 1, \ldots, n+1)$. The pairs (A_i, B_i) satisfy our hypotheses, so choose the ϕ_i as there and then let $\begin{cases} O_i = \{x : \phi_i(x) < 1/2\} \\ P_i = \{x : \phi_i(x) > 1/2\} \end{cases}$. Note that $\bigcap_1^{n+1} (X - (O_i \cup P_i)) = \bigcap_1^{n+1} \phi_i^{-1}(1/2) = \emptyset$, hence that $X = \bigcup_1^{n+1} O_i \cup \bigcup_1^{n+1} P_i$. Put

$V_i = P_i$ $(i \leq n+1)$, $V_{n+2} = U_{n+2} \cap \bigcup\limits_1^{n+1} O_i$—then $\{V_1, \ldots, V_{n+2}\}$ is a precise open refinement of $\{U_1, \ldots, U_{n+2}\}$ such that $\bigcap\limits_1^{n+2} V_i = \emptyset$. The sublemma therefore implies that $\dim X \leq n$.]

The characterization of $\dim X \leq n$ given by the lemma extends to the completely regular situation so long as it is formulated in terms of disjoint pairs (A_i, B_i) of zero sets.

When the context dictates, we shall abuse the notation and write $\mathbf{S}^n$ for the frontier of $[0,1]^{n+1}$.

ALEXANDROFF'S CRITERION The statement $\dim X \leq n$ is true iff every closed subset $A \subset X$ has the EP w.r.t. $\mathbf{S}^n$.

[Necessity: Given $f \in C(A, \mathbf{S}^n)$: $f = (f_1, \ldots, f_{n+1})$, let $\begin{cases} A_i = \{x : f_i(x) = 0\} \\ B_i = \{x : f_i(x) = 1\} \end{cases}$—then A is the union $\bigcup\limits_i (A_i \cup B_i)$ and the preceding lemma is applicable to the pairs (A_i, B_i). The corresponding $\phi_i : X \to [0,1]$ combine to determine a continuous function $\phi : X \to [0,1]^{n+1}$, the restriction of which to A defines an element $\psi \in C(A, \mathbf{S}^n)$. Put $H(x,t) = (1-t)\psi(x) + tf(x)$ $((x,t) \in IA)$—then $H \in C(IA, \mathbf{S}^n)$, so ψ and f are homotopic. On the other hand, $\mathbf{S}^n$ is a retract of $[0,1]^{n+1}$ punctured at its center $(1/2, \ldots, 1/2)$. Since $\bigcap\limits_1^{n+1} \phi_i^{-1}(1/2) = \emptyset$, it follows that ψ has an extension $\Psi \in C(X, \mathbf{S}^n)$. But A has the HEP w.r.t. $\mathbf{S}^n$ (cf. p. 6–41), therefore f has an extension $F \in C(X, \mathbf{S}^n)$.

Sufficiency: Consider an arbitrary collection $\{(A_i, B_i) : i = 1, \ldots, n+1\}$ of $n+1$ pairs of disjoint closed subsets of X. Put $A = \bigcup\limits_i (A_i \cup B_i)$. Choose $f_i \in C(A, [0,1])$ such that $\begin{cases} f_i | A_i = 0 \\ f_i | B_i = 1 \end{cases}$ and then combine the f_i to determine a continuous function $f : A \to \mathbf{S}^n$. By assumption, f has an extension $F \in C(X, \mathbf{S}^n)$. Write ϕ_i for the ith component of F—then $\phi_i | A = f_i$ and $\bigcap\limits_1^{n+1} \phi_i^{-1}(1/2) = \emptyset$. That $\dim X \leq n$ is thus a consequence of the preceding lemma.]

EXAMPLE Take for X the long ray L^+—then $\dim X = 1$.

[Since $\dim X > 0$, one need only show that $\dim X \leq 1$. But real valued continuous functions are constant on "tails", so Alexandroff's criterion is applicable.]

FACT Let X be a compact Hausdorff space. Suppose that $X = \bigcup\limits_1^{\infty} A_j$, where the A_j are closed subspaces of X such that $\forall\, i \neq j : \dim(A_i \cap A_j) < n$—then each A_j has the EP w.r.t. $\mathbf{S}^n$.

[Recall that if X is a connected compact Hausdorff space admitting a disjoint decomposition $\bigcup\limits_1^{\infty} A_j$ by closed subspaces A_j, then $A_j = X$ for some j.]

Application: Because the identity map $\mathbf{S}^n \rightarrow \mathbf{S}^n$ cannot be extended continuously over $[0,1]^{n+1}$, $\mathbf{R}^{n+1}$ cannot be covered by a sequence $\{K_j\}$ of compact sets such that $\forall\ i \neq j : \dim(K_i \cap K_j) < n$.

[Note: With more work, one can do better in that "compact" can be replaced by "closed" (cf. p. 19-24).]

The compactness assumption on X in the preceding result is essential. Example: Take for X a one dimensional connected locally compact subspace of the plane admitting a disjoint decomposition $\bigcup\limits_1^{\infty} A_j$ by nonempty closed proper subspaces A_j, fix two indices $i \neq j$, and consider the continuous function $f : A_i \cup A_j \rightarrow \mathbf{S}^0$ which is 0 on A_i and 1 on A_j.

Using Alexandroff's criterion, Cantwell[†] proved that the statement $\dim X \leq n$ is true iff the closed unit ball in $BC(X, \mathbf{R}^{n+1})$ is the convex hull of its extreme points ($n = 1, 2, \ldots$).

[Note: Let X be a nonempty CRH space—then the extreme points of the closed unit ball in $BC(X, \mathbf{R}^{n+1})$ are the functions whose range is a subset of $\mathbf{S}^n$ and it is always true that the closed unit ball in $BC(X, \mathbf{R}^{n+1})$ is the closed convex hull of its extreme points ($n = 1, 2, \ldots$), a purely topological assertion. By contrast, the closed unit ball in $BC(X)$ is the closed convex hull of its extreme points iff $\dim X = 0$.]

In the completely regular situation, there is only a partial analog to Alexandroff's criterion.

(1) Suppose that every zero set $A \subset X$ has the EP w.r.t. $\mathbf{S}^n$—then $\dim X \leq n$. Proof: Since for any pair (A, B) of disjoint zero sets there exists a continuous function $f : X \rightarrow [0,1]$ such that $\begin{cases} f|A = 0 \\ f|B = 1 \end{cases}$, the argument used in the normal case can be transcribed in the obvious way.

(2) Suppose that $\dim X \leq n$—then every subset $A \subset X$ which has the EP w.r.t. $[0,1]$ has the EP w.r.t. $\mathbf{S}^n$. Proof: Since $\dim X = \dim \beta X$, βA, the closure of A in βX, has the EP w.r.t. $\mathbf{S}^n$.

[Note: This need not be true if A is a zero set. Example: Take, after Terasawa (cf. p. 19-11), $X = X_1 \cup X_2$, where $\dim X = 1$ and X_1 and X_2 are zero sets with $\begin{cases} \dim X_1 = 0 \\ \dim X_2 = 0 \end{cases}$—then either X_1 or X_2 fails to have the EP w.r.t. $[0,1]$ (otherwise $\dim X = \max\{\dim X_1, \dim X_2\}$). To be specific, assume that it is X_1. Put $A = X_1$ and choose a continuous function $\phi : A \rightarrow [0,1]$ that does not extend to a continuous function $\Phi : X \rightarrow [0,1]$—then $f = (\phi, 0)$ is a continuous function $A \rightarrow \mathbf{S}^1$ that does not extend to a continuous function $F : X \rightarrow \mathbf{S}^1$.]

Let Y be a topological space—then a map $f \in C(X,Y)$ is said to be universal if $\forall\ g \in C(X,Y)\ \exists\ x \in X : f(x) = g(x)$. A universal map is clearly surjective. Note too that if there is a universal map $X \rightarrow Y$, then every element of $C(Y,Y)$ must have a fixed point.

[†] *Proc. Amer. Math. Soc.* 19 (1968), 821-825.

LEMMA A continuous function $f : X \to [0,1]^{n+1}$ is universal iff the restriction $f^{-1}(\mathbf{S}^n) \to \mathbf{S}^n$ has no extension $F \in C(X, \mathbf{S}^n)$.

[Necessity: To get a contradiction, suppose that there exists a continuous function $F : X \to \mathbf{S}^n$ which agrees with f on $f^{-1}(\mathbf{S}^n)$ and then postcompose F with the antipodal map $\mathbf{S}^n \to \mathbf{S}^n$.

Sufficiency: To get a contradiction, suppose that there exists a continuous function $g : X \to [0,1]^{n+1}$ such that $f(x) \neq g(x)$ for every $x \in X$ and define a continuous function $F : X \to \mathbf{S}^n$ by setting $F(x)$ equal to the intersection of $\mathbf{S}^n$ with the ray containing $f(x)$ which emanates from $g(x)$.]

It therefore follows that $\dim X \geq n$ iff there exists a universal map $f : X \to [0,1]^n$. Example: $\dim[0,1]^n \geq n$. Indeed, the Brouwer fixed point theorem says that the identity map $[0,1]^n \to [0,1]^n$ is universal. Example: $\dim[0,1]^n \geq n \Rightarrow \dim \mathbf{R}^n \geq n$.

The equivalence $\dim X \geq n$ iff there exists a universal map $f : X \to [0,1]^n$ holds for any completely regular X.

LEMMA Let A be a closed subset of X. Suppose that $\dim B \leq n$ for every closed subset $B \subset X$ which does not meet A—then each $f \in C(A, \mathbf{S}^n)$ has an extension $F \in C(X, \mathbf{S}^n)$.

[Choose an open $U \supset A$ and a $\phi \in C(U, \mathbf{S}^n)$ such that $\phi | A = f$. Choose an open $V : A \subset V \subset \overline{V} \subset U$—then $\overline{V} - V$ is closed in $X - V$, so Alexandroff's criterion says there exists a $\Phi \in C(X - V, \mathbf{S}^n) : \Phi | \overline{V} - V = \phi | \overline{V} - V$. Consider the function $F \in C(X, \mathbf{S}^n)$ defined by $F(x) = \begin{cases} \phi(x) & (x \in \overline{V}) \\ \Phi(x) & (x \in X - V) \end{cases}$.]

CONTROL LEMMA Let A be a closed subset of X. Suppose that $\dim A \leq n$ and that $\dim B \leq n$ for every closed subset $B \subset X$ which does not meet A—then $\dim X \leq n$.

[Fix a closed subset $A_0 \subset X$ and take an $f_0 \in C(A_0, \mathbf{S}^n)$. Claim: f_0 has an extension $f \in C(A \cup A_0, \mathbf{S}^n)$. Assuming that $A \cap A_0 \neq \emptyset$, in view of Alexandroff's criterion, the restriction $f_0 | A \cap A_0$ has an extension $F_0 \in C(A, \mathbf{S}^n)$. Define $f \in C(A \cup A_0, \mathbf{S}^n)$ piecewise: $\begin{cases} f|A = F_0 \\ f|A_0 = f_0 \end{cases}$. Now let B be a closed subset of X disjoint from $A \cup A_0$. By hypothesis, $\dim B \leq n$ so the lemma implies that f has an extension $F \in C(X, \mathbf{S}^n)$. But $F|A_0 = f_0$. Invoke Alexandroff's criterion to conclude that $\dim X \leq n$.]

Suppose that $A \subset X$ is closed—then the quotient X/A is a normal Hausdorff space and it follows from the control lemma that $\dim X = \max\{\dim A, \dim X/A\}$.

[Note: If A is a closed G_δ, then $X - A$ is an open F_σ, thus is normal, and $\dim X/A = \dim(X - A)$.]

The position of quotients in the completely regular situation is complicated by the fact that X/A need not be completely regular even under favorable circumstances, e.g., when A has the EP w.r.t. $[0,1]$ or A is closed. Still, $\dim X/A$ is meaningful (cf. p. 19–2) and nothing more than that is really needed.

Given a nonempty $A \subset X$, write $*_A$ for the image of A under the projection $p : X \to X/A$.

LEMMA Let X be a nonempty CRH space. Suppose that A is a nonempty subspace of X—then $\dim X/A \leq \dim X$.

[Assume that $\dim X \leq n$. Take a finite cozero set covering $\mathcal{U} = \{U_1, \ldots, U_k\}$ of X/A. Choose a continuous function $\phi : X/A \to [0,1]$ such that $\phi^{-1}(]0,1]) = \bigcap_i \{U_i : *_A \in U_i\}$. Let $q = \phi(*_A)$. Put $V_0 = \{x : \phi(x) > q/2\}$, $V_i = U_i - \{x : \phi(x) \geq q\}$ $(i > 0)$—then $\mathcal{V} = \{V_0, \ldots, V_k\}$ is a finite cozero set refinement of $\mathcal{U}$ and $*_A \notin V_i$ $(i > 0)$. The collection $p^{-1}(\mathcal{V}) = \{p^{-1}(V_0), \ldots, p^{-1}(V_k)\}$ is a finite cozero set covering of X, hence has a precise cozero set refinement $\mathcal{W} = \{W_0, \ldots, W_k\}$ of order $\leq n + 1$, which in turn has a precise zero set refinement $\mathcal{Z} = \{Z_0, \ldots, Z_k\}$ of order $\leq n + 1$. Since Z_i and $X - W_i$ are disjoint zero sets, there exists a continuous function $\phi_i : X \to [0,1]$ with $\begin{cases} \phi_i | Z_i = 1 \\ \phi_i | X - W_i = 0 \end{cases}$. But $A \subset Z_0$ and $A \cap W_i = \emptyset$ $(i > 0)$. Therefore each ϕ_i factors through X/A to give a continuous function $\psi_i : X/A \to [0,1]$. The collection $\{\psi_i^{-1}(]0,1])\}$ is a finite cozero set refinement of $\mathcal{U}$ of order $\leq n + 1$.]

LEMMA Let X be a nonempty CRH space. Suppose that A is a nonempty subspace of X which has the EP w.r.t. $[0,1]$—then $\dim X = \max\{\dim A, \dim X/A\}$.

[The point here is that every finite cozero set covering of A is refined by the restriction to A of a finite cozero set covering of X (cf. §6, Proposition 4).]

The relation $\dim X = \max\{\dim A, \dim X/A\}$ need not hold if A is merely $\mathcal{Z}$-embedded in X. Indeed, Pol[†] has constructed an example of a completely regular X having the following properties: (i) $\dim X > 0$; (ii) $X = X_1 \cup X_2$, where X_1 and X_2 are zero sets with $\begin{cases} \dim X_1 = 0 \\ \dim X_2 = 0 \end{cases}$; (iii) $\begin{cases} X_1 = U_1 \cup D \\ X_2 = U_2 \cup D \end{cases}$, where U_1 and U_2 are cozero sets and D is discrete; (iv) $U_1 \cup U_2$ is a countable dense subset of X. Consider $A = U_1 \cup U_2$.

PROPOSITION 7 Suppose that $X = Y \cup Z$, where Y and Z are normal—then $\dim X \leq \dim Y + \dim Z + 1$.

[There is nothing to prove if either $\dim Y = \infty$ or $\dim Z = \infty$, so assume that $\dim Y \leq r$ and $\dim Z \leq s$. Owing to the control lemma, it will be enough to show that

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[†] *Fund. Math.* **102** (1979), 29–43.

$\dim \overline{Y} \leq r + s + 1$. Let $\mathcal{U} = \{U_i\}$ be a finite open covering of $\overline{Y}$. Since $\dim Y \leq r$, there exists a collection $\mathcal{V} = \{V_i\}$ of open subsets of $\overline{Y}$ such that $V_i \subset U_i$, $Y \subset \bigcup\limits_i V_i$, and $\operatorname{ord}(\{Y \cap V_i\}) \leq r + 1$. Put $D = \overline{Y} - \bigcup\limits_i V_i$. Because $\dim D \leq s$, there exists a closed covering $\mathcal{A} = \{A_i\}$ of D of order $\leq s + 1$ such that $A_i \subset U_i$. Without changing the order, expand $\mathcal{A}$ to a collection $\mathcal{W} = \{W_i\}$ of open subsets of $\overline{Y}$ such that $A_i \subset W_i \subset U_i$. The union $\mathcal{V} \cup \mathcal{W}$ covers $\overline{Y}$, refines $\mathcal{U}$, and is of order $\leq r + 1 + s + 1$.]

[Note: When X is metrizable, there is another way to argue. Assume: $\left\{\begin{matrix} \dim Y = r \\ \dim Z = s \end{matrix}\right.$ — then every closed subset of $\left\{\begin{matrix} Y \\ Z \end{matrix}\right.$ has the EP w.r.t. $\left\{\begin{matrix} \mathbf{S}^s \\ \mathbf{S}^r \end{matrix}\right.$, thus every closed subset of X has the EP w.r.t. $\mathbf{S}^s * \mathbf{S}^r = \mathbf{S}^{s+r+1}$ (cf. p. 6–43).]

By way of an application, suppose that X is hereditarily normal and $X = \bigcup\limits_0^n X_i$, where $\forall\, i : \dim X_i \leq 0$—then $\dim X \leq n$.

This remark can be used to prove that $\dim \mathbf{R}^n \leq n$, from which $\dim \mathbf{R}^n = n$ (cf. p. 19–18). Thus suppose that $n \geq 1$ and that $0 \leq m \leq n$. Denote by $\mathbf{Q}_m^n$ the subspace of $\mathbf{R}^n$ consisting of all points with exactly m rational coordinates—then $\mathbf{R}^n = \mathbf{Q}_0^n \cup \cdots \cup \mathbf{Q}_n^n$. Claim: $\forall\, m$, $\dim \mathbf{Q}_m^n = 0$. This is immediate if $m = n$ (cf. Proposition 2), so assume that $m < n$. For any choice of m distinct natural numbers $i_1, \ldots, i_m$, each $\leq n$, and any choice of m rational numbers $r_1, \ldots, r_m$, the space $\prod\limits_{i=1}^n R_i$, where $R_{i_j} = \{r_j\}$ for $j = 1, \ldots, m$ and $R_i = \mathbf{R}$ for $i \neq i_j$, is a closed subspace of $\mathbf{R}^n$. Therefore $\mathbf{Q}_m^n \cap \prod\limits_{i=1}^n R_i$ is a closed subspace of $\mathbf{Q}_m^n$. On the other hand, $\mathbf{Q}_m^n \cap \prod\limits_{i=1}^n R_i$ is homeomorphic to the subspace of $\mathbf{R}^{n-m}$ consisting of all points with irrational coordinates, hence $\dim(\mathbf{Q}_m^n \cap \prod\limits_{i=1}^n R_i) = 0$ (cf. Proposition 2). Since the collection of all sets of the form $\mathbf{Q}_m^n \cap \prod\limits_{i=1}^n R_i$ is a countable closed covering of $\mathbf{Q}_m^n$, the countable union lemma implies that $\dim \mathbf{Q}_m^n = 0$.

FUNDAMENTAL THEOREM OF DIMENSION THEORY The topological dimension of $\mathbf{R}^n$ is exactly n.

One consequence is the evaluation $\dim[0,1]^n = n$. Corollary: Take $X = \mathbf{S}^n$—then $\dim X = n$. In fact, $X = X_1 \cup X_2$, where X_1 and X_2 are closed and homeomorphic to $[0,1]^n$.

Another consequence is the evaluation $\left\{\begin{matrix} \dim(\mathbf{Q}_0^n \cup \cdots \cup \mathbf{Q}_m^n) = m \\ \dim(\mathbf{Q}_m^n \cup \cdots \cup \mathbf{Q}_n^n) = n - m \end{matrix}\right.$.

EXAMPLE [Assume CH] Take $X = [0,1]^n$—then the topological dimension of X in any Kunen modification of its euclidean topology is $n-1$ (cf. p. 19–14).

FACT Let X and Y be normal. Let $A \rightarrow X$ be a closed embedding and let $f : A \rightarrow Y$ be a continuous function. Assume: $\dim X \leq n$ & $\dim Y \leq n$—then $\dim(X \sqcup_f Y) \leq n$.

[Use the control lemma ($X \sqcup_f Y$ is a normal Hausdorff space (cf. p. 3–1)).]

Application: If X is obtained from a normal A by attaching n-cells, then $\dim X = n$ provided that $\dim A \leq n$ and the index set is not empty.

[X contains an embedded copy of $\mathbf{B}^n$ which is strongly paracompact, thus a priori, $\dim X \geq n$ (cf. p. 19–12).]

EXAMPLE (CW Complexes) Let X be a CW complex—then by the countable union lemma, $\dim X = \sup \dim X^{(n)}$ and $\forall\, n, \dim X^{(n)} \leq n$. Therefore the combinatorial dimension of X is equal to the topological dimension of X.

FACT Suppose that X is normal. Let $\mathcal{A} = \{A_j : j \in J\}$ be an absolute closure preserving closed covering of X such that $\forall\, j, \dim A_j \leq n$—then $\dim X \leq n$, hence $\dim X = \sup \dim A_j$.

[Use Alexandroff's criterion. Let A be a closed subset of X, take an $f \in C(A, \mathbf{S}^n)$, and let $\mathcal{F}$ be the set of continuous functions F that are extensions of f and have domains of the form $A \cup X_I$, where $X_I = \bigcup\limits_i A_i$ $(I \subset J)$. Order $\mathcal{F}$ by writing $F' \leq F''$ iff F'' is an extension of F'. Every chain in $\mathcal{F}$ has an upper bound, so by Zorn, $\mathcal{F}$ has a maximal element F_0. But the domain of F_0 is necessarily all of X and $F_0|A = f$.]

EXAMPLE (Vertex Schemes) Let $K = (V, \Sigma)$ be a vertex scheme—then one can attach to K its combinatorial dimension $\dim K$, as well as the topological dimensions of $|K|$ (Whitehead topology) and $|K|_b$ (barycentric topology). The claim is that these are all equal. Note that in any event, if σ is an n-simplex of K, then $\dim|\sigma| = n$, so, $|\sigma|$ being a closed subspace of both $|K|$ and $|K|_b$, $\begin{cases} \dim|K| \geq \dim K \\ \dim|K|_b \geq \dim K \end{cases}$. Regarding the inequalities in the opposite direction, first observe that $\{|\sigma|\}$ is an absolute closure preserving closed covering of $|K|$, thus in this case the preceding result is immediately applicable. Turning to $|K|_b$, $\{|\sigma|\}$ is still closure preserving. To exploit this, consider the n-skeleton $K^{(n)}$. Assertion: $\forall\, n, \dim|K^{(n)}|_b \leq n$. Obviously, $\dim|K^{(0)}|_b = 0$. Suppose that $n \geq 1$ and $\dim|K^{(n-1)}|_b \leq n-1$. Let Σ_n be the set of n-simplexes of K. The collection $\{\langle\sigma\rangle : \sigma \in \Sigma_n\}$ is an open covering of $|K^{(n)}|_b - |K^{(n-1)}|_b$. Write $\langle\sigma\rangle = \bigcup\limits_j A_{\sigma j}$, where the $A_{\sigma j} \subset |\sigma|$ are compact. The collection $\{A_{\sigma j} : \sigma \in \Sigma_n\}$ is discrete. Let A_j be its union—then $\dim A_j \leq n$. Finish the induction via the countable union lemma: $|K^{(n)}|_b = |K^{(n-1)}|_b \cup \bigcup\limits_j A_j$.

[Note: It is therefore a corollary that the combinatorial dimension of $|K|$ viewed as a CW complex is equal to $\dim K$.]

Let X be an n-manifold. Since compact subsets of a nonempty CRH space have the EP w.r.t. $[0,1]$ and since X contains a compact subset homeomorphic to $[0,1]^n$, of necessity $\dim X \geq n$, the euclidean dimension of X. To reverse the inequality $\dim X \geq n$ when X is paracompact or, equivalently, metrizable (cf. §1, Proposition 11), one can assume that X is connected. But then X is second countable (cf. p. 1–2), thus admits a covering by a countable collection of closed sets, each of topological dimension n, so $\dim X \leq n$.

[Note: Using the combinatorial principal $\diamondsuit$, Fedorchuk[†] has constructed a perfectly normal n-manifold X such that $n < \dim X$.]

LEMMA $\mathbf{R}^n$ is homogeneous with respect to countable dense subsets, i.e., if A and B are two countable dense subsets of $\mathbf{R}^n$, then there exists a homeomorphism $f : \mathbf{R}^n \to \mathbf{R}^n$ such that $f(A) = B$.

PROPOSITION 8 Let X be a subspace of $\mathbf{R}^n$—then $\dim X = n$ iff X has a nonempty interior.

[Suppose that the interior of X is empty. Since $\mathbf{R}^n - X$ is dense in $\mathbf{R}^n$, there exists a countable set $A \subset \mathbf{R}^n - X : \overline{A} = \mathbf{R}^n$. Choose a homeomorphism $f : \mathbf{R}^n \to \mathbf{R}^n$ such that $f(A) = \mathbf{Q}_n^n$—then $f(X) \subset \bigcup\limits_{m<n} \mathbf{Q}_m^n$, which gives $\dim X \leq n-1$.]

It follows from this result that if X is a subspace of $[0,1]^n$ or $\mathbf{S}^n$, then $\dim X = n$ iff X has a nonempty interior.

SUBLEMMA Suppose that X is Lindelöf. Let $\mathcal{O} = \{O\}$ be a basis for X—then for every pair (A, B) of disjoint closed subsets of X there exists an open set $P \subset X$ and a sequence $\{O_j\} \subset \mathcal{O}$ such that $A \subset P \subset \overline{P} \subset X - B$ and $\operatorname{fr} P \subset \bigcup\limits_j \operatorname{fr} O_j$.

[Given $x \in X$, choose a neighborhood $O_x \in \mathcal{O}$ of x such that either $A \cap \overline{O}_x = \emptyset$ or $B \cap \overline{O}_x = \emptyset$. Let $\{O_j\}$ be a countable subcover of $\{O_x\}$. Divide $\{O_j\}$ into two subcollections $\{O_i'\}$ and $\{O_i''\}$ according to whether $\overline{O}_j$ does or does not meet A. Put $\begin{cases} P_i = O_i' - \bigcup\limits_{j<i} \overline{O}_j'' \\ Q_i = O_i'' - \bigcup\limits_{j\leq i} \overline{O}_j' \end{cases}$—then $\begin{cases} P = \bigcup\limits_i P_i \\ Q = \bigcup\limits_i Q_i \end{cases}$ are disjoint open subsets of X and $A \subset P \subset \overline{P} \subset X - B$, with $\operatorname{fr} \overline{P} \subset X - (P \cup Q)$. Let $x \in X - (P \cup Q)$. Denote by S the first element of the

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[†] *Topology Appl.* **54** (1993), 221–239; see also *Math. Sbornik* **186** (1995), 151–162.

sequence $\overline{O}'_1$, $\overline{O}''_1$, $\overline{O}'_2$, $\overline{O}''_2, \ldots$ that contains x. If $S = \overline{O}'_i$, then $x \notin P_i$ and $x \notin \overline{O}''_j$ $(j < i)$, so $x \in \operatorname{fr} O'_i$; if $S = \overline{O}''_i$, then $x \notin Q_i$ and $x \notin \overline{O}'_j$ $(j \leq i)$, so $x \in \operatorname{fr} O''_i$. Therefore $x \in \bigcup\limits_i \operatorname{fr} O'_i \cup \bigcup\limits_i \operatorname{fr} O''_i$ or still, $x \in \bigcup\limits_j \operatorname{fr} O_j$.]

LEMMA Suppose that X is Lindelöf. Let $\mathcal{O} = \{O\}$ be a basis for X such that $\forall\, O : \dim \operatorname{fr} O \leq n - 1$—then $\dim X \leq n$.

[Let $\mathcal{U} = \{U_i\}$ be a finite open covering of X; let $\mathcal{A} = \{A_i\}$ be a precise closed refinement of $\mathcal{U}$. Use the sublemma and for each i, choose an open set $P_i \subset X$ and a sequence $\{O_{i,j}\} \subset \mathcal{O}$: $A_i \subset P_i \subset \overline{P}_i \subset U_i$ and $\operatorname{fr} P_i \subset \bigcup\limits_j \operatorname{fr} O_{i,j}$. Put $D = \bigcup\limits_i \operatorname{fr} P_i$. The countable union lemma implies that $\dim D \leq n - 1$, so there exists a collection $\mathcal{V} = \{V_i\}$ of open subsets of X such that $\overline{V}_i \subset U_i$, $D \subset \bigcup\limits_i V_i$, and $\operatorname{ord}(\{\overline{V}_i\}) \leq n$. Write B_i in place of $\overline{P}_i - (\cup\mathcal{V} \cup \bigcup\limits_{j<i} P_j)$. Since the B_i are pairwise disjoint, it follows that the collection $\{B_i\} \cup \{\overline{V}_i\}$ is a finite closed refinement of $\mathcal{U}$ of order $\leq n + 1$.]

PROPOSITION 9 Let U be a nonempty, nondense open subset of $\mathbf{R}^n$—then $\dim \operatorname{fr} U = n - 1$.

[Suppose that U is bounded. In this case, U has a basis consisting of sets homeomorphic to itself, so if $\dim \operatorname{fr} U < n - 1$, then by the lemma, $\dim U \leq n - 1$, a contradiction.

