G, Γ, G/T

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§1. ASSOCIATIVE ALGEBRAS

1: DEFINITION An associative algebra over $\mathbb{C}$ is a finite dimensional vector space $A$ over $\mathbb{C}$ equipped with a bilinear map

$$\mu : A \times A \to A, \ (x,y) \mapsto \mu(x,y) \equiv xy$$

such that $(xy)z = x(yz)$.

2: DEFINITION An associative algebra $A$ is said to be unital if there exists an element $e \in A$ with the property that $xe = ex = x$ for all $x \in A$.

[Note: Such an $e$ is called an identity element and is denoted by $1_A$.]

3: N.B. Identity elements are unique.

4: EXAMPLE Let $V$ be a finite dimensional vector space over $\mathbb{C}$ -- then $\text{Hom}(V)$ (the set of all $\mathbb{C}$-linear maps of $V$) is a unital associative algebra over $\mathbb{C}$ (multiplication being composition of linear transformations and identity element $\text{id}_V$).

Let $A$ be an associative algebra over $\mathbb{C}$.

5: DEFINITION A representation of $A$ is a pair $(\rho, V)$, where $V$ is a finite dimensional vector space over $\mathbb{C}$ and $\rho : A \to \text{Hom}(V)$ is a morphism of associative algebras.

[Note: If $A$ is unital, then it will be assumed that $\rho(1_A) = \text{id}_V$, thus is a morphism of unital associative algebras.]

6: DEFINITION Let $(\rho, V)$ be a representation of $A$ -- then a linear subspace $U \subset V$ is said to be $\rho$-invariant if $\forall x \in A, \ \rho(x)U \subset U$. 
7: N.B. A $\rho$-invariant subspace $U \subset V$ gives rise to two representations of $A$, viz. by restricting to $U$ and passing to the quotient $V/U$.

8: DEFINITION A representation $(\rho, V)$ of $A$ is irreducible if $V \neq \{0\}$ and if the only $\rho$-invariant subspaces are $\{0\}$ and $V$.

9: NOTATION Given a representation $(\rho, V)$ of $A$, put
$$\text{Ker}(\rho) = \{x \in A : \rho(x) = 0\}.$$  

10: N.B. $\text{Ker}(\rho)$ is a two-sided ideal in $A$.

11: DEFINITION A representation $(\rho, V)$ of $A$ is faithful if $\text{Ker}(\rho) = \{0\}$.

12: DEFINITION Let $(\rho, V)$, $(\sigma, W)$ be representations of $A$ -- then an intertwining operator is a $C$-linear map $T : V \rightarrow W$ such that $T\rho(x) = \sigma(x)T$ for all $x \in A$.

13: NOTATION $I_A(\rho, \sigma)$ is the set of intertwining operators between $(\rho, V)$ and $(\sigma, W)$.

14: EXAMPLE Let $(\rho, V)$ be a representation of $A$ and suppose that $U \subset V$ is a $\rho$-invariant subspace -- then the inclusion map $U \rightarrow V$ is an intertwining operator, as is the quotient map $V \rightarrow V/U$.

15: DEFINITION Representations $(\rho, V)$ and $(\sigma, W)$ of $A$ are equivalent if there exists an invertible operator in $I_A(\rho, \sigma)$, in which case we write
$$(\rho, V) \simeq (\sigma, W) \quad \text{(or } \rho \simeq \sigma).$$

16: NOTATION $\hat{A}$ is the set of equivalence classes of irreducible representations of $A$. 
17: EXAMPLE Take $A = \text{Hom}(V)$, where $V$ is a finite dimensional complex vector space -- then up to equivalence, the only irreducible representation of $\text{Hom}(V)$ is the representation $\rho$ given by

$$\rho(T)v = Tv \ (T \in \text{Hom}(V)).$$
1. REPRESENTATION THEORY

Let $A$ be a unital associative algebra over $\mathbb{C}$.

1: **THEOREM** Let $(\rho, V), (\sigma, W)$ be irreducible representations of $A$ -- then

$$\dim I_A(\rho, \sigma) = \begin{cases} 1 & \text{if } (\rho, V) \cong (\sigma, W) \\ 0 & \text{if } (\rho, V) \not\cong (\sigma, W). \end{cases}$$

2: **THEOREM** Let $(\rho, V)$ be an irreducible representation of $A$ -- then $\rho(A) = \text{Hom}(V)$.

3: **DEFINITION** A representation $(\rho, V)$ of $A$ is completely reducible if for every $\rho$-invariant subspace $V_1 \subset V$ there exists a $\rho$-invariant subspace $V_2 \subset V$ such that $V = V_1 \oplus V_2$.

4: **LEMMA** Suppose that $(\rho, V)$ is a representation of $A$ -- then $(\rho, V)$ is completely reducible iff there is a decomposition

$$V = V_1 \oplus \cdots \oplus V_s,$$

where each $V_i$ is $\rho$-invariant and irreducible.

5: **LEMMA** Suppose that $(\rho, V)$ is a representation of $A$ -- then $(\rho, V)$ is completely reducible iff there is a decomposition

$$V = U_1 + \cdots + U_t,$$

where each $U_j$ is $\rho$-invariant and irreducible.

6: **DEFINITION** Let $V$ be a finite dimensional vector space over $\mathbb{C}$. Given
2.

a subset $\mathcal{S}$ of $\text{Hom}(V)$, put

$$\text{Com}(\mathcal{S}) = \{ T \in \text{Hom}(V) : Ts = sT \ \forall \ s \in \mathcal{S} \},$$

the commutant of $\mathcal{S}$.

7: N.B. $\text{Com}(\mathcal{S})$ is a unital associative algebra over $\mathbb{C}$.

8: THEOREM Suppose that $V$ is a finite dimensional vector space over $\mathbb{C}$ and let $V \subset \text{Hom}(V)$ be an associative algebra over $\mathbb{C}$ with identity $\text{id}_V$. Assume: $V$ is completely reducible per the canonical action of $V$ — then

$$\text{Com}(\text{Com}(V)) = V.$$  

[Note: A priori, $V \subset \text{Com}(\text{Com}(V))$.]

9: NOTATION Let $(\rho, V)$ be a completely reducible representation of $A$. Given $\delta \in \hat{A}$, put

$$V_\delta = \sum_{U \in V : [U] = \delta} U,$$

the subspaces $U$ being $\rho$-invariant and irreducible, $[U]$ standing for the equivalence class in $\hat{A}$ determined by $U$.

10: THEOREM Let $(\rho, V)$ be a completely reducible representation of $A$ and let

$$V = V_1 \oplus \cdots \oplus V_s$$

be a decomposition, where each $V_i$ is $\rho$-invariant and irreducible — then $\forall \ \delta \in \hat{A}$,

$$V_\delta = \bigoplus_{[V_i] = \delta} V_i.$$
thus

\[ V = \bigoplus_{\delta \in \hat{A}} V_\delta. \]

[Note: An empty sum is taken to be zero.]

11: DEFINITION The decomposition

\[ V = \bigoplus_{\delta \in \hat{A}} V_\delta \]

is the primary decomposition of \( V \) and \( V_\delta \) is the \( \delta \)-isotypic subspace of \( V \).

12: DEFINITION The cardinality \( m_V(\delta) \) of

\[ \{i: [V_i] = \delta\} \]

is the multiplicity of \( \delta \) in \( V \).

13: NOTATION Given \( \delta \in \hat{A} \), let \( U(\delta) \) be an element in the class \( \delta \).

14: LEMMA

\[ m_V(\delta) = \dim I_A(U(\delta), V) = \dim I_A(V, U(\delta)). \]
§3. CHARACTERS

Let $A$ be a unital associative algebra over $\mathbb{C}$.

1: DEFINITION Let $(\rho, V)$ be a representation of $A$ -- then its character is the linear functional

$$\chi_{\rho}: A \rightarrow \mathbb{C}$$

given by the prescription

$$\chi_{\rho}(x) = \text{tr}(\rho(x)) \quad (x \in A).$$

2: LEMMA

$$\chi_{\rho}(1_A) = \dim V.$$

3: LEMMA \forall x, y \in A,

$$\chi_{\rho}(xy) = \chi_{\rho}(yx).$$

4: DEFINITION Let $(\rho, V)$ be a representation of $A$ -- then a composition series for $\rho$ is a sequence of $\rho$-invariant subspaces

$$\{0\} = V_0 \subset V_1 \subset \cdots \subset V_s = V$$

such that

$$\{0\} \neq V_i/V_{i-1} \quad (i = 1, \ldots, s)$$

is irreducible.

5: LEMMA Composition series exist.

6: DEFINITION The semisimplification of $(\rho, V)$ is the direct sum
2. \[ V_{ss} = \bigoplus_{i=1}^{s} (V_i/V_{i-1}) \]
equipped with the canonical operations.

7: **DEFINITION** The irreducible quotients \( V_i/V_{i-1} \) are the composition factors of \((\rho,V)\).

Let \( \rho_{ss} \) be the representation of \( A \) per \( V_{ss} \) and let \( \rho_i \) be the representation of \( A \) per \( V_i/V_{i-1} \).

8: **LEMMA**

\[ \chi_{\rho_{ss}} = \sum_{i=1}^{s} \chi_{\rho_i} = \chi_{\rho}. \]

9: **LEMMA** Suppose that \((\sigma_1,U_1), \ldots, (\sigma_r,U_r)\) are irreducible representations of \( A \). Assume: \((\sigma_k,U_k)\) is not equivalent to \((\sigma_l,U_l)\) \((k \neq l)\) -- then the set \(\{\chi_{\sigma_1}, \ldots, \chi_{\sigma_r}\}\) is linearly independent.

10: **SCHOLIUM** The composition factors in a composition series for \( \rho \) are unique up to isomorphism and order of appearance and \((\rho_{ss},V_{ss})\) is uniquely determined by \( \chi_{\rho} \) up to isomorphism.
1.

§4. SIMPLE AND SEMISIMPLE ALGEBRAS

Let $A$ be a unital associative algebra over $\mathbb{C}$.

1: DEFINITION $A$ is simple if the only two-sided ideals in $A$ are $\{0\}$ and $A$.

2: LEMMA If $V$ is a finite dimensional vector space over $\mathbb{C}$, then $\text{Hom}(V)$ is simple.

3: THEOREM If $A$ is simple, then there is a finite dimensional vector space $V$ over $\mathbb{C}$ such that $A \simeq \text{Hom}(V)$.

4: DEFINITION $A$ is semisimple if it is a finite direct sum of simple algebras.

Accordingly, if $A$ is semisimple, then there is a finite set $L$, finite dimensional complex vector spaces $V_{\lambda}$ ($\lambda \in L$), and an isomorphism

$$\phi: A \to \bigoplus_{\lambda \in L} \text{Hom}(V_{\lambda}).$$

Denote by $E_{\lambda}$ the element

$$0 \oplus \cdots \oplus \text{id}_{V_{\lambda}} \oplus \cdots \oplus 0$$

and define a representation $(\rho_{\lambda}, V_{\lambda})$ by the prescription

$$\rho_{\lambda}(x) = \phi(x)E_{\lambda} \quad (x \in A).$$

5: LEMMA The $(\rho_{\lambda}, V_{\lambda})$ are irreducible.
2.

6: **THEOREM** Every irreducible representation of $\Lambda$ is equivalent to some $(\rho, V)$. 

7: **N.B.** Therefore 

$$\Lambda \leftrightarrow L,$$

so the term "$\lambda$-isotypic subspace" makes sense.

Put 

$$e_\lambda = \phi^{-1}(E_\lambda).$$

Then $e_\lambda$ is a central idempotent and 

$$\sum_{\lambda \in L} e_\lambda = 1_{\Lambda}.$$

8: **THEOREM** Suppose that $\Lambda$ is semisimple and let $(\rho, V)$ be a representation of $\Lambda$ -- then its $\lambda$-isotypic subspace is $\rho(e_\lambda) V$ and 

$$V = \bigoplus_{\lambda \in L} \rho(e_\lambda) V$$

is the primary decomposition of $V$.

9: **LEMMA** Let $\Lambda$ be a unital associative algebra over $\mathbb{C}$ and let $(\rho, V)$ be a completely reducible representation of $\Lambda$ -- then $\rho(\Lambda)$ is semisimple.

10: **THEOREM** Let $\Lambda$ be a unital associative algebra over $\mathbb{C}$ -- then the following conditions are equivalent:

1. The left regular representation $(L, \Lambda)$ of $\Lambda$ is completely reducible ($L(x)y = xy$).

2. Every representation of $\Lambda$ is completely reducible.

3. $\Lambda$ is a semisimple algebra.
3.

[1 => 3: L(A) is semisimple (cf. #9). On the other hand, $A \simeq L(A)$, $L$ being faithful.

3 => 2: Quote #3 and §2, #4.

2 => 1: Obvious.]

THEOREM Every representation of a semisimple algebra is uniquely determined by its character up to isomorphism.
1. \textit{GROUP ALGEBRAS}

1: \textbf{NOTATION} If $X$ is a finite set, then $|X|$ is the cardinality of $X$ and $C(X)$ is the vector space of complex valued functions on $X$.

2: \textbf{N.B.} The functions \{$\delta_x : x \in X$\}, where

\[
\delta_x(y) = \begin{cases} 
1 & (x = y) \\
0 & (x \neq y),
\end{cases}
\]

constitute a basis for $C(X)$. Therefore

\[
\dim C(X) = |X|
\]

and every $f \in C(X)$ admits a decomposition

\[
f = \sum_{x \in X} f(x) \delta_x.
\]

In particular: If $1_X$ is the function on $X$ which is $= 1$, then

\[
1_X = \sum_{x \in X} \delta_x.
\]

Let $G$ be a finite group.

3: \textbf{DEFINITION} Given $f, g \in C(G)$, their \textit{convolution} $f \ast g$ is the element of $C(G)$ defined by the rule

\[
(f \ast g)(x) = \sum_{y \in G} f(xy^{-1})g(y)
\]

\[
= \sum_{y \in G} f(y)g(y^{-1}x).
\]
2.

[Note: \( \forall x, y \in G, \]
\[ \delta_x \ast \delta_y = \delta_{xy}. \]

4: LEMMA \( C(G) \) is an associative algebra over \( \mathbb{C} \).

5: N.B. If \( e \) is the identity in \( G \), then \( \delta_e \) is the identity in \( C(G) \), which is therefore unital.

6: LEMMA The center of \( C(G) \) consists of those \( f \) such that
\[ f(x) = f(\text{yxy}^{-1}) \quad (x, y \in G). \]

[Note: In other words, the center of \( C(G) \) consists of those \( f \) that are constant on conjugacy classes, the so-called class functions.]

E.g.: \( \forall x \in G \), the function
\[ \sum_{y \in G} \delta_{\text{yxy}^{-1}} \]
is a class function.

[Given \( z \) in \( G \),
\[ (\sum_{y \in G} \delta_{\text{yxy}^{-1}}) \ast \delta_z = \sum_{y \in G} \delta_{\text{zyx}(zy)^{-1}z} \]
\[ = \sum_{y \in G} \delta_{\text{zy}(zy)^{-1}z} \]
\[ = \sum_{y \in G} \delta_{\text{zyxy}^{-1}} \]
\[ = \delta_z \ast (\sum_{y \in G} \delta_{\text{yxy}^{-1}}). \]
7: DEFINITION A representation of $G$ is a pair $(\pi, V)$, where $V$ is a finite dimensional vector space over $\mathbb{C}$ and $\pi: G \to GL(V)$ is a morphism of groups.

8: SCHOLIUM Let $V$ be a finite dimensional vector space over $\mathbb{C}$.

- Every representation $\pi: G \to GL(V)$ extends to a representation $\rho$ of $C(G)$ on $V$, viz.
  $$\rho(f) = \sum_{x \in G} f(x)\pi(x).$$

- Every representation $\rho: C(G) \to \text{Hom}(V)$ restricts to a representation $\pi$ of $G$ on $V$, viz.
  $$\pi(x) = \rho(\delta_x).$$
  [Note: If $\pi$ is given, it is customary to denote its extension "$\rho$" by $\pi$ as well.]

9: LEMMA Let $W \subset V$ be a linear subspace -- then $W$ is invariant under $G$ iff $W$ is invariant under $C(G)$.

10: LEMMA An operator $T \in \text{Hom}(V)$ commutes with the action of $G$ iff it commutes with the action of $C(G)$.

11: THEOREM $C(G)$ is semisimple.

PROOF Let $(\rho, V)$ be a representation of $C(G)$ and suppose that $V_\perp \subset V$ is a $\rho$-invariant subspace. Fix a linear complement $U$ per $V_\perp: V = V_\perp \oplus U$. Let $P: V \to V_\perp$ be the corresponding projection and put
$$Q = \frac{1}{|G|} \sum_{x \in G} \pi(x)P\pi(x)^{-1}.$$
Then $Q$ is a projection with range $V_1$. In addition, $\forall y \in G,$

$$\pi(y)Q = \frac{1}{|G|} \sum_{x \in G} \pi(xy)p\pi^{-1}(x)$$

$$= \frac{1}{|G|} \sum_{x \in G} \pi(x)p\pi^{-1}(y^{-1}x)$$

$$= \frac{1}{|G|} \sum_{x \in G} \pi(x)p\pi^{-1}(x)p(y)$$

$$= Q\pi(y).$$

Consequently, $\forall v \in V,$

$$\pi(y)(id_V - Q)v = \pi(y)(v - Qv)$$

$$= \pi(y)v - \pi(y)Qv$$

$$= \pi(y)v - Q\pi(y)v$$

$$= (id_V - Q)\pi(y)v,$$

thus the range $V_2$ of $id_V - Q$ is a $\rho$-invariant complement per $V_1$. It therefore follows that every representation of $C(G)$ is completely reducible, hence $C(G)$ is semisimple (cf. I, §4, #10).

12: DEFINITION

- The left translation representation $L$ of $G$ on $C(G)$ is the prescription

$$L(x)f(y) = f(x^{-1}y) (\Rightarrow L(x)f = \delta_{x^{-1}} f).$$

- The right translation representation of $G$ on $C(G)$ is the prescription

$$R(x)f(y) = f(yx) (\Rightarrow R(x)f = f \ast \delta_{x^{-1}}).$$

13: N.B. Since $C(G)$ is semisimple, both $L$ and $R$ are completely reducible.
5.

14: REMARK There is also a representation $\pi_{L,R}$ of $G \times G$ on $C(G)$, namely

$$(\pi_{L,R}(x_1, x_2)f)(x) = f(x^{-1}_1 xx_2).$$

And it too is completely reducible ($C(G \times G)$ is semisimple).

15: DEFINITION Let $(\pi_1, V_1)$, $(\pi_2, V_2)$ be representations of $G$ --- then an intertwining operator is a $C$-linear map $T: V_1 \to V_2$ such that $T\pi_1(x) = \pi_2(x)T$ for all $x \in G$.

16: NOTATION $I_G(\pi_1, \pi_2)$ is the set of intertwining operators between $(\pi_1, V_1)$ and $(\pi_2, V_2)$.

17: N.B. On the basis of the definitions,

$$I_G(\pi_1, \pi_2) = I_{C(G)}(\rho_1, \rho_2).$$

18: LEMMA Let $(\pi_1, V_1)$, $(\pi_2, V_2)$ be irreducible representations of $G$ and let $T \in I_G(\pi_1, \pi_2)$ --- then either $T$ is zero or it is an isomorphism.

19: LEMMA Suppose that $(\pi, V)$ is an irreducible representation of $G$ and suppose that $T \in I_G(\pi, \pi)$ --- then $T$ is a scalar multiple of $id_V$.

20: DEFINITION Representations $(\pi_1, V_1)$ and $(\pi_2, V_2)$ of $G$ are equivalent if there exists an invertible operator in $I_G(\pi_1, \pi_2)$, in which case we write

$$(\pi_1, V_1) \simeq (\pi_2, V_2) \quad \text{or} \quad \pi_1 \simeq \pi_2.$$
6.

[Note: By convention, the zero representation of \( G \) on \( V = \{0\} \) is not to be viewed as irreducible.]

22: N.B. There is a one-to-one correspondence

\[
\hat{G} \simeq C(G).
\]

In the sequel, \( \Pi \) stands for an element of \( \hat{G} \) with representation space \( V(\Pi) \) of dimension \( d_\Pi \). Without loss of generality, it can be assumed moreover that \( \Pi \) is unitary with respect to a \( G \)-invariant inner product \( < , >_\Pi \) on \( V(\Pi) \).

[Recall the argument. Start with an inner product \( < , > \) on \( V(\Pi) \) and put

\[
<v_1, v_2>_\Pi = \frac{1}{|G|} \sum_{x \in G} <\Pi(x)v_1, \Pi(x)v_2>.
\]

APPENDIX

Let \((\pi_1, V_1), (\pi_2, V_2)\) be unitary representations of \( G \). Suppose that there exists an invertible

\[
T \in I_G(\pi_1, \pi_2).
\]

Then there exists a unitary

\[
U \in I_G(\pi_1, \pi_2).
\]

[Let \( T = U|T| \) be the polar decomposition of \( T \) — then \( \forall x \in G, \)

\[
|T|\pi_1(x) = \pi_1(x)|T|.
\]

Therefore

\[
U\pi_1(x)U^{-1} = |T|^{-1}\pi_1(x)|T|^{-1} = \pi_2(x).
\]
1.

§2. CONTRAGREIDENTS AND TENSOR PRODUCTS

1: NOTATION Given a finite dimensional vector space $V$ over $C$, let $V^*$ be its dual and denote by

$$\begin{align*}
V^* \times V &\to C \\
(v^*, v) &\mapsto \langle v^*, v \rangle \quad (= v^*(v))
\end{align*}$$

the evaluation pairing.

Let $G$ be a finite group.

2: DEFINITION Suppose that $\pi: G \to GL(V)$ is a representation -- then its contragredient is the representation $\pi^*: G \to GL(V^*)$ defined by requiring that $\forall x \in G$, $\pi^*(x)v^* = v^* \circ \pi(x^{-1}) \quad (v^* \in V^*)$,

thus $\forall v \in V$,

$$\langle \pi^*(x)v^*, v \rangle = \langle v^*, \pi(x^{-1})v \rangle.$$  

3: N.B. The identification $(V^*)^* \simeq V$ leads to an equivalence $(\pi^*)^* \simeq \pi$.

4: LEMMA $(\pi, V)$ is irreducible iff $(\pi^*, V^*)$ is irreducible.

5: CONVENTION Given $(\Pi, V(\Pi))$ in $\hat{G}$, take

$$V(\Pi^*) = V(\Pi)^*, \Pi^*(x) = \Pi(x^{-1})^\tau.$$

6: NOTATION Given finite dimensional vector spaces $V_1, V_2$ over $C$, let $V_1 \otimes V_2$ be their tensor product.
Let $G$ be a finite group.

**7: DEFINITION** Suppose that $\pi_1: G \rightarrow GL(V_1), \pi_2: G \rightarrow GL(V_2)$ are representations -- then their **tensor product** is the representation $\pi_1 \otimes \pi_2: G \rightarrow GL(V_1 \otimes V_2)$ defined by requiring that $\forall x \in G$,

$$(\pi_1 \otimes \pi_2)(x)(v_1 \otimes v_2) = \pi_1(x)v_1 \otimes \pi_2(x)v_2.$$ 

Let $(\pi_1, V_1), (\pi_2, V_2)$ be representations of $G$ -- then the prescription

$$\pi_{1,2}(x)T = \pi_2(x)T\pi_1(x^{-1}) \quad (T \in \text{Hom}(V_1, V_2))$$

defines a representation $\pi_{1,2}$ of $G$ on $\text{Hom}(V_1, V_2)$.

**8: RAPPEL** There is a canonical isomorphism

$$\Theta: V_2 \otimes V_1^* \cong \text{Hom}(V_1, V_2).$$

[Send $v_2 \otimes v_1^*$ to the linear transformation $T(v_2, v_1^*): v_1 \rightarrow v_1^*(v_1)v_2$.]

Consider

$$\pi_2(x)v_2 \otimes \pi_1^*(x)v_1^* \in V_2 \otimes V_1^*.$$ 

Then the corresponding element of $\text{Hom}(V_1, V_2)$ is the assignment

$$v_1 \rightarrow (\pi_1^*(x)v_1^*)(v_1)\pi_2(x)v_2$$ 

$$= v_1^*(\pi_1(x^{-1})v_1)\pi_2(x)v_2$$ 

$$= \pi_2(x)(v_1^*(\pi_1(x^{-1})v_1))v_2$$
3.

\[ \pi_2(x)T(v_2, v_1^*) \pi_1(x^{-1})v_1. \]

9: **LEMMa** \( \pi_2 \otimes \pi_1^* \) is equivalent to \( \pi_{1,2} \).

[The isomorphism \( \Theta \) intertwines \( \pi_2 \otimes \pi_1^* \) and \( \pi_{1,2} \): \( \forall x \in G \),

\[ \Theta \circ (\pi_2(x) \otimes \pi_1^*(x)) = \pi_{1,2}(x) \circ \Theta. \]

Let \( G_1, G_2 \) be finite groups.

10: **DEFINITION** Suppose that \( \pi_1: G_1 \to GL(V_1), \pi_2: G \to GL(V_2) \) are representations -- then their **outer tensor product** is the representation \( \pi_1 \otimes \pi_2: G_1 \times G_2 \to GL(V_1 \otimes V_2) \) defined by requiring that \( \forall x_1 \in G_1, \forall x_2 \in G_2 \),

\[ (\pi_1 \otimes \pi_2)(x_1, x_2) = \pi_1(x_1) \otimes \pi_2(x_2). \]

11: **N.B.** If \( G_1 = G_2 = G \), then the restriction of the outer tensor product \( \pi_1 \otimes \pi_2 \) to the diagonal subgroup

\[ \{(x, x): x \in G \} \]

of \( G \times G \) is the tensor product \( \pi_1 \otimes \pi_2 \).

12: **REMARK** Take \( G_1 = G_2 = G \) and define a representation \( \pi_{1,2} \) of \( G \times G \) on \( Hom(V_1, V_2) \) via the prescription

\[ \pi_{1,2}(x, y)T = \pi_2(x)T\pi_1(y^{-1}) \quad (T \in Hom(V_1, V_2)). \]

Then \( \pi_2 \otimes \pi_1^* \) is equivalent to \( \pi_{1,2} \).
4.

13: **Lemma** If \( \pi_1 \) and \( \pi_2 \) are irreducible, then \( \pi_1 \otimes \pi_2 \) is irreducible.

[To begin with,

\[ C(G_1 \times G_2) \approx C(G_1) \otimes C(G_2) \]

and

\[ \text{Hom}(V_1 \otimes V_2) \approx \text{Hom}(V_1) \otimes \text{Hom}(V_2) \].

Now make the passage

\[
\begin{bmatrix}
\pi_1 \rightarrow \rho_1 \\
\pi_2 \rightarrow \rho_2
\end{bmatrix}
\]

Then

\[
\begin{bmatrix}
\rho_1(C(G_1)) = \text{Hom}(V_1) \\
\rho_2(C(G_2)) = \text{Hom}(V_2)
\end{bmatrix}
\]

(cf. I, §2, #2).

Conversely:

14: **Theorem** Every irreducible representation of \( G_1 \times G_2 \) is equivalent to an outer tensor product \( \pi_1 \otimes \pi_2 \).

15: **Scholium**

\[ G_1 \times G_2 \approx \hat{G}_1 \times \hat{G}_2. \]
§3. FOURIER TRANSFORMS

Let $G$ be a finite group.

1. **DEFINITION** Given $f \in C(G)$, its **Fourier transform** $\hat{f}$ is that element of

$$\hat{f} : \bigoplus_{\pi \in \hat{G}} \text{Hom}(V(\pi))$$

whose $\pi$-component is

$$\hat{f}(\pi) = \sum_{x \in G} f(x) \pi(x) (= \pi(f)).$$

E.g.: $\forall x \in G,$

$$\hat{\delta_x}(\pi) = \pi(x).$$

2. **LEMMA** $\forall f_1, f_2 \in C(G),$ 

$$f_1 \ast f_2 (\pi) = f_1(\pi)f_2(\pi).$$

3. **EXAMPLE** $\forall x \in G,$

$$L(x)f (\pi) = \delta_x \ast f (\pi) = \pi(x)\hat{f}(\pi)$$

$$R(x)f (\pi) = f \ast \delta_{x^{-1}} (\pi) = \hat{f}(\pi)\pi(x^{-1}).$$

4. **THEOREM** The Fourier transform

$$\hat{\cdot} : C(G) \to \bigoplus_{\pi \in \hat{G}} \text{Hom}(V(\pi))$$

is an algebra isomorphism.
2.

5: APPLICATION

\[ |G| = \sum_{\Pi \in \hat{G}} d_{\Pi}^2. \]

[In fact,

\[ \dim C(G) = |G| \text{ and } \dim \text{Hom}(V_{\Pi}) = d_{\Pi}^2. \]

As it stands, \( C(G) \) is a unital associative algebra over \( \mathbb{C} \). But more is true:

\( C(G) \) is a \( * \)-algebra, i.e., admits a conjugate linear antiautomorphism \( f \to f^* \)
given by \( f^*(x) = f(x^{-1}) \) (\( x \in G \)).

Each \( T \in \text{Hom}(V_{\Pi}) \) has an adjoint \( T^* \) per \( \langle \cdot , \cdot \rangle_{\Pi} \): \( \forall v_1, v_2 \in V_{\Pi}, \)

\[ \langle T v_1, v_2 \rangle_{\Pi} = \langle v_1, T^* v_2 \rangle_{\Pi}. \]

Therefore

\[ \bigoplus_{\Pi \in \hat{G}} \text{Hom}(V_{\Pi}) \]

admits a conjugate linear antiautomorphism by using the arrow \( T \to T^* \) on each summand.

6: N.B. It can and will be assumed that

\[ V(\Pi^*) = V(\Pi) \]

(cf. §2, #5),

\[ \Pi^*(x) = \Pi(x^{-1}) \]

hence in terms of adjoints

\[ \Pi(x)^* = \Pi(x)^{-1} = \Pi(x^{-1}) = \Pi^*(x). \]
7: LEMMA The Fourier transform

\[ \hat{\Lambda} : C(G) \to \bigoplus_{\Pi \in \hat{G}} \text{Hom}(V(\Pi)) \]

preserves the *-operations: \( \forall f \in C(G), \)

\[ f^* = (\hat{f})^*. \]

8: INVERSION FORMULA Given \( f \in C(G), \forall x \in G, \)

\[ f(x) = \frac{1}{|G|} \sum_{\Pi \in \hat{G}} d_{\Pi} \text{tr}(\Pi(x^{-1})\hat{f}(\Pi)). \]

In particular:

\[ f(e) = \frac{1}{|G|} \sum_{\Pi \in \hat{G}} d_{\Pi} \text{tr}(\hat{f}(\Pi)). \]

9: PARSEVAL IDENTIY Given \( f_1, f_2 \in C(G), \)

\[ \sum_{x \in G} f_1(x)f_2(x^{-1}) = \frac{1}{|G|} \sum_{\Pi \in \hat{G}} d_{\Pi} \text{tr}(\hat{f}_1(\Pi)\hat{f}_2(\Pi)). \]

**PROOF** Put \( f = f_1 * f_2 \) -- then

\[ f(e) = \sum_{x \in G} f_1(x)f_2(x^{-1}). \]

On the other hand,

\[ \frac{1}{|G|} \sum_{\Pi \in \hat{G}} d_{\Pi} \text{tr}(\hat{f}_1(\Pi)\hat{f}_2(\Pi)) \]

\[ = \frac{1}{|G|} \sum_{\Pi \in \hat{G}} d_{\Pi} \text{tr}(\hat{f}_1 * f_2(\Pi)) \]
= (f_1 \ast f_2)(e) = f(e).

10: **COMPLETENESS PRINCIPLE** If \( f \in C(G) \) and if \( \hat{f}(\Pi) = 0 \) for all \( \Pi \), then \( f = 0 \).
§4. CLASS FUNCTIONS

Let G be a finite group.

1: DEFINITION Let \((\pi, V)\) be a representation of \(G\) — then its character is the function

\[ \chi_{\pi} : G \to \mathbb{C} \]

given by the prescription

\[ \chi_{\pi}(x) = \text{tr}(\pi(x)) \quad (x \in G). \]

2: N.B. It is clear that characters are class functions and that equivalent representations have equal characters.

3: LEMMA \( \forall x \in G, \)

\[ \chi_{\pi}(x^{-1}) = \overline{\chi_{\pi}(x)}. \]

4: N.B.

\[ \chi_{\pi^*} = \overline{\chi_{\pi}}. \]

5: LEMMA Let

\[ \begin{align*}
\pi_1 : G &\to GL(V_1) \\
\pi_2 : G &\to GL(V_2)
\end{align*} \]

be representations of \(G\) — then the character of

\[ (\pi_1 \otimes \pi_2, V_1 \otimes V_2) \]

is \( \chi_{\pi_1} \chi_{\pi_2}. \)
2.

[For the record, the character of

\[(\pi_1 \oplus \pi_2, V_1 \oplus V_2)\]

is \(\chi_{\pi_1} + \chi_{\pi_2}\), implying thereby that a nonnegative integral linear combination of characters is again a character.]

6: EXAMPLE \(\pi_{1,2}\) is equivalent to \(\pi_2 \otimes \pi_1^*\) (cf. §2, #9), hence

\[
\chi_{\pi_{1,2}} = \chi_{\pi_2} \otimes \pi_1^* = \chi_{\pi_2} \chi_{\pi_1}^* = \chi_{\pi_2} \overline{\chi_{\pi_1}}.
\]

7: DEFINITION The character of an irreducible representation is called an irreducible character.

[Note: The zero function (i.e., the additive identity of \(C(G)\)) is a character but it is not an irreducible character (cf. §1, #21).]

8: N.B. The irreducible characters are thus the \(\chi_{\Pi}(\Pi \in \hat{G})\).

9: FIRST ORTHOGONALITY RELATION Let \(\Pi_i, \Pi_j \in \hat{G}\) -- then

\[
\frac{1}{|G|} \sum_{x \in G} \chi_i(x) \chi_j(x^{-1}) = \delta_{ij},
\]

where for short

\[
\chi_i = \chi_{\Pi_i}, \chi_j = \chi_{\Pi_j}.
\]

10: NOTATION Given \(x \in G\), write \(C(x)\) for its conjugacy class and \(G_x\) for its centralizer.

11: RAPPEL The number of conjugates of \(x\) in \(G\) is \([G;G_x]\), i.e.,

\[
|C(x)| = [G;G_x].
\]
3.  

[Note: The class equation for $G$ is the relation]

$$|G| = \sum_{i} [G:G_{x_{i}}],$$

one $x_{i}$ having been chosen from each conjugacy class.

12: SECOND ORTHOGONALITY RELATION Let $x_{1}, x_{2} \in G$ -- then

$$\sum_{\pi \in \hat{G}} \chi_{\pi}(x_{1})\chi_{\pi}(x_{2}^{-1}) = \begin{cases} |G_{x}| & \text{if } x_{1} = x_{2} \\ 0 & \text{if } C(x_{1}) \neq C(x_{2}). \end{cases}$$

[Note:

$$|G_{x}| = \frac{|G|}{[G:G_{x}]} = \frac{|G|}{|C(x)|}.$$]

13: NOTATION Given $f, g \in C(G)$, put

$$\langle f, g \rangle_{G} = \frac{1}{|G|} \sum_{x \in G} f(x)\overline{g(x)},$$

the canonical inner product on $C(G)$.

14: EXAMPLE $\forall \pi_{1}, \pi_{2} \in \hat{G},$

$$\langle \chi_{\pi_{1}}, \chi_{\pi_{2}} \rangle_{G} = \begin{cases} 1 & \text{if } \pi_{1} = \pi_{2} \\ 0 & \text{if } \pi_{1} \neq \pi_{2}. \end{cases}$$

15: SCHOLIUM The irreducible characters form an orthonormal set, thus are linearly independent (cf. I, §3, #9).
Recall now that the Fourier transform

\[ \hat{\lambda}: \mathcal{C}(G) \to \bigoplus_{\Pi \in \hat{G}} \text{Hom}(V(\Pi)) \]

is an algebra isomorphism. Since the center of each \( \text{Hom}(V(\Pi)) \) consists of scalar multiples of the identity operator, it follows that an \( f \in \mathcal{C}(G) \) is a class function iff \( \forall \, \Pi \in \hat{G}, \)

\[ \hat{f}(\Pi) = C_\Pi \text{id}_{V(\Pi)} \left( C_\Pi \in \mathcal{C} \right). \]

**16: INVERSION FORMULA** Given a class function \( f \in \mathcal{C}(G), \forall \, x \in G, \)

\[ f(x) = \sum_{\Pi \in \hat{G}} \langle f, \chi_\Pi \rangle_G \chi_\Pi(x). \]

**PROOF**

\[ f(x) = \frac{1}{|G|} \sum_{\Pi \in \hat{G}} d_\Pi \text{tr} (\Pi(x^{-1}) \hat{f}(\Pi)) \]

\[ = \frac{1}{|G|} \sum_{\Pi \in \hat{G}} d_\Pi c_\Pi \chi_\Pi(x^{-1}) \]

\[ = \frac{1}{|G|} \sum_{\Pi \in \hat{G}} d_\Pi c_\Pi \chi_\Pi(x). \]

Fix \( \Pi_0 \in \hat{G} \) -- then

\[ \langle f, \chi_{\Pi_0} \rangle_G = \frac{1}{|G|} \sum_{\Pi \in \hat{G}} d_\Pi C_\Pi \langle \chi_{\Pi'}, \chi_{\Pi_0} \rangle_G \]

\[ = \frac{1}{|G|} \sum_{\Pi \in \hat{G}} d_\Pi C_\Pi \langle \chi_{\Pi'}, \chi_{\Pi_0} \rangle_G \]

\[ = \frac{1}{|G|} d_{\Pi_0} C_{\Pi_0}. \]
17: \textbf{N.B.} \quad \forall x \in G, \quad \sum_{\Pi \in \hat{G}} <f, \chi_{\Pi} \rangle_G \chi_{\Pi}(x) = \sum_{\Pi \in \hat{G}} \overline{f} \chi_{\Pi}(x) \chi_{\Pi}(x) = \sum_{\Pi \in \hat{G}} \langle \overline{f}, \chi_{\Pi} \rangle_G \chi_{\Pi}(x).

The preceding discussion makes it clear that a class function \( f \) is a character iff \( <f, \Pi \rangle_G \) is a nonnegative integer for all \( \Pi \in \hat{G} \).

18: \textbf{NOTATION} \quad \text{CON}(G) \text{ is the set of conjugacy classes of } G.

19: \textbf{SCHOLIUM} \quad \text{The dimension of the space of class functions is equal to} \quad |\text{CON}(G)| \quad \text{or still, is equal to} \quad |\hat{G}|.

20: \textbf{NOTATION} \quad \text{Given } C \in \text{CON}(G), \text{ let } \chi_C \text{ be the characteristic function of } C:

\[ \chi_C(x) = \begin{cases} 1 & \text{if } x \in C \\ 0 & \text{if } x \notin C. \end{cases} \]

21: \textbf{LEMMA} \quad \chi_C = \sum_{Y \in C} \delta_Y.

22: \textbf{N.B.} \quad \text{The } \chi_C \ (C \in \text{CON}(G)) \text{ are a basis for the class functions on } G \quad \text{(as are the } \chi_{\Pi} \ (\Pi \in \hat{G})).
23: **Lemma** Let $C_1, C_2, \ldots$ be the elements of $\text{CON}(G)$ — then there are nonnegative integers $m_{i,j,k}$ such that

$$\chi_{C_i} \chi_{C_j} = \sum_{k} m_{i,j,k} \chi_{C_k}.$$  

[Note: Fixing an $x_k \in C_k$, qualitatively $m_{i,j,k}$ is the number of ordered pairs $(x,y)$ with $x \in C_i$, $y \in C_j$ and $xy = x_k$ while quantitatively]

$$m_{i,j,k} = \frac{|C_i| \cdot |C_j|}{|G|} \sum_{\Pi \in \hat{G}} \frac{\chi_{\Pi}(x_i) \chi_{\Pi}(x_j) \chi_{\Pi}(x_k)}{d_{\Pi}}.$$  

24: **Notation** Given $\Pi \in \hat{G}$, put

$$e_{\Pi} = \chi_{\Pi}^{-1}(E_{\Pi}) \quad (E_{\Pi} \in \text{Hom}(V(\Pi))) \quad (\text{cf. I, §4}).$$

25: **Lemma**

$$e_{\Pi} = \frac{d_{\Pi}}{|G|} \sum_{y \in G} x_{\Pi}(y)^{-1} \delta_{y}.$$  

[Note: In brief,]

$$e_{\Pi} = \frac{d_{\Pi}}{|G|} \chi_{\Pi^*}.$$  

26: **Lemma**

$$e_{\Pi_1} \ast e_{\Pi_2} =
\begin{cases}
0 & \text{if } \Pi_1 \neq \Pi_2 \\
e_{\Pi} & \text{if } \Pi_1 = \Pi_2
\end{cases}$$

27: **Lemma**

$$\delta_e = \sum_{\Pi \in \hat{G}} e_{\Pi}.$$
§5. DECOMPOSITION THEORY

Let $G$ be a finite group.

1. **CONSTRUCTION** Suppose that $G$ operates on a finite set $S$, hence for each $x \in G$ there is given a bijection $s \rightarrow x \cdot s$ of $S$ satisfying the identities

$$e \cdot s = s, \quad x \cdot (y \cdot s) = (xy) \cdot s.$$ 

Let $V = \{f : S \rightarrow \mathbb{C}\}$ and define a representation $\pi : G \rightarrow \text{GL}(V)$ by

$$\pi(x)f(s) = f(x^{-1} \cdot s).$$

Then

$$\chi_\pi(x) = |\{s \in S : x \cdot s = s\}|.$$

2. **EXAMPLE** Take $S = G$ and write $x \cdot y = xy$ -- then the role of $V$ is played by $C(G)$ and the role of $\pi$ is played by $L$ (the left translation representation of $G$ (cf. §1, §12)), hence

$$\chi_L(x) = |\{y \in G : xy = y\}|,$$

which is $|G|$ if $x = e$ and is 0 otherwise.

3. **EXAMPLE** Take $S = G$ but replace $G$ by $G \times G$, the action being $(x_1, x_2) \cdot y = x_1yx_2^{-1}$ -- then the associated representation is $\pi_{L,R}$ (cf. §1, §14) and

$$\chi_{\pi_{L,R}}(x_1, x_2) = |\{y \in G : x_1yx_2^{-1} = y\}|$$

$$= |\{y \in G : x_1 = yx_2y^{-1}\}|$$

$$= |G_{x_1}|$$

if $x_1$ and $x_2$ are conjugate and is 0 otherwise.
2.

4: DEFINITION Let \((\pi, V)\) be a representation of \(G\) -- then by complete reducibility, there is a direct sum decomposition

\[
\pi = \bigoplus_{\pi \in \hat{G}} m(\pi, \pi) \pi,
\]

the nonnegative integer \(m(\pi, \pi)\) being the multiplicity of \(\pi\) in \(\pi\).

5: LEMMA \(\forall \pi \in \hat{G},\)

\[
m(\pi, \pi) = \langle \chi_{\pi}, X_\pi \rangle_G.
\]

6: N.B.

\[
dim I_G(\pi, \pi) = m(\pi, \pi).
\]

7: REMARK The operator

\[
P_\pi = \frac{d_\pi}{|G|} \sum_{x \in G} \chi_\pi(x) \chi_\pi(x)^*(-x)
\]

is the projection onto the \(\pi\)-isotypic subspace of \(V\).

8: THEOREM Each \(\pi \in \hat{G}\) is contained in \(L\) with multiplicity \(d_\pi\).

PROOF In fact,

\[
m(\pi, L) = \langle X_\pi, X_L \rangle_G
\]

\[
= \frac{1}{|G|} \sum_{x \in G} \chi_\pi(x) \chi_L(x)
\]

\[
= \frac{1}{|G|} \chi_\pi(e) |G|
\]

\[
= \chi_\pi(e) = d_\pi.
\]
9: N.B. It is a corollary that

\[ |G| = \sum_{\pi \in \hat{G}} d_{\pi}^2 \quad \text{(cf. §3, #5)}. \]

10: LEMMA Let \((\pi_1, V_1), (\pi_2, V_2)\) be representations of \(G\). Assume: \(\chi_{\pi_1} = \chi_{\pi_2}\) -- then \((\pi_1, V_1) \approx (\pi_2, V_2)\).

PROOF \(\forall \pi \in \hat{G},\)

\[ <\chi_{\pi'}, \chi_{\pi_1}>_G = <\chi_{\pi'}, \chi_{\pi_2}>_G \]

or still, \(\forall \pi \in \hat{G},\)

\[ m(\pi, \pi_1) = m(\pi, \pi_2), \]

from which the assertion.

11: IRREDUCIBILITY CRITERION A representation \(\pi: G \to GL(V)\) is irreducible iff \(<\chi_{\pi'}, \chi_{\pi}>_G = 1.\)

PROOF The necessity is implied by the first orthogonality relations and the sufficiency follows upon noting that

\[ <\chi_{\pi'}, \chi_{\pi}>_G = \sum_{\pi \in \hat{G}} m(\pi, \pi^2). \]

Let \(G_1, G_2\) be finite groups and let

\[
\begin{align*}
\Pi_1: & G_1 \to GL(V_1) \\
\Pi_2: & G_2 \to GL(V_2)
\end{align*}
\]
4.

be irreducible representations of $G_1$, $G_2$ -- then the character $\chi_{\Pi_1 \otimes \Pi_2}$ of

$$(\Pi_1 \otimes \Pi_2, V_1 \otimes V_2)$$

is the function

$$(x_1, x_2) \rightarrow \chi_{\Pi_1}(x_1)\chi_{\Pi_2}(x_2) \quad (x_1 \in G_1, x_2 \in G_2).$$

12: LEMMA $\Pi_1 \otimes \Pi_2$ is irreducible (cf. §2, #13).

PROOF It is a question of applying the irreducibility criterion. Thus

$$\langle \chi_{\Pi_1 \otimes \Pi_2}, \chi_{\Pi_1 \otimes \Pi_2} \rangle_{G \times G}$$

$$= \frac{1}{|G_1 \times G_2|} \sum_{(x_1, x_2) \in G_1 \times G_2} \chi_{\Pi_1}(x_1)\chi_{\Pi_2}(x_2)\overline{\chi_{\Pi_1}(x_1)}\overline{\chi_{\Pi_2}(x_2)}$$

$$= \frac{1}{|G_1|} \sum_{x_1 \in G_1} \chi_{\Pi_1}(x_1)\overline{\chi_{\Pi_1}(x_1)} \cdot \frac{1}{|G_2|} \sum_{x_2 \in G_2} \chi_{\Pi_2}(x_2)\overline{\chi_{\Pi_2}(x_2)}$$

$$= \langle \chi_{\Pi_1}, \chi_{\Pi_1} \rangle_{G_1} \cdot \langle \chi_{\Pi_2}, \chi_{\Pi_2} \rangle_{G_2}$$

$$= 1.$$ 

13: REMARK The cardinality of $G_1 \times G_2$ is $|\text{CON}(G_1 \times G_2)|$ (cf. §4, #19).

But

$$|\text{CON}(G_1 \times G_2)| = |\text{CON}(G_1)| \cdot |\text{CON}(G_2)|$$

and the preceding considerations produce

$$|\text{CON}(G_1)| \cdot |\text{CON}(G_2)|$$

pairwise distinct irreducible characters of $G_1 \times G_2$. Therefore every irreducible
representation of $G_1 \times G_2$ is equivalent to an outer tensor product $\Pi_1 \otimes \Pi_2$, where $\Pi_1 \in \hat{G}_1$, $\Pi_2 \in \hat{G}_2$ (cf. §2, #14).
§6. INTEGRABILITY

1: DEFINITION An algebraic integer is a complex number \( \lambda \) which is a root of a polynomial of the form

\[
x^n + a_{n-1}x^{n-1} + \ldots + a_0,
\]

where \( a_i \in \mathbb{Z} \) (0 \( \leq \) \( i \leq \) n-1).

[Note: Equivalently, an algebraic integer is a complex number \( \lambda \) which is a zero of \( \det(A - \lambda I) \) for some square matrix \( A \) with entries in \( \mathbb{Z} \).]

2: N.B. The rational algebraic integers are precisely the elements of \( \mathbb{Z} \).

3: LEMMA If \( \mu, \nu \) are algebraic integers, then \( \mu + \nu \) and \( \mu \nu \) are also algebraic integers.

Therefore the set of algebraic integers is a subring of \( \mathbb{C} \).

4: EXAMPLE Roots of unity are algebraic integers.

Let \( G \) be a finite group.

5: LEMMA Let \((\pi, V)\) be a representation of \( G \), \( \chi_\pi \) its character -- then

\( \forall x \in G, \chi_\pi (x) \) is an algebraic integer.

[This is because \( \chi_\pi (x) \) is a finite sum of roots of unity.]

The center of \( \mathbb{C}(G) \) (i.e., the class function) is a unital commutative associative algebra over \( \mathbb{C} \), thus its irreducible representations are just
2.

Homomorphisms into \( \mathbb{C} \) and are indexed by the \( \Pi \in \hat{G} \), say \( \omega_{\Pi} \) with

\[
\omega_{\Pi_1}(e_{\Pi_2}) = \delta_{\Pi_1,\Pi_2}.
\]

[Note: The \( e_{\Pi} \) (\( \Pi \in \hat{G} \)) are a basis for the class functions on \( G \).]

6: THEOREM \( \forall C \in \text{CON}(G), \omega_{\Pi}(C) \) is an algebraic integer.

PROOF In the notation of §4, #23,

\[
\chi_{C_i} \chi_{C_j} = \sum_m m_{i,j,k} \chi_{C_k},
\]

hence

\[
\omega_{\Pi}(\chi_{C_i})\omega_{\Pi}(\chi_{C_j})
\]

\[
= \omega_{\Pi}(\chi_{C_i} \chi_{C_j})
\]

\[
= \sum_k m_{i,j,k} \omega_{\Pi}(\chi_{C_k})
\]

\[=>\]

\[
\sum_k (m_{i,j,k} - \delta_{j,k} \omega_{\Pi}(\chi_{C_i})) \omega_{\Pi}(\chi_{C_k}) = 0.
\]

But this means that \( \omega_{\Pi}(\chi_{C_i}) \) is an eigenvalue of the matrix \( A_i \) whose \((j,k)\)th entry is \( m_{i,j,k} \) or still, is a zero of

\[
\det(A_i - \xi I),
\]

thus is an algebraic integer.

7: LEMMA \( \forall C \in \text{CON}(G), \)
Now fix $\Pi_0 \in \hat{G}$ — then

$$
\omega_{\Pi_0} (\chi_C) = \omega_{\Pi_0} \left( \frac{1}{|G_x|} \sum_{\Pi \in \hat{G}} \frac{|G|}{d_\Pi} \chi_\Pi (x) e_\Pi \right)
$$

$$
= \frac{|G|}{|G_x|} \sum_{\Pi \in \hat{G}} \frac{1}{d_\Pi} \chi_\Pi (x) \omega_{\Pi_0} (e_\Pi)
$$

$$
= \frac{|G|}{|G_x|} \frac{\chi_\Pi_0 (x)}{d_{\Pi_0}} \omega_{\Pi_0} = \frac{|C|}{d_{\Pi_0}} \chi_{\Pi_0} (x).
$$
Consequently, \( \forall C \subseteq \text{CON}(G), \)

\[
\frac{|C|}{d_{\Pi}} \chi_\Pi(x) \quad (x \in C)
\]

is an algebraic integer.

8: \textbf{THEOREM} \( \forall \Pi \in \hat{G}, \)

\[
\frac{|G|}{d_{\Pi}} \in \mathbb{Z}.
\]

PROOF In view of §4, #9,

\[
|G| = \sum_{x \in G} \chi_\Pi(x)\chi_\Pi(x^{-1}).
\]

Given \( C \subseteq \text{CON}(G), \) fix an \( x_C \in C \) -- then

\[
|G| = \sum_{C \subseteq \text{CON}(G)} |C| \chi_\Pi(x_C)\chi_\Pi(x_C^{-1})
\]

\[
\Rightarrow \frac{|G|}{d_{\Pi}} = \sum_{C \subseteq \text{CON}(G)} \left( \frac{|C|}{d_{\Pi}} \chi_\Pi(x_C)\right)\chi_\Pi(x_C^{-1}),
\]

hence \( \frac{|G|}{d_{\Pi}} \) is a rational algebraic integer, hence is an integer.

In other words, the \( d_{\Pi} \) divide \( |G| \).

9: \textbf{THEOREM} If \( A \) is an abelian normal subgroup of \( G \), then the \( d_{\Pi} \) divide \( [G:A] \).

10: \textbf{APPLICATION} Let \( Z(G) \) be the center of \( G \) -- then the \( d_{\Pi} \) divide \( [G:Z(G)] \).
1.

§7. **INDUCED CLASS FUNCTIONS**

Let $G$ be a finite group, $\Gamma \subseteq G$ a subgroup.

1: **NOTATION** $\text{CL}(G)$ is the subspace of $C(G)$ comprised of the class functions and $\text{CL}(\Gamma)$ is the subspace of $C(\Gamma)$ comprised of the class functions.

2: **NOTATION** Extend a function $\phi \in C(\Gamma)$ to a function $\phi^* \in C(G)$ by writing

$$
\phi^*(x) = \begin{cases} 
\phi(x) & \text{if } x \in \Gamma \\
0 & \text{if } x \notin \Gamma.
\end{cases}
$$

3: **NOTATION** Given a class function $\phi \in \text{CL}(\Gamma)$, put

$$(i_{\Gamma} + G\phi)(x) = \frac{1}{|\Gamma|} \sum_{y \in G} \phi^*(yxy^{-1})$$

$$= \frac{1}{|\Gamma|} \sum_{y \in G, yxy^{-1} \in \Gamma} \phi(yxy^{-1}).$$

4: **LEMMA**

$$i_{\Gamma} \to G\phi \in \text{CL}(G),$$

the **induced** class function.

5: **N.B.** Therefore

$$i_{\Gamma} \to G: \text{CL}(\Gamma) \to \text{CL}(G).$$

[Note:

$$i_{\Gamma} \to G\phi = ci_{\Gamma} + G\phi (c \in C), i_{\Gamma} \to G(\phi_1 + \phi_2) = i_{\Gamma} \to G\phi_1 + i_{\Gamma} \to G\phi_2$$]
but in general,

\[ i_\Gamma \rightarrow G(\phi_1 \phi_2) = (i_\Gamma \rightarrow G\phi_1)(i_\Gamma \rightarrow G\phi_2). \]

The arrow of restriction \( C(G) \rightarrow C(\Gamma) \) leads to a map

\[ r_G \rightarrow \Gamma : CL(G) \rightarrow CL(\Gamma). \]

And:

6: FROBENIUS RECIPROCITY Let \( \phi \in CL(\Gamma), \psi \in CL(G) \) -- then

\[ \langle i_\Gamma \rightarrow G\phi, \psi \rangle_G = \langle \phi, r_G \rightarrow \Gamma \psi \rangle_\Gamma. \]

PROOF

\[ \langle i_\Gamma \rightarrow G\phi, \psi \rangle_G = \frac{1}{|G|} \sum_{x \in G} (i_\Gamma \rightarrow G\phi)(x) \overline{\psi(x)} \]

\[ = \frac{1}{|\Gamma|} \frac{1}{|G|} \sum_{y \in G} \sum_{x \in G} \phi(yx^{-1}) \overline{\psi(x)} \]

\[ = \frac{1}{|\Gamma|} \frac{1}{|G|} \sum_{y \in G} \sum_{x \in G} \phi(x) \overline{\psi(y^{-1}xy)} \]

\[ = \frac{1}{|G|} \sum_{y \in G} \frac{1}{|\Gamma|} \sum_{\gamma \in \Gamma} \phi(y) \overline{\psi(y)} \]

\[ = \frac{1}{|G|} \sum_{y \in G} \langle \phi, r_G \rightarrow \Gamma \psi \rangle_\Gamma \]

\[ = \langle \phi, r_G \rightarrow \Gamma \phi \rangle_\Gamma. \]

7: APPLICATION If \( \phi \) is a character of \( \Gamma \), then \( i_\Gamma \rightarrow G\phi \) is a character of \( G \).

[If \( \chi \) is a character of \( G \), then \( r_G \rightarrow \Gamma \chi \) is a character of \( \Gamma \), hence}
3.

\[ \langle \phi, r_G \to \Gamma \rangle \]

is a nonnegative integer for all \( \Pi \in \hat{G} \) or still,

\[ \langle i_\Gamma \to G^\phi, \chi_\Pi \rangle \]

is a nonnegative integer for all \( \Pi \in \hat{G} \) which implies that \( i_\Gamma \to G^\phi \) is a character of \( G \) (cf. §4, §17 ff.).

8: **LEMMA** Let \( \phi \in CL(\Gamma), \psi \in CL(G) \) — then

\[ i_\Gamma \to G((r_G \to \Gamma)^\psi) = \psi(i_\Gamma \to G^\phi). \]

**PROOF** From the definitions,

\[ i_\Gamma \to G((r_G \to \Gamma)^\psi)(x) \]

\[ = \frac{1}{|\Gamma|} \sum_{y \in G} r_G \to \Gamma^{\psi} \circ (yxy^{-1})^\phi(yxy^{-1}) \]

\[ = \frac{1}{|\Gamma|} \sum_{y \in G} \psi(yxy^{-1}) \circ (yxy^{-1}) \]

\[ = \frac{1}{|\Gamma|} \sum_{y \in G} \psi(x) \circ (yxy^{-1}) \]

\[ = \psi(x) \frac{1}{|\Gamma|} \sum_{y \in G} \circ (yxy^{-1}) \]

\[ = \psi(x) (i_\Gamma \to G^\phi)(x). \]

9: **APPLICATION** The image of \( i_\Gamma \to G \) is an ideal in \( CL(G) \).

Write

\[ G = \bigoplus_{k=1}^{n} x_k^\Gamma. \]
10: **Lemma** For any $\phi \in \text{CL}(\Gamma)$,

$$\left( i_\Gamma \rightarrow G\phi \right)(x) = \sum_{k=1}^{n} \phi(x_k^{-1}xx_k).$$

**Proof** In fact,

$$\left( i_\Gamma \rightarrow G\phi \right)(x) = \frac{1}{|\Gamma|} \sum_{y \in G} \phi(yxy^{-1})$$

$$= \frac{1}{|\Gamma|} \sum_{y \in G} \phi(y^{-1}xy)$$

$$= \frac{1}{|\Gamma|} \sum_{\gamma \in \Gamma} \sum_{k=1}^{n} \phi(\gamma^{-1}x_k^{-1}xx_k\gamma).$$

There are then two possibilities.

- $\gamma^{-1}x_k^{-1}xx_k \gamma \not\in \Gamma$
  
  $\Rightarrow x_k^{-1}xx_k \not\in \Gamma$

  $\Rightarrow \phi(x_k^{-1}xx_k) = 0 = \phi(x_k^{-1}xx_k)$.

- $\gamma^{-1}x_k^{-1}xx_k \gamma \in \Gamma$

  $\Rightarrow x_k^{-1}xx_k \in \Gamma$

  $\Rightarrow \phi(x_k^{-1}xx_k) = \phi(x_k^{-1}xx_k)$

  $= \phi(x_k^{-1}xx_k) = \phi(x_k^{-1}xx_k)$.

Therefore the sum $\frac{1}{|\Gamma|} \sum_{\gamma \in \Gamma} \phi(x_k^{-1}xx_k)$ disappears, leaving

$$\sum_{k=1}^{n} \phi(x_k^{-1}xx_k).$$
Note: If instead, 

\[ G = \bigoplus_{k=1}^{n} x_k' \]

then for any \( \phi \in \text{CL}(\pi) \),

\[ (i_{\pi} \rightarrow G^\phi)(x) = \sum_{k=1}^{n} \phi(x_k x_k^{-1}) \]

**Example**: Let \( S \) be a transitive \( G \)-set, \( \pi \) the associated representation (cf. §5, #1). Fix a point \( s \in S \) and let \( G_s \) be its stabilizer — then

\[ \chi_{\pi} = i_{G_s} \rightarrow G^1_{G_s} \]

where \( 1_{G_s} \in \text{CL}(G_s) \) is \( \equiv 1 \).

[Take \( S = \{1, \ldots, n\} \) and \( s = 1 \). Write

\[ G = \bigoplus_{k=1}^{n} x_k G_s \]

with \( x_k \cdot 1 = k \) — then

\[ (i_{G_s} \rightarrow G^1_{G_s})(x) = \sum_{k=1}^{n} 1_{G_s}(x_k^{-1} x_k) \]

\[ = \sum_{k, x_k^{-1} x_k \in G_s} 1 \]

\[ = \sum_{k, (x_k^{-1} x_k) \cdot 1 = 1} 1 \]

\[ = \sum_{k, (x_k \cdot 1) \cdot 1 = x_k \cdot 1} 1 \]
6.

\[= \sum_{k, x : k \cdot k = k} 1\]

\[= |\{k \in S : x \cdot k = k\}|\]

\[= \chi_\pi(x) \quad \text{(cf. §5, #1).} \]

[Note: Here is a "for instance". Take \( S = G/\Gamma \) and write]

\[G/\Gamma = \bigsqcup_{k=1}^{n} x_k \Gamma.\]

Then \( G/\Gamma \) is a transitive \( G \)-set and

\[Gx_k \Gamma = x_k \Gamma x_k^{-1}.\]

In particular: Take \( x_k = 1 \) to get

\[\chi_\pi = i_\Gamma \to G^{-1} \Gamma,\]

thus at a given \( x \in G \), \((i_\Gamma \to G^{-1} \Gamma)(x)\) is the number of left cosets of \( \Gamma \) in \( G \) fixed by \( x \).]

12: LEMMA Suppose that \( \Gamma_1 \subseteq \Gamma_2 \subseteq G \). Let \( \phi_1 \in \text{CL}(\Gamma_1) \) -- then

\[i_{\Gamma_2} \to G(i_{\Gamma_1} \to \Gamma_2 \phi_1) = i_{\Gamma_1} \to G\phi_1.\]

PROOF Both sides of the putative equality are class functions, thus it suffices to show that

\[\langle i_{\Gamma_2} \to G(i_{\Gamma_1} \to \Gamma_2 \phi_1), \chi_{\pi} \rangle_G = \langle i_{\Gamma_1} \to G\phi_1, \chi_{\pi} \rangle_G\]

for all \( \pi \in \hat{G} \). But the LHS equals

\[\langle i_{\Gamma_1} \to \Gamma_2 \phi_1, \pi_G \to \Gamma_2 \chi_{\pi} \rangle_{\Gamma_2}.\]
which is the RHS.

13: NOTATION Given $x \in G$, put

$$\Gamma^x = x \Gamma x^{-1} = \{xyx^{-1} : y \in \Gamma\}. $$

The range of

$$i_\Gamma \rightarrow G: \text{CL}(\Gamma) \rightarrow \text{CL}(G)$$

is contained in the subspace $S_\Gamma$ of CL(G) consisting of those class functions $f \in \text{CL}(G)$ that vanish on

$$G = \bigcup_{x \in G} \Gamma^x. $$

14: LEMMA

$$i_\Gamma \rightarrow G \text{CL}(\Gamma) = S_\Gamma. $$

PROOF Assume not, thus

$$i_\Gamma \rightarrow G \text{CL}(\Gamma) \neq S_\Gamma. $$

Then there exists a nonzero $f \in S_\Gamma$, which is orthogonal to all functions in

$$i_\Gamma \rightarrow G \text{CL}(\Gamma): \forall \phi \in \text{CL}(\Gamma),$$

$$<i_\Gamma \rightarrow G^\phi, f> = 0$$

or still, $\forall \phi \in \text{CL}(\Gamma),$

$$<\phi, r_G \rightarrow \Gamma^f> = 0.$$. 


Now take \( \phi = r_G \to \Gamma^f \) to get

\[ \langle r_G \to \Gamma^f, r_G \to \Gamma^f \rangle_{\Gamma} = 0, \]

hence \( r_G \to \Gamma^f = 0 \), i.e., \( f \) vanishes on \( \Gamma \). But \( f \in \text{CL}(G) \), so \( \forall x \in G, f \) vanishes on \( \Gamma^x \). Since \( f \in S_\Gamma \), it then follows that \( f \) vanishes on \( G \): \( f \equiv 0 \), contradicting the supposition that \( f \) is nonzero.

15: APPLICATION The image of \( i_{\Gamma} \to G \) is an ideal in \( \text{CL}(G) \) (cf. #9).

Let \( \phi \in \text{CL}(\Gamma) \). Given \( x \in G \), define \( \phi^x \in \text{C}(\Gamma^x) \) by

\[ \phi^x(y) = \phi(x^{-1}yx) \quad (y = x\gamma x^{-1}, \gamma \in \Gamma) \]

\[ = \phi(x^{-1}x\gamma x^{-1}x) \]

\[ = \phi(\gamma). \]

16: LEMMA

\( \phi^x \in \text{CL}(\Gamma^x) \).

PROOF Let

\[ y_1 = xy_1 x^{-1}, \quad y_2 = xy_2 x^{-1}. \]

Then

\[ \phi^x(y_1 y_2^{-1}) \]

\[ = \phi(x^{-1}y_1 y_2^{-1}x) \]

\[ = \phi(x^{-1}x\gamma_1 x^{-1}x\gamma_2 x^{-1}x) \]

\[ = \phi(x^{-1}x\gamma_1 x^{-1}x\gamma_2 x^{-1}x) \]

\[ = \phi(x^{-1}x\gamma_1 x^{-1}x\gamma_2 x^{-1}x) \]

\[ = \phi(x^{-1}x\gamma_1 x^{-1}x\gamma_2 x^{-1}x) \]

\[ = \phi(x^{-1}x\gamma_1 x^{-1}x\gamma_2 x^{-1}x) \]

\[ = \phi(x^{-1}x\gamma_1 x^{-1}x\gamma_2 x^{-1}x) \]

\[ = \phi(x^{-1}x\gamma_1 x^{-1}x\gamma_2 x^{-1}x) \]

\[ = \phi(x^{-1}x\gamma_1 x^{-1}x\gamma_2 x^{-1}x) \]

\[ = \phi(x^{-1}x\gamma_1 x^{-1}x\gamma_2 x^{-1}x) \]

\[ = \phi(x^{-1}x\gamma_1 x^{-1}x\gamma_2 x^{-1}x) \]
9.

\[ = \phi(y_1y_2y_1^{-1}) \]

\[ = \phi(y_2) = \phi^x(y_2). \]

17: LEMMA \( \forall x \in G \) and \( \forall \phi \in CL(\Gamma) \),

\[ i_x^\Gamma \rightarrow G = i_\Gamma \rightarrow G^\phi. \]

PROOF Write

\[ G = \bigoplus_{k=1}^{n} x_k \Gamma = \bigoplus_{k=1}^{n} xx_kx^{-1}x. \]

Then (cf. #10)

\[ (i_x^\Gamma \phi^x)(y) = \sum_{k=1}^{n} \phi^x((xx_kx^{-1})^{-1}y(xx_kx^{-1})) \]

\[ = \sum_{k=1}^{n} \phi^x(xx_k^{-1}x^{-1}yxx_kx^{-1}) \]

\[ = \sum_{k=1}^{n} \phi(xx_k^{-1}xx_kx^{-1}x) \]

\[ = \sum_{k=1}^{n} \phi(x^{-1}(xx_k^{-1}x^{-1}yxx_kx^{-1})) \]

\[ = (i_\Gamma \rightarrow G^\phi)(x^{-1}yx) \]

\[ = (i_\Gamma \rightarrow G^\phi)(y) \quad (\text{cf. #4}). \]
§8. MACKEY THEORY

Let $G$ be a finite group, let $\Gamma_1, \Gamma_2 \subseteq G$ be subgroups, and let

$$G = \bigcup_{s \in S} \Gamma_1 s \Gamma_2$$

be a double coset decomposition of $G$. Given $s \in S$, put

$$\Gamma_2(s) = \Gamma_2^s \cap \Gamma_1 = s \Gamma_2 s^{-1} \cap \Gamma_1.$$ 

**1: LEMMA** Let

$$\Gamma_1 = \bigcup_{t \in T(s)} t \Gamma_2(s)$$

be a left coset decomposition of $\Gamma_1$ -- then

$$\Gamma_1 s \Gamma_2 = \left( \bigcup_{t \in T(s)} t \Gamma_2(s) s \right) \Gamma_2$$

$$= \bigcup_{t \in T(s)} t \Gamma_2(s) s \Gamma_2$$

$$= \bigcup_{t \in T(s)} t \Gamma_2(s) (s \Gamma_2 s^{-1}) s$$

$$= \bigcup_{t \in T(s)} t (s \Gamma_2 s^{-1}) s$$

$$= \bigcup_{t \in T(s)} t s \Gamma_2$$

is a partition of $\Gamma_1 s \Gamma_2$.

**PROOF** Suppose that

$$t_1 s \Gamma_2 \cap t_2 s \Gamma_2 \neq \emptyset \quad (t_1 \neq t_2).$$
2.

so

\[ t_1 s = t_2 s \gamma_2 \ (\gamma_2 \in \Gamma_2) \].

Then

\[ t_1 = t_2 s \gamma_2 s^{-1} \Rightarrow t_2^{-1} t_1 \in \Gamma_2^s. \]

Meanwhile

\[ t_1, t_2 \in \Gamma_1 \Rightarrow t_2^{-1} t_1 \in \Gamma_1. \]

Therefore

\[ t_2^{-1} t_1 \in \Gamma_2^s \cap \Gamma_1 = \Gamma_2(s) \]

\[ \Rightarrow t_1 = t_2. \]

Let \( R(s) = \{ ts : t \in T(s) \} \equiv T(s)s \) and let

\[ R = \bigcup_{s \in S} R(s). \]

2: LEMMA \( R \) is a set of left coset representatives of \( \Gamma_2 \) in \( G \).

PROOF Let \( x \in G \) -- then

\[ x \in \Gamma_1^s \Gamma_2 \ (\exists s \in S) \]

\[ \Rightarrow x = ts\gamma_2 \ (\exists t \in T(s)) \]

\[ \Rightarrow x = r\gamma_2 \ (r \in R(s), \ r = ts). \]

Therefore

\[ G = \bigcup_{r \in R} r \Gamma_2. \]
Suppose now that
\[ x \in r_2 \cap r'_2 . \]

Then
\[ x = r_2 y_2 = r'_2 y'_2 \quad (r \in R(s), \ r' \in R(s')) \]
\[ \implies x = t s y_2 \quad (t \in T(s)) \]
\[ \implies x = t' s' y'_2 \quad (t' \in T(s')). \]

But
\[ t \in T(s) \implies t \in \Gamma_1 \]
\[ t' \in T(s') \implies t' \in \Gamma_1 \]
\[ \implies x \in \Gamma_1 s_2 \cap \Gamma_1 s'_2 \implies s = s' \]
\[ \implies t s y_2 = t' s' y'_2 \]
\[ \implies t s = t' s' y_2^{-1} = t' s y'_2 \]
\[ \implies t = t' \implies r = r'. \]

Given \( \phi \in \text{CL}(\Gamma_2) \), put
\[ \phi_s = r_t^s \Gamma_2 \phi \Gamma_2 (s) . \]

Here, by definition (cf. §7, #16), \( \phi_s \in \text{CL}(\Gamma_2^s) \), where
\[ \phi_s(y) = \phi(y_2) \quad (y = s y_2 s^{-1}, \ y_2 \in \Gamma_2) . \]
3: THEOREM Under the above assumptions,

\[ r_G \cdot \Gamma_1 (i_{\Gamma_2} \cdot G^\phi) = \sum_{s \in S} i_{\Gamma_2}(s) \cdot \Gamma_1^s. \]

PROOF Since

\[ G = \bigsqcup_{r \in R} r\Gamma_2, \]

\[ \forall x \in G, \]

\[ (i_{\Gamma_2} \cdot G^\phi)(x) = \sum_{r \in R} \phi(r^{-1}xr) \quad (cf. \S 7, \#10), \]

so \( \forall \gamma_1 \in \Gamma_1 \),

\[ (r_G \cdot \Gamma_1 (i_{\Gamma_2} \cdot G^\phi))(\gamma_1) \]

\[ = \sum_{r \in R} \phi(r^{-1}\gamma_1 r) \]

\[ = \sum_{r \in R, r^{-1}\gamma_1 \in \Gamma_2} \phi(r^{-1}\gamma_1 r) \]

\[ = \sum_{s \in S, r \in R(s), r^{-1}\gamma_1 \in \Gamma_2} \phi(s^{-1}t^{-1}\gamma_1 ts) \]

\[ = \sum_{s \in S, t \in T(s), t^{-1}\gamma_1 \in \Gamma_2} \phi(s^{-1}t^{-1}\gamma_1 t) \]

\[ = \sum_{s \in S} \sum_{t \in T(s)} \phi_s(t^{-1}\gamma_1 t) \]
4: LEMMA Let $\psi \in \text{CL}(\Gamma_1)$, $\phi \in \text{CL}(\Gamma_2)$ — then

$$\langle i_{\Gamma_1} \rightarrow G^\psi, i_{\Gamma_2} \rightarrow G^\phi \rangle_G$$

$$= \sum_{s \in S} \langle r_{\Gamma_1} \rightarrow \Gamma_1(s)^\psi, i_{\Gamma_2} \rightarrow \Gamma_1(s)^\phi \rangle_{\Gamma_1(s)}.$$ 

PROOF Taking into account §7, #6,

$$\langle i_{\Gamma_1} \rightarrow G^\psi, i_{\Gamma_2} \rightarrow G^\phi \rangle_G$$

$$= \langle \psi, r_{\Gamma_1} \rightarrow \Gamma_1(i_{\Gamma_2} \rightarrow G^\phi) \rangle_{\Gamma_1}$$

$$= \langle \psi, \sum_{s \in S} i_{\Gamma_2(s)} \rightarrow \Gamma_1(s)^\phi \rangle_{\Gamma_1}$$

$$= \sum_{s \in S} \langle \psi, i_{\Gamma_2(s)} \rightarrow \Gamma_1(s)^\phi \rangle_{\Gamma_1}$$

$$= \sum_{s \in S} \langle i_{\Gamma_2(s)} \rightarrow \Gamma_1(s)^\phi, \psi \rangle_{\Gamma_1(s)}$$

$$= \sum_{s \in S} \langle r_{\Gamma_1} \rightarrow \Gamma_2(s)^\psi, i_{\Gamma_2(s)} \rightarrow \Gamma_2(s)^\phi \rangle_{\Gamma_2(s)}.$$ 

5: NOTATION Given a subgroup $\Gamma \subset G$, let $1_\Gamma$ stand for the function $\Gamma \to G$ which is $1$, that is, the character of the trivial one-dimensional representation of $\Gamma$. 
6: EXAMPLE Take $\Gamma_1 = \Gamma_2 = \Gamma$ -- then

$$<i_{\Gamma} \rightarrow G^1_{\Gamma}, i_{\Gamma} \rightarrow G^1_{\Gamma}>_G$$

$$= |\Gamma \backslash G/\Gamma|.$$ 

Therefore $i_{\Gamma} \rightarrow G^1_{\Gamma}$ is not irreducible if $|\Gamma \backslash G/\Gamma| > 1$ (cf. §5, #11).

[Note: $i_{\Gamma} \rightarrow G^1_{\Gamma}$ is a character of $G$ (cf. §7, #7).]
§9. INDUCED REPRESENTATIONS

Let $G$ be a finite group, $\Gamma \subset G$ a subgroup.

1: CONSTRUCTION Let $(\theta,E)$ be a unitary representation of $\Gamma$ and denote by $E^G_{\Gamma,\theta}$ the space of all $E$-valued functions $f$ on $G$ such that $f(xy) = \theta(\gamma^{-1}) f(x)$ $(x \in G, \gamma \in \Gamma)$ — then the prescription

$$\left(\text{Ind}_{\Gamma,\theta}^G(x)f\right)(y) = f(x^{-1}y)$$

defines a representation $\text{Ind}_{\Gamma,\theta}^G$ of $G$ on $E^G_{\Gamma,\theta}$, the representation of $G$ induced by $\theta$.

2: N.B. The inner product

$$\langle f,g\rangle_\theta = \frac{1}{|G|} \sum_{x \in G} \langle f(x),g(x) \rangle_E$$

equips $E^G_{\Gamma,\theta}$ with the structure of a Hilbert space and $\text{Ind}_{\Gamma,\theta}^G$ is a unitary representation.

3: EXAMPLE Take $\Gamma = \{e\}$ and take $\theta$ to be the trivial representation of $\Gamma$ on $E = \mathbb{C}$ — then $E^G_{\Gamma,\theta} = \mathbb{C}(G)$ and

$$\text{Ind}_{\Gamma,\theta}^G = L,$$

the left translation representation of $G$ (cf. §1, #12).

4: EXAMPLE Take $\Gamma = G$ and let $(\pi,V)$ be a unitary representation of $G$. Define a linear bijection

$$T: V^G_{G,\pi} \to V$$
by sending \( f \) to \( f(e) \) -- then \( \forall \, x \in G \),

\[
T(\text{Ind}_{G, \pi}^G(x)f) = (\text{Ind}_{G, \pi}^G(x)f)(e)
\]

\[
= f(x^{-1}e) = f(x^{-1}) = f(ex^{-1})
\]

\[
= \pi(x)f(e) = \pi(x)(Tf).
\]

Therefore

\[
T \circ \text{Ind}_{G, \pi}^G = \pi \circ T.
\]

I.e.:

\[
T \in I_G(\text{Ind}_{G, \pi}^G)
\]

is an invertible intertwining operator, thus \( \text{Ind}_{G, \pi}^G \) is equivalent to \( \pi \).

[Note: \( T \) is unitary. In fact,

\[
\langle f, g \rangle_{\pi} = \frac{1}{|G|} \sum_{x \in G} \langle f(x), g(x) \rangle_V
\]

\[
= \frac{1}{|G|} \sum_{x \in G} \langle f(ex), g(ex) \rangle_V
\]

\[
= \frac{1}{|G|} \sum_{x \in G} \langle \pi(x^{-1})f(e), \pi(x^{-1})g(e) \rangle_V
\]

\[
= \frac{1}{|G|} \sum_{x \in G} \langle f(e), g(e) \rangle_V
\]

\[
= \langle f(e), g(e) \rangle_V
\]

\[
= \langle T(f), T(g) \rangle_V.
\]

5: LEMMA The dimension of \( E^G_{\rho, \eta} \) equals

\[
\frac{|G|}{|\Gamma|} \dim E.
\]
3.

PROOF Write

\[ G = \prod_{k=1}^{n} x_k^r, \]

where \( n = \frac{|G|}{|\Gamma|} \), and define a bijection

\[ \Lambda: E^G_{\Gamma, \theta} \to \bigoplus_{k=1}^{n} E \]

by the stipulation that

\[ \Lambda f = (f(x_1), \ldots, f(x_n)), \]

from which the assertion.

For any character \( \chi \) of \( G \) and for any conjugacy class \( C \in \text{CON}(G) \), write \( \chi(C) \) for the common value of \( \chi(x) \) (\( x \in C \)) (and analogously if \( G \) is replaced by \( \Gamma \)).

Fixing \( C \), the intersection \( C \cap \Gamma \) is a union of elements of \( \text{CON}(\Gamma) \), say

\[ C \cap \Gamma = \bigcup_{\ell} C_{\ell}. \]

[Note: If \( C \cap \Gamma = \emptyset \), then the sum that follows is empty and its value is 0.]

6: THEOREM Set \( \pi = \text{Ind}_{\Gamma, \theta}^G \) -- then

\[ \chi_\pi(C) = \frac{|G|}{|\Gamma|} \sum_{\ell} \frac{|C_{\ell}|}{|C|} \chi_\theta(C_{\ell}). \]

PROOF If \( \chi_C \) is the characteristic function of \( C \) (cf. §4, #20), then \( \chi_C = \sum_{Y \subseteq C} \delta_Y \) (cf. §4, #21). Denoting by \( \rho \) the canonical extension of \( \pi \) to \( C(G) \), it thus follows that

\[ \chi_\pi(C) = \frac{1}{|C|} \text{tr}(\rho(\chi_C)). \]
Fix an orthonormal basis $\phi_1, \ldots, \phi_m$ in $E$ and in $E^G_{\Gamma, \theta}$, let

$$f_j(x) = \begin{cases} \left(\frac{|G|}{|\Gamma|}\right)^{1/2} \theta(\gamma^{-1}) \phi_j & (x = \gamma \in \Gamma) \\ 0 & (x \not\in \Gamma) \end{cases}$$

- The $f_j (1 \leq j \leq m)$ are an orthonormal set in $E^G_{\Gamma, \theta}$.
- The $\rho(x_k)f_j$ $(1 \leq k \leq n, 1 \leq j \leq m)$ are an orthonormal basis for $E^G_{\Gamma, \theta}$.

Proceeding

$$\text{tr}(\rho(\chi_C)) = \sum_{k=1}^{n} \sum_{j=1}^{m} \langle \rho(\chi_C) \rho(x_k)^{-1} \rho(x_k) f_j, f_j \rangle_{\theta}$$

$$= \sum_{k=1}^{n} \sum_{j=1}^{m} \langle \rho(x_k)^{-1} \rho(\chi_C) \rho(x_k) f_j, f_j \rangle_{\theta}$$

$$= \sum_{k=1}^{n} \sum_{j=1}^{m} \langle \rho(\chi_C) f_j, f_j \rangle_{\theta}$$

$$= n \sum_{j=1}^{m} \langle \rho(\chi_C) f_j, f_j \rangle_{\theta}$$

$$= [G:\Gamma] \sum_{j=1}^{m} \langle \rho(\chi_C) f_j, f_j \rangle_{\theta}$$

$$= \frac{|G|}{|\Gamma|} \sum_{j=1}^{m} \langle \rho(\chi_C) f_j, f_j \rangle_{\theta}$$

But

$$\langle \rho(\chi_C) f_j, f_j \rangle_{\theta} = \sum_{\gamma \in \mathcal{C}} \rho(\delta_{\gamma}) f_j, f_j \rangle_{\theta}$$
Therefore

\[ \text{tr}(\rho(\chi_G)) = \left| \frac{|G|}{|\Gamma|} \right| \sum_{j=1}^{m} \sum_{\gamma \in \Gamma} \langle \rho(\delta_{\gamma}) f_j, f_j \rangle_{\theta} \]

But \( \forall \gamma_0 \in \Gamma \),

\[ \langle \rho(\gamma_0) f_j, f_j \rangle_{\theta} = \frac{1}{|\Gamma|} \sum_{\gamma \in \Gamma} \langle \rho(\gamma) f_j, f_j \rangle_{\theta} \]

\[ = \frac{1}{|\Gamma|} \sum_{\gamma \in \Gamma} \left| \frac{|G|}{|\Gamma|} \right| \langle \theta(\gamma^{-1}) \gamma_0, \theta(\gamma^{-1}) \phi_j \rangle_E \]

\[ = \frac{1}{|\Gamma|} \sum_{\gamma \in \Gamma} \left| \frac{|G|}{|\Gamma|} \right| \langle \theta(\gamma^{-1}) \gamma_0, \theta(\gamma^{-1}) \phi_j \rangle_E \]

\[ = \frac{1}{|\Gamma|} \sum_{\gamma \in \Gamma} \left| \frac{|G|}{|\Gamma|} \right| \langle \theta(\gamma_0) \phi_j, \phi_j \rangle_E = \langle \theta(\gamma_0) \phi_j, \phi_j \rangle_E \]

Therefore

\[ \text{tr}(\rho(\chi_G)) \]

\[ = \left| \frac{|G|}{|\Gamma|} \right| \sum_{j=1}^{m} \sum_{\ell \in \Gamma} \langle \theta(\gamma) \phi_j, \phi_j \rangle_E \]

\[ = \left| \frac{|G|}{|\Gamma|} \right| \sum_{\ell \in \Gamma} \sum_{\gamma \in \Gamma} \sum_{j=1}^{m} \langle \theta(\gamma) \phi_j, \phi_j \rangle_E \]

\[ = \left| \frac{|G|}{|\Gamma|} \right| \sum_{\ell \in \Gamma} \text{tr}(\theta(\gamma)) \]
\[ \frac{1}{|\Gamma|} \sum_{\gamma \in C_\ell} \chi_\theta(\gamma) \]

\[ = \frac{1}{|\Gamma|} \sum_{\ell} |C_\ell| \chi_\theta(C_\ell) \]

I.e.:

\[ \chi_\pi(C) = \frac{1}{|\Gamma|} \sum_{\ell} |C_\ell| \chi_\theta(C_\ell) \]

[Note: If \( \theta \) is the trivial representation of \( \Gamma \) on \( E = C \), then \( \chi_\theta = 1_\Gamma \) (the function \( \equiv 1 \)) and matters reduce to

\[ \chi_\pi(C) = \frac{1}{|\Gamma|} \frac{|\text{cn} \Gamma|}{|C|} \]

7: N.B. Take \( C = \{e\} \):

\[ \chi_\pi(e) = \frac{1}{|\Gamma|} \chi_\theta(e) \]

\[ = \]

\[ \dim E^G_{\Gamma, \theta} = \frac{1}{|\Gamma|} \dim E \quad \text{(cf. \#5).} \]

8: LEMMA Set \( \pi = \text{Ind}^G_{\Gamma, \theta} \) -- then for any class function \( f \in \text{CL}(G) \),

\[ \langle \chi_\pi, f \rangle_G = \langle \chi_\theta, f \rangle_{\Gamma}. \]

PROOF

\[ \langle \chi_\pi, f \rangle_G = \frac{1}{|G|} \sum_{x \in G} \chi_\pi(x) \overline{f(x)} \]

\[ = \frac{1}{|G|} \sum_{C \in \text{CON}(G)} |C| \chi_\pi(C) \overline{f(C)} \]
7. 

\[ \text{\[G\]} \sum_{\mathcal{C} \in \text{CON}(G)} |\mathcal{C}| \prod_{\mathcal{C}} \prod_{\mathcal{C}} \chi_\theta(C_\mathcal{C}) \overline{f(C)} \]

= \frac{1}{|\mathcal{F}|} \sum_{\mathcal{C} \in \text{CON}(\mathcal{G})} \prod_{\mathcal{C}} \prod_{\mathcal{C}} \chi_\theta(C_\mathcal{C}) \overline{f(C)}

= \frac{1}{|\mathcal{F}|} \sum_{\mathcal{C} \in \text{CON}(\mathcal{F})} \prod_{\mathcal{C}} \prod_{\mathcal{C}} \chi_\theta(C_\mathcal{C}) \overline{f(C)}

= \frac{1}{|\mathcal{F}|} \sum_{\mathcal{C} \in \text{CON}(\mathcal{F})} \chi_\theta(\gamma) \overline{f(\gamma)} = \langle \chi_\theta, f \rangle_{\mathcal{F}}.

[Note: One cannot simply quote §7, #6... .]

9: APPLICATION Take \( f = \chi_\Pi \) (\( \Pi \in \hat{G} \)) and suppose that \( \theta \) is irreducible -- then the multiplicity of \( \Pi \) in \( \text{Ind}_{\Gamma, \theta}^G \) equals the multiplicity of \( \theta \) in the restriction of \( \Pi \) to \( \Gamma \) (cf. §5, #5).

10: THEOREM Set \( \pi = \text{Ind}_{\Gamma, \theta}^G \) -- then

\[ i_\Gamma \to G X_\theta = \chi_\pi. \]

PROOF The function

\[ i_\Gamma \to G X_\theta \]

is a class function on \( G \), as is \( \chi_\pi \), thus it suffices to show that \( \forall \Pi \in \hat{G}, \)

\[ \langle i_\Gamma \to G X_\theta, \chi_\Pi \rangle_G = \langle \chi_\pi, \chi_\Pi \rangle_G. \]

But

\[ \langle i_\Gamma \to G X_\theta, \chi_\Pi \rangle_G \]

= \frac{1}{|G|} \sum_{x \in G} (i_\Gamma \to G X_\theta)(x) \chi_\Pi(x)
\[
= \frac{1}{|G|} \frac{1}{|G|} \sum_{x \in G} \sum_{y \in G} x_\theta (yxy^{-1}) \overline{x_{\Pi}(x)} \\
= \frac{1}{|G|} \frac{1}{|G|} \sum_{x \in G} \sum_{y \in G} x_\theta (x) \overline{x_{\Pi}(y^{-1}x)} \\
= \frac{1}{|G|} \frac{1}{|G|} \sum_{x \in G} \sum_{y \in G} x_\theta (x) \overline{x_{\Pi}(x)} \\
= \frac{1}{|G|} \sum_{x \in G} x_\theta (x) \overline{x_{\Pi}(x)} \\
= \frac{1}{|G|} \sum_{y \in G} x_\theta (y) \overline{x_{\Pi}(y)} \\
= \langle x_\theta, x_{\Pi} | \Gamma \rangle = \langle x_{\pi_1}, x_{\pi_2} \rangle_G \quad \text{(cf. \#8)}.
\]

11: N.B. It is this result that provides the link with the machinery developed in §7 and §8.

Suppose that $\Gamma_1 \subset \Gamma_2 \subset G$ are subgroups. Let $(\theta_1, E_1)$ be a unitary representation of $\Gamma_1$ -- then one can form $\text{Ind}_{\Gamma_1, \theta_1}^G$. On the other hand, one can first form $\theta_2 = \text{Ind}_{\Gamma_1, \theta_1}^{\Gamma_2} \text{Ind}_{\Gamma_1, \theta_1}^{\Gamma_2, \theta_2}$.

12: INDUCTION IN STAGES

\[
\pi_1 = \text{Ind}_{\Gamma_1, \theta_1}^G \approx \text{Ind}_{\Gamma_2, \theta_2}^G \equiv \pi_2.
\]

[Apply §7, §12:

\[x_{\pi_1} = x_{\pi_2} \]
[Note: Characters determine representations up to equivalence (cf. §5, #10).]

13: **LEMMA** If \((\theta_1, E_1), (\theta_2, E_2)\) are unitary representations of \(\Gamma\), then

\[
\text{Ind}_{\Gamma, \theta_1}^{\Gamma} \oplus \theta_2 \cong \text{Ind}_{\Gamma, \theta_1}^{\Gamma} \oplus \text{Ind}_{\Gamma, \theta_2}^{\Gamma}.
\]

14: **N.B.** Consequently, \(\text{Ind}_{\Gamma, \theta}^{\Gamma}\) cannot be irreducible unless \(\theta\) itself is irreducible (cf. §10, #3).

Let \(G_1, G_2\) be finite groups, let \(\Gamma_1 \subset G_1\), \(\Gamma_2 \subset G_2\) be subgroups.

Put

\[G = G_1 \times G_2, \quad \Gamma = \Gamma_1 \times \Gamma_2 .\]

15: **LEMMA** If

\[
\begin{align*}
\chi_1 & \text{ is a character of } \Gamma_1, \\
\chi_2 & \text{ is a character of } \Gamma_2, 
\end{align*}
\]

then \(\chi_1 \chi_2\) is a character of \(\Gamma\) and

\[
i_{\Gamma} \rightarrow G \chi_1 \chi_2 = (i_{\Gamma_1} \rightarrow G_1 \chi_1)(i_{\Gamma_2} \rightarrow G_2 \chi_2).
\]
1.

§10. IRREDUCIBILITY OF $\text{Ind}_\Gamma^G$

Let $G$ be a finite group.

1: DEFINITION Let $(\pi_1, V_1)$, $(\pi_2, V_2)$ be unitary representations of $G$ -- then $\pi_1$ and $\pi_2$ are disjoint if they have no common nonzero unitarily equivalent subrepresentations.

2: LEMMA $\pi_1$ and $\pi_2$ are disjoint iff $\chi_{\pi_1}$ and $\chi_{\pi_2}$ are orthogonal:

$$<\chi_{\pi_1}, \chi_{\pi_2}>_G = 0.$$

3: THEOREM Let $\Gamma$ be a subgroup of $G$, $(\theta, E)$ an irreducible unitary representation of $\Gamma$ -- then $\text{Ind}_\Gamma^G$ is irreducible iff for every $x \in G-\Gamma$, the unitary representations

$$\gamma \rightarrow \theta(\gamma), \gamma \rightarrow \theta(x^{-1}\gamma x)$$

of the subgroup

$$\Gamma(x) = \Gamma^x \cap \Gamma \quad (\Gamma^x = x\Gamma x^{-1})$$

are disjoint.

PROOF Set $\pi = \text{Ind}_\Gamma^G$ -- then on general grounds, $\pi$ is irreducible iff

$$<\chi_\pi, \chi_\pi>_G = 1 \quad (\text{cf. } \S5, \#11).$$

I.e.: Iff

$$<i_\Gamma \rightarrow G\chi_\theta, i_\Gamma \rightarrow G\chi_\theta>_G = 1 \quad (\text{cf. } \S9, \#10)$$

or still, iff

$$<\chi_\theta, r_G \rightarrow \Gamma(i_\Gamma \rightarrow G\chi_\theta)>_\Gamma = 1 \quad (\text{cf. } \S7, \#6)$$
2.

or still, iff

\[ \langle \chi_{\theta}, \sum_{s \in S} i_{\Gamma}(s) \cdot \Gamma'(\chi_{\theta}) s' \rangle_{\Gamma} \]

\[ = \sum_{s \in S} \langle \chi_{\theta}, i_{\Gamma}(s) \cdot \Gamma'(\chi_{\theta}) s' \rangle_{\Gamma} = 1 \quad \text{(cf. §8, #3)}. \]

Here \( S = \Gamma \setminus G/\Gamma \) and it can be assumed that one element of the sum is \( s = e \) in which case \( (\chi_{\theta})_s = \chi_{\theta'}, \Gamma(s) = \Gamma \), hence

\[ \langle \chi_{\pi}', \chi_{\pi} \rangle_G \]

\[ = \langle \chi_{\theta}', \chi_{\theta} \rangle_{\Gamma} + \sum_{s \in \Gamma \setminus G/\Gamma \setminus \Gamma \setminus G/\Gamma} \langle \chi_{\theta}', i_{\Gamma}(s) \cdot \Gamma'(\chi_{\theta}) s' \rangle_{\Gamma} \]

\[ = \langle \chi_{\theta}', \chi_{\theta} \rangle_{\Gamma} + \sum_{s \in \Gamma \setminus G/\Gamma \setminus \Gamma \setminus G/\Gamma} \langle r_{\Gamma} \cdot \Gamma(s) \chi_{\theta}', (\chi_{\theta})_s \Gamma'(s) \rangle_{\Gamma(s)} \quad \text{(cf. §7, #6)} \]

\[ = 1 + \sum_{s \in \Gamma \setminus G/\Gamma \setminus \Gamma \setminus G/\Gamma} \langle r_{\Gamma} \cdot \Gamma(s) \chi_{\theta}', (\chi_{\theta})_s \Gamma'(s) \rangle_{\Gamma(s)} \quad \text{(cf. §5, #11)}. \]

Each term

\[ \langle r_{\Gamma} \cdot \Gamma(s) \chi_{\theta}', (\chi_{\theta})_s \Gamma'(s) \rangle_{\Gamma(s)} \]

is nonnegative and per \( \Gamma(s) \),

\[ r_{\Gamma} \cdot \Gamma(s) \chi_{\theta} \] is the character of \( \gamma + \theta(\gamma) \)

\[ (\chi_{\theta})_s \] is the character of \( \gamma + \theta(s^{-1} \gamma s) \).

If now \( \pi = \text{Ind}^G_{\Gamma, \theta} \) is irreducible, then \( \langle \chi_{\pi}', \chi_{\pi} \rangle_G = 1 \), thus \( \forall s \in \Gamma \setminus G/\Gamma \) (\( s \not\in \Gamma \)),

\[ r_{\Gamma} \cdot \Gamma(s) \chi_{\theta} \text{ and } (\chi_{\theta})_s \]
are orthogonal. Since $S$ can be chosen so that it contains any given element of $G - \Gamma$, the disjointness claim is manifest. Conversely, the orthogonality of

$$r_\Gamma \cdot \Gamma(s)x_\theta \text{ and } (x_\theta)_s$$

for all $s \in \Gamma \setminus \Gamma (s \notin \Gamma)$ forces $\langle x_\pi, x_\pi \rangle_G = 1$. 
§11. BURNSIDE RINGS

Let $G$ be a finite group.

1: **DEFINITION** Let $\chi_1, \ldots, \chi_t$ be the characters of the irreducible unitary representations of $G$ — then the **character ring** $X(G)$ is the free abelian group on generators $\chi_1, \ldots, \chi_t$ under pointwise addition and multiplication with unit $1_G$ (cf. §8, #5).

[Note: Recall that $t = |G| = |\text{CON}(G)| = \dim \text{CL}(G).$]

2: **N.B.** The pointwise sum or product of two characters is a character and the canonical arrow

$$X(G) \otimes \mathbb{Z} \to \text{CL}(G)$$

is an isomorphism.

3: **DEFINITION** An element of $X(G)$ is called a **virtual character**.

4: **LEMA** A class function $f \in \text{CL}(G)$ is a virtual character iff $<f, \chi_{\pi}>_G \in \mathbb{Z}$ for all $\pi \in \hat{G}$.

5: **REMARK** The values of a virtual character are algebraic integers (cf. §6, #5), hence $X(G)$ is a proper subring of $\text{CL}(G)$.

[Note: On the other hand, a class function whose values are algebraic integers need not be a virtual character.]

6: **NOTATION** Let $H$ be a collection of subgroups of $G$ with the property that

$$H \in H \& H' \subset H \Rightarrow H' \in H,$$
in which case \( H \) is termed a hereditary class.

Given \( H \), let \( X(G; H) \) be the additive subgroup of \( X(G) \) spanned by the

\[
i_H \to G^1 \quad (H \in H).
\]

**Lemma** \( X(G; H) \) is a subring of \( X(G) \).

**Proof** Let \( H_1, H_2 \in H \) -- then the claim is that

\[
(i_{H_1} \to G^1_{H_1})(i_{H_2} \to G^1_{H_2}) \in H.
\]

Put

\[
\chi = (i_{H_1} \to G^1_{H_1})
\]

and write

\[
\chi(i_{H_2} \to G^1_{H_2})
\]

\[
= i_{H_2} \to G((r_G \to H_2) \chi)_{H_2} \quad \text{(cf. §7, #8)}
\]

\[
= i_{H_2} \to G((r_G \to H_2) \chi).
\]

Then, thanks to §8, #3, there are subgroups \( K_1, \ldots, K_r \) of \( H_2 \) such that

\[
r_G \to H_2 \chi = \sum_{\ell=1}^r i_{K_\ell} \to H_2 \ 1_{K_\ell}.
\]

Therefore

\[
i_{H_2} \to G((r_G \to H_2) \chi)
\]

\[
= i_{H_2} \to G(\sum_{\ell=1}^r i_{K_\ell} \to H_2 \ 1_{K_\ell})
\]
3. 

\[ r \sum_{\ell=1}^{r} i_{H_2} + G(i_{K_\ell} \rightarrow H_2 ^{1_{K_\ell}}) \]

\[ = \sum_{\ell=1}^{r} i_{K_\ell} + G^{1_{K_\ell}} \quad (\text{cf. §7, #12}) \]

\[ \in X(G;H). \]

8: **DEFINITION** \( X(G;H) \) is the **Burnside ring** of \( G \) associated with the hereditary class \( H \).

[Note: It is not a priori evident that \( 1_G \in X(G;H) \).]