Suppose that U is not bounded. Fix a point x in the interior of the complement of U and choose an open ball B centered at x which is entirely contained therein. The associated inversion $\mathbf{R}^n - \{x\} \rightarrow \mathbf{R}^n - \{x\}$ carries U onto a nonempty open set $O \subset B$. Obviously, $\operatorname{fr} O - \{x\}$ is homeomorphic to $\operatorname{fr} U$. On the other hand, by the above, $\dim \operatorname{fr} O = n - 1$. So, from the control lemma, $\dim \operatorname{fr} U = n - 1$.]

LEMMA The following conditions are equivalent.

(1) X can be disconnected by a closed subset of topological dimension $\leq n$.

(2) X contains a nonempty, nondense open subset whose frontier has topological dimension $\leq n$.

(3) $X = A \cup B$, where A and B are closed proper subsets of X such that $\dim(A \cap B) \leq n$.

Take $X = \mathbf{R}^n$—then, in view of Proposition 9, $\mathbf{R}^n$ cannot be disconnected by a closed subset of topological dimension $\leq n - 2$. The same is true of $[0, 1]^n$ of $\mathbf{S}^n$.

Let X be a LCH space. Suppose that X is connected and locally connected—then X is said to be n-solid $(n \geq 1)$ if for every $x \in X$ and for every neighborhood U of x there is a connected relatively compact neighborhood V of x such that $\begin{cases} \overline{V} \subset U \\ \dim \overline{V} \geq n \end{cases}$ and $\overline{V}$ cannot be disconnected by a closed subset of topological dimension $\leq n-2$. Examples: $\mathbf{R}^n, [0,1]^n$, and $\mathbf{S}^n$ are n-solid.

[Note: A LCH space X that is both connected and locally connected is necessarily 1-solid. Specialization of the argument infra then leads to the conclusion that X does not admit a disjoint decomposition $\bigcup\limits_1^\infty A_j$ by nonempty closed proper subspaces A_j. If X is compact, then the assumption of local connectedness is unnecessary but simple examples show that it is not superfluous in general.]

FACT Suppose that X is n-solid and perfectly normal—then X cannot be covered by a sequence $\{A_j\}$ of nonempty closed proper subsets such that $\forall\, i \neq j : \dim(A_i \cap A_j) \leq n-2$.

[Proceed by contradiction, so $X = \bigcup\limits_1^\infty A_j$, where the A_j satisfy the conditions set forth above. Claim: There exists a sequence $\{x_0, x_1, \ldots\} \subset X$ subject to: (1) $x_i \in V_i, V_i$ as in the definition of "n-solid"; (2) $\forall\, j : \overline{V}_i \not\subset A_j$; (3) $\overline{V}_i \subset \overline{V}_{i-1}$; (4) $\overline{V}_i \cap A_i = \emptyset$. Here $\begin{cases} V_{-1} = X \\ A_0 = \emptyset \end{cases}$. Granted the claim, $\bigcap\limits_0^\infty \overline{V}_i = \emptyset$, an impossibility. The x_i can be constructed inductively. Start by fixing an index j_0 such that the interior of A_{j_0} is not empty (Baire). Choose a point x_0 in the frontier of the interior of A_{j_0} and take a neighborhood V_0 of x_0 as in the definition of "n-solid"—then the pair (x_0, V_0) satisfies (1)–(4). Given x_i and V_i $(i > 0)$, look at a component Y of $V_i - A_{i+1}$. Show that Y is not a subset of any A_j and then get x_{i+1} and V_{i+1} by repeating the process used to get x_0 and V_0.]

[Note: Proposition 5 is tacitly used at several points. When $n = 1$, the assumption of perfect normality plays no role, hence can be dropped.]

LEMMA Let X be a closed subspace of $\mathbf{R}^n$; let $x \in X$—then x belongs to the frontier of X iff x has a neighborhood basis $\{U\}$ in X such that $\forall\, U : X - U$ has the EP w.r.t. $\mathbf{S}^{n-1}$.

[Necessity: Let x be an element of the frontier of X. Assuming that x is the origin, put $U = X \cap \epsilon\mathbf{B}^n$ $(\epsilon > 0)$. To simplify, take $\epsilon = 1$. Fix a point $x_0 \in \mathbf{B}^n - X$ and write r_0 for the radial retraction $\mathbf{D}^n - \{x_0\} \to \mathbf{S}^{n-1}$. Choose an $f \in C(X - U, \mathbf{S}^{n-1})$. Since $A = (X - U) \cap \mathbf{S}^{n-1}$ is a closed subset of $\mathbf{S}^{n-1}$, Alexandroff's criterion implies that $f|A$ can be extended to a continuous function $g : \mathbf{S}^{n-1} \to \mathbf{S}^{n-1}$. The function $F : X \to \mathbf{S}^{n-1}$ defined by $\begin{cases} F|X - U = f \\ F|U = g \circ r_0 \end{cases}$ is then a continuous extension of f to X.

Sufficiency: Let x be an element of the interior of X. Assuming that x is the origin, fix an $\epsilon > 0 : \epsilon\mathbf{D}^n \subset X$. Let U be a neighborhood of x in $X : U \subset \epsilon\mathbf{B}^n$—then the claim is that there exists an $f \in C(X - U, \mathbf{S}^{n-1})$ that has no extension $F \in C(X, \mathbf{S}^{n-1})$. To see this, identify the frontier of $\epsilon\mathbf{D}^n$ with $\mathbf{S}^{n-1}$ and consider the projection $X - U \to \mathbf{S}^{n-1}$

determined by x which, if extendible, would lead to a retraction of $\epsilon\mathbf{D}^n$ onto its frontier.]

Let X and Y be closed subspaces of $\mathbf{R}^n$—then the characterization provided by the lemma tells us that any homeomorphism $f : X \rightarrow Y$ necessarily carries the frontier of X onto the frontier of Y.

THEOREM OF INVARIANCE OF DOMAIN Let U be an open subset of $\mathbf{R}^n$—then every continuous injective map $U \rightarrow \mathbf{R}^n$ is an open embedding.

This result does not extend to an infinite dimensional normed linear space X. Indeed, for such an X, there always exists an embedding $f : X \rightarrow X$ that is not open and there always exists a bijective continuous map $f : X \rightarrow X$ that is not a homeomorphism (van Mill[†]).

FACT Let $f : \mathbf{R}^n \rightarrow \mathbf{R}^n$ be continuous and locally one-to-one. Assume that $\|f(x)\| \rightarrow \infty$ as $\|x\| \rightarrow \infty$—then $f(\mathbf{R}^n) = \mathbf{R}^n$.

Let X and Y be n-manifolds; let $\begin{cases} U \subset X \\ V \subset Y \end{cases}$ and suppose that $f : U \rightarrow V$ is a homeomorphism—then from the domain invariance of $\mathbf{R}^n$, U open in $X \Rightarrow V$ open in Y. Corollary: Homeomorphic topological manifolds have the same euclidean dimension.

Let X be a CRH space. Suppose that $\dim X = n$ $(n \geq 1)$—then X is said to be a Cantor n-space if X cannot be disconnected by a closed subset of topological dimension $\leq n - 2$. Since $\dim \emptyset = -1$, a Cantor n-space is necessarily connected. For example, $\mathbf{R}^n$ is a Cantor n-space. So too are $[0,1]^n$ and $\mathbf{S}^n$. The tubular arrangement

$$\bigcup_1^\infty\left(\left[-\frac{1}{2n-1},-\frac{1}{2n}\right]\times[-1,1]\right)\cup\bigcup_1^\infty\left(\left[-\frac{1}{2n},-\frac{1}{2n+1}\right]\times\left[-\frac{1}{n},\frac{1}{n}\right]\right)\cup([0,1]\times[-1,1])$$

is a Cantor 2-space. It remains connected after removal of the origin but what's left is no longer path connected.

FACT Suppose that X is compact, with $\dim X = n$ $(n \geq 1)$—then X contains a Cantor n-space, thus X has a component of topological dimension n.

[There exists a closed subset $A \subset X$ and a continuous function $f : A \rightarrow \mathbf{S}^{n-1}$ that has no continuous extension $F : X \rightarrow \mathbf{S}^{n-1}$. Use Zorn and construct a closed subset $B_f \subset X$ such that (i) f does not have a continuous extension to $A \cup B_f$ and (ii) f does have a continuous extension to $A \cup B$ for each closed proper subset B of B_f. In view of condition (i), $\dim B_f = n$. Claim: B_f is a Cantor n-space. Assume not and write $B_f = B' \cup B''$, where B' and B'' are closed proper subsets of B_f with $\dim(B' \cap B'') \leq n-2$. On account of condition (ii), f has a continuous extension $\begin{cases} f' \text{ to } A \cup B' \\ f'' \text{ to } A \cup B'' \end{cases}$. Therefore f has a continuous extension to $A \cup B_f$ (cf. Proposition 15). Contradiction.]

[†] *Proc. Amer. Math. Soc.* **101** (1987), 173–180.

[Note: One cannot expect in general that a noncompact X will contain a compact Cantor n-space. Reason: For each $n \geq 1$, there exists a zero dimensional X of topological dimension n (consider an "n-dimensional" variant of Dowker's Example "M").]

Suppose that X is compact and perfectly normal, with $\dim X = n$ ($n \geq 1$). Denote by C_X the union of all Cantor n-spaces in X—then $\dim(X - C_X) \leq \dim X$ but if $n > 1$ equality can obtain even when X is metrizable (Pol[†]).

FACT Suppose that X is a compact connected homogeneous ANR of topological dimension $n \geq 1$—then X is a Cantor n-space.

[Note: Is such an X actually an n-manifold? This is true if $n = 1$ or 2 (Bing-Borsuk[‡]) but is a mystery if $n > 2$. The three dimensional case is related to the Poincaré conjecture (Jakobsche[‖]).]

MARDEŠIĆ FACTORIZATION LEMMA Let X and Y be compact Hausdorff spaces—then for every $f \in C(X,Y)$ there exists a compact Hausdorff space Z with $\begin{cases} \dim Z \leq \dim X \\ \text{wt}\, Z \leq \text{wt}\, Y \end{cases}$ and functions $\begin{cases} g \in C(X,Z) \\ h \in C(Z,Y) \end{cases}$ such that $f = h \circ g$ and $g(X) = Z$.

[Assume that $\dim X = n$ is finite and $\text{wt}\, Y \geq \omega$. Fix a basis $\mathcal{V}$ for Y of cardinality $\text{wt}\, Y$. Denote by $\mathbf{V}$ the collection of all finite open coverings of Y made up of members of $\mathcal{V}$ and put $\mathbf{U}_0 = f^{-1}(\mathbf{V})$. Inductively define a sequence $\mathbf{U}_1, \mathbf{U}_2, \ldots$ of collections of finite open coverings of X by assigning to each pair $\begin{cases} \mathcal{U}' \\ \mathcal{U}'' \end{cases} \in \mathbf{U}_{i-1}$ a finite open covering $\mathcal{U}$ of X of order $\leq n+1$ that is a star refinement of both $\mathcal{U}'$ and $\mathcal{U}''$ and write $\mathbf{U}_i$ for $\{\mathcal{U}\}$. The declaration $x \sim y$ iff $y \in [x] \equiv \bigcap\limits_1^\infty \bigcap \{\text{st}(x,\mathcal{U}) : \mathcal{U} \in \mathbf{U}_i\}$ is an equivalence relation on X and for any open set $U \subset X$ and any $[x] \subset U, \exists\, \mathcal{U}_x \in \mathbf{U}_{i_x}$:

$$[x] \subset \text{st}(x,\mathcal{U}_x) \subset \bigcup_{\text{st}(x,\mathcal{U}_x)} [y] \subset \text{st}(\text{st}(x,\mathcal{U}_x),\mathcal{U}_x) \subset U.$$

[†] *Fund. Math.* **136** (1990), 127–131.

[‡] *Ann. of Math.* **81** (1965), 100–111.

[‖] *Fund. Math.* **106** (1980), 127–134.

Therefore the union of the equivalence classes that are contained in U is open in X. Give $Z = X/\sim$ the quotient topology. Since the projection $g : X \rightarrow Z$ is a closed map, Z is a compact Hausdorff space. By construction, f is constant on equivalence classes so there is a continuous factorization $f = h \circ g$. Assign to each $\mathcal{U} = \{U\}$ in $\mathbf{U}_i$ the collection $\mathcal{U}^* = \{U^*\}$, where $U^* = Z - g(X - U)$—then $\mathcal{U}^*$ is a finite open covering of Z of order $\leq n+1$. Moreover, every finite open covering $\mathcal{P} = \{P\}$ of Z has a refinement of the form $\mathcal{U}^*$, hence $\dim Z \leq n$. In fact, $\forall\ x \in X\ \exists\ P_x \in \mathcal{P} : [x] \subset g^{-1}(P_x)$. Choose $\mathcal{U}_x \in \mathbf{U}_{i_x} : O_x \equiv \text{st}(\text{st}(x, \mathcal{U}_x), \mathcal{U}_x) \subset g^{-1}(P_x)$. Let $\{O_{x_j}\}$ be a finite subcover of $\{O_x\}$. Take a $\mathcal{U} \in \mathbf{U}_i$ that refines the $\mathcal{U}_{x_j}$ and consider the associated $\mathcal{U}^*$. Finally, the collection $\bigcup_1^\infty \bigcup \{\mathcal{U}^* : \mathcal{U} \in \mathbf{U}_i\}$ is a basis for Z of cardinality $\leq \text{wt}\, Y$.]

PROPOSITION 10 X has a compactification ΔX such that $\begin{cases} \dim \Delta X \leq \dim X \\ \text{wt}\, \Delta X \leq \text{wt}\, X \end{cases}$.

[Assume that $\text{wt}\, X \geq \omega$. Choose an embedding $X \rightarrow [0,1]^{\text{wt}\, X}$ and denote by f its extension $\beta X \rightarrow [0,1]^{\text{wt}\, X}$. Apply the Mardešić factorization lemma to get a compact Hausdorff space ΔX and functions $\begin{cases} g \in C(\beta X, \Delta X) \\ h \in C(\Delta X, [0,1]^{\text{wt}\, X}) \end{cases}$: $\begin{cases} \dim \Delta X \leq \dim \beta X = \dim X \\ \text{wt}\, \Delta X \leq \text{wt}\, [0,1]^{\text{wt}\, X} = \text{wt}\, X \end{cases}$ and $f = h \circ g$ $(g(\beta X) = \Delta X)$. Look at $g|X$.]

Since the normality of X was not used in the proof, Proposition 10 is true in the completely regular situation.

FACT For every integer $n \geq 0$ and for every cardinal $\kappa \geq \omega$, there exists a compact Hausdorff space $K(n,\kappa) : \begin{cases} \dim K(n,\kappa) \leq n \\ \text{wt}\, K(n,\kappa) \leq \kappa \end{cases}$ having the property that if X is a nonempty CRH space of topological dimension $\leq n$ and weight $\leq \kappa$, then there is an embedding $X \rightarrow K(n,\kappa)$.

[Consider the collection $\{X_i : i \in I\}$ of all subspaces $X_i \subset [0,1]^\kappa$, where $\dim X_i \leq n$. Let f be the natural map $\coprod_i X_i \rightarrow [0,1]^\kappa$. Work with βf.]

Does every subspace $X \subset \mathbf{R}^n$ have a dimension preserving compactification that embeds in $\mathbf{R}^n$? This is an open question.

A set $S \subset \mathbf{R}^n$ is said to be in general position if every subset $T \subset S$ of cardinality $\leq n+1$ is geometrically independent.

LEMMA $\mathbf{R}^n$ contains a countable dense set in general position.

Suppose that X is second countable—then there is an embedding $X \rightarrow \mathbf{R}^\omega$. If $\dim X = n$, then one can say more: There is an embedding $X \rightarrow \mathbf{R}^{2n+1}$.

Start with an initial reduction: Take X compact (cf. Proposition 10). Fix a compatible metric d on X. Attach to each $f \in C(X, \mathbf{R}^{2n+1})$ its "injectivity deviation"

$$\operatorname{dev} f = \sup\{\operatorname{diam} f^{-1}(p) : p \in \mathbf{R}^{2n+1}\}.$$

Given $\epsilon > 0$, put $D_\epsilon = \{f : \operatorname{dev} f < \epsilon\}$. Claim: $\forall\ \epsilon > 0$, D_ϵ is open and dense in $C(X, \mathbf{R}^{2n+1})$. Admit this—then $\bigcap\limits_1^\infty D_{1/k}$ is dense in $C(X, \mathbf{R}^{2n+1})$ (Baire), thus is nonempty. But $\bigcap\limits_1^\infty D_{1/k}$ is the set of embeddings $X \to \mathbf{R}^{2n+1}$.

(1) D_ϵ is open in $C(X, \mathbf{R}^{2n+1})$. Proof: Let $f \in D_\epsilon$. Choose $r : \operatorname{dev} f < r < \epsilon$. Set $A_r = \{(x, y) : d(x, y) \geq r\}$. Call δ_f the minimum of $\frac{1}{2}\|f(x) - f(y)\|$ on A_r—then $\{g : \|f - g\| < \delta_f\} \subset D_\epsilon$.

(2) D_ϵ is dense in $C(X, \mathbf{R}^{2n+1})$. Proof: Fix $f \in C(X, \mathbf{R}^{2n+1})$. Given $\delta > 0$, let $\mathcal{U} = \{U_i\}$ be a finite open covering of X of order $\leq n + 1 : \forall\ i$, $\begin{cases} \operatorname{diam} U_i < \epsilon/2 \\ \operatorname{diam} f(U_i) < \delta/2 \end{cases}$ and denote by $\{\kappa_i\}$ a partition of unity on X subordinate to $\mathcal{U}$. Choose a point $x_i \in U_i$ and then choose a point $p_i \in \mathbf{R}^{2n+1}$ within $\delta/2$ of $f(x_i)$, using the lemma to arrange matters so that in addition $\{p_i\}$ is in general position. Put $g = \sum\limits_i \kappa_i p_i$—then

$$f(x) - g(x) = \sum_i \kappa_i(x)(f(x_i) - p_i) + \sum_i \kappa_i(x)(f(x) - f(x_i)),$$

hence $\|f - g\| < \delta$. There remains the verification: $g \in D_\epsilon$. For this, it need only be shown that if $g(x) = g(y)$, then $\exists\ i : x, y \in U_i$. Consider the relation $\sum\limits_i(\kappa_i(x) - \kappa_i(y))p_i = 0$. Because the order of $\mathcal{U}$ is $\leq n + 1$, at most $2n + 2$ of these terms are nonzero. However, $\sum\limits_i(\kappa_i(x) - \kappa_i(y)) = 0$, from which $\kappa_i(x) - \kappa_i(y) = 0\ \forall\ i$, $\{p_i\}$ being in general position. But $\exists\ i : \kappa_i(x) > 0$. Therefore both x and y belong to U_i.

EMBEDDING THEOREM Every second countable normal Hausdorff space of topological dimension n can be embedded in $\mathbf{R}^{2n+1}$.

EXAMPLE The exponent "$2n + 1$" is sharp. Indeed, if $K = (V, \Sigma)$, where $\#(V) = 2n + 3$ and Σ is the set of all nonempty subsets of V, then $|K^{(n)}|$ cannot be embedded in $\mathbf{R}^{2n}$.

[Assuming the contrary, work with the cone $\Gamma|K^{(n)}|$ of $|K^{(n)}|$ (which would embed in $\mathbf{R}^{2n+1}$) and construct a continuous function $f : \mathbf{S}^{2n+1} \to \mathbf{R}^{2n+1}$ that does not fuse antipodal points, in violation of the Borsuk-Ulam theorem.]

EXAMPLE Suppose that X and Y are second countable normal Hausdorff spaces of finite topological dimension—then the coarse join $X *_c Y$ is a second countable normal Hausdorff space of finite topological dimension. In fact, there exist positive integers p and q such that X embeds in $\mathbf{S}^p$ and Y embeds in $\mathbf{S}^q$. Therefore $X *_c Y$ embeds in $\mathbf{S}^p *_c \mathbf{S}^q = \mathbf{S}^{p+q+1}$.

Suppose that X is a second countable compact Hausdorff space of topological dimension $n > 1$—then, from the proof of the embedding theorem, the set of embeddings $X \rightarrow \mathbf{R}^{2n+1}$ is dense in $C(X, \mathbf{R}^{2n+1})$. What can be said about the set of embeddings $X \rightarrow \mathbf{R}^{2n}$? Answer: This set can be empty (cf. supra) or nonempty and nowhere dense (cf. infra) or nonempty and dense. As regards the latter point, there is a characterization (Krasinkiewicz[†], Spiez[‡]): The set of embeddings $X \rightarrow \mathbf{R}^{2n}$ is dense in $C(X, \mathbf{R}^{2n})$ iff $\dim(X \times X) < 2n$. Examples of spaces satisfying this condition are given in §20 (cf. p. 20–20).

[Note: It can happen that $\forall\, \epsilon > 0\ \exists\, f \in C(X, \mathbf{R}^{2n})$ with $\operatorname{dev} f < \epsilon$ and yet X does not embed in $\mathbf{R}^{2n}$. Here is an example when $n = 1$. Identify $\mathbf{R}^2$ with the set of (x, y, z) in $\mathbf{R}^3 : z = 0$. Put $A = \bigcup\limits_1^\infty (1/n)\mathbf{S}^1$, $B = \{(x, 0, 0) : |x| \leq 1\} \cup \{(0, y, 0) : |y| \leq 1\}$, $C = \{(0, 0, z) : 0 \leq z \leq 1\}$ and set $X = A \cup B \cup C$. Given $\epsilon > 0$, select $k : 1/2k < \epsilon$. Denote by X_k the quotient X/K, K the subset of $A \cup B$ consisting of those points whose distance from the origin is $\leq 1/2k$. Let p be the projection $X \rightarrow X_k$, choose an embedding $f_k : X_k \rightarrow \mathbf{R}^2$ and consider $f = f_k \circ p$. Nevertheless, X cannot be embedded in $\mathbf{R}^2$.]

EXAMPLE The set of embeddings $[0, 1]^n \rightarrow \mathbf{R}^{2n}$ is nonempty and nowhere dense in $C([0, 1]^n, \mathbf{R}^{2n})$.

[Show that there exists a function $f_0 \in C([0, 1]^n, \mathbf{R}^{2n})$ and an $\epsilon_0 > 0$ such that if $f \in C([0, 1]^n, \mathbf{R}^{2n})$ and if $\|f_0 - f\| < \epsilon_0$, then f is not one-to-one.]

FACT Suppose that X is a second countable normal Hausdorff space of topological dimension n. Equip the function space $C(X, \mathbf{R}^{2n+1})$ with the limitation topology—then the set of embeddings $X \rightarrow \mathbf{R}^{2n+1}$ contains a dense G_δ in $C(X, \mathbf{R}^{2n+1})$.

Suppose that $\dim X = n$—then there is a closed embedding $X \rightarrow \mathbf{R}^{2n+1}$ iff X is second countable and locally compact. For X_∞ is second countable and $\dim X = \dim X_\infty$ (by the control lemma). Embed X_∞ in $\mathbf{R}^{2n+1}$. Add to $\mathbf{R}^{2n+1}$ a point at infinity and remove the point corresponding to $X_\infty - X$. This gives another copy of $\mathbf{R}^{2n+1}$ containing X as a closed subset.

Put $\mathbf{N}_n^{2n+1} = \mathbf{Q}_0^{2n+1} \cup \cdots \cup \mathbf{Q}_n^{2n+1}$, the subspace of $\mathbf{R}^{2n+1}$ consisting of all points with at most n rational coordinates—then $\dim \mathbf{N}_n^{2n+1} = n$.

LEMMA Every second countable normal Hausdorff space of topological dimension n can be embedded in $\mathbf{N}_n^{2n+1}$.

[†] *Fund. Math.* **133** (1989), 247–253.

[‡] *Fund. Math.* **134** (1990), 105–115; see also *Fund. Math.* **135** (1990), 127–145.

[The complement $\mathbf{R}^{2n+1} - \mathbf{N}_n^{2n+1}$ has the form $\bigcup_1^\infty H_k$, where $\forall\ k$, H_k is a plane of euclidean dimension n. Take X compact, let $D_{1/k}(H_k) = D_{1/k} \cap \{f : f(X) \cap H_k = \emptyset\}$, and consider $\bigcap_1^\infty D_{1/k}(H_k)$.]

Application: Every second countable normal Hausdorff space of topological dimension n can be written as a union of $n + 1$ subspaces, each of topological dimension ≤ 0.

[Note: Filippov[†] has constructed an example of a compact perfectly normal X : $\dim X = 1$, which cannot be written as a union $X_1 \cup X_2$, where $\begin{cases} \dim X_1 = 0 \\ \dim X_2 = 0 \end{cases}$.]

When $n = 0$, the space $\mathbf{N}_n^{2n+1}$ becomes the set of irrationals, the latter being homeomorphic to $\mathbf{N}^\omega$. The Cantor cube C_ω embeds in $\mathbf{N}^\omega$ and, as has been noted on p. 19–3, if $\begin{cases} \dim X = 0 \\ \text{wt}\, X \leq \omega \end{cases}$, then X embeds in C_ω. There is a higher dimensional counterpart to this in that one can construct a compact subspace $\mathbf{M}_n^{2n+1} \subset \mathbf{R}^{2n+1}$ of topological dimension n which embeds in $\mathbf{N}_n^{2n+1}$ and has the property that if $\begin{cases} \dim X = n \\ \text{wt}\, X \leq \omega \end{cases}$, then X embeds in $\mathbf{M}_n^{2n+1}$. In a word: Subdivide $[0,1]^{2n+1}$ into cubes of side length 1/3, retain those that meet the n-faces of $[0,1]^{2n+1}$, repeat the process on each element of their union K_0 and continue to the limit: $\mathbf{M}_n^{2n+1} = \bigcap_0^\infty K_i$ (Bothe[‡]).

Denote by $\mathbf{N}_n(\kappa)$ the subspace of $\mathbf{S}(\kappa)^\omega$ consisting of those points which have at most n nonzero rational coordinates—then $\begin{cases} \text{wt}\, \mathbf{N}_n(\kappa) = \kappa \\ \dim \mathbf{N}_n(\kappa) = n \end{cases}$.

FACT Every metrizable space X of weight $\leq \kappa$ and of topological dimension $\leq n$ can be embedded in $\mathbf{N}_n(\kappa)$.

[Note: By comparison, recall that every metrizable space X of weight $\leq \kappa$ can be embedded in $\mathbf{S}(\kappa)^\omega$ (cf. p. 6–37).]

Suppose that X is metrizable (completely metrizable) of weight κ. Equip the function space $C(X, \mathbf{S}(\kappa)^\omega)$ with the limitation topology—then Pol[‖] has shown that the set of embeddings (closed embeddings) $X \to \mathbf{S}(\kappa)^\omega$ contains a dense G_δ in $C(X, \mathbf{S}(\kappa)^\omega)$.

Can one characterize dim by a set of axioms on the class $\mathcal{E}$, the subspaces of euclidean spaces? The answer is "yes".

[†] *Soviet Math. Dokl.* **11** (1970), 687–691.

[‡] *Fund. Math.* **52** (1963), 209–224; see also Bestvina, *Memoirs Amer. Math. Soc.* **380** (1988), 1–110.

[‖] *Topology Appl.* **39** (1991), 189–204.

Consider a function $d : \mathcal{E} \to \{-1, 0, 1, \ldots\}$ subject to:

(d$_1$) (Normalization Axiom) $d(\emptyset) = -1$, $d([0,1]^n) = n$ $(n = 0, 1, \ldots)$.

(d$_2$) (Topological Invariance Axiom) If $X, Y \in \mathcal{E}$ and are homeomorphic, then $d(X) = d(Y)$.

(d$_3$) (Monotonicity Axiom) If $X, Y \in \mathcal{E}$ with $X \subset Y$, then $d(X) \leq d(Y)$.

(d$_4$) (Countable Union Axiom) If $X \in \mathcal{E}$ is the union of a sequence of closed subspaces X_i, then $d(X) \leq \sup\limits_i d(X_i)$.

(d$_5$) (Compactification Axiom) If $X \in \mathcal{E}$, then there is a compactification $\widetilde{X} \in \mathcal{E}$ of X such that $d(X) = d(\widetilde{X})$.

(d$_6$) (Decomposition Axiom) If $X \in \mathcal{E}$ and $d(X) = n$, then there exist $n+1$ sets $X_i \subset X$ such that $X = \bigcup\limits_0^n X_i$ and $\forall\, i$, $d(X_i) \leq 0$.

Hayashi[†] has shown that these axioms are independent and serve to characterize the topological dimension dim on the class $\mathcal{E}$.

[Note: The key here is the last axiom on the list. The first five are satisfied by the cohomological dimension $\dim_G$ with respect to a nonzero finitely generated abelian group G.]

While it is not true in general that an arbitrary normal X of topological dimension n can be written as a union of $n+1$ normal subspaces, each of topological dimension ≤ 0, there is nevertheless a partial substitute in that every neighborhood finite open covering of X of order $\leq n+1$ has an open refinement that can be written as a union of $n+1$ collections, each of order ≤ 1. This is a consequence of the following statement.