9: **CRITERION** Let \( R \) be a ring of \( \mathbb{Z} \)-valued functions on a finite set \( X \) under pointwise operations. Suppose that for each \( x \in X \) and each prime \( p \) there exists \( f \in R \) such that \( f(x) \neq 0 \mod p \) -- then \( 1_X \in R \).

[Attach to each \( x \in X \) the ideal

\[ I_x = \{ f(x) : f \in R \} \subset \mathbb{Z}. \]

Then, in view of the assumption, \( I_x = \mathbb{Z} \) so there exists \( f_x \in R \) such that \( f_x(x) = 1 \), hence

\[ \prod_{x \in X} (1 - f_x) = 0. \]

Now expand the product to get 1 as a sum of elements of \( R \).]

Let \( G \) be a finite group.

10: **DEFINITION** Let \( p \) be a prime -- then \( G \) is a \( p \)-**group** if every element \( x \in G \) has order a power of \( p \).

[Note: Every \( p \)-group is nilpotent.]
11: LEMMA \( G \) is a \( p \)-group iff \( |G| \) is a power of \( p \).

12: DEFINITION Let \( p \) be a prime -- then a subgroup \( P \) of \( G \) is a Sylow \( p \)-subgroup of \( G \) if it is a maximal \( p \)-subgroup of \( G \).

13: THEOREM

- Sylow \( p \)-subgroups exist.
- All Sylow \( p \)-subgroups are conjugate.
- Every \( p \)-subgroup is contained in a Sylow \( p \)-subgroup.

14: N.B. The number of Sylow \( p \)-subgroups of \( G \) is a divisor of \( |G| \).

15: DEFINITION Given a prime \( p \), a finite group \( H \) is \( p \)-elementary if it is the direct product of a cyclic group \( C \) of order prime to \( p \) and a \( p \)-group \( P \).

[Note: Accordingly, \( C \) and \( P \) are normal subgroups, \( C \cap P = \{e\} \), and \( H = CP \).]

16: LEMMA Subgroups of \( p \)-elementary groups are again \( p \)-elementary, hence the \( p \)-elementary subgroups of \( G \) constitute a hereditary class \( E_p(G) \).

17: DEFINITION A finite group \( H \) is elementary if it is \( p \)-elementary for some prime \( p \).

18: NOTATION Put

\[
E(G) = \bigcup_{p \in \mathbb{P}} E_p(G).
\]

19: N.B. Since \( E(G) \) is a hereditary class, one can form its Burnside ring \( X(G; E(G)) \).
5.

20: **DEFINITION** Given a prime \( p \), a group \( H \) is \( p \)-semielementary if it is the semidirect product of a cyclic subgroup \( C \) of order prime to \( p \) and a \( p \)-group \( P \).

[Note: Accordingly, \( C \) is a normal subgroup, \( C \cap P = \{e\} \), and \( H = CP \).]

21: **LEMMA** Subgroups of \( p \)-semielementary groups are again \( p \)-semielementary, hence the \( p \)-elementary subgroups of \( G \) constitute a hereditary class \( SE_p(G) \).

22: **DEFINITION** A finite group \( H \) is **semielementary** if it is \( p \)-semielementary for some prime \( p \).

23: **NOTATION** Put

\[
SE(G) = \bigcup_p SE_p(G).
\]

24: **N.B.** Since \( SE(G) \) is a hereditary class, one can form its Burnside ring \( X(G; SE(G)) \).

25: **LEMMA**

\[
l_G \in X(G; SE(G)),
\]

i.e., there exist integers \( a_H(H \in SE(G)) \) such that

\[
l_G = \sum_{H \in SE(G)} a_H (i_H \rightarrow G^H).
\]

**PROOF** It suffices to show that the ring \( X(G; SE(G)) \) satisfies the assumptions of \#9: For every \( x \in G \) and for every prime \( p \), there exists a group \( H_{x,p} \in SE(G) \) such that

\[
(i_H \rightarrow G^H)(x) \not\equiv 0 \mod p.
\]

This said, factor the order of \( x \) as \( p^a n \) (\( p \nmid n \)) and let \( C = \langle x^p^a \rangle \) (hence \( |C| = n \),...
hence is prime to $p$). Let $N$ be the normalizer of $C$ in $G$, let $P$ be a Sylow $p$-subgroup of $N$ containing $x$, and let $H_{x,p} \subseteq H = CP$ — then $H$ is $p$-semielementary and the claim is that

$$(i_{H} + C_{H}(x)) \neq 0 \mod p.$$ 

By definition,

$$(i_{H} + C_{H}(x)) = \frac{1}{|H|} \sum_{y \in G} yC_{H}(x) y^{-1}.$$ 

But

$$yC_{H}(x) y^{-1} \in H \Rightarrow yC_{H}(x) y^{-1} \subseteq H$$

$$\Rightarrow yC_{H}(x) y^{-1} = C \Rightarrow y \in N.$$ 

Therefore

$$(i_{H} + C_{H}(x)) = (i_{N} + C_{N}(x)).$$

the term on the right being the number of left cosets of $H$ in $N$ fixed by $x$ (cf. §7, §11). Since $C$ is a normal subgroup of $N$ and since $C \subseteq H$, it follows that $C$ must fix the left cosets of $H$ in $N$. Thus the $x$-orbits have cardinality dividing $p^a$, thus each nontrivial $x$-orbit has cardinality divisible by $p$. On the other hand, the number of left cosets of $H$ in $N$ is prime to $p$ ($H$ contains a Sylow $p$-subgroup of $N$). Combining these facts then leads to the conclusion that the number of left cosets of $H$ in $N$ fixed by $x$ is prime to $p$, i.e.,

$$(i_{H} + C_{H}(x)) \neq 0 \mod p.$$ 

26: DEFINITION A monomial character of a finite group is a character of degree 1.
27: DEFINITION A finite group $H$ is said to be an $M$-group if each irreducible character of $H$ is induced by a monomial character of a subgroup of $H$.

28: THEOREM Suppose that $H$ is a finite group which is a semidirect product of an abelian normal subgroup and a nilpotent group (in particular, a $p$-group) -- then $H$ is an $M$-group.

29: APPLICATION $p$-elementary groups and $p$-semielementary groups are $M$-groups.
Let $G$ be a finite group.

1. CHARACTERIZATION OF CHARACTERS A class function $f \in \text{CL}(G)$ is a virtual character (i.e., belongs to $X(G)$) iff for every $H \in \text{E}(G)$,

$$r_G \to_H f \in X(H).$$

2. INDUCTION PRINCIPLE A class function $f \in \text{CL}(G)$ is a virtual character (i.e., belongs to $X(G)$) iff there exist elementary subgroups $H_i$, monomial characters $\lambda_i$ of $H_i$, and integers $a_i$ ($1 \leq i \leq n$) such that

$$f = \sum_{i=1}^{n} a_i (i_{H_i} \to G \lambda_i).$$

These are the main results. Turning to their proofs, let $R$ be the ring with unit $1_G$ whose elements are the class functions $f$ on $G$ such that

$$r_G \to_H f \in X(H)$$
for all $H \in \text{E}(G)$ and let $L$ be the subgroup of $X(G)$ spanned over $\mathbb{Z}$ by characters of the form $i_{H} \to G \lambda$, where $\lambda$ is a monomial character of some $H \in \text{E}(G)$.

3. LEMMA Statements 1 and 2 are equivalent to $L = R$.

[Note: Obviously,

$$L \subseteq X(G) \subseteq R.$$
]

4. LEMMA $L$ is an ideal in $R$.

PROOF Let $\Lambda \in L$, say

$$\Lambda = \sum_{i} a_i (i_{H_i} \to G \lambda_i).$$
and let \( \psi \in R \) -- then

\[
\psi \lambda = \sum_i a_i \psi(i_{H_i} \to G^\lambda_i)
\]

\[
= \sum_i a_i (i_{H_i} \to G((r_G \to H_i) \psi\lambda_i)) \quad \text{(cf. §7, #8)}.
\]

Since

\[
r_G \to H_i \psi \in X(H_i),
\]

there exist integers \( b_{ij} \) such that

\[
r_G \to H_i \psi = \sum_j b_{ij} \xi_{ij},
\]

\( \xi_{ij} \) running through the irreducible characters of \( H_i \), hence

\[
\psi \lambda = \sum_{i,j} a_i b_{ij} (i_{H_i} \to G^\xi_{ij}_i).
\]

But elementary groups are M-groups (cf. §11, #29), so \( \xi_{ij} \) is induced by a monomial character of some subgroup of \( H_i \). Taking into account that \( E(G) \) is a hereditary class, apply §7, #12 to conclude that \( \psi \lambda \in L \). Therefore \( L \) is an ideal in \( R \).

[Note: Operations in \( R \) are pointwise and, of course, \( R \) is commutative.]

Matters thus reduce to showing that \( l_G \in L \). To this end, suppose that it were possible to write

\[
l_G = \sum_k c_k (i_{H_k} \to G \chi_k),
\]

where \( c_k \in Z \) and \( \chi_k \) is a character of some proper subgroup \( H_k \) of \( G \). Inductively, it can be assumed that #2 holds for \( H_k \), hence that \( \chi_k \) can be written as a \( Z \)-linear combination of induced monomial characters from elements of \( E(H_k) \). But then \( l_G \in L \), as desired.
3.

[Note: Nothing need be done if G is elementary to begin with (it being automatic that \( l_G \in L \)).]

5: LEMMA If G is not elementary, then \( l_G \) can be written as a \( \mathbb{Z} \)-linear combination of induced characters from proper subgroups of G.

Case 1: G is not selementary, thus \( G \not\in SE(G) \) and the \( H \in SE(G) \) are proper subgroups. The contention then follows from \( \S 11, \# 25 \).

Case 2: G is selementary: \( G \in SE(G) \), say \( G = CP \) for some prime \( p \). Let \( N \) be the normalizer of \( P \) in \( G \), hence \( N = (C \cap N) \times P \) is \( p \)-elementary and it can be assumed that \( N \neq G \) (otherwise G is elementary and there is nothing to prove). Write

\[
i_N \rightarrow l_N = a_0 l_G + \sum_{i>0} a_i \chi_i,
\]

where the \( \chi_i \neq l_G \) are irreducible characters and the \( a_i \) are positive integers.

6: N.B.

\[
a_0 = \langle i_N \rightarrow l_N, l_G \rangle_G = \langle l_N, l_G \rangle_N \times P \quad (\text{cf. } \S 7, \# 6)
\]

\[
= \langle l_N, l_N \rangle_N = 1.
\]

7: N.B. \( \chi_i(e) > 1 \) for all \( i > 0 \).

[Suppose that \( \chi_i(e) = 1 \) (\( \exists i \)). Write

\[
\text{res}_{G \rightarrow N} \chi_i = c_i l_N + \chi
\]

for some character \( \chi \) orthogonal to \( l_N \) -- then]
4.

\[ c = <l_N, \text{res}_{G \to N^\chi_i}>_N \]

\[ = <i_N \to G^N, ^1_N, \chi_i>^G \] (cf. §7, #6)

\[ = a_i. \]

And

\[ l = \chi_i(e) = a_i + \chi(e) \]

\[ \Rightarrow a_i = l = \text{res}_{G \to N^\chi_i} = l_N. \]

Recall now that the kernel \( K_i \) of \( \chi_i \) is the proper normal subgroup of \( G \) consisting of those \( x \in G \) such that \( \chi_i(x) = \chi_i(e) \) or still, consisting of those \( x \in G \) such that \( \chi_i(x) = 1 \), thus \( N \subset K_i \) (since \( \text{res}_{G \to N^\chi_i} = l_N \)). But this is impossible: \( P \) is a Sylow \( p \)-subgroup of \( K_i \), so \( G = K_i \) (cf. infra).

[Note: Let \( x \in G \) -- then both \( P \) and \( xPx^{-1} \) are Sylow \( p \)-subgroups of \( K_i \), hence \( \text{for some } k \in K_i \) which implies that \( kx \in N \subset K_i \), thereby forcing \( x \in K_i \), so \( G = K_i \).]}

Return to the formula

\[ i_N \to G^N = l_G + \sum_{i>0} a_i \chi_i. \]

Since \( \chi_i(e) > 1 \) for all \( i > 0 \), the \( \chi_i \) are not monomial. On the other hand, \( G = CP \) is semielementary, thus is an \( M \)-group, thus each \( \chi_i \) is induced by a monomial character \( \lambda_i \) of some proper subgroup \( H_i \) of \( G \). Therefore

\[ l_G = i_N \to G^N - \sum_{i>0} a_i (i_{H_i} \to G^\lambda_i), \]

which completes the proof of #5.
§13. GROUPS OF LIE TYPE

Let $k$ be a finite field.

1: **DEFINITION** A $k$-group is a linear algebraic group defined over $k$.

[Note: A $k$-subgroup of a $k$-group is a subgroup which is a $k$-group.]

2: **NOTATION** Given $k$-groups $A$, $B$, $C$, ..., denote their group of $k$-rational points $A(k)$, $B(k)$, $C(k)$, ..., by $A$, $B$, $C$, ....

Let $G$ be a connected reductive $k$-group.

3: **DEFINITION** $G$ is said to be a group of Lie type.

4: **N.B.** $G$ is, of course, finite and it is possible to compute $|G|$ explicitly.

5: **DEFINITION** A maximal closed connected solvable subgroup of $G$ is called a Borel subgroup.

[Note: The conditions "closed" and "connected" can be omitted from the definition.]

6: **LEMMA**

- Any two Borel subgroups of $G$ are conjugate.
- Every element of $G$ belongs to some Borel subgroup of $G$.
- Every closed subgroup of $G$ containing a Borel subgroup is equal to its own normalizer and is connected.
- Any two closed subgroups of $G$ containing the same Borel subgroup and conjugate in $G$ are equal.
7: N.B. Since $k$ is finite, $G$ is quasi-split, hence contains a Borel subgroup defined over $k$.

[Note: Any two such are $G$-conjugate.]

Let $B$ be a Borel $k$-subgroup of $G$, let $T \subseteq B$ be a maximal torus of $G$ defined over $k$, and put

$$N = N_{G/B}.$$ 

8: LEMMA $N$ is a $k$-subgroup of $G$.

9: NOTATION Set

$$W = N/T.$$ 

10: LEMMA

$$W \cong N/T.$$ 

[Note: $B \cap N = T \Rightarrow B \cap N = T$.]

11: LEMMA $W$ is a finite Coxeter group.

[Note: Spelled out, $W$ admits a finite system of generators $w_1, \ldots, w_\ell$ ($w_i \neq 1$ and $w_i \neq w_j$ for $i \neq j$) subject to the relations

$$w_i^2 = 1, \quad (w_i w_j)^{m_{ij}} = 1 \quad (i \neq j),$$

where $m_{ij}$ is the order of $w_i w_j$ ($i \neq j$).]

12: BRUHAT LEMMA

$$G = \bigsqcup_{w \in W} BwB.$$
13: **DEFINITION** A closed subgroup $P$ of $G$ is parabolic if it contains a Borel subgroup of $G$.

14: **LEMMA** Let $P_1, P_2$ be parabolic $k$-subgroups of $G$ -- then $P_1 = P_2$ iff $P_1 = P_2$.

15: **NOTATION** Given a parabolic $k$-subgroup of $G$, denote its unipotent radical by $U$.

[Note: Recall that $P$ is the normalizer of $U$.]

17: **DEFINITION** Let $P$ be a parabolic $k$-subgroup of $G$ -- then a closed connected reductive $k$-subgroup $L$ of $P$ is a **Levi subgroup** of $P$ if $P$ is the semidirect product $LU$ (hence $P = LU$).

18: **LEMMA** Levi subgroups of $P$ exist and any two such are conjugate by a unique element of $U$.

19: **N.B.** $L$ is a group of Lie type.

20: **LEMMA** Let $P_1, P_2$ be parabolic $k$-subgroups of $G$ -- then the following conditions are equivalent.

- $P_1 \cap U_2 \subset U_1$, $P_2 \cap U_1 \subset U_2$
- $P_1$ and $P_2$ have a common Levi subgroup.

21: **APPLICATION**

$$U_1 = U_2 \Rightarrow P_1 = P_2.$$
[For under these circumstances, \(P_1\) and \(P_2\) have a common Levi subgroup \(U_1\), thus
\[
P_1 = LU_1 = LU_2 = P_2,
\]
so one can quote #14.]

22: **DEFINITION** Let \(P_1, P_2\) be parabolic \(k\)-subgroups of \(G\) -- then \(P_1\) and \(P_2\) are said to be **associate** if there exists an \(x \in G\) such that \(P_1\) and \(xP_2x^{-1}\) have a common Levi subgroup.

23: **N.B.** The relation determined by "to be associate" is an equivalence relation on the set of parabolic \(k\)-subgroups of \(G\).

24: **LEMMA** If \(P_1, P_2\) are not associate, then \(\forall x \in G, P_1, xP_2x^{-1}\) are not associate.

[If there exists \(x \in G\) such that \(P_1\) and \(xP_2x^{-1}\) are associate, then there exists \(y \in G\) such that \(P_1\) and \(yP_2x^{-1}y^{-1}\) have a common Levi subgroup, thus \(P_1\) and \(P_2\) are associate, contradiction.]

25: **LEMMA** Let \(P_1, P_2\) be parabolic \(k\)-subgroups of \(G\). Assume: \(P_1\) and \(P_2\) are associate -- then \(|P_1| = |P_2|\).

[There is no loss of generality in supposing that \(P_1\) and \(P_2\) have a common Levi subgroup \(L\), thereby reducing matters to the claim that \(|U_1| = |U_2|\).]

26: **DESCENT** Fix a parabolic \(k\)-subgroup \(P \subset G\) and let \(L \subset P\) be a Levi subgroup -- then there is a 1-to-1 correspondence between the set of parabolic \(k\)-subgroups of \(G\) contained in \(P\) and the set of parabolic \(k\)-subgroups of \(L\).
5.

- Given a parabolic $k$-subgroup $E' \subset P$, write $E' = U'y'$ and put $*E = E' \cap U$ -- then $*E$ is a parabolic $k$-subgroup of $U$ with unipotent radical $*U = U' \cap U$.

- Given a parabolic $k$-subgroup $*E$ of $U$, write $*E = *Uy$ and put $L' = *L$, $y' = *uu$ -- then $E' = L'y'$ is a parabolic $k$-subgroup of $G$ such that $E' \subset P$.

The bijection in question is the assignment $E' \rightarrow *E$.

27: N.B. $E'$ and $E''$ are conjugate by an element of $G$ iff $E' \cap L$ and $E'' \cap L$ are conjugate by an element of $L$.

APPENDIX

LEMMA Suppose that $P_1 = L_1u_1$ and $P_2 = L_2u_2$ are associate -- then $L_1$ and $L_2$ are conjugate by an element of $G$.

(Choose $x \in G$ such that $P_1$ and $xP_2x^{-1}$ have a common Levi subgroup $L$. Choose $u_1 \in U_1$:

$$u_1^{-1}u_1^{-1} = u_1.$$  

Choose $xu_2x^{-1} \in xU_2x^{-1}$:

$$xu_2x^{-1}L_1xu_2x^{-1} = xu_2x^{-1}L_1xu_2x^{-1}$$

Then

$$u_2x^{-1}L_1xu_2x^{-1} = u_2$$

$$L = xu_2L_2u_2x^{-1}$$

$\Rightarrow$
6.

\[ L_1 = u_1 L_{\bar{u}_1} \]

\[ = u_1 x u_2^{-1} L_{\bar{u}_2} u_2 x^{-1} u_1^{-1}. \]
§14. HARISH-CHANDRA THEORY

Let $k$ be a finite field, $G$ a connected reductive $k$-group.

1: DEFINITION Let $P$ be a parabolic $k$-subgroup of $G$, then $P$ is termed a cuspidal subgroup of $G$.

2: NOTATION Given a cuspidal subgroup $P = LU$ of $G$ and an $f \in C(G)$, let

$$f_p(x) = \sum_{u \in U} f(xu) \quad (x \in G).$$

[Note: If $P = G$, then $f_G(x) = f(x) \quad (x \in G).$]

3: DEFINITION Let $f \in C(G)$ -- then $f$ is said to be a cusp form if $f_p = 0$ for all $P \neq G$.

4: NOTATION Write $^0C(G)$ for the set of cusp forms and put

$$^0CL(G) = CL(G) \cap ^0C(G).$$

5: LEMMA $^0C(G)$ is a linear subspace of $C(G)$.

6: LEMMA $^0C(G)$ is stable under left translations, hence is a left ideal in $C(G)$.

7: REMARK If $G$ is a torus, then $^0C(G) = C(G)$.

8: NOTATION Given $f \in C(G)$, write $f_p \sim 0$ if

$$\sum_{\ell \in L} f_p(x\ell) \overline{\phi(\ell)} = 0$$

for all $\phi \in ^0C(L)$ and all $x \in G$. 
2.

[Note: Bear in mind that $L$ is a group of Lie type (cf. §13, #19).]

9: N.B. Matters are independent of the choice of $L$ in $P$.

10: LANGLANDS PRINCIPLE If $f_P \sim 0$ for all cuspidal subgroups $P$ of $G$ (including $P = G$), then $f = 0$.

PROOF Proceed by induction on the semisimple $k$-rank $s$ of $G$, the case $s = 0$ being trivial (because then $G$ is anisotropic, there is only one $P$, viz. $P = G$, and $L = G$, $^0C(L) = C(G)$...). So assume that $s$ is positive and let $P = LU$ be for the moment a proper cuspidal subgroup, thus $U \neq \{e\}$ and the semisimple $k$-rank of $L$ is strictly smaller than that of $G$. Using now §13, #26, let $*P = *L*U$ be a cuspidal subgroup of $L$ — then $P' = L'U' = *LU'$ is a cuspidal subgroup of $G$ contained in $P$. Freeze $x \in G$ and put $g(l) = f_P(xl) (l \in L)$:

$$g_{*P}(l) = \sum_{u \in *U} g(l*u)$$
$$= \sum_{u \in *U} f_P(xl*u)$$
$$= \sum_{u \in *U} \sum_{u' \in U} f(xluu')$$
$$= \sum_{u' \in U'} f(xlu')$$
$$= f_{P'}(xl).$$

But by assumption,

$$\sum_{*l \in *L} f_{P'}(xl*l) \phi(*l) = 0$$
for all $\phi \in ^0C(*L)$ or still,

$$\sum_{*L \in *L} g_{*P}(l*L) \overline{\phi(l*L)} = 0$$

for all $\phi \in ^0C(*L)$. The induction hypothesis then implies that $g = 0$, hence

$$f_p(x) = g(e) = 0.$$ 

Therefore $f$ is a cusp form ($x \in G$ being arbitrary), i.e., $f \in ^0C(G)$. Finally,

$$f_G \sim 0 \Rightarrow \sum_{y \in G} f(xy) \overline{\phi(y)} = 0$$

for all $\phi \in ^0C(G)$ and all $x \in G$. Take $x = e$ to conclude that

$$\sum_{y \in G} f(y) \overline{\phi(y)} = 0$$

for all $\phi \in ^0C(G)$ and then take $\phi = f$ to conclude that

$$<f,f>_G \sim 0 \Rightarrow f = 0.$$

11: NOTATION Given a cuspidal subgroup $P = LU$ of $G$, let $C(G;P)$ be the subspace of $C(G)$ consisting of those $f$ such that

(i) $f(xu) = f(x)$ ($x \in G$, $u \in U$)

and

(ii) $l \rightarrow f(xl) \in ^0C(L)$ ($x \in G$, $l \in L$).

[Note: $C(G;P)$ is stable under left translations, hence is a left ideal in $C(G)$.]

12: EXAMPLE

$$C(G;G) = ^0C(G).$$

13: SUBLEMMA Fix $P$ -- then $\forall f \in C(G;P)$ and $\forall g \in C(G)$,
4.

\[ <f, g_p>_G = |U| <f, g>_p. \]

**PROOF**

\[ <f, g_p>_G = \frac{1}{|G|} \sum_{x \in G} f(x) \overline{g_p(x)} \]

\[ = \frac{1}{|G|} \sum_{x \in G} \sum_{u \in U} f(x) \overline{g(xu)} \]

\[ = \frac{1}{|G|} \sum_{u \in U} \sum_{x \in G} f(x) \overline{g(xu)} \]

\[ = \frac{1}{|G|} \sum_{u \in U} \sum_{x \in G} f(xu) \overline{g(xu)} \]

\[ = \sum_{u \in U} \frac{1}{|G|} \sum_{x \in G} f(x) \overline{g(x)} \]

\[ = \sum_{u \in U} <f, g>_G = |U| <f, g>_G. \]

14: **RAPPEL** Let \( H \) be a finite dimensional complex Hilbert space -- then a subset \( M \subset H \) is total if \( M_{\text{fin}} = H \), this being the case iff \( M^\perp = \{0\} \).

[Note: Subspaces of \( H \) are necessarily closed... .]

Put

\[ M = \bigcup_{P} C(G; P). \]

15: **LEMMA** \( C(G) \) is spanned by the \( f \in M \).

**PROOF** It suffices to show that if for some \( g \in C(G) \), we have \( <f, g>_G = 0 \)
for all \( f \in C(G;P) \) and for all cuspidal \( P \), then \( g = 0 \). And to this end, it need only be established that \( g_p \sim 0 \) for all cuspidal \( P \) (cf. #10). So fix \( x \in G \) and let \( \phi \in \mathscr{O}C(L) \). Define \( f \in C(G) \) as follows:

\[
\begin{align*}
f(y) &= 0 \text{ if } y \notin xP \\
f(xlu) &= \phi(\ell) \quad (\ell \in L, \ u \in U).
\end{align*}
\]

Then \( f \in C(G;P) \), so

\[
0 = \langle f,g \rangle_G = \frac{1}{|U|} \langle f,g_p \rangle_G
\]

\[
\implies 0 = \frac{1}{|U|} \langle g_p,f \rangle_G
\]

\[
= \frac{1}{|U|} \sum_{y \in G} g_p(y) \overline{f(y)}
\]

\[
= \frac{1}{|U|} \sum_{y \in xP} g_p(y) \overline{f(y)}
\]

\[
= \frac{1}{|U|} \sum_{\ell,u} g_p(xlu) \overline{f(xlu)}
\]

\[
= \frac{1}{|U|} \sum_{\ell,u} g_p(x\ell) \overline{\phi(\ell)}
\]

\[
= \sum_{u} \sum_{\ell \in L} g_p(x\ell) \overline{\phi(\ell)}
\]

\[
= \sum_{\ell \in L} g_p(x\ell) \overline{\phi(\ell)}.
\]

Therefore \( g_p \sim 0 \).
6.

16: CONVENTION Cuspidal subgroups \(P_1, P_2\) are said to be associate if this is the case of \(P_1 \equiv P_2\).

17: LEMMA If \(P_1, P_2\) are associate, then

\[ C(G; P_1) = C(G; P_2). \]

18: LEMMA If \(P_1, P_2\) are not associate, then \(C(G; P_1), C(G; P_2)\) are orthogonal.

Let \(P_1, \ldots, P_r\) be a set of representatives for the association classes of cuspidal subgroups of \(G\).

19: THEOREM There is an orthogonal decomposition

\[ C(G) = \bigoplus_{i=1}^{r} C(G; P_i). \]

20: N.B. \#17, \#18 can be established without the use of representation theory but its introduction leads to another approach.

21: LEMMA Let \(\pi \in \hat{G}\) -- then \(\chi_{\pi}\) is a cusp form iff \(\forall\) cuspidal \(P \neq G\),

\[ m(\pi, \text{Ind}_U^G \theta) = 0, \]

where \(\theta\) is the trivial representation of \(U\) on \(E = \mathbb{C}\). I.e.: Iff

\[ \langle \chi_{\pi}, \text{i}_U G \rangle = \langle \chi_G \rangle = \langle u \chi_{\pi}, \text{i}_U G \rangle \]

\[ = \langle \text{i}_U \chi_G, u \rangle = m(\theta, \pi | U) \]

\[ = 0. \]
22: LEMMA Let \( \Pi \in \hat{G} \) -- then \( \chi_\Pi \) is a cusp form iff \( \forall \) cuspidal \( P \neq G \),
\[
\sum_{u \in U} \Pi(u) = 0.
\]

23: N.B. Let \( V(\Pi)_U = \{ v \in V(\Pi) : \sum_{u \in U} \Pi(u)v = 0 \} \).

Then \( \chi_\Pi \) is a cusp form iff \( \forall \) cuspidal \( P \neq G \),
\[
V(\Pi) = V(\Pi)_U.
\]

24: DEFINITION Let \( \Pi \in \hat{G} \) -- then \( \Pi \) is said to be in the discrete series
if its character \( \chi_\Pi \) is a cusp form.

25: NOTATION \( ^0\hat{G} \) is the subset of \( \hat{G} \) consisting of those \( \Pi \) in the discrete series.

Given \( P = LU \) and \( \Theta \in ^0\hat{L} \), one can lift \( \Theta \) to \( P \) and form \( \text{Ind}^G_P\Theta \) with character
\[
i_P + G\chi_\Theta \quad (\text{cf. } \S 9, \#10).
\]

26: THEOREM Let \( \Pi \in \hat{G} - ^0\hat{G} \) -- then there exists a proper cuspidal \( P = LU \)
and a \( \Theta \in ^0\hat{L} \) such that \( \Pi \) occurs as a subrepresentation of \( \text{Ind}^G_P\Theta \):
\[
\langle \chi_\Pi, \chi_\pi \rangle_G \neq 0 \quad (\pi = \text{Ind}^G_P\Theta) \quad (\text{cf. } \S 5, \#5).
\]

PROOF Proceed by induction on the semisimple k-rank \( s \) of \( G \), there being nothing
to prove if \( s = 0 \), so assume that \( s > 0 \) -- then there exists a proper cuspidal
\( P = LU \) such that \( V(\Pi) \neq V(\Pi)_U \). Claim: \( V(\Pi)_U \) is \( P \)-invariant: \( \forall \ell_0 \in L, \forall u_0 \in U, \)
\[ \forall v \in V(\Pi): \Sigma \Pi(u)v = 0, \quad u \in U \]

\[
\begin{align*}
\Sigma \Pi(u)\Pi(l_0u_0)v \\
\quad u \in U \\
&= \sum \Pi(u)\Pi(l_0u_0)v \\
&= \sum \Pi(l_0\Pi^{-1}(l_0u_0)u_0)v \\
&= \Pi(l_0)\left( \sum \Pi(l_0^{-1}u_0)\Pi(u_0)v \right) \\
&= \Pi(l_0)\left( \sum \Pi(u)\Pi(u_0)v \right) \\
&= \Pi(l_0)\sum \Pi(u_0)v \\
&= \Pi(l_0)\sum \Pi(u)v \\
&= \Pi(l_0)0 = 0.
\end{align*}
\]

Consequently, \( P \) operates on the quotient \( V(\Pi)/V(\Pi)_U \). Moreover, its restriction to \( U \) is trivial: \( \forall u_0 \in U, \forall v \in V(\Pi), \)

\[
\begin{align*}
\sum \Pi(u)(\Pi(u_0)v - v) \\
\quad u \in U \\
&= \sum \Pi(u_0)v - \sum \Pi(u)v \\
&= \sum \Pi(u)v - \sum \Pi(u)v \\
&= 0
\end{align*}
\]
On the other hand, while its restriction to $L$ need not be irreducible, there is in any event an $L$-invariant subspace $V$ of $V(\Pi)$ containing $V(\Pi)_U$ such that the quotient representation $\Theta$ of $L$ on 

$V(\Pi)/V(\Pi)_U/V/V(\Pi)_U$

is irreducible. Pass now to $\text{Ind}^G_{P,\Theta}$ and note that $\Pi$ occurs as a subrepresentation of $\text{Ind}^G_{P,\Theta}$ (see below). Accordingly, if $\Theta \in \hat{\hat{0}}L$, then we are done. If, however, $\Theta \not\in \hat{\hat{0}}L$, then, thanks to the induction hypothesis, there exists a proper cuspidal subgroup $*P = *L*U$ of $L$ and a discrete series representation $\Theta$ of $*L$ such that $\Theta$ occurs as a subrepresentation of $\text{Ind}^L_{*P,*\Theta}$. Form $P' = L'U' = *L*UU$, view $\Theta$ as a representation of $P'$ trivial on $U' = *UU$, and utilize the induction in stages rule (cf. §9, §12)

$$\text{Ind}^G_{P',\Theta} \simeq \text{Ind}^G_{P,\Theta}, \text{Ind}^L_{*P,*\Theta}$$

to conclude that $\Pi$, which occurs as a subrepresentation of $\text{Ind}^G_{P,\Theta}$, must actually occur as a subrepresentation of $\text{Ind}^G_{P',\Theta}$:

$$\Theta \subset \text{Ind}^L_{*P,*\Theta} \Rightarrow \Pi \subset \text{Ind}^G_{P,\Theta} \subset \text{Ind}^G_{P,\text{Ind}^L_{*P,*\Theta}}.$$

[Note: To confirm that $I_G(\Pi,\text{Ind}^G_{P,\Theta}) \neq 0$,]
10. define an intertwining operator

\[ T: \mathcal{V}(\Pi) \rightarrow \mathbb{P}_p, \theta \]

by assigning to each \( v \in \mathcal{V} \) the function

\[ f_v: G \rightarrow \mathcal{V}(\Pi)/\mathcal{V} \]

given by the prescription

\[ f_v(x) = \Pi(x^{-1})v + \mathcal{V}. \]

This result reduces the problem of describing the elements of \( \hat{G} \) into two parts.

- Isolate the discrete series (Deligne-Lusztig theory).
- Explicate the decomposition of \( \text{Ind}_{P,\theta}^G \) and determine its irreducibility (Howlett-Lehrer theory).

We shall pass in silence on the first of these points (for a recent survey, consult arXiv:1404.0861) and settle for a summary on the second (cf. §15).

27: LEMMA The canonical representation of \( G \) on \( C(G;P) \) is equivalent to

\[ \bigoplus \text{Ind}_{P,\theta}^G \]

where \( \theta \) runs through the elements of \( \hat{\mathcal{L}} \).

28: NOTATION Given a parabolic \( k \)-subgroup \( P \) of \( G \), let \( \mathcal{C}(P) \) be the subspace of \( C(P) \) consisting of those \( f \) which are invariant to the right under \( U \) and have the property that the function on \( P/U \) thereby defined belongs to \( \mathcal{C}(P/U) \).

29: LEMMA Let \( P_1, P_2 \) be parabolic \( k \)-subgroups of \( G \) and let

\[ f_1 \in \mathcal{C}(P_1), f_2 \in \mathcal{C}(P_2). \]
Then

\[ \langle r_{P_1} \rightarrow P_1 \cap P_2, f_1 \rangle, \quad r_{P_2} \rightarrow P_2 \cap P_1, f_2 \rangle_{P_1 \cap P_2} = 0 \]

unless \( P_1 \) and \( P_2 \) have a common Levi subgroup \( L \).

PROOF Ignoring constant factors (signified by \( \hat{=} \)), we have

\[ \langle r_{P_1} \rightarrow P_1 \cap P_2, f_1 \rangle, \quad r_{P_2} \rightarrow P_2 \cap P_1, f_2 \rangle_{P_1 \cap P_2} \]

\[ \hat{=} \Sigma_{P_1 \cap P_2} f_1(x) f_2(x) \]

\[ \hat{=} \Sigma_{P_1 \cap P_2 / U_1} f_1(x) f_2(x) \]

\[ \hat{=} \Sigma_{P_1 \cap P_2 / U_1 \cap U_2} f_1(x) \Sigma_{P_1 \cap U_2 / U_1 \cap U_2} f_1(xu). \]

Let \( \pi_1 : P_1 \rightarrow P_1 / U_1 \approx L_1 \) be the canonical projection -- then \( *P = \pi_1 (P_1 \cap P_2) \) is a cuspidal subgroup of \( L_1 \) with unipotent radical \( *U = P_1 \cap U_2 / U_1 \cap U_2 \). Given \( x \in P_1 \cap P_2 \), write \( x = \ell_1 u_1 \) \( (\ell_1 \in L_1, u_1 \in U_1) \), thus

\[ f_1(xu) = f_1(\ell_1 u_1 u) \]

\[ = f_1(\ell_1 u_1 u^{-1}), \]

so

\[ \Sigma_{P_1 \cap U_2 / U_1 \cap U_2} f_1(xu) \]

\[ \hat{=} \Sigma_{*U} f_1(\ell_1 u) = 0 \]

unless \( *U = \{e\} \), i.e., unless
12.

\[ P_1 \cap U_2 = U_1 \cap U_2 \subseteq U_1. \]

Switching roles leads to

\[ P_2 \cap U_1 = U_2 \cap U_1 \subseteq U_2. \]

Therefore the relevant integrals vanish unless \( P_1 \) and \( P_2 \) have a common Levi subgroup (cf. §13, #20).

30: APPLICATION Assume: \( P_1 \) and \( P_2 \) are not associate -- then

\[ \langle r_{P_1} \rightarrow P_1 \cap P_2, r_{P_2} \rightarrow P_2 \cap P_1 \rangle _{P_1 \cap P_2} = 0. \]

31: THEOREM Let \( P_1 = L_1 U_1, P_2 = L_2 U_2 \) be cuspidal subgroups of \( G \). Suppose that \( P_1 \) and \( P_2 \) are not associate -- then \( \forall \theta_1 \in \hat{\theta}_{L_1}, \forall \theta_2 \in \hat{\theta}_{L_2}, \)

\[ \pi_1 = \text{Ind}_{P_1, \theta_1}^G \text{ and } \pi_2 = \text{Ind}_{P_2, \theta_2}^G \]

are disjoint:

\[ \langle i_{P_1} \rightarrow G^{\chi_{\theta_1}}, i_{P_2} \rightarrow G^{\chi_{\theta_2}} \rangle _G = 0 \quad \text{(cf. §10, #2)}. \]

PROOF In the notation of §8, #4,

\[ \langle i_{P_1} \rightarrow G^{\chi_{\theta_1}}, i_{P_2} \rightarrow G^{\chi_{\theta_2}} \rangle _G \]

\[ = \sum_{s \in S} \langle r_{P_1} \rightarrow P_2(s)^{\chi_{\theta_1}} \rangle \langle (\chi_{\theta_2})^s \rangle _{P_2(s)} \]

\[ = \sum_{s \in S} \langle r_{P_1} \rightarrow P_2(s)^{\chi_{\theta_1}}, r_{P_2(s)} \rightarrow P_2(s)^{\chi_{\theta_2}} \rangle _{P_2(s)} \]
where
\[ P_2(s) = P_2^s \cap P_1 (= sP_2s^{-1} \cap P_1). \]

But \( P_1 \) and \( P_2 \) are not associate, hence \( P_{\leq 1} \) and \( sP_2s^{-1} \) are not associate (cf. §13, #24). Therefore each of the terms in the sum \( \sum_{s \in S} \) must vanish (cf. #30).

32: NOTATION Given a parabolic \( k \)-subgroup \( P \) of \( G \) and a Levi subgroup \( L \subset P \), put

\[ W_L = N_G(L)/L. \]

33: N.B. If \( L' \) is another Levi subgroup of \( P \), then there is a unique \( u \in U \) such that \( L' = uL' u^{-1} \), hence there is a canonical isomorphism

\[ W_L \cong W_{L'}. \]

Set

\[ W_L = W_L(k) \ (= N_G(L)/L). \]

Then each \( w \in W_L \) can be represented by an element \( n_w \in N_G(L) \).

34: LEMMA The arrow

\[ W_L \rightarrow P\backslash G/P \]

given by

\[ w \rightarrow Pn_wP \]

is injective.

35: LEMMA \( W_L \) operates on \( ^0 C(P) \).
36: REDUCTION PRINCIPLE Let \( P_{1}, P_{2} \) be parabolic \( k \)-subgroups of \( G \) and let

\[ f_{1} \in \overline{\text{CL}}(P_{1}), f_{2} \in \overline{\text{CL}}(P_{2}). \]

Assume: \( P_{1} \) and \( P_{2} \) have a common Levi subgroup \( L' \) -- then

\[ \langle i_{P_{1}} \rightarrow G_{1}^{-1}, i_{P_{2}} \rightarrow G_{2}^{-1} \rangle \]

\[ = \sum_{w \in W_{L}} \langle r_{P_{1}} \rightarrow L_{1}^{-1}, r_{P_{2}} \rightarrow L_{2}^{-1} \rangle L'. \]

PROOF In the notation of §8, #4,

\[ \langle i_{P_{1}} \rightarrow G_{1}^{-1}, i_{P_{2}} \rightarrow G_{2}^{-1} \rangle \]

\[ = \sum_{s \in S} \langle r_{P_{1}} \rightarrow P_{2}(s)_{1}^{-1}, r_{P_{2}} \rightarrow P_{2}(s)_{2}^{-1} \rangle P_{2}(s) \]

\[ = \sum_{s \in S} \langle r_{P_{1}} \rightarrow P_{2}(s)_{1}^{-1}, r_{P_{2}} \rightarrow P_{2}(s)_{2}^{-1} \rangle P_{2}(s) \]

where

\[ P_{2}(s) = P_{2}^{s} \cap P_{1} ( = sP_{2}^{-1} \cap P_{1} ). \]

The only nonzero terms in the sum are those for which \( P_{1} \) and \( sP_{2}^{-1} \) have a common Levi subgroup \( L' \) (cf. #31). Choose \( u_{1} \in U_{1} \) such that \( u_{1}1u_{1}^{-1} = L' \). Next

\[ L' \subset sP_{2}^{-1} \Rightarrow s^{-1}L' \subset s^{-1} \subset P_{2}. \]

Choose \( u_{2} \in U_{2} \) such that \( u_{2}L_{2}^{-1} = s^{-1}L' \subset s^{-1} \)

\[ L_{2} = su_{2}L_{2}^{-1}s^{-1}. \]
On the other hand, 

\[ u_1 s u_2 \in N_G(L) \].

Therefore the double cosets \( P_1 \backslash G/P_2 \) that intervene are those containing an element of \( N_G(L) \), so

\[
\langle i_{P_1} \to G f_1', i_{P_2} \to G f_2' \rangle_G
\]

\[ = \sum_{w \in w_L} \langle r_{P_1 + P_2(w)} f_1', r_{P_2 + P_2(w)} (w \cdot f_2) \rangle_{P_2(w)}. \]

Noting that \( L = w_L w^{-1} \subset w_{P_2} w^{-1} \) is a Levi subgroup of \( w_{P_2} w^{-1} \), write

\[ P_2(w) = P_2 \cap P_1 = P_1 \cap w_{P_2} w^{-1} \]

\[ = L \cdot (L \cap w_{U_2} w^{-1}) \cdot (U_1 \cap L) \cdot (U_1 \cap w_{U_2} w^{-1}) \]

with uniqueness of expression -- then

\[ L \cap w_{U_2} w^{-1} = \{e\}, \quad U_1 \cap L = \{e\} \]

and

\[
\langle r_{P_1} \to P_2(w) f_1', r_{P_2 + P_2(w)} (w \cdot f_2) \rangle_{P_2(w)} \]

\[ = \frac{1}{|P_2(w)|} \sum_{x, u} f_1(xu) (w \cdot f_2)(xu), \]
where the sum runs over all \( x \in L \) and all \( u \in U_1 \cap wU_2w^{-1} \). Since \( f_1 \) and \( w \cdot f_2 \) are invariant to the right under \( U_1 \cap wU_2w^{-1} \), the above expression equals

\[
\frac{|U_1 \cap wU_2w^{-1}|}{|P_2(w)|} \sum_{x} f_1(x) \frac{v(x)}{(w \cdot f_2)(x)}
\]

\[
= \frac{|U_1 \cap wU_2w^{-1}|}{|P_2(w)|} |L| \langle r_{P_1} + L \mathbf{f}_1, r_{P_2} + L (w \cdot \mathbf{f}_2) \rangle_L.
\]

And

\[
\frac{|U_1 \cap wU_2w^{-1}|}{|P_2(w)|} |L|
\]

\[
= \frac{|U_1 \cap wU_2w^{-1}| \cdot |L|}{|L| \cdot |U_1 \cap wU_2w^{-1}|} = 1.
\]

37: SUBLEMMA Let \( H \) be a Hilbert space and let \( x, y \in H \). Assume:

\[
\langle x, x \rangle = \langle x, y \rangle = \langle y, y \rangle.
\]

Then \( x = y \).

PROOF In fact,

\[
\langle x - y, x - y \rangle = \langle x, x \rangle + \langle y, y \rangle - \langle x, y \rangle - \langle y, x \rangle
\]

\[
= \langle x, y \rangle + \langle x, y \rangle - \langle x, y \rangle - \langle x, y \rangle
\]

\[
= \langle x, y \rangle - \langle x, y \rangle = \langle x, y \rangle - \langle x, y \rangle = 0.
\]

38: APPLICATION If

\[
r_{P_1} + \mathbf{f}_1 = r_{P_2} + \mathbf{f}_2.
\]
then

\[ i_{P_1} \rightarrow G^{f_1} = i_{P_2} \rightarrow G^{f_2}. \]

[It follows from \#36 that]

\[ <i_{P_1} \rightarrow G^{f_1}, i_{P_1} \rightarrow G^{f_1}>_{G} \]

\[ = <i_{P_1} \rightarrow G^{f_1}, i_{P_2} \rightarrow G^{f_2}>_{G} \]

\[ = <i_{P_2} \rightarrow G^{f_2}, i_{P_2} \rightarrow G^{f_2}>_{G}. \]

39: NOTATION Given a cuspidal subgroup \( P = LU \) of \( G \) and a \( \theta \in \hat{G} \), let

\[ W_{L}(\theta) = \{ w \in W_{L} : w \cdot \chi_{\theta} = \chi_{\theta} \}. \]

40: THEOREM

\[ <i_{P} \rightarrow G^{\chi_{\theta}}, i_{P} \rightarrow G^{\chi_{\theta}}>_{G} = |W_{L}(\theta)|. \]

[In \#36, take \( P_1 = P_2 = P \) and note that]

\[ <r_{P} \rightarrow L^{\chi_{\theta}}, r_{P} \rightarrow L^{(w \cdot \chi_{\theta})}>_{L} \]

equals 1 if \( w \cdot \chi_{\theta} = \chi_{\theta} \) and equals 0 if \( w \cdot \chi_{\theta} \neq \chi_{\theta} \).

Let \( P \) be the set of parabolic \( k \)-subgroups of \( G \). Decompose \( P \) into association classes: \( P = \bigsqcup C \). Given \( C \), take a \( \Xi \in C \) and denote by \( \hat{G}(C) \) the subset of \( \hat{G} \) comprised of those \( \pi \) which occur as a subrepresentation of

\[ \text{Ind}_{P,\Theta}^{G}, \]

for some \( \Theta \in \hat{G}. \)
41: Lemma $G(C)$ is independent of the choice of $P \in C$.

Proof The theory does not change if $P$ is replaced by $x^{-1}Px$ ($x \in G$), so if $P_1, P_2$ are associate, then there is no loss of generality in assuming that $P_1$ and $P_2$ have a common Levi subgroup $L$, thus $[L \subset P_1, L \subset P_2]$. Given $\theta \in \hat{G}$, lift

\[
\begin{align*}
\theta &\to P_1, \text{ call it } \theta_1, \\
\theta &\to P_2, \text{ call it } \theta_2.
\end{align*}
\]

Then

\[
\begin{align*}
i_{P_1} \rightarrow G^{\chi_{\theta_1}} \text{ is the character of } \text{Ind}^G_{P_1, \theta_1} \\
i_{P_2} \rightarrow G^{\chi_{\theta_2}} \text{ is the character of } \text{Ind}^G_{P_2, \theta_2}.
\end{align*}
\]

But

\[
\begin{align*}
r_{P_1}L^{\chi_{\theta_1}} &\equiv \chi_{\theta} \\
r_{P_2}L^{\chi_{\theta_2}} &\equiv \chi_{\theta}
\end{align*}
\]

$\Rightarrow$

\[
i_{P_1}G^{\chi_{\theta_1}} = i_{P_2}G^{\chi_{\theta_2}} \quad \text{(cf. #38)}.
\]

Therefore

\[
\text{Ind}^G_{P_1, \theta_1} \cong \text{Ind}^G_{P_2, \theta_2}.
\]

42: Lemma If $C_1 \neq C_2$, then

\[
\hat{G}(C_1) \cap \hat{G}(C_2) = \emptyset \quad \text{(cf. #30)}.
\]
Accordingly:

43: **THEOREM** There is a disjoint decomposition

\[ \hat{G} = \bigsqcup_{C} \hat{G}(C). \]

44: **NOTATION** Given \( P \in \mathcal{P} \), let \([P]\) be the association class to which \( P \) belongs.

45: **EXAMPLE** Take \( P = G \) -- then the elements of \( \hat{G}([G]) \) comprise the discrete series for \( G \), i.e., \( \hat{G}([G]) = \hat{G} \).

46: **EXAMPLE** Take \( P = \mathcal{B} \) -- then the elements of \( \hat{G}([\mathcal{B}]) \) comprise the principal series for \( G \).

47: **REMARK** \( W_L \) operates on \( \hat{0}^L \), hence \( \hat{0}^L \) breaks up into \( W_L \)-orbits. Let \( \varnothing_1, \varnothing_2 \in \hat{0}^L \) -- then there are two possibilities.

- If \( \varnothing_1, \varnothing_2 \) are on the same \( W_L \)-orbit, then

\[ \text{Ind}_{P, \varnothing_1}^G \approx \text{Ind}_{P, \varnothing_2}^G. \]

- If \( \varnothing_1, \varnothing_2 \) are not on the same \( W_L \)-orbit, then

\[ \text{Ind}_{P, \varnothing_1}^G \text{ and Ind}_{P, \varnothing_2}^G \]

are disjoint.
1.