DECOMPOSITION LEMMA Let $\mathcal{U} = \{U_i : i \in I\}$ be a neighborhood finite open covering of X of order $\leq n+1$—then there exists an open covering $\mathcal{V}$ of X which can be represented as a union of $n+1$ collections $\mathcal{V}_0, \ldots, \mathcal{V}_n$, where $\mathcal{V}_j = \{V_{i,j} : i \in I\}$ consists of pairwise disjoint open sets such that $\forall\, i : V_{i,j} \subset U_i$.

[There is nothing to prove if $n = 0$. Proceeding by induction, assume the validity of the assertion for all normal spaces and for all neighborhood finite open coverings of order $< n+1$ $(n \geq 1)$. Choose a precise open refinement $\mathcal{O} = \{O_i : i \in I\}$ of $\mathcal{U} = \{U_i : i \in I\} : \forall\, i, A_i \equiv \overline{O}_i \subset U_i$. Put $\mathcal{F} = \{F : F \subset I \ \&\ \#(F) = n+1\}$. Assign to each $F \in \mathcal{F} : U_F = \bigcap\limits_{i \in F} U_i$ and $\begin{cases} O_F = \bigcap\limits_{i \in F} O_i \\ A_F = \bigcap\limits_{i \in F} A_i \end{cases}$. Select a point $i_F \in F$ and let $V_{i,n} = \bigcup\{U_F :$

[†] *Topology Appl.* **37** (1990), 83–92.

$i_F = i\}$—then the order of $\mathcal{V}_n = \{V_{i,n} : i \in I\}$ is ≤ 1 and $\forall\, i : V_{i,n} \subset U_i$. The subspace $Y = X - \bigcup\limits_F O_F$ is closed, hence normal. Since the order of the neighborhood finite open covering $\{Y \bigcap O_i : i \in I\}$ of Y is $\leq n$, there exists an open covering $\mathcal{V}'$ of Y which can be represented as a union of n collections $\mathcal{V}'_0, \ldots, \mathcal{V}'_{n-1}$, where $\mathcal{V}'_j = \{V'_{i,j} : i \in I\}$ consists of pairwise disjoint open sets such that $\forall\, i : V'_{i,j} \subset Y \cap O_i$. The subspace $Z = X - \bigcup\limits_F A_F$ is open ($\{A_F\}$ is neighborhood finite) and is contained in Y. For $j = 0, \ldots, n-1$, let $V_{i,j} = Z \bigcap V'_{i,j}$ and $\mathcal{V}_j = \{V_{i,j} : i \in I\}$. Consideration of the union $\mathcal{V} = \bigcup\limits_0^n \mathcal{V}_j$ completes the induction.]

PROPOSITION 11 Suppose that $\dim X \leq n$. Let $\mathcal{U} = \{U_i : i \in I\}$ be a neighborhood finite open covering of X—then there exist sequences $\begin{cases} \mathcal{V}_0, \mathcal{V}_1, \ldots \\ \mathcal{W}_0, \mathcal{W}_1, \ldots \end{cases}$ of discrete collections of open subsets $\mathcal{V}_j = \{V_{i,j} : i \in I\}$ & $\mathcal{W}_j = \{W_{i,j} : i \in I\}$ of X such that any $n+1$ of the $\mathcal{V}_j$ cover X and $\forall\, i : \overline{V}_{i,j} \subset W_{i,j} \subset U_i$.

[Bearing in mind Proposition 6, normality and the decomposition lemma provide us with the $\mathcal{V}_j$ and $\mathcal{W}_j$ for $j \leq n$. Now argue by induction, assuming that the $\mathcal{V}_j$ and $\mathcal{W}_j$ have been defined for $j \leq m-1$, $m-1$ being $\geq n$. Assign to each $M \subset \{0, \ldots, m-1\}$ of cardinality n the closed subset $A_M = X - \bigcup\limits_{j \in M} \cup \mathcal{V}_j$—then the A_M are pairwise disjoint because any $n+1$ of the $\mathcal{V}_j$ cover X. Determine open $\begin{cases} V_M \\ W_M \end{cases} : A_M \subset V_M \subset \overline{V}_M \subset W_M$, where $M' \neq M'' \Rightarrow \overline{W}_{M'} \cap \overline{W}_{M''} = \emptyset$. Select a point $j_M \leq m-1 : j_M \notin M$. Note that $A_M \subset \cup \mathcal{V}_{j_M}$. Put $\begin{cases} V_{i,m} = \bigcup\limits_M V_M \cap V_{i,j_M} \\ W_{i,m} = \bigcup\limits_M W_M \cap W_{i,j_M} \end{cases}$. The associated collections $\mathcal{V}_m$ and $\mathcal{W}_m$ are discrete and open with $\overline{V}_{i,m} \subset W_{i,m} \subset U_i$. And since any n of the $\mathcal{V}_j$ $(j \leq m-1)$ cover $X - \bigcup\limits_M A_M$, any $n+1$ of the $\mathcal{V}_j$ $(j \leq m)$ cover X.]

The Kolmogorov superposition theorem, which resolved Hilbert's 13th problem in the negative, says that for each $n \geq 1$ there exist functions $\phi_1, \ldots, \phi_{2n+1}$ in $C([0,1]^n)$ such that every $f \in C([0,1]^n)$ can be represented in the form $f = \sum\limits_i g_i \circ \phi_i$ for certain $g_i \in C(\mathbf{R})$ (depending on f). Objective: Isolate the dimension theoretic content of this result.

Suppose that X is a second countable compact Hausdorff space. Let $\phi_i \in C(X)$ $(i = 1, \ldots, k)$—then the collection $\{\phi_i\}$ is said to be basic if for every $f \in C(X)$ there exist continuous functions $g_i : \mathbf{R} \to \mathbf{R}$ such that $f = \sum\limits_i g_i \circ \phi_i$. A basic embedding of X in $\mathbf{R}^k$ is an embedding $X \to \mathbf{R}^k$ corresponding to a basic collection $\{\phi_i\}$. So, e.g., according to Kolmogorov, $X = [0,1]^n$ can be basically embedded in $\mathbf{R}^{2n+1}$.

BASIC EMBEDDING THEOREM Every second countable compact Hausdorff space of topological dimension n can be basically embedded in $\mathbf{R}^{2n+1}$.

[Note: Sternfeld[†] has shown that if $\dim X = n$ $(n > 1)$, then X cannot be basically embedded in $\mathbf{R}^{2n}$. Example: Let $X = \{(x,0) : |x| \leq 1\} \cup \{(0,y) : |y| \leq 1\}$—then $\dim X = 1$ and X can be basically embedded in $\mathbf{R}^2$.]

The proof of the basic embedding theorem is not a general position argument. It depends instead on Proposition 11 and some elementary functional analysis.

There is a simple interpretation of what it means for $\{\phi_i\}$ to be basic in terms of the dual $C(X)^*$ of $C(X)$. Thus put $Y_i = \phi_i(X)$ and let $Y = \coprod_i Y_i$—then the collection $\{\phi_i\}$ determines a bounded linear operator $T : C(Y) \to C(X)$, viz. $T(g_1, \ldots, g_k) = \sum_i g_i \circ \phi_i$, with adjoint $T^* : C(X)^* \to C(Y)^*$, viz. $T^*\mu = \sum_i \mu_i$, μ_i the image of μ under ϕ_i. Note that $\|T^*\mu\| = \sum_i \|\mu_i\|$. Obviously, $\{\phi_i\}$ is basic iff T is surjective or still, iff $\exists\, \lambda : 0 < \lambda \leq 1$ such that $\forall\, \mu \in C(X)^*\ \exists\, i : \|\mu_i\| \geq \lambda\|\mu\|$. When this occurs, call $\{\phi_i\}$ λ-basic.

Fix a compatible metric d on X. Given a finite discrete collection $\mathcal{U} = \{U\}$ of open subsets of X, we shall write $d(\mathcal{U})$ for $\sup\{\operatorname{diam} U : U \in \mathcal{U}\}$ and agree that a function $\phi \in C(X)$ separates $\mathcal{U}$ if $\forall\, U \neq V$ in $\mathcal{U} : \phi(\overline{U}) \cap \phi(\overline{V}) = \emptyset$.

LEMMA Let $\phi_i \in C(X)$ $(i = 1, \ldots, k)$. Suppose that $\forall\, \epsilon > 0$ and $\forall\, i$, there exists a finite discrete collection $\mathcal{U}_i$ of open subsets of X with $d(\mathcal{U}_i) < \epsilon$ such that ϕ_i separates $\mathcal{U}_i$ and

$$\forall\, x \in X : \sum_i \operatorname{ord}(x, \mathcal{U}_i) \geq \left[\frac{k}{2}\right] + 1.$$

Then $\{\phi_i\}$ is $1/k$-basic.

[The set of $\mu \in C(X)^*$ for which $\operatorname{spt}(\mu^+) \cap \operatorname{spt}(\mu^-) = \emptyset$ is dense in $C(X)^*$ (Hahn plus regularity). Therefore take a $\mu \in C(X)^*$ of norm one, assume that $\epsilon = d(\operatorname{spt}(\mu^+), \operatorname{spt}(\mu^-)) > 0$, and choose the $\mathcal{U}_i$ accordingly. If as usual $|\mu| = \mu^+ + \mu^-$, then $|\mu|$ is a probability measure on X and $\sum_i |\mu|(\cup\, \mathcal{U}_i) \geq [k/2] + 1$, implying that for some i_0, $|\mu|(\cup \mathcal{U}_{i_0}) \geq (1/k)([k/2] + 1) \geq 1/2 + 1/2k$. On the other hand, $\forall\, U \in \mathcal{U}_{i_0}, |\mu|(U) = |\mu(U)|$, thus $|\mu|(\cup\, \mathcal{U}_{i_0}) = \sum_U |\mu(U)|$ and so $\|\mu_{i_0}\| \geq 1/2 + 1/2k - |\mu|(X - \cup\, \mathcal{U}_{i_0}) \geq 1/k$.]

Let $\mathcal{U}(p)$ be a finite discrete collection of open subsets of X with $d(\mathcal{U}(p)) < 1/p$ $(p = 1, 2, \ldots)$. Claim: There exists a dense set of $\phi \in C(X)$ separating $\mathcal{U}(p)$ for infinitely many p. To see this, let Φ_q be the set of $\phi \in C(X)$ separating $\mathcal{U}(p)$ for some $p \geq q$ ($q = 1, 2, \ldots$)—then it need only be shown that $\forall\ q$, Φ_q is open and dense in $C(X)$ (consider $\bigcap\limits_1^\infty \Phi_q$ and quote Baire).

[†] *Israel J. Math.* **50** (1985), 13–53; see also Levin, *Israel J. Math.* **70** (1990), 205–218.

(1) Φ_q is open in $C(X)$. Proof: Let $\phi \in \Phi_q$. Choose p per ϕ. Let $2\epsilon = \inf\{\text{dis}(\phi(\overline{U}), \phi(\overline{V})) : U \neq V \text{ in } \mathcal{U}(p)\}$. Suppose that $\|\phi - f\| < \epsilon/4$—then $U \neq V$ in $\mathcal{U}(p) \Rightarrow \text{dis}(f(\overline{U}), f(\overline{V})) > \epsilon$.

(2) Φ_q is dense in $C(X)$. Proof: Fix $f \in C(X)$. Given $\epsilon > 0$, choose $p \geq q$: $\text{osc}(f|\overline{U}) < \epsilon/2$ ($U \in \mathcal{U}(p)$). Define a continuous function $g : \cup\overline{U} \rightarrow \mathbf{R}$ by picking distinct constants c_U: $\begin{cases} g|\overline{U} = c_U \\ \|f|\overline{U} - g|\overline{U}\| < \epsilon \end{cases}$. Use Tietze and extend $f|\cup\overline{U} - g$ to an $h \in C(X)$: $\|h\| < \epsilon$. Put $\phi = f - h$: $\phi \in \Phi_q$ & $\|f - \phi\| < \epsilon$.

To prove the basic embedding theorem, take $k = 2n + 1$—then, in view of Proposition 11, there exist finite discrete collections $\mathcal{U}_i(p)$ ($i = 1, \ldots, k$) of open subsets of X with $d(\mathcal{U}_i(p)) < 1/p$ ($p = 1, 2, \ldots$) such that for each p the union of any $n + 1$ of the $\mathcal{U}_i(p)$ is a covering of X, so

$$\forall\ x \in X : \sum_i \text{ord}(x, \mathcal{U}_i(p)) \geq \left[\frac{k}{2}\right] + 1.$$

Thanks to the preceding remarks, it is possible to select integers $p_1 < p_2 < \cdots$ and functions $\phi_i \in C(X)$ ($i = 1, \ldots, k$) having the property that ϕ_i separates $\mathcal{U}_i(p_j)$ ($j = 1, 2, \ldots$). Apply the lemma and conclude that $\{\phi_i\}$ is $1/k$-basic ($k = 2n + 1$).

When $X = [0,1]^n$, one can explicate, at least to some extent, the analytic structure of the ϕ_i. Precisely put: Given rationally independent real numbers $r_1, \ldots, r_n$, there exist increasing continuous functions $\psi_1, \ldots, \psi_{2n+1}$ on $[0,1]$ such that the

$$\phi_i(x_1, \ldots, x_n) = \sum\nolimits_{j=1}^n r_j \psi_i(x_j) \ (1 \leq i \leq 2n + 1)$$

constitute a $1/k$-basic collection ($k = 2n + 1$). Moreover, the g_i can be chosen independently of i, so $\forall\ f \in C([0,1]^n)$ there exists a $g \in C(\mathbf{R})$:

$$f(x_1, \ldots, x_n) = \sum\nolimits_{i=1}^{2n+1} g\left(\sum\nolimits_{j=1}^n r_j \psi_i(x_j)\right).$$

[Note: The "inner functions" can even be taken in $\text{Lip}_1([0,1])$. Reason: There exists a homeomorphism $\iota : [0,1] \rightarrow [0,1]$ such that $\forall\ i, \psi_i \circ \iota \in \text{Lip}_1([0,1])$. Consider, e.g., the inverse to the assignment $x \rightarrow C(x + \sum\limits_i(\psi_i(x) - \psi_i(0)))$, where C is the reciprocal of $1 + \sum\limits_i(\psi_i(1) - \psi_i(0))$.]

To avoid trivialities, assume that $n > 1$. There are then three steps to the proof.

(I) For $p = 1, 2, \ldots$, partition $[0,1]$ into p closed subintervals I of length $1/p$ indexed by the natural order and for $1 \leq i \leq k$, let $\mathcal{I}_i(p)$ denote the collection of closed subintervals of $[0,1]$ obtained by removing from $[0,1]$ the interior of those I whose index is congruent to $i \bmod k$. Write $C_i(p)$ for the set of all products $C_i(p) = I_1(p) \times \cdots \times I_n(p) : \forall\, j, I_j(p) \in \mathcal{I}_i(p)$. It is clear that $\mathcal{C}_i(p)$ is a discrete collection of closed n-cubes in $[0,1]^n$. Furthermore, every $x \in [0,1]^n$ belongs to at least $[k/2]+1 \equiv n+1$ of the $\cup\mathcal{C}_i(p)$.

(II) Let Ψ stand for the set of increasing continuous functions on $[0,1]$, equipped with the uniform norm. Attach to each $\epsilon > 0 : 0 < \epsilon < 1/2k$, and to each $f \in C([0,1]^n) : ||f|| \neq 0$, the set $\Omega_f(\epsilon)$ of all $\{\psi_i\} \in \Psi^k$ for which there exists an $h \in C(\mathbf{R}) : ||h|| \leq ||f||$ & $||f - \sum\limits_i h(\sum\limits_j r_j\psi_i)|| < (1-\epsilon)||f||$. Claim: $\Omega_f(\epsilon)$ is open and dense. Of course, only the density is at issue. And for this, it suffices to fix a nonempty open $\Omega \subset \Psi^k$ and show that $\Omega \cap \Omega_f(\epsilon) \neq \emptyset$. Let $\Psi^k(p)$ be the subset of Ψ^k consisting of the $\{\psi_i\}$ such that $\forall\, i : \psi_i$ is constant on the elements of $\mathcal{I}_i(p)$. Choose $p \gg 0 : \Omega \cap \Psi^k(p) \neq \emptyset$ & $\operatorname{osc}(f|C_i(p)) < \epsilon||f|| \ \forall\, C_i(p) \in \mathcal{C}_i(p)$. Fix $\{\psi_i\} \in \Omega \cap \Psi^k(p)$. Because the r_j are rationally independent, there is no loss of generality in supposing that $\phi_i \equiv \sum\limits_j r_j\psi_i$ takes different values on different elements of $\mathcal{C}_i(p)$ and that in addition these values are distinct for distinct i. We shall now construct an $h \in C(\mathbf{R})$ in terms of the ϕ_i and deduce that $\{\psi_i\} \in \Omega_f(\epsilon)$. Call M_i the value of f at the center of $C_i(p)$. Let $h(\phi_i(C_i(p))) = 2\epsilon M_i$ and extend h continuously to all of $\mathbf{R} : ||h|| \leq 2\epsilon||f||$. Using the fact that every $x \in [0,1]^n$ belongs to at least $n+1$ of the $\cup\mathcal{C}_i(p)$, one has

$$|f(x) - \sum_i h(\phi_i(x))| \leq (1 - 2(n+1)\epsilon)|f(x)| + 2(n+1)\epsilon^2||f|| + 2n\epsilon||f||$$

$$\leq (1 - 2\epsilon + 2(n+1)\epsilon^2)||f|| < (1-\epsilon)||f||.$$

Therefore $\{\psi_i\} \in \Omega_f(\epsilon)$.

(III) Let $D = \{f_d\}$ be a countable dense subset of $C([0,1]^n)$, not containing the zero function—then $\bigcap\limits_1^\infty \Omega_{f_d}(\epsilon)$ is dense in Ψ^k (Baire). Fix $\{\psi_i\} \in \bigcap\limits_1^\infty \Omega_{f_d}(\epsilon)$. Let $f \in C([0,1]^n) : ||f|| \neq 0$. Choose $f_d \in D : ||(1-\epsilon/4)f - f_d|| < (\epsilon/4)||f||$, so $\begin{cases} ||f_d|| \leq ||f|| \\ ||f - f_d|| < (\epsilon/2)||f|| \end{cases}$ and choose $h_d \in C(\mathbf{R}) : ||h_d|| \leq ||f_d||$ & $||f_d - \sum\limits_i h_d(\sum\limits_j r_j\psi_i)|| < (1-\epsilon)||f_d||$. Conclusion: $\exists\, h = \gamma(f) \in C(\mathbf{R})$ such that $||h|| \leq ||f||$ & $||f - \sum\limits_i h(\sum\limits_j r_j\psi_i)|| \ (1-\epsilon/2)||f||$. Recursively define a sequence $\chi_0, \chi_1, \ldots$ in $C([0,1]^n)$ by $\chi_0 = f$, $\chi_{m+1} = \chi_m - \sum\limits_i h_m(\sum\limits_j r_j\psi_i)$, where $h_m = \gamma(\chi_m)$ $(\gamma(0) = 0)$. The series $\sum\limits_0^\infty h_m$ is uniformly convergent, thus its sum g is continuous and satisfies the relation $f = \sum\limits_i g(\sum\limits_j r_j\psi_i)$.

[Note: Let $C^1([0,1]^n)$ be the set of continuously differentiable functions on $[0,1]^n$—then Kaufman† has shown that for $n > 1$, no finite subset of $C^1([0,1]^n)$ can be basic.]

FACT There exist real valued continuous functions ϕ_i $(i = 1, \ldots, 2n+1)$ on $\mathbf{R}^n$ such that $\forall\, f \in BC(\mathbf{R}^n)\ \exists\, g \in C(\mathbf{R}) : f = \sum\limits_i g \circ \phi_i$.

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† *Proc. Amer. Math. Soc.* **46** (1974), 360–362.

[Note: This result remains true if $\mathbf{R}^n$ is replaced by a noncompact second countable LCH space X of topological dimension n.]

If X and Y are nonempty normal Hausdorff spaces, what is the relation between $\dim(X \times Y)$ and $\begin{cases} \dim X \\ \dim Y \end{cases}$? An initial difficulty is that $X \times Y$ need not be normal so formally $\dim(X \times Y)$ can be undefined.

This is not a serious problem. Reason: $X \times Y$ is at least completely regular, therefore in this context $\dim(X \times Y)$ is meaningful (cf. p. 19–2).

Examples: (1) Take $X = Y =$ Sorgenfrey line—then X is perfectly normal and paracompact but $X \times X$ is not normal (cf. p. 5–11); (2) Take $X = [0, \Omega[$, $Y = [0, \Omega]$—then X is normal and Y is compact but $X \times Y$ is not normal; (3) Take $X =$ Michael line, $Y = \mathbf{P}$—then X is paracompact and Y is metrizable but $X \times Y$ is not normal (cf. p. 6–8 ff.); (4) Take $X =$ Rudin's Dowker space, $Y = [0,1]$—then $X \times [0,1]$ is not normal.

Here are some conditions on X and Y that ensure that the product $X \times Y$ is normal.

(1) Suppose that X is perfectly normal (perfectly normal and paracompact) and Y is metrizable—then $X \times Y$ is perfectly normal (perfectly normal and paracompact).

(2) Suppose that X is normal and countably compact and Y is metrizable—then $X \times Y$ is normal.

(3) Suppose that X is normal and countably paracompact and Y is metrizable and σ-locally compact—then $X \times Y$ is normal.

(4) Suppose that X is paracompact and Y is paracompact and σ-locally compact—then $X \times Y$ is paracompact.

[Note: A CRH space is said to be σ-locally compact if it can be written as a countable union of closed locally compact subspaces. Example: Every CW complex is σ-locally compact.]

If enough pathology is built into X and Y, then it can happen that $\dim X + \dim Y < \dim(X \times Y)$. Examples illustrating the point are given below. Because of this, one looks instead for conditions on X and Y that serve to force $\dim(X \times Y) \leq \dim X + \dim Y$.

PRODUCT THEOREM Suppose that X is normal and Y is paracompact and σ-locally compact. Assume: $X \times Y$ is normal—then $\dim(X \times Y) \leq \dim X + \dim Y$.

[Note: Tacitly, $X \neq \emptyset$ & $Y \neq \emptyset$.]

The inequality in the product theorem can be strict even if X and Y are compact ARs (Dranishnikov[†]).

The proof of the product theorem is carried out in stages under the supposition that $\begin{cases} n = \dim X \\ m = \dim Y \end{cases} < \infty.$

PROPOSITION 12 Suppose that both X and Y are compact—then $\dim(X \times Y) \leq \dim X + \dim Y$.

[Let $\mathcal{W}$ be a finite open covering of $X \times Y$. Choose finite open coverings $\begin{cases} \mathcal{U} \\ \mathcal{V} \end{cases}$ of $\begin{cases} X \\ Y \end{cases} : \mathcal{U} \times \mathcal{V}$ refines $\mathcal{W}$. Attach to $\begin{cases} \mathcal{U} \\ \mathcal{V} \end{cases}$ sequences $\begin{cases} \mathcal{O}_0, \mathcal{O}_1, \ldots \\ \mathcal{P}_0, \mathcal{P}_1, \ldots \end{cases}$ of discrete collections of open subsets of $\begin{cases} X \\ Y \end{cases}$ having the properties delineated in Proposition 11. In particular: Each $x \in X$ can fail to belong to at most n of the $\cup \mathcal{O}_k$ and each $y \in Y$ can fail to belong to at most m of the $\cup \mathcal{P}_k$. The union $\mathcal{O}_0 \times \mathcal{P}_0 \bigcup \cdots \bigcup \mathcal{O}_{n+m} \times \mathcal{P}_{n+m}$ is therefore an open refinement of $\mathcal{U} \times \mathcal{V}$ of order $\leq n + m + 1$.]

If X and Y are compact and metrizable and if $f : X \rightarrow Y$ is continuous and surjective, then there exists a Baire class one function $g : Y \rightarrow X$ such that $f \circ g = \mathrm{id}_Y$ (Engelking[‡]). Since $g \circ f$ is a function of the first Baire class, its graph is a G_δ in $X \times X$, which implies that the range of g, viz. $\{x : g(f(x)) = x\}$, is a G_δ in X that intersects each fiber of f in exactly one point.

EXAMPLE Let $\mathcal{K}$ be the collection of all nonempty closed subsets of $[0,1] \times [0,1]$ equipped with the Vietoris topology, so $\mathcal{K}$ is compact and metrizable. Write p for the vertical projection—then the collection $\mathcal{C}$ of all compact connected subsets of $[0,1] \times [0,1]$ that meet both $p^{-1}(0)$ and $p^{-1}(1)$ is a closed subspace of $\mathcal{K}$, hence is compact. Therefore there exists a continuous surjection Γ from the Cantor set $C \subset [0,1]$ to $\mathcal{C}$. Because $C \times C$ is homeomorphic to C, one can assume that the fibers of Γ have cardinality 2^ω. If now $X = \bigcup \{p^{-1}(t) \cap \Gamma(t) : t \in C\}$, then X is a compact subspace of $[0,1] \times [0,1]$ and $f \equiv p|X : X \rightarrow C$ is surjective. From the remark above, there exists a Baire class one function $g : C \rightarrow X$ such that $f \circ g = \mathrm{id}_C$. Define $\phi : C \rightarrow [0,1]$ by $g(t) = (t, \phi(t)) : \phi$ is a function of the first Baire class and its graph gr_ϕ is a G_δ in X that intersects each fiber of f in exactly one point. Consequently, gr_ϕ is completely metrizable, thus is a G_δ in $C \times [0,1]$. Note too that gr_ϕ is totally disconnected and intersects each element of $\mathcal{C}$ in a set of cardinality 2^ω. Claim: $\dim \mathrm{gr}_\phi = 1$. In fact, by Proposition 12, $\dim \mathrm{gr}_\phi \leq \dim C + \dim[0,1] = 0 + 1 = 1$.

[†] *Soviet Math. Dokl.* **37** (1988), 769–773.

[‡] *Bull. Acad. Polon. Sci.* **16** (1968), 277–282.

To see that $\dim \mathrm{gr}_\phi > 0$, write q for the horizontal projection, put $\begin{cases} A = \mathrm{gr}_\phi \cap q^{-1}([0,1/7]) \\ B = \mathrm{gr}_\phi \cap q^{-1}([6/7,1]) \end{cases}$ and let U be any open subset of gr_ϕ : $\begin{cases} A \subset U \\ B \cap \overline{U} = \emptyset \end{cases}$ —then $\#(\mathrm{fr}\, U) = 2^\omega$.

[Note: Working instead with $[0,1]^{n+1} = [0,1] \times [0,1]^n$, one can modify the preceding construction and produce an example of a second countable completely metrizable totally disconnected space of topological dimension n. Such a space cannot contain a compact Cantor n-space (cf. p. 19–25).]

FACT Let X and Y be nonempty CRH spaces. Suppose that $X \times Y$ is strongly paracompact—then $\dim(X \times Y) \leq \dim X + \dim Y$.

[View $X \times Y$ as a subspace of $\beta X \times \beta Y$ to get $\dim(X \times Y) \leq \dim(\beta X \times \beta Y)$ (cf. p. 19–12), which is $\leq \dim \beta X + \dim \beta Y$ (cf. Proposition 12) or still, $\leq \dim X + \dim Y$ (cf. Proposition 1).]

[Note: Is it sufficient that $X \times Y$ be paracompact? The answer is unknown.]

Application: Suppose that X and Y are second countable and metrizable—then $\dim(X \times Y) \leq \dim X + \dim Y$.

EXAMPLE Take for X the subspace of l^2 consisting of all sequences $\{x_n\}$, with x_n rational—then $\dim X = 1$. But X is homeomorphic to $X \times X$, so $\dim(X \times X) = 1$, which is $< 2 = \dim X + \dim X$.

[Note: Given any $n \in \mathbf{N}$, there exists an $X \subset \mathbf{R}^{n+1}$ such that $\dim X = \dim(X \times X) = n$ (Anderson-Keisler[†]).]

FACT Let X and Y be nonempty CRH spaces. Suppose that X and Y are infinite and $X \times Y$ is pseudocompact—then $\dim(X \times Y) \leq \dim X + \dim Y$.

[Glicksberg's theorem says that if X and Y are infinite CRH spaces, then the product $X \times Y$ is pseudocompact iff $\beta(X \times Y) = \beta X \times \beta Y$, the equal sign meaning that the two compactifications of $X \times Y$ are equivalent (and not just homeomorphic). Recall that the product of two pseudocompact spaces need not be pseudocompact but this will be the case if one of the factors is compactly generated. Example: $\dim([0,\Omega[\times[0,\Omega]) = 0$.]

PROPOSITION 13 Suppose that X is a CW complex and Y is compact—then $\dim(X \times Y) \leq \dim X + \dim Y$.

[Argue by induction on $\dim X$. There is nothing to prove if $\dim X = 0$. If $\dim X > 0$, then, since the combinatorial and topological dimensions of X coincide (cf. p. 19–21), $X = X^{(n)}$. Thus one can write $X = X^{(n-1)} \cup \bigcup_1^\infty A_j$, where each A_j is closed and expressible as a disjoint union $\bigcup\limits_i K_{i,j}$, $\{K_{i,j}\}$ being a discrete collection of compacta, with $\dim K_{i,j} \leq n$. From the induction hypothesis, $\dim(X^{(n-1)} \times Y) \leq \dim X^{(n-1)} + \dim Y \leq n - 1 + m$. On the other hand, Proposition 12 implies that $\dim(K_{i,j} \times Y) \leq \dim K_{i,j} + \dim Y \leq n + m$, so $\dim(A_j \times Y) \leq n + m$. Now apply the countable union lemma.]