§15. HOWLETT-LEHRER THEORY

In view of §14, #40,

\[ \langle i_P \rightarrow 1, i_P \rightarrow G \lambda \rangle_G = |W_L(\theta)|. \]

And on general grounds (cf. §5, #11), Ind_{P,\theta}^G is irreducible iff

\[ \langle i_P \rightarrow 1, i_P \rightarrow G \lambda \rangle_G = 1. \]

1: DEFINITION \( \theta \) is unramified if \( |W_L(\theta)| = 1. \)

2: THEOREM Ind_{G,\theta}^P is irreducible iff \( \theta \) is unramified.

To discuss the decomposability of Ind_{P,\theta}^G, note that \( \Pi \in \hat{G} \) occurs as a sub-representation of Ind_{P,\theta}^G iff

\[ \langle \chi_\Pi, i_P \rightarrow G \lambda \rangle_G \neq 0. \]

one-to-one correspondence

3: LEMMA There is a correspondence between the \( \Pi \in \hat{G} \) such that

\[ \langle \chi_\Pi, i_P \rightarrow G \lambda \rangle_G \neq 0 \]

and the irreducible representations \( \rho \) of

\[ I_G(\text{Ind}_{P,\theta}^G, \text{Ind}_{P,\theta}^G). \]

and if \( \Pi \leftrightarrow \rho \), then

\[ \chi_\rho(1) = \langle \chi_\Pi, i_P \rightarrow G \lambda \rangle_G' \]

the positive integer on the right being the multiplicity

\[ m(\Pi, \text{Ind}_{P,\theta}^G) \]

of \( \Pi \) in \( \text{Ind}_{P,\theta}^G. \)
4: THEOREM The semisimple algebra

\[ I_G(\text{Ind}_P^G, \text{Ind}_P^G) \]

is isomorphic to the semisimple algebra

\[ C(\mathcal{W}_L(\Theta)) \].

The irreducible components of \( \text{Ind}_P^G \) are therefore parameterized by the elements of \( \mathcal{W}_L(\Theta) \): If \( \omega \in \mathcal{W}_L(\Theta) \) and if \( \Pi(\omega) \in \hat{G} \) is the irreducible component of \( \text{Ind}_P^G \) corresponding to \( \omega \), then

\[ \langle \chi_{\Pi(\omega)}, i_P \to G \chi_\Theta \rangle_G = \chi_\omega(1), \]

the dimension of the representation space of \( \omega \).
1.

§16. MODULE LANGUAGE

Let $G$ be a finite group, $\Gamma \subset G$ a subgroup. View $C(G)$ as a left $C(G)$-module and as a right $C(\Gamma)$-module.

1: CONSTRUCTION Let $\theta: \Gamma \to GL(E)$ be a representation of $\Gamma$ — then the tensor product

$$C(G) \otimes_{C(\Gamma)} E$$

is a left $C(G)$-module or, equivalently, a representation, the representation $\text{Ind}^G_{\Gamma, \theta}$ of $G$ induced by $\theta$.

2: N.B. The left action is given by

$$\left( \sum_{x \in G} f(x) \delta_x \right) \otimes_Y (\delta_y \otimes X)$$

$$= \sum_{x \in G} \delta_{xy} \otimes f(x)X \quad (X \in E)$$

and from the definitions, $\forall \gamma \in \Gamma$,

$$\delta_x \delta_{xy} \otimes X = \delta_x \otimes \theta(\gamma)X \quad (X \in E).$$

3: LEMMA Write

$$G = \bigsqcup_{k=1}^n x_k \Gamma.$$ 

Then as a vector space

$$\text{Ind}^G_{\Gamma, \theta} = \bigoplus_{k=1}^n \delta_{x_k} \otimes E.$$ 

[Note: The summand $$\delta_{x_k} \otimes E = \{\delta_{x_k} \otimes X : X \in E\}$$]
\[
\approx E(\delta_x \otimes X \leftrightarrow X)
\]
is the transform of \( \delta_x \otimes E \approx E \) under the action of \( \delta_x \):

\[
\delta_x (\delta_x \otimes X) = \delta_x \otimes X.
\]

The following result justifies the notation and the terminology.

4: \textbf{THEOREM} Set \( \pi = \text{Ind}_{\Gamma}^G \theta \) -- then

\[
i_{\Gamma} \cdot \gamma \theta = \chi_{\pi} \quad \text{(cf. §9, §10)}.
\]

\textbf{PROOF} Let \( X_1, \ldots, X_d \) be a basis for \( E \) and define \( \theta_{ij}(\gamma) \) by

\[
\theta(\gamma)X_j = \sum_i \theta_{ij}(\gamma)X_i.
\]

Equip \( C(G) \otimes C(\Gamma) \) \( E \) with the basis \( \{ \delta_{x_1} \otimes X_1, \ldots, \delta_{x_1} \otimes X_d, \delta_{x_2} \otimes X_1, \ldots, \delta_{x_2} \otimes X_d, \ldots, \delta_{x_n} \otimes X_1, \ldots, \delta_{x_n} \otimes X_d \} \)

and write \( xx_k = x_{\ell} \gamma \) -- then

\[
\pi(x) (\delta_{x_k} \otimes X_j) = \delta_{xx_k} \otimes X_j
\]

\[
= \delta_{x_{\ell} \gamma} \otimes X_j
\]

\[
= \delta_{x_{\ell}} \delta_{x_\gamma} \otimes X_j
\]

\[
= \delta_{x_{\ell}} \otimes \theta(\gamma)X_j
\]

\[
= \delta_{x_{\ell}} \otimes \sum_i \theta_{ij}(\gamma)X_i
\]
Define $\hat{\theta}$ on $G$ by $\hat{\theta}(\gamma) = \Theta_{ij}(\gamma)$ ($\gamma \in \Gamma$) and $\hat{\theta}(x) = 0_d$ if $x \not\in \Gamma$ ($0_d$ the zero $d$-by-$d$ matrix), thus the block matrix representing $\pi(x)$ is

$$
\begin{pmatrix}
\hat{\theta}(x_1^{-1}xx_1) & \hat{\theta}(x_1^{-1}xx_2) & \cdots & \hat{\theta}(x_1^{-1}xx_n) \\
\hat{\theta}(x_2^{-1}xx_1) & \hat{\theta}(x_2^{-1}xx_2) & \cdots & \hat{\theta}(x_2^{-1}xx_n) \\
\vdots & \vdots & \ddots & \vdots \\
\hat{\theta}(x_n^{-1}xx_1) & \hat{\theta}(x_n^{-1}xx_2) & \cdots & \hat{\theta}(x_n^{-1}xx_n)
\end{pmatrix}
$$

Taking the trace

$$
\chi_\pi(x) = \text{tr}(\pi(x)) = \sum_{k=1}^{n} \text{tr}(\hat{\theta}(x_k^{-1}xx_k))
$$

$$
= \sum_{k=1}^{n} \chi_{\theta}(x_k^{-1}xx_k)
$$

$$
= (i_\Gamma \to G^x\theta)(x)
$$

finishes the proof.

5. NOTATION MOD($\Gamma$) is the category of left $C(\Gamma)$-modules and MOD($G$) is the category of left $C(G)$-modules.
[Note: All data is over $\mathbb{C}$ and finite dimensional.]

6: **N.B.** Morphisms are intertwining operators.

7: **SCHOLIUM** The assignment

$$ (\theta, E) \rightarrow \text{Ind}_{\Gamma, \theta}^G $$

defines a functor

$$ \text{MOD}(\Gamma) \rightarrow \text{MOD}(G). $$

8: **NOTATION** Given a representation $(\pi, V)$ of $G$, denote its restriction to $\Gamma$ by $\text{Res}_{\Gamma, \pi}^G$.

9: **SCHOLIUM** The assignment

$$ (\pi, V) \rightarrow \text{Res}_{\Gamma, \pi}^G $$

defines a functor

$$ \text{MOD}(G) \rightarrow \text{MOD}(\Gamma). $$

Here now are the fundamental formalities.

10: **LEMMA**

$$ I_G(\text{Ind}_{\Gamma, \theta}^G (\pi, V)) \approx I_\Gamma((\theta, E), \text{Res}_{\Gamma, \theta}^G). $$

11: **SLOGAN** The restriction functor is a right adjoint for the induction functor.

12: **LEMMA**

$$ I_G((\pi, V), \text{Ind}_{\Gamma, \theta}^G) \approx I_\Gamma(\text{Res}_{\Gamma, \pi}^G, (\theta, E)). $$
13: SLOGAN The restriction functor is a left adjoint for the induction functor.

Moving on:

14: DEFINITION Let $\theta: \Gamma \to \text{GL}(E)$ be a representation of $\Gamma$ -- then

$$\text{Inv}_\Gamma(E) = \{X \in E: \theta(\gamma)X = X \forall \gamma \in \Gamma\}$$

is the set of $\Gamma$-invariants per $E$.

15: DEFINITION Let $\theta: \Gamma \to \text{GL}(E)$ be a representation of $\Gamma$ -- then

$$\text{CoInv}_\Gamma(E) = E/I_\Gamma E$$

is the set of coinvariants per $E$.

[Note: $I_\Gamma \subset C(\Gamma)$ is the augmentation ideal, thus $I_\Gamma E$ stands for the set of all finite sums $\sum_i \theta(\gamma_i)X_i (\delta_i \in I_\Gamma, X_i \in E)$.]  

Specialize and assume that $G$ is a group of Lie type (cf. §13, #3).

16: NOTATION Given a cuspidal subgroup $P = LU$ of $G$,  

$$\text{Inf}_{L,P}^{E}: \text{MOD}(L) \to \text{MOD}(P)$$

is the inflation functor.

[In other words, given a representation $(\theta,E)$ of $L$, $\text{Inf}_{L,P}^E \theta$ is the lift of $\theta$ to $P$, i.e., $E$ viewed as a left $C(P)$-module with trivial $U$-action.]  

17: DEFINITION The composite

$$\text{Ind}_P^G \circ \text{Inf}_{L,P}^E$$
defines a functor

\[ R^G_{L,P} : \text{MOD}(L) \to \text{MOD}(G) \]

termed Harish-Chandra induction.

18: THEOREM If \( P_1 = LU_1, P_2 = LU_2 \) are cuspidal subgroups of \( G \), then the functors

\[
\begin{array}{c}
R^G_{L,P_1} \\
\downarrow \\
R^G_{L,P_2}
\end{array}
\]

are naturally isomorphic.

[Note: Accordingly, the left \( C(G) \)-module isomorphism class of \( R^G_{L,P}(\theta, E) \) depends only on \( \theta \) (it being independent of the particular cuspidal subgroup \( P = LU \)).]

19: LEMMA

\[
\text{Inv}_U(M) \]

10: SLOGAN The composite of restriction followed by the taking of invariants is a right adjoint for Harish-Chandra induction.
21: LEMMA

\[ I_G((\pi, V), R_{L, P}^G(\theta, E)) \]

\[ \cong I_P(\text{Res}_{P, \pi}^G, \text{Inf}_{L, P}(\theta)) \]

\[ \cong I_L(\text{CoInv}_U(\text{Res}_{P, \pi}^G), (\theta, E)). \]

[Note: For any left \( C(P) \)-module \( M \), the set \( \text{CoInv}_U(M) \) is canonically a left \( C(L) \)-module.]

22: SLOGAN The composite of restriction followed by the taking of coinvariants is a left adjoint for Harish-Chandra induction.

23: SUBLEMMA For any left \( C(P) \)-module \( M \),

\[ \text{Inv}_U(M) \cong \text{CoInv}_U(M). \]

24: SCHOLIUM The left and right adjoint of Harish-Chandra induction are naturally isomorphic.

25: DEFINITION Harish-Chandra restriction \( \ast_{R_{L, P}^G} \) is the left and right adjoint of Harish-Chandra induction.

26: LEMMA

\[ \ast_{R_{L, P}^G((\pi, V))} = e_U V, \]

where

\[ e_U = \frac{1}{\# U} \sum_{u \in U} \pi(u). \]
27: THEOREM If $P_1 = LU_1$, $P_2 = LU_2$ are cuspidal subgroups of $G$, then the functors

\[ *_{RG L,P_1} \]

\[ *_{RG L,P_2} \]

are naturally isomorphic.

[Note: Accordingly, the left $C(L)$-module class of $*_{RL,P}(\pi,V)$ depends only on $\pi$ (it being independent of the particular cuspidal parabolic subgroup $P = LU$).]

APPENDIX

Let $P = LU$ be a cuspidal subgroup of $G$.

DEFINITION Given $\phi \in CL(L)$, define $\tilde{\phi} \in C(P)$ by the rule

\[ \tilde{\phi}(lu) = \phi(l). \]

LEMMA $\tilde{\phi}$ is a class function, i.e.,

\[ \tilde{\phi} \in CL(P). \]

PROOF The claim is that $\forall \, p \in P, \forall \, p_1 \in P$,

\[ \tilde{\phi}(pp_1p^{-1}) = \tilde{\phi}(p_1). \]

Write $p = lu$, $p_1 = l_1u_1$ -- then

\[ \tilde{\phi}(pp_1p^{-1}) = \tilde{\phi}(lu_l_1u_1^{-1}l^{-1}) \]

\[ = \tilde{\phi}(lu_1l^{-1}_1u_1^{-1}l^{-1}) \]

\[ = \tilde{\phi}(lu_1l^{-1}_1u_1^{-1}l^{-1}v) \quad (v = lu^{-1}_1l^{-1} \in U) \]
Thus there is an arrow

\[ \text{CL}(L) \to \text{CL}(P) \to \text{CL}(G), \]

namely

\[ \phi \to \tilde{\phi} \to i_P \to G\tilde{\phi}. \]

On the other hand, there is an arrow

\[ \text{CL}(G) \to \text{CL}(L), \]

namely

\[ \psi \to \psi_P|_L \equiv \sigma_G \to L\psi_P. \]

[Note: \( \forall \ell \in L, \forall \ell_1 \in L, \]

\[ \sum_{u \in U} \psi(\ell \ell_1 \ell^{-1}u) = \sum_{u \in U} \psi(\ell \ell_1 \ell^{-1}u \ell \ell^{-1}) \]

\[ = \sum_{u \in U} \psi(\ell_1 \ell^{-1}u \ell) \]

\[ = \sum_{u \in U} \psi(\ell_1 u). \]
LEMMA Let \( \phi \in \text{CL}(L) \), \( \psi \in \text{CL}(G) \) -- then

\[
<i_P \rightarrow G^\phi, \psi_G > = <\phi, \Gamma_G \rightarrow L^\psi_p >_L.
\]

PROOF

\[
<i_P \rightarrow G^\phi, \psi_G > = \frac{1}{|G|} \sum_{x \in G} (i_P \rightarrow G^\phi)(x) \overline{\psi(x)}
\]

\[
= \frac{1}{|G|} \sum_{y \in G} \frac{1}{|P|} \sum_{x \in G} \tilde{\phi} (xy^{-1}) \overline{\psi(x)}
\]

\[
= \frac{1}{|G|} \sum_{y \in G} \frac{1}{|P|} \sum_{x \in G} \tilde{\phi} (xy^{-1}) \overline{\psi(x)}
\]

\[
= \frac{1}{|G|} \sum_{y \in G} \frac{1}{|P|} \sum_{x \in G} \tilde{\phi} (x) \overline{\psi(y^{-1}xy)}
\]

\[
= \frac{1}{|G|} \sum_{y \in G} \frac{1}{|P|} \sum_{p \in P} \tilde{\phi} (p) \overline{\psi(yp^{-1})}
\]

\[
= \frac{1}{|G|} \sum_{y \in G} \frac{1}{|P|} \sum_{\ell, u} \tilde{\phi} (\ell u) \overline{\psi(yluy^{-1})}
\]

\[
= \frac{1}{|G|} \sum_{y \in G} \frac{1}{|P|} \sum_{\ell, u} \phi (\ell) \overline{\psi(yluy^{-1})}
\]

\[
= \frac{1}{|G|} \sum_{y \in G} \frac{1}{|P|} \sum_{\ell, u} \phi (\ell) \overline{\psi(\ell u)}
\]

\[
= \frac{1}{|P|} \sum_{\ell, u} \phi (\ell) \overline{\psi(\ell u)}
\]

\[
= \frac{1}{|L||U|} \sum_{\ell, u} \phi (\ell) \overline{\psi(\ell u)}
\]

\[
= \frac{1}{|L|} \sum_{\ell} \phi (\ell) \frac{1}{|U|} \sum_{u \in U} \overline{\psi(\ell u)}
\]
\[ = \frac{1}{|L|} \sum_{\ell} \phi(\ell) \overline{\psi_p(\ell)} \]

\[ = \langle \phi, \mathcal{I}_G \rightarrow L \psi_p \rangle L. \]
§1. ORBITAL SUMS

Let $G$ be a finite group.

1: DEFINITION Given $f \in C(G)$ and $\gamma \in G$, put

$$O(f, \gamma) = \sum_{x \in G} f(x \gamma x^{-1}),$$

the orbital sum of $f$ at $\gamma$.

2: LEMMA The function $O(f)$ defined by the assignment

$$\gamma \mapsto O(f, \gamma)$$

is a class function on $G$, i.e., is an element of $\text{CL}(G)$.

3: LEMMA There is an expansion

$$O(f, \gamma) = \sum_{\Pi \in G} \text{tr}(\Pi^*(f)) \chi_{\Pi}(\gamma),$$

where

$$\Pi^*(f) = \sum_{x \in G} f(x) \Pi^*(x).$$

PROOF Since $O(f)$ is a class function, $\forall \gamma \in G,$

$$O(f, \gamma) = \sum_{\Pi \in G} <O(f), \chi_{\Pi}^G> \chi_{\Pi}(\gamma) \quad (\text{cf. II, §4, #17}).$$

But

$$<O(f), \chi_{\Pi}^G> \neq \frac{1}{|G|} \sum_{\gamma \in G} O(f, \gamma) \chi_{\Pi}(\gamma)$$

$$= \frac{1}{|G|} \sum_{\gamma \in G} \sum_{x \in G} f(x \gamma x^{-1}) \chi_{\Pi}(\gamma)$$

$$= \frac{1}{|G|} \sum_{x \in G} \sum_{\gamma \in G} f(x \gamma x^{-1}) \chi_{\Pi}(\gamma)$$
2.

\[
= \frac{1}{|G|} \sum_{x \in G} \sum_{\gamma \in G} f(\gamma) \chi_{\Pi}(x^{-1} \gamma x)
\]

\[
= \frac{1}{|G|} |G| \sum_{\gamma \in G} f(\gamma) \chi_{\Pi}(\gamma)
\]

\[
= \sum_{\gamma \in G} f(\gamma) \chi_{\Pi}(\gamma)
\]

\[
= \sum_{x \in G} f(x) \chi_{\Pi}(x)
\]

\[
= \text{tr}(\Pi^*(f)).
\]

[Note: Recall that
\[\chi_{\Pi^*} = \chi_{\Pi} \quad (\text{cf. II, §4, #4}).\]

4: \textbf{N.B.} In terms of the Fourier transform,
\[\Pi^*(f) = \hat{f}(\Pi^*) \Rightarrow \text{tr}(\Pi^*(f)) = \text{tr}(\hat{f}(\Pi^*)).\]
§2. THE LOCAL TRACE FORMULA

Let G be a finite group.

1. NOTATION Denote by \( \pi_{L,R} \) the representation of \( G \times G \) on \( C(G) \) given by

\[
(\pi_{L,R}(x_1,x_2)f)(x) = f(x^{-1}xx_2) \quad \text{(cf. II, §1, #14)}.
\]

Define a linear bijection

\[ T : C(G) \rightarrow C(G \times G/G) \]

via the prescription

\[ Tf(x_1,x_2) = f(x_1x_2^{-1}). \]

2. N.B. Embed \( G \) diagonally into \( G \times G \) -- then \( \forall x \in G \),

\[
Tf((x_1,x_2)(x,x)) = Tf(x_1x_2x) = f(x_1xx^{-1}x_2^{-1})
\]

\[ = f(x_1x_2^{-1}) = Tf(x_1,x_2). \]

3. NOTATION Set

\[ L_G \times G/G = \text{Ind}^{G \times G}_G \theta, \]

where \( \theta \) is the trivial representation of \( G \) on \( E = \mathbb{C} \).

4. LEMMA

\[ T \in I_G \times G(\pi_{L,R'} L_G \times G/G). \]

PROOF \( \forall x_1,x_2 \in G, \forall f \in C(G), \)
2.

\[(T_{\pi_{L,R}}(x_1, x_2)f)(y_1, y_2)\]

\[= (\pi_{L,R}(x_1, x_2)f)(y_1y_2^{-1})\]

\[= f(x_1^{-1}y_1y_2^{-1}x_2).\]

And

\[(L_G \times G/G(x_1, x_2)Tf)(y_1, y_2)\]

\[= Tf((x_1, x_2)^{-1}(y_1, y_2))\]

\[= Tf(x_1^{-1}, x_2^{-1})(y_1, y_2)\]

\[= Tf(x_1^{-1}y_1, x_2^{-1}y_2)\]

\[= f(x_1^{-1}y_1y_2^{-1}x_2).\]

5: N.B. \(T\) is unitary: \(\forall f, g \in C(G),\)

\[<Tf, Tg>_G \times G = <f, g>_G.\]

[By definition,]

\[<Tf, Tg>_G \times G\]

\[= \frac{1}{|G \times G|} \sum_{(x_1, x_2) \in G \times G} \bar{g}(x_1^{-1}x_2) f(x_1x_2^{-1})\]

\[= \frac{1}{|G \times G|} \sum_{x_1 \in G} \sum_{x_2 \in G} \bar{g}(x_1^{-1}x_2) f(x_1x_2^{-1})\]

\[= \frac{1}{|G \times G|} \sum_{x_1 \in G} \sum_{x_2 \in G} \bar{g}(x_1^{-1}x_2) f(x_1x_2^{-1})\]
3. 

\[
= \frac{1}{|G \times G|} \sum_{x_1 \in G} \sum_{x_2 \in G} f(x_1 x_2) g(x_1 x_2)
\]

\[
= \frac{1}{|G \times G|} \sum_{x_1 \in G} \sum_{x_2 \in G} f(x_2) g(x_2)
\]

\[
= \frac{|G|}{|G \times G|} \sum_{x \in G} f(x) g(x)
\]

\[
= \frac{1}{|G|} \sum_{x \in G} f(x) g(x) = \langle f, g \rangle_G.
\]

6: **NOTATION** Given an \(x \in G\), write \(C(x)\) for its conjugacy class and \(C_x\) for its centralizer (cf. II, §4, #10).

7: **EXAMPLE** \(\forall f \in C(G)\) and \(\forall \gamma \in G\),

\[
0(f, \gamma) = \sum_{x \in G} f(\gamma x \gamma^{-1}) = |G| \sum_{x \in C(\gamma)} f(x).
\]

8: **LEMMA** Abbreviate \(\chi_{I_{L,R}}\) to \(\chi_{L,R}\) -- then

\[
\chi_{L,R}(x_1, x_2) = \begin{cases} 
|G_x| & (x = x_1 = x_2) \\
0 & (C(x_1) \neq C(x_2)).
\end{cases}
\]

[Work instead with the character of \(I_G \times G/G\) and apply II, §7, #11.]

Given \(f_1, f_2 \in C(G)\), define \(f \in C(G \times G)\) by

\[
f(x_1, x_2) = f_1(x_1) f_2(x_2),
\]
and let
\[ \pi_{L,R}(f) = x_1 \sum_{x \in G} x_2 \sum_{x \in G} f_1(x_1) f_2(x_2) \pi_{L,R}(x_1, x_2). \]

Then \( \forall \phi \in C(G) \),
\[ (\pi_{L,R}(f) \phi)(x) = x_1 \sum_{x \in G} x_2 \sum_{x \in G} f_1(x_1) f_2(x_2) \phi(x^{-1} x_2) \]
\[ = \sum_{y \in G} K_f(x, y) \phi(y), \]
where
\[ K_f(x, y) = \sum_{z \in G} f_1(xz) f_2(zy). \]

Therefore \( \pi_{L,R}(f) \) is an integral operator on \( C(G) \) (a.k.a. \( L^2(G) \)...) with kernel \( K_f(x, y) \).

9: LEMMA The \( \sqrt{|G|} \delta_x \) (\( x \in G \)) constitute an orthonormal basis for \( C(G) \).

10: LEMMA \( \forall f = f_1 f_2 \),
\[ \text{tr}(\pi_{L,R}(f)) = \sum_{x \in G} K_f(x, x). \]

PROOF In fact,
\[ \text{tr}(\pi_{L,R}(f)) = \sum_{x \in G} \langle \pi_{L,R}(f) \sqrt{|G|} \delta_x, \sqrt{|G|} \delta_x \rangle_G \]
\[ = |G| \sum_{y \in G} \frac{1}{|G|} \sum_{y \in G} (\pi_{L,R}(f) \delta_x)(y) \delta_x(y) \]
\[ = \sum_{x \in G} (\pi_{L,R}(f) \delta_x)(x) \]
\[ = \sum_{x \in G} \sum_{x_1 \in G} x_2 \sum_{x_2 \in G} f_1(x_1) f_2(x_2) \delta_x(x_1^{-1} x_2). \]
= \sum_{x \in G} \sum_{z \in G} f_1(xz)f_2(zy)

= \sum_{x \in G} K_f(x,x).

Enumerate the elements of \( \text{CON}(G) \), say
\[
\text{CON}(G) = \{C_1, \ldots, C_n\}.
\]

For each \( i \), fix a \( \gamma_i \in C_i \) (\( 1 \leq i \leq n \)).

11: Lemma \( \forall f = f_1f_2' \),
\[
\sum_{x \in G} K_f(x,x) = \sum_{i=1}^{n} \frac{1}{|G|} \cdot 0(f_1, \gamma_i) \cdot 0(f_2, \gamma_i).
\]

Proof: Start with the LHS:
\[
\sum_{x \in G} K_f(x,x) = \sum_{x \in G} \sum_{y \in G} f_1(y)f_2(y^{-1}yx)
\]
\[
= \sum_{y \in G} F(y),
\]
where
\[
F(y) = \sum_{x \in G} f_1(y)f_2(y^{-1}yx).
\]

Using now §4, #2 below, we have
\[
\sum_{y \in G} F(y) = \sum_{i=1}^{n} \frac{1}{|G|} \cdot 0(F, \gamma_i).
\]

And
\[
0(F, \gamma_i) = \sum_{x \in G} F(x \gamma_i x^{-1}).
\]
\[ \begin{align*}
&= \sum_{x \in G} \sum_{y \in G} f_1(x, y^{-1}) f_2(y^{-1} x, y^{-1}) \\
&= \sum_{x \in G} f_1(x, y^{-1}) \sum_{y \in G} f_2(y, y^{-1}) \\
&= \sum_{x \in G} f_1(x, y^{-1}) \sum_{y \in G} f_2(y, y^{-1}) \\
&= 0(f_1, y) \cdot 0(f_2, y) \\
&= 0(f_1, y) \cdot 0(f_2, y).
\end{align*} \]

**12: Lemma** \( \forall f = f_1 f_2 \),

\[ \sum_{x \in G} K_f(x, x) = \sum_{\Pi \in G} \text{tr}(\hat{f}_1(\Pi)) \text{tr}(\hat{f}_2(\Pi^*)). \]

**Proof** Write

\[ \sum_{x \in G} K_f(x, x) = \sum_{x \in G} \sum_{y \in G} f_1(y) f_2(y^{-1} x) \]

\[ = \sum_{y \in G} f_1(y) \sum_{x \in G} f_2(y x^{-1}) \]

\[ = \sum_{y \in G} f_1(y) 0(f_2, y) \]

\[ = \sum_{y \in G} f_1(y) \sum_{\Pi \in G} \text{tr}(\Pi^* f_2(\Pi)) \chi_{\Pi}(y) \quad (\text{cf. \S1, \#3}) \]

\[ = \sum_{\Pi \in G} \left( \sum_{y \in G} f_1(y) \chi_{\Pi}(y) \right) \text{tr}(\Pi^* f_2(\Pi)) \]

\[ = \sum_{\Pi \in G} \text{tr}(\Pi(f_1)) \text{tr}(\Pi^* f_2(\Pi)) \]

\[ = \sum_{\Pi \in G} \text{tr}(\hat{f}_1(\Pi)) \text{tr}(\hat{f}_2(\Pi^*)). \]
13: DEFINITION Given \( f = f_1 f_2 \), the local trace formula is the relation

\[ \sum_{\Pi \in G} \text{tr}(\hat{f}_1(\Pi)) \text{tr}(\hat{f}_2(\Pi^*)) \]

\[ = \sum_{i=1}^{n} \frac{1}{|G_{\gamma_i}|} \theta(f_1, \gamma_i) \theta(f_2, \gamma_i). \]

14: EXAMPLE Suppose that \( f_1 = f_2 \) is real valued, call it \( \phi \) -- then

\[ \text{tr}(\hat{\phi}(\Pi^*)) = \sum_{x \in G} \phi(x) \chi_{\Pi^*}(x) \]

\[ = \sum_{x \in G} \phi(x) \chi_{\Pi}(x) \]

\[ = \sum_{x \in G} \bar{\phi}(x) \chi_{\Pi}(x) \]

\[ = \sum_{x \in G} \phi(x) \chi_{\Pi}(x) \]

\[ = \text{tr}(\hat{\phi}(\Pi)). \]

Therefore

\[ \sum_{\Pi \in G} \text{tr}(\hat{\phi}(\Pi)) \text{tr}(\hat{\phi}(\Pi)) \]

\[ = \sum_{\Pi \in G} |\text{tr}(\hat{\phi}(\Pi))|^2 \]

\[ = \sum_{i=1}^{n} \frac{1}{|G_{\gamma_i}|} \theta(\phi, \gamma_i)^2. \]

[Note: Specialize and take \( f_1 = f_2 = \delta_e \) -- then

\[ \text{tr}(\hat{\delta}_e(\Pi)) = \chi_{\Pi}(e) = d_\Pi \]
8.

and

\[ 0(\delta_e, \gamma_i) = 0 (\gamma_i \neq e) \]

while

\[ 0(\delta_e, e) = \sum_{x \in G} \delta_e(xe^{-1}) = \sum_{x \in G} \delta_e(e) = |G|. \]

Consequently,

\[ \sum_{II \in G} d_{II}^2 = \frac{1}{|G|^2} |G|^2 = |G| \quad (\text{cf. II, §3, #5 and II, §5, #9}). \]

From the definitions,

\[ \text{tr}(\hat{f}_1(\Pi)) = \sum_{x \in G} f_1(x) \chi_{\Pi}(x) = |G| <\chi_1, \chi_{\Pi} >_G \]

\[ \text{tr}(\hat{f}_2(\Pi^*)) = \sum_{x \in G} f_2(x) \chi_{\Pi^*}(x) = |G| <\chi_2, \chi_{\Pi^*} >_G. \]

Therefore

\[ \sum_{II \in G} \text{tr}(\hat{f}_1(\Pi)) \text{tr}(\hat{f}_2(\Pi^*)) \]

\[ = |G|^2 \sum_{II \in G} <\chi_1, \chi_{\Pi^*} >_G <\chi_2, \chi_{\Pi^*} >_G \]

\[ = |G|^2 \sum_{II \in G} <\chi_1, \chi_{\Pi^*} >_G <\chi_2, \chi_{\Pi^*} >_G. \]

15: N.B. Assume in addition that \( f_1 \) and \( f_2 \) are class functions. Write

\[ f_1(x) = \sum_{II \in G} <\chi_1, \chi_{\Pi} >_G \chi_{\Pi}(x) \]

\[ (\text{cf. II, §4, #17}). \]

\[ f_2(x) = \sum_{II \in G} <\chi_2, \chi_{\Pi} >_G \chi_{\Pi}(x) \]
Then

\[ \langle f_1, \overline{f_2} \rangle_G = \sum_{\pi \in G} \langle f_1, \chi_{\pi} \rangle_G \overline{\langle f_2, \chi_{\pi} \rangle_G} \]

(first orthogonality relations)

\[ = \sum_{\pi \in G} \langle f_1, \chi_{\pi} \rangle_G \overline{\langle f_2, \chi_{\pi^*} \rangle_G}. \]

On the other hand,

\[ \langle f_1, \overline{f_2} \rangle_G = \frac{1}{|G|} \sum_{x \in G} f_1(x) \overline{f_2(x)} \]

\[ = \frac{1}{|G|} \sum_{i=1}^{n} \sum_{x \in C_i} f_1(x) \overline{f_2(x)} \]

\[ = \frac{1}{|G|} \sum_{i=1}^{n} |C_i| f_1(y_i) \overline{f_2(y_i)} \]

\[ = \sum_{i=1}^{n} \frac{|C_i|}{|G|} f_1(y_i) \overline{f_2(y_i)} \]

\[ = \sum_{i=1}^{n} \frac{1}{|G|_{Y_i}} f_1(y_i) \overline{f_2(y_i)} \]

\[ \Rightarrow \]

\[ |G|^2 \langle f_1, \overline{f_2} \rangle_G \]

\[ = \sum_{i=1}^{n} \frac{1}{|G|_{Y_i}} |G| f_1(y_i) \overline{|G| f_2(y_i)} \]

\[ = \sum_{i=1}^{n} \frac{1}{|G|_{Y_i}} \delta(f_1, y_i) \delta(f_2, y_i). \]
The irreducible representations of $G \times G$ are the outer tensor products

$$\Pi_1 \otimes \Pi_2 \ (\Pi_1, \Pi_2 \in \hat{G}) \quad (\text{cf. II, } \S 5, \#13).$$

Moreover,

$$\chi_{\Pi_1 \otimes \Pi_2} = \chi_{\Pi_1} \chi_{\Pi_2}.$$

Consider now the direct sum decomposition

$$L_G \times G/G = \bigoplus_{\Pi_1, \Pi_2 \in \hat{G}} \mathfrak{m}(\Pi_1 \otimes \Pi_2, L_G \times G/G)\Pi_1 \otimes \Pi_2.$$

Then

$$\text{tr}(L_G \times G/G(f))$$

$$= \sum_{\Pi_1, \Pi_2 \in \hat{G}} \mathfrak{m}(\Pi_1 \otimes \Pi_2, L_G \times G/G)\text{tr}(\Pi_1 (f_1))\text{tr}(\Pi_2 (f_2)).$$

I.e. (cf. #4):

$$\text{tr}(L_{L,R}(f))$$

$$= \sum_{\Pi_1, \Pi_2 \in \hat{G}} \mathfrak{m}(\Pi_1 \otimes \Pi_2, L_G \times G/G)\text{tr}(\Pi_1 (f_1))\text{tr}(\Pi_2 (f_2)).$$

I.e. (cf. #12):

$$\sum_{\Pi \in \hat{G}} \sum_{\Pi_1, \Pi_2 \in \hat{G}} \mathfrak{m}(\Pi_1 \otimes \Pi_2, L_G \times G/G)\text{tr}(\Pi_1 (f_1))\text{tr}(\Pi_2 (f_2)).$$

Therefore, thanks to I, §3, #9,

$$\mathfrak{m}(\Pi_1 \otimes \Pi_2, L_G \times G/G)$$
must vanish unless $\Pi_1 = \Pi, \Pi_2 = \Pi^*$, in which case the coefficient is equal to 1.

16: SCHOLIUM

$$\Pi_{L,R} \approx \bigoplus_{\Pi G} \Pi \oplus \Pi^*.$$
§3. THE GLOBAL PRE-TRACE FORMULA

Let $G$ be a finite group, $\Gamma \subseteq G$ a subgroup.

1. NOTATION Set

$$L_{G/\Gamma} = \text{Ind}^G_{\Gamma, \theta},$$

where $\theta$ is the trivial representation of $\Gamma$ on $E = \mathbb{C}$.

[Note: Accordingly, $\chi_{\theta} = 1_{\Gamma}$ and $E^G_{\Gamma, \theta} = C(G/\Gamma)$.]  

2. EXAMPLE In the special case when $\Gamma = \{e\}$, $L_{G/\Gamma} = L$, the left translation representation of $G$ on $C(G)$ (cf. II, §1, #12).

3. N.B. The pair $(G \times G, G)$ figuring in §2 is an instance of the overall setup.

Given $f \in C(G)$, $\phi \in C(G/\Gamma)$, we have

$$(L_{G/\Gamma}(f)\phi)(x)$$

$$= \sum_{y \in G} f(y) (L_{G/\Gamma}(y)\phi)(x)$$

$$= \sum_{y \in G} f(y) \phi(y^{-1}x)$$

$$= \sum_{y \in G} f(xy^{-1}) \phi(y)$$

$$= \sum_{y \in G \atop xy^{-1} \in \Gamma} \frac{1}{|\Gamma|} \sum_{\gamma \in \Gamma} \phi(\gamma y)$$

$$= \sum_{y \in G \atop xy^{-1} \in \Gamma} \frac{1}{|\Gamma|} \sum_{\gamma \in \Gamma} f(xy^{-1}) \phi(\gamma y)$$
2.

\[
= \sum_{\gamma \in \Gamma} \frac{1}{|\Gamma|} \sum_{\gamma \in G} f(xy^{-1})\phi(y)
\]

\[
= \sum_{\gamma \in \Gamma} \frac{1}{|\Gamma|} \sum_{\gamma \in G} f(\gamma y^{-1})\phi(y)
\]

\[
= \sum_{\gamma \in G} \left( \frac{1}{|\Gamma|} \sum_{\gamma \in \Gamma} f(\gamma y^{-1})\right)\phi(y)
\]

\[
= \sum_{\gamma \in G} K_f(x, \gamma)\phi(y),
\]

where

\[
K_f(x, \gamma) = \frac{1}{|\Gamma|} \sum_{\gamma \in \Gamma} f(\gamma y^{-1}).
\]

To summarize:

4: **Lemma** \( \forall f \in C(G), \forall \phi \in C(G/\Gamma), \forall x \in G, \)

\[
(L_{G/\Gamma}(f) \phi)(x) = \sum_{\gamma \in G} K_f(x, \gamma)\phi(y),
\]

where

\[
K_f(x, \gamma) = \frac{1}{|\Gamma|} \sum_{\gamma \in \Gamma} f(\gamma y^{-1}).
\]

Write

\[
G = \bigsqcup_{k=1}^{n} x_k \Gamma.
\]

Then for any \( f \in C(G), \)

\[
\sum_{x \in G} f(x) = \sum_{k=1}^{n} \sum_{\gamma \in \Gamma} f(x_k \gamma),
\]

thus for any \( \phi \in C(G/\Gamma), \)

\[
\sum_{x \in G} \phi(x) = |\Gamma| \sum_{k=1}^{n} \phi(x_k).
\]
3.

5: RAPPEL (cf. II, §9, #2) The Hilbert space structure on $C(G/\Gamma)$ is defined by the inner product

$$<\phi, \psi>_\theta = \frac{1}{|G|} \sum_{x \in G} \phi(x) \overline{\psi(x)}$$

$$= \frac{\lvert \Gamma \rvert}{|G|} \sum_{k=1}^{n} \phi(x_k) \overline{\psi(x_k)}.$$

6: NOTATION Define functions $\delta_k \in C(G/\Gamma)$ by the rule

$$\delta_k(x_{\ell}y) = \delta_{k\ell} \quad (1 \leq k, \ell \leq n).$$

7: LEMMA The

$$\Delta_k = \left(\frac{|G|}{|\Gamma|}\right)^{1/2} \delta_k$$

constitute an orthonormal basis for $C(G/\Gamma)$.

PROOF A given $\phi \in C(G/\Gamma)$ admits the decomposition

$$\phi = \sum_{k=1}^{n} \phi(x_k) \delta_k.$$ 

In addition,

$$<\Delta_k, \Delta_\ell>_\theta = \frac{\lvert \Gamma \rvert}{|G|} \sum_{j=1}^{n} \Delta_k(x_j) \overline{\Delta_\ell(x_j)}$$

$$= \frac{\lvert \Gamma \rvert}{|G|} \frac{|G|}{|\Gamma|} = 1$$

if $k = \ell$ and is 0 otherwise.

8: LEMMA $\forall f \in C(G)$,

$$\text{tr}(L_{G/\Gamma}(f)) = \sum_{x \in G} K_f(x,x).$$
PROOF In fact,

\[
\text{tr}(L_{G/\Gamma}(f)) = \sum_{k=1}^{n} \left< L_{G/\Gamma}(f) \Delta_k, \Delta_k \right>_\theta
\]

\[
= \sum_{k=1}^{n} \left| \frac{|G|}{|\Gamma|} \right| \sum_{\ell=1}^{n} L_{G/\Gamma}(f) \delta_k(x_{\ell}) \delta_k(x_{\ell})
\]

\[
= \sum_{k=1}^{n} (L_{G/\Gamma}(f) \delta_k)(x_k)
\]

\[
= \sum_{k=1}^{n} \sum_{\gamma \in \Gamma} f(x_k \gamma^{-1}) \delta_k(y)
\]

\[
= \sum_{k=1}^{n} \sum_{\ell=1}^{n} \sum_{\gamma \in \Gamma} f(x_k \gamma^{-1} x_{\ell}^{-1}) \delta_k(x_{\ell})
\]

\[
= \sum_{k=1}^{n} \sum_{\gamma \in \Gamma} f(x_k \gamma^{-1} x_k^{-1})
\]

\[
= \sum_{k=1}^{n} \sum_{\gamma \in \Gamma} f(x_k \gamma x_k^{-1})
\]

\[
= \frac{1}{|\Gamma|} \sum_{k=1}^{n} |\Gamma| f(x_k \gamma x_k^{-1})
\]

\[
= \frac{1}{|\Gamma|} \sum_{k=1}^{n} \sum_{\gamma \in \Gamma} f(x_k \gamma x_k^{-1})
\]

\[
= \frac{1}{|\Gamma|} \sum_{x \in G} \sum_{\gamma \in \Gamma} f(x \gamma x^{-1})
\]
5.

\[= \sum_{x \in G} \frac{1}{|\Gamma|} \sum_{y \in \Gamma} f(xy^{-1})\]

\[= \sum_{x \in G} K_f(x, x).\]

9: EXAMPLE Take \( \Gamma = G \) -- then \( \forall f \in C(G), \)

\[\text{tr}(L_G/G(f)) = \sum_{x \in G} \frac{1}{|G|} \sum_{y \in G} f(xy^{-1})\]

\[= \frac{1}{|G|} \sum_{x \in G} \sum_{y \in G} f(xy^{-1})\]

\[= \frac{1}{|G|} \sum_{x \in G} \sum_{y \in G} f(y)\]

\[= \frac{|G|}{|G|} \sum_{y \in G} f(y) = \sum_{x \in G} f(x).\]

10: EXAMPLE Fix \( C \in \text{CON}(G) \) and \( x \in C \) -- then

\[|C| \chi_{L_G/\Gamma}(x) = \frac{|G|}{|\Gamma|} |C \cap \Gamma| \quad (\text{cf. II, } \S 9, \# 6).\]

[Work with \( f = \chi_C \), thus]

\[\text{tr}(L_G/\Gamma(\chi_C)) = \sum_{y \in G} \chi_C(y) \chi_{L_G/\Gamma}(y)\]

\[= \sum_{y \in C} \chi_C(y) \chi_{L_G/\Gamma}(y)\]

\[= |C| \chi_{L_G/\Gamma}(x).\]

Meanwhile

\[\text{tr}(L_G/\Gamma(\chi_C)) = \sum_{y \in G} K \chi_C(y, y)\]
On general grounds, there is a direct sum decomposition

\[ L_{G/\Gamma} = \bigoplus_{\Pi \in \hat{G}} m(\Pi, L_{G/\Gamma}) \Pi. \]

[Note:

\[ m(\Pi, L_{G/\Gamma}) \neq 0 \]

iff the restriction of \( \Pi \) to \( \Gamma \) contains the trivial representation \( \theta \) of \( \Gamma \) on \( E = \mathbb{C} \) (cf. II, §9, #9) (but see below (cf. #14)).]

11: SCHOLIUM \( \forall f \in C(G), \)

\[ \text{tr}(L_{G/\Gamma}(f)) = \sum_{\Pi \in \hat{G}} m(\Pi, L_{G/\Gamma}) \text{tr}(\hat{f}(\Pi)). \]

[Note: Explicated,

\[ \text{tr}(\hat{f}(\Pi)) = \sum_{x \in G} f(x) \chi_\Pi(x) = \text{tr}(\Pi(f)). \]

12: DEFINITION Given \( f \in C(G) \), the global pre-trace formula is the relation

\[ \sum_{\Pi \in \hat{G}} m(\Pi, L_{G/\Gamma}) \text{tr}(\hat{f}(\Pi)) \]

\[ = \sum_{x \in G} K_f(x, x) = \sum_{x \in G} \frac{1}{|\Gamma|} \sum_{\gamma \in \Gamma} f(x \gamma x^{-1}). \]
13: APPLICATION Take $\Gamma = \{e\}$ -- then
\[
\sum_{\Pi \in G} m(\Pi, L_G/\Gamma) \text{tr}(\hat{f}(\Pi))
\]
becomes
\[
\sum_{\Pi \in G} d_{\Pi} \text{tr}(\hat{f}(\Pi))
\]
while
\[
\sum_{x \in G} \frac{1}{|\Gamma|} \sum_{\gamma \in \Gamma} f(x\gamma^{-1})
\]
becomes
\[
|G| f(e).
\]
I.e.:
\[
f(e) = \frac{1}{|G|} \sum_{\Pi \in G} d_{\Pi} \text{tr}(\hat{f}(\Pi)),
\]
the so-called "Plancherel theorem" for $G$.

14: APPLICATION Fix $\Pi_0 \in \hat{G}$ and take $f = \overline{\chi_{\Pi_0}}$.

- $\Pi \neq \Pi_0$

$\Rightarrow \text{tr}(\Pi (\overline{\chi_{\Pi_0}})) = \sum_{x \in G} \overline{\chi_{\Pi_0}(x)} \chi_{\Pi_0}(x)$

$= 0$.

- $\Pi = \Pi_0$

$\Rightarrow \text{tr}(\Pi_0 (\overline{\chi_{\Pi_0}})) = \sum_{x \in G} \overline{\chi_{\Pi_0}(x)} \chi_{\Pi_0}(x)$

$= |G|$.

Therefore
\[
\sum_{\Pi \in G} m(\Pi, L_G/\Gamma) \text{tr}(\Pi (\overline{\chi_{\Pi_0}}))
\]
8.

reduces to

\[ |G|m(\Pi_0, L_G/\Gamma). \]

On the other hand,

\[ \sum_{x \in G} K_f(x, x) = \sum_{x \in G} \frac{1}{| \Gamma |} \sum_{\gamma \in \Gamma} \chi_{\Pi_0}^{\gamma}(x^{-1} \gamma) \]

\[ = \sum_{x \in G} \frac{1}{| \Gamma |} \sum_{\gamma \in \Gamma} \chi_{\Pi_0}^{\gamma}(\gamma) \]

\[ = \frac{|G|}{| \Gamma |} \sum_{\gamma \in \Gamma} \chi_{\Pi_0}^{\gamma}(\gamma) \]

\[ = |G|\langle 1_{\Gamma}, \chi_{\Pi_0}^{\gamma} \rangle_{\Gamma} \]

\[ = |G|m(\theta, \Pi_0|\Gamma). \]

So

\[ |G|m(\Pi_0, L_G/\Gamma) = |G|m(\theta, \Pi_0|\Gamma) \]

\[ \Rightarrow \]

\[ m(\Pi_0, L_G/\Gamma) = m(\theta, \Pi_0|\Gamma). \]

[Note: As above, \( \theta \) is the trivial representation of \( \Gamma \) on \( E = \mathbb{C} \).]

15: N.B. Take \( \Gamma = \{e\} \) — then

\[ m(\theta, \Pi_0|\Gamma) = d_{\Pi_0}, \]

hence

\[ m(\Pi_0, L_G/\Gamma) = d_{\Pi_0} \quad (\text{cf. II, §5, #8}). \]
§4. THE GLOBAL TRACE FORMULA

Let $G$ be a finite group, $\Gamma \subset G$ a subgroup.

1: NOTATION For any $\gamma \in \Gamma$,

- $G_{\gamma} = \text{centralizer of } \gamma \text{ in } G$
- $\Gamma_{\gamma} = \text{centralizer of } \gamma \text{ in } \Gamma$.

Given an $f \in C(G)$, we have

$$\text{tr}(L_{G/\Gamma}(f)) = \sum_{x \in G} K_f(x,x),$$

where

$$K_f(x,x) = \frac{1}{|\Gamma|} \sum_{\gamma \in \Gamma} f(x\gamma x^{-1}) \quad \text{(cf. §3, #8)}.$$

Enumerate the elements of $CON(\Gamma)$, say

$$CON(\Gamma) = \{c_1, \ldots, c_n\}.$$

For each $i$, fix a $\gamma_i \in c_i$ ($1 \leq i \leq n$).

2: LEMMA $\forall f \in C(G),$

$$\sum_{x \in G} K_f(x,x) = \sum_{i=1}^{n} \frac{1}{|\Gamma_{\gamma_i}|} \text{tr}(f, \gamma_i).$$

PROOF Write

$$\Gamma = \bigvee_{k \in k} \gamma_i, k \Gamma_{\gamma_i}.$$

Then $\forall x \in G$,

$$K_f(x,x) = \frac{1}{|\Gamma|} \sum_{\gamma \in \Gamma} f(x\gamma x^{-1})$$
Therefore

\[ \sum_{x \in G} K_F(x, x) = \frac{1}{|\Gamma|} \sum_{i=1}^{n} \sum_{k \in \Gamma} \sum_{y \in \Gamma} f(x, y, x^{-1}) \]

Write

\[ G = \prod_{k} x_i, k^\gamma_i. \]

Then

\[ \sum_{x \in G} K_F(x, x) \]

\[ = \frac{1}{|\Gamma|} \sum_{i=1}^{n} [\Gamma : \Gamma_i] \sum_{k, \eta_i \in \Gamma_i} f(x_i, k, \eta_i, \eta_i^{-1}, x_i, k) \]

\[ = \frac{1}{|\Gamma|} \sum_{i=1}^{n} [\Gamma : \Gamma_i] [\Gamma_i] \sum_{k} f(x_i, k, \gamma_i, x_i^{-1}, k) \]

\[ = \sum_{i=1}^{n} \frac{[\Gamma : \Gamma_i] [\Gamma_i]}{|\Gamma|} f(x_i, k, \gamma_i, x_i^{-1}, k) \]

\[ = \sum_{i=1}^{n} \sum_{k} f(x_i, k, \gamma_i, x_i^{-1}). \]
Write

\[ \begin{align*}
G &= \prod_{\ell} \prod_{\gamma_{\ell}} y_{\ell_{\gamma_{\ell}}} \\
G_{\gamma} &= \prod_{m} z_{m_{\gamma}}^{\gamma_{m}}
\end{align*} \]

=>

\[ G = \prod_{\ell} \prod_{m} y_{\ell_{\gamma}z_{m_{\gamma}}}^{\gamma_{m}}. \]

Then

\[ \sum_{x \in G} K_f(x, x) \]

\[ = \sum_{i=1}^{n} \sum_{\ell} \sum_{m} f(y_{\ell_{\gamma_{\ell}}z_{m_{\gamma}}}^{\gamma_{m}}, \gamma_{\ell_{\gamma_{\ell}}}z_{m_{\gamma}}^{\gamma_{m}, \ell}) \]

\[ = \sum_{i=1}^{n} [G_{\gamma_{i}} : \gamma_{i}] \sum_{\ell} f(y_{\ell_{\gamma_{\ell}}y_{\gamma_{i}}}^{\gamma_{i}}y_{\gamma_{i}}, \ell) \]

\[ = \sum_{i=1}^{n} \left| \frac{G_{\gamma_{i}}}{\gamma_{i}} \right| \sum_{\ell} f(y_{\ell_{\gamma_{\ell}}y_{\gamma_{i}}}^{\gamma_{i}}, \ell) \]

\[ = \sum_{i=1}^{n} \frac{1}{\left| \frac{G_{\gamma_{i}}}{\gamma_{i}} \right|} O(f, \gamma_{i}). \]

\[ 3: \text{ N.B. } \forall \gamma, \]

\[ O(f, \gamma) = |G_{\gamma}| \sum_{x \in G/G_{\gamma}} f(xyx^{-1}), \]

the sum on the right being taken over a set of representatives for the left cosets of \( G_{\gamma} \) in \( G \).
4. EXAMPLE Take $\Gamma = G$ — then $\forall f \in C(G)$,

\[
\sum_{i=1}^{n} \frac{1}{|G_{\gamma_i}|} \sum_{x \in G} f(x\gamma_i^{-1})
\]

\[
= \sum_{i=1}^{n} \frac{1}{|G_{\gamma_i}|} \left| G_{\gamma_i} \right| \sum_{y \in G/G_{\gamma_i}} f(y\gamma_i^{-1})
\]

\[
= \sum_{i=1}^{n} \sum_{y \in G/G_{\gamma_i}} f(y\gamma_i^{-1})
\]

\[
= \sum_{i=1}^{n} \sum_{y \in C_{\gamma_i}} f(y)
\]

\[
= \sum_{x \in G} f(x) \quad \text{(cf. §3, #9)}.
\]

5. EXAMPLE Suppose that $f \in CL(G)$ — then

\[
\sum_{x \in G} K_f(x,x) = \sum_{x \in G} \frac{1}{|\Gamma|} \sum_{\gamma \in \Gamma} f(x\gamma^{-1})
\]

\[
= \sum_{x \in G} \frac{1}{|\Gamma|} \sum_{\gamma \in \Gamma} f(\gamma)
\]

\[
= \frac{|G|}{|\Gamma|} \sum_{\gamma \in \Gamma} f(\gamma).
\]

In the other direction,

\[
\sum_{i=1}^{n} \frac{1}{|\Gamma_{\gamma_i}|} \sum_{x \in G} f(x\gamma_i^{-1}) = \sum_{i=1}^{n} \frac{1}{|\Gamma_{\gamma_i}|} \sum_{x \in G} f(x\gamma_i^{-1})
\]
Therefore
\[
\frac{1}{|\Gamma|} \sum_{\gamma \in \Gamma} f(\gamma) = \frac{n}{\sum_{i=1}^{\gamma_{1}}} \frac{f(\gamma_{1})}{|\Gamma_{\gamma_{1}}|}.
\]

6: DEFINITION Given \( f \in C(G) \), the global trace formula is the relation
\[
\sum_{\pi \in \hat{G}} m(\pi, L_{G/\Gamma}) \text{tr}(\hat{f}(\pi)) = \frac{n}{\sum_{i=1}^{\gamma_{1}}} \frac{1}{|\Gamma_{\gamma_{1}}|} \Theta(f, \gamma_{1}) \quad (\text{cf. §3, #12}).
\]

7: EXAMPLE (POISSON SUMMATION) Take \( G \) abelian and identify \( \hat{G} \) with the character group of \( G \): \( \Pi \leftrightarrow \chi \), hence
\[
\hat{f}(\chi) = \sum_{x \in G} f(x) \chi(x).
\]
Consider now the sum
\[
\sum_{\chi \in \hat{G}} m(\chi, L_{G/\Gamma}) \hat{f}(\chi).
\]
Let \( \Gamma^{\perp} = \{ \chi \in \hat{G} : \chi(\gamma) = 1 \ \forall \ \gamma \in \Gamma \} \) -- then
\[
\begin{align*}
\text{if } \chi \in \Gamma^{\perp} \Rightarrow m(\chi, L_{G/\Gamma}) = 1 \\
\text{if } \chi \not\in \Gamma^{\perp} \Rightarrow m(\chi, L_{G/\Gamma}) = 0
\end{align*}
\]
(cf. §3, #14).
Therefore matters on the "spectral side" reduce to
\[
\sum_{\chi \in \Gamma^{\perp}} \hat{f}(\chi).
\]
And on the "geometric side",
\[
\frac{1}{|\Gamma|} \sum_{\gamma \in \Gamma} O(f, \gamma) = \frac{1}{|\Gamma|} \sum_{\gamma \in \Gamma} O(f, \gamma) = \frac{1}{|\Gamma|} \sum_{\gamma \in \Gamma} f(x \gamma x^{-1}) = \frac{1}{|\Gamma|} \sum_{\gamma \in \Gamma} |G| f(\gamma) = \frac{|G|}{|\Gamma|} \sum_{\gamma \in \Gamma} f(\gamma).
\]

Therefore
\[
\frac{1}{|G|} \sum_{\chi \in \Gamma} \hat{f}(\chi) = \frac{1}{|\Gamma|} \sum_{\gamma \in \Gamma} f(\gamma).
\]

Each element \(\zeta\) in the center \(Z(\Gamma)\) of \(\Gamma\) determines a one-element conjugacy class \(\{\zeta\}\).

8. DEFINITION The central contribution to the global trace formula is the subsum
\[
\sum_{\zeta \in Z(\Gamma)} \frac{1}{|\Gamma|} O(f, \zeta)
\]
of
\[
\sum_{i=1}^{n} \frac{1}{|\Gamma_{Y_i}|} O(f, \gamma_i).
\]

Accordingly,
\[
\sum_{\zeta \in Z(\Gamma)} \frac{1}{|\Gamma|} O(f, \zeta) = \frac{1}{|\Gamma|} \sum_{\zeta \in Z(\Gamma)} O(f, \zeta)
\]
\[
\begin{align*}
    &= \frac{1}{|\Gamma|} \sum_{\zeta \in \mathbb{Z}(\Gamma)} \sum_{x \in G} f(x\zeta x^{-1}) \\
    &= \frac{1}{|\Gamma|} \sum_{\zeta \in \mathbb{Z}(\Gamma)} |G| f(\zeta) \\
    &= \frac{|G|}{|\Gamma|} \sum_{\zeta \in \mathbb{Z}(\Gamma)} f(\zeta).
\end{align*}
\]
1.

§1. UNITARY REPRESENTATIONS

Let $G$ be a compact group.

1: NOTATION \[ d_G \] is normalized Haar measure on $G$:
\[ \int_G 1 d_G(x) = 1. \]

2: LEMMA
\[ L^1(G) \supset L^2(G) \supset C(G) \]
and

- $\forall f \in L^2(G)$, $||f||_2 \geq ||f||_1$.
- $\forall f \in C(G)$, $||f||_2 \leq ||f||_\infty$.

3: N.B. The convolution operator
\[ \ast : L^2(G) \times L^2(G) \to C(G) \]
is given by
\[ (f \ast g)(x) = \int_G f(xy^{-1})g(y) d_G(y) \]
\[ = \int_G f(y)g(y^{-1}x) d_G(y). \]

4: DEFINITION A unitary representation of $G$ on a Hilbert space $H$ is a homomorphism $\pi : G \to \text{UN}(H)$ from $G$ to the unitary group $\text{UN}(H)$ of $H$ such that $\forall a \in H$, the map
\[ x \mapsto \pi(x)a \]
of $G$ into $H$ is continuous.
2.

5: DEFINITION

- The left translation representation of $G$ on $L^2(G)$ is the prescription
  \[ L(x)f(y) = f(x^{-1}y). \]
- The right translation representation of $G$ on $L^2(G)$ is the prescription
  \[ R(x)f(y) = f(yx). \]

[Note: Both $L$ and $R$ are unitary.]

6: N.B. There is also a unitary representation $\pi_{L,R}$ of $G \times G$ on $L^2(G)$, namely

\[ (\pi_{L,R}(x_1,x_2)f)(x) = f(x_1^{-1}xx_2). \]

7: DEFINITION A unitary representation $\pi$ of $G$ on a Hilbert space $H \neq \{0\}$ is irreducible if the only closed subspaces of $H$ which are invariant under $\pi$ are $\{0\}$ and $H$.

8: THEOREM Let $\pi$ be a unitary representation of $G$ — then $\pi$ is the Hilbert space direct sum of finite dimensional irreducible unitary representations.

9: APPLICATION Every irreducible unitary representation of $G$ is finite dimensional.

10: NOTATION $\hat{G}$ is the set of unitary equivalence classes of irreducible unitary representations of $G$.

[Note: Generically, $\Pi \in \hat{G}$ with representation space $V(\Pi)$ and $d_\Pi = \dim V(\Pi)$ is its dimension.]
11: N.B. Let \( \pi \) be a unitary representation of \( G \) -- then there exist cardinal numbers \( n_\Pi (\Pi \in \hat{G}) \) such that

\[
\pi = \bigoplus_{\Pi \in \hat{G}} n_\Pi \Pi.
\]

12: EXAMPLE Take \( \pi = L \) -- then

\[
L = \bigoplus_{\Pi \in \hat{G}} d_\Pi \Pi.
\]

[Note: There is also an analog of A, III, §2, #16.]

13: THEOREM \( \forall x \in G (x \neq e) , \exists \) an irreducible unitary representation \( \Pi \) such that \( \Pi (x) \neq \text{id} \) (Gelfand-Raikov).

14: APPLICATION

\[
\bigcap_{\Pi \in \hat{G}} \ker \Pi = \{e\}.
\]

15: LEMMA Given \( \Pi \in \hat{G} \), suppose that \( A \in \text{Hom}(V(\Pi), V(\Pi)) \) has the property that \( \forall x \in G, \)

\[
A\Pi(x) = \Pi(x)A.
\]
Then \( A \) is a scalar multiple of the identity (Schur), call it \( \lambda_A \).
§2. EXPANSION THEORY

Let $G$ be a compact group.

1: DEFINITION Let $\pi$ be a finite dimensional unitary representation of $G$ then its **character** is the function

$$\chi_\pi : G \to \mathbb{C}$$

given by the prescription

$$\chi_\pi(x) = \text{tr}(\pi(x)) \ (x \in G).$$

2: DEFINITION The character of an irreducible unitary representation is called an **irreducible character**.

3: LEMMA Let $\Pi_1, \Pi_2 \in \hat{G}$ and suppose that $\Pi_1 \neq \Pi_2$ then

$$\langle \chi_{\Pi_1}, \chi_{\Pi_2} \rangle = 0.$$

4: LEMMA Let $\Pi \in \hat{G}$ then

$$\langle \chi_{\Pi}, \chi_{\Pi} \rangle = 1.$$

5: DEFINITION A continuous complex valued function $\varphi$ on $G$ is of **positive type** if for all $x_1, \ldots, x_n \in G$ and $\lambda_1, \ldots, \lambda_n \in \mathbb{C},$

$$\sum_{i,j=1}^{n} \lambda_i \overline{\lambda_j} \varphi(x_i^{-1} x_j) \geq 0.$$

6: N.B. The sum of two functions of positive type is of positive type and a positive scalar multiple of a function of positive type is of positive type.
7: **Lemma** If $\pi: G \to \text{UN}(H)$ is a unitary representation and if $a \in H$, then

$$\phi(x) = \langle \pi(x)a, a \rangle \quad (x \in G)$$

is of positive type.

[Note:

$$||\phi||_\infty = \langle a, a \rangle.$$

]

8: **Example** $\forall \Pi \in \hat{G}$, $\chi_\Pi$ is of positive type.

[Fix an orthonormal basis $v_1, \ldots, v_n$ in $V(\Pi)$ -- then

$$\chi_\Pi(x) = \langle \Pi(x)v_1, v_1 \rangle + \cdots + \langle \Pi(x)v_n, v_n \rangle,$$

from which the assertion.]

9: **Notation** Given $\Pi \in \hat{G}$ and $f \in L^2(G)$, put

$$\Pi(f) = \int_G f(x)\Pi(x)d_G(x).$$

10: **Lemma** $\forall f_1, f_2 \in L^2(G)$,

$$\Pi(f_1 \ast f_2) = \Pi(f_1) \circ \Pi(f_2).$$

11: **Notation** Given $f \in L^2(G)$, define $f^* \in L^2(G)$ by

$$f^*(x) = \sqrt{\frac{1}{f(x^{-1})}} \left(= \overline{f(x)} \right).$$

12: **Lemma** $\forall f \in L^2(G)$, $\forall v_1, v_2 \in V(\Pi)$,

$$\langle \Pi(f)v_1, v_2 \rangle = \langle v_1, \Pi(f^*)v_2 \rangle,$$

i.e.,

$$\Pi(f)^* = \Pi(f^*).$$
3.

**PROOF**

\[
\langle \Pi(f) v_1, v_2 \rangle = \int_G f(x) \langle \Pi(x) v_1, v_2 \rangle d_G(x)
\]

\[
= \int_G \langle v_1, f(x) \Pi(x^{-1}) v_2 \rangle d_G(x)
\]

\[
= \int_G \langle v_1, f(x^{-1}) \Pi(x) v_2 \rangle d_G(x)
\]

\[
= \int_G \langle v_1, f^*(x) \Pi(x) v_2 \rangle d_G(x)
\]

\[
= \langle v_1, \int_G f^*(x) \Pi(x) v_2 d_G(x) \rangle
\]

\[
= \langle v_1, \Pi(f^*) v_2 \rangle.
\]

**13: THEOREM** Let \( f \in L^2(G) \) -- then

\[
\int_G |f(x)|^2 d_G(x) = \sum_{\Pi \in G} d_{\Pi} \text{tr}(\Pi(f) \Pi(f^*)).
\]

**14: THEOREM** Let \( f \in L^2(G) \) -- then

\[f = \sum_{\Pi \in G} d_{\Pi} (f \ast \chi_{\Pi}),\]

the series converging in \( L^2(G) \).

**15: THEOREM** Let

\[f \in \text{span}_C(L^2(G) \ast L^2(G)) \subset C(G).\]

Then

\[f(e) = \sum_{\Pi \in G} d_{\Pi} \text{tr}(\Pi(f)).\]
PROOF Put $f = f_1 * f_2$:

$$f(e) = \int_G f_1(x^{-1})f_2(x)d_G(x)$$

$$= \int_G f_1(x^{-1})f_2(x)d_G(x)$$

$$= \int_G f_2(x)f_1(x^{-1})d_G(x)$$

$$= \int_G f_2(x)f_1(x)d_G(x)$$

$$= \langle f_2, f_1 \rangle$$

$$= \sum_{\pi \in \hat{G}} d_\pi \text{tr}(\pi(f_1 * f_2))$$

$$= \sum_{\pi \in \hat{G}} d_\pi \text{tr}(\pi(f)).$$

[Note: This is the so-called "Plancherel theorem" for $G$ (cf. A, III, §3, #13).

16: N.B. The foregoing may fail if $f$ is only assumed to be continuous (e.g., take $G = S^1$...).

17: DEFINITION A function $f \in L^2(G)$ is said to be an $L^2$ class function if

$$f(x) = f(yxy^{-1})$$

for almost all $x$ and all $y$.}
\textbf{5.} \hspace{2cm} \\

\textbf{18:} \forall \Pi \in \hat{G}, \chi_{\Pi} \text{ is an } L^2 \text{ class function.} \\

\textbf{19:} \textbf{THEOREM} Suppose that } f \in L^2(G) \text{ is an } L^2 \text{ class function -- then} \\
\hspace{1cm} f = \sum_{\Pi \in \hat{G}} \left< f, \chi_{\Pi} \right> \chi_{\Pi}, \\
\text{the series converging in } L^2(G), \text{ and} \\
\hspace{1cm} ||f||^2 = \sum_{\Pi \in \hat{G}} |\left< f, \chi_{\Pi} \right>|^2. \\

\textbf{20:} \textbf{SCHOLIUM} The \{\chi_{\Pi} : \Pi \in \hat{G}\} constitute an orthonormal basis for the set of } L^2 \text{ class functions.} \\

\textbf{21:} \textbf{NOTATION} Write } C(G)_{\text{fin}}(L) \text{ for the set of } G\text{-finite functions in } C(G) \text{ per } L:\\\n\hspace{2cm} f \in C(G)_{\text{fin}}(L) \iff \dim\{L(x)f : x \in G\}_{\text{fin}}^{<\infty}. \\

\textbf{22:} \textbf{NOTATION} Write } C(G)_{\text{fin}}(R) \text{ for the set of } G\text{-finite functions in } C(G) \text{ per } R:\\\n\hspace{2cm} f \in C(G)_{\text{fin}}(R) \iff \dim\{R(x)f : x \in G\}_{\text{fin}}^{<\infty}. \\

\textbf{23:} \textbf{LEMMA} \\
\hspace{1cm} C(G)_{\text{fin}}(L) = C(G)_{\text{fin}}(R). \\

\textbf{24:} \textbf{NOTATION} Write } C(G)_{\text{fin}} \text{ unambiguously for the } G\text{-finite functions per either action.} \\
\hspace{2cm} \text{Recalling §1, #6, } \pi_{L,R} \text{ operates on } C(G)_{\text{fin}} \text{ and it turns out that}
6.

\[ \text{C(G)} \text{fin} = \bigoplus_{\Pi \in \hat{G}} V(\Pi^*) \otimes V(\Pi). \]

Here the identification sends an element

\[ v^* \otimes v \in V(\Pi^*) \otimes V(\Pi) \]

to

\[ f \in \text{C(G)} \text{fin'}, \]

where

\[ f(x) = v^*(\Pi (x^{-1}) v). \]

[Note:

\[ L^2(G) \simeq \bigoplus_{\Pi \in \hat{G}} V(\Pi^*) \otimes V(\Pi). \]

25: **THEOREM** \( \text{C(G)} \text{fin} \) is dense in \( \text{C(G)}. \)

26: **THEOREM** \( \text{C(G)} \text{fin} \) is dense in \( L^2(G). \)

27: **DEFINITION** A function \( f \in \text{C(G)} \) is said to be a **continuous class function** if \( f(x) = f(yxy^{-1}) \) for all \( x,y \in G \) (written \( f \in \text{CL(G)}. \))

28: **EXAMPLE** \( \forall \Pi \in \hat{G}, \chi_\Pi \) is a continuous class function: \( \chi_\Pi \in \text{CL(G)}. \)

29: **THEOREM** The span of the \( \chi_\Pi \) (\( \Pi \in \hat{G} \)) equals the set of continuous class functions in \( \text{C(G) fin'}. \)

30: **THEOREM** The span of the \( \chi_\Pi \) (\( \Pi \in \hat{G} \)) is dense in the set of continuous class functions.
§3. STRUCTURE THEORY

Let $G$ be a compact group.

1: NOTATION $G^0 \leq G$ is the connected component of the identity of $G$.

2: LEMMA $G^0$ is a closed normal subgroup of $G$.

3: LEMMA The quotient $G/G^0$ is compact and totally disconnected.

4: DEFINITION A topological group possessing a neighborhood of the identity which does not contain a nontrivial subgroup is said to be a group with no small subgroups.

5: RAPPEL A Lie group has no small subgroups.

6: THEOREM The following conditions on a compact group $G$ are equivalent.
   - $G$ is a Lie group.
   - $G$ has no small subgroups.
   - $G$ has a faithful finite dimensional representation.

7: REMARK Every compact group is the projective limit of compact Lie groups.

Let $G$ be a compact Lie group.

8: N.B. Every finite group (discrete topology) is a compact Lie group.

9: EXAMPLE The product $\prod_{n=1}^{\infty} SU(n)$ is a compact group but it is not a Lie group.
10: EXAMPLE The p-adic integers
\[ Z_p = \lim_{n \to \infty} (\mathbb{Z}/p^n \mathbb{Z}) \]
are a compact group but they are not a Lie group.

11: DEFINITION A torus is a compact Lie group which is isomorphic to \( \mathbb{R}^n/\mathbb{Z}^n \cong (\mathbb{R}/\mathbb{Z})^n \) for some \( n \geq 0 \).
[Note: The nonnegative integer \( n \) is called the rank of the torus.]

12: THEOREM Every compact abelian Lie group is isomorphic to the product of a torus and a finite abelian group.

13: DEFINITION A compact Lie group is topologically cyclic if it contains an element whose powers are dense.

14: LEMMA Every torus \( T \) is topologically cyclic.
[Note: There are infinitely many topologically cyclic elements in \( T \) and their totality has full measure in any Haar measure on \( T \).]

15: THEOREM A compact Lie group is topologically cyclic iff it is isomorphic to the product of a torus and a finite cyclic group.

Let \( G \) be a compact Lie group, \( g \) its Lie algebra.

16: LEMMA \( G^0 \) is an open normal subgroup of \( G \).

Therefore the compact quotient \( G/G^0 \) is discrete, hence is a finite group, the group of components of \( G \).
17: **NOTATION** \( Z(G) \) is the center of \( G \), \( Z(G)^0 \subset Z(G) \) is the connected component of the identity element of \( Z(G) \).

18: N.B. In general, \( Z(G) \) is not connected (consider \( SU(3) \)).

19: **THEOREM** Assume that \( G \) is connected -- then \( Z(G)^0 \) is a compact abelian Lie subgroup of \( G \) and its Lie algebra is the center of \( g \), i.e., the ideal
\[ \{ X \in g : [X,Y] = 0 \ \forall \ Y \in g \} . \]

20: **DEFINITION**
- A Lie algebra is **simple** if it is noncommutative and has no proper nontrivial ideals.
- A Lie algebra is **semisimple** if it is noncommutative and has no proper nontrivial commutative ideals.
- A Lie algebra is **reductive** if it is the direct sum of an abelian Lie algebra and a semisimple Lie algebra.

21: N.B. A Lie group is simple, semisimple, or reductive if this is the case of its Lie algebra.

22: **LEMMA** A semisimple Lie algebra has a trivial center (it being a commutative ideal).

23: **LEMMA** A semisimple Lie algebra can be decomposed as a finite direct sum of simple ideals.

24: **DEFINITION** If \( G \) and \( H \) are Lie groups and if \( H \) is a subgroup of \( G \), then \( H \) is a **Lie subgroup** of \( G \) if the arrow \( H \rightarrow G \) of inclusion is continuous.
[Note: If $G$ is a Lie group and if $H$ is a closed subgroup of $G$, then $H$ is a Lie group.]

25: N.B. A Lie subgroup of a compact Lie group needn’t be compact nor carry the relative topology.

26: THEOREM Let $G$ be a compact Lie group and let $H$ be a semisimple connected Lie subgroup of $G$ -- then as a subset of $G$, $H$ is closed, and as a Lie subgroup of $G$, $H$ carries the relative topology.

27: NOTATION
- $z(g)$ is the center of $g$.
- $\mathfrak{g}_{ss}$ is the ideal in $g$ spanned by $[g, g]$.

28: LEMMA $\mathfrak{g}_{ss}$ is a semisimple Lie algebra.

29: THEOREM Let $G$ be a compact Lie group -- then
\[ \mathfrak{g} = z(g) \oplus \mathfrak{g}_{ss}, \]
thus $\mathfrak{g}$ is reductive or still, $G$ is reductive.

30: NOTATION $G_{ss}$ is the analytic subgroup of $G$ corresponding to $\mathfrak{g}_{ss}$.

31: NOTATION $G^*$ is the commutator subgroup of $G$, i.e., the subgroup of $G$ generated by the
\[ xyz^{-1}y^{-1} \quad (x, y \in G). \]
[Note: $G^*$ is necessarily normal.]
5.

32: THEOREM Assume that $G$ is connected -- then $G^*$ is a compact connected Lie subgroup of $G$ with Lie algebra $g_{ss'}$, so $G^* = G_{ss'}$, hence is semisimple.

33: THEOREM Assume that $G$ is connected -- then $G$ is the commuting product $Z(G)^0G_{ss}$.

34: THEOREM Assume that $G$ is connected -- then

$$G \cong (Z(G)^0 \times G_{ss})/\Delta,$$

where

$$\Delta \cong Z(G)^0 \cap G_{ss}$$

is embedded in $Z(G)^0 \times G_{ss}$ via the arrow $z \mapsto (z^{-1}, z)$.

[Note: Spelled out, there is an exact sequence

$$\{1\} \rightarrow Z(G)^0 \cap G_{ss} \xrightarrow{\iota} Z(G)^0 \times G_{ss} \xrightarrow{\mu} G \rightarrow \{1\},$$

where

$$\iota(z) = (z^{-1}, z), \mu(z, x) = zx.$$]

35: N.B. Structurally, $Z(G)^0$ is a torus and

$$Z(G_{ss}) = Z(G) \cap G_{ss}$$

is a finite abelian group.

36: SCHOLIUM Assume that $G$ is connected -- then $G$ is semisimple iff $Z(G)$ is finite.

[Note: Here is another way to put it: $G$ is semisimple iff $G = G_{ss}$ or still,
6.

iff $G = G^*$. To see that connectedness is essential, consider the 8 element quaternion group $\{ \pm 1, \pm i, \pm j, \pm k \}$ — then its commutator group is $\{ \pm 1 \}$.

37: **EXAMPLE** The center of $G/Z(G)$ is trivial, so $G/Z(G)$ (which is connected) is semisimple.

There are simple ideals $h_i \subset g_{ss}$ such that

$$g_{ss} = \bigoplus_{i=1}^{r} h_i$$

with $[h_i, h_j] = 0$ for $i \neq j$ and such that the span of $[h_i, h_j] = h_i$.

Put $H_i = \exp h_i$.

38: **LEMMA** $H_i$ is a compact connected normal Lie subgroup of $G_{ss}$ and its Lie algebra is $h_i$ (hence $H_i$ is simple).

39: **LEMMA** A proper compact normal Lie subgroup of $H_i$ is necessarily discrete, finite, and central.

40: **LEMMA** There is a decomposition

$$G_{ss} = H_1 \cdots H_r$$

where $H_i$ and $H_j$ commute ($i \neq j$).

41: **N.B.** The differential of the arrow

$$H_1 \times \cdots \times H_r \to G_{ss}$$

defined by the rule

$$(x_1, \ldots, x_r) \mapsto x_1 \cdots x_r$$
is the identity map, thus its kernel \( \Delta \) is discrete and normal, thus finite and central as well, so

\[
G_{ss} \cong (H_1 \times \cdots \times H_r)/\Delta.
\]

**APPENDIX**

Let \( G \) be a compact connected Lie group.

**DEFINITION** \( G \) is **tall** if for each positive integer \( n \), there are but finitely many elements of \( \hat{G} \) of degree \( n \).

**THEOREM** \( G \) is semisimple iff \( G \) is tall.

**REMARK** If \( G \) is not semisimple, then \( G \) possesses infinitely many nonisomorphic irreducible representations of degree 1.
Let $G$ be a compact Lie group, $g$ its Lie algebra.

1: LEMMA Every connected abelian subgroup $A \subset G$ is contained in a maximal connected abelian subgroup $T \subset G$.

2: N.B. $T$ is compact.

[In fact, $T$ is connected and abelian.]

3: DEFINITION A maximal torus $T \subset G$ is a maximal connected abelian subgroup of $G$.

[Note: $T$ is a torus... .]

4: THEOREM Assume that $G$ is connected and let $T_1 \subset G$, $T_2 \subset G$ be maximal tori -- then $\exists x \in G$ such that $xT_1x^{-1} = T_2$.

5: THEOREM Assume that $G$ is connected and let $T \subset G$ be a maximal torus -- then

$$G = \bigcup_{x \in G} xTx^{-1}.$$ 

6: APPLICATION The exponential map $\exp : g \rightarrow G$ is surjective.

[Every element of $G$ belongs to a maximal torus and the exponential map of a torus is surjective.]

7: LEMMA Assume that $G$ is connected and let $T \subset G$ be a maximal torus -- then the centralizer of $T$ in $G$ is $T$ itself.
8: APPLICATION The center of $G$ is contained in $T$, i.e., $Z(G) \subseteq T$.

[Note: More is true, viz.

$$Z(G) = \bigcap_T$$

the intersection being taken over all maximal tori in $G$.]

9: LEMMA Assume that $G$ is connected and let $T \subseteq G$ be a maximal torus — then $T$ is a maximal abelian subgroup.

10: REMARK A maximal abelian subgroup need not be a maximal torus.

[In $SO(3)$, there is a maximal abelian subgroup which is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^2$, hence is not a maximal torus.]

11: NOTATION Given a torus $T \subseteq G$, let $N(T)$ be its normalizer in $G$.

12: LEMMA The quotient $N(T)/T$ is finite iff $T$ is a maximal torus.

Let $G$ be a compact connected Lie group, $T \subseteq G$ a maximal torus.

13: DEFINITION The Weyl group of $T$ in $G$ is the quotient

$$W = N(T)/T.$$  

14: N.B. Different choices of $T$ give rise to isomorphic Weyl groups.

Fix a maximal torus $T \subseteq G$ — then $N(T)$ operates on $T$ by conjugation:

$$N(T) \times T \to T$$

$$(n, t) \mapsto ntn^{-1}.$$
Since $T$ operates trivially on itself, there is an induced operation of the Weyl group:

$$W \times T \rightarrow T.$$ 

[Note: The action of $W$ is on the left, thus the orbit space is denoted by $W\backslash T$.]

15: **Lemma** The canonical homomorphism $W \rightarrow \text{Aut } T$ is injective.

16: **Lemma** Two elements of $T$ are conjugate in $G$ iff they lie on the same orbit under the action of $W$.

17: **Rappel** Let $G$ be a compact group and let $X$ be a Hausdorff topological space on which $G$ operates to the left -- then the action arrow

$$G \times X \rightarrow X$$

is a closed map. Equip the orbit space $G\backslash X$ with the quotient topology and let $\pi:X \rightarrow G\backslash X$ be the projection. Then:

- $G\backslash X$ is a Hausdorff space.
- $X$ is compact iff $G\backslash X$ is compact.
- $\pi:X \rightarrow G\backslash X$ is open, closed, and proper.

18: **Example** $W\backslash T$ is a compact Hausdorff space.

19: **Notation** $\text{CON}(G)$ is the set of conjugacy classes of $G$.

Geometrically, $\text{CON}(G)$ is the orbit space under the action of $G$ on itself via inner automorphisms:

$$\begin{align*}
\text{CON}(G) & : G \times G \rightarrow G \\
& (x, y) \rightarrow xyx^{-1}.
\end{align*}$$
4.

It carries the quotient topology per the projection \( G \to \text{CON}(G) \) under which it is a compact Hausdorff space.

20: RAPPEL A one-to-one continuous map from a compact Hausdorff space \( X \) onto a Hausdorff space \( Y \) is a homeomorphism.

21: THEOREM The arrow \( \text{W}\backslash \text{T} \to \text{CON}(G) \) which sends the \( \text{W}\)-orbit \( \text{W}t \) of \( t \in \text{T} \) to the conjugacy class of \( t \in \text{T} \) in \( G \) is a (well defined) homeomorphism.

[The map is injective (cf. #16), continuous (see below), and surjective (cf. #5), so #20 is applicable.]

[Note: To check the continuity of the arrow \( \text{W}\backslash \text{T} \to \text{CON}(G) \), bear in mind that \( \text{W}\backslash \text{T} \) has the quotient topology, thus it suffices to check the continuity of the composition

\[
\text{T} \to \text{W}\backslash \text{T} \to \text{CON}(G).
\]

But this map is just the restriction to \( \text{T} \) of the arrow \( \text{G} \to \text{CON}(G) \).]

22: NOTATION
- Given \( f \in \text{C}(\text{T}) \) and \( w \in \text{W} \), \( w \cdot f \) is the function in \( \text{C}(\text{T}) \) defined by the rule
  \[
  (w \cdot f)(t) = f(n^{-1}tn) \quad (w = nT).
  \]
- Given \( f \in \text{C}(\text{G}) \) and \( x \in \text{G} \), \( x \cdot f \) is the function in \( \text{C}(\text{G}) \) defined by the rule
  \[
  (x \cdot f)(y) = f(x^{-1}yx).
  \]
23: N.B. These rules define operations

\[
\begin{array}{c}
W \times C(T) \rightarrow C(T) \\
G \times C(G) \rightarrow C(G)
\end{array}
\]

with associated invariants

\[
\begin{array}{c}
C(W \backslash T) = C(T)^W \\
CL(G) = C(G)^G.
\end{array}
\]

[Note: \(CL(G)\) is the subspace of \(C(G)\) comprised of the continuous class functions (cf. §2, #27) or still, the space \(C(CON(G))\).]

24: LEMMA The arrow

\[f \mapsto f|_T\]

of restriction defines an isomorphism

\[CL(G) \rightarrow C(T)^W.\]
§5. REGULARITY

Let \( G \) be a connected Lie group with Lie algebra \( \mathfrak{g} \). Consider the polynomial

\[
\det((t + 1) - \text{Ad}(x)) = \sum_{i=0}^{n} D_i(x)t^i \quad (x \in G),
\]

where \( t \) is an indeterminate and \( n = \dim G \). The \( D_i \) are real analytic functions on \( G \) and \( D_n = 1 \). Let \( \ell \) be the smallest positive integer such that \( D_\ell \neq 0 \) — then \( \ell \) is called the rank of \( G \) and an element \( x \in G \) is said to be singular or regular according to whether \( D_\ell(x) = 0 \) or not.

1: NOTATION \( G^\text{reg} \) is the set of regular elements in \( G \).

2: LEMMA \( G^\text{reg} \) is an open, dense subset of \( G \) while its complement, the set of singular elements, is a set of Haar measure zero (right or left).

3: N.B. \( G^\text{reg} \) is inner automorphism invariant and stable under multiplication by elements from the center of \( G \).

From this point forward, assume that \( G \) is a compact connected Lie group.

4: LEMMA The set of singular elements in \( G \) is a finite union of submanifolds of \( G \), each of dimension \( \leq \dim G - 3 \).

[Note: Therefore \( G^\text{reg} \) is path connected.]

5: RAPPEL The fundamental group of a connected Lie group is abelian.

Fix a maximal torus \( T \).
6: LEMMA The quotient $G/T$ is simply connected.

7: LEMMA The induced map $\pi_1(T) \to \pi_1(G)$ is surjective.

PROOF Consider the exact sequence

$$\pi_1(T) \to \pi_1(G) \to \pi_1(G/T)$$

arising from the fibration $T \to G \to G/T$.

8: THEOREM $\pi_1(G)$ is a finitely generated abelian group.

[Note: If $G$ is semisimple, then $\pi_1(G)$ is finite, thus its universal covering group $\tilde{G}$ is compact.]

9: LEMMA An element $x \in G$ is regular iff $x$ lies in a unique maximal torus.

Put

$$T_{\text{reg}} = T \cap G_{\text{reg}}.$$

10: THEOREM

$$G_{\text{reg}} = \bigcup_{x \in G} xT_{\text{reg}}x^{-1}.$$ 

11: THEOREM The map

$$\nu: G/T \times T_{\text{reg}} \to G_{\text{reg}}$$

that sends

$$(xT, t) \to xtx^{-1}$$

is a surjective, $|W|$-to-one local diffeomorphism.

[To verify the "$|W|"-to-one" claim, observe first that $\forall w \in W (w = nT)$,
3.

\[ \kappa(x^{-1}T, n^{-1}T) = x_n^{-1} \cdot n^{-1} \cdot n x^{-1} = x t x^{-1} = \kappa(xT, t), \] 

hence

\[ |\kappa^{-1}(xtx^{-1})| \geq |W|. \]

In the opposite direction, suppose that

\[ x t x^{-1} = y s y^{-1} (t, s \in T_{\text{reg}}). \]

Then there is a \( w \in W \) such that

\[ s = n t n^{-1} (w = n T) \) (cf. §4, #16)

from which

\[ x t x^{-1} = y n t n^{-1} y^{-1}, \]

so \( x^{-1}y n \in G_t \), the centralizer of \( t \) in \( G \). But

\[ t \in T_{\text{reg}} \Rightarrow G_t^0 = T \]

which implies that conjugation by \( x^{-1}y n \) preserves \( T \) (\( G_t^0 \) being the identity component of \( G_t \)), i.e.,

\[ n' \equiv x^{-1}y n \in N \]

\[ \Rightarrow \]

\[ (y T, s) = (x(x^{-1}y n)n^{-1}T, n t n^{-1}) = (x n' n^{-1}T, n t n^{-1}) = (x n' n^{-1}T, n(n^{-1}y^{-1}x t(x^{-1}y n)n^{-1}) = (x(n' n^{-1})T, (n' n^{-1})^{-1}t(n' n^{-1})) \in \kappa^{-1}(xtx^{-1}). \]
4.

[Note: \( G_+^0 \) is a compact connected Lie group and \( T \subset G_+^0 \) is a maximal torus.]

If \( T \neq G_+^0 \), \( \exists z \in G_+^0: \ zTz^{-1} \neq T \) (cf. §4, #5 (applied to \( G_+^0 \))). But then

\[
t = ztz^{-1} \in zTz^{-1},
\]

contradicting the regularity of \( t \) (cf. #9).]

Let \( g \) be the Lie algebra of \( G \), \( t \) the Lie algebra of \( T \). Since \( G \) is compact, there is a positive definite symmetric bilinear form on \( g \) which is invariant under the adjoint representation:

\[
\text{Ad}: G \to \text{Aut} g.
\]

Denote by \( g/t \) the orthogonal complement of \( t \) in \( g \) -- then \( g/t \) is stable under \( \text{Ad} T \), which gives rise to an induced action

\[
\text{Ad}_{G/T} : T \to \text{Aut} g/t.
\]

Denoting by \( I_{G/T} \) the identity map \( g/t \to g/t \), one may then attach to each \( t \in T \) the endomorphism

\[
\text{Ad}_{G/T}(t^{-1}) - I_{G/T}
\]

of \( g/t \).

12: **Lemm**a The determinant of

\[
\text{Ad}_{G/T}(t^{-1}) - I_{G/T}
\]

is positive on the subset of \( T \) comprised of the topologically cyclic elements.

13: **Integration Formula** For any continuous function \( f \in C(G) \),

\[
\int_G f(x) d_G(x)
\]
\[ = \frac{1}{|W|} \int_T \left| \det (\text{Ad}_{G/T}(t^{-1}) - I_{G/T}) \int_G f(x t x^{-1}) d_G(x) \right| d_T(t). \]

[Note: \( d_G(x) \) is normalized Haar measure on \( G \) and \( d_T(t) \) is normalized Haar measure on \( T \).]

14. SCHOLIUM For any continuous class function \( f \in \text{Cl}(G) \),

\[ \int_G f(x) d_G(x) \]

\[ = \frac{1}{|W|} \int_T \det (\text{Ad}_{G/T}(t^{-1}) - I_{G/T}) f(t) d_T(t). \]

APPENDIX

Consider the polynomial

\[ \det(t - \text{ad}(X)) = \sum_{i=0}^{n} d_i(X) t^i \quad (X \in g), \]

where \( t \) is an indeterminate and \( n = \text{dim } g \). The \( d_i \) are polynomial functions on \( g \) and \( d_n = 1 \). Let \( \ell \) be the smallest positive integer such that \( d_\ell \neq 0 \) -- then \( \ell \) is called the rank of \( g \) and an element \( X \in g \) is said to be singular or regular according to whether \( d_\ell(X) = 0 \) or not.

N.B. The rank of \( g \) equals the rank of \( G \), both being equal to the dimension of \( t \).

NOTATION \( g^{\text{reg}} \) is the set of regular elements in \( g \).

LEMMA \( g^{\text{reg}} \) is an open, dense subset of \( g \).
NOTATION $\mathcal{G} \equiv \text{Int } g$ is the adjoint group of $g$.

[Note: Recall that the arrow

$$\text{Ad}: G \rightarrow \mathcal{G}$$

is surjective with kernel $Z(G)$, so

$$G/Z(G) \cong \mathcal{G}.\]"

Put

$$t^{\text{reg}} = t \cap g^{\text{reg}}.$$ 

THEOREM

$$g^{\text{reg}} = \bigcup_{x \in G} x(t^{\text{reg}}).$$
§6. WEIGHTS AND ROOTS

Let G be a compact connected semisimple Lie group, T ⊂ G a maximal torus. Denote their respective Lie algebras by \( g, t \) and let \( g_c, t_c \) stand for their complexifications.

Suppose that \((\pi, V)\) is a representation of G — then \( V \) can be equipped with a \( G \)-invariant inner product, thus rendering matters unitary.

1: **LEMMA** \( d\pi \) is skew-adjoint on \( g \) (hence self-adjoint on \( \sqrt{-1} g \)).

[Given \( X \in g \), apply \( \frac{d}{dt}|_{t=0} \) to

\[ \langle \pi(\exp t)v_1, \pi(\exp t)v_2 \rangle = \langle v_1, v_2 \rangle \]

to get

\[ \langle d\pi(X)v_1, v_2 \rangle + \langle v_1, d\pi(X)v_2 \rangle = 0. \]

2: N.B. \( \forall X \in g, \pi(\exp X) = e^{d\pi(X)} \).

3: **LEMMA** \( V \) is simultaneously diagonalizable under the action of \( t_c \).

[This is because

\[ \{d\pi(H) : H \in t_c\} \]

is a commuting family of normal operators.]

Consequently, there is a finite set \( \phi(V) \subset t^*_c - \{0\} \), the elements of which being the weights of \( V \), such that

\[ V = v^0 \oplus \bigoplus_{\lambda \in \phi(V)} v^\lambda, \]
where
\[ V^0 = \{ v \in V : d\pi(H)v = 0 \} \quad (H \in t) \]
and
\[ V^\lambda = \{ v \in V : d\pi(H)v = \lambda(H)v \} \quad (H \in t). \]

4: **LEMMA** Fix a \( \lambda \in \phi(V) \) -- then \( \lambda \) is purely imaginary and \( \lambda |\sqrt{-1} \) is purely real.

5: **N.B.** Given \( t \in T \), choose \( H \in t \) such that \( t = \exp H \) -- then \( \forall v \in V^\lambda \),
\[ \pi(t)v = \pi(\exp H)v = e^{d\pi(H)}v = e^{\lambda(H)}v. \]

6: **RAPPEL** Denote by \( I_x \) the inner automorphism \( y \mapsto x y x^{-1} \) attached to \( x \in G \) -- then the adjoint representation of \( G \) is the homomorphism \( \text{Ad}: G \to \text{Aut} g \) defined by the rule
\[ \text{Ad}(x) = (dI_x)_e \]
and the adjoint representation of \( g \) is the homomorphism \( \text{ad}: g \to \text{End} g \) defined by the rule
\[ \text{ad}(X) = (d\text{Ad})_e(X). \]

7: **N.B.** \( \forall X, Y \in g \),
\[ \text{ad}(X)Y = [X, Y]. \]

- For each \( x \in G \), extend the domain of \( \text{Ad}(x) \) from \( g \) to \( g_c \) by complex linearity.
- For each \( X \in g \), extend the domain of \( \text{ad}(X) \) from \( g \) to \( g_c \) by complex linearity.
3.

8: LEMMA (Ad, g_c) is a representation of G with differential (ad, g_c).

Take now V = g_c, let π = Ad, and abbreviate (g_c)^α to g^α (α ∈ ϕ(g_c)) -- then

\[ g^0 = t_c \] and there is a weight space decomposition

\[ g_c = g^0 \oplus \bigoplus_{\alpha \in \phi(g_c)} g^\alpha. \]

9: TERMINOLOGY The elements α ∈ ϕ(g_c) are called the roots of the pair (g_c, t_c).

10: N.B.

\[ g^\alpha = \{X ∈ g_c : [H, X] = \alpha(H)X (H ∈ t_c)\}. \]

11: LEMMA ∀ α ∈ ϕ(g_c), ∀ λ ∈ ϕ(V) ∪ {0},

\[ d\pi(g^\alpha)v^\lambda ∈ V^{\alpha+\lambda}. \]

PROOF Let H ∈ t_c, X^\alpha ∈ g^\alpha, v_\lambda ∈ V^\lambda -- then

\[ d\pi(H)d\pi(X^\alpha)v_\lambda \]

\[ = (d\pi(X^\alpha)d\pi(H) + [d\pi(H), d\pi(X^\alpha)])v_\lambda \]

\[ = (d\pi(X^\alpha)d\pi(H) + \alpha(H)d\pi(X^\alpha))v_\lambda \]

\[ = (\lambda(H) + \alpha(H))d\pi(X^\alpha)v_\lambda \]

\[ ⇒ \]

\[ d\pi(X^\alpha)v_\lambda ∈ V^{\alpha+\lambda}. \]
4.

[Note: Take $\lambda = 0$ to see that

$$d\pi(g^\alpha)\mathbb{V}^0 \subset \mathbb{V}^\alpha.$$  

12: APPLICATION \forall \alpha, \beta \in \Phi(g_c) \cup \{0\},

$$[g^\alpha, g^\beta] \subset g^{\alpha + \beta}.$$

13: LEMMA Let $\langle , \rangle$ be an Ad $G$ invariant inner product on $g_c$ -- then for all $\alpha, \beta \in \Phi(g_c) \cup \{0\}$,

$$\langle g^\alpha, g^\beta \rangle = 0 \text{ if } \alpha + \beta \neq 0.$$  

14: LEMMA \forall \alpha \in \Phi(g_c), \dim g^\alpha = 1 \text{ and the only multiples of } \alpha \text{ in } \Phi(g_c) \text{ are } \pm \alpha.$$

15: NOTATION $\sigma: g_c \rightarrow g_c$ is the map that sends $Z = X + \sqrt{-1}Y$ to $\bar{Z} = X - \sqrt{-1}Y \ (X,Y \in g)$.

16: LEMMA $\sigma$ is an R-linear involution which preserves the bracket, i.e.,

$$\sigma([Z_1,Z_2]) = [\sigma Z_1, \sigma Z_2] \quad (Z_1, Z_2 \in g_c).$$

17: N.B. \forall $\alpha \in \Phi(g_c)$,

$$\sigma g^\alpha = g^{-\alpha}.$$  

18: RAPPEL The Killing form of $g_c$ is the bilinear form $B: g_c \times g_c \rightarrow \mathbb{C}$ given by

$$B(Z_1, Z_2) = \text{tr}(\text{ad}(Z_1) \circ \text{ad}(Z_2)).$$
19: PROPERTIES

- \( \forall x \in G, \forall z_1, z_2 \in g_c \),

\[
B(\text{Ad}(x)z_1, \text{Ad}(x)z_2) = B(z_1, z_2).
\]

- \( \forall z, z_1, z_2 \in g_c \),

\[
B(\text{ad}(z)z_1, z_2) = -B(z_1, \text{ad}(z)z_2).
\]

20: N.B. The prescription

\[
\langle z_1, z_2 \rangle_0 = -B(z_1, oz_2)
\]

is an Ad G invariant inner product on \( g_c \).

Every \( \alpha \in \Phi(g_c) \) is determined by its restriction to either \( t \) or \( \sqrt{-1} t \), so \( \alpha \) can be viewed as an element of \((\sqrt{-1} t)^*\) (purely real) or of \( t^* \) (purely imaginary).

21: CONSTRUCTION \( B \) induces an isomorphism between \( \sqrt{-1} t \) and \((\sqrt{-1} t)^*\) as follows: Given \( \lambda \in (\sqrt{-1} t)^* \), define \( H_\lambda \in \sqrt{-1} t \) by the relation

\[
\lambda(H) = B(H, H_\lambda) \quad (H \in \sqrt{-1} t).
\]

[Note: \( B \) is negative definite on \( t \times t \), hence \( B \) is a real inner product on the real vector space \( \sqrt{-1} t \) and for \( \lambda_1, \lambda_2 \in (\sqrt{-1} t)^* \), one writes

\[
B(\lambda_1, \lambda_2) = B(H_{\lambda_1}, H_{\lambda_2}).
\]

22: DEFINITION The vector \( H_\alpha \in \sqrt{-1} t \) is called the root vector associated with \( \alpha \).