[†] *Proc. Amer. Math. Soc.* **18** (1967), 709–713.

STACKING LEMMA Let X and Y be nonempty CRH spaces. Suppose that Y is compact—then for every numerable open covering $\mathcal{W}$ of $X \times Y$, there exists a numerable open covering $\mathcal{U} = \{U_i : i \in I\}$ of X and $\forall\, i \in I$, a finite open covering $\mathcal{V}_i = \{V_{i,j} : j \in J_i\}$ of Y such that the collection $\{U_i \times \mathcal{V}_i : i \in I\}$ refines $\mathcal{W}$.

[The assertion is trivial if X is paracompact. In general, there exists a metric space Z, an open covering $\mathcal{Z}$ of Z, and a continuous function $f : X \times Y \rightarrow Z$ such that $f^{-1}(\mathcal{Z})$ refines $\mathcal{W}$ (cf. p. 1–25). Define $e : C(Y, Z) \times Y \rightarrow Z$ by $e(\phi, y) = \phi(y)$—then $e^{-1}(\mathcal{Z})$ is a numerable open covering of $C(Y, Z) \times Y$. Since $C(Y, Z) \times Y$ is paracompact, one can find a numerable open covering $\mathcal{O} = \{O_i : i \in I\}$ of $C(Y, Z)$ and $\forall\, i \in I$, a finite open covering $\mathcal{V}_i = \{V_{i,j} : j \in J_i\}$ of Y such that the collection $\{O_i \times \mathcal{V}_i : i \in I\}$ refines $e^{-1}(\mathcal{Z})$. Put $F(x)(y) = f(x, y) : F \in C(X, C(Y, Z))$ & $f = e \circ (F \times \mathrm{id}_Y)$. Consider $\mathcal{U} = \{U_i : i \in I\}$, where $U_i = F^{-1}(O_i)$.]

[Note: The complete regularity of X plays no role in the proof.]

To establish the product theorem, first employ the countable union lemma and make the obvious reductions to the case when Y is compact. This done, let $\mathcal{W}$ be a finite open covering of $X \times Y$. According to the stacking lemma, there exists a neighborhood finite open covering $\mathcal{U} = \{U_i : i \in I\}$ of X and for each $i \in I$, a finite open covering $\mathcal{V}_i = \{V_{i,j} : j \in J_i\}$ of Y such that the collection $\{U_i \times \mathcal{V}_i : i \in I\}$ refines $\mathcal{W}$. Fix a precise open refinement $\mathcal{O} = \{O_i : i \in I\}$ of $\mathcal{U}$ of order $\leq n + 1$ (cf. Proposition 6)—then $\dim |N(\mathcal{O})| \leq n$, $N(\mathcal{O})$ the nerve of $\mathcal{O}$. Choose an $\mathcal{O}$-map f, i.e., a continuous function $f : X \rightarrow |N(\mathcal{O})|$ with the property that $\forall\, O_i \in \mathcal{O} : (b_{O_i} \circ f)^{-1}\ (]0, 1]) \subset O_i$ (cf. p. 5–3). Put $F = f \times \mathrm{id}_Y$. Since $\dim(|N(\mathcal{O})| \times Y) \leq n + m$ (cf. Proposition 13), the open covering $\{b_{O_i}^{-1}(]0, 1]) \times \mathcal{V}_i : i \in I\}$ of $|N(\mathcal{O})| \times Y$ has an open refinement $\mathcal{P}$ of order $\leq n + m + 1$. Consider $F^{-1}(\mathcal{P})$.

The product theorem holds if X is merely completely regular. Indeed, once the reductions to the case "Y compact" have been carried out, the argument proceeds as when X is normal. The reductions depend in turn on the countable union lemma which retains its validity in the completely regular situation provided the subspaces in question have the EP w.r.t. $[0, 1]$ (cf. p. 19–13). Two results are relevant for the transition.

LEMMA Let X be a topological space. Let B be a compact subspace of a CRH space Y—then $X \times B$, as a subspace of $X \times Y$, has the EP w.r.t. $[0,1]$.

[Recalling that $B \subset Y$ has the EP w.r.t. $[0,1]$ (cf. p. 6–4), let $\mathcal{O}$ be a finite numerable open covering of $X \times B$. Use the stacking lemma and construct a numerable open covering $\mathcal{W}$ of $X \times Y$ such that $\mathcal{W} \cap (X \times B)$ is a refinement of $\mathcal{O}$. Apply §6, Proposition 4 (the proof of sufficiency does not require a cardinality assumption on $\mathcal{W}$).]

LEMMA Let X be a topological space. Let B be a closed subspace of a paracompact LCH space Y—then $X \times B$, as a subspace of $X \times Y$, has the EP w.r.t. $[0,1]$.

[Note: Paracompactness of Y alone is not enough. Example: Take $X = \mathbf{P}$, $Y =$ Michael line and $B = \mathbf{Q}$—then $X \times B$, as a subspace of $X \times Y$, does not have the EP w.r.t. $[0,1]$. One can, however, drop local compactness if some other assumption on Y is imposed, e.g., stratifiability.]

Its utility notwithstanding, there are limitations to the product theorem. For example, it is not necessarily applicable if both factors are metrizable. However, this possibility (and others) can be readily placed in a general framework.

Let X and Y be nonempty CRH spaces—then a cozero set rectangle in $X \times Y$ is a set of the form $U \times V$, where $\begin{cases} U \\ V \end{cases}$ is a cozero set in $\begin{cases} X \\ Y \end{cases}$.

LEMMA $X \times Y$ is $\mathcal{Z}$-embedded in $X \times \beta Y$ iff every cozero set in $X \times Y$ can be written as the union of a collection of cozero set rectangles $U \times V$, where $\{U\}$ is σ-neighborhood finite.

[Use the stacking lemma and the fact that the union of a σ-neighborhood finite collection of cozero sets is a cozero set.]

[Note: $X \times Y$ is $\mathcal{Z}$-embedded in $\beta X \times \beta Y$ iff every cozero set in $X \times Y$ can be written as the union of a countable collection of cozero set rectangles $U \times V$.]

The following conditions are equivalent.

(a) Every cozero set in $X \times Y$ can be written as the union of a collection of cozero set rectangles $U \times V$, where $\{U\}$ is σ-neighborhood finite.

(b) Given any $f \in C(X \times Y)$ and any $\epsilon > 0$, there exists a covering of $X \times Y$ by cozero set rectangles $U \times V$ such that $\text{osc}(f|U \times V) < \epsilon$ and $\{U\}$ is σ-neighborhood finite.

[(a) $\Rightarrow$ (b): Fix a sequence of open intervals $]a_n, b_n[$, each of length $< \epsilon/2 : \mathbf{R} = \bigcup\limits_1^\infty\,]a_n, b_n[$—then $X \times Y = \bigcup\limits_1^\infty f^{-1}(]a_n, b_n[)$. Write $f^{-1}(]a_n, b_n[)$ as the union of a collection of cozero set rectangles $U_i \times V_i$, where $\{U_i : i \in I_n\}$ is σ-neighborhood finite. Obviously, $\text{osc}(f|U_i \times V_i) < \epsilon$ and $\bigcup\limits_1^\infty \{U_i : i \in I_n\}$ is σ-neighborhood finite.

(b) $\Rightarrow$ (a): Take an $f \in C(X \times Y)$. Pick a cozero set rectangle covering $\mathcal{W}_n = \{U \times V\}$ of $X \times Y$ such that $\operatorname{osc}(f|U \times V) < 1/n$ and $\{U\}$ is σ-neighborhood finite. Denote by $\mathcal{W}_n(f)$ the subset of $\mathcal{W}_n$ consisting of the $U \times V$ that are contained in $X \times Y - Z(f)$—then $\bigcup_1^\infty \mathcal{W}_n(f)$ covers $X \times Y - Z(f)$.]

Assume: Every open subset of $\begin{cases} X \\ Y \end{cases}$ is $\mathcal{Z}$-embedded in $\begin{cases} X \\ Y \end{cases}$—then (a) and (b) above are equivalent to the following conditions.

(a)$_{\mathcal{Z}}$ Every cozero set in $X \times Y$ can be written as the union of a collection of open rectangles $U \times V$, where $\{U\}$ is σ-neighborhood finite.

(b)$_{\mathcal{Z}}$ Given any $f \in C(X \times Y)$ and any $\epsilon > 0$, there exists a covering of $X \times Y$ by open rectangles $U \times V$ such that $\operatorname{osc}(f|U \times V) < \epsilon$ and $\{U\}$ is σ-neighborhood finite.

[That (a) $\Rightarrow$ (a)$_{\mathcal{Z}}$ is clear, as is (a)$_{\mathcal{Z}}$ $\Rightarrow$ (b)$_{\mathcal{Z}}$. To prove that (b)$_{\mathcal{Z}}$ $\Rightarrow$ (b), let $f \in C(X \times Y)$ and $\epsilon > 0$ but with $\operatorname{osc}(f|U \times V) < \epsilon/2$. The assumption on $\begin{cases} X \\ Y \end{cases}$ implies that the interior of $\begin{cases} \overline{U} \\ \overline{V} \end{cases}$ is a cozero set in $\begin{cases} X \\ Y \end{cases}$. The corresponding collection of cozero set rectangles thereby produced covers $X \times Y$ and the oscillation of f on any one of them is $< \epsilon$.]

In a CRH space, every open subset is $\mathcal{Z}$-embedded iff every open subset which is the interior of its closure is cozero. The latter property is evidently a weakening of perfect normality and, e.g., is possessed by an arbitrary product of metrizable spaces (Ščepin[†]) but not by $[0, \Omega[$ or $\beta\mathbf{R}$.

LEMMA Suppose that X is metrizable and that every open subset of Y is $\mathcal{Z}$-embedded in Y—then $X \times Y$ is $\mathcal{Z}$-embedded in $X \times \beta Y$.

[It suffices to check (b)$_{\mathcal{Z}}$, so let $f \in C(X \times Y)$ and $\epsilon > 0$. Enumerate $\mathbf{Q} : \{q_n\}$ and put $I_n =]q_n - \epsilon/3, q_n + \epsilon/3[$. Fix a σ-neighborhood finite basis $\{U\}$ for X. Let $Y(U, n)$ be the subset of Y made up of those points which admit a neighborhood $V : f(U \times V) \subset I_n$—then $Y(U, n)$ is open in Y, $\operatorname{osc}(f|U \times Y(U, n)) < \epsilon$, and since $\forall\ (x, y) \in X \times Y\ \exists\ q_n \in \mathbf{Q} : |f(x, y) - q_n| < \epsilon/6$, the open rectangles $U \times Y(U, n)$ cover $X \times Y$.]

FACT Let X and Y be nonempty CRH spaces. Suppose that $X \times Y$ is $\mathcal{Z}$-embedded in $X \times \beta Y$—then $\dim(X \times Y) \leq \dim X + \dim Y$.

[Simply note that $\dim(X \times Y) \leq \dim(X \times \beta Y)$ (cf. p. 19–13), which, by the product theorem, is $\leq \dim X + \dim \beta Y = \dim X + \dim Y$.]

Application: Suppose that X and Y are metrizable—then $\dim(X \times Y) \leq \dim X + \dim Y$.

[†] *Soviet Math. Dokl.* **17** (1976), 152–155; see also Blair-Swardson, *Topology Appl.* **36** (1990), 73–92.

EXAMPLE Let X and Y be nonempty M complexes—then $X \times_k Y$ is an M complex and $\dim(X \times_k Y) \leq \dim X + \dim Y$.

[Assume first that X is an M_n space and Y is an M_m space, proceeding by induction on $n + m$.]

That dim is monotonic on $\mathcal{Z}$-embedded subspaces is the key to the preceding method. But one can get away with even less. In general, a subspace A of a topological space X is said to be weakly $\mathcal{Z}$-embedded in X if for any cozero set O in A there exists a σ-neighborhood finite collection $\{O_i : i \in I\}$ of cozero sets O_i in A, each of which is the intersection of A with a cozero set in X, such that $O = \bigcup_i O_i$.

LEMMA Let X be a nonempty CRH space. Suppose that A is a weakly $\mathcal{Z}$-embedded subspace of X—then $\dim A \leq \dim X$.

Let X and Y be nonempty CRH spaces—then $X \times Y$ is said to be rectangular if every cozero set in $X \times Y$ can be written as the union of a σ-neighborhood finite collection of cozero set rectangles $U \times V$. If $X \times Y$ is $\mathcal{Z}$-embedded in $X \times \beta Y$, then $X \times Y$ is rectangular (the converse is false).

EXAMPLE Suppose that X and Y are paracompact Hausdorff spaces satisfying Arhangel'skiĭ's condition—then $X \times Y$ is rectangular.

FACT Let X and Y be nonempty CRH spaces. Suppose that $X \times Y$ is rectangular—then $\dim(X \times Y) \leq \dim X + \dim Y$.

[Indeed, $X \times Y$, as a subspace of $\beta X \times \beta Y$, is weakly $\mathcal{Z}$-embedded.]

EXAMPLE Rectangularity of $X \times Y$ is not a necessary condition for the validity of the relation $\dim(X \times Y) \leq \dim X + \dim Y$.

(1) (The Sorgenfrey Plane) Let X be the Sorgenfrey line—then X is zero dimensional and Lindelöf, hence $\dim X = 0$ (cf. Proposition 2). The Sorgenfrey plane $X \times X$ is zero dimensional but not normal and is "asymmetrical" in that every line with negative slope is discrete but every line with positive slope is homeomorphic to X. Moreover, it is not rectangular as may be seen by considering the points on or above the line $x + y = 1$. Still, $\dim(X \times X) = 0$. As a preliminary, show that if O is any open subset of $X \times X$, then there exists a sequence of clopen sets O_n such that $O \subset \bigcup_n O_n \subset \overline{O}$ and from this deduce that every cozero set in $X \times X$ is a countable union of clopen sets (cf. p. 19–4).

(2) (The Michael Line × The Irrationals) Let X be the Michael line—then X is hereditarily paracompact, hence hereditarily normal, so it follows from the control lemma that $\dim X = 0$. The product $X \times \mathbf{P}$ is zero dimensional but not normal. Nor is it rectangular: Otherwise, $\mathbf{P}$ would be an F_σ in $\mathbf{R}$. However, one can show that $\dim(X \times \mathbf{P}) = 0$.

Let X and Y be nonempty CRH spaces—then $X \times Y$ is said to be piecewise rectangular if every cozero set in $X \times Y$ can be written as the union of a σ-neighborhood finite collection $\{W\}$, where each W is a clopen subset of some cozero set rectangle $U \times V$. In this terminology, Pasynkov[†] proved that if $\begin{cases} \dim X = 0 \\ \dim Y = 0 \end{cases}$, then $\dim(X \times Y) = 0$ iff $X \times Y$ is piecewise rectangular.

[Note: For every pair of positive integers (n, m), Tsuda[‡] has constructed a normal $\begin{cases} X : \dim X = n \\ Y : \dim Y = m \end{cases}$ for which $X \times Y$ is also normal with $\dim(X \times Y) = n + m$ but such that $X \times Y$ is not piecewise rectangular.]

EXAMPLE [Assume CH] There exist nonempty perfectly normal locally compact $\begin{cases} X \\ Y \end{cases}$: $X \times Y$ is a perfectly normal LCH space and $\dim X + \dim Y < \dim(X \times Y)$. For this, use the notation of the example following Proposition 12, letting Δ_C be the diagonal of C in C^2, which will then be identified with C when convenient. Transfer the topology on gr_ϕ back to C to get a second countable completely metrizable topology τ_ϕ on C finer than the euclidean topology τ.

Claim: There exists a second countable metrizable topology Λ on C^2 finer than the euclidean topology τ^2 with $\Lambda|\Delta_C = \tau_\phi$ & $\Lambda|C^2 - \Delta_C = \tau^2|C^2 - \Delta_C$ such that every element of Λ containing a point $(x, x) \in \Delta_C$ also contains the intersection with C^2 of two disjoint open disks, tangent to Δ_C at (x, x).

[Fix a countable basis $\{U_i\}$ for τ_ϕ. Since ϕ is Baire one, each U_i is a euclidean $F_\sigma : U_i = \bigcup\limits_1^\infty A_{ij}, A_{ij}$ τ-closed. Enumerate the $A_{ij} : \{K_n\}$. Given $r > 0$, let $K_n(r)$ be the union of all $B_r \cap C^2$, where B_r is an open disk of radius r tangent to Δ_C at some point of K_n. Recursively determine a sequence of positive real numbers $r_n : r_n > r_{n+1}$ & $\lim r_n = 0$, subject to $K_n \cap K_m = \emptyset \Rightarrow K_n(r_n) \cap K_m(r_m) = \emptyset$. Put $O_i = \bigcup\{K_n(2^{-i} r_n) : K_n \subset U_i\}$. Consider the topology on C^2 generated by the O_i and a countable basis for the euclidean topology on $C^2 - \Delta_C$.]

Construct Kunen modifications τ' and τ'' of τ such that $\tau' \times \tau''$ is a perfectly normal locally compact topology finer than Λ whose restriction $\tau' \times \tau''|\Delta_C$ is a Kunen modification of τ_ϕ (cf. p. 1–16). In so doing, work with an enumeration $\{x_\alpha : \alpha < \Omega\}$ of C, letting $\{C_\alpha : \alpha < \Omega\}$ be an enumeration of the countable subsets of C^2 such that $\forall\, \alpha : C_\alpha \subset \{x_\beta : \beta < \alpha\}^2$. While $\tau' \times \tau''$ is not a Kunen modification of Λ, local compactness is, of course, automatic. As for perfect normality, the essential preliminary is that $\forall\, S \subset C^2\ \exists\, \alpha < \Omega : \mathrm{cl}_\Lambda(S) \cap \{x_\beta : \beta > \alpha\}^2 = \mathrm{cl}_{\tau' \times \tau''}(S) \cap \{x_\beta : \beta > \alpha\}^2$. This said, let $S \subset C^2$ be $\tau' \times \tau''$-closed and choose a sequence $\{O_n\}$ of Λ-open sets: $\mathrm{cl}_\Lambda(S) = \bigcap\limits_n O_n = \bigcap\limits_n \mathrm{cl}_\Lambda(O_n)$—then $\exists\, \alpha < \Omega : \mathrm{cl}_\Lambda(S) \cap \{x_\beta : \beta > \alpha\}^2 = \bigcap\limits_n \mathrm{cl}_{\tau' \times \tau''}(O_n) \cap \{x_\beta : \beta > \alpha\}^2$. On the other hand, for each $\beta \leq \alpha$ there are countable collections $\begin{cases} \{P_n'(\beta)\} \\ \{P_n''(\beta)\} \end{cases}$ of $\tau' \times \tau''$-open sets: $\begin{cases} (C \times \{x_\beta\}) \cap (C^2 - S) \subset \bigcup\limits_n P_n'(\beta) \\ (\{x_\beta\} \times C) \cap (C^2 - S) \subset \bigcup\limits_n P_n''(\beta) \end{cases}$

[†] *London Math. Soc. Lecture Notes* **93** (1985), 227–250.

[‡] *Canad. Math. Bull.* **30** (1987), 49–56.

& $\begin{cases} \text{cl}_{\tau' \times \tau''}(P_n'(\beta)) \cap S = \emptyset \\ \text{cl}_{\tau' \times \tau''}(P_n''(\beta)) \cap S = \emptyset \end{cases}$. Form $\begin{cases} O_n'(\beta) = C^2 - \text{cl}_{\tau' \times \tau''}(P_n'(\beta)) \\ O_n''(\beta) = C^2 - \text{cl}_{\tau' \times \tau''}(P_n''(\beta)) \end{cases}$ and combine the $\begin{cases} O_n'(\beta) \\ O_n''(\beta) \end{cases} (\beta \leq \alpha)$ with the O_n to obtain a single countable collection $\{U_n\}$ of $\tau' \times \tau''$-open sets: $S = \bigcap\limits_n U_n = \bigcap\limits_n \text{cl}_{\tau' \times \tau''}(U_n)$.

Claim: Let $\begin{cases} X = (C, \tau') \\ Y = (C, \tau'') \end{cases}$—then $\begin{cases} \dim X = 0 \\ \dim Y = 0 \end{cases}$ (cf. p. 19–14) and $\dim(X \times Y) > 0$.

[It is enough to show that $\Delta_C \subset (C \times C, \tau' \times \tau'')$ has positive topological dimension. Return to C, which thus carries three topologies, namely τ, τ_ϕ, and $\tau^* \equiv \tau' \times \tau''|C$, a Kunen modification of τ_ϕ. Let $\begin{cases} A = \phi^{-1}([0, 1/7]) \\ B = \phi^{-1}([6/7, 1]) \end{cases}$; let $\begin{cases} A^* = \phi^{-1}([0, 1/3]) \\ B^* = \phi^{-1}([2/3, 1]) \end{cases}$. To arrive at a contradiction, suppose that O^* is a τ^*-clopen set: $\begin{cases} A^* \subset O^* \\ B^* \cap O^* = \emptyset \end{cases}$. If the bar denotes closure in τ_ϕ and if $V = C - \overline{O^*}$, then $\begin{cases} A \cap \overline{V} = \emptyset \\ B \subset V \end{cases}$ & $\#(\text{fr}\, V) > \omega$. But $\text{fr}\, V \subset \overline{O^*} \cap \overline{C - O^*}$ and $\#(\overline{O^*} \cap \overline{C - O^*}) \leq \omega$.]

[Note: CH is not necessary here. Examples of this type exist in ZFC (Przymusiński[†]), the main difference being that the product $X \times Y$ is not perfectly normal but rather is a normal countably paracompact LCH space.]

One final point: The product theorem holds if X is an arbitrary nonempty topological space. In fact, if $A \subset X$ has the EP w.r.t. $[0, 1]$, then its image cr$A$ in cr$X$ "is" the complete regularization of A and as such has the EP w.r.t. $[0, 1]$, so $\dim A = \dim \text{cr} A \leq \dim \text{cr} X = \dim X$ (cf. p. 19–2). The countable union lemma is therefore applicable provided the $A_j \subset X$ have the EP w.r.t. $[0, 1]$ (cf. p. 19–13). It is then easy to fall back to the completely regular case since for any LCH space Y, $\text{cr}(X \times Y) = \text{cr}X \times Y$.

LEMMA Suppose that X is a compact Hausdorff space. Let $f, g \in C(X, \mathbf{S}^n)$ and put $D = \{x : f(x) \neq g(x)\}$. Assume: $\dim D \leq n - 1$—then $f \simeq g$.

[Since ID is an F_σ in IX, hence is normal, it follows from the product theorem that $\dim ID \leq n$. Set $Y = i_0 X \cup I(X - D) \cup i_1 X$ and define $h : Y \rightarrow \mathbf{S}^n$ by $\begin{cases} h(x, 0) = f(x) \\ h(x, 1) = g(x) \end{cases}$ & $h(x, t) = f(x) = g(x)$—then h is continuous and has a continuous extension $H \in C(IX, \mathbf{S}^n)$ (cf. p. 19–18).]

PROPOSITION 14 Let $f, g \in C(X, \mathbf{S}^n)$ and put $D = \{x : f(x) \neq g(x)\}$. Assume: $\dim D \leq n - 1$—then $f \simeq g$.

[The subset of βX on which $\beta f \neq \beta g$ can be written as a countable union $\bigcup\limits_1^\infty \overline{U_j}$, each U_j being open in βX. And: $\dim(\overline{U_j} \cap X) \leq n - 1 \Rightarrow \dim \overline{U_j} = \dim \beta(\overline{U_j} \cap X) \leq n - 1 \Rightarrow \dim \bigcup\limits_1^\infty \overline{U_j} \leq n - 1$, thus from the lemma, $\beta f \simeq \beta g$.]

[†] *Proc. Amer. Math. Soc.* **76** (1979), 315–321; see also Tsuda, *Math. Japon.* **27** (1982), 177–195.

Application: If $\dim X \leq n-1$, then $[X, \mathbf{S}^n] = *$.

FACT Suppose that X is normal and $\dim X$ is finite—then the natural map $[\beta X, \mathbf{S}^n] \to [X, \mathbf{S}^n]$ is bijective if $n > 1$ but if $n = 1$ and X is connected, there is an exact sequence $0 \to C(X)/BC(X) \to [\beta X, \mathbf{S}^1] \to [X, \mathbf{S}^1] \to 0$.

[To discuss the second assertion, observe that X connected iff βX connected and form the commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & C(\beta X)/\mathbf{Z} \arrow[r, "\exp"] \arrow[d] & C(\beta X, \mathbf{S}^1) \arrow[r] \arrow[d] & {[\beta X, \mathbf{S}^1]} \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & C(X)/\mathbf{Z} \arrow[r, "\exp"'] & C(X, \mathbf{S}^1) \arrow[r] & {[X, \mathbf{S}^1]} \arrow[r] & 0 \end{tikzcd}.$$

Since the rows are exact and the middle vertical arrow is an isomorphism, the ker-coker lemma gives $\ker([\beta X, \mathbf{S}^1] \to [X, \mathbf{S}^1]) \approx \operatorname{coker}(C(\beta X)/\mathbf{Z} \to C(X)/\mathbf{Z}) \approx C(X)/BC(X)$. As for the need of the connectedness assumption, take $X = \mathbf{N} : \dim \mathbf{N} = 0 \Rightarrow [\mathbf{N}, \mathbf{S}^1] = * = [\beta \mathbf{N}, \mathbf{S}^1]$.]

[Note: The exact sequence $0 \to C(X)/BC(X) \to [\beta X, \mathbf{S}^1] \to [X, \mathbf{S}^1] \to 0$ translates to $0 \to C(X)/BC(X) \to \check{H}^1(\beta X) \to \check{H}^1(X) \to 0$. Because the quotient $C(X)/BC(X)$ is torsion free and divisible when nontrivial, it follows that if X is not pseudocompact, then $\check{H}^1(\beta X) \approx \oplus \mathbf{Q}$ and is in fact uncountable. Proof: Let $f : X \to \mathbf{R}$ be an unbounded continuous function, put $f_r = r \cdot f$ $(r \in \mathbf{R})$ and consider the $f_r + BC(X)$. Example: $\check{H}^1(\beta \mathbf{R}) \approx C(\mathbf{R})/BC(\mathbf{R})$.]

Let Y be a connected CW space—then Bartik[†] has shown that the arrow $[\beta X, Y] \to [X, Y]$ is bijective for every nonempty CRH space X with $\dim X$ finite iff $\pi_1(Y)$ is finite and $\forall\, q > 1$, $\pi_q(Y)$ is finitely generated or still, iff $\pi_1(Y)$ is finite and Y has the homotopy type of a connected CW complex K such that $\forall\, n, K^{(n)}$ is finite (cf. p. 5–23).

Application: Suppose that π is a finitely generated abelian group. Let X be a nonempty CRH space of finite topological dimension—then $\forall\, n > 1, \check{H}^n(\beta X; \pi) \approx \check{H}^n(X; \pi)$.

EXAMPLE Take $X = Y = \mathbf{P}^\infty(\mathbf{C})$—then $\dim X = \infty$ and the natural map $[\beta X, X] \to [X, X]$ is not surjective (consider id_X).

DOWKER EXTENSION THEOREM Let X be normal with $\dim X \leq n+1$ $(n \geq 1)$ and let A be a closed subspace of X. Suppose that $f \in C(A, \mathbf{S}^n)$—then $\exists\, F \in C(X, \mathbf{S}^n) : F|A = f$ iff $f^*(\check{H}^n(\mathbf{S}^n)) \subset i^*(\check{H}^n(X))$, $i : A \to X$ the inclusion.

[The argument splits into two parts.

[†] *Quart. J. Math.* **29** (1978), 77–91; see also Calder-Siegel, *Trans. Amer. Math. Soc.* **235** (1978), 245–270 and *Proc. Amer. Math. Soc.* **78** (1980), 288–290.

($n = 1$) In this case, $[X, A; \mathbf{S}^1, s_1] \approx \check{H}^1(X, A)$, so one can proceed directly (A has the HEP w.r.t. $\mathbf{S}^1$ (cf. p. 6–41)).

($n > 1$) To reduce to the compact situation, use the fact that the extendability of $f : A \to \mathbf{S}^n$ to X is equivalent to the extendability of $\beta f : \beta A \to \mathbf{S}^n$ to βX and consider the commutative diagram

$$\begin{tikzcd}
\check{H}^n(\beta X) \arrow[r] \arrow[d] & \check{H}^n(\beta A) \arrow[d] & \check{H}^n(\mathbf{S}^n) \arrow[l] \arrow[d, equal] \\
\check{H}^n(X) \arrow[r] & \check{H}^n(A) & \check{H}^n(\mathbf{S}^n) \arrow[l]
\end{tikzcd}$$

.]

DOWKER CLASSIFICATION THEOREM Let X be normal with $\dim X \leq n$ ($n \geq 1$) and let A be a closed subspace of X. Fix a generator $\iota \in \check{H}^n(\mathbf{S}^n, s_n; \mathbf{Z})$—then the assignment $[f] \to f^*\iota$ defines a bijection $[X, A; \mathbf{S}^n, s_n] \to \check{H}^n(X, A; \mathbf{Z})$.

[Show that $\forall\, n > 1, [\beta X, \beta A; \mathbf{S}^n, s_n] \approx [X, A; \mathbf{S}^n, s_n]$.]

PROPOSITION 15 Suppose that $X = A \cup B$, where A and B are closed. Let $\begin{cases} f \in C(A, \mathbf{S}^n) \\ g \in C(B, \mathbf{S}^n) \end{cases}$ and put $D = \{x \in A \cap B : f(x) \neq g(x)\}$. Assume: $\dim D \leq n - 1$—then $\exists \begin{cases} F \in C(X, \mathbf{S}^n) : F|A = f \\ G \in C(X, \mathbf{S}^n) : G|A = g \end{cases}$ & $F \simeq G$.

[Using Proposition 14, fix a homotopy $h : I(A \cap B) \to \mathbf{S}^n$ such that $\begin{cases} h(x, 0) = f(x) \\ h(x, 1) = g(x) \end{cases}$ ($x \in A \cap B$). Since $A \cap B$ as a subspace of $\begin{cases} A \\ B \end{cases}$ has the HEP w.r.t. $\mathbf{S}^n$, there exist continuous functions $\begin{cases} \phi : IA \to \mathbf{S}^n \\ \psi : IB \to \mathbf{S}^n \end{cases}$ with $\begin{cases} \phi(x, 0) = f(x) & (x \in A) \\ \psi(x, 1) = g(x) & (x \in B) \end{cases}$ and $\phi|I(A \cap B) = h = \psi|I(A \cap B)$. Define $H \in C(IX, \mathbf{S}^n)$ by $\begin{cases} H|IA = \phi \\ H|IB = \psi \end{cases}$ and consider $\begin{cases} F(x) = H(x, 0) \\ G(x) = H(x, 1) \end{cases}$ ($x \in X$).]

FACT Let A be a closed subset of X and let $f \in C(A, \mathbf{S}^n)$. Assume: $X = \bigcup\limits_1^\infty O_j$, where the O_j are open, $\dim \operatorname{fr} O_j \leq n - 1$, and $\forall\, j$, f has a continuous extension to $A \cup \overline{O}_j$—then $\exists\, F \in C(X, \mathbf{S}^n) : F|A = f$.

Suppose that $\dim X = n$ is positive. Let $f : X \to [0, 1]^n$ be universal—then the restriction $f^{-1}(\mathbf{S}^{n-1}) \to \mathbf{S}^{n-1}$ has no continuous extension to X, thus is essential. Put $X_f = X/f^{-1}(\mathbf{S}^{n-1})$, identify $\mathbf{S}^n$ with $[0, 1]^n/\mathbf{S}^{n-1}$ and let $F_f : X_f \to \mathbf{S}^n$ be the induced map.

LEMMA F_f is essential, hence $\dim X_f = n$.

[Put $A = f^{-1}(\mathbf{S}^{n-1})$—then there is a commutative diagram

$$\begin{tikzcd}
(X, A) \arrow[r] \arrow[d, "f"'] & (X/A, *_A) \arrow[d, "F_f"] \\
([0, 1]^n, \mathbf{S}^{n-1}) \arrow[r] & (\mathbf{S}^n, s_n)
\end{tikzcd}$$

.

$(n = 1)$ To get a contradiction, assume that F_f is inessential. Choose $\phi \in C(X_f) : F_f(x) = \exp(2\pi i \phi(x))$. Since $F_f(x) = 1$ only if $x = *_A$, $\phi(x) \in \mathbf{Z}$ only if $x = *_A$. Normalize and take $\phi(*_A) = 0$. Let $S = f^{-1}(0) \cup p^{-1}(\phi^{-1}(]0,1[))$. Noting that $f(x) = \phi(p(x)) \bmod 1$, write $S = f^{-1}([0,1/2]) \cap p^{-1}(\phi^{-1}([0,1/2])) \cup p^{-1}(\phi^{-1}([1/2,1]))$ to see that S is closed and write $S = f^{-1}([0,1/2[) \cap p^{-1}(\phi^{-1}(]-1/4,1/2[)) \cup p^{-1}(\phi^{-1}(]1/4,1[))$ to see that S is open. The characteristic function of the complement of S is thus a continuous extension to X of the restriction $f^{-1}(\{0,1\}) \to \{0,1\}$.

$(n > 1)$ The commutative diagram

$$\begin{tikzcd}
\check{H}^n(\mathbf{S}^n, s_n) \arrow[r] & \check{H}^n([0,1]^n, \mathbf{S}^{n-1}) \arrow[r, "f^*"] & \check{H}^n(X, A) & \\
& \check{H}^{n-1}(\mathbf{S}^{n-1}) \arrow[u] \arrow[r] & \check{H}^{n-1}(A) \arrow[u] & \check{H}^{n-1}(X) \arrow[l]
\end{tikzcd}$$

displays the data (cf. p. 20–1). In view of the Dowker extension theorem, f^* is not the zero homomorphism. Since the arrow $\check{H}^n(\mathbf{S}^n, s_n) \to \check{H}^n([0,1]^n, \mathbf{S}^{n-1})$ is an isomorphism, it follows that F_f is essential.]

Suppose that IX is normal—then by the product theorem, $\dim IX \leq \dim X + 1$. One can also go the other way: $\dim IX \geq \dim X + 1$. This is obvious if $\dim X = 0$, so assume that $\dim X = n$ is positive. Claim: $\dim IX_f \geq n + 1$. Indeed, if $\dim IX_f \leq n$, then Alexandroff's criterion would imply that the continuous function $\phi : i_0 X_f \cup i_1 X_f \to \mathbf{S}^n$ defined by $\begin{cases} \phi(x,0) = F_f(x) \\ \phi(x,1) = s_n \end{cases}$ $(x \in X_f)$ has a continuous extension to IX_f, meaning that F_f is homotopic to a constant map and this contradicts the lemma. Now write $X - f^{-1}(\mathbf{S}^{n-1}) = \bigcup\limits_1^\infty A_j$, where the A_j are closed subspaces of X. Let $*_f$ be the image of $f^{-1}(\mathbf{S}^{n-1})$ in X_f—then $X_f = \{*_f\} \cup \bigcup\limits_1^\infty A_j \Rightarrow IX_f = I\{*_f\} \cup \bigcup\limits_1^\infty IA_j \Rightarrow \exists\, j : \dim IX_f = \dim IA_j \Rightarrow \dim IX \geq \dim IA_j = \dim IX_f \geq n + 1 = \dim X + 1$.

Application: Suppose that $X \times [0,1]^m$ is normal—then $\dim(X \times [0,1]^m) = \dim X + m$.

PROPOSITION 16 Suppose that X is normal and Y is a CW complex. Assume: $X \times Y$ is normal—then $\dim(X \times Y) = \dim X + \dim Y$.

[If B is a compact subspace of Y which is homeomorphic to $[0,1]^m$, where $m = \dim Y$, then $\dim(X \times B) = \dim X + m$.]

[Note: The same conclusion obtains if Y is a metrizable topological manifold.]

EXAMPLE Let X and Y be nonempty CW complexes—then $X \times_k Y$ is a CW complex and $\dim(X \times_k Y) = \dim(X \times Y)$.

PROPOSITION 17 Suppose that X is normal with $\dim X = 1$ and Y is paracompact and σ-locally compact. Assume: $X \times Y$ is normal—then $\dim(X \times Y) = \dim X + \dim Y$.

[Switch the roles of X and Y and reduce to the case when X is compact. Since $\dim Y = 1$, there exist disjoint closed sets $\begin{cases} B' \subset Y \\ B'' \subset Y \end{cases}$ such that $\overline{V} - V \neq \emptyset$ for any open $V \subset Y : B' \subset V \subset Y - B''$. Arguing as above, it need only be shown that $\dim(X_f \times Y) \geq n + 1$ $(n > 0)$. If instead $\dim(X_f \times Y) \leq n$, define a continuous function $\phi : X_f \times (B' \cup B'') \rightarrow \mathbf{S}^n$ by $\begin{cases} \phi(x,y) = F_f(x) & ((x,y) \in X_f \times B') \\ \phi(x,y) = s_n & ((x,y) \in X_f \times B'') \end{cases}$ and use Alexandroff's criterion to get a continuous extension $\Phi : X_f \times Y \rightarrow \mathbf{S}^n$. Let $V \subset Y$ be the set of all y with the property that the section $\Phi_y : \begin{cases} X_f \rightarrow \mathbf{S}^n \\ x \rightarrow \Phi(x,y) \end{cases}$ is essential—then $B' \subset V \subset Y - B''$ and V is clopen, X_f being compact. Contradiction.]

EXAMPLE Take, after Anderson-Keisler (cf. p. 19–38), an $X \subset \mathbf{R}^2 : \dim X = \dim(X \times X) = 1$—then $\dim \beta(X \times X) = 1$ but $\dim(\beta X \times \beta X) = \dim \beta X + \dim \beta X = 2$ (cf. Proposition 17).

While there is no reason to suppose that X_f is completely regular if X is, nevertheless the lemma and Propositions 16 and 17 are still true in this setting, although some changes in the proofs are necessary (Morita[†]). Consider, e.g., Proposition 17. Having made the reduction and the switch (so X is compact and $\dim Y = 1$), choose a continuous function $h : Y \rightarrow [0,1]$ such that $\overline{V} - V \neq \emptyset$ for any open $V \subset Y : h^{-1}(0) \subset V \subset Y - h^{-1}(1)$. Define $H : X_f \times Y \rightarrow [0,1]^{n+1}$ by $H(x,y) = (1 - h(y))F_f(x) + h(y)s_n$. If $\dim(X \times Y) \leq n$ (where $n \geq 1$), then $\dim(X_f \times Y) \leq n$, therefore H is not universal. Accordingly (cf. p. 19–18), $\exists\, \Phi \in C(X_f \times Y, \mathbf{S}^n) : \begin{cases} \Phi(x,y) = F_f(x) & (y \in h^{-1}(0)) \\ \Phi(x,y) = s_n & (y \in h^{-1}(1)) \end{cases}$ and this suffices.

EXAMPLE Let X be an arbitrary nonempty topological space—then $\dim IX = \dim \mathrm{cr} IX = \dim I\mathrm{cr}X = \dim \mathrm{cr}X + 1 = \dim X + 1$. This fact can be used to compute $\dim \Gamma X$ and $\dim \Sigma X$, both of which have the value $\dim X + 1$. Observe first that the two lemmas on p. 19–19 hold "in general". Therefore $\dim X + 1 = \dim IX = \max\{\dim i_1 X, \dim IX/i_1 X\} = \max\{\dim X, \dim \Gamma X\} = \dim \Gamma X$. And then $\dim \Gamma X = \max\{\dim X, \dim \Gamma X/X\} = \max\{\dim X, \Sigma X\} = \dim \Sigma X$. Corollary: If $f : X \rightarrow Y$ is a continuous function and if M_f is its mapping cylinder, then $\dim M_f = \max\{1 + \dim X, \dim Y\}$.

[Note: Recall that a cofibered subspace has the EP w.r.t. $\mathbf{R}$, hence w.r.t. $[0,1]$ (cf. p. 6–40).]

LEMMA Let X be normal. Suppose that there exists a sequence $\mathcal{U}_1, \mathcal{U}_2, \ldots$ of open coverings of X such that $\mathcal{U}_{i+1}$ is a refinement of $\mathcal{U}_i$, the collection $\{\mathrm{st}(U, \mathcal{U}_i) : U \in \mathcal{U}_i \ (i = 1, 2, \ldots)\}$ is a basis for X, and $\forall\, i : \mathrm{ord}(\mathcal{U}_i) \leq n + 1$—then $\dim X \leq n$.

[†] *Fund. Math.* **87** (1975), 31–52.

[Let $\mathcal{U} = \{U_1, \ldots, U_k\}$ be a finite open covering of X. Denote by X_i the union of all $U \in \mathcal{U}_i$: $\text{st}(U, \mathcal{U}_i)$ is contained in some element of $\mathcal{U}$. Each X_i is open; moreover, $X = \bigcup\limits_i X_i$. Fix a map $f_i^{i+1} : \mathcal{U}_{i+1} \rightarrow \mathcal{U}_i$ such that $\forall\, U \in \mathcal{U}_{i+1} : f_i^{i+1}(U) \supset U$. Set $f_i^i = \text{id}_{\mathcal{U}_i}$ and for $i < j$, put $f_i^j = f_i^{i+1} \circ \ldots \circ f_{j-1}^j$. Introduce

$$\mathcal{U}(j) = \{U \in \mathcal{U}_j : U \cap X_j \neq \emptyset\} \text{ and } \mathcal{V}(j) = \{U \in \mathcal{U}(j) : U \cap (\bigcup_{i<j} X_i) = \emptyset\}.$$

Obviously, $\mathcal{V}(j) \subset \mathcal{U}(j) \subset \mathcal{U}_j$ and $j' \neq j'' \Rightarrow \mathcal{V}(j') \cap \mathcal{V}(j'') = \emptyset$. Given $U \in \mathcal{U}(j)$, let $i(U)$ be the smallest integer $i \leq j : f_i^j(U) \cap X_i \neq \emptyset$, so $f_{i(U)}^j(U) \in \mathcal{V}(i(U))$. Corresponding to any $V \in \mathcal{V}(i)$ is the open set

$$V^* = \bigcup_{j \geq i} \bigcup \{U \cap X_j : U \in \mathcal{U}(j), f_i^j(U) = V \;\&\; i(U) = i\}.$$

Note that $V^* \subset V$ and $\forall\, U \in \mathcal{U}(j)$, $U \cap X_j \subset f_{i(U)}^j(U)^*$. In addition, $\exists\, U \in \mathcal{U}_i$: $U \cap V \neq \emptyset$ and $\exists\, k(V) \leq k : V \subset \text{st}(U, \mathcal{U}_i) \subset U_{k(V)}$, hence $V^* \subset U_{k(V)}$. The collection $\mathcal{V}^* = \{V^* : V \in \bigcup\limits_i \mathcal{V}(i)\}$ is therefore an open refinement of $\mathcal{U}$. The claim then is that $\text{ord}(\mathcal{V}^*) \leq n + 1$. To this end, consider a generic nonempty intersection $V_1^* \cap \cdots \cap V_p^*$, where $V_1 \in \mathcal{V}(i_1), \ldots, V_p \in \mathcal{V}(i_p)$ are distinct elements of $\bigcup\limits_i \mathcal{V}(i)$. Take an x in $V_1^* \cap \cdots \cap V_p^*$ and choose $j : x \in X_j - \bigcup\limits_{i<j} X_i$ $(\Rightarrow i_1 \leq j, \ldots, i_p \leq j)$. From the definitions, there exist

$$U_1 \in \mathcal{U}(j_1) : \begin{cases} f_{i_1}^{j_1}(U_1) = V_1 \\ i(U_1) = i_1 \end{cases} \& \; x \in U_1 \cap X_{j_1}, \ldots, U_p \in \mathcal{U}(j_p) : \begin{cases} f_{i_p}^{j_p}(U_p) = V_p \\ i(U_p) = i_p \end{cases} \& \; x \in U_p \cap X_{j_p}.$$

But $x \in f_j^{j_1}(U_1) \cap \cdots \cap f_j^{j_p}(U_p)$ and since $f_j^{j_1}(U_1), \ldots, f_j^{j_p}(U_p)$ are all different, the claim is thus seen to follow from the fact that $\text{ord}(\mathcal{U}_j) \leq n + 1$.]

Application: Let X be normal. Suppose that X admits a development $\{\mathcal{U}_i\}$ such that $\{\mathcal{U}_i\}$ is a star sequence and $\forall\, i : \text{ord}(\mathcal{U}_i) \leq n + 1$—then $\dim X \leq n$.

PASYNKOV FACTORIZATION LEMMA Suppose that X is normal and Y is metrizable—then for every $f \in C(X, Y)$ there exists a metrizable space Z with $\begin{cases} \dim Z \leq \dim X \\ \text{wt}\, Z \leq \text{wt}\, Y \end{cases}$ and functions $\begin{cases} g \in C(X, Z) \\ h \in C(Z, Y) \end{cases}$ such that $f = h \circ g$ with h uniformly continuous and $g(X) = Z$.

[Assume that $\dim X = n$ is finite and $\text{wt}\, Y \geq \omega$. Fix a sequence $\{\mathcal{V}_i\}$ of neighborhood finite open coverings of Y such that $\forall\, i : \#(\mathcal{V}_i) \leq \text{wt}\, Y$, arranging matters so that the diameter of each $V \in \mathcal{V}_i$ is $< 1/i$. Inductively construct a star sequence $\{\mathcal{U}_i\}$ of neighborhood finite open coverings of X such that $\forall\, i : \begin{cases} \operatorname{ord}(\mathcal{U}_i) \leq n+1 \\ \#(\mathcal{U}_i) \leq \operatorname{wt} Y \end{cases}$ and $\mathcal{U}_i$ is a star refinement of $f^{-1}(\mathcal{V}_i)$. Justification: Quote Proposition 6 and recall §1, Proposition 13 (the proof of which allows one to say that the cardinality of $\mathcal{U}_i$ remains $\leq \operatorname{wt} Y$). Let δ be a continuous pseudometric on X associated with $\{\mathcal{U}_i\}$ as on p. 6–37. The claim is that one can take for Z the metric space X_δ obtained from X by identifying points at zero distance from one another. Granted this, it is clear what g and h have to be. Denote by $X(\delta)$ the set X equipped with the topology determined by δ. Given $U \in \mathcal{U}_i$, write $U(\delta)$ for its interior in $X(\delta)$ and put $\mathcal{U}_i(\delta) = \{U(\delta) : U \in \mathcal{U}_i\}$—then $\{\mathcal{U}_i(\delta)\}$ is a development for $X(\delta)$ and is a star sequence such that $\forall\, i : \operatorname{ord}(\mathcal{U}_i(\delta)) \leq n+1$. The projection $p : X(\delta) \rightarrow Z$ is an open map (every open subset of $X(\delta)$ is p-saturated), thus $\mathcal{W}_i \equiv p(\mathcal{U}_i(\delta))$ is an open covering of Z. Furthermore, $\{\mathcal{W}_i\}$ is a development for Z and is a star sequence such that $\forall\, i : \operatorname{ord}(\mathcal{W}_i) \leq n+1$. Therefore $\dim Z \leq n$. As for the assertion $\operatorname{wt} Z \leq \operatorname{wt} Y$, note that the $\mathcal{W}_i$ are point finite and the collection $\bigcup\limits_1^\infty \{\operatorname{st}(z, \mathcal{W}_i) : z \in Z\}$ is a basis for Z.]

There are two related results, applicable to pairs (X, A).

(A) Suppose that X is normal and Y is metrizable of weight $\leq \kappa$. Let A be a subspace of X having the EP w.r.t. $\mathcal{B}(\kappa)$—then for every $f \in C(A, Y)$ there exists a metrizable space Z_A of weight $\leq \kappa$ and functions $\begin{cases} g \in C(X, Z_A) \\ h_A \in C(g(A), Y) \end{cases}$ such that $f = h_A \circ (g|A)$ with h_A uniformly continuous and $g(X) = Z_A$.

[Argue as in §6, Proposition 15 (proof of sufficiency).]

(X/A) Suppose that X is normal and Y is metrizable of weight $\leq \kappa$. Let A be a closed subspace of $X : \dim(X/A) \leq n$—then for every $f \in C(X, Y)$ there exists a metrizable space Z of weight $\leq \kappa$ and functions $\begin{cases} g \in C(X, Z) \\ h \in C(Z, Y) \end{cases}$ such that $f = h \circ g$ with h uniformly continuous and $g(X) = Z$, $\dim(Z - \overline{g(A)}) \leq n$.

[This is the relative version of the Pasynkov factorization lemma. The proof is the same as for the absolute case modulo the following remark: Every neighborhood finite open covering $\mathcal{U} = \{U_i : i \in I\}$ of X has a neighborhood finite open refinement $\mathcal{O}$ such that the order of the collection $\{O, \operatorname{st}(A, \mathcal{O}) : O \in \mathcal{O}\ \&\ O \cap A = \emptyset\}$ is $\leq n+1$. Proof: Assuming that the U_i are cozero sets, let $\mathcal{Z} = \{Z_i : i \in I\}$ be a precise zero set refinement of $\mathcal{U}$ (cf. p. 1–25). Define $I_0 = \{i \in I : U_i \cap A \neq \emptyset\}$ and put $\begin{cases} Z_0 = \bigcup\{Z_i : i \in I_0\} \\ U_0 = \bigcup\{U_i : i \in I_0\} \end{cases}$—then $\begin{cases} Z_0 \\ U_0 \end{cases}$ is a $\begin{cases} \text{zero set} \\ \text{cozero set} \end{cases}$ (cf. p. 1–24). Choose $\phi \in C(X, [0,1]) : Z_0 = \phi^{-1}(0)$ & $X - U_0 = \phi^{-1}(1)$. Let $X_0 = \{x : \phi(x) \leq 1/2\}$. Since A is contained in Z_0 and Z_0 is contained in the interior of X_0, the collection $\{U_i - X_0, U_0 : i \in I - I_0\}$ is the inverse image of a neighborhood finite cozero set covering of X/A under the projection $X \to X/A$. Therefore there exists a neighborhood finite cozero set covering $\{O_i, O_0 : i \in I - I_0\}$ of X whose order does not exceed $n+1$ such that $O_i \subset U_i - X_0 (i \in I - I_0)$ and $A \subset O_0 \subset U_0$. If $\mathcal{O} = \{O_i : i \in I - I_0\} \cup \{O_0 \cap U_i : i \in I_0\}$, then $O_0 = \text{st}(A, \mathcal{O})$ and $\mathcal{O}$ is a neighborhood finite cozero set refinement of $\mathcal{U}$ with the stated property.]

PROPOSITION 18 Suppose that X is normal and Y is completely metrizable of weight $\le \kappa$ and locally n-connected (n-connected and locally n-connected). Let A be a closed subspace of X having the EP w.r.t. $\mathcal{B}(\kappa)$. Assume: $\dim X/A \le n+1$—then A has the NEP (EP) w.r.t. Y.

[Take an $f \in C(A, Y)$ and write $f = h_A \circ (g|A)$. Since $g \in C(X, Z_A)$ and since $\text{wt}Z_A \le \kappa$, g can in turn be factored: $g = \psi \circ \phi$, where $\begin{cases} \phi \in C(X, Z) \\ \psi \in C(Z, Z_A) \end{cases}$. Here, of course, $\dim(Z - \overline{\phi(A)}) \le n+1$. On the other hand, $h_A \circ (\psi|\phi(A))$ is uniformly continuous, hence extends to a continuous function $H_A : \overline{\phi(A)} \to Y$. Now apply the results of Dugundji cited on p. 6–15.]

PROPOSITION 19 Suppose that IX is normal and Y is completely metrizable of weight $\le \kappa$ and locally n-connected. Let A be a closed subspace of X having the EP w.r.t. $\mathcal{B}(\kappa)$. Assume: $\dim X/A \le n$—then A has the HEP w.r.t. Y.

[Let $f : i_0X \cup IA \to Y$ be continuous. Since $i_0X \cup IA$, as a subspace of IX, has the EP w.r.t. $\mathcal{B}(\kappa)$ (cf. §6, Proposition 16) and since $\dim IX/i_0X \cup IA \le \dim IX/IA \le \dim I(X/A) \le \dim X/A + 1 \le n+1$, Proposition 18 implies that there exists a cozero set $O \subset IX : O \supset i_0X \cup IA$ and a continuous function $g : O \to Y$ extending f. Fix a cozero set $U \subset X : IA \subset IU \subset O$. Choose $\phi \in C(X, [0,1]) : \begin{cases} \phi|A = 1 \\ \phi|X - U = 0 \end{cases}$. Define $F \in C(IX, Y)$ by $F(x,t) = g(x, \phi(x)t) : F$ is a continuous extension of f.]

The normality of X can be dispensed with in Pasynkov's factorization lemma: Everything goes through in the completely regular situation.

[Note: Pasynkov's factorization lemma is then valid for an arbitrary topological space as may be seen by passing to its complete regularization.]

As for Propositions 18 and 19, they too are true if X is a nonempty CRH space. The assumption that A is closed was made only to ensure that the quotient X/A is normal. Therefore it can be dropped. Likewise, the assumption that IX is normal was made only to use the product theorem. This, however, is of no real consequence, as the product theorem holds for an arbitrary nonempty topological space (cf. p. 19–44).

For another application of these methods, suppose that Y is completely metrizable of weight $\leq \kappa$ and is n-connected and locally n-connected. Assume: $\dim X/A \leq n$. Let $\begin{cases} f : X \rightarrow Y \\ g : X \rightarrow Y \end{cases}$ be continuous functions such that $f|A \simeq g|A$—then $f \simeq g$. Corollary: If X is κ-collectionwise normal with $\dim X \leq n$, then $[X, Y] = *$.

FACT Suppose that X is a nonempty metrizable space. Let A be a nonempty closed subspace of $X : \dim(X - A) = 0$—then there exists a retraction $r : X \rightarrow A$.

A compact connected ANR Y is said to be a test space for dimension n $(n \geq 1)$ provided that the statement $\dim X \leq n$ is true iff every closed subset $A \subset X$ has the EP w.r.t. Y. Example: $\mathbf{S}^n$ is a test space for dimension n (Alexandroff's criterion).

[Note: No compact connected AR Y can be a test space for dimension n.]

LEMMA Let $\begin{cases} Y' \\ Y'' \end{cases}$ be compact connected ANRs of the same homotopy type—then Y' is a test space for dimension n iff Y'' is a test space for dimension n.

[If X is normal and $A \subset X$ is closed, then A has the HEP w.r.t. $\begin{cases} Y' \\ Y'' \end{cases}$ (cf. p. 6–41).]

A finite wedge $\bigvee \mathbf{S}^n$ of n-spheres is a test space for dimension n. Indeed, $\bigvee \mathbf{S}^n$ is a compact connected ANR of topological dimension n. Moreover, $\bigvee \mathbf{S}^n$ is $(n-1)$-connected (since for $n > 1$, $\pi_q(\bigvee \mathbf{S}^n) = \oplus \pi_q(\mathbf{S}^n)$ $(q < 2n - 1)$), thus Proposition 18 implies that if $\dim X \leq n$ and if $A \subset X$ is closed, then A has the EP w.r.t. $\bigvee \mathbf{S}^n$. Here it is necessary to recall that A has the EP w.r.t. $\mathcal{B}(\omega)$ (cf. p. 6–37). On the other hand, there is a retraction $r : \bigvee \mathbf{S}^n \rightarrow \mathbf{S}^n$ so if $A \subset X$ is closed and has the EP w.r.t. $\bigvee \mathbf{S}^n$ then A has the EP w.r.t. $\mathbf{S}^n$, from which $\dim X \leq n$.

TEST SPACE THEOREM Let Y be a compact connected ANR of topological dimension n $(n \geq 1)$—then Y is a test space for dimension n iff Y has the homotopy type of a finite wedge of n-spheres.

[Only the necessity need be dealt with. There are two cases: $n = 1$ or $n > 1$. If $n = 1$, then $\pi_1(Y) \neq 1$ (otherwise, Y would be an AR), hence Y has the homotopy type of a finite wedge of 1-spheres (cf. p. 6–21). If $n > 1$, then for $q > n$, $H_q(Y) = 0$ (cf. p. 6–21) and Y must be $(n-1)$-connected (cf. p. 6–15 & p. 19–18). Accordingly, by Hurewicz, $H_q(Y) = 0$ $(0 < q < n)$ and $H_n(Y) = \pi_n(Y)$, a nontrivial finitely generated free abelian group. Picking a set of base point preserving maps $\mathbf{S}^n \rightarrow Y$ which generate $\pi_n(Y)$ then leads to a homology equivalence $\bigvee \mathbf{S}^n \rightarrow Y$ that, by the Whitehead theorem, is a homotopy equivalence.]

If Y is a compact connected ANR which is a test space for dimension n, then $\dim Y \geq n$ (look at the proof of the test space theorem). There are test spaces for dimension n of topological dimension $n+k$ ($k > 0$). Consider, e.g., $[0,1]^{n+k} \vee \mathbf{S}^n$.

EXAMPLE Let $\alpha \in \pi_{n+k}(\mathbf{S}^n)$ ($k > 0, n \geq 1$). Choose a representative $f \in \alpha$ and put $Y_\alpha = \mathbf{D}^{n+k+1} \sqcup_f \mathbf{S}^n$—then Y_α is a compact connected ANR (cf. p. 6–29) with $\dim Y_\alpha = n + k + 1$ (cf. p. 19–21) and Dranishnikov[†] has shown that Y_α is a test space for dimension n.

[Note: The preceding considerations break down if $k = 0$. Example: $\mathbf{P}^2(\mathbf{R})$ is not a test space for dimension 1.]

[†] *Tsukuba J. Math.* 14 (1990), 247–262.

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§20. COHOMOLOGICAL DIMENSION THEORY

Cohomological dimension theory enables one to associate with each nonempty normal Hausdorff space X and every nonzero abelian group G a topological invariant $\dim_G X \in \{0, 1, \ldots\} \cup \{\infty\}$ called its cohomological dimension with respect to G. It turns out that $\dim_{\mathbf{Z}} X = \dim X$ if $\dim X < \infty$ (but this can fail if $\dim X = \infty$) and when X is a CW complex, $\dim_G X = \dim X \ \forall\, G \neq 0$.