23: LEMMA The roots span \((\sqrt{-1} t)^*\) and the root vectors span \( \sqrt{-1} t \).
24: LEMMA Let $X_\alpha \in g^\alpha, X_{-\alpha} \in g^{-\alpha}$, then

$$[X_\alpha, X_{-\alpha}] = B(X_\alpha, X_{-\alpha})^H_\alpha.$$ 

PROOF First of all,

$$[g^\alpha, g^{-\alpha}] \subset g^{\alpha-\alpha} = g^0 = t_c$$ (cf. #12),

thus

$$[X_\alpha, X_{-\alpha}] \in t_c.$$ 

Proceeding, $\forall H \in t_c$,

$$B([X_\alpha, X_{-\alpha}], H)$$

$$= - B([X_{-\alpha}, X_\alpha], H)$$

$$= - B(\text{ad}(X_{-\alpha})X_\alpha, H)$$

$$= B(X_{-\alpha}, \text{ad}(X_{-\alpha})H)$$

$$= B(X_\alpha, [X_{-\alpha}, H])$$

$$= - B(X_\alpha, [H, X_{-\alpha}])$$

$$= - B(X_\alpha, -\alpha(H)X_{-\alpha})$$

$$= \alpha(H)B(X_\alpha, X_{-\alpha})$$

$$= B(H, H_\alpha)B(X_\alpha, X_{-\alpha})$$

$$= B(H_\alpha, H)B(X_\alpha, X_{-\alpha})$$

$$= B(B(X_\alpha, X_{-\alpha})H_\alpha, H)$$

$$\Rightarrow$$

$$[X_\alpha, X_{-\alpha}] = B(X_\alpha, X_{-\alpha})^H_\alpha.$$
25: NOTATION Put

\[ h_{\alpha} = 2 \frac{H_{\alpha}}{B(H_{\alpha}, H_{\alpha})}. \]

Then \( \alpha(h_{\alpha}) = 2. \)

26: N.B. \( \forall \lambda \in (\sqrt{-1} \ t)^*, \)

\[ \lambda(h_{\alpha}) = \lambda(2 \frac{H_{\alpha}}{B(H_{\alpha}, H_{\alpha})}) \]
\[ = \lambda(2 \frac{H_{\alpha}}{B(\alpha, \alpha)}) \]
\[ = 2 \frac{\lambda(\alpha)}{B(\alpha, \alpha)} \]
\[ = 2 \frac{B(H_{\alpha}, H_{\lambda})}{B(\alpha, \alpha)} \]
\[ = 2 \frac{B(H_{\lambda}, H_{\alpha})}{B(\alpha, \alpha)} \]
\[ = 2 \frac{B(\alpha, \alpha)}{B(\alpha, \alpha)} \]

and analogously, \( \forall H \in \sqrt{-1} \ t, \)

\[ \alpha(H) = 2 \frac{B(H, h_{\alpha})}{B(h_{\alpha}, h_{\alpha})}. \]

27: NORMALIZATION Scale the data and choose \( e_{\alpha} \in g^\alpha, f_{\alpha} \in g^{-\alpha} \) such that

\[ [e_{\alpha}, f_{\alpha}] = h_{\alpha}, \]
hence

\[ [h, e] = 2e, \]

\[ [h, f] = -2f. \]

Consequently,

\[ \text{span}\{h, e, f\} \approx \mathfrak{sl}(2, \mathbb{C}), \]

where

\[
\begin{align*}
  h^\alpha \leftrightarrow h &= \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \\
  e^\alpha \leftrightarrow e &= \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \\
  f^\alpha \leftrightarrow f &= \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}.
\end{align*}
\]

28: N.B. Under this correspondence,

\[
\text{su}(2) \approx \text{span}_\mathbb{R}\{\sqrt{-1} h, e - f, \sqrt{-1} (e + f)\}
\]

\[ \approx \mathfrak{s}_\alpha, \]

where

\[
\sqrt{-1} h^\alpha \leftrightarrow \sqrt{-1} h = \begin{bmatrix} \sqrt{-1} & 0 \\ 0 & -\sqrt{-1} \end{bmatrix}
\]

and

\[
\begin{align*}
  e^\alpha - f^\alpha \leftrightarrow e - f &= \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \\
  \sqrt{-1}(e^\alpha + f^\alpha) \leftrightarrow \sqrt{-1}(e + f) &= \begin{bmatrix} 0 & -\sqrt{-1} \\ \sqrt{-1} & 0 \end{bmatrix}.
\end{align*}
\]

29: LEMMA The analytic subgroup \( S_\alpha \) of \( G \) with Lie algebra \( \mathfrak{s}_\alpha \) is compact and isomorphic to \( \text{SU}(2) \) or \( \text{SU}(2)/\mathbb{Z}_2 \).
30: LEMMA Let $(\pi, V)$ be a unitary representation of $G$ -- then $\forall \lambda \in \phi(V)$, $\lambda(h_{\alpha}) \in \mathbb{Z}$.

PROOF In $SU(2)$, $e^{2\pi \sqrt{-1}} h = I$. This said, let $\phi_{\alpha} : SU(2) \to G$ be the arrow realizing the preceding setup and consider $\pi \circ \phi_{\alpha}$:

$$I = \pi(\phi_{\alpha}(e^{2\pi \sqrt{-1}} h))$$
$$= \pi(e^{2\pi \sqrt{-1} h})$$
$$= \pi(e)$$
$$= e^{2\pi \sqrt{-1} \alpha} = e^{\alpha}.$$

On the other hand, $\forall \mathbf{v} \in V^\lambda$,

$$\mathbf{v} = e^{2\pi \sqrt{-1} \lambda(h_{\alpha})} \mathbf{v}$$
$$= e^{2\pi \sqrt{-1} \lambda(h_{\alpha})} \mathbf{v} \quad (\text{cf. #5}).$$

Therefore $\lambda(h_{\alpha}) \in \mathbb{Z}$. 
1. 

57. LATTICES

Let $V$ be a finite dimensional vector space over $\mathbb{R}$.

1: **DEFINITION** A lattice in $V$ is an additive subgroup $L \subset V$ such that

- $L$ is closed;
- $L$ is discrete;
- $L$ spans $V$.

2: **EXAMPLE** $\mathbb{Z}^n$ is a lattice in $\mathbb{R}^n$.

3: **DEFINITION** A basis for a lattice $L \subset V$ is a set $\{e_1, \ldots, e_n\} \subset L$ ($n = \dim V$) such that

$$L = \{ \sum_{i=1}^{n} k_i e_i ; k_i \in \mathbb{Z} \}.$$

4: **LEMMA** Every lattice has a basis.

5: **DEFINITION** If $L, K$ are lattices in $V$, then $L$ is a sublattice of $K$ if $L$ is a subset of $K$.

6: **LEMMA** If $L$ is a sublattice of $K$, then $K/L$ is a finite group $G$. Moreover, there is a one-to-one correspondence between the subgroups $H \subset G$ and the lattices $L \subset M \subset K$, viz.

$$\pi(M) = H \text{ and } M = \pi^{-1}(H),$$

where $\pi : K \rightarrow K/L$ is the projection.

7: **NOTATION** Given a lattice $L \subset V$, let

$$L^* = \{ v^* \in V^* ; v^*(x) \in \mathbb{Z} \ \forall \ x \in L \}. $$
8. **Lemma** \( L^* \) is a lattice in \( V^* \), the dual of \( L \).

Let \( \{e_1, \ldots, e_n\} \) be a basis for a lattice \( L \subset V \). Define \( \{f_1, \ldots, f_n\} \) by

\[
f_j(e_i) = \delta_{ij}.
\]

9. **Lemma** \( \{f_1, \ldots, f_n\} \) is a basis for \( L^* \).

10. **Application**

\[
L^{**} \cong L.
\]

[In fact, the condition

\[
f_j(e_i) = \delta_{ij}
\]

is symmetric in \( f \) and \( e \).]

11. **Lemma** Suppose that \( L \) is a sublattice of \( K \) — then \( K^* \subset L^* \) and

\[
L^*/K^* \cong \widehat{K/L}.
\]

**Proof** The first point is obvious. As for the second, define a homomorphism

\[
\rho: L^* \to \widehat{K/L}
\]

by stipulating that

\[
\rho(\ell^*)(x + L) = \exp(2\pi i \ell^*(x)).
\]

Then the kernel of \( \rho \) is \( K^* \), so \( \rho \) induces an injection \( L^*/K^* \to \widehat{K/L} \), thus

\[
|L^*/K^*| \leq |\widehat{K/L}| = |K/L|.
\]

But then by duality,

\[
|L^*/K^*| \geq |K^{**}/L^{**}| = |K/L|.
\]

Let \( G \) be a compact connected semisimple Lie group, \( T \subset G \) a maximal torus.
12: CONVENTION Identify $(\sqrt{-1} t)^\ast$ with $\sqrt{-1} t$ and let $L$ be a lattice in $(\sqrt{-1} t)^\ast$ -- then its dual is the lattice $L^\ast \subset \sqrt{-1} t$ specified via the prescription

$$\{ H \in \sqrt{-1} t : \lambda(H) \in \mathbb{Z} \land \lambda \in L \}.$$ 

13: DEFINITION The root lattice is the lattice $L_{rt}$ in $(\sqrt{-1} t)^\ast$ generated by the $\alpha \in \phi(g_c)$.

14: DEFINITION The weight lattice is the lattice $L_{wt}$ in $(\sqrt{-1} t)^\ast$ given by

$$\{ \lambda \in (\sqrt{-1} t)^\ast : \lambda(h_\alpha) \in \mathbb{Z} \land \alpha \in \phi(g_c) \}.$$ 

15: LEMMA $L_{rt}$ is a sublattice of $L_{wt}$.

Given a character $\chi : T \to S^1$, there is a commutative diagram

\[
\begin{array}{ccc}
T & \xrightarrow{d\chi} & \sqrt{-1} R \\
\downarrow{\exp} & & \downarrow{\exp} \\
T & \xrightarrow{\chi} & S^1
\end{array}
\]

and the arrow $\chi \circ d\chi$ implements an identification of $\hat{T}$ with the lattice

$$d\hat{T} \equiv \{ \lambda \in (\sqrt{-1} t)^\ast : \lambda|\exp^{-1}(e) \subset 2\pi \sqrt{-1} \mathbb{Z} \}.$$ 

Here

$$d\chi \in \text{Hom}_R(t, \sqrt{-1} R)$$

which we shall view as an element of

$$\text{Hom}_R(\sqrt{-1} t, R)$$
by writing
\[ d\lambda(\sqrt{-1} H) = \sqrt{-1} d\lambda(H) \quad (H \in t). \]

[Note: \( \sqrt{-1} R \) is the Lie algebra of \( S^1 \), the exponential map \( \exp:\sqrt{-1} R \to S^1 \) being the usual exponential function \( \sqrt{-1} \theta \to e^{\sqrt{-1} \theta} \).]

16: **Lemma** \( L_{rt} \) is a sublattice of \( \hat{dT} \) and \( \hat{dT} \) is a sublattice of \( L_{wt} \).

17: **Theorem**

- \( Z(G) \cong \hat{dT}/L_{rt} \)

- \( \pi_1(G) \cong L_{wt}/\hat{dT} \).
Let $G$ be a compact connected semisimple Lie group, $T \subset G$ a maximal torus, $\Phi(g_c)$ the roots of the pair $(g_c, t_c)$.

1: **Definition** A subset $\Psi$ of $\Phi(g_c)$ is a **simple system** of roots if it is a vector space basis for $(\sqrt{-1} t)^*$ and has the property that every root can be written as a linear combination

$$\sum_{\alpha \in \Psi} n_\alpha \alpha,$$

where the $n_\alpha$ are integers all of the same sign.

2: **Definition** The elements in a simple system of roots are said to be **simple**.

3. **N.B.** Simple systems exist (cf. infra).

4: **Construction** Let $\Psi$ be a simple system of roots.

- The **positive roots** per $\Psi$ is the set

$$\Phi^+ = \{ \beta \in \Phi(g_c) : \beta = \sum_{\alpha \in \Psi} n_\alpha \alpha \ (n_\alpha \in \mathbb{Z}_{\geq 0}) \}.$$

- The **negative roots** per $\Psi$ is the set

$$\Psi^- = \{ \beta \in \Phi(g_c) : \beta = \sum_{\alpha \in \Psi} n_\alpha \alpha \ (n_\alpha \in \mathbb{Z}_{\leq 0}) \}.$$

Accordingly,

$$\Phi(g_c) = \Phi^+ \uplus \Phi^-.$$
5: DEFINITION
- The connected components of

$$\left(\sqrt{-1} \mathfrak{t}\right)^* = \bigcup_{\alpha \in \Phi(\mathfrak{g}_C)} \alpha^1$$

are called the Weyl chambers of $$\left(\sqrt{-1} \mathfrak{t}\right)^*$$. 
- The connected components of

$$\sqrt{-1} \mathfrak{t} = \bigcup_{\alpha \in \Phi(\mathfrak{g}_C)} \mathfrak{h}_\alpha^1$$

are called the Weyl chambers of $$\sqrt{-1} \mathfrak{t}$$.

6: DEFINITION
- If $$C \subset \left(\sqrt{-1} \mathfrak{t}\right)^*$$ is a Weyl chamber, then $$\alpha \in \Phi(\mathfrak{g}_C)$$ is said to be C-positive if $$B(C,\alpha) > 0$$ and C-negative if $$B(C,\alpha) < 0$$.
- If $$C \subset \sqrt{-1} \mathfrak{t}$$ is a Weyl chamber, then $$\alpha \in \Phi(\mathfrak{g}_C)$$ is said to be C-positive if $$B(C,h_\alpha) > 0$$ and C-negative if $$B(C,h_\alpha) < 0$$.

7: DEFINITION
- If $$C \subset \left(\sqrt{-1} \mathfrak{t}\right)^*$$ is a Weyl chamber and if $$\alpha$$ is C-positive, then $$\alpha$$ is decomposable w.r.t. C if there exist $$\beta, \gamma \in \Phi(\mathfrak{g}_C)$$ such that $$\alpha = \beta + \gamma$$ (otherwise, $$\alpha$$ is indecomposable w.r.t. C).
- If $$C \subset \sqrt{-1} \mathfrak{t}$$ is a Weyl chamber and if $$\alpha$$ is C-positive, then $$\alpha$$ is decomposable w.r.t. C if there exist $$\beta, \gamma \in \Phi(\mathfrak{g}_C)$$ such that $$\alpha = \beta + \gamma$$ (otherwise, $$\alpha$$ is indecomposable w.r.t. C).

8: NOTATION
- Given a Weyl chamber $$C \subset \left(\sqrt{-1} \mathfrak{t}\right)^*$$, let $$\psi(C)$$ be the subset of $$\Phi(\mathfrak{g}_C)$$
comprised of those \( \alpha \) which are \( \mathcal{C} \)-positive and indecomposable.

- Given a Weyl chamber \( \mathcal{C} \subset \sqrt{-1} \mathfrak{t} \), let \( \psi(\mathcal{C}) \) be the subset of \( \psi(\mathcal{C}) \) comprised of those \( \alpha \) which are \( \mathcal{C} \)-positive and indecomposable.

9: LEMMA In either case, \( \psi(\mathcal{C}) \) is a simple system of roots.

10: NOTATION
- Given a simple system of roots \( \psi \), let
  \[
  \mathcal{C}(\psi) = \{ \lambda \in (\sqrt{-1} \mathfrak{t})^* : B(\lambda, \alpha) > 0 \quad \forall \alpha \in \psi \}.
  \]
- Given a simple system of roots \( \psi \), let
  \[
  \mathcal{C}(\psi) = \{ H \in \sqrt{-1} \mathfrak{t} : B(H, \alpha) > 0 \quad \forall \alpha \in \psi \}.
  \]

11: LEMMA In either case, \( \mathcal{C}(\psi) \) is a Weyl chamber.

12: THEOREM
- There is a one-to-one correspondence between the simple systems of roots and the Weyl chambers of \( (\sqrt{-1} \mathfrak{t})^* \):
  \[
  \begin{array}{ccc}
  \psi & \rightarrow & \mathcal{C}(\psi) \\
  \mathcal{C} & \rightarrow & \psi(\mathcal{C}).
  \end{array}
  \]
- There is a one-to-one correspondence between the simple systems of roots and the Weyl chambers of \( \sqrt{-1} \mathfrak{t} \):
  \[
  \begin{array}{ccc}
  \psi & \rightarrow & \mathcal{C}(\psi) \\
  \mathcal{C} & \rightarrow & \psi(\mathcal{C}).
  \end{array}
  \]
13: **Lemma** The action of $W$ on $\sqrt{-1} \ t$ and $(\sqrt{-1} \ t)^*$ is faithful, i.e., $w \in W$ acts trivially iff $w$ is the identity element.

**Proof** Suppose that $\text{Ad}(n)$ ($n \in N$) is the identity on $t$ and consider the commutative diagram

$$
\begin{array}{ccc}
\text{Ad}(n) & : & t \\
\exp & & \exp \\
\downarrow & & \downarrow \\
T & \rightarrow & T \\
& & I_n \\
\end{array}
$$

Then

$$
\exp t = T
$$

and $\forall X \in t$,

$$
I_n(\exp X) = n(\exp X)n^{-1} = \exp(\text{Ad}(n)X) = \exp X.
$$

Therefore $n$ centralizes $T$, hence $n \in T$ (cf. §4, #7), i.e., $n$ represents the identity element of $W$.

14: **Lemma** $W$ preserves $\phi(q_c)$ and $wh_\alpha = h_{w\alpha}$ ($w \in W$).

15: **Notation**

- Given $\alpha \in \phi(q_c)$, define

$$
r_\alpha : (\sqrt{-1} \ t)^* \rightarrow (\sqrt{-1} \ t)^*
$$

by

$$
r_\alpha(\lambda) = \lambda - 2 \frac{B(\lambda, \alpha)}{B(\alpha, \alpha)} \alpha = \lambda - \lambda(h_\alpha) \alpha.
$$
5.

- Given $\alpha \in \Phi(g_C)$, define
  
  $$r_{h_{\alpha}} : \sqrt{-1} t + \sqrt{-1} t$$

  by

  $$r_{h_{\alpha}}(H) = h - 2 \frac{B(h_{\alpha}, h_{\alpha})}{B(h_{\alpha}, h_{\alpha})} h_{\alpha} = H - \alpha(H) h_{\alpha}.$$  

  [Note: Geometrically, $r_{\alpha}$ is the reflection of $(\sqrt{-1} t)^*$ across the hyperplane perpendicular to $\alpha$ and $r_{h_{\alpha}}$ is the reflection of $\sqrt{-1} t$ across the hyperplane perpendicular to $h_{\alpha}$.)]

16: NOTATION Depending on the context, $W(\Phi(g_C))$ is the group generated by

$$\{r_{\alpha} : \alpha \in \Phi(g_C)\} \text{ or } \{r_{h_{\alpha}} : h_{\alpha} \in \Phi(g_C)\}.$$

17: N.B. $W(\Phi(g_C))$ operates on $t^*$ and $t$ (extension by complex linearity).

18: LEMMA $\forall \alpha \in \Phi(g_C), \exists n_{\alpha} \in N(T)$ such that the action of $n_{\alpha}$ on $(\sqrt{-1} t)^*$ is given by $r_{\alpha}$ and the action of $n_{\alpha}$ on $\sqrt{-1} t$ is given by $r_{h_{\alpha}}$.

19: THEOREM

- Per $(\sqrt{-1} t)^*$, $W \cong W(\Phi(g_C))$.

- Per $\sqrt{-1} t$, $W \cong W(\Phi(g_C))$.

[Note: It follows from #18 that in either case,

$$W(\Phi(g_C)) \subset W,$$

so the crux is the reversal of this.]
20: LEMMA W operates simply transitively on the set of Weyl chambers in \((\sqrt{-1} t)^*\) or \(\sqrt{-1} t\).

[Note: In other words, there is exactly one element of the Weyl group mapping a given Weyl chamber onto another one.]

21: N.B. It is a corollary that \(|W|\) is the cardinality of the set of Weyl chambers.

22: EXAMPLE Given a Weyl chamber \(C\) (be it in \((\sqrt{-1} t)^*\) or \(\sqrt{-1} t\)), there exists a unique element \(w^\circ \in W\) which maps \(C\) to its negative \(-C\), hence \(w^\circ \Psi(C) = -\Psi(C)\).

[Note: In general, \(-e \notin W\).]

23: THEOREM Let \(C \subset (\sqrt{-1} t)^*\) or \(C \subset \sqrt{-1} t\) be a Weyl chamber -- then its closure \(\overline{C}\) is a fundamental domain for the action of \(W\), i.e., \(\overline{C}\) meets each \(W\) orbit exactly once.

Fix a Weyl chamber \(C \subset (\sqrt{-1} t)^*\) and thereby determine the simple system \(\Psi(C)\), hence \(\Phi^+\).

24: NOTATION \(W(C)\) is the subgroup of \(W(\Phi(g_C))\) generated by the \(r_\alpha(\alpha \in \Psi(C))\).

25: LEMMA

\[W(C) = W(\Phi(g_C)).\]

26: NOTATION Given \(w \in W(\Phi(g_C))\), let \(\ell(w)\) be the smallest \(k\) such that
w can be factored as a product $r_{\alpha_1} \cdots r_{\alpha_k}$, where the $\alpha_i \in \Psi(C)$ (set $\ell(w) = 0$ if $w = e$).

[Note: $\ell(w)$ is referred to as the length of $w$.]

27: **LEMMA** $\ell(w)$ is the number of $\alpha \in \Phi^+$ such that $w\alpha \in \Phi^-$.

28: **APPLICATION** If $w\Phi^+ = \Phi^+$, then $w = e$.

29: **N.B.** The assignment

$$w \rightarrow \det(w) = (-1)^{\ell(w)} \in \{\pm 1\}$$

is a character of $W$.

30: **LEMMA** If $\lambda \in \mathfrak{l}_{\mathfrak{w}t}$, then $\forall w \in W$, $\lambda - w\lambda \in \mathfrak{l}_{\mathfrak{r}t}$.

**PROOF** This is obvious if $w = r_{\alpha}$ for some $\alpha \in \Psi(C)$. In general, $w = r_{\alpha_1} \cdots r_{\alpha_k}$ ($k = \ell(w)$) and one can write

$$\lambda - w\lambda = (\lambda - r_k(\lambda)) + (r_k(\lambda) - r_{k-1}(r_k(\lambda))) + \cdots.$$ 

Let $\alpha_1, \ldots, \alpha_\ell$ be an enumeration of the elements of $\Psi(C)$.

[Note: Recall that $\ell$ is the rank of $G$ or still, the dimension of $\mathfrak{T}$ or still, the dimension of $\mathfrak{g} - \mathfrak{t}$ or still, the dimension of $(\sqrt{-1} \mathfrak{t})^*$.]

31: **DEFINITION** The fundamental weights are the $\omega_i \in \mathfrak{l}_{\mathfrak{w}t}$ per the prescription

$$2 \frac{B(\omega_i, \alpha_j)}{B(\alpha_j, \alpha_j)} = \delta_{ij} \quad (1 \leq i, j \leq \ell).$$

32: **LEMMA** The set $\{\omega_1, \ldots, \omega_\ell\}$ is a basis for $\mathfrak{l}_{\mathfrak{w}t}$. 
33: **DEFINITION** A weight \( \lambda \in L_{wt} \) is said to be **dominant** if \( B(\lambda, \alpha) \geq 0 \) for all \( \alpha \in \Psi(C) \).

34: **N.B.** To say that \( \lambda \in L_{wt} \) is dominant amounts to saying that \( \lambda \in \overline{C} \) (the closure of \( C \)).

35: **LEMMA** A weight \( \lambda \in L_{wt} \) is dominant iff it is a linear combination with nonnegative integral coefficients of the \( \omega_i \).

36: **NOTATION** Put

\[
\rho = \frac{1}{2} \sum_{\alpha \in \Phi^+} \alpha.
\]

37: **N.B.** Ultimately, \( \rho \) depends on the choice of \( C \).

38: **LEMMA** \( \forall w \in W, \)

\[
w \rho = \rho - \sum_{\alpha \in \Phi^+, w^{-1} \alpha \in \Phi^-} \alpha.
\]

39: **APPLICATION** \( \forall \alpha \in \Psi(C), \)

\[r_\alpha(\rho) = \rho - \alpha.\]

[Note: \( \forall \alpha \in \Psi(C), \)

\[r_\alpha(\Phi^+ - \{\alpha\}) = \Phi^+ - \{\alpha\}.\]

40: **LEMMA**

\[
\rho = \omega_1 + \cdots + \omega_\ell.
\]
PROOF Given $\alpha_i \in \Psi(C)$,

\[
\alpha_i = \rho - r_{\alpha_i}(\rho)
\]

\[
= \rho - (\rho - 2 \frac{B(\rho, \alpha_i)}{B(\alpha_i, \alpha_i)})
\]

\[
= 2 \frac{B(\rho, \alpha_i)}{B(\alpha_i, \alpha_i)} \alpha_i
\]

\[
\Rightarrow 2 \frac{B(\rho, \alpha_i)}{B(\alpha_i, \alpha_i)} = 1 \Rightarrow \rho \in \mathcal{L}_{\text{wt}} \text{ (see below)}.
\]

Now write

\[
\rho = n_1 \omega_1 + \cdots + n_\ell \omega_\ell.
\]

Then

\[
1 = 2 \frac{B(\rho, \alpha_j)}{B(\alpha_j, \alpha_j)} = 2 \frac{B(\sum n_i \omega_i, \alpha_j)}{B(\alpha_j, \alpha_j)}
\]

\[
= \sum_{i} n_i 2 \frac{B(\omega_i, \alpha_j)}{B(\alpha_j, \alpha_j)}
\]

\[
= \sum_{i} n_i \delta_{ij} = n_i \Rightarrow 1 = n_i.
\]

Therefore

\[
\rho = \omega_1 + \cdots + \omega_\ell.
\]

41: N.B. It follows that $\rho$ is a dominant weight.
LEMMA Suppose that \( \lambda \in (\sqrt{-1} b)^* \) has the property that
\[
\frac{B(\lambda, \alpha_i)}{B(\alpha_i, \alpha_i)} \in \mathbb{Z} \quad (i = 1, \ldots, \ell).
\]
Then \( \lambda \in I_{\text{wt}} \).

PROOF It is a question of showing that \( \forall \alpha \in \phi^+ \),
\[
\lambda(h_\alpha) = 2 \frac{B(\lambda, \alpha)}{B(\alpha, \alpha)} \in \mathbb{Z}.
\]
To this end, let \( \alpha = \sum_{i=1}^{n} n_i \alpha_i \in \phi^+ \) and proceed by induction on \( |\alpha| = \sum_{i=1}^{\ell} n_i \),
the level of \( \alpha \). The case \( |\alpha| = 1 \) is the hypothesis, so assume that the assertion is true for all levels \( < |\alpha| \). Choose \( \alpha_i \) such that \( B(\alpha, \alpha_i) > 0 \), hence
\[
\beta = r_{\alpha_i}(\alpha) = \alpha - 2 \frac{B(\alpha, \alpha_i)}{B(\alpha_i, \alpha)} \alpha_i
\]
is positive and has level \( < |\alpha| \), thus
\[
2 \frac{B(\lambda, \alpha)}{B(\alpha, \alpha)} = 2 \frac{B(r_{\alpha_i}(\lambda), \beta)}{B(\beta, \beta)}
\]
\[
= 2 \frac{B(\lambda, \beta)}{B(\beta, \beta)} - 2 \frac{B(\lambda, \alpha_i)}{B(\alpha_i, \alpha_i)} \frac{B(\alpha_i, \beta)}{B(\beta, \beta)}
\]
is an integer.
§9. DESCENT

Let \( G \) be a compact connected semisimple Lie group, \( T \subset G \) a maximal torus, \( \phi(\mathfrak{g}_C) \) the roots of the pair \((\mathfrak{g}_C, \mathfrak{t}_C)\), \( C \subset (\sqrt{-1} \mathfrak{t})^* \) a Weyl chamber, \( \psi(=\psi(C)) \) the simple system of roots thereby determined, and \( \phi^+ (\phi^-) \) the positive (negative) roots per \( \psi \).

1: RAPPEL Given a character \( \chi: T \to \mathbb{S}^1 \), there is a commutative diagram

\[
\begin{array}{ccc}
\mathfrak{t} & \xrightarrow{d\chi} & \sqrt{-1} \mathbb{R} \\
\exp \downarrow & & \exp \\
T & \xrightarrow{\chi} & \mathbb{S}^1
\end{array}
\]

and the arrow \( \chi \to d\chi \) implements an identification of \( \hat{T} \) with the lattice

\[ d\hat{T} \equiv \{ \lambda \in (\sqrt{-1} \mathfrak{t})^*: \lambda | \exp^{-1}(e) \subset 2\pi\sqrt{-1} \mathbb{Z} \}. \]

2: N.B. \( d\hat{T} \) is a sublattice of \( L_{wt} \) and

\[ \pi_1(G) \simeq L_{wt}/d\hat{T} \quad (cf. \; \S7, \; \#17). \]

[Note: Therefore \( L_{wt} = d\hat{T} \) iff \( G \) is simply connected.]

3: NOTATION Each \( \lambda \in d\hat{T} \) determines a character \( \xi_\lambda \in \hat{T} \) such that

\[ \xi_\lambda(\exp H) = e^{\lambda(H)} \quad (H \in \mathfrak{t}). \]

4: DEFINITION A function \( f: \mathfrak{t} \to \mathbb{C} \) descends to \( T \) if it factors through the exponential map, i.e., if \( f(H + Z) = f(H) \; \forall \; H \in \mathfrak{t} \) and \( \forall \; Z \in \mathfrak{t} \) such that \( \exp Z = e \).
2.

If \( f : t \to C \) descends to \( T \), then there is a function \( F : T \to C \) such that
\[
F(\exp H) = f(H) \quad (H \in t).
\]

5: EXAMPLE Given \( \lambda \in dT \), the function \( H \mapsto e^{\lambda(H)} \) descends to \( T \) \((F = \xi_{\lambda})\).

6: EXAMPLE Put
\[
\rho = \frac{1}{2} \sum_{\alpha \in \Phi^+} \alpha \quad (\text{cf. } \S 8, \#36).
\]
Then \( \forall w \in W \),
\[
w^p - \rho \in L_{r^t} \subset dT \quad (\text{cf. } \S 8, \#38),
\]
thus the function
\[
H \mapsto e^{(w_p - \rho)(H)}
\]
descends to \( T \) \((F = \xi_{w^p - \rho})\).

7: N.B. It is not claimed nor is it true in general that the function
\[
H \mapsto e^\rho(H)
\]
descends to \( T \).

8: DEFINITION \( \Delta : t \to C \) is the function
\[
H \mapsto \prod_{\alpha \in \Phi^+} (e^{\alpha(H)/2} - e^{-\alpha(H)/2}) \quad (H \in t).
\]
[Note: \( \alpha/2 \) need not belong to \( L_{r^t} \).]

9: LEMMA
\[
\Delta = e^\rho \prod_{\alpha \in \Phi^+} (1 - e^{-\alpha}).
\]
Therefore \( \Delta \) descends to \( T \) iff \( e^\rho \) descends to \( T \).
10: LEMMA $|\Delta|^2$ descends to $T$.

PROOF $\forall H \in \mathfrak{t},$

$$|\Delta(H)|^2 = \Delta(H) \overline{\Delta(H)}$$

$$= e^{\rho(H)} \prod_{\alpha \in \Phi^+} (1 - e^{-\alpha(H)}) e^{\rho(H)} \prod_{\alpha \in \Phi^+} (1 - e^{-\alpha(H)})$$

$$= e^{\rho(H)} \prod_{\alpha \in \Phi^+} (1 - e^{-\alpha(H)}) e^{-\rho(H)} \prod_{\alpha \in \Phi^+} (1 - e^{-\alpha(H)})$$

$$= \prod_{\alpha \in \Phi^+} (1 - e^{-\alpha(H)}) (1 - e^{-\alpha(H)})$$

$$= \prod_{\alpha \in \Phi^+} |1 - e^{-\alpha(H)}|^2,$$

which descends to $T$.

11: LEMMA $\forall t \in T,$

$$\det(\Ad_{G/T}(t^{-1}) - I_{G/T}) = |\Delta(t)|^2.$$

PROOF The complexification of $\mathfrak{g}/\mathfrak{t}$ is the direct sum of the $\mathfrak{g}^\alpha$ on which $t \in T$ acts by $\xi_\alpha(t)$ in the adjoint representation, so

$$\det(\Ad_{G/T}(t^{-1}) - I_{G/T})$$

$$= \prod_{\alpha \in \Phi(g_C)} (\xi_\alpha(t^{-1}) - 1)$$
4.

\[ \prod_{\alpha \in \Phi(g_C)} (1 - \xi_{-\alpha}(t)) = \prod_{\alpha \in \Phi} |1 - \xi_{-\alpha}(t)|^2. \]

[Note: The number of roots is even.]

12: INTEGRATION FORMULA For any continuous function \( f \in C(G) \),

\[ \int_G f(x) d_G(x) = \frac{1}{|W|} \int_T |\Delta(t)|^2 \int_G f(x t x^{-1}) d_G(x) d_T(t) \quad (cf. \S 5, \#13). \]

13: SCHOLIUM For any continuous class function \( f \in CL(G) \),

\[ \int_G f(x) d_G(x) = \frac{1}{|W|} \int_T |\Delta(t)|^2 f(t) d_T(t) \quad (cf. \S 5, \#14). \]

14: REMARK Let \( t \in T \) -- then \( t \in T_{\text{reg}} \) iff

\[ |\Delta(t)|^2 \neq 0 \]

or still, iff

\[ \prod_{\alpha \in \Phi} |1 - \xi_{-\alpha}(t)|^2 \neq 0. \]

15: N.B. Let \( H \in t \) -- then

\[ |\Delta(e^H)|^2 = 2^{\phi(g_C)} \prod_{\alpha \in \Phi} \sin^2 \left( \frac{\alpha(H)}{2\sqrt{-1}} \right). \]

[Note: Bear in mind that \( \alpha(H) \in \sqrt{-1} \mathbb{R} \).]

16: NOTATION Let

\[ \Xi = \{ H \in t : \forall \alpha \in \Phi(g_C), \alpha(H) \notin 2\pi \sqrt{-1} \mathbb{Z} \}. \]
17: **Lemma** $\Xi$ is open and dense in $\mathfrak{t}$. Moreover,
\[
\exp \Xi = \mathfrak{t}^{\text{reg}}.
\]

18: **Rappel** The inclusion $\mathfrak{t} \to G$ induces a bijection between the orbits of $W$ in $\mathfrak{t}$ and the conjugacy classes of $G$ (cf. §4, #16). Consequently, class functions on $G$ are the "same thing" as $W$-invariant functions on $\mathfrak{t}$.

19: **Notation** Given $\lambda \in \mathfrak{t}^\wedge$, define $\psi_\lambda : \Xi \to \mathbb{C}$ by setting
\[
\psi_\lambda(H) = \frac{\sum_{w \in W} \det(w) e^{w(\lambda + \rho)}(H)}{\Delta(H)} \quad (H \in \Xi).
\]

20: **Lemma** $\forall \, w \in W$
\[
w(\Delta) = (-1)^{\ell(w)} \Delta.
\]

Recalling that $\det(w) = (-1)^{\ell(w)}$ (cf. §8, #29), it therefore follows that $\psi_\lambda$ is a $W$-invariant function on $\Xi$.

Next, $\forall \, H \in \Xi$,
\[
\psi_\lambda(H) = \frac{\sum_{w \in W} \det(w) e^{w(\lambda + \rho) - \rho}(H)}{\prod_{\alpha \in \Phi^+} (1 - e^{-\alpha(H)})}.
\]

Since
\[
e^{w(\lambda + \rho) - \rho}(H) = e^{w\lambda(H)} e^{(wp - \rho)}(H),
\]
the numerator of this fraction descends to $\mathfrak{t}$ (cf. #5 and #6). The same also goes for the denominator which is nonzero on $\Xi$. Accordingly, $\psi_\lambda$ descends to a $W$-invariant function on $\mathfrak{t}^{\text{reg}}$, hence extends to a class function on $G^{\text{reg}}$ (cf. §5, #10), denoted still by $\psi_\lambda$. 
1.

§10. CHARACTER THEORY

Let $G$ be a compact connected semisimple Lie group, $T \subseteq G$ a maximal torus, and maintain the assumptions/notation of §9.

1: THEOREM Suppose given a $\Pi \in \hat{G}$, then there is a $\lambda_\Pi \in \mathfrak{d}T$ subject to $\lambda_\Pi + \rho \in \mathbb{C}$ such that $\forall \ x \in G^{\text{reg}},$

$$\chi_\Pi(x) = \psi_{\lambda_\Pi}(x).$$

The proof proceeds by a series of lemmas.

2: NOTATION Given $\gamma \in \mathbb{C}$, define $A_\gamma : \mathfrak{t} \to \mathbb{C}$ by

$$A_\gamma(H) = \sum_{w \in W} \det(w) e^{\gamma(H)}.$$

Rephrased, the claim becomes the assertion that

$$\chi_\Pi(\exp H) A(H) = \psi_{\lambda_\Pi + \rho}(H) \quad (H \in \mathbb{C})$$

for some $\lambda_\Pi \in \mathfrak{d}T$ subject to $\lambda_\Pi + \rho \in \mathbb{C}$.

3: NOTATION $\mathfrak{d}T(C)$ is the subset of $\mathfrak{d}T$ consisting of those $\lambda$ such that $\lambda + \rho \in \mathbb{C}$, say $\mathfrak{d}T(C) = \{ \lambda_k \}$.

[Note: It turns out that $\mathfrak{d}T(C) = \hat{\mathfrak{d}T} \cap \mathbb{C}$ (cf. #9).]

4: LEMMA There exist integers $m_k$ such that $\forall \ H \in \mathbb{C},$

$$\chi_\Pi(\exp H) A(H) = \sum_k m_k A_{\lambda_k + \rho}(H).$$
2.

[Note: The point of departure is the fact that $\chi_\Pi|T$ decomposes as a finite sum

$$\sum_{\lambda \in \Delta T} n_\lambda \xi_\lambda \quad (n_\lambda \in \mathbb{Z}_{\geq 0}).$$

Proceeding,

$$1 = \int_G |\chi_\Pi(x)|^2 d_G(x)$$

$$= \frac{1}{|W|} \int_T |\Delta(t)|^2 |\chi_\Pi(t)|^2 d_T(t) \quad (\text{cf. §9, #13}).$$

5: N.B. The function

$$|\sum m_k \lambda_k |^2$$

descends to $T$ (because $|\Delta|^2$ descends to $T$ (cf. §9, #10)).

Therefore

$$1 = \frac{1}{|W|} \int_T |\sum m_k \lambda_k |^2 d_T(t).$$

6: Lemma The function

$$A_{\lambda_k + \rho} \overline{A_{\lambda_k', + \rho}}$$

$$= (e^{-\rho} A_{\lambda_k + \rho})(e^{-\rho} A_{\lambda_k', + \rho})$$

descends to $T$ (cf. §9, #6).

Therefore

$$\frac{1}{|W|} \int_T A_{\lambda_k + \rho} \overline{A_{\lambda_k', + \rho}} d_T(t)$$
\[
= \frac{1}{|W|} \int_T \left( e^{-\rho A_{\lambda_k+\rho}} \right) \left( e^{-\rho A_{\lambda_{k'}+\rho}} \right) d_T(t)
\]

\[
= \frac{1}{|W|} \sum_{w,w' \in W} \det (w w') \int_T \xi_w(\lambda_k+\rho) - \rho \xi_{w'}(\lambda_{k'}+\rho) + \rho d_T(t).
\]

And

\[
\int_T \xi_w(\lambda_k+\rho) - \rho \xi_{-(w' (\lambda_{k'}+\rho) - \rho)} d_T(t)
\]

\[= 1 \iff w(\lambda_k+\rho) - \rho = w'(\lambda_{k'}+\rho) - \rho\]

\[\iff w(\lambda_k+\rho) = w'(\lambda_{k'}+\rho)\]

\[\iff w = w' \text{ and } k = k'\]

but is zero otherwise.

Therefore

\[
\frac{1}{|W|} \int_T A_{\lambda_k+\rho} A_{\lambda_{k'}+\rho} d_T(t) = \begin{cases} 
1 & \text{if } k = k' \\
0 & \text{if } k \neq k'.
\end{cases}
\]

Matters then reduce to the equation

\[1 = \sum_k m_k^2\]

However, the \(m_k \in \mathbb{Z}\), hence all but one are zero. Consequently, there is a \(\lambda_\Pi \in dT\) subject to \(\lambda_\Pi + \rho \in C\) such that \(\forall H \in \Xi\),

\[\chi_\Pi(\exp H) \Delta(H) = \pm A_{\lambda_\Pi+\rho}(H).\]

7: LEMMA The \(A_\gamma \ (\gamma \in C)\) are linearly independent over \(\mathbb{Z}\).
4.

[Given $\gamma, \gamma' \in \mathbb{C}$,

$$\langle A_\gamma, A_{\gamma'} \rangle = \begin{cases} 1 & \text{if } \gamma = \gamma' \\ 0 & \text{if } \gamma \neq \gamma', \end{cases}$$

the inner product $\langle , \rangle$ being by definition the multiplicity of the "zero weight" in

$$\frac{1}{|W|} \left[ \sum_{w \in W} \det(w) e^{w\gamma} \right] \left[ \sum_{w' \in W} \det(w') e^{-w'\gamma'} \right]$$

$$= \frac{1}{|W|} \sum_{w,w' \in W} \det(ww') e^{w\gamma - w'\gamma'}.$$ 

But

$$w\gamma - w'\gamma' = 0 \Rightarrow \gamma = w^{-1}w'\gamma'$$

$$\Rightarrow w = w' \Rightarrow \gamma = \gamma',$$

so the number of solutions is $|W|$ if $\gamma = \gamma'$ and is zero otherwise.]

3: APPLICATION The linear function $\lambda + \rho \in \mathbb{C}$ is unique.

9: LEMMA Let $\lambda \in \mathbb{C}$ -- then

$$\lambda + \rho \in \mathbb{C} \Leftrightarrow \lambda \in \bar{\mathbb{C}}.$$ 

PROOF $\forall \alpha \in \mathfrak{C}$,

$$2 \frac{B(\rho, \alpha)}{B(\alpha, \alpha)} = 1 \quad (\text{cf. } \S 8, \#40)$$

and

$$2 \frac{B(\lambda, \alpha)}{B(\alpha, \alpha)} \in \mathbb{Z} \quad (\lambda \in \mathbb{C}, \alpha \in \mathbb{C}).$$
The stated equivalence then follows upon writing

\[ \frac{2B(\lambda+\rho,a_i)}{B(a_i,a_i)} = 2 \frac{B(\rho,a_i)}{B(a_i,a_i)} + 2 \frac{B(\lambda,a_i)}{B(a_i,a_i)} \]

\[ = 1 + 2 \frac{B(\lambda,a_i)}{B(a_i,a_i)}. \]

10: APPLICATION

\[ \lambda_\Pi + \rho \in C \Rightarrow \lambda_\Pi \in \overline{C}. \]

Return now to the expression

\[ \chi_\Pi(\exp H)\Delta(H) = \pm \lambda_\Pi^{\Pi+\rho}(H) \]

valid for \( H \in \Xi \), the objective then being to establish that it is the plus sign which obtains.

11: LEMMA \( \forall \, H \in \mathfrak{t}, \)

\[ \Delta(H) = \sum_{w \in W} \det(w) e^{wp}(H). \]

[Note: There is no vicious circle here in that the formula can be derived by direct (albeit somewhat tedious) manipulation, the derivation being independent of the preceding considerations (but consistent with the final outcome).]

From this it follows that \( \forall \, H \in \Xi, \)

\[ \pm \chi_\Pi(\exp H) = \frac{\sum_{w \in W} \det(w) e^{wp}(H)}{\sum_{w \in W} \det(w) e^{wp}(H)} \]

\[ = \frac{w(\lambda_\Pi^{\Pi+\rho})(H)}{w(\lambda_\Pi^{\Pi+\rho})(H)}. \]
12: NOTATION Define $H_{\rho} \in \sqrt{-1} t$ by the relation

$$\rho(H) = B(H, H_{\rho}) (H \in \sqrt{-1} t) \quad (\text{cf. \S6, \#21}).$$

13: LEMMA $\sqrt{-1} t H_{\rho} \in \mathbb{E}$ for small positive $t$.

14: LEMMA

$$\lim_{t \to 0} \chi_{\Pi}(\exp \sqrt{-1} t H_{\rho}) = d_{\Pi}. \quad [\text{For } \chi_{\Pi}|T \text{ is continuous and } d_{\Pi} = \chi_{\Pi}(e).]$$

15: APPLICATION

$$d_{\Pi} = \lim_{t \to 0} \chi_{\Pi}(\sqrt{-1} t H_{\rho}).$$

16: SUBLEMMA $\forall w \in W$,

$$w(\lambda_{\Pi} + \rho)(\sqrt{-1} t H_{\rho})$$

$$= \sqrt{-1} t (\lambda_{\Pi} + \rho)(w^{-1}H_{\rho})$$

$$= \sqrt{-1} t B(H_{\lambda_{\Pi} + \rho}, w^{-1}H_{\rho})$$

$$= \sqrt{-1} t B(wh_{\lambda_{\Pi} + \rho}, w^{-1}H_{\rho})$$

$$= \sqrt{-1} t \rho(wh_{\lambda_{\Pi} + \rho})$$

$$= (w^{-1}\rho)(\sqrt{-1} t H_{\lambda_{\Pi} + \rho}).$$
17: LEMMA

$$
\lim_{t \to 0} \lambda_{\Pi} (\sqrt{-1} t H_\rho) = \frac{\prod_{\alpha \in \Phi^+} B(\lambda_{\Pi}^+ + \rho, \alpha)}{\prod_{\alpha \in \Phi^+} B(\rho, \alpha)}.
$$

PROOF Write

$$
\sum_{w \in W} \det(w) e^{w(\lambda_{\Pi} + \rho) (\sqrt{-1} t H_\rho)}
$$

$$
= \sum_{w \in W} \det(w) e^{(w^{-1} \rho) (\sqrt{-1} t H_{\lambda_{\Pi}^+ + \rho})}
$$

$$
= \sum_{w \in W} \det(w^{-1}) e^{(w^{-1} \rho) (\sqrt{-1} t H_{\lambda_{\Pi}^+ + \rho})}
$$

$$
= \sum_{w \in W} \det(w) e^{(w \rho) (\sqrt{-1} t H_{\lambda_{\Pi}^+ + \rho})}
$$

$$
= \sum_{w \in W} \det(w) e^{\Delta (\sqrt{-1} t H_{\lambda_{\Pi}^+ + \rho})}
$$

$$
= \prod_{\alpha \in \Phi^+} (e^{\alpha (\sqrt{-1} t H_{\lambda_{\Pi}^+ + \rho})/2} - e^{-\alpha (\sqrt{-1} t H_{\lambda_{\Pi}^+ + \rho})/2})
$$

$$
= \prod_{\alpha \in \Phi^+} e^{\sqrt{-1} t \alpha (H_{\lambda_{\Pi}^+ + \rho}) + o(1)}
$$

$$
= (\sqrt{-1} t) |\phi^+| \prod_{\alpha \in \Phi^+} B(\lambda_{\Pi}^+ + \rho, \alpha) + o(1).
$$

Analogously,

$$
\sum_{w \in W} \det(w) e^{w \rho (\sqrt{-1} t H_\rho)}
$$

$$
= \prod_{\alpha \in \Phi^+} B(\lambda_{\Pi}^+ + \rho, \alpha) + o(1).
$$
Taking the limit as $t \to 0$ then finishes the proof.

18: N.B. Both $\rho$ and $\lambda_\Pi + \rho$ belong to $C$, thus $\forall \alpha \in \Phi^+$,
\[ B(\rho, \alpha) > 0 \text{ and } B(\lambda_\Pi + \rho, \alpha) > 0, \]
so
\[ \lim_\limits{t \to 0} \frac{\Psi_{\lambda_\Pi} (\sqrt{t} t H \rho)}{\lambda_\Pi} > 0. \]

19: APPLICATION
\[ d_\Pi = \lim_\limits{t \to 0} \frac{\Psi_{\lambda_\Pi} (\sqrt{t} H \rho) t + \Omega}{\lambda_\Pi}. \]
I.e.: The plus sign prevails.

20: SCHOLIUM
\[ d_\Pi = \frac{\prod_{\alpha \in \Phi^+} B(\lambda_\Pi + \rho, \alpha)}{\prod_{\alpha \in \Phi^+} B(\rho, \alpha)}. \]

21: LEMMA The arrow from $\hat{G}$ to $dT \cap \hat{C}$ that sends $\Pi$ to $\lambda_\Pi$ is well-defined (cf. #8) and injective.

PROOF Given $\Pi_1, \Pi_2 \in \hat{G}$, suppose that $\lambda_\Pi_1 = \lambda_\Pi_2$ -- then $\lambda_\Pi_1 + \rho = \lambda_\Pi_2 + \rho$, hence
\[ \Psi_{\lambda_\Pi_1} = \Psi_{\lambda_\Pi_2}, \]
which implies that $\chi_{\Pi_1} = \chi_{\Pi_2}$ on $G^{reg}$ or still, by continuity, $\chi_{\Pi_1} = \chi_{\Pi_2}$ on $G$, so
$\Pi_1 = \Pi_2$. 
22: LEMMA The arrow from \( \hat{G} \) to \( d \hat{T} \cap \hat{C} \) that sends \( \Pi \) to \( \lambda_\Pi \) is surjective.