Let G be an abelian group—then for any topological pair (X, A), $\check{H}^n(X, A; G)$ is the nth Čech cohomology group of (X, A) with coefficients in G calculated per numerable open coverings (rather than arbitrary open coverings).

[Note: As was shown by Morita[†], $[X, A; K(G, n), k_{G,n}] \approx \check{H}^n(X, A; G)$ (cf. p. 5–30) which, however, need not be true if the usual definition of "$\check{H}^n$" is employed (Bredon[‡]). Bear in mind that when $n = 0$, the agreement is that $K(G, 0) = G$ (discrete topology).]

LEMMA If A is a nonempty subspace of X, then $\forall\, n \geq 1$, $\check{H}^n(X, A; G) \approx \check{H}^n(X/A; G)$.

Let A be a subspace of X—then A is said to be numerably embedded in X if for every numerable open covering $\mathcal{O}$ of A there exists a numerable open covering $\mathcal{U}$ of X such that $\mathcal{U} \cap A$ is a refinement of $\mathcal{O}$ (cf. §6, Proposition 15). Example: If X is a collectionwise normal Hausdorff space, then every closed subspace A of X is numerably embedded in X (cf. p. 6–37).

LEMMA Suppose that A is numerably embedded in X—then $\forall\, G$, there is a long exact sequence $\cdots \to \check{H}^{n-1}(A; G) \to \check{H}^n(X, A; G) \to \check{H}^n(X; G) \to \check{H}^n(A; G) \to \cdots$.

Remark: If $G = \mathbf{Z}$ (or, more generally, is finitely generated), one can get away with less, viz. it suffices that A have the EP w.r.t. $\mathbf{R}$.

[Note: Working with countable numerable open coverings, an appeal to Proposition 4 in §6 leads to the definition of the coboundary operator $\check{H}^{n-1}(A) \to \check{H}^n(X, A)$.]

Example: If X is a normal Hausdorff space and if $A \subset X$ is closed, then there is a long exact sequence $\cdots \to \check{H}^{n-1}(A) \to \check{H}^n(X, A) \to \check{H}^n(X) \to \check{H}^n(A) \to \cdots$.

FACT Suppose that A is numerably embedded in X—then IA is numerably embedded in IX.

[†] *Fund. Math* **87** (1975), 31–52.

[‡] *Proc. Amer. Math. Soc.* **19** (1968), 396–398.

It is known that $\check{H}^*(—;G)$, restricted to the full subcategory of $\mathbf{TOP}^2$ whose objects are the pairs (X, A), where A is closed and numerably embedded in X, satisfies the seven axioms of Eilenberg-Steenrod for a cohomology theory (Watanabe[†]).

PROPOSITION 1 Let X be a nonempty normal Hausdorff space. Assume: $\dim X \leq n$—then $\check{H}^q(X;G) = 0$ $(q > n)$.

[This is a consequence of the definitions (cf. §19, Proposition 6).]

PROPOSITION 1 (bis) Let A be a nonempty closed subspace of X. Assume: $\dim X/A \leq n$—then $\check{H}^q(X,A;G) = 0$ $(q > n)$.

If X is a locally contractible paracompact Hausdorff space (e.g., a CW complex or an ANR), then $\forall\, n$, $\check{H}^n(X;G) \approx H^n(X;G)$. In general, though, Čech cohomology and singular cohomology can differ even if X is compact Hausdorff (Barratt-Milnor[‡]).

[Note: Proposition 1 is a key property of Čech cohomology. It is not shared by singular cohomology.]

Fix an abelian group $G \neq 0$ and let X be a nonempty normal Hausdorff space. Consider the following statement.

($\dim_G X \leq n$) There exists an integer $n = 0, 1, \ldots$ such that $\check{H}^q(X,A;G) = 0$ $(q > n)$ for all closed subsets A of X.

If $\dim_G X \leq n$ is true for some n, then the cohomological dimension of X with respect to G, denoted by $\dim_G X$, is the smallest value of n for which $\dim_G X \leq n$.

[Note: By convention, $\dim_G X = -1$ when $X = \emptyset$ or when $G = 0$. If the statement $\dim_G X \leq n$ is false for every n, then we put $\dim_G X = \infty$.]

EXAMPLE Let X be a metrizable compact Hausdorff space of finite topological dimension, K a simply connected CW complex—then $\dim_{H_q(K)} X \leq q\ \forall\, q \geq 1$ iff $\dim_{\pi_q(K)} X \leq q\ \forall\, q \geq 1$ and both are equivalent to every closed subset $A \subset X$ having the EP w.r.t. K (Dranishnikov[‖]). Example: One can take $K = M(G,n)$ $(n \geq 2)$ (realized as a simply connected CW complex) provided that $\dim_G X \leq n$.

PROPOSITION 2 Suppose that $\dim X \leq n$—then $\dim_G X \leq n$.

[In fact, for $A \neq \emptyset$, $\dim X \leq n \Rightarrow \dim X/A \leq n$ (cf. p. 19–18) $\Rightarrow \check{H}^q(X,A;G) = 0$ $(q > n)$ (cf. Proposition 1 (bis)) $\Rightarrow \dim_G X \leq n$ (Proposition 1 covers the case when $A = \emptyset$).]

[†] *Glas. Mat.* **22** (1987), 187–238; see also *SLN* **1283** (1987), 221–239.

[‡] *Proc. Amer. Math. Soc.* **13** (1962), 293–297.

[‖] *Math. Sbornik* **74** (1993), 47–56; see also Dydak, *Trans. Amer. Math. Soc.* **337** (1993), 219–234.

PROPOSITION 3 Suppose that $\dim X < \infty$—then $\dim_{\mathbf{Z}} X = \dim X$.

[In view of Proposition 2, $\dim_{\mathbf{Z}} X \leq \dim X$. Now argue by contradiction and assume that $\dim_{\mathbf{Z}} X \leq n$, $\dim X = n + 1$. Choose a universal map $f : X \rightarrow [0,1]^{n+1}$ (cf. p. 19–18)—then on the basis of the Dowker extension theorem, the arrow $\check{H}^{n+1}([0,1]^{n+1}, \mathbf{S}^n; \mathbf{Z}) \xrightarrow{f^*} \check{H}^{n+1}(X, f^{-1}(\mathbf{S}^n); \mathbf{Z})$ is not the zero homomorphism. But $\dim_{\mathbf{Z}} X \leq n \Rightarrow \check{H}^{n+1}(X, f^{-1}(\mathbf{S}^n); \mathbf{Z}) = 0$.]

Application: If the topological dimension of X is finite, then $\forall\, G$, $\dim_G X \leq \dim_{\mathbf{Z}} X$.

[Note: For any compact Hausdorff space X (possibly of infinite topological dimension), one has $\dim_G X \leq \dim_{\mathbf{Z}} X$ (immediate from the universal coefficient theorem (cf. infra)).]

EXAMPLE The validity of the relation $\dim_{\mathbf{Z}} X = \dim X$ depends on the assumption that $\dim X < \infty$. Indeed, Dranishnikov[†] has given an example of a compact metric space X such that $\dim X = \infty$, while $\dim_{\mathbf{Z}} X < \infty$.

[Note: According to Watanabe[‡], $\dim_{\mathbf{Z}} X = \dim X$ if X is a compact ANR (no restriction on $\dim X$).]

There is not a great deal that can be said about $\dim_G X$ if X is merely normal, so we shall restrict ourselves in what follows to paracompact X and begin with a review of Čech cohomology in this situation (all open coverings thus being numerable).

MAYER-VIETORIS SEQUENCE Let X be a paracompact Hausdorff space. Suppose that A, B are closed subsets of X with $X = A \cup B$—then the sequence $\cdots \rightarrow \check{H}^n(X;G) \rightarrow \check{H}^n(A;G) \oplus \check{H}^n(B;G) \rightarrow \check{H}^n(A \cap B;G) \rightarrow \check{H}^{n+1}(X;G) \rightarrow \cdots$ is exact.

BOCKSTEIN SEQUENCE Let X be a paracompact Hausdorff space, A a closed subset. Suppose that $0 \rightarrow G' \rightarrow G \rightarrow G'' \rightarrow 0$ is a short exact sequence of abelian groups—then there is a long exact sequence $\cdots \rightarrow \check{H}^n(X,A;G') \rightarrow \check{H}^n(X,A;G) \rightarrow \check{H}^n(X,A;G'') \rightarrow \check{H}^{n+1}(X,A;G') \rightarrow \cdots$.

UNIVERSAL COEFFICIENT THEOREM Let X be a compact Hausdorff space, A a closed subset—then there is a split short exact sequence $0 \rightarrow \check{H}^n(X,A;\mathbf{Z}) \otimes G \rightarrow \check{H}^n(X,A;G) \rightarrow \operatorname{Tor}(\check{H}^{n+1}(X,A;\mathbf{Z}), G) \rightarrow 0$.

[†] *Math. Sbornik* **63** (1989), 539–545; see also Chigogidze, *Inverse Spectra*, North Holland (1996), 100–116.

[‡] *Proc. Amer. Math. Soc.* **123** (1995), 2883–2885.

KÜNNETH FORMULA Let X be a paracompact Hausdorff space, A a closed subset; let Y be a compact Hausdorff space, B a closed subset—then $\check{H}^n((X,A) \times (Y,B);G) \approx \bigoplus_{q=0}^{n} \check{H}^q(X,A;\check{H}^{n-q}(Y,B;G))$.

[Note: The product $X \times Y$ is a paracompact Hausdorff space and, as usual, $(X,A) \times (Y,B) = (X \times Y, X \times B \cup A \times Y)$.]

Let X be a paracompact Hausdorff space of finite topological dimension. Suppose that G is finitely generated—then Bartik[†] has shown that for every closed subset A of X, the arrow $\check{H}^n(\beta X, \beta A; G) \to \check{H}^n(X,A;G)$ is surjective for $n = 1$ and bijective for $n > 1$.

[Note: More is true if G is finite: The arrow $\check{H}^n(\beta X, \beta A; G) \to \check{H}^n(X,A;G)$ is bijective $\forall\, n \geq 0$.]

EXAMPLE Let X be a paracompact Hausdorff space of finite topological dimension—then $\dim_G X \leq \dim_G \beta X$ provided that G is finitely generated.

[This is clear if $\dim_G X \leq 0$, so let $n = \dim_G X$ be positive and choose a closed subset A of X such that $\check{H}^n(X,A;G) \neq 0$. By the above, $\check{H}^n(\beta X, \beta A; G) \neq 0$, thus $n \leq \dim_G \beta X$.]

Notation: Let X be a paracompact Hausdorff space, $A \subset X$ a closed subset. Given $e \in \check{H}^n(X;G)$, write $e|A$ for the image of e under the arrow $\check{H}^n(X;G) \to \check{H}^n(A;G)$.

RESTRICTION PRINCIPLE Let e be an element of $\check{H}^n(X;G)$. Assume: $e|A = 0$—then $\exists$ an open $U \supset A : e|\overline{U} = 0$.

EXTENSION PRINCIPLE Suppose that $\alpha \in \check{H}^n(A;G)$—then $\exists$ an open $U \supset A$ and an $e \in \check{H}^n(\overline{U};G) : e|A = \alpha$.

These two principles date back to Wallace[‡] who used them to establish the following result.

RELATIVE HOMEOMORPHISM THEOREM Let $\begin{cases} X \\ Y \end{cases}$ be paracompact Hausdorff spaces; let $\begin{cases} A \subset X \\ B \subset Y \end{cases}$ be closed subsets. Suppose given a closed map $f : (X,A) \to (Y,B)$ such that $f|X - A$ is a homeomorphism of $X - A$ onto $Y - B$—then $f^* : \check{H}^n(Y,B;G) \to \check{H}^n(X,A;G)$ is an isomorphism.

[†] *Quart. J. Math.* **29** (1978), 77–91.

[‡] *Duke Math J.* **19** (1952), 177–182.

Application: Let X be a paracompact Hausdorff space; let $\begin{cases} A \\ B \end{cases} \subset X$ be closed subsets—then the arrow $\check{H}^n(A \cup B, A) \rightarrow \check{H}^n(B, A \cap B)$ induced by the inclusion $(B, A \cap B) \rightarrow (A \cup B, A)$ is an isomorphism.

It is possible to expand the level of generality and incorporate sheaves (of abelian groups) into the theory. While this is definitely of interest, I shall limit the discussion to a few elementary observations.

Let X be a paracompact Hausdorff space. Given a sheaf $\mathcal{F} \neq 0$ on X, write $\dim_{\mathcal{F}} X \leq n$ if $\exists$ an integer $n = 0, 1, \ldots$ such that $H^q(X; \mathcal{F}|U) = 0$ $(q > n)$ for all open subsets U of X. Example: $\dim X \leq n \Rightarrow \dim_{\mathcal{F}} X \leq n$ (cf. Proposition 2).

Remark: Let $G \neq 0$ be an abelian group, $\mathbf{G}$ the constant sheaf on X determined by G—then $\forall$ closed subset $A \subset X$, $\check{H}^n(X, A; G) \approx H^n(X; \mathbf{G}|X - A)$ (Godement[†]).

FACT Let $\mathcal{F} \neq 0$ be a sheaf on X—then $\dim_{\mathcal{F}} X \leq n$ iff $\mathcal{F}$ admits a soft resolution $0 \rightarrow \mathcal{F} \rightarrow \mathcal{S}^0 \rightarrow \mathcal{S}^1 \rightarrow \cdots \rightarrow \mathcal{S}^n$ of length n.

LEMMA Let $\{\mathcal{F}_\alpha\}$ be a collection of soft subsheaves of a sheaf $\mathcal{F}$ which is directed by inclusion. Assume: $\mathcal{F} = \operatorname{colim} \mathcal{F}_\alpha$—then $\mathcal{F}$ is soft.

FACT Let $\{\mathcal{F}_\alpha\}$ be a collection of subsheaves of a sheaf $\mathcal{F}$ which is directed by inclusion. Assume: $\mathcal{F} = \operatorname{colim} \mathcal{F}_\alpha$—then $\dim_{\mathcal{F}} X \leq n$ if $\forall\, \alpha$, $\dim_{\mathcal{F}_\alpha} X \leq n$, hence $\dim_{\mathcal{F}} X \leq \sup \dim_{\mathcal{F}_\alpha} X$.

[Work with the canonical (=Godement) resolution of the $\mathcal{F}_\alpha$.]

If $\mathcal{F} = \mathcal{F}' \oplus \mathcal{F}''$, then $H^n(X; \mathcal{F}) = H^n(X; \mathcal{F}') \oplus H^n(X; \mathcal{F}'')$. Therefore $\dim_{\mathcal{F}'} X \leq n$ & $\dim_{\mathcal{F}''} X \leq n \Rightarrow \dim_{\mathcal{F}} X \leq n$.

Suppose now that $\{\mathcal{F}_i\}$ is a collection of sheaves indexed by a set I. Given a finite subset $F \subset I$, put $\mathcal{F}_F = \bigoplus\limits_{i \in F} \mathcal{F}_i$—then $\mathcal{F} \equiv \bigoplus\limits_i \mathcal{F}_i = \operatorname{colim} \mathcal{F}_F$. So, under the assumption that $\dim_{\mathcal{F}_i} X \leq n \ \forall\, i$, one has $\dim_{\mathcal{F}} X \leq n$ as well.

Fix an abelian group G and let X be a paracompact Hausdorff space—then X is said to satisfy Okuyama's condition at n if $\forall\, q \geq n$ and each closed subset A of X, the arrow $\check{H}^q(X; G) \rightarrow \check{H}^q(A; G)$ is surjective.

SUBLEMMA Suppose that X satisfies Okuyama's condition at n—then every closed subspace A of X satisfies Okuyama's condition at n.

[†] *Théorie des Faisceaux*, Hermann (1964), 234–236.

[Given a closed subset B of A, consider the commutative triangle
$$\begin{tikzcd} \check{H}^q(X;G) \arrow[r] \arrow[d] & \check{H}^q(B;G) \\ \check{H}^q(A;G) \arrow[ur] & \end{tikzcd}$$
.]

LEMMA Suppose that X satisfies Okuyama's condition at n. Let $\begin{cases} A \\ B \end{cases}$ be closed subspaces of X and let e be an element of $\check{H}^q(X;G)$ such that $\begin{cases} e|A = 0 \\ e|B = 0 \end{cases}$ for some $q > n$—then $e|A \cup B = 0$.

[Consider the Mayer-Vietoris sequence $\cdots \rightarrow \check{H}^{q-1}(A;G) \oplus \check{H}^{q-1}(B;G) \xrightarrow{i} \check{H}^{q-1}(A \cap B;G) \rightarrow \check{H}^q(A \cup B;G) \xrightarrow{j} \check{H}^q(A;G) \oplus \check{H}^q(B;G) \rightarrow \cdots$. Thanks to the sublemma, i is surjective. Therefore j is injective. But $j(e|A \cup B) = 0$, so $e|A \cup B = 0$.]

PROPOSITION 4 Let X be a paracompact Hausdorff space—then $\dim_G X \leq n$ iff X satisfies Okuyama's condition at n.

[Necessity: Inspect the exact sequence $\cdots \rightarrow \check{H}^q(X,A;G) \rightarrow \check{H}^q(X;G) \rightarrow \check{H}^q(A;G) \rightarrow \check{H}^{q+1}(X,A;G) \rightarrow \cdots$.

Sufficiency: Fix $q \geq n$—then since $\check{H}^q(X;G) \rightarrow \check{H}^q(A;G)$ is surjective, $\check{H}^{q+1}(X,A;G) \rightarrow \check{H}^{q+1}(X;G)$ is injective, thus it need only be shown that $\check{H}^{q+1}(X;G) = 0$. Take an $e \in \check{H}^{q+1}(X;G)$. Because $\check{H}^{q+1}(\{x\};G) = 0$, $\exists$ a neighborhood U_x of x such that $e|\overline{U}_x = 0$ (restriction principle) and by paracompactness, the open covering $\{U_x : x \in X\}$ admits a σ-discrete closed refinement $\mathcal{A} = \bigcup\limits_k \mathcal{A}_k$. Put $A_k = \cup\mathcal{A}_k$ and inductively determine a sequence $\{U_k\}$ of open sets: $A_k \cup \overline{U}_{k-1} \subset U_k$ & $e|\overline{U}_k = 0$, where $U_0 = \emptyset$. Noting that $e|A_k = 0 \ \forall \ k$, proceed as follows. First, $\exists$ an open $U_1 \supset A_1 : e|\overline{U}_1 = 0$, hence $e|A_2 \cup \overline{U}_1 = 0$ (apply the preceding lemma). Assuming that $U_k \supset A_k \cup \overline{U}_{k-1}$ with $e|\overline{U}_k = 0$ has been constructed, one has again $e|A_{k+1} \cup \overline{U}_k = 0$, so $\exists$ an open set $U_{k+1} : U_{k+1} \supset A_{k+1} \cup \overline{U}_k = 0$ & $e|\overline{U}_{k+1} = 0$, which pushes the induction forward. Now let $W_k = \overline{U}_k - U_{k-1} : W_k$ is closed, $e|W_k = 0$, and $X = \bigcup\limits_k W_k$. On the other hand, the collections $\{W_1, W_3, \ldots\}$, $\{W_2, W_4, \ldots\}$ are discrete. Therefore the restriction of e to their respective unions must vanish, thus from the lemma, $e = 0$.]

Notation: Write $K(G,q)$ for an Eilenberg-MacLane space of type (G,q) realized as an ANR in NES(paracompact) (cf. p. 6–43).

PROPOSITION 5 Let X be a paracompact Hausdorff space—then X satisfies Okuyama's condition at n iff every closed subset $A \subset X$ has the EP w.r.t. $K(G,q) \ \forall \ q \geq n$.

[There are two points: (1) $\check{H}^q(X;G) \approx [X, K(G,q)]$, $\check{H}^q(A;G) \approx [A, K(G,q)]$; (2) A has the HEP w.r.t. $K(G,q)$ (cf. p. 6–46).]

Application: Let X be a paracompact Hausdorff space—then $\dim_G X \leq n$ iff every closed subset $A \subset X$ has the EP w.r.t. $K(G,q)$ $\forall\, q \geq n$.

PROPOSITION 6 The following conditions on a paracompact Hausdorff space X are equivalent. $(1)_n$ $\forall$ closed $A \subset X : \check{H}^q(X,A;G) = 0$ $(q > n)$; $(2)_n$ $\forall$ closed $A \subset X$: $\check{H}^{n+1}(X,A;G) = 0$; $(3)_n$ $\forall$ closed $A \subset X : \check{H}^n(X;G) \twoheadrightarrow \check{H}^n(A;G)$.

[Trivially, $(1)_n \Rightarrow (2)_n$, $(2)_n \Rightarrow (3)_n$. And: $(4)_n \Rightarrow (3)_{n+1}$, $(3)_n \wedge (4)_n \Rightarrow (2)_n$, where $(4)_n$ is the condition $\check{H}^{n+1}(A;G) = 0$ $\forall$ closed $A \subset X$. In addition, $(1)_n = \bigwedge\limits_{q \geq n} (2)_q$. Suppose that $(3)_n$ holds—then the claim is that $(3)_q \wedge (4)_q$ holds $\forall\, q \geq n$, hence that $(1)_n$ holds. Here is the pattern for the argument: $(3)_n \Rightarrow (4)_n \Rightarrow (3)_{n+1} \Rightarrow (4)_{n+1} \Rightarrow \cdots$. Therefore one has to show that $(3)_q \Rightarrow (4)_q$ $\forall\, q \geq n$. But $(3)_q$ gives $\check{H}^{q+1}(X;G) = 0$ (see the proof of sufficiency in Proposition 4) and since $(3)_q$ is inherited by A, $\check{H}^{q+1}(A;G) = 0$ too.]

Application: Let X be a paracompact Hausdorff space—then $\dim_G X \leq n$ iff every closed subset $A \subset X$ has the EP w.r.t. $K(G,n)$.

[Note: This result is the cohomological counterpart to Alexandroff's criterion. If X is compact or stratifiable, then one can take $K(G,n)$ as a CW complex (cf. p. 6–43).]

EXAMPLE Suppose that X is an ANR and let $G = \prod\limits_i G_i$ be the direct product of abelian groups $G_i \neq 0$—then $\dim_G X = \sup \dim_{G_i} X$.

[Since each G_i is a direct summand of G, $\dim_G X \geq \dim_{G_i} X$ $\forall\, i$, so if $\sup \dim_{G_i} X = \infty$, we are done. Assume therefore that $\sup \dim_{G_i} X = n$. Consider the product $Y = \prod\limits_i K(G_i, n)$—then every closed subset $A \subset X$ has the EP w.r.t. Y, hence every closed subset $A \subset X$ has the EP w.r.t. $|\sin Y|$ (cf. p. 6–46). But $|\sin Y|$ is a CW complex and, as such, is an Eilenberg-MacLane space of type (G,n).]

PROPOSITION 7 Let X be a nonempty paracompact Hausdorff space—then $\dim X = 0$ iff $\dim_G X = 0$ $\forall\, G \neq 0$.

[By Proposition 2, $\dim X = 0 \Rightarrow \dim_G X = 0$. Conversely, since G (discrete topology) $= K(G,0) \in \text{NES(paracompact)}$ contains $\mathbf{S}^0$ as a retract (G being nontrivial), every closed subset $A \subset X$ has the EP w.r.t. $\mathbf{S}^0$, hence $\dim X \leq 0$ (Alexandroff's criterion).]

Examples: $\forall\, G \neq 0$, (1) $\dim_G[0,1] = 1$; (2) $\dim_G \mathbf{R} = 1$; (3) $\dim_G \mathbf{S}^1 = 1$.

EXAMPLE Let X be a paracompact Hausdorff space of finite topological dimension—then $\dim_G \beta X \leq \dim_G X$ provided that G is finitely generated.

[It suffices to show that $\dim_G X \leq n \Rightarrow \dim_G \beta X \leq n$. This is trivial if $X = \emptyset$ or $G = 0$, so take X nonempty and G nonzero. Because $\dim \beta X = \dim X$ (cf. §19, Proposition 1), from Proposition 7, $\dim_G X = 0 \Rightarrow \dim X = 0 \Rightarrow \dim \beta X = 0 \Rightarrow \dim_G \beta X = 0$. Suppose now that n is positive and let A be a closed subset of βX. Claim: $\check{H}^{n+1}(A;G) = 0$, which is enough (cf. Proposition 6: $(1)_n \Leftrightarrow (4)_n$). To verify this, fix an $\alpha \in \check{H}^{n+1}(A;G)$ and, using the extension principle, choose an open $U \supset A$ and an $e \in \check{H}^{n+1}(\overline{U};G) : e|A = \alpha$. Since $\beta(\overline{U} \cap X) = \mathrm{cl}_{\beta X}(\overline{U} \cap X) = \overline{U}$, $\check{H}^{n+1}(\overline{U};G) \approx \check{H}^{n+1}(\overline{U} \cap X;G)$ (cf. p. 20–4). But $\dim_G X \leq n \Rightarrow \check{H}^{n+1}(\overline{U} \cap X;G) = 0$, so $e = 0$, thus $\alpha = 0$.]

[Note: Consequently, under the stated hypotheses on X and G, $\dim_G X = \dim_G \beta X$ (cf. p. 20–4).]

Remark: If the topological dimension of X is infinite, then one can find examples for which $\dim_{\mathbf{Z}} X \neq \dim_{\mathbf{Z}} \beta X$ (Dranishnikov[†]).

EXAMPLE For any nonempty paracompact Hausdorff space X, $\dim_{\mathbf{Z}} X = 1$ iff $\dim X = 1$.

[If $\dim_{\mathbf{Z}} X = 1$, then every closed subset $A \subset X$ has the EP w.r.t. $\mathbf{S}^1 = K(\mathbf{Z},1)$, hence $\dim X \leq 1$ (Alexandroff's criterion) and $\dim X = 0$ is untenable (cf. Proposition 7).]

PROPOSITION 8 Let X be a paracompact Hausdorff space—then for any closed subspace A of X, $\dim_G A \leq \dim_G X$.

EXAMPLE Let X be a paracompact LCH space—then $\dim_G X = \sup \dim_G K$, where $K \subset X$ is compact.

[Since $\dim_G X \geq \dim_G K \ \forall\, K$ (cf. Proposition 8), $\sup \dim_G K = \infty \Rightarrow \dim_G X = \infty$. So suppose that $\sup \dim_G K = n$. Write $X = \bigcup_i K_i$, where K_i is compact and $\{K_i : i \in I\}$ is neighborhood finite. Well order I and deduce by transfinite induction that every closed subset $A \subset X$ has the EP w.r.t. $K(G,n)$, hence that $\dim_G X \leq n$.]

FACT Let X be a closed subset of $\mathbf{R}^n$—then $\dim X = n-1$ iff $\dim_G X = n-1 \ \forall\, G \neq 0$.

PROPOSITION 9 Let X be a paracompact Hausdorff space. Suppose that $X = \bigcup_1^\infty A_j$, where the A_j are closed subspaces of X such that $\forall\, j$, $\dim_G A_j \leq n$—then $\dim_G X \leq n$, hence $\dim_G X = \sup \dim_G A_j$.

[Fix a closed subset $A \subset X$ and a continuous function $f : A \to K(G,n)$. Put $U_0 = A$ and $F_0 = f$—then since $\dim_G A_1 \leq n$, $F_0|U_0 \cap A_1$ has a continuous extension $\Phi_0 : A_1 \to K(G,n)$. Define a continuous function $f_1 : U_0 \cup A_1 \to K(G,n)$ by $f_1|U_0 = F_0$ &

[†] *C.R. Acad. Bulgare Sci.* **41** (1988), 9–10; see also, Dydak-Walsh, *Proc. Amer. Math. Soc.* **113** (1991), 1155–1162 and Dydak, *Topology Appl.* **50** (1993), 1–10.

$f_1|A_1 = \Phi_0$. Recalling that $K(G,n) \in \text{NES(paracompact)}$, choose an open $U_1 \supset U_0 \cup A_1$ and a continuous function $F_1 : \overline{U}_1 \rightarrow K(G,n)$ such that $F_1|U_0 \cup A_1 = f_1$. Since $\dim_G A_2 \leq n$, $F_1|\overline{U}_1 \cap A_2$ has a continuous extension $\Phi_1 : A_2 \rightarrow K(G,n)$. Define a continuous function $f_2 : \overline{U}_1 \cup A_2 \rightarrow K(G,n)$ by $f_2|\overline{U}_1 = F_1$ & $f_2|A_2 = \Phi_1$. Choose an open $U_2 \supset \overline{U}_1 \cup A_2$ and a continuous function $F_2 : \overline{U}_2 \rightarrow K(G,n)$ such that $F_2|\overline{U}_1 \cup A_2 = f_2$. Continue the process to get a sequence of open sets $U_j\ (j \geq 1) : \overline{U}_j \cup A_{j+1} \subset U_{j+1}$ and a sequence of continuous functions $F_j : \overline{U}_j \rightarrow K(G,n)\ (j \geq 1) : F_{j+1}|\overline{U}_j = F_j$. Finally, if $F : X \rightarrow K(G,n)$ is defined by $F|\overline{U}_j = F_j$, then F is a continuous extension of f ($X = \bigcup_j \overline{U}_j$ has the final topology corresponding to the inclusions $\overline{U}_j \rightarrow X$).]