PROOF Fix a \( \lambda \in d \hat{T} \cap \hat{C} \) --- then

\[
\int_G |\mathcal{U}_\lambda(x)|^2 d_G(x) = \frac{1}{|W|} \int_{\mathfrak{m} \text{reg}} |\Delta(t)|^2 |\mathcal{U}_\lambda(t)|^2 d_T(t)
\]

\[
= \frac{1}{|W|} \int_T A_{\lambda+\rho} \frac{A_{\lambda+\rho}}{A_\lambda} d_T(t)
\]

\[
= 1.
\]

Therefore \( \mathcal{U}_\lambda \) is an \( L^2 \) class function (cf. §2, #17). Now fix a \( \Pi_0 \in \hat{G} \):

\[
\langle \mathcal{U}_\lambda, \chi_{\Pi_0} \rangle
\]

\[
= \int_G \mathcal{U}_\lambda(x) \overline{\chi_{\Pi_0}(x)} d_G(x)
\]

\[
= \frac{1}{|W|} \int_{\mathfrak{m} \text{reg}} |\Delta(t)|^2 |\mathcal{U}_\lambda(t)\overline{\chi_{\Pi_0}(t)}| d_T(t)
\]

\[
= \frac{1}{|W|} \int_{\mathfrak{m} \text{reg}} |\Delta(t)|^2 |\mathcal{U}_\lambda(t)\overline{\chi_{\lambda_0}(t)}| d_T(t)
\]

\[
= \begin{cases} 
1 & \text{if } \lambda_{\Pi_0} = \lambda \\
0 & \text{if } \lambda_{\Pi_0} \neq \lambda
\end{cases}
\]

\[
=> \mathcal{U}_\lambda = \sum_{\Pi \in \hat{G}} \langle \mathcal{U}_\lambda, \chi_{\Pi} \rangle \chi_{\Pi} \quad \text{(cf. §2, #19)}
\]

\[
= \chi_{\Pi}
\]

for a unique \( \Pi \in \hat{G} \) with \( \lambda_{\Pi} = \lambda \).
23: SCHOLIUM

\[
\begin{align*}
\hat{G} & \leftrightarrow d\hat{T} \cap \bar{C} \\
\Pi & \leftrightarrow \lambda_\Pi.
\end{align*}
\]

[Note: \(W\) operates on \(d\hat{T}\) and \(d\hat{T} \cap \bar{C}\) is a fundamental domain for this action (cf. §8, #23), hence \(\hat{G}\) is parametrized by the orbits of \(W\) in \(d\hat{T}\).]

24: N.B.

\[
\lambda_{\Pi^*} = -\omega^\bullet \lambda_\Pi \quad (\text{cf. §8, #22}).
\]

25: REMARK It is clear that if \(\lambda_\Pi = 1_G\), then \(\lambda_\Pi = 0\).

26: LEMMA In the restriction of \(\chi_\Pi\) to \(T\), \(\xi_\lambda_\Pi\) occurs with multiplicity 1.

APPENDIX

There are two directions in which the theory can be extended.

- Drop the assumption that \(G\) is semisimple and work with an arbitrary compact connected Lie group.
- Drop the assumption that \(G\) is connected and work with an arbitrary compact Lie group.

As regards the first point, no essential difficulties are encountered. As regards the second point, however, there are definitely some subtleties (see Chapter 1 of D. Vogan's book "Unitary Representations of Reductive Lie Groups").

NOTATION Let \(G\) be a compact semisimple Lie group, \(T \subset G\) a maximal torus,
C \subset \sqrt{-1} \text{ is a Weyl chamber and let}

\[ N_G(C) = \{ x \in G : \text{Ad}(x)C \subset C \}. \]

**Lemma**

\[ N_G(C) \cap G^0 = T, \quad N_G(C)G^0 = G. \]

Therefore

\[ G/G^0 \cong N_G(C)/T. \]

N.B. Each element of G is conjugate to an element of \( N_G(C) \).
§11. THE INVARIANT INTEGRAL

Let $G$ be a compact connected semisimple Lie group, $T \subset G$ a maximal torus etc.

1: NOTATION Set

$$\pi = \prod_{\alpha \in \Phi^+} \alpha \equiv \prod_{\alpha > 0} \alpha.$$ 

2: LEMMA $\pi$ is a homogeneous polynomial of degree $r(= \vert \Phi^+ \vert)$ and $\forall w \in W,$ 

$$w\pi = \det(w)\pi.$$ 

3: LEMMA If $p$ is a homogeneous polynomial such that $\forall w \in W,$ 

$$wp = \det(w)p,$$ 

then $p$ can be written as $\pi P,$ where $P$ is a homogeneous $W$-invariant polynomial.

4: N.B. $P = 0$ if $\deg p < r$ and $P = C$ (a constant) if $\deg p = r.$

5: DEFINITION Given $f \in C^\infty(g)$ and $H \in t,$ put 

$$\phi_f(H) = \pi(H) \int_G f(\text{Ad}(x)H)d_G(x),$$ 

the invariant integral of $f$ at $H.$

6: FUNCTIONAL EQUATION $\forall w \in W \ (w = nT),$ 

$$\phi_f(wH) = \pi(wh) \int_G f(\text{Ad}(x)H)d_G(x)$$

$$= \det(w)\pi(H) \int_G f(\text{Ad}(x)\text{Ad}(n)H)d_G(x)$$

$$= \det(w)\pi(H) \int_G f(\text{Ad}(xn)H)d_G(x)$$

$$= \det(w)\pi(H) \int_G f(\text{Ad}(x)H)d_G(x)$$
2.

\[ = \det(w) \phi_f(H). \]

7: LEMMA

\[ f \in C^\infty(g) \Rightarrow \phi_f \in C^\infty(t). \]

8: LEMMA

\[ f \in C_c^\infty(g) \Rightarrow \phi_f \in C_c^\infty(t). \]

9: LEMMA

\[ f \in C(g) \Rightarrow \phi_f \in C(t). \]

[If \( D \in \mathcal{D}(g) \) is a polynomial differential operator, then there exist a finite number of elements \( D_1, \ldots, D_p \in \mathcal{D}(g) \) and analytic functions \( a_1, \ldots, a_p \) on \( G \) such that
\[ \forall x \in G, \]
\[ Ad(x)D = \sum_{i=1}^{p} a_i(x)D_i. \]

[Note: An automorphism of \( g \) extends to an automorphism of \( \mathcal{D}(g) \).]

10: NOTATION Set

\[ \tilde{\pi} = \prod_{\alpha \in \Phi^+} H_\alpha = \prod_{\alpha > 0} H_\alpha. \]

11: N.B. \( \theta(\tilde{\pi})(\pi) \) is a constant (explicated infra).

[The point is that \( \pi \) is a homogeneous polynomial of degree \( r \) and \( \theta(\tilde{\pi}) \) is a polynomial differential operator of degree \( r \).]

12: RAPPEL For the record,

\[ \theta^H_\alpha(f) \bigg|_H = \frac{d}{dt} f(H + tH_\alpha) \bigg|_{t=0}. \]
In particular, if $f$ is linear, then

$$3\alpha(H)(f) = f(H_al),$$
a constant.]

Put

$$F(H) = \int_G f(Ad(x)H) d_G(x) \quad (H \in \mathfrak{h}).$$

Then

$$\left(3\alpha(\tilde{\pi}) \circ \pi \right) F|_{H=0} = \left.F(H; 3\alpha(\tilde{\pi}) \circ \pi) \right|_{H=0}
= 3\alpha(\tilde{\pi})(\pi) F(0)
= 3\alpha(\tilde{\pi})(\pi)f(0).$$

13: THEOREM

$$3\alpha(\tilde{\pi})(\pi) = |W| \prod_{\alpha > 0} B(\rho, \alpha).$$

PROOF The sum

$$\sum_{w \in W} \det(w)(wp)^k$$
is a homogeneous polynomial of degree $k$ which transforms according to the determinant per the action of $W$, hence vanishes if $0 \leq k < r$ but if $k = r$,

$$\frac{1}{r!} \sum_{w \in W} \det(w)(wp)^r = C(\rho) \pi$$

for some constant $C(\rho)$ (cf. #4). To calculate $C(\rho)$, note that $\rho^r$ is a homogeneous polynomial of degree $r$, thus $3\alpha(\tilde{\pi})(\rho)^r$ is a constant, so

$$3\alpha(\tilde{\pi})(\det(w)(wp)^r)$$

$$= w(3\alpha(\tilde{\pi})(\rho)^r)$$
Therefore, on the one hand,

\[ \mathfrak{A}(\tilde{\pi}) \left( \frac{1}{r!} \sum_{w \in \mathcal{W}} \det(w) (wp)^r \right) \]

\[ = \frac{1}{r!} \sum_{w \in \mathcal{W}} \mathfrak{A}(\tilde{\pi}) (\det(w) (wp)^r) \]

\[ = \frac{1}{r!} |W| r! \prod_{\alpha > 0} B(\rho, \alpha) \]

\[ = |W| \prod_{\alpha > 0} B(\rho, \alpha), \]

while on the other

\[ \mathfrak{A}(\tilde{\pi}) \left( \frac{1}{r!} \sum_{w \in \mathcal{W}} \det(w) (wp)^r \right) \]

\[ = C(\rho) \mathfrak{A}(\tilde{\pi}) (\pi). \]

Consequently,

\[ |W| \prod_{\alpha > 0} B(\rho, \alpha) = C(\rho) \mathfrak{A}(\tilde{\pi}) (\pi) \]

\[ \Rightarrow \]

\[ \frac{1}{r!} \sum_{w \in \mathcal{W}} \det(w) (wp)^r \]

\[ = \frac{|W| \prod_{\alpha > 0} B(\rho, \alpha)}{\mathfrak{A}(\tilde{\pi}) (\pi)} \pi. \]
Let $H = \sqrt{-1} \ t_H$, (cf. §10, #13) and write $\lim$ in place of $\lim_{t \to 0}$:

$$1 = \frac{\Delta(H)}{\Delta(H)}$$

$$= \frac{\sum_{w \in \mathcal{W}} \det(w) e^{wp(H)}}{\prod_{\alpha > 0} (e^{\alpha(H)/2} - e^{-\alpha(H)/2})}$$

$$= \lim_{H \to 0} \frac{\sum_{w \in \mathcal{W}} \det(w) e^{wp(H)}}{\prod_{\alpha > 0} (e^{\alpha(H)/2} - e^{-\alpha(H)/2})}$$

$$= \lim_{H \to 0} \left[ \frac{\sum_{w \in \mathcal{W}} \det(w) e^{wp(H)}}{\prod_{\alpha > 0} (e^{\alpha(H)/2} - e^{-\alpha(H)/2})} \times \frac{\sum_{w \in \mathcal{W}} \det(w) e^{wp(H)}}{\prod_{\alpha > 0} (e^{\alpha(H)/2} - e^{-\alpha(H)/2})} \right]$$

$$= \lim_{H \to 0} \frac{\sum_{w \in \mathcal{W}} \det(w) e^{wp(H)}}{\prod_{\alpha > 0} (e^{\alpha(H)/2} - e^{-\alpha(H)/2})}$$

which upon expansion of the exponentials equals

$$\lim_{H \to 0} (C(\rho) + o(1)) = C(\rho)$$

$$=>$$
6.

\[ 1 = C(\rho) = \frac{|W| \prod_{\alpha > 0} B(\rho, \alpha)}{\delta(\pi)(\pi)} \]

=>

\[ \delta(\pi)(\pi) = |W| \prod_{\alpha > 0} B(\rho, \alpha). \]

14: APPLICATION Given \( f \in C^\infty(g) \),

\[ \phi_f(0; \delta(\pi)) = (\delta(\pi) \phi_f)(0) \]

\[ = (|W| \prod_{\alpha > 0} B(\rho, \alpha)) f(0). \]
§12. PLANCHEREL

Keeping to the overall setup of §11, assume in addition that $G$ is simply connected, so

$$L_w = d\hat{T} \quad \text{(cf. §7, #17)}$$

and $e^\Theta$ descends to $T$, so does $\Delta$, thus

$$\Delta(t) = \xi_\rho(t) \prod_{\alpha > 0} (1 - \xi_\alpha(t^{-1})) \quad (t \in T).$$

1: NOTATION Put

$$w = L_w, \quad w^+ = L_w \cap \tilde{C}.$$  

2: N.B. The elements of $w^+$ are the dominant weights (cf. §8, #34).

3: NOTATION Given $\Lambda \in w^+$, $\Pi_\Lambda$ is the irreducible unitary representation of $G$ associated with $\Lambda$, $\chi_\Lambda$ its character,

$$d_\Lambda = \frac{\prod_{\alpha > 0} B(\Lambda + \rho, \alpha)}{\prod_{\alpha > 0} B(\rho, \alpha)}$$

its dimension (cf. §10, #20).

4: N.B. On $T_{\text{reg}}$,

$$\chi_\Lambda(t) \Delta(t) = \sum_{w \in W} \det(w) \xi_w(\Lambda + \rho)(t).$$

It is well known that

$$C^\infty(G) * C^\infty(G) = C^\infty(G),$$
so on the basis of §2, #15, the Plancherel theorem is in force:

\[ f(e) = \sum_{\Pi \in G} d_{\Pi} \text{tr}(\Pi(f)) \]

\[ = \sum_{\Pi \in G} d_{\Pi} \int_{\mathcal{G}} f(x) \chi_{\Pi}(x) d_{\Pi}(x) \]

or still,

\[ f(e) = \sum_{\Lambda \in \mathcal{W}^+} \mathcal{d}_{\Lambda} \int_{\mathcal{G}} f(x) \chi_{\Lambda}(x) d_{\Lambda}(x). \]

Our objective now will be to give another proof of this relation which is independent of the factorization theory for \( C^\infty(G) \) but hinges instead on the result formulated in §11, #13.

5: NOTATION Given \( f \in C^\infty(G) \) and \( t \in T \), put

\[ F_f(t) = \Delta(t) \int_{\mathcal{G}} f(xt^{-1}) d_G(x), \]

the invariant integral of \( f \) at \( t \).

6: LEMMA

\[ F_f \in C^\infty(T). \]

Owing to §9, #12,

\[ \int_{\mathcal{G}} f(x) d_G(x) = \frac{1}{|W|} \int_{\mathcal{T}} |\Delta(t)|^2 \int_{\mathcal{G}} f(xt^{-1}) d_G(x) d_T(t) \]

which equals

\[ \frac{1}{|W|} \int_{\mathcal{T}} \zeta(T) \Delta(t) \int_{\mathcal{G}} f(xt^{-1}) d_G(x) d_T(t) \]

or still,

\[ \frac{1}{|W|} \int_{\mathcal{T}} \zeta(T) F_f(t) d_T(t) \]
or still,
\[
\frac{(-1)^r}{|W|} \sum_{\Lambda \in W^+} \mathcal{F}_{ \Lambda}(t) \mathcal{F}_{ \Lambda}(t) d_T(t) \quad (r = |\mathcal{F}|).
\]

Therefore
\[
\sum_{\Lambda \in W^+} \mathcal{F}_{ \Lambda}(t) \mathcal{F}_{ \Lambda}(t) d_T(t) = \frac{(-1)^r}{|W|} \sum_{\Lambda \in W^+} \mathcal{F}_{ \Lambda}(t) \mathcal{F}_{ \Lambda}(t) d_T(t).
\]

7: LEMMA \( \forall w \in W, \)
\[
\det(w) \prod_{\alpha > 0} B(\Lambda + \rho, \alpha) = \prod_{\alpha > 0} B(w(\Lambda + \rho), \alpha).
\]

Proceeding,
\[
\frac{(-1)^r}{|W|} \sum_{\Lambda \in W^+} \mathcal{F}_{ \Lambda}(t) \mathcal{F}_{ \Lambda}(t) d_T(t)
\]
\[
= \frac{(-1)^r}{|W|} \sum_{\Lambda \in W^+} \mathcal{F}_{ \Lambda}(t) \mathcal{F}_{ \Lambda}(t) d_T(t)
\]
\[
= \frac{(-1)^r}{|W|} \sum_{\Lambda \in W^+} \mathcal{F}_{ \Lambda}(t) \mathcal{F}_{ \Lambda}(t) d_T(t)
\]
\[
= \frac{(-1)^r}{|W|} \sum_{\Lambda \in W^+} \mathcal{F}_{ \Lambda}(t) \mathcal{F}_{ \Lambda}(t) d_T(t)
\]
\[
= \frac{(-1)^r}{|W|} \sum_{\Lambda \in W^+} \mathcal{F}_{ \Lambda}(t) \mathcal{F}_{ \Lambda}(t) d_T(t)
\]
\[
= \frac{(-1)^r}{|W|} \sum_{\Lambda \in W^+} \mathcal{F}_{ \Lambda}(t) \mathcal{F}_{ \Lambda}(t) d_T(t)
\]
\[
= \frac{(-1)^r}{|W|} \sum_{\Lambda \in W^+} \mathcal{F}_{ \Lambda}(t) \mathcal{F}_{ \Lambda}(t) d_T(t)
\]
\[
= \frac{(-1)^r}{|W|} \sum_{\Lambda \in W^+} \mathcal{F}_{ \Lambda}(t) \mathcal{F}_{ \Lambda}(t) d_T(t)
\]
the $\lambda \in \mathcal{W}$ for which $\bigwedge_{\alpha>0} B(\lambda, \alpha) = 0$ making no contribution.

8: REMARK The elements $\lambda \in \mathcal{W}$ such that $w\lambda \neq \lambda$ when $w \neq e$ ($w \in \mathcal{W}$) are in a one-to-one correspondence with the pairs $(\Lambda, w) \in \mathcal{W}^+ \times \mathcal{W}$ via the arrow $(\Lambda, w) \rightarrow w(\Lambda + \rho)$.

To isolate $f(e)$, put $\check{f}(X) = f(\exp X)$ ($X \in \mathfrak{g}$) -- then $\check{f} \in C^\infty(\mathfrak{g})$ and $\forall H \in \mathfrak{t},$

$$F_f(\exp H) = \Delta(\exp H) \int_G f(x(\exp H)x^{-1})d_G(x)$$

$$= \Delta(\exp H) \int_G f(\exp(\text{Ad}(x)H))d_G(x)$$

$$= \Delta(\exp H) \int_G \check{f}(\text{Ad}(x)H)d_G(x).$$

9: LEMMA Let $\lambda$ be a linear function on $\mathfrak{t}_C$ -- then there exists a unique $\text{Ad} \ G$ invariant analytic function $\Gamma^\lambda$ on $\mathfrak{g}$ such that $\forall H \in \mathfrak{t},$

$$\Gamma^\lambda(H) \eta(H) = \sum_{w \in \mathcal{W}} \det(w)e^{\lambda(H)}.$$ 

10: APPLICATION Take $\lambda = \rho$ -- then there exists a unique $\text{Ad} \ G$ invariant analytic function $\Gamma^\rho$ on $\mathfrak{g}$ such that $\forall H \in \mathfrak{t},$

$$\Gamma^\rho(H) \eta(H) = \sum_{w \in \mathcal{W}} \det(w)e^{\rho(H)}$$

$$= \Delta(H) \quad \text{(cf. } \S 10, \#11).$$

Therefore

$$\Delta(\exp H) \int_G \check{f}(\text{Ad}(x)H)d_G(x)$$

$$= \Gamma^\rho(H) \eta(H) \int_G \check{f}(\text{Ad}(x)H)d_G(x)$$
= \pi (H) \Gamma \rho (H) \int \frac{\mathcal{Y}}{f} (\text{Ad}(x) H) d_G(x) \\
= \pi (H) \int \Gamma \rho (H) \frac{\mathcal{Y}}{f} (\text{Ad}(x) H) d_G(x) \\
= \pi (H) \int \Gamma \rho (\text{Ad}(x) H) \frac{\mathcal{Y}}{f} (\text{Ad}(x) H) d_G(x) \\
= \phi \frac{\mathcal{Y}}{f} (H).

Summary: \forall H \in \mathfrak{t},

\mathcal{F}_f(\exp H) = \phi \frac{\mathcal{Y}}{f} (H).

\textbf{II: SUBLEMMA} In \{H: \pi(H) \neq 0\},

\Gamma \rho (0) = \lim_{H \to 0} \frac{\Delta(H)}{\pi (H)}

= \lim_{H \to 0} \frac{\prod_{\alpha > 0} (e^{\alpha(H)/2} - e^{-\alpha(H)/2})}{\prod_{\alpha > 0} \alpha(H)}

= \lim_{H \to 0} \prod_{\alpha > 0} \left[ \frac{e^{\alpha(H)/2} - e^{-\alpha(H)/2}}{\alpha(H)} \right]

= 1.

Next

\mathcal{F}_f(\exp H) = \phi \frac{\mathcal{Y}}{f} (H)

=>

\mathcal{E}(\tilde{\mathcal{N}}) \mathcal{F}_f(\exp H) = \mathcal{E}(\tilde{\mathcal{N}}) \phi \frac{\mathcal{Y}}{f} (H)
And

\( (\Gamma_\rho \check{f}) (0) = \Gamma_\rho (0) \hat{f}(0) = \check{f}(0) = f(e). \)

Therefore

\[
 f(e) = \frac{1}{|W|} \prod_{\alpha > 0} B(\rho, \alpha) \lim_{H \to 0} F_f(\exp H; \varphi(\pi)).
\]

12: NOTATION Given \( \lambda \in \mathcal{W} \), put

\[
\hat{F}_f(\lambda) = \int_T F_f(t) \xi_\lambda(t) d_\lambda(t),
\]

the Fourier transform of \( F_f \).

13: N.B. Assume that the Haar measure on \( \hat{T} \) is normalized so that Fourier inversion is valid (thus each \( \lambda \in \mathcal{W} \) is assigned mass 1).

Write

\[
\lim_{H \to 0} F_f(\exp H; \varphi(\pi))
\]

\[
= \int_{\hat{T}} \hat{F}_f(\lambda) \lim_{H \to 0} \xi_\lambda(\exp H; \varphi(\pi)) d_\lambda(\lambda)
\]

\[
= \int_{\hat{T}} \hat{F}_f(\lambda) \lim_{H \to 0} \varphi(\pi) e^{-\lambda(H)} d_\lambda(\lambda)
\]
7.

\[
(-1)^r \sum_{\lambda \in \mathbb{W}} \frac{1}{|W|} \prod_{\alpha > 0} B(\rho, \alpha) \chi_\lambda(x) d_\lambda(x),
\]

the relation at issue.
§13. DETECTION

Let \( G \) be a compact group.

1: DEFINITION The character ring \( X(G) \) is the free abelian group generated by the irreducible characters of \( G \) (i.e., by the \( \chi_\Pi \) \((\Pi \in \hat{G}\)) under pointwise addition and multiplication with unit \( 1_G \).

2: DEFINITION An element of \( X(G) \) is called a \textit{virtual} character.

3: NOTATION \( CL(G) \) is the subspace of \( C(G) \) comprised of the continuous class functions (cf. §2, #27).

4: LEMMA A class function \( f \in CL(G) \) is a virtual character of \( G \) iff

\[
\langle f, \chi_\Pi \rangle = \int_G f(x) \overline{\chi_\Pi(x)} d_G(x) \in \mathbb{Z}
\]

for all \( \Pi \in \hat{G} \).

PROOF The condition is obviously necessary. As for its sufficiency, we have

\[
||f||^2 = \sum_{\Pi \in \hat{G}} \langle f, \chi_\Pi \rangle^2 \quad (\text{cf. \S2, \#19}),
\]

hence

\[
\langle f, \chi_\Pi \rangle = 0
\]

for all but finitely many \( \chi_\Pi \), say \( \chi_{\Pi_1}, \ldots, \chi_{\Pi_n} \) and then

\[
f = \sum_{i=1}^{n} \langle f, \chi_{\Pi_i} \rangle \chi_{\Pi_i} \quad \text{(ibid.).}
\]

[Note: A priori, this is an equality in the \( L^2 \)-sense, hence is valid almost everywhere. But both sides are continuous, thus the equality is valid everywhere.]
Let $G$ be a compact connected Lie group.

5: NOTATION $\mathcal{C}^\infty(G)$ is the set of $C^\infty$ class functions.

6: RAPPEL The characters of $G$ belong to $\mathcal{C}^\infty(G)$.

7: N.B. Therefore $X(G)$ is a subring of the ring of $C^\infty$ functions on $G$.

8: REMARK Per #4, suppose that $f \in \mathcal{C}(G)$ has the property that

$$<f, \chi_\Pi> \in \mathbb{Z}$$

for all $\Pi \in \hat{G}$ -- then it follows after the fact that $f \in \mathcal{C}^\infty(G)$.

Let $T \subset G$ be a maximal torus and assign to the symbol $X(T)$ the obvious interpretation.

9: RAPPEL The arrow

$$f \rightarrow f|_T$$

of restriction defines an isomorphism

$$\mathcal{C}(G) \rightarrow C(T)^\mathbb{W}$$  (cf. §4, #24).

10: APPLICATION Restriction to $T$ induces an injective homomorphism

$$X(G) \rightarrow X(T)^\mathbb{W}.$$  

Take a $\phi \in X(T)^\mathbb{W}$ and let $f \in \mathcal{C}(G)$ be the class function that restricts to $\phi$.

11: LEMMA $f$ is a virtual character of $G$, i.e., $f \in X(G)$.

PROOF With #4 in view, write

$$<f, \chi_\Pi> = \int_G f(x)\overline{\chi_\Pi(x)}d_G(x)$$
3.

\[ \frac{1}{|W|} \int_T |\Delta(t)|^2 \phi(t) \chi_H^*(t) d_T(t) = \frac{1}{|W|} \int_T \Delta(t) \phi(t) \overline{\chi_H(t)} d_T(t) \]

\[ \in \frac{1}{|W|} (|W|Z) = Z. \]

12: SCHOLIUM

\[ X(G) = X(T)^W. \]

13: N.B. Rephrased, a continuous class function \( f: G \to \mathbb{C} \) is a virtual character of \( G \) iff \( f|T \) is a virtual character of \( T \).

14: THEOREM Let \( f \in \text{CL}(G) \) -- then \( f \in X(G) \) iff its restriction to every finite elementary subgroup of \( G \) is a virtual character.

PROOF To establish the nontrivial assertion, let \( H \subset G \) be a finite subgroup -- then the assumption on \( f \) coupled with A, II, §12, #1 implies that \( f|H \in X(H) \).

Matters can thus be reinforced, the assumption on \( f \) becoming that its restriction to every finite subgroup of \( G \) is a virtual character and, thanks to what has been said above, one might just as well work with \( T \) rather than \( G \). Choose a sequence \( H_1 \subset H_2 \subset \cdots \) of finite subgroups of \( T \) whose union is dense in \( T \) -- then \( \forall \chi \in X(T), \)

\[ \langle f, \chi \rangle_T = \int_T f \overline{\chi} = \lim_{n \to \infty} \left( \frac{1}{|H_n|} \sum_{h \in H_n} f(h) \overline{\chi(h)} \right) \]

\[ = \lim_{n \to \infty} \langle f, \chi \rangle_{H_n}. \]

But

\[ f|H_n \in X(H_n) \]

\[ \Rightarrow \langle f, \chi \rangle_{H_n} \in \mathbb{Z} \]

\[ \Rightarrow \langle f, \chi \rangle_T \in \mathbb{Z}. \]
§14. INDUCTION

Let $G$ be a finite group, $\Gamma \subset G$ a subgroup.

1: RAPPEL There is an arrow

$$i_{\Gamma} \to G : \text{CL}(\Gamma) \to \text{CL}(G)$$

which sends characters of $\Gamma$ to characters of $G$ (cf. A, II, §9, #10), thus induces an arrow

$$X(\Gamma) \to X(G).$$

2: N.B. If

$$G = \bigoplus_{k=1}^{n} x_k \Gamma$$

and if $\phi \in \text{CL}(\Gamma)$ is a class function, then

$$(i_{\Gamma} \to G \phi)(x) = \sum_{k=1}^{n} \phi(x_k^{-1}xx_k) \quad (\text{cf. A, II, §7, #10}),$$

i.e.,

$$(i_{\Gamma} \to G \phi)(x) = \sum_{k, x_k^{-1}xx_k \in \Gamma} \phi(x_k^{-1}xx_k).$$

Let $G$ be a compact Lie group, $\Gamma \subset G$ a closed Lie subgroup.

3: NOTATION Given an $x \in G$, write $(G/\Gamma)^x$ for the fixed point set of the action of $x$ on $G/\Gamma$.

4: LEMMA A coset $y\Gamma$ in $G/\Gamma$ lies in $(G/\Gamma)^x$ iff $y^{-1}xy \in \Gamma$.
5: **Lemma** If cosets $y_1 \Gamma, y_2 \Gamma$ lie in the same connected component of $(G/\Gamma)^X$, then $y_2 \Gamma = y_1 y \Gamma$ for some $y$ in the centralizer of $x$.

6: **N.B.** If $\phi \in CL(\Gamma)$ is a class function and if $y_2 = y y_1$, then

$$
\phi(y_2^{-1} x y_2) = \phi(y_1^{-1} y^{-1} x y y_1^{-1})
$$

$$
= \phi(y_1^{-1} y^{-1} x y_1^{-1})
$$

$$
= \phi(y_1^{-1} x y_1^{-1}).
$$

Let $C_1, \ldots, C_m$ be the connected components of $(G/\Gamma)^X$, thus

$$(G/\Gamma)^X = \bigsqcup_{j=1}^{m} C_j,$$

let $\chi(C_j)$ be the Euler characteristic of $C_j$, and fix elements

$$y_1 \Gamma \in C_1, \ldots, y_m \Gamma \in C_m.$$

7: **Notation** Given a class function $\phi \in CL(\Gamma)$, put

$$
(i_\Gamma \to G\phi)(x) = \sum_{j=1}^{m} \chi(C_j) \phi(y_j^{-1} x y_j).
$$

8: **Lemma**

$$i_\Gamma \to G\phi \in CL(G),$$

the induced class function.

9: **N.B.** Therefore

$$i_\Gamma : CL(\Gamma) \to CL(G).$$
10: REMARK The definition of $i_\Gamma \to G^\phi$ is independent of the choice of representatives $y_j \Gamma$ for the components of $(G/\Gamma)^x$ but it is not quite obvious that $i_\Gamma \to G^\phi$ is continuous.

11: RECONCILIATION Take the case when $G$ and $\Gamma \subset G$ are finite. Write

$$G = \biguplus_{k=1}^n x_k \Gamma.$$ 

Then, as recalled in #2,

$$(i_\Gamma \to G^\phi)(x) = \Sigma \phi(x_k^{-1} xx_k)_{k, x_k^{-1} xx_k \in \Gamma}$$

which, in view of #4, is equal to

$$\Sigma \phi(y^{-1} xy)_{y \Gamma \in G/\Gamma} \quad \text{xy} \Gamma = y \Gamma$$

or still, is equal to

$$\Sigma \phi(y^{-1} xy)_{y \Gamma \in (G/\Gamma)^x}.$$ 

But here the $C_j$ are points, say

$$C_j = \{y_j \Gamma\} \implies (G/\Gamma)^x = \{\{y_1 \Gamma\}, \ldots, \{y_m \Gamma\}\},$$

so $\chi(C_j) = 1$, thus

$$\sum_{j=1}^m \chi(C_j) \phi(y_j^{-1} xy_j) = \sum_{j=1}^m \phi(y_j^{-1} xy_j) = \sum_{y \Gamma \in (G/\Gamma)^x} \phi(y^{-1} xy).$$
12: **RAPPEL** A compact connected Lie group of positive dimension has zero Euler characteristic, so the connected components of a compact Lie group of positive dimension have zero Euler characteristic.

13: **EXAMPLE** Take $\Gamma = \{ e \}$, let $\phi = 1_{\Gamma}$, and assume that $\dim G > 0$ -- then $(G/\Gamma)^X$ is empty of $x \neq e$, hence for such $x$, 

$$(i_{\Gamma} \to G^\phi)(x) = 0,$$

but if $x = e$, then $(G/\Gamma)^e = G$ and 

$$ (i_{\Gamma} \to G^\phi)(e) = \sum_{j=1}^{m} \chi(C_j) \phi(y_j^{-1} e y_j) $$ 

$$ = \left( \sum_{j=1}^{m} \chi(C_j) \right) \phi(e) $$ 

$$ = 0. $$

Therefore 

$$ i_{\Gamma} \to G^\phi = 0. $$

14: **DEFINITION** A closed subgroup $H$ of $G$ is **generic** if it is topologically cyclic and of finite index in its normalizer.

[Note: Let $G$ be a compact connected Lie group, $T \subset G$ a maximal torus -- then $T$ is generic.]

15: **DEFINITION** An element $x \in G$ is **generic** if it generates a generic subgroup of $G$.

[Note: Let $G$ be a compact connected Lie group -- then a generic element is necessarily regular.]
16: LEMMA The generic elements are dense in $G$.

17: THEOREM Suppose that $x \in G$ is generic -- then

$$|(G/\Gamma)^x| > \infty$$

and

$$(i_{\Gamma} \rightarrow G^\phi)(x) = \sum_{y \Gamma \in (G/\Gamma)^x} \phi(y^{-1}xy).$$

18: EXAMPLE Take $\Gamma = \{e\}$, let $\phi = 1_{\Gamma}$, and assume that $\dim G > 0$ -- then at every generic element of $G$,

$$(i_{\Gamma} \rightarrow G^\phi)(x) = 0,$$

hence by continuity (in conjunction with #16),

$$i_{\Gamma} \rightarrow G^\phi = 0 \quad (\text{cf. } #13).$$

Let $G$ be a compact Lie group, let $\Gamma_1, \Gamma_2 \subset G$ be closed Lie subgroups, and let

$$G = \bigcup_{s \in S} \Gamma_1 s \Gamma_2$$

be a double coset decomposition of $G$.

19: N.B.

$$\Gamma_1 \backslash G/\Gamma_2$$

is the orbit space per the action of $\Gamma_1$ by left translation on $G/\Gamma_2$.

Write

$$\Gamma_1 \backslash G/\Gamma_2 = \bigsqcup_{s \in S} U_s,$$

where each $U_s$ is a connected component of one orbit type for the action of $\Gamma_1$ on
G/Γ₂. Fix elements xₛ ∈ G such that Γ₁xₛΓ₂ ∈ Us and for each s let

\[ \phi_s : \text{CL}(Γ₂) \rightarrow \text{CL}(Γ₁) \]

denote the following composite: Take a \( \phi \in \text{CL}(Γ₂) \) and form \( \phi^S \equiv \phi \circ \iota_{-1} \) (a class function on \( x_sΓ₂x_s^{-1} \)), then restrict \( \phi^S \) to \( Γ₂(s) = x_sΓ₂x_s^{-1} \cap Γ₁ \), call it \( \phi_s \), and finally apply \( i_{Γ₂(s)} \rightarrow Γ₁ \). I.e.:

\[ \phi_s(\phi) = i_{Γ₂(s)} \rightarrow Γ₁ \phi_s. \]

20: THEOREM As maps from \( \text{CL}(Γ₂) \) to \( \text{CL}(Γ₁) \),

\[ r_1 ⊕ i_{Γ₂} \rightarrow r_1 \rightarrow G = \sum_{s ∈ S} \chi^♯(U_s) \phi_s, \]

where for each \( s ∈ S \),

\[ \chi^♯(U_s) = \chi(\tilde{U}_s) - \chi(\tilde{U}_s - U_s). \]

21: N.B. When \( G \) and \( Γ₁, Γ₂ \subset G \) are finite, matters reduce to A, II, §8, #3.

Here is a sketch of the proof.

1. Fix a class function \( \phi \in \text{CL}(Γ₂) \) and a \( γ₁ ∈ Γ₁ \).

2. Let \( \tilde{U}_s \subset G/Γ₂ \) denote the inverse image of \( U_s \) under the projection to \( Γ₁ \backslash G/Γ₂ \), thus

\[ G/Γ₂ = \bigsqcup_{s ∈ S} \tilde{U}_s. \]

3. Let \( C_1, ..., C_m \) be the connected components of \( (G/Γ₂)^γ₁ \), thus
For each pair \((s,j)\), put
\[ V_{s,j} = (\Gamma_1 \cdot x_{s}^{\Gamma_2}) \cap C_j \subset \tilde{U}_s \cap C_j \subset G/\Gamma_2. \]

The arrows
\[ V_{s,j} \to \tilde{U}_s \cap C_j \to U_s \]
are a fibration sequence, hence by the multiplicativity of the Euler characteristic,
\[ \chi(C_j) = \sum_{s \in S} \chi(V_{s,j}) \chi(U_s). \]

Fix elements \(\gamma_{s,j} \in \Gamma_1\) such that
\[ \gamma_{s,j} x_{s}^{\Gamma_2} \in V_{s,j}. \]
Then in particular,
\[ \gamma_{s,j} x_{s}^{\Gamma_2} \in (G/\Gamma_2)^{\gamma_1} \]
\[ \Rightarrow \]
\[ x_{s}^{\gamma_{s,j}^{-1} \gamma_1} \gamma_{s,j} x_{s} \in \Gamma_2 \]
\[ \Rightarrow \]
\[ \gamma_{s,j}^{-1} \gamma_1 \gamma_{s,j} \in x_{s}^{\Gamma_2} x_{s}^{-1}, \]
the domain of \(\phi^S\).

7. From the definitions,
\[ (r_G \to \Gamma_1 (i_{\Gamma_2} \to G^\phi)) (\gamma_1) \]
\[ = \sum_{j=1}^{m} \chi(C_j) \phi(y_j^{-1} \gamma_1 y_j) \]
8. The isotropy subgroup of the action of \( \Gamma_1 \) on \( x_s \Gamma_2 \in G/\Gamma_2 \) is

\[
\Gamma_2(s) = x_s \Gamma_2 x_s^{-1} \cap \Gamma_1.
\]

And

\[
(\Gamma_1/\Gamma_2(s)) \gamma_1 \simeq (\Gamma_1 \cdot x_s \Gamma_2) \gamma_1
\]

\[
= \bigcup_{j=1}^{m} V_{s,j} \subset G/\Gamma_2.
\]

9. Given \( s \in S \),

\[
= \sum_{j=1}^{m} \chi(V_{s,j}) \phi^s(\gamma_s j^{-1} \gamma_s j^{-1})
\]

\[
= (i_{\Gamma_2}(s) \to \Gamma_1^\phi)(\gamma_1).
\]

10. Therefore

\[
(\kappa G \to \Gamma_1 (i_{\Gamma_2} \to G^\phi))(\gamma_1)
\]

\[
= \sum_{s \in S} \chi^s(U_s)(i_{\Gamma_2}(s) \to \Gamma_1^\phi)(\gamma_1)
\]

\[
= \sum_{s \in S} \chi^s(U_s) \phi_s(\phi)(\gamma_1),
\]

the contention.
22: THEOREM The arrow 
\[ i_\Gamma \to G : \text{CL}(\Gamma) \to \text{CL}(G) \]

sends virtual characters to virtual characters, thus induces an arrow 
\[ X(\Gamma) \to X(G). \]

PROOF Recall first that this is true when G is finite (cf. #1). In general, let \( \chi \in X(\Gamma) \) -- then to conclude that 
\[ i_\Gamma \to G\chi \in X(G), \]
it suffices to show that its restriction to every finite subgroup \( H \) of \( G \) is a virtual character (cf. §13, #14). So consider 
\[ r_G \to H(i_\Gamma \to G\chi) \]
or still, take in the above \( \Gamma_1 = H, \Gamma_2 = \Gamma, \phi = \chi, \) and consider
\[ \sum_{s \in S} \chi'(U_s)\phi_s(\chi). \]

Here 
\[ \phi_s(\chi) = i \quad \chi_s' \quad x_s^{-1} \cap H \to \chi_s' \]
where \( \chi_s \) is the restriction of \( \chi \) to \( \Gamma(s) \equiv x_s^{-1}x_s^{-1} \cap H, \) a finite group. But now 
\[ \chi_s \in X(\Gamma(s)) \Rightarrow i_{\Gamma(s)} \to H\chi \in X(H), \]
which finishes the proof.

23: N.B. If G is finite, then the arrow 
\[ i_\Gamma \to G : \text{CL}(\Gamma) \to \text{CL}(G) \]
sends characters of \( \Gamma \) to characters of \( G \) but this need not be true if \( \dim G > 0 \)
10.

(cf. #13) \( l \Gamma \) is a character of \( \Gamma \) but the induced class function

\[
i_{\Gamma} + G_{\Gamma}
\]

is identically zero, a virtual character, not a character).

24: **REMARK** Let \( G \) be a compact connected semisimple Lie group, \( T \subset G \) a maximal torus -- then

\[
\hat{G} \longleftrightarrow dT \cap \mathbb{C}
\]

(cf. §10, #23).

While the theory developed above gives rise to an arrow

\[
X(T) \rightarrow X(G),
\]

it does not respect the foregoing parameterization which can only be accomplished by a more sophisticated version of the preceding process.

**APPENDIX**

There is a different approach to induction which is suggested by A, II, §9, #1.

So let \( G \) be a compact Lie group, \( \Gamma \subset G \) a closed Lie subgroup.

**CONSTRUCTION** Let \((\theta, E)\) be a finite dimensional unitary representation of \( \Gamma \) and denote by \( E_{\Gamma, \theta}^G \) the space of all \( E \)-valued measurable functions \( f \) on \( G \) such that \( f(xy) = \theta(y^{-1})f(x) (x \in G, y \in \Gamma) \) subject to

\[
\int_{\Gamma} ||f||^{2}d_{G/\Gamma} < \infty.
\]

Then the prescription

\[
(Ind_{\Gamma, \theta}^G(x)f)(y) = f(x^{-1}y)
\]

defines a representation \( Ind_{\Gamma, \theta}^G \) of \( G \) on \( E_{\Gamma, \theta}^G \), the representation of \( G \) induced by \( \theta \).
N.B. The inner product

\[ \langle f, g \rangle_\theta = \int_{G/\Gamma} \langle f, g \rangle d_{G/\Gamma} \]

equips \( E_{\Gamma, \theta}^G \) with the structure of a Hilbert space and \( \text{Ind}_{\Gamma, \theta}^G \) is a unitary representation.

EXAMPLE Take \( \theta \) to be the trivial representation of \( \Gamma \) on \( E = C \) -- then

\[ E_{\Gamma, \theta}^G = L^2(G/\Gamma). \]

[Note: When \( \Gamma = \{e\} \), \( E_{\Gamma, \theta}^G = L^2(G) \) and

\[ \text{Ind}_{\Gamma, \theta}^G = L, \]

the left translation representation of \( G \) (cf. §1, #5).]
1. ORBITAL INTEGRALS

Let G be a compact group.

1: DEFINITION Given $f \in C(G)$ and $\gamma \in G$, put

$$0(f, \gamma) = \int_G f(x\gamma x^{-1}) d_G(x),$$

the orbital integral of $f$ at $\gamma$.

2: LEMMA The function $0(f)$ defined by the assignment

$$\gamma \rightarrow 0(f, \gamma)$$

is a continuous class function on $G$, i.e., is an element of $CL(G)$.

3: RAPPEL If $f \in C(G)_{\text{fin}}$, then

$$\langle f, \chi_\Pi \rangle = 0$$

for all but finitely many $\Pi$.

4: LEMMA Suppose that $f \in C(G)_{\text{fin}}$, then $\forall \gamma \in G$,

$$0(f, \gamma) = \sum_{\Pi \in G} \text{tr}(\Pi^*(f)) \chi_\Pi(\gamma)$$

(cf. A, III, §1, #3),

the sum on the right being finite.

PROOF Apply I, §2, #19 to get

$$0(f) = \sum_{\Pi \in G} \langle 0(f), \chi_\Pi \rangle \chi_\Pi,$$

where the series converges in $L^2(G)$. But

$$\langle 0(f), \chi_\Pi \rangle = \int_G 0(f, \gamma) \overline{\chi_\Pi(\gamma)} d_G(\gamma)$$
Therefore \( \langle 0, \chi_{\Pi} \rangle = 0 \) for all but finitely many \( \Pi \), thus the almost everywhere equality

\[
0(\mathbf{f}) = \sum_{\Pi \in \mathcal{G}} \langle 0(\mathbf{f}), \chi_{\Pi} \rangle \chi_{\Pi}
\]

is that of two continuous functions, thus is valid everywhere. Finally, from the definitions,

\[
\langle \mathbf{f}, \chi_{\Pi} \rangle = \int_G \mathbf{f}(x) \overline{\chi_{\Pi}(x)} \, d_G(x)
\]

\[
= \int_G \mathbf{f}(x) \chi_{\Pi^*}(x) \, d_G(x)
\]

\[
= \int_G \mathbf{f}(x) \text{tr}((\Pi^*)(x)) \, d_G(x)
\]

\[
= \text{tr}(\Pi^*(\mathbf{f})).
\]
§2. KERNELS

Let \((X, \mu), (Y, \nu)\) be \(\sigma\)-finite measure spaces.

1: NOTATION Given \(K \in L^2(X \times Y)\), define \(T_K : L^2(Y) \to L^2(X)\) by

\[
(T_K \phi)(x) = \int_Y K(x, y) \phi(y) \, d\nu(y).
\]

2: THEOREM The map \(K \to T_K\) is a linear isometry of \(L^2(X \times Y)\) onto \(L^2_{\text{HS}}(L^2(Y), L^2(X))\).

3: NOTATION Given

\[
\begin{align*}
K_1 &\in L^2(X \times Y) \\
K_2 &\in L^2(Y \times Z),
\end{align*}
\]

define their convolution

\[
K_1 \ast K_2 \in L^2(X \times Z)
\]

by

\[
(K_1 \ast K_2)(x, z) = \int_Y K_1(x, y) K_2(y, z) \, d\nu(y).
\]

[Note: The underlying measure-theoretic assumption is again \(\sigma\)-finiteness (which is needed infra for Fubini).]

4: THEOREM

\[
T_{K_1} \circ T_{K_2} = T_{K_1 \ast K_2}.
\]
5: APPLICATION Take $X = Y = Z$ -- then

\[
T_{K_1} : L^2(X) \to L^2(X)
\]

\[
T_{K_2} : L^2(X) \to L^2(X)
\]

are Hilbert-Schmidt, hence

\[
T_{K_1 \ast K_2} : L^2(X) \to L^2(X)
\]

is trace class.

6: LEMMA Take $X = Y = Z$ and put $K = K_1 \ast K_2$ -- then

\[
\text{tr}(T_K) = \int_X K(x,x) d\mu(x).
\]

PROOF

\[
\text{tr}(T_K) = \text{tr}(T_{K_1} \circ T_{K_2})
\]

\[
= \langle T_{K_2} \rho \ast_{K_1} \rangle_{HS}
\]

\[
= \langle K_2, K_1^\ast \rangle
\]

\[
= \int_X \int_X K_2(y, x) \overline{K_1(y, x)} d\mu(y) d\mu(x)
\]

\[
= \int_X \int_X K_2(y, x) \overline{K_1(x, y)} d\mu(y) d\mu(x)
\]

\[
= \int_X \int_X K_1(x, y) K_2(y, x) d\mu(y) d\mu(x)
\]

\[
= \int_X K_1 \ast K_2(x, x) d\mu(x)
\]
= \int_X K(x,x) \, d\mu(x).

7: REMARK It can happen that \( K_1 = K_2 \) a.e. (so \( T_{K_1} = T_{K_2} \)), yet

\[ \int_X K_1(x,x) \, d\mu(x) \neq \int_X K_2(x,x) \, d\mu(x). \]

[E.g.: Take \( X = Y = [0,1] \), \( K_1 \equiv 0 \), \( K_2 = \chi_\Delta \) (\( \Delta \) the diagonal).]

8: THEOREM Let \( X \) be a locally compact Hausdorff space, \( \mu \) a \( \sigma \)-finite
Radon measure on \( X \). Suppose that \( K \in L^2(X \times X) \) is separately continuous and \( T_K \)
is trace class -- then the function

\[ x \mapsto K(x,x) \]
is integrable on \( X \) and

\[ \text{tr}(T_K) = \int_X K(x,x) \, d\mu(x). \]

APPENDIX

LEMMA Let \( M \) be a compact \( C^\infty \) manifold, \( \mu \) a smooth measure on \( M \), \( T:L^2(M) \to C^{2k}(M) (k > \frac{1}{4} \text{ dim } M) \) -- then \( T \) is trace class.

PROOF Let \( \Delta \) be a Laplacian on \( M \) and write

\[ T = (1-\Delta)^{-k}(1-\Delta)^{k}T. \]

Then

\[ (1-\Delta)^{k}T(L^2(M)) \subset C(M) \subset L^\infty(M) \]

so \( (1-\Delta)^{k}T \) is Hilbert-Schmidt. On the other hand, by Sobolev theory,

\[ (1-\Delta)^{-k}L^2(M) \subset H^{2k}(M) \subset C(M), \]

thus \( (1-\Delta)^{-k} \) is also Hilbert-Schmidt.
§3. THE LOCAL TRACE FORMULA

Let $G$ be a compact group.

1: NOTATION Denote by $\pi_{L,R}$ the representation of $G \times G$ on $L^2(G)$ given by

$$(\pi_{L,R}(x_1,x_2)f)(x) = f(x_1^{-1}xx_2) \quad \text{(cf. A, III, §2, #1).}$$

2: LEMMA $\pi_{L,R}$ is unitary.

PROOF

$$||\pi_{L,R}(x_1,x_2)f||^2$$

$$= \int_G |(\pi_{L,R}(x_1,x_2)f)(x)|^2 d_G(x)$$

$$= \int_G |f(x_1^{-1}xx_2)|^2 d_G(x)$$

$$= \int_G |f(x)|^2 d_G(x) = ||f||^2. $$

Given $f_1,f_2 \in C(G)$, define $f \in C(G \times G)$ by

$$f(x_1,x_2) = f_1(x_1)f_2(x_2),$$

and let

$$\pi_{L,R}(f) = \int_G \int_G f_1(x_1)f_2(x_2) \pi_{L,R}(x_1,x_2) d_G(x_1)d_G(x_2).$$

Then $\forall \phi \in L^2(G),$

$$(\pi_{L,R}(f)\phi)(x) = \int_G \int_G f_1(x_1)f_2(x_2)\phi(x_1^{-1}xx_2)d_G(x_1)d_G(x_2)$$

$$= \int_G K_f(x,y)\phi(y)d_G(y),$$
2.

where

\[ K_f(x,y) = \int_G f_1(xz) f_2(yz) d_G(z). \]

Therefore \( \pi_{L,R}(f) \) is an integral operator on \( L^2(G) \) with kernel \( K_f(x,y) \).