Proposition 9 is the analog for cohomological dimension of the countable union lemma for topological dimension but there are instances where the parallel breaks down. Here is a case in point. Suppose that $X = Y \cup Z$ is metrizable—then $\dim X \leq \dim Y + \dim Z + 1$ (cf. §19, Proposition 7). The situation for cohomological dimension is more complicated.

(R) For any ring R with unit, $\dim_R X \leq \dim_R Y + \dim_R Z + 1$.

(G) For any abelian group $G \neq 0$, $\dim_G X \leq \dim_G Y + \dim_G Z + 2$.

[Note: These estimates cannot be improved. See Dydak[†] for details and references.]

FACT Suppose that X is a paracompact Hausdorff space. Let $\mathcal{A} = \{A_j : j \in J\}$ be an absolute closure preserving closed covering of X such that $\forall\, j$, $\dim_G A_j \leq n$—then $\dim_G X \leq n$, hence $\dim_G X = \sup \dim_G A_j$.

LEMMA If K is a finite CW complex, then $\dim_G K = \dim K \ \forall\, G \neq 0$.

[On general grounds, $\dim_G K \leq \dim K$ (cf. Proposition 2). Taking $K \neq \emptyset$, let $n = \dim_G K > 0$ (cf. Proposition 7), and suppose that $k = \dim K > n$. Fix a k-cell $e \subset K$ and let $\mathbf{S}^n$ be an n-sphere contained in e. Since $G \neq 0$, $\exists$ a map $f : \mathbf{S}^n \rightarrow K(G,n)$ which induces a nontrivial homomorphism $\mathbf{Z} = \pi_n(\mathbf{S}^n) \rightarrow \pi_n(K(G,n)) = G$. But f admits a continuous extension $K \rightarrow K(G,n)$. Therefore $\pi_n(f)$ is trivial, $\mathbf{S}^n$ being contractible in K. Contradiction.]

EXAMPLE Let X be a CW complex—then the collection $\{K\}$ of finite subcomplexes of X is an absolute closure preserving closed covering of X, thus $\dim X = \sup \dim K$ (cf. p. 19–21). On the other hand, $\forall\, G \neq 0$, $\dim_G X = \sup \dim_G K$ (cf. supra) and by the lemma, $\dim_G K = \dim K$. Therefore $\dim_G X = \dim X$.

Examples: $\forall\, G \neq 0$, (1) $\dim_G [0,1]^n = n$; (2) $\dim_G \mathbf{R}^n = n$; (3) $\dim_G \mathbf{S}^n = n$.

[†] *Trans. Amer. Math. Soc.* **348** (1996), 1647–1661.

EXAMPLE Let X be a paracompact n-manifold—then $\forall\, G \neq 0$, $\dim_G X = n$ (cf. p. 19–22).

PROPOSITION 10 Let X be a paracompact Hausdorff space. Assume: X is hereditarily paracompact—then for any subspace Y of X, $\dim_G Y \leq \dim_G X$.

PROPOSITION 11 Let X be a paracompact Hausdorff space. Suppose that Y is a strongly paracompact subspace—then $\dim_G Y \leq \dim_G X$.

EXAMPLE Suppose that X contains an embedded copy of $\mathbf{R}^n$—then $\forall\, G \neq 0$, $\dim_G X \geq n$.

LEMMA Let X be a nonempty paracompact Hausdorff space, Y a nonempty compact Hausdorff space. Assume: $\left\{\begin{matrix}\dim X\\ \dim Y\end{matrix}\right. < \infty$—then $\forall\, G \neq 0$, $\dim_G(X \times Y)$ is the largest integer n such that $\check{H}^n((A', A) \times (B', B); G) \neq 0$ for certain closed sets $\left\{\begin{matrix}A \subset A' \subset X\\ B \subset B' \subset Y\end{matrix}\right.$.

[By the product theorem, $\dim(X \times Y) \leq \dim X + \dim Y$, so Proposition 2 implies that $\dim_G(X \times Y)$ is finite. This said, to prove the lemma, it suffices to show that whenever $m > n$ and $\check{H}^m((A', A) \times (B', B); G) = 0$ for all closed sets $\left\{\begin{matrix}A \subset A' \subset X\\ B \subset B' \subset Y\end{matrix}\right.$, then $\dim_G(X \times Y) \leq n$. Thus let $W \subset X \times Y$ be closed. Fix a continuous function $f : W \to K(G, n)$—then $\exists$ an open $U \supset W : f$ is continuously extendable over U. The open covering $\mathcal{W} = \{U, X \times Y - W\}$ is numerable, hence by the stacking lemma, there exists a neighborhood finite open covering $\mathcal{U} = \{U_i : i \in I\}$ of X and $\forall\, i \in I$, a finite open covering $\mathcal{V}_i = \{V_{i,j} : j \in J_i\}$ of Y such that the collection $\{U_i \times \mathcal{V}_i : i \in I\}$ refines $\mathcal{W}$. Choose a neighborhood finite open covering $\mathcal{O} = \{O_\lambda : \lambda \in \Lambda\}$ of X of order $\leq \dim X + 1$ such that $\{\mathrm{st}(x, \mathcal{O}) : x \in X\}$ is a refinement of $\mathcal{U}$ (cf. §19, Proposition 6). Given $\xi = (\lambda_1, \ldots, \lambda_p) \in \Lambda^p$, put $A_\xi = X - \bigcup\limits_{\lambda \in \Lambda} \{O_\lambda : \lambda \neq \lambda_i \ (1 \leq i \leq p)\}$ if $\bigcap\limits_{i=1}^{p} O_{\lambda_i} \neq \emptyset$, otherwise put $A_\xi = \emptyset$. The covering $\mathcal{A} = \bigcup\limits_{p=1}^{d} \mathcal{A}_p$, where $\mathcal{A}_p = \{A_\xi : \xi \in \Lambda^p\}$ and $d = \dim X + 1$, is a neighborhood finite closed refinement of $\mathcal{U}$. For each $A_\xi \in \mathcal{A}$, determine $U_{i(\xi)} \in \mathcal{U} : A_\xi \subset U_{i(\xi)}$. Let $\mathcal{B}_i = \{B_{i,j} : j \in J_i\}$ be a precise closed refinement of $\mathcal{V}_i$. The collection $\{A_\xi \times B_{i(\xi),j} : A_\xi \in \mathcal{A}, j \in J_{i(\xi)}\}$ is therefore a neighborhood finite closed refinement of $\mathcal{W} = \{U, X \times Y - W\}$. Set $A_0 = \bigcup\{A_\xi \times B_{i(\xi),j} : W \cap (A_\xi \times B_{i(\xi),j}) \neq \emptyset\}$. Since $W \subset A_0 \subset U$, $\exists$ a continuous function $f_0 : A_0 \to K(G, n)$ such that $f_0 | W = f$. Now put $A_p = A_0 \cup \{A_\xi \times Y : \xi \in \Lambda^p\}$ $(1 \leq p \leq d)$ and assume that f_0 has a continuous extension $f_{p-1} : A_{p-1} \to K(G, n)$ for some $p \geq 1$. Claim: f_{p-1} can be continuously extended over A_p. To see this, note first that $\xi, \xi' \in \Lambda^p$ & $\xi \neq \xi' \Rightarrow A_\xi \cap A_{\xi'} \in \mathcal{A}_{p-1}$, so it will be enough to establish that $f_{p-1,\xi} \equiv f_{p-1} | (A_{p-1} \cap (A_\xi \times Y))$ is continuously extendable over $A_\xi \times Y$ for each $\xi \in \Lambda^p$. Write $J_{i(\xi)} = \{j : 1 \leq j \leq j_{i(\xi)}\}$. Suppose inductively that

$f_{p-1,\xi}$ has been continuously extended over $(A_{p-1} \cap (A_\xi \times Y)) \cup \bigcup_{1 \leq j \leq k-1} A_\xi \times B_{i(\xi),j}$ for some $k \leq j_{i(\xi)}$. Let $A' = A_\xi$, $A = A_\xi \cap (\bigcup_{\xi' \in \Lambda^{p-1}} A_{\xi'})$, $B' = B_{i(\xi),k}$, $B = \bigcup_{1 \leq j \leq k-1} B_{i(\xi),k} \cap B_{i(\xi),j} \cup \bigcup\{B_{i(\xi),k} \cap B_{i(\xi'),j} : W \cap (A_{\xi'} \times B_{i(\xi'),j}) \neq \emptyset\}$—then from the exact sequence $\cdots \to \check{H}^n(A' \times B'; G) \to \check{H}^n(A' \times B \cup A \times B'; G) \to \check{H}^{n+1}((A', A) \times (B', B); G) \to \cdots$, it follows that the arrow $\check{H}^n(A' \times B'; G) \to \check{H}^n(A' \times B \cup A \times B'; G)$ is surjective. Accordingly, every continuous function $A' \times B \cup A \times B' \to K(G, n)$ can be extended to a continuous function $A' \times B' \to K(G, n)$. In particular: $f_{p-1,\xi}$ is continuously extendable over $(A_{p-1} \cap (A_\xi \times Y)) \cup \bigcup_{1 \leq j \leq k} A_\xi \times B_{i(\xi),j}$, which completes the induction. Consequently, f_{p-1} extends to a continuous function $f_p : A_p \to K(G, n)$, hence by induction once again, f extends to a continuous function $f_d : X \times Y \to K(G, n)$.]

PROPOSITION 12 Let X be a nonempty paracompact Hausdorff space, Y a nonempty compact Hausdorff space. Assume: $\begin{cases} \dim X \\ \dim Y \end{cases} < \infty$—then $\forall\, G \neq 0$, $\dim_G(X \times Y)$ is the largest integer n such that $\check{H}^n((X, A) \times (Y, B); G) \neq 0$ for certain closed sets $\begin{cases} A \subset X \\ B \subset Y \end{cases}$.

[Suppose that $n = \dim_G(X \times Y)$. Using the lemma, choose closed sets $\begin{cases} A \subset A' \subset X \\ B \subset B' \subset Y \end{cases}$ such that $\check{H}^n((A', A) \times (B', B); G) \neq 0$. Put $C = A' \times B' \cup X \times B \cup A \times Y$—then $(A' \times B') \cap (X \times B \cup A \times Y) = A' \times B \cup A \times B'$, thus by the relative homeomorphism theorem, $\check{H}^n(C, X \times B \cup A \times Y; G) \approx \check{H}^n(A' \times B', A' \times B \cup A \times B'; G) \neq 0$. Consider the exact sequence $\cdots \to \check{H}^n((X, A) \times (Y, B); G) \to \check{H}^n(C, X \times B \cup A \times Y; G) \to \check{H}^{n+1}(X \times Y, C; G) \to \cdots$ corresponding to the triple $(X \times Y, C, X \times B \cup A \times Y)$. Since $n = \dim_G(X \times Y)$, $\check{H}^{n+1}(X \times Y, C; G) = 0$, hence $\check{H}^n((X, A) \times (Y, B); G) \neq 0$.]

Application: Under the preceding hypotheses on $X \& Y$, $\dim_G(X \times Y) \leq n$ iff $\check{H}^q((X, A) \times (Y, B); G) = 0 \; \forall\, q > n$ and for all closed sets $\begin{cases} A \subset X \\ B \subset Y \end{cases}$.

EXAMPLE With X & Y as in Proposition 12, suppose that $\exists\, k : \dim_{\check{H}^{k-i}(Y, B; G)} X \leq i \; \forall\, i \geq 0$ and all closed subsets $B \subset Y$—then $\dim_G(X \times Y) \leq k$.

[It is a question of verifying that $\check{H}^l((X, A) \times (Y, B); G) = 0 \; \forall\, l \geq k+1$. But by the Künneth formula, $\check{H}^l((X, A) \times (Y, B); G) \approx \bigoplus_{q=0}^{l} \check{H}^q(X, A; \check{H}^{l-q}(Y, B; G)) \approx \check{H}^q(X, A; \check{H}^{k-(q-l+k)}(Y, B; G)) = 0$.]

EXAMPLE With X & Y as in Proposition 12, suppose that $\dim_{\check{H}^m(Y, B; G)} X \geq n$ for some closed subset $B \subset Y$—then $\dim_G(X \times Y) \geq n + m$.

[Choose a closed subset $A \subset X : \check{H}^n(X, A; \check{H}^m(Y, B; G)) \neq 0 \Rightarrow \check{H}^{n+m}((X, A) \times (Y, B); G) \approx \bigoplus_{q=0}^{n+m} \check{H}^q(X, A; \check{H}^{n+m-q}(Y, B; G)) \neq 0$, hence $\dim_G(X \times Y) \geq n + m$.]

PROPOSITION 13 Let X be a nonempty paracompact Hausdorff space of finite topological dimension—then $\forall\, G \neq 0$, $\dim_G IX = \dim_G X + 1$.

[$\dim_G IX \geq \dim_G X + 1$: Choose a closed subset $A \subset X : \check{H}^n(X, A; G) \neq 0$, where $n = \dim_G X$. Applying the Künneth formula, we have $\check{H}^{n+1}((X, A) \times ([0,1], \{0,1\}); G) \approx \bigoplus_{q=0}^{n+1} \check{H}^q(X, A; \check{H}^{n+1-q}([0,1], \{0,1\}; G)) \approx \check{H}^n(X, A; \check{H}^1([0,1], \{0,1\}; G)) \approx \check{H}^n(X, A; G) \neq 0$, which implies that $\dim_G IX \geq \dim_G X + 1$.

$\dim_G X + 1 \geq \dim_G IX$: Fix $m \geq n + 2$ ($n = \dim_G X$) and let $\begin{cases} A \subset X \\ B \subset I \end{cases}$ be closed ($I = [0,1]$). Utilization of the Künneth formula then gives $\check{H}^m((X, A) \times (I, B); G) \approx \check{H}^m(X, A; \check{H}^0(I, B; G)) \oplus \check{H}^{m-1}(X, A; \check{H}^1(I, B; G))$. Case 1: $B = \emptyset$. Here, $\check{H}^0(I, \emptyset; G) = G$, $\check{H}^1(I, \emptyset; G) = 0$, hence $\check{H}^m((X, A) \times (I, B); G) = 0$. Case 2: $B \neq \emptyset$. Here, $\check{H}^0(I, B; G) = 0$, $\check{H}^1(I, B; G) = \check{H}^1(I, B; \mathbf{Z}) \otimes G$ (by the universal coefficient theorem), hence $\check{H}^m((X, A) \times (I, B); G) \approx \check{H}^{m-1}(X, A; \check{H}^1(I, B; \mathbf{Z}) \otimes G) = 0$ (cf. Proposition 18 ($m - 1 \geq n + 1$)). Therefore $\dim_G X + 1 \geq \dim_G IX$.]

Application: Let X be a nonempty paracompact Hausdorff space, Y a nonempty CW complex. Assume: $\begin{cases} \dim X \\ \dim Y \end{cases} < \infty$—then $\forall\, G \neq 0$, $\dim_G(X \times Y) = \dim_G X + \dim_G Y$ ($= \dim_G X + \dim Y$ (cf. p. 20–9)).

[If B is a compact subspace of Y which is homeomorphic to $[0,1]^m$, where $m = \dim_G Y$, then $\dim_G(X \times B) = \dim_G X + m = \dim_G X + \dim_G Y$.]

[Note: Y is paracompact and σ-locally compact, thus $X \times Y$ is paracompact (cf. p. 19–36).]

FACT Let X be a nonempty paracompact Hausdorff space, Y a nonempty compact Hausdorff space. Assume: $\dim X < \infty$ & $\dim Y = 0$—then $\forall\, G \neq 0$, $\dim_G(X \times Y) = \dim_G X$.

[It is clear that $\dim_G(X \times Y) \geq \dim_G X$ (cf. Proposition 8). With $n = \dim_G X$, fix $m \geq n+1$ and let $\begin{cases} A \subset X \\ B \subset Y \end{cases}$ be closed. From the Künneth formula, $\check{H}^m((X, A) \times (Y, B); G) \approx \bigoplus_{q=0}^{m} \check{H}^q(X, A; \check{H}^{m-q}(Y, B; G))$. But $\dim Y = 0 \Rightarrow \dim_G Y = 0$ (cf. Proposition 2), so $\check{H}^{m-q}(Y, B; G) = 0$ if $q \leq m - 1$, thus $\check{H}^m((X, A) \times (Y, B); G) \approx \check{H}^m(X, A; \check{H}^0(Y, B; G)) \approx \check{H}^m(X, A; \check{H}^0(Y, B; \mathbf{Z}) \otimes G) = 0$ (cf. Proposition 18). Therefore $\dim_G(X \times Y) \leq n$.]

PROPOSITION 14 Let X be a paracompact Hausdorff space. Suppose that $\{G_\alpha\}$ is a collection of subgroups of an abelian group G which is directed by inclusion. Assume: $G = \operatorname{colim} G_\alpha$—then $\dim_G X \leq n$ if $\forall\, \alpha$, $\dim_{G_\alpha} X \leq n$, hence $\dim_G X \leq \sup \dim_{G_\alpha} X$.

[This is a special case of the generalities on p. 20–5.]

DIRECT SUM CRITERION Let X be a paracompact Hausdorff space—then $\dim_{\bigoplus_i G_i} X = \sup \dim_{G_i} X$.

[Apply Proposition 14 (cf. p. 20–5).]

EXAMPLE Since $\widehat{\mathbf{Z}}_p/\mathbf{Z}_p$ is a vector space over $\mathbf{Q}$, $\dim_{\widehat{\mathbf{Z}}_p/\mathbf{Z}_p} X = \dim_{\mathbf{Q}} X$.

PROPOSITION 15 Let X be a paracompact Hausdorff space. Suppose that $0 \to G' \to G \to G'' \to 0$ is a short exact sequence of abelian groups—then $\dim_G X \leq \max\{\dim_{G'} X, \dim_{G''} X\}$, $\dim_{G'} X \leq \max\{\dim_G X, \dim_{G''} X + 1\}$, and $\dim_{G''} X \leq \max\{\dim_G X, \dim_{G'} X - 1\}$.

[Use the Bockstein sequence.]

EXAMPLE (Bockstein's Inequalities) Let X be a paracompact Hausdorff space and fix a prime p.

(BO$_1$) $\dim_{\mathbf{Z}/p\mathbf{Z}} X = \dim_{\mathbf{Z}/p^n\mathbf{Z}} X$.

[From the short exact sequence $0 \to \mathbf{Z}/p^n\mathbf{Z} \to \mathbf{Z}/p^{n+1}\mathbf{Z} \to \mathbf{Z}/p\mathbf{Z} \to 0$, it follows that $\dim_{\mathbf{Z}/p^{n+1}\mathbf{Z}} X \leq \max\{\dim_{\mathbf{Z}/p^n\mathbf{Z}} X, \dim_{\mathbf{Z}/p\mathbf{Z}} X\}$ and $\dim_{\mathbf{Z}/p\mathbf{Z}} X \leq \max\{\dim_{\mathbf{Z}/p^{n+1}\mathbf{Z}} X, \dim_{\mathbf{Z}/p^n\mathbf{Z}} X - 1\}$. Now argue by induction.]

(BO$_2$) $\dim_{\mathbf{Z}/p^\infty\mathbf{Z}} X \leq \dim_{\mathbf{Z}/p\mathbf{Z}} X$.

[Since $\mathbf{Z}/p^\infty\mathbf{Z} = \operatorname{colim} \mathbf{Z}/p^n\mathbf{Z}$, Proposition 14 implies that $\dim_{\mathbf{Z}/p^\infty\mathbf{Z}} X \leq \sup \dim_{\mathbf{Z}/p^n\mathbf{Z}} X = \dim_{\mathbf{Z}/p\mathbf{Z}} X$.]

(BO$_3$) $\dim_{\mathbf{Z}/p\mathbf{Z}} X \leq \dim_{\mathbf{Z}/p^\infty\mathbf{Z}} X + 1$.

[Consider the short exact sequence $0 \to \mathbf{Z}/p\mathbf{Z} \to \mathbf{Z}/p^\infty\mathbf{Z} \xrightarrow{p} \mathbf{Z}/p^\infty\mathbf{Z} \to 0$.]

(BO$_4$) $\dim_{\mathbf{Z}/p\mathbf{Z}} X \leq \dim_{\mathbf{Z}_p} X$.

[Consider the short exact sequence $0 \to \mathbf{Z}_p \xrightarrow{p} \mathbf{Z}_p \to \mathbf{Z}/p\mathbf{Z} \to 0$.]

(BO$_5$) $\dim_{\mathbf{Q}} X \leq \dim_{\mathbf{Z}_p} X$.

[Consider the short exact sequence $0 \to \mathbf{Z}_p \to \mathbf{Q} \to \mathbf{Z}/p^\infty\mathbf{Z} \to 0$.]

[Note: In addition, $\dim_{\mathbf{Z}_p} X \leq \max\{\dim_{\mathbf{Q}} X, \dim_{\mathbf{Z}/p^\infty\mathbf{Z}} X + 1\}$, $\dim_{\mathbf{Z}/p^\infty\mathbf{Z}} X \leq \max\{\dim_{\mathbf{Q}} X, \dim_{\mathbf{Z}_p} X - 1\}$.]

Warning: Bockstein's inequalities are used without citation in the sequel.

FACT Let X be a compact Hausdorff space. Suppose that $0 \to G' \to G \to G'' \to 0$ is a short exact sequence of abelian groups. Assume: G'' is torsion free—then $\dim_G X = \max\{\dim_{G'} X, \dim_{G''} X\}$.

EXAMPLE Let X be a compact Hausdorff space—then $\dim_{\mathbf{Z}_p} X = \dim_{\widehat{\mathbf{Z}}_p} X$.

[From the short exact sequence $0 \to \mathbf{Z}_p \to \widehat{\mathbf{Z}}_p \to \widehat{\mathbf{Z}}_p/\mathbf{Z}_p \to 0$, we have $\dim_{\widehat{\mathbf{Z}}_p} X = \max\{\dim_{\mathbf{Z}_p} X, \dim_{\widehat{\mathbf{Z}}_p/\mathbf{Z}_p} X\}$. But $\dim_{\widehat{\mathbf{Z}}_p/\mathbf{Z}_p} X = \dim_{\mathbf{Q}} X$ and $\dim_{\mathbf{Q}} X \leq \dim_{\mathbf{Z}_p} X$.]

A Bockstein function is a function D defined on $\{\mathbf{Q}\} \cup \bigcup_p \{\mathbf{Z}_p, \mathbf{Z}/p\mathbf{Z}, \mathbf{Z}/p^\infty\mathbf{Z}\}$ with values in $\mathbf{Z}_{\geq 0} \cup \{\infty\}$ such that $D(\mathbf{Z}/p^\infty\mathbf{Z}) \leq D(\mathbf{Z}/p\mathbf{Z})$, $D(\mathbf{Z}/p\mathbf{Z}) \leq D(\mathbf{Z}/p^\infty\mathbf{Z}) + 1$, $D(\mathbf{Z}/p\mathbf{Z}) \leq D(\mathbf{Z}_p)$, $D(\mathbf{Q}) \leq D(\mathbf{Z}_p)$, $D(\mathbf{Z}_p) \leq \max\{D(\mathbf{Q}), D(\mathbf{Z}/p^\infty\mathbf{Z}) + 1\}$, $D(\mathbf{Z}/p^\infty\mathbf{Z}) \leq \max\{D(\mathbf{Q}), D(\mathbf{Z}_p) - 1\}$, and D is $\equiv 0$ if $D(G) = 0$ $(\exists\, G)$ (cf. Proposition 7).

Example: Every nonempty paracompact Hausdorff space X gives rise to a Bockstein function D_X, viz. $D_X(G) = \dim_G X$.

DRANISHNIKOV'S[†] REALIZATION THEOREM Given a Bockstein function D, $\exists$ a metrizable compact Hausdorff space X such that $D = D_X$ and $\dim X = \sup D$.

EXAMPLE The fundamental compacta are those metrizable compact Hausdorff spaces which realize the Bockstein functions defined by the table below.

D	$\mathbf{Z}_p$	$\mathbf{Z}/p\mathbf{Z}$	$\mathbf{Z}/p^\infty\mathbf{Z}$	$\mathbf{Q}$	$\mathbf{Z}_q$	$\mathbf{Z}/q\mathbf{Z}$	$\mathbf{Z}/q^\infty\mathbf{Z}$
$\Phi(\mathbf{Q}, n)$	n	1	1	n	n	1	1
$\Phi(\mathbf{Z}_p, n)$	n	n	n	n	n	1	1
$\Phi(\mathbf{Z}/p\mathbf{Z}, n)$	n	n	$n-1$	1	1	1	1
$\Phi(\mathbf{Z}/p^\infty\mathbf{Z}, n)$	n	$n-1$	$n-1$	1	1	1	1

[Note: Here p, q are primes, q runs over all primes $\neq p$, and $\Phi(G, n)$ is the Bockstein function corresponding to the pair (G, n), where $G = \mathbf{Q}$, $\mathbf{Z}_p$, $\mathbf{Z}/p\mathbf{Z}$, $\mathbf{Z}/p^\infty\mathbf{Z}$.]

Notation: Given an abelian group G, G_{tor} is its torsion subgroup and $G_{\text{tor}}(p)$ is the p-primary component of G_{tor} (so $G_{\text{tor}} \approx \bigoplus_p G_{\text{tor}}(p)$).

[Note: Accordingly, for a paracompact Hausdorff space X, $\dim_{G_{\text{tor}}} X = \sup \dim_{G_{\text{tor}}(p)} X$ (direct sum criterion).]

Given an abelian group G, its Bockstein basis $\sigma(G)$ is the subset of $\{\mathbf{Q}\} \cup \bigcup_p \{\mathbf{Z}_p, \mathbf{Z}/p\mathbf{Z}, \mathbf{Z}/p^\infty\mathbf{Z}\}$ defined as follows.

($\mathbf{Q}$) $\mathbf{Q} \in \sigma(G)$ iff $G/G_{\text{tor}} \neq 0$.

($\mathbf{Z}_p$) $\mathbf{Z}_p \in \sigma(G)$ iff G/G_{tor} is not divisible by p.

($\mathbf{Z}/p\mathbf{Z}$) $\mathbf{Z}/p\mathbf{Z} \in \sigma(G)$ iff $G_{\text{tor}}(p)$ is not divisible by p.

[†] *Siberian Math. J.* **29** (1988), 24–29, **30** (1989), 74–79, and **32** (1991), 145–147; see also Dydak, *Topology Appl.* **65** (1995), 1–7.

($\mathbf{Z}/p^\infty\mathbf{Z}$) $\mathbf{Z}/p^\infty\mathbf{Z} \in \sigma(G)$ iff $G_{\text{tor}}(p) \neq 0$ is divisible by p.

Examples: (1) $\sigma(\mathbf{Q}) = \{\mathbf{Q}\}$; (2) $\sigma(\mathbf{Z}_p) = \{\mathbf{Q}, \mathbf{Z}_p\}$; (3) $\sigma(\mathbf{Z}/p\mathbf{Z}) = \{\mathbf{Z}/p\mathbf{Z}\}$; (4) $\sigma(\mathbf{Z}/p^\infty\mathbf{Z}) = \{\mathbf{Z}/p^\infty\mathbf{Z}\}$; (5) $\sigma(\mathbf{Z}) = \{\mathbf{Q}\} \cup \bigcup\limits_p \{\mathbf{Z}_p\}$; (6) $\sigma(\widehat{\mathbf{Z}}_p) = \{\mathbf{Q}, \mathbf{Z}_p\}$.

Remark: $\forall\, G \neq 0$, $\sigma(G)$ is nonempty. Indeed, if $G \neq G_{\text{tor}}$, then $\mathbf{Q} \in \sigma(G)$ and if $G = G_{\text{tor}}$, then $\exists\, p : G_{\text{tor}}(p) \neq 0$, so either $\mathbf{Z}/p\mathbf{Z} \in \sigma(G)$ or $\mathbf{Z}/p^\infty\mathbf{Z} \in \sigma(G)$.

LEMMA Given an abelian group G, $\sigma(G) = \sigma(G/G_{\text{tor}}) \cup \bigcup\limits_p \sigma(G_{\text{tor}}(p))$.

FACT If $G_{\text{tor}}(p)$ is not divisible by p, then $\exists\, n \geq 1 : \mathbf{Z}/p^n\mathbf{Z}$ is a direct summand of G.

FACT If $G_{\text{tor}}(p) \neq 0$ is divisible by p, then $G_{\text{tor}}(p) \approx \oplus \mathbf{Z}/p^\infty\mathbf{Z}$ and $G_{\text{tor}}(p)$ is a direct summand of G.

PROPOSITION 16 Let X be a paracompact Hausdorff space. Suppose that $G \neq 0$ is torsion—then $\dim_G X = \sup\limits_{H \in \sigma(G)} \dim_H X$.

[From what has been said above, one can assume that $G = G(p)$ ($\exists\, p$).

($\mathbf{Z}/p\mathbf{Z}$) If $\mathbf{Z}/p\mathbf{Z} \in \sigma(G)$, then $\dim_{\mathbf{Z}/p\mathbf{Z}} X = \max\limits_{H \in \sigma(G)} \dim_H X$. But $\mathbf{Z}/p^n\mathbf{Z}$ is a direct summand of G for some $n \geq 1$, thus $\dim_G X \geq \dim_{\mathbf{Z}/p^n\mathbf{Z}} X = \dim_{\mathbf{Z}/p\mathbf{Z}} X$. On the other hand, G is a colimit of its finite subgroups. As these are direct sums of groups of the form $\mathbf{Z}/p^k\mathbf{Z}$, $\dim_G X \leq \dim_{\mathbf{Z}/p\mathbf{Z}} X$ by Proposition 14.

($\mathbf{Z}/p^\infty\mathbf{Z}$) In this case, G is isomorphic to a direct sum of copies of $\mathbf{Z}/p^\infty\mathbf{Z}$ and the direct sum criterion is applicable.]

PROPOSITION 17 Let X be a paracompact Hausdorff space—then for any $G \neq 0$, $\dim_G X = \max\{\dim_{G/G_{\text{tor}}} X, \dim_{G_{\text{tor}}} X\}$.

[The short exact sequence $0 \to G_{\text{tor}} \to G \to G/G_{\text{tor}} \to 0$ leads to the inequalities $\dim_G X \leq \max\{\dim_{G_{\text{tor}}} X, \dim_{G/G_{\text{tor}}} X\}$, $\dim_{G/G_{\text{tor}}} X \leq \max\{\dim_G X, \dim_{G_{\text{tor}}} X - 1\}$ (cf. Proposition 15), thus it suffices to prove that $\dim_G X \geq \dim_{G_{\text{tor}}} X$. But if $\mathbf{Z}/p\mathbf{Z} \in \sigma(G)$, then $\mathbf{Z}/p^n\mathbf{Z}$ is a direct summand of G ($\exists\, n \geq 1$), while if $\mathbf{Z}/p^\infty\mathbf{Z} \in \sigma(G)$, then $\mathbf{Z}/p^\infty\mathbf{Z}$ is a direct summand of G. Therefore $\dim_G X \geq \dim_{G_{\text{tor}}} X$ (cf. Proposition 16).]

PROPOSITION 18 Let X be a paracompact Hausdorff space—then $\dim_{G \otimes K} X \leq \dim_G X$ for any two abelian groups G & K.

[This is obvious if either G or K is trivial, so assume that $G \neq 0$ & $K \neq 0$.

(I) $K = \mathbf{Z}^k$ ($k \geq 1$). Here $G \otimes \mathbf{Z}^k$ is a direct sum of copies of G, thus the direct sum criterion is applicable.

(II) $K = \mathbf{Z}/p^k\mathbf{Z}$ $(k \geq 1)$. Case 1: $G_{\text{tor}}(p) = 0$. Since $G \otimes \mathbf{Z}/p^k\mathbf{Z} = G/p^kG$, the exactness of $0 \to G \xrightarrow{p^k} G \to G/p^kG \to 0$ gives $\dim_{G\otimes K} X \leq \dim_G X$ (cf. Proposition 15). Case 2: $G_{\text{tor}}(p) \neq 0$. There are two possibilities: $\mathbf{Z}/p\mathbf{Z} \in \sigma(G)$ or $\mathbf{Z}/p^\infty\mathbf{Z} \in \sigma(G)$. If $\mathbf{Z}/p\mathbf{Z} \in \sigma(G)$, then $\dim_{\mathbf{Z}/p\mathbf{Z}} X \leq \dim_G X$ (cf. Proposition 17). And: $\dim_{G\otimes K} X \leq \dim_{\mathbf{Z}/p\mathbf{Z}} X$ ($G\otimes\mathbf{Z}/p^k\mathbf{Z}$ is p-torsion and $\mathbf{Z}/p\mathbf{Z} \in \sigma(G\otimes\mathbf{Z}/p^k\mathbf{Z})$ (see the proof of Proposition 16)). If $\mathbf{Z}/p^\infty\mathbf{Z} \in \sigma(G)$, then $G = G_{\text{tor}}(p) \oplus H$, where $G \approx \oplus\mathbf{Z}/p^\infty\mathbf{Z}$, so $G \otimes K = H \otimes K$. Because $H_{\text{tor}}(p) = 0$, it follows that $\dim_{G\otimes K} X = \dim_{H\otimes K} X \leq \dim_H X \leq \dim_G X$.

(III) Taking into account the direct sum criterion, parts I and II cover the case when K is finitely generated. Finally, an arbitrary K is a colimit of its finitely generated subgroups, thus this situation can be handled by an appeal to Proposition 14.]

EXAMPLE If $G \neq G_{\text{tor}}$, then $\dim_{\mathbf{Q}} X \leq \dim_G X$.

[Proposition 18 implies that $\dim_{G\otimes\mathbf{Q}} X \leq \dim_G X$. But $G \otimes \mathbf{Q}$ contains $\mathbf{Q}$ as a direct summand.]

EXAMPLE Suppose that X is an ANR—then $\dim_{\mathbf{Z}/p\mathbf{Z}} X = \dim_{\widehat{\mathbf{Z}}_p} X$.

[Since $\widehat{\mathbf{Z}}_p \otimes \mathbf{Z}/p\mathbf{Z} \approx \widehat{\mathbf{Z}}_p/p\widehat{\mathbf{Z}}_p \approx \mathbf{F}_p$ and $\mathbf{Z}/p\mathbf{Z} \in \sigma(\mathbf{F}_p)$, one has $\dim_{\mathbf{Z}/p\mathbf{Z}} X \leq \dim_{\widehat{\mathbf{Z}}_p\otimes\mathbf{Z}/p\mathbf{Z}} X \leq \dim_{\widehat{\mathbf{Z}}_p} X$. To establish the inequality in the other direction, put $G = \prod\limits_1^\infty \mathbf{Z}/p^n\mathbf{Z}$—then $\dim_{\mathbf{Z}/p\mathbf{Z}} X = \dim_G X$ (cf. p. 20–7) and $\dim_G X \geq \dim_{\widehat{\mathbf{Z}}_p} X$ (G/G_{tor} is not divisible by p).]

[Note: If X is compact, then $\dim_{\mathbf{Z}/p\mathbf{Z}} X = \dim_{\mathbf{Z}_p} X$ (cf. p. 20–13).]

EXAMPLE Suppose that X is an ANR—then $\dim_{\mathbf{Q}} X \leq \dim_G X \ \forall\, G \neq 0$.

BOCKSTEIN THEOREM Let X be a compact Hausdorff space—then for any $G \neq 0$, $\dim_G X = \sup\limits_{H\in\sigma(G)} \dim_H X$.

[One can suppose for this that G is torsion free (cf. Propositions 16 and 17), hence that the elements of $\sigma(G)$ are $\mathbf{Q}$ and the $\mathbf{Z}_p : pG \neq G$. We then claim that $\dim_G X \leq n$ iff $\dim_{\mathbf{Q}} X \leq n$ & $\dim_{\mathbf{Z}_p} X \leq n \ \forall\, p : pG \neq G$. Indeed, for a given closed subset A of X, by the universal coefficient theorem, $\check{H}^{n+1}(X,A;G) = 0$ iff $\check{H}^{n+1}(X,A;\mathbf{Z}) \otimes G = 0$ or still, iff $\check{H}^{n+1}(X,A;\mathbf{Z}) \otimes \mathbf{Q} = 0$ & $\check{H}^{n+1}(X,A;\mathbf{Z}) \otimes \mathbf{Z}_p = 0 \ \forall\, p : pG \neq G$, i.e., iff $\check{H}^{n+1}(X,A;\mathbf{Q}) = 0$ & $\check{H}^{n+1}(X,A;\mathbf{Z}_p) = 0 \ \forall\, p : pG \neq G$, as claimed.]

[Note: The compactness assumption on X in the Bockstein theorem can be relaxed to "paracompact & σ-locally compact" (Goto[†]). However the Bockstein theorem is not true for an arbitrary metrizable X, even if X has finite topological dimension (Dranishnikov-Repovš-Ščepin[†]).]

[†] *Sci. Rep. Tokyo Kyoiku Daigaku Sect. A* **10** (1969), 17–23.

To illustrate the Bockstein theorem, take $G = \mathbf{Z}$. Since $\sigma(\mathbf{Z}) = \{\mathbf{Q}\} \cup \bigcup_p \{\mathbf{Z}_p\}$ and $\dim_{\mathbf{Q}} X \leq \dim_{\mathbf{Z}_p} X \ \forall \ p$, it follows that $\dim_{\mathbf{Z}} X = \dim_{\mathbf{Z}_p} X \ (\exists \ p)$.

[Note: If $\dim X < \infty$, then $\dim X = \dim_{\mathbf{Z}} X$ (cf. Proposition 3) and either $\dim X - 1 \leq \dim_{\mathbf{Q}} X$ or $\dim X - 1 \leq \dim_{\mathbf{Z}/p\mathbf{Z}} X \ (\exists \ p)$. Thus fix $p : \dim_{\mathbf{Z}} X = \dim_{\mathbf{Z}_p} X$. There are now two possibilities: $\dim_{\mathbf{Z}_p} X = \dim_{\mathbf{Q}} X$, from which $\dim X - 1 \leq \dim_{\mathbf{Q}} X$ or $\dim_{\mathbf{Q}} X < \dim_{\mathbf{Z}_p} X$, from which $\dim_{\mathbf{Z}_p} X \leq \max\{\dim_{\mathbf{Q}} X, \dim_{\mathbf{Z}/p^\infty\mathbf{Z}} X + 1\} = \dim_{\mathbf{Z}/p^\infty\mathbf{Z}} X + 1 \leq \dim_{\mathbf{Z}/p\mathbf{Z}} X + 1 \Rightarrow \dim X - 1 \leq \dim_{\mathbf{Z}/p\mathbf{Z}} X$.]

EXAMPLE If X is a compact ANR, then $\dim X = \dim_{\mathbf{Z}/p\mathbf{Z}} \ (\exists \ p)$.

[For $\dim_{\mathbf{Z}} X = \dim_{\mathbf{Z}_p} X \ (\exists \ p)$ and, as noted above, $\dim_{\mathbf{Z}_p} X = \dim_{\mathbf{Z}/p\mathbf{Z}} X$. But here $\dim_{\mathbf{Z}} X = \dim X$ (cf. p. 20–3).]

EXAMPLE Let $\begin{cases} X \\ Y \end{cases}$ be compact Hausdorff spaces. Assume: $\dim_G X \leq n$—then $\dim_{\check{H}^i(Y;G)} X \leq n + 1 \ \forall \ i \geq 0$.

[Consider the short exact sequence $0 \rightarrow \check{H}^i(Y;\mathbf{Z}) \otimes G \rightarrow \check{H}^i(Y;G) \rightarrow \mathrm{Tor}(\check{H}^{i+1}(Y;\mathbf{Z}),G) \rightarrow 0$ coming from the universal coefficient theorem. By Proposition 18, $\dim_{\check{H}^i(Y;\mathbf{Z})\otimes G} X \leq \dim_G X \leq n$, so it suffices to show that $\dim_{\mathrm{Tor}(\check{H}^{i+1}(Y;\mathbf{Z}),G)} X \leq n + 1$ (cf. Proposition 15). Assuming that $\mathrm{Tor}(\check{H}^{i+1}(Y;\mathbf{Z}),G) \neq 0$, $\exists \ p : G_{\mathrm{tor}}(p) \neq 0$, hence either $\mathbf{Z}/p\mathbf{Z} \in \sigma(G)$ or $\mathbf{Z}/p^\infty\mathbf{Z} \in \sigma(G)$. But $\dim_{\mathbf{Z}/p\mathbf{Z}} X \leq \dim_G X$ & $\dim_{\mathbf{Z}/p^\infty\mathbf{Z}} X \leq \dim_G X$ (Bockstein theorem). And: $\dim_{\mathbf{Z}/p^\infty\mathbf{Z}} X \leq \dim_{\mathbf{Z}/p\mathbf{Z}} X$, $\dim_{\mathbf{Z}/p\mathbf{Z}} X \leq \dim_{\mathbf{Z}/p^\infty\mathbf{Z}} X + 1 \leq n + 1$.]

FACT Let X be a paracompact Hausdorff space—then for any $G \neq 0$, $\max\{\dim_G X, \dim_{\mathbf{Q}} X + 1\} \geq \sup\limits_{H \in \sigma(G)} \dim_H X$.

[Take G torsion free and consider the case when $H = \mathbf{Z}_p$ ($pG \neq G$). One has $\dim_{\mathbf{Z}_p} X \leq \max\{\dim_{\widehat{\mathbf{Z}}_p} X, \dim_{\widehat{\mathbf{Z}}_p/\mathbf{Z}_p} X + 1\} = \max\{\dim_{\widehat{\mathbf{Z}}_p} X, \dim_{\mathbf{Q}} X + 1\}$. Moreover, $\dim_G X \leq n \Rightarrow \dim_{\widehat{\mathbf{Z}}_p} X \leq n$.]

PROPOSITION 19 Let $\begin{cases} X \\ Y \end{cases}$ be nonempty compact Hausdorff spaces. Assume: $\begin{cases} \dim X \\ \dim Y \end{cases} < \infty$—then $\dim_G(X \times Y) \leq \dim_G X + \dim_G Y$ if G is torsion free.

[With $n = \dim_G X$ & $m = \dim_G Y$, put $k = n + m : \dim_G(X \times Y) \leq k$ if $\dim_{\check{H}^{k-i}(Y,B;G)} X \leq i \ \forall \ i \geq 0$ and all closed subsets $B \subset Y$ (cf. p. 20–11). Case 1:

[†] *Topology Proc.* **18** (1993), 57–73.

$i \leq n - 1$. Since $k - i \geq m + 1$, we have $\check{H}^{k-i}(Y, B; G) = 0$. Case 2: $i \geq n$. By the universal coefficient theorem, $\check{H}^{k-i}(Y, B; G) \approx \check{H}^{k-i}(Y, B; \mathbf{Z}) \otimes G$, hence $\dim_{\check{H}^{k-i}(Y,B;G)} X \leq \dim_G X \leq i$ (cf. Proposition 18).]

[Note: This inequality is also true if $G = \mathbf{Z}/p\mathbf{Z}$. For $\sigma(\check{H}^{k-i}(Y, B; G)) \subset \{\mathbf{Z}/p\mathbf{Z}, \mathbf{Z}/p^\infty\mathbf{Z}\}$ and by the Bockstein theorem, $\dim_{\check{H}^{k-i}(Y,B;\mathbf{Z}/p\mathbf{Z})} X = \dim_{\mathbf{Z}/p\mathbf{Z}} X$ (because $\dim_{\mathbf{Z}/p^\infty\mathbf{Z}} X \leq \dim_{\mathbf{Z}/p\mathbf{Z}} X$).]

LEMMA Let $\left\{\begin{matrix} X \\ Y \end{matrix}\right.$ be nonempty compact Hausdorff spaces. Assume: $\left\{\begin{matrix} \dim X \\ \dim Y \end{matrix}\right. < \infty$—then $\dim_G(X \times Y) \geq \dim_G X + \dim_G Y$ if G is a field.

[Let $n = \dim_G X$, $m = \dim_G Y$ and choose closed subsets $A \subset X$, $B \subset Y$ such that $\check{H}^n(X, A; G) \neq 0$, $\check{H}^m(Y, B; G) \neq 0$. The universal coefficient theorem then gives $\check{H}^n(X, A; \check{H}^m(Y, B; G)) \approx \check{H}^n(X, A; \mathbf{Z}) \otimes \check{H}^m(Y, B; G)$. But $\check{H}^m(Y, B; G) \approx \oplus G$, so $\check{H}^n(X, A; \check{H}^m(Y, B; G)) \neq 0$, which means that $\dim_{\check{H}^m(Y,B;G)} X \geq n$, thus $\dim_G (X \times Y) \geq n + m$ (cf. p. 20–11).]

PROPOSITION 20 Let $\left\{\begin{matrix} X \\ Y \end{matrix}\right.$ be nonempty compact Hausdorff spaces. Assume: $\left\{\begin{matrix} \dim X \\ \dim Y \end{matrix}\right. < \infty$—then $\dim_G(X \times Y) = \dim_G X + \dim_G Y$ for any field G.

[This is implied by Proposition 19 and the lemma.]

PROPOSITION 21 Let $\left\{\begin{matrix} X \\ Y \end{matrix}\right.$ be nonempty compact Hausdorff spaces. Assume: $\left\{\begin{matrix} \dim X \\ \dim Y \end{matrix}\right. < \infty$—then $\forall\, G \neq 0$, $\dim_G(X \times Y) \leq \dim_G X + \dim_G Y + 1$.

[With $n = \dim_G X$ & $m = \dim_G Y$, put $k = n + m + 1$: $\dim_G(X \times Y) \leq k$ if $\dim_{\check{H}^{k-i}(Y,B;G)} X \leq i \ \forall\, i \geq 0$ and all closed subsets $B \subset Y$ (cf. p. 20–11). The case $i \leq n$ being trivial, suppose that $i \geq n + 1$. Taking $j \geq i$ and $A \subset X$ closed, repeated use of the universal coefficient theorem leads to $\check{H}^j(X, A; \check{H}^{k-i}(Y, B; G)) \approx \check{H}^j(X, A; \mathbf{Z}) \otimes \check{H}^{k-i}(Y, B; G) \oplus \operatorname{Tor}(\check{H}^{j+1}(X, A, \mathbf{Z}), \check{H}^{k-i}(Y, B, G)) \approx \check{H}^j(X, A; \mathbf{Z}) \otimes [\check{H}^{k-i}(Y, B; \mathbf{Z}) \otimes G \oplus \operatorname{Tor}(\check{H}^{k-i+1}(Y, B; \mathbf{Z}), G)] \oplus \operatorname{Tor}(\check{H}^{j+1}(X, A, \mathbf{Z}), \check{H}^{k-i}(Y, B; \mathbf{Z}) \otimes G \oplus \operatorname{Tor}(\check{H}^{k-i+1}(Y, B, \mathbf{Z}), G)) \approx [\check{H}^j(X, A, \mathbf{Z}) \otimes \check{H}^{k-i}(Y, B; \mathbf{Z}) \otimes G \oplus \operatorname{Tor}(\check{H}^{j+1}(X, A; \mathbf{Z}), \check{H}^{k-i}(Y, B, \mathbf{Z}) \otimes G)] \oplus [\check{H}^j(X, A, \mathbf{Z}) \otimes \operatorname{Tor}(\check{H}^{k-i+1}(Y, B; \mathbf{Z}), G) \oplus \operatorname{Tor}(\check{H}^{j+1}(X, A; \mathbf{Z}), \operatorname{Tor}(\check{H}^{k-i+1}(Y, B; \mathbf{Z}), G))] \approx \check{H}^j(X, A; \check{H}^{k-i}(Y, B; \mathbf{Z}) \otimes G) \oplus \check{H}^j(X, A; \operatorname{Tor}(\check{H}^{k-i+1}(Y, B; \mathbf{Z}), G))$. By Proposition 18, $\dim_{\check{H}^{k-i}(Y,B;\mathbf{Z}) \otimes G} X \leq \dim_G X < i$, so $\check{H}^j(X, A; \check{H}^{k-i}(Y, B; \mathbf{Z}) \otimes G) = 0$. On the other hand, $\dim_{\operatorname{Tor}(\check{H}^{k-i+1}(Y,B;\mathbf{Z}),G)} X \leq \dim_G X + 1 \leq i$ (imitate the argument used in the second example on p. 20–17), thus $\check{H}^j(X, A; \operatorname{Tor}(\check{H}^{k-i+1}(Y, B; \mathbf{Z}), G)) = 0$. Therefore $\dim_{\check{H}^{k-i}(Y,B;G)} X \leq i$, as desired.]

Let X, Y be nonempty compact Hausdorff spaces of finite topological dimension.

FACT $\dim_{\mathbf{Z}/p^\infty\mathbf{Z}}(X \times Y) = \dim_{\mathbf{Z}/p^\infty\mathbf{Z}} X + \dim_{\mathbf{Z}/p^\infty\mathbf{Z}} Y$ if $\dim_{\mathbf{Z}/p^\infty\mathbf{Z}} X = \dim_{\mathbf{Z}/p\mathbf{Z}} X$ or $\dim_{\mathbf{Z}/p^\infty\mathbf{Z}} Y = \dim_{\mathbf{Z}/p\mathbf{Z}} Y$, otherwise $\dim_{\mathbf{Z}/p^\infty\mathbf{Z}}(X \times Y) = \dim_{\mathbf{Z}/p^\infty\mathbf{Z}} X + \dim_{\mathbf{Z}/p^\infty\mathbf{Z}} Y + 1 = \dim_{\mathbf{Z}/p\mathbf{Z}}(X \times Y) - 1$.

[If the second eventuality obtains, then $\dim_{\mathbf{Z}/p^\infty\mathbf{Z}} X < \dim_{\mathbf{Z}/p\mathbf{Z}} X$ & $\dim_{\mathbf{Z}/p^\infty\mathbf{Z}} Y < \dim_{\mathbf{Z}/p\mathbf{Z}} Y \Rightarrow \dim_{\mathbf{Z}/p\mathbf{Z}} X + \dim_{\mathbf{Z}/p\mathbf{Z}} Y - 1 = \dim_{\mathbf{Z}/p\mathbf{Z}}(X \times Y) - 1$ (cf. Proposition 20) $\leq \dim_{\mathbf{Z}/p^\infty\mathbf{Z}}(X \times Y) \leq \dim_{\mathbf{Z}/p^\infty\mathbf{Z}} X + \dim_{\mathbf{Z}/p^\infty\mathbf{Z}} Y + 1$ (cf. Proposition 21)$=$ $(\dim_{\mathbf{Z}/p^\infty\mathbf{Z}} X + 1) + (\dim_{\mathbf{Z}/p^\infty\mathbf{Z}} Y + 1) - 1 = \dim_{\mathbf{Z}/p\mathbf{Z}} X + \dim_{\mathbf{Z}/p\mathbf{Z}} Y - 1$.]

FACT $\dim_{\mathbf{Z}_p}(X \times Y) = \dim_{\mathbf{Z}_p} X + \dim_{\mathbf{Z}_p} Y$ if $\dim_{\mathbf{Z}/p^\infty\mathbf{Z}} X = \dim_{\mathbf{Z}_p} X$ and $\dim_{\mathbf{Z}/p^\infty\mathbf{Z}} Y = \dim_{\mathbf{Z}_p} Y$, otherwise $\dim_{\mathbf{Z}_p}(X \times Y) = \max\{\dim_{\mathbf{Q}}(X \times Y), \dim_{\mathbf{Z}/p^\infty\mathbf{Z}}(X \times Y) + 1\}$.

[If the first eventuality obtains, then $\dim_{\mathbf{Z}_p} X + \dim_{\mathbf{Z}_p} Y \geq \dim_{\mathbf{Z}_p}(X \times Y)$ (cf. Proposition 19) which is $\geq \dim_{\mathbf{Z}/p\mathbf{Z}}(X \times Y) = \dim_{\mathbf{Z}/p\mathbf{Z}} X + \dim_{\mathbf{Z}/p\mathbf{Z}} Y$ (cf. Proposition 20), which is $\geq \dim_{\mathbf{Z}/p^\infty\mathbf{Z}} X + \dim_{\mathbf{Z}/p^\infty\mathbf{Z}} Y = \dim_{\mathbf{Z}_p} X + \dim_{\mathbf{Z}_p} Y$.]

EXAMPLE Given m, n, and q such that $n \leq m < q \leq n + m$, $\exists$ metrizable compact Hausdorff spaces $X_m, X_n : \dim X_m = m$, $\dim X_n = n$, and $\dim(X_m \times X_n) = q$.

[Specify two Bockstein functions D_m, D_n by the following table

$\mathbf{Z}_2$	$\mathbf{Z}/2\mathbf{Z}$	$\mathbf{Z}/2^\infty\mathbf{Z}$	$\mathbf{Q}$	$\mathbf{Z}_p$	$\mathbf{Z}/p\mathbf{Z}$	$\mathbf{Z}/p^\infty\mathbf{Z}$
m	1	1	m	m	1	1
n	n	$n-1$	$q-m$	$q-m$	$q-m$	$q-m$

and consider the metrizable compact Hausdorff spaces produced by the Dranishnikov realization theorem.]

PROPOSITION 22 Let X be a nonempty compact Hausdorff space of finite topological dimension. Assume: $\dim X = \dim_{\mathbf{Q}} X$ or $\dim X = \dim_{\mathbf{Z}/p\mathbf{Z}} X$ ($\exists$ p)—then $\dim X^n = n \cdot \dim X$.

[If $\dim X = \dim_G X$, where $G = \mathbf{Q}$ or $\mathbf{Z}/p\mathbf{Z}$ ($\exists$ p), then $n \cdot \dim X \geq \dim X^n$ (product theorem) $\geq \dim_G X^n$ (cf. Proposition 2)$= n \cdot \dim_G X$ (cf. Proposition 20)$= n \cdot \dim X$.]

EXAMPLE If X is a compact ANR of finite topological dimension, then $\dim X^n = n \cdot \dim X$.
[This is because $\dim X = \dim_{\mathbf{Z}/p\mathbf{Z}}$ ($\exists$ p) (cf. p. 20–17).]

FACT Let X be a nonempty compact Hausdorff space of finite topological dimension. Assume: $\dim X > \dim_G X$ for $G = \mathbf{Q}$ and $G = \mathbf{Z}/p\mathbf{Z}$ ($\forall$ p)—then $\dim X^n = n \cdot \dim X - (n - 1)$.

EXAMPLE Suppose that X realizes the Bockstein function $\Phi(\mathbf{Z}/p^\infty\mathbf{Z}, n)$ (cf. p. 20–14)—then $\dim X = n$ and X satisfies the assumption of the preceding result. Therefore $\dim(X \times X) = 2n - 1 < 2n$ (cf. p. 19–29).

PROPOSITION 23 Let $\begin{cases} X \\ Y \end{cases}$ be nonempty compact Hausdorff spaces. Assume: $\begin{cases} \dim X \\ \dim Y \end{cases} < \infty$—then $\forall\, G, K \neq 0$, $\dim_{G\otimes K}(X \times Y) \leq \dim_G X + \dim_K Y$.

[Take $k = \dim_G X + \dim_K Y$ and show that $\dim_{\check{H}^{k-i}(Y,B;G\otimes K)} X \leq i \ \forall\, i \geq 0$ and all closed subsets $B \subset Y$ (cf. p. 20–11).]

Application: Under the assumptions of the preceding proposition, $\dim_R(X \times Y) \leq \dim_R X + \dim_R Y$ for any ring R with unit.

[In fact, R is a retract of $R \otimes_{\mathbf{Z}} R$, thus is a direct summand, so $\dim_R(X \times Y) \leq \dim_{R\otimes_{\mathbf{Z}} R}(X \times Y) \leq \dim_R X + \dim_R Y$.]

PROPOSITION 24 Let $\begin{cases} X \\ Y \end{cases}$ be nonempty compact Hausdorff spaces. Assume: $\begin{cases} \dim X \\ \dim Y \end{cases} < \infty$—then $\forall\, G, K \neq 0$, $\dim_{\mathrm{Tor}(G,K)}(X \times Y) \leq \dim_G X + \dim_K Y + 1$.

[Since $\mathrm{Tor}(G, K) = \mathrm{Tor}(G_{\mathrm{tor}}, K_{\mathrm{tor}})$, one can assume that G and K are torsion (cf. Proposition 17). Making the obvious reductions, one can assume further that G and K are p-primary (tacitly, $\mathrm{Tor}(G,K) \neq 0$). Case 1: $\mathrm{Tor}(G,K)$ is not divisible by p. In this situation, either G or K is not divisible by p. And: $\dim_{\mathrm{Tor}(G,K)}(X \times Y) = \dim_{\mathbf{Z}/p\mathbf{Z}}(X \times Y)$ (Bockstein theorem) $\leq \dim_{\mathbf{Z}/p\mathbf{Z}} X + \dim_{\mathbf{Z}/p\mathbf{Z}} Y$. But either $\dim_{\mathbf{Z}/p\mathbf{Z}} X = \dim_G X$ or $\dim_{\mathbf{Z}/p\mathbf{Z}} Y = \dim_K Y$ and at worst, $\dim_{\mathbf{Z}/p\mathbf{Z}} X \leq \dim_G X + 1$ & $\dim_{\mathbf{Z}/p\mathbf{Z}} Y \leq \dim_K Y + 1$, G & K being p-primary. Case 2: $\mathrm{Tor}(G,K)$ is divisible by p. Here, $\dim_{\mathrm{Tor}(G,K)}(X \times Y) = \dim_{\mathbf{Z}/p^\infty\mathbf{Z}}(X \times Y)$ (Bockstein theorem) $\leq \dim_{\mathbf{Z}/p^\infty\mathbf{Z}} X + \dim_{\mathbf{Z}/p^\infty\mathbf{Z}} Y + 1$. But $\dim_{\mathbf{Z}/p^\infty\mathbf{Z}} X \leq \dim_G X$ & $\dim_{\mathbf{Z}/p^\infty\mathbf{Z}} Y \leq \dim_K Y$, G & K being p-primary.]

§ 20

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