3: CONSTRUCTION

- Given \( f_1 \in C(G) \), put

\[ K_{f_1}(x,y) = f_1(xy^{-1}) \quad (x,y \in G). \]

Then

\[ K_1 \in L^2(G \times G) \quad (K_1 = K_{f_1}) \]

and

\[ (T_{K_1} \phi)(x) = \int_G K_1(x,y) \phi(y) d_G(y) \]

\[ = \int_G f(xy^{-1}) \phi(y) d_G(y) \]

\[ = \int_G f(y) \phi(y^{-1}x) d_G(y) \]

\[ = \int_G f(y) (L(y) \phi)(x) d_G(y) \]

\[ = (L(f) \phi)(x). \]

- Given \( f_2 \in C(G) \), put

\[ K_{f_2}(x,y) = f_2(x^{-1}y) \quad (x,y \in G). \]

Then

\[ K_2 \in L^2(G \times G) \quad (K_2 = K_{f_2}) \]
and

\[(T_{K_2}\phi)(x) = \int_G K_2(x,y)\phi(y)d_G(y)\]

\[= \int_G f_2(x^{-1}y)\phi(y)d_G(y)\]

\[= \int_G f_2(y)(R(x)\phi)(y)d_G(y)\]

\[= (R(f)\phi)(x).\]

4: LEMMA Let \(f_1, f_2 \in C(G)\) and let \(f = f_1f_2\) --- then \(K_f = K_{f_1} * K_{f_2}\).

PROOF

\[(K_1 * K_2)(x,y) = \int_G K_1(x,z)K_2(z,y)d_G(z)\]

\[= \int_G f_1(xz^{-1})f_2(z^{-1}y)d_G(z)\]

\[= \int_G f_1(xz)f_2(zy)d_G(z).\]

Since the kernels of

\[
\begin{array}{c}
T_{K_1} : L^2(G) \rightarrow L^2(G) \\
T_{K_2} : L^2(G) \rightarrow L^2(G)
\end{array}
\]

are square integrable, it follows that these operators are Hilbert-Schmidt. But
4. \[ T_{K_1} \circ T_{K_2} = T_{K_1 \ast K_2} \] (cf. §2, §4).

Therefore \( T_{K_1 \ast K_2} \) is trace class, i.e., \( T_{K_f} \) is trace class, i.e., \( \pi_{L,R}(f) \) is trace class.

5: **Lemma**

\[ \text{tr}(\pi_{L,R}(f)) = \int_G K_f(x,x)d_G(x) \] (cf. §2, §6).

6: **RappeL** Let

\[ f \in \text{span}_c(L^2(G) \ast L^2(G)) \subseteq C(G). \]

Then

\[ f(e) = \sum_{\Pi \in G} \text{tr}(\Pi(f)) \] (cf. I, §2, §15).

We have

\[ K_f(x,y) = \int_G f_1(xz)f_2(zy)d_G(z) \]

\[ = \int_G f_1(u)f_2(x^{-1}uy)d_G(u) \]

\[ = \int_G f_1(u)f_{2,x,y}(u)d_G(u). \]

Put now

\[ F_{x,y}(v) = \int_G f_1(u)f_{2,x,y}(v^{-1}u)d_G(u). \]

Then

\[ F_{x,y}(v) = \int_G f_1(u)f_{2,x,y}(u^{-1}v)d_G(u) \]

\[ = f_1 \ast f_{2,x,y}(v) \]
\[ K_f(x, y) = F_{x, y}(e) \]

\[ = \sum_{\Pi \in G} d_{\Pi} \text{tr}(\Pi(K_{x, y})) \]

\[ = \sum_{\Pi \in G} d_{\Pi} \text{tr}(\Pi(f_1 \ast \bar{f}_2, x, y)) \]

\[ = \sum_{\Pi \in G} d_{\Pi} \text{tr}(\Pi(f_1 \Pi(\delta_x \ast \bar{f}_2, x, y)) \Pi(f_2)) \]

\[ = \sum_{\Pi \in G} d_{\Pi} \text{tr}(\Pi(f_1 \Pi(\delta_x \Pi(\delta_y^{-1})), x, y) \Pi(f_2)) \]

\[ = \sum_{\Pi \in G} d_{\Pi} \text{tr}(\Pi(y^{-1}) \Pi(f_1) \Pi(x) \Pi(f_2)) \]

\[ \Rightarrow \]

\[ \text{tr}(\pi_{L, R}(f)) = \int_{G} K_{\pi_{L, R}(f)}(x) d_{G}(x) \]

\[ = \sum_{\Pi \in G} d_{\Pi} \text{tr}(\Pi(f_1) \Pi(x^{-1}) \Pi(f_1) \Pi(x) d_{G}(x)) \cdot \Pi(f_2) \]

\[ = \sum_{\Pi \in G} d_{\Pi} \text{tr}(\Pi(f_1) \Pi(x) \Pi(f_1) \Pi(x^{-1}) d_{G}(x)) \cdot \Pi(f_2). \]

7: SUBLEMMA \( \forall \phi \in C(G) \), the operator

\[ \int_{G} \Pi(x) \Pi(\phi) \Pi(x^{-1}) d_{G}(x) \]
6.

intertwines $\Pi$, hence is a scalar multiple of the identity (cf. I, §1, #15), call it $\lambda_\phi$.

8: N.B.

$$\lambda_\phi = \int_G \Pi(x) \Pi(\phi) \Pi(x^{-1}) d_G(x)$$

$\Rightarrow$

$$\lambda_\phi d_\Pi = \text{tr} \left( \int_G \Pi(x) \Pi(\phi) \Pi(x^{-1}) d_G(x) \right)$$

$\Rightarrow$

$$\lambda_\phi = \frac{\text{tr} (\Pi(\phi))}{d_\Pi}.$$

Therefore

$$\text{tr} (\pi_{L,R}(f))$$

$$= \sum_{\Pi \in G} d_\Pi \text{tr} \left( (\int_G \Pi(x) \Pi(f_1) \Pi(x^{-1}) d_G(x)) \circ \Pi(f_2) \right)$$

$$= \sum_{\Pi \in G} d_\Pi \lambda_\phi f_1 \text{tr} (\Pi(f_2))$$

$$= \sum_{\Pi \in G} d_\Pi \frac{\text{tr} (\Pi(f_1))}{d_\Pi} \text{tr} (\Pi(f_2))$$

$$= \sum_{\Pi \in G} \text{tr} (\Pi(f_1)) \text{tr} (\Pi(f_2))$$

$$= \sum_{\Pi \in G} J(\Pi, f)$$
if

\[ J(\Pi, f) = \text{tr}(\Pi(f_1)) \text{tr}(\Pi(f_2)). \]

9: \textbf{SUBLEMMA} \forall \phi \in \text{C(G)},

\[ \text{tr}(\Pi(\phi)) = \text{tr}(\int_G \phi(x) \Pi(x) d_G(x)) \]

\[ = \text{tr}(\int_G \phi(x^{-1}) \Pi(x) d_G(x)) \]

\[ = \text{tr}(\int_G \phi(x) \Pi(x^{-1}) d_G(x)) \]

\[ = \text{tr}(\int_G \phi(x) \Pi^*(x) d_G(x)) \]

\[ = \text{tr}(\Pi^*(\phi)). \]

10: \textbf{N.B.} Consequently,

\[ J(\Pi, f) = \text{tr}(\Pi(f_1)) \text{tr}(\Pi^*(f_2)). \]

There is another way to manipulate

\[ \int_G K_f(x, x) d_G(x) \]

which then leads to a second formula for

\[ \text{tr}(\Pi_{L, R}(f)). \]

To wit:

\[ \int_G K_f(x, x) d_G(x) \]

\[ = \int_G \int_G f_1(xzx^{-1}) f_2(z) d_G(z) d_G(x) \]
or still, for any $y \in G$,

$$\int_G \int_G f_1(xzx^{-1})f_2(zyy^{-1})d_G(z)d_G(x).$$

Now multiply through by $d_G(y)$ and integrate with respect to $y$:

$$\text{tr}(\pi_{L,R}(f)) = \int_G \text{tr}(\pi_{L,R}(f))d_G(y)$$

$$= \int_G \int_G f_1(xzx^{-1})f_2(zyy^{-1})d_G(z)d_G(x)d_G(y)$$

$$= \int_G (\int_G f_1(xzx^{-1})d_G(x))(\int_G f_2(zyy^{-1})d_G(y))d_G(z)$$

$$= \int_G \delta(f_1,z)\delta(f_2,z)d_G(z).$$

11: DEFINITION Given $f = f_1f_2$, the local trace formula is the relation

$$\sum_{\Pi \in \mathbb{G}} J(\Pi,f)$$

$$= \sum_{\Pi \in \mathbb{G}} \text{tr}(\Pi(f_1))\text{tr}(\Pi^*(f_2))$$

$$= \int_G \delta(f_1,z)\delta(f_2,z)d_G(z).$$

Let $G$ be a compact connected semisimple Lie group, $T \subset G$ a maximal torus.

12: RAPPEL For any continuous function $f \in C(G)$,

$$\int_G f(x)d_G(x) = \frac{1}{|W|} \int_T |\Delta(t)|^2 \int_G f(xtx^{-1})d_G(x)d_T(t) \quad \text{(cf. I, §9, #12)}$$

or still,

$$\int_G f(x)d_G(x) = \frac{1}{|W|} \int_T |\Delta(t)|^2 \delta(f,t)d_T(t).$$
As above, let $f = f_1 f_2$ -- then

$$
\int_G K_f(x,x) \, d_G(x)
$$

$$
= \int_G \int_G f_1(xzx^{-1}) f_2(z) \, d_G(z) \, d_G(x)
$$

$$
= \int_G \frac{1}{|W|} \int_T |\Delta(t)|^2 \int_G f_1(xyty^{-1} x^{-1}) f_2(yty^{-1}) \, d_G(y) \, d_T(t) \, d_G(x)
$$

$$
= \frac{1}{|W|} \int_T |\Delta(t)|^2 \int_G (\int_G f_1(xyty^{-1} x^{-1}) \, d_G(x)) f_2(yty^{-1}) \, d_T(t) \, d_G(x)
$$

$$
= \frac{1}{|W|} \int_T |\Delta(t)|^2 \int_G (\int_G f_1(xtx^{-1}) \, d_G(x)) (\int_G f_2(yty^{-1}) \, d_G(y)) \, d_T(t)
$$

$$
= \frac{1}{|W|} \int_T |\Delta(t)|^2 \varphi(f_1,t) \varphi(f_2,t) \, d_T(t).
$$
§1. TOPOLOGICAL TERMINOLOGY

1: DEFINITION A topological space $X$ is **compact** if every open cover of $X$ has a finite subcover.

2: DEFINITION A topological space $X$ is **locally compact** if every point in $X$ has a neighborhood basis consisting of compact sets.

3: LEMMA A Hausdorff space $X$ is locally compact iff every point in $X$ has a compact neighborhood.

4: APPLICATION Every compact Hausdorff space is locally compact.

5: EXAMPLE $\mathbb{R}$ is a locally compact Hausdorff space.

6: EXAMPLE $\mathbb{Q}$ is a Hausdorff space but it is not locally compact ($\mathbb{Q}$ is first category while a locally compact Hausdorff space is second category).

7: LEMMA An open subset of a locally compact Hausdorff space is locally compact.

8: LEMMA A closed subset of a locally compact Hausdorff space is locally compact.

9: LEMMA In a locally compact Hausdorff space, the intersection of an open set with a closed set is locally compact.

10: EXAMPLE The semiclosed, semiopen interval $[0, 1]$ is locally compact. 
[In fact, 
$$[0, 1] = \mathbb{R} \setminus (1, 1] \cap [0, 1]$$]
11: DEFINITION A topological group is a Hausdorff topological space $G$ equipped with a group structure such that the function from $G \times G$ to $G$ defined by $(x,y) \mapsto xy^{-1}$ is continuous or still, as is equivalent:

- The function $G \times G \to G$ that sends $(x,y)$ to $xy$ is continuous.
- The function $G \to G$ that sends $x$ to $x^{-1}$ is continuous.

If $G$ is a topological group and if $H \subset G$ is a subgroup, then the set $G/H$ is to be given the quotient topology.

12: LEMMA The space $G/H$ is Hausdorff iff $H$ is closed.

13: DEFINITION A locally compact (compact) group is a topological group $G$ that is both locally compact (compact) and Hausdorff.

14: LEMMA If $G$ is a locally compact group and if $H$ is a closed subgroup, then $G/H$ is a locally compact Hausdorff space.

15: LEMMA If $G$ is a locally compact group and if $H$ is a closed normal subgroup, then $G/H$ is a locally compact group.

16: LEMMA If $G$ is a locally compact group and if $H$ is a locally compact subgroup, then $H$ is closed in $G$.

17: LEMMA If $G$ is a locally compact group, then a subgroup $H$ is open iff the quotient $G/H$ is discrete.

18: LEMMA If $G$ is a compact group, then a subgroup $H$ is open iff the quotient $G/H$ is finite.
19: **Lemma** If $G$ is a locally compact group, then every open subgroup of $G$ is closed and every finite index closed subgroup of $G$ is open.

20: **Definition** A topological space $X$ is **totally disconnected** if the connected components of $X$ are singletons.

21: **Example** $\mathbb{Q}$ is totally disconnected.

22: **Lemma** If $G$ is a totally disconnected locally compact group, then $\{e\}$ has a neighborhood basis consisting of open-compact subgroups.

23: **Lemma** If $G$ is a totally disconnected compact group, then $\{e\}$ has a neighborhood basis consisting of open-compact normal subgroups.

24: **Definition** A topological space $X$ is **0-dimensional** if every point of $X$ has a neighborhood basis consisting of open-closed sets.

25: **Example** $\mathbb{Q}$ is 0-dimensional.

26: **Lemma** A locally compact Hausdorff space is 0-dimensional iff it is totally disconnected.

[Note: In such a space, every point has a neighborhood basis consisting of open-compact sets.]

27: **Remark** It is false that the continuous image of a 0-dimensional locally compact Hausdorff space is again 0-dimensional.

[To see this, recall that every compact metric space is the continuous image of the Cantor set.]
28. **Lemma** If $G$ is a locally compact $0$-dimensional group and if $H$ is a closed subgroup of $G$, then $G/H$ is $0$-dimensional.

29. **Lemma** A $0$-dimensional $T_1$ space is totally disconnected.

30. **Remark** There are totally disconnected metric spaces which are not $0$-dimensional.
§2. INTEGRATION THEORY

Let $X$ be a locally compact Hausdorff space.

1: DEFINITION A Radon measure is a measure $\mu$ defined on the Borel $\sigma$-algebra of $X$ subject to the following conditions.

1. $\mu$ is finite on compacta, i.e., for every compact set $K \subset X$, $\mu(K) < \infty$.
2. $\mu$ is outer regular, i.e., for every Borel set $A \subset X$, $\mu(A) = \inf_{U \supseteq A} \mu(U)$, where $U \subset X$ is open.
3. $\mu$ is inner regular, i.e., for every open set $A \subset X$, $\mu(A) = \sup_{K \subseteq A} \mu(K)$, where $K \subset X$ is compact.

2: RAPPEL If $X$ is a locally compact Hausdorff space and if $X$ is second countable, then for any open subset $U \subset X$, there exist compact sets $K_1 \subset K_2 \subset \ldots$ such that $U = \bigcup_{n=1}^{\infty} K_n$.

3: APPLICATION If $(X, \mu)$ is a Radon measure space and if $X$ is second countable, then $X$ is $\sigma$-finite.

4: RIESZ REPRESENTATION THEOREM Let $X$ be a locally compact Hausdorff space. Suppose that $\Lambda: C_c(X) \to \mathbb{C}$ is a positive linear functional -- then there exists a unique Radon measure $\mu$ on $X$ such that $\forall f \in C_c(X)$,

$$\Lambda f = \int_X f(x) \, d\mu(x).$$
Let $G$ be a locally compact group.

5: **Definition.** A left Haar measure on $G$ is a Radon measure $\mu_G \neq 0$ which is left invariant, i.e., $\forall x \in G$ and $\forall$ Borel set $A \subset G$, $\mu_G(xA) = \mu_G(A)$.

[Note: Equivalently, a Radon measure $\mu \neq 0$ is a left Haar measure on $G$ if $\forall f \in C_c(G)$ and $\forall y \in G$,
\[
\int_G f(yx) \, d\mu(x) = \int_G f(x) \, d\mu(x).
\]

6: **Theorem.** $G$ admits a left Haar measure and if $\mu_G^1$, $\mu_G^2$ are two such, then $\mu_G^1 = c \mu_G^2$ ($\exists c > 0$).

7: **Lemma.** Every nonempty open subset of $G$ has positive left Haar measure.

8: **Lemma.** Every compact subset of $G$ has finite left Haar measure.

9: **N.B.** The definition of a right Haar measure on $G$ is analogous.

Given $x \in G$ and a Borel set $A \subset X$, let
\[
\mu_{G,x}(A) = \mu_G(Ax).
\]

Then $\mu_{G,x}$ is a left Haar measure on $G$:
\[
\mu_{G,x}(yA) = \mu_G(yAx) = \mu_G(Ax) = \mu_{G,x}(A).
\]

The uniqueness of left Haar measure now implies that there is a unique positive real number $\Delta_G(x)$ such that
\[
\mu_{G,x} = \Delta_G(x) \mu_G.
\]

10: **Lemma.** $\Delta_G : G \to \mathbb{R}_>^\times$ is independent of the choice of $\mu$. 

11: Lemma $\Delta_G : G \rightarrow \mathbb{R}_0^\times$ is a continuous homomorphism.

12: Definition $\Delta_G$ is called the modular function of $G$.

So, $\forall f \in C_c(G)$ and $\forall y \in G$,

$$\int_G f(xy^{-1}) d\mu_G(x) = \Delta_G(y) \int_G f(x) d\mu_G(x).$$

13: Lemma $\forall f \in C_c(G)$,

$$\int_G \hat{f}(x) d\mu_G(x) = \int_G \frac{f(x)}{\Delta_G(x)} d\mu_G(x).$$

[Note: As usual, $\hat{f}(x) = f(x^{-1})$.]

14: N.B. The positive linear functional that assigns to each $f \in C_c(G)$ the common value of the two members of this equality is a right Haar integral.

15: Lemma If $\phi : G \rightarrow G$ is a topological automorphism, then there is a unique positive real number $\delta_G(\phi)$ such that $\forall f \in C_c(G)$,

$$\int_G f(\phi^{-1}(x)) d\mu_G(x) = \delta_G(\phi) \int_G f(x) d\mu_G(x).$$

[The positive linear functional $f \mapsto \int_G f(\phi^{-1}(x)) d\mu_G(x)$ is a left Haar integral.]

[Note: The arrow $\phi \mapsto \delta_G(\phi)$ is a homomorphism: $\delta_G(\phi \phi_2) = \delta_G(\phi_1) \delta_G(\phi_2)$.]

16: Example If $V$ is a real finite dimensional vector space and if
T:V \to V is an invertible linear transformation, then per "Lebesgue measure",
\[ \int_V f(T^{-1}(x)) \, dx = |\det T| \int_V f(x) \, dx, \]
so here
\[ \delta_V(T) = |\det T|. \]

17: EXAMPLE Define \( I_y : G \to G \) by \( I_y(x) = yxy^{-1} \) -- then
\[ \int_G f(I_y^{-1}(x)) \, d\mu_G(x) = \int_G f(y^{-1}xy) \, d\mu_G(x) \]
\[ = \int_G f(xy) \, d\mu_G(x) \]
\[ = \Delta_G(y) \int_G f(x) \, d\mu_G(x), \]
which implies that
\[ \delta_G(I_y) = \Delta_G(y^{-1}). \]

18: LEMMA If \( \phi:G \to G \) is a topological automorphism, then \( \forall y \in G, \)
\[ \Delta_G(\phi(y)) = \Delta_G(y). \]
[On the one hand,
\[ \int_G f(\phi(xy^{-1})) \, d\mu_G(x) \]
\[ = \Delta_G(y) \int_G f(\phi(x)) \, d\mu_G(x) \]
\[ = \Delta_G(y) \delta_G(\phi^{-1}) \int_G f(x) \, d\mu_G(x) \]
and, on the other hand,
\[ \int_G f(\phi(xy^{-1})) \, d\mu_G(x) \]
Therefore

\[ \Delta_G(y) = \Delta_G(\phi(y)). \]

19: **Lemma** If \( G_1, G_2 \) are locally compact groups and if \( \mu_{G_1}, \mu_{G_2} \) are left Haar measures per \( G_1, G_2 \), then \( \mu_{G_1} \times \mu_{G_2} \) is a left Haar measure per \( G_1 \times G_2 \) and

\[
\Delta_{G_1 \times G_2}(x_1, x_2) = \Delta_{G_1}(x_1) \Delta_{G_2}(x_2).
\]

Let \( G \) be a locally compact group, \( X \) and \( Y \) two closed subgroups of \( G \).

20: **Definition** The pair \((X, Y)\) is admissible if the following conditions are satisfied.

- The intersection \( X \cap Y \) is compact.
- The multiplication \( X \times Y \to G \) is an open map.
- The set of products \( XY \) exhausts \( G \) up to a set of Haar measure 0 (left or right).

21: **Example** Using the notation of #19, work with \( G_1 \times G_2 \) and take

\[ X = G_1 \times \{e_2\}, \ Y = \{e_1\} \times G_2 \] -- then the pair \((X, Y)\) is admissible.

22: **Theorem** Suppose that the pair \((X, Y)\) is admissible. Fix left Haar
measures \( \mu_X, \mu_Y \) on \( X, Y \) — then there is a unique left Haar measure \( \mu_G \) on \( G \) such that \( \forall f \in C_c(G) \),

\[
    \int_G f d\mu_G = \int_X \int_Y f(xy) \frac{\Delta_G(y)}{\Delta_Y(y)} d\mu_X(x) d\mu_Y(y).
\]

23: N.B. Specializing the setup to that of \#21 leads back to \#19.

[Note that thereby cancelling the factor in the denominator.]

24: LEMMA If \( G \) is a locally compact group and if \( H \subset G \) is a closed normal subgroup, then \( \Delta_G|_H = \Delta_H \).

25: APPLICATION In the setup of \#22, assume in addition that \( Y \) is normal — then \( \forall f \in C_c(G) \),

\[
    \int_G f d\mu_G = \int_X \int_Y f(xy) d\mu_X(x) d\mu_Y(y).
\]

[Note: Given \( x \in X \), the restriction

\[
    - \quad I_{x^{-1}}: Y \to Y
    - \quad x^{-1}: y \mapsto x^{-1}yx
\]

is an automorphism of \( Y \) and

\[
    \Delta_G(xy) = \Delta_X(x) \Delta_Y(y) \delta_Y(I_{x^{-1}}).]
\]
Let $G$ be a locally compact group, $X$ and $Y$ two closed subgroups of $G$.

26: **DEFINITION** $G$ is the **topological semidirect product** of $X$ and $Y$ if every element $z \in G$ can be expressed in a unique manner as a product $z = xy$ ($x \in X$, $y \in Y$) and if the multiplication $X \times Y \to G$ is a homeomorphism.

27: **N.B.** A priori, the multiplication $X \times Y \to G$ is a continuous bijection, thus the condition is satisfied if the multiplication $X \times Y \to G$ is an open map, this being automatic whenever $G$ is second countable.

[Under these circumstances, $G$ is the union of a sequence of compact sets (cf. #2), so the same is true of $X \times Y$. But $G$ is a locally compact Hausdorff space, hence is a Baire space.]

[Note: If $A$ is a Baire space and if $\{A_n : n \in \mathbb{N}\}$ is a closed covering of $A$, then at least one $A_n$ must contain an open set.]

If $G$ is the topological semidirect product of $X$ and $Y$, then $X \cap Y = \{e\}$ and the pair $(X,Y)$ is admissible. Therefore the theory is applicable in this situation.

28: **N.B.** In general, the arrow $(x,y) \to xy$ is not an isomorphism of groups but this will be the case if every element of $X$ commutes with every element of $Y$ or, equivalently, if $X$ and $Y$ are normal subgroups of $G$, i.e., if $G$ is the topological direct product of $X$ and $Y$. 
§3. UNIMODULARITY

Let $G$ be a locally compact group, $\mu_G$ a left Haar measure on $G$.

1: **DEFINITION** $G$ is unimodular if $\Delta_G \equiv 1$.

2: **N.B.** $G$ is unimodular iff $\mu_G$ is a right Haar measure on $G$.

3: **EXAMPLE** Take for $G$ the group of all real matrices of the form

\[
\begin{pmatrix}
1 & x \\
0 & y
\end{pmatrix}
\quad (y \neq 0) \quad \text{-- then}
\]

\[
\Delta_G \begin{pmatrix}
1 & x \\
0 & y
\end{pmatrix} = |y|,
\]

thus $G$ is not unimodular.

4: **LEMMA** $G$ is unimodular iff $\forall f \in C_c(G)$,

\[
\int_G f(x^{-1})d\mu_G(x) = \int_G f(x)d\mu_G(x) \quad (\text{cf. §2, #13}).
\]

5: **LEMMA**

- Every locally compact abelian group is unimodular.
- Every compact group is unimodular.
- Every discrete group is unimodular.

6: **LEMMA** Every locally compact group that coincides with its closed commutator subgroup is unimodular.
7: **Lemma** Every open subgroup of a unimodular locally compact group is unimodular.

8: **Lemma** Every closed normal subgroup of a unimodular locally compact group is unimodular.

[Note: A closed subgroup of a unimodular locally compact group is not necessarily unimodular.]

9: **Lemma** Let $G$ be a locally compact group, $Z(G)$ its center -- then $G$ is unimodular iff $G/Z(G)$ is unimodular.

Let $G$ be a locally compact group, $H \triangleleft G$ a closed subgroup ($H$ is then a locally compact subgroup) (cf. §1, #8).

10: **Definition** $H$ is a **cocompact** subgroup if the quotient $G/H$ is compact.

11: **Lemma** If $G$ admits a unimodular cocompact subgroup $H \triangleleft G$, then $G$ is unimodular.
§4. INTEGRATION ON HOMOGENEOUS SPACES

Let $G$ be a locally compact group, $H \subset G$ a closed subgroup.

1: N.B. The quotient $G/H$ is a locally compact Hausdorff space (cf. §1, #14).

Fix left Haar measures

\[
\begin{array}{c}
\mu_G \text{ on } G \\
\mu_H \text{ on } H.
\end{array}
\]

2: NOTATION Given $f \in C_c(G)$, define $f^H \in C_c(G/H)$ by the rule

\[f^H(xH) = \int_H f(xy) \, d\mu_H(y).\]

3: LEMMA The arrow $f \rightarrow f^H$ sends $C_c(G)$ onto $C_c(G/H)$.

4: DEFINITION A Radon measure $\mu \neq 0$ on the Borel $\sigma$-algebra of $G/H$ is said to be an invariant measure if $\forall x \in G$ and $\forall$ Borel set $A \subset G/H$, $\mu(xA) = \mu(A)$.

[Note: If $H = \{e\}$, then "invariant measure" = "left Haar measure".]

5: THEOREM There exists an invariant measure $\mu_{G/H}$ on $G/H$ iff $\Delta_G \mid H = \Delta_H$ and when this is so, $\mu_{G/H}$ is unique up to a positive scalar factor.

[Note: Matters are automatic if $H$ is compact or if both $G$ and $H$ are unimodular.]
2.

6: **N.B.** If $H$ is a normal closed subgroup of $G$, then $\Delta_G|H = \Delta_H$.

[For a left Haar measure on $G/H$ is an invariant measure.]

7: **THEOREM** There is a unique choice for $\mu_{G/H}$ such that $\forall f \in \mathcal{C}_c(G)$,

$$\int_G f(x) d\mu_G(x) = \int_{G/H} f^H(x) d\mu_{G/H}(x) \quad (x = xH).$$

[Note: Bear in mind that $\mu_G, \mu_H$ have been fixed at the beginning.]

8: **N.B.** This formula is valid for all $f \in L^1(G)$.

9: **LEMMA** Let $H_1 \subset G$, $H_2 \subset G$ be closed subgroups of $G$ with $H_1 \subset H_2$ --

then $G/H_2$ and $H_2/H_1$ admit finite invariant measures iff $G/H_1$ admits a finite

invariant measure.

10: **APPLICATION** If $G/H$ has a finite invariant measure and if $H$ is

unimodular, then $G$ is unimodular.

[Let $K$ be the kernel of $\Delta_G$ -- then $\Delta_G|H = \Delta_H \equiv 1$, thus $H \subset K$ and so $G/K$

has a finite invariant measure (as does $K/H$). But $G/K$ is a locally compact group.

Therefore $G/K$ is actually a compact group (its Haar measure being finite) and this

implies that $\Delta_G(G)$ is a compact subgroup of $\mathbb{R}_0^\times$, hence $\Delta_G(G) = \{1\}$, i.e., $G$ is

unimodular.]

11: **N.B.** Suppose that $H \subset G$ is a unimodular cocompact subgroup -- then

$G/H$ admits a finite invariant measure $\mu_{G/H}$.

[In fact, $G$ is necessarily unimodular (cf. §3, #11), from which the existence
of $\mu_{G/H}$. But $\mu_{G/H}$ is Radon, hence finite on compacta, hence in particular,

$$\mu_{G/H}(G/H) < \infty.$$ 

[Note: Take $G = SL(2,\mathbb{R})$ and let

$$H = \{X : X = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \in SL(2,\mathbb{R})\}.$$ 

Then $G$ is unimodular but $H$ is not unimodular. Therefore $G/H$ does not admit an invariant measure even though $H$ is a cocompact subgroup.]

12: LEMMA Let $H_1 \subset G, H_2 \subset G$ be closed subgroups of $G$ with $H_1$ normalizing $H_2$ and $H_1 H_2$ closed in $G$ -- then the following are equivalent.

- $H_1 H_2 / H_1$ admits a finite invariant measure.
- $H_2 / H_1 \cap H_2$ admits a finite invariant measure.

[Note: There is a commutative diagram

$$
\begin{array}{cccc}
H_2 & & & H_2 \\
\downarrow & & \downarrow & \\
H_2 / H_1 \cap H_2 & \overset{\phi}{\longrightarrow} & H_1 H_2 / H_1,
\end{array}
$$

where

$$\phi(x_2(H_1 \cap H_2)) = x_2 H_1.$$ 

The vertical arrows are continuous and open. Therefore the bottom horizontal arrow is a homeomorphism.]

13: APPLICATION Suppose that $G$ is the topological semidirect product
of $X$ and $Y$ (cf. §2, #26) and take $Y$ normal -- then $G = XY$ and $X \cap Y = \{e\}$.

Therefore $G/X$ has a finite invariant measure iff $Y$ has a finite invariant measure.
§5. INTEGRATION ON LIE GROUPS

Suppose that \( M \) is an orientable \( n \)-dimensional \( C^\infty \) manifold which we take to be second countable. Let \( \omega \) be a positive \( n \)-form on \( M \) -- then the theory leads to a positive linear functional

\[ f \to f_M f_\omega \quad (f \in C^\infty_c(M)) \]

from which a Radon measure \( \mu_\omega \).

Assume now that \( G \) is a Lie group with Lie algebra \( g \). Let \( L_x: G \to G \) be left translation \( y \to xy \) by \( x \).

1: DEFINITION A differential form \( \omega \) on \( G \) is left invariant if \( \forall x \in G \), \( L_{x}^*\omega = \omega \).

2: NOTATION Given \( X \in g \), let \( \tilde{X} \) be the corresponding left invariant vector field on \( G \).

Let \( n = \dim G = \dim g \) and fix a basis \( X_1, \ldots, X_n \) for \( g \). Define 1-forms \( \omega^1, \ldots, \omega^n \) on \( G \) by the condition \( \omega^i(X_j) = \delta^i_j \).

3: LEMMA The \( \omega^i \) are left invariant.

Put
\[ \omega = \omega^1 \wedge \ldots \wedge \omega^n. \]

Then \( \forall x \in G \),
\[ L_{x}^*\omega = L_{x}^*(\omega^1 \wedge \ldots \wedge \omega^n) \]
\[ = L_{x}^*\omega^1 \wedge \ldots \wedge L_{x}^*\omega^n. \]
2.

\[ = \omega^1 \wedge \ldots \wedge \omega^n = \omega. \]

I.e.: \( \omega \) is a left invariant \( n \)-form on \( G \).

4: LEMMA \( \omega \) is nowhere vanishing on \( G \).

5: LEMMA \( G \) can be oriented so as to render \( \omega \) positive. 

[Note: The orientation of \( G \) depends on the choice of a basis for \( g \). If \( Y_1, \ldots, Y_n \) is another basis, then the resulting orientation of \( G \) does not change iff the linear transformation \( X_i \to Y_i \) \( (1 \leq i \leq n) \) has positive determinant.]

6: SCHOLIUM The assignment 

\[ f \to \int_G f \omega \quad (f \in C_c(G)) \]

is a positive linear functional.

7: LEMMA The Radon measure \( \mu_\omega \) is a left Haar measure.

PROOF \( \forall x \in G, L_x : G \to G \) is an orientation preserving diffeomorphism, so \( \forall f \in C_c(G), \)

\[ \int_G f d\mu_\omega = \int_G f \omega = \int_G (f \circ L_x^*) dx \omega \]

\[ = \int_G (f \circ L_x) \omega = \int_G (f \circ L_x) d\mu_\omega. \]

8: REMARK Any subset \( S \) of \( G \) which is contained in an at most countable union of smooth images of \( C^\infty \) manifolds of dimension \( < \dim G \) has zero left Haar measure.

9: THEOREM \( \forall x \in G, \)

\[ \Delta_G(x) = \frac{1}{|\det \text{Ad}(x)|^1}. \]
10: EXAMPLE Every connected nilpotent Lie group $N$ is unimodular.

[If $X \in \mathfrak{n}$ (the Lie algebra of $N$), then $\text{ad}(X)$ is nilpotent, thus $\text{tr}(\text{ad}(X)) = 0$ and so

$$\det \text{Ad}(\exp X) = \det e^{\text{ad}(X)}$$

$$= e^{\text{tr}(\text{ad}(X))} = 1.$$]

11: LEMMA A 1-dimensional representation of a connected semisimple Lie group is trivial.

12: APPLICATION The restriction of $\Delta_G$ to any semisimple analytic subgroup of $G$ is $\equiv 1$.

13: THEOREM Suppose that $G$ is a reductive Lie group in the Harish-Chandra class -- then $G$ is unimodular.

PROOF First decompose $G$ as the product $^0G \times V$, where $V$ is a central vector group (possibly trivial) and

$$^0G = \bigcap_X \ker \chi,$$

the $\chi$ running through the set of continuous homomorphisms $G \to \mathbb{R}_{>0}^X$. This done, take for a left Haar measure on $G$ the product of the left Haar measures on $^0G$ and $V$. Since $V$ is unimodular, it will be enough to deal with $^0G$ (cf. §2, #19).

Fix a maximal compact subgroup $K$ of $G$ -- then $K$ is a maximal compact subgroup of $^0G$ and $^0G = KG_{ss}$, thus $\forall k \in K$, $\forall x \in G_{ss}$,

$$\Delta_{^0G}(kx) = \Delta_{^0G}(k)\Delta_{^0G}(x) = 1 \cdot 1 = 1.$$
[Note: $G_{SS}$ is the analytic subgroup of $G$ corresponding to $g_{SS}$ (the ideal in $g$ spanned by $[g,g]$). It is closed and normal.]

Maintaining the supposition that $G$ is a reductive Lie group in the Harish-Chandra class, consider an Iwasawa decomposition $G = KAN$.

14: N.B. $N$ is a normal subgroup of $AN$ and $AN$ is the topological semi-direct product of $A$ and $N$.

[Note: $AN$ is second countable so there are no technical issues.]

15: Lemma

$$\Delta_{AN}(an) = \frac{1}{|\text{det } Ad(an)|} = \frac{1}{e^{2\rho(\log a)}}.$$  

[Note: Here $2\rho$ is the sum of the positive roots of $(g,a)$ counted with multiplicities.]

Since the pair $(K,AN)$ is admissible and since $\Delta_{G} = 1$, it follows from §2, #22 that $\forall f \in C_{c}(G)$,

$$\int_{G} fd\mu_{G} = \int_{K \times AN} f(kan) \frac{\Delta_{G}(an)}{\Delta_{AN}(an)} d\mu_{K}(k) d\mu_{AN}(an)$$

$$= \int_{K \times AN} f(kan) \frac{1}{\Delta_{AN}(an)} d\mu_{K}(k) d\mu_{AN}(an)$$

$$= \int_{K \times AN} f(kan) e^{2\rho(\log a)} d\mu_{K}(k) d\mu_{AN}(an)$$

$$= \int_{K \times A \times N} f(kan) e^{2\rho(\log a)} d\mu_{K}(k) d\mu_{A}(a) d\mu_{N}(n).$$
[Note: To be completely precise, fix left Haar measures \( \mu_K, \mu_A, \mu_N \) on \( K, A, N \) -- then there is a unique determination of the left Haar measure \( \mu_G \) on \( G \) such that for any \( f \in C_c(G) \), the function
\[
(k,a,n) \to f(kan)
\]
lies in
\[
C_c(K \times A \times N)
\]
and
\[
\int_G f d\mu_G = \int_{K \times A \times N} f(kan) e^{2\rho(\log a)} d\mu_K(k) d\mu_A(a) d\mu_N(n).
\]

16: Lemma
\[
\Delta_{AN}(an) = \Delta_A(a) \Delta_N(n) \delta_n(I_{a^{-1}}) \quad (\text{cf. } §2, \#25)
\]
\[
= \delta_N(I_{a^{-1}}).
\]

[Note: \( A \) is abelian and \( N \) is nilpotent... .]

So, \( \forall f \in C_c(G) \),
\[
\int_{K \times N \times A} f(kna) d\mu_K(k) d\mu_N(n) d\mu_A(a)
\]
\[
= \int_{K \times N \times A} f(ka^{-1}na) d\mu_K(k) d\mu_N(n) d\mu_A(a)
\]
\[
= \int_{K \times N \times A} f(kaI_{a^{-1}}(n)) d\mu_K(k) d\mu_N(n) d\mu_A(a)
\]
\[
= \int_{K \times N \times A} f(kan) \delta_n(I_{a}) d\mu_K(k) d\mu_A(a) d\mu_N(n) \quad (\text{cf. } §2, \#15)
\]
\[
= \int_{K \times A \times N} f(kan) \Delta_{AN}(a^{-1}n) d\mu_K(k) d\mu_A(a) d\mu_N(n)
\]
6.

\[ = \int_{K \times A \times N} f(kan) e^{2\rho(\log a)} d\mu_K(k) d\mu_A(a) d\mu_N(n) \]

\[ = \int_G f d\mu_G. \]

[Note: As a corollary,]

\[ = \int_{K \times N \times A} f(kan) d\mu_K(k) d\mu_N(n) d\mu_A(a) \]

\[ = \int_{K \times N \times A} f(kan) d\mu_K(k) d\mu_N(n) d\mu_A(a) \quad (\text{cf. } \S3, \#4) \]

\[ = \int_G f d\mu_G = \int_G f d\mu_G', \]

\( G \) being unimodular (cf. \#13).]

Let \( M \) be the centralizer of \( a \) in \( K \) and put \( \tilde{N} = \emptyset N \) -- then the map

\( (\tilde{n}, m, a, n) + \tilde{\text{man}} \)

is an open bijection of \( \tilde{N} \times M \times A \times N \) onto an open submanifold \( \tilde{N}\text{MAN} \subset G \).

17: LEMMA The complement of \( \tilde{N}\text{MAN} \) in \( G \) is a set of Haar measure 0.

[Using the Bruhat decomposition, the said complement is seen to be a finite union of smooth images of \( C^\infty \) manifolds of dimension < \( \text{dim } G \) so one can quote \#8.]

The pair \( (\tilde{N}, \text{MAN}) \) is therefore admissible, hence \( \forall f \in C_c(G) \) (cf. \$2,\#22),

\[ = \int_{\tilde{N} \times \text{MAN}} f(\tilde{\text{man}}) \frac{\Delta_G(\text{man})}{\Delta_{\text{MAN}}(\text{man})} d\mu_N(\tilde{n}) d\mu_{\text{MAN}}(\text{man}) \]
$$= f_{\tilde{N} \times \text{MAN}} f(\tilde{\text{man}}) \frac{1}{\Lambda_{\text{MAN}}(\text{man})} d\mu_{\tilde{N}}(\tilde{n}) d\mu_{\text{MAN}}(\text{man})$$

$$= f_{\tilde{N} \times \text{MAN}} f(\tilde{\text{man}}) e^{2\rho(\log a)} d\mu_{\tilde{N}}(\tilde{n}) d\mu_{\text{MAN}}(\text{man})$$

$$= f_{\tilde{N} \times M \times A \times N} f(\tilde{\text{man}}) e^{2\rho(\log a)} d\mu_{\tilde{N}}(\tilde{n}) d\mu_{M}(m) d\mu_{A}(a) d\mu_{N}(n).$$

18: **RAPPEL** Let $V$ be a finite dimensional real Hilbert space -- then the canonical Haar measure $dV$ on $V$ is that in which the parallelepiped determined by an orthonormal basis has unit measure.

[Spelled out, if $\{X_1, \ldots, X_n\}$ is an orthonormal basis for $V$ and if $Q$ is the set of all $X = \sum_{i=1}^{d} c_i X_i$ ($c_i \in \mathbb{R}$) with $0 \leq c_i \leq 1$, then

$$\int_{Q} dV = 1.$$]

[Note: Matters are independent of the particular choice of an orthonormal basis since the transition matrix between any two such is orthogonal, hence the absolute value of its determinant is 1.]

19: **SUBLEMMA** Let $V$ be a finite dimensional real Hilbert space; let $V_1 \subset V$, $V_2 \subset V$ be subspaces. Suppose that $T:V_1 \to V_2$ is a bijective linear transformation -- then $\forall \phi \in C_c(V_2)$,

$$\int_{V_2} \phi \ dV_2 = |\det T| \int_{V_1} \phi \circ T \ dV_1,$$

where the determinant is computed relative to an orthonormal basis in $V_1$ and an orthonormal basis in $V_2$. 
20: **N.B.** Symbolically,

\[ dV_2 = |\det T| \, dV_1. \]

21: **CONVENTION** Extend the Killing form on \( g_{ss} \times g_{ss} \) to a nondegenerate symmetric bilinear form \( B : g \times g \to \mathbb{R} \) with the following properties:

- \( B \) is \( \text{Ad} \, G \) invariant.
- \( B \) is \( \theta \)-invariant.
- \( k \) and \( p \) are orthogonal under \( B \).
- \( B \) is positive definite on \( p \) and negative definite on \( k \).

22: **N.B.** The bilinear form

\[ (X,Y)_\theta = -B(X,\theta Y) \quad (X,Y \in g) \]

equips \( g \) with the structure of a real Hilbert space.

Relative to this data, any subspace \( \mathfrak{l} \) of \( g \) carries a canonical Haar measure \( dl \), an instance being the Lie algebra \( \mathfrak{l} \) of a closed Lie subgroup \( L \) of \( G \).

23: **EXAMPLE** \( k \) and \( p \) are orthogonal and \( dg = dk dp \).

[Note: The orthogonal projections \( E_k, E_p \) of \( g \) onto \( k, p \) are given by

\[
\begin{bmatrix}
E_k &=& \frac{1 + \theta}{2} \\
E_p &=& \frac{1 - \theta}{2}
\end{bmatrix}
\]

respectively.]

24: **CONSTRUCTION** Choose an open neighborhood \( N_0 \) of \( 0 \) in \( \mathfrak{l} \) and an open
neighborhood $N_e$ of $e$ in $L$ such that $\exp$ is an analytic diffeomorphism of $N_0$ onto $N_e$. Normalize the left Haar measure $\mu_L$ on $L$ in such a way that $\forall f \in C_c(N_e)$,

$$\int_{N_e} f d\mu_L = \int_{N_0} F d1,$$

where

$$F(X) = f(\exp X) \left| \det \left[ \frac{I-e^{-ad(x)}}{ad(x)} \right] \right|.$$

This fixes $\mu_L$ uniquely, call it $dL$, and its definition is independent of the choice of $N_0$.

25: N.B. If $L$ is compact, put

$$\text{vol}(L) = \int_L dL$$

and term $\frac{1}{\text{vol}(L)} dL$ the normalized Haar measure of $L$.

Now write after Iwasawa $G = KAN$, thus $\forall f \in C_c(G)$,

$$\int_G f d\mu_G = \int_K \times A \times N f(\text{kan}) e^{2\rho(\log a)} du_K(k) du_A(a) du_N(n).$$

On the right hand side, take

$$du_K(k) = \frac{1}{\text{vol}(K)} dk, \quad du_A(a) = da, \quad du_N(n) = dN.$$

Then these choices determine $d\mu_G$ uniquely, denote it by the symbol $d_{st}G$ and refer to it as the standard Haar measure of $G$.

26: LEMMA

$$dG = 2^{-\frac{1}{2} \dim N} e^{2\rho(\log a)} dK dA dN.$$
PROOF It suffices to show that
\[ \frac{1}{2} \dim N \]
\[ d_{\text{st}} G = e^{2\rho (\log a)} \left( \frac{dK}{\text{vol}(K)} \right) d\text{Ad}N \]
\begin{align*}
\frac{1}{\text{vol}(K)} & \quad \frac{-\frac{1}{2} \dim N}{2} e^{2 \rho (\log a)} dK \mathrm{d} \alpha d\mathbb{N} \\
= \frac{1}{\text{vol}(K)} & \quad \frac{\frac{1}{2} \dim N}{2} d \mathbb{G}.
\end{align*}
§1. TRANSVERSALS

Let $G$ be a locally compact group.

1: SUBLEMMA Fix $x \in G$ -- then for any open neighborhood $U$ of $e$ there exists an open neighborhood $V$ of $x$ such that $V^{-1}V \subset U$.

2: DEFINITION A subgroup $\Gamma \subset G$ is a discrete subgroup if the relative topology on $\Gamma$ is the discrete topology.

3: LEMMA A subgroup $\Gamma \subset G$ is discrete iff there exists an open neighborhood $U$ of $e$ (in $G$) such that $\Gamma \cap U = \{e\}$.

4: THEOREM Suppose that $\Gamma \subset G$ is a discrete subgroup -- then $\Gamma$ is closed in $G$, hence $G/\Gamma$ is a locally compact Hausdorff space (cf. I, §1, #14).

5: EXAMPLE
   - Take $G = \mathbb{R}$, $\Gamma = \mathbb{Z}$.
   - Take $G = \mathbb{A}$, $\Gamma = \mathbb{Q}$.
   - Take $G = \mathbb{I}$, $\Gamma = \mathbb{Q}^\times$.

6: LEMMA Let $\Gamma$ be a discrete subgroup of $G$ -- then there exists an open neighborhood $U_0$ of $e$ such that $U_0 \gamma \cap U_0 = \emptyset$ for all $\gamma \neq e$ in $\Gamma$.

   PROOF First choose $U$ per #3. This done, choose $V$ per #1 (with $x = e$) and put $U_0 = V$. Assume now that $u'_0 \in U_0 \gamma \cap U_0$, thus $u'_0 = u_0 \gamma$ ($\exists u_0 \in U_0$), so

   $$\gamma = u_0^{-1}u'_0 \in U_0^{-1}U_0 = V^{-1}U \subset U$$

   $$\Rightarrow \gamma = e.$$
7: SUBLEMMA Let $H$ be a closed subgroup of $G$ and give $G/H$ the quotient topology -- then the projection $\pi:G \to G/H$ is an open map.

[Let $U \subset G$ be a nonempty open set, the claim being that $\pi(U) \subset G/H$ is a nonempty open set. But $\pi(U)$ is open iff $\pi^{-1}(\pi(U))$ is open. And

$$\pi^{-1}(\pi(U)) = UH = \bigcup_{h \in H} Uh$$

which is a union of open sets.]

8: THEOREM Suppose that $\Gamma \subset G$ is a discrete subgroup -- then the projection $\pi:G \to G/\Gamma$ is a local homeomorphism.

PROOF Fix $x \in G$ and choose $U_0$ per #6 to get an open neighborhood $xU_0$ of $x$ with the property that $\forall \gamma \neq e$ in $\Gamma$,

$$xU_0 \cap xU_0 = x(U_0 \cap U_0) = \emptyset.$$  

Therefore the arrow $xU_0 \to \pi(xU_0)$ is a continuous bijection, hence is a homeomorphism (cf. #7).

9: DEFINITION Let $\Gamma$ be a discrete subgroup of $G$ -- then a Borel subset $U \subset G$ is a transversal for $G/\Gamma$ if the restriction of $\pi$ to $U$ is bijective.

10: N.B. In other words, a transversal $U$ for $G/\Gamma$ is a Borel subset of $G$ which meets each coset exactly once.

11: THEOREM Suppose that $\Gamma \subset G$ is a discrete subgroup. Assume: $G$ is second countable -- then $G/\Gamma$ admits a transversal $U$.

12: REMARK A transversal $U$ for $G/\Gamma$ gives rise to a unique section $\tau:G/\Gamma \to U \subset G$ ($\pi \circ \tau = \text{id}$) which is Borel measurable if $G$ is second countable.
3.

13: N.B. Tacitly, Lie groups are assumed to be second countable (cf. I, §5), hence σ-compact (cf. I, §2, #2).

[Note: Still, in this situation it is not claimed (nor is it true in general) that smooth sections exist.]

14: EXAMPLE Take $G = \mathbb{R}$, $\Gamma = \mathbb{Z}$ -- then $[0,1]$ is a transversal for $\mathbb{R}/\mathbb{Z}$.

15: EXAMPLE Take $G = A$, $\Gamma = Q$ -- then $\prod_{p} \mathbb{Z}_p \times [0,1]$ is a transversal for $A/Q$.

16: EXAMPLE Take $G = I$, $\Gamma = Q^x$ -- then $\prod_{p} \mathbb{Z}_p^x \times R^x_0$ is a transversal for $I/Q^x$.

17: CONVENTION The Haar measure on a discrete group $\Gamma$ is the counting measure:

$$f_\Gamma f(\gamma) d_\Gamma (\gamma) = \sum_{\gamma \in \Gamma} f(\gamma).$$

[Note: $\Gamma$ is unimodular (being discrete).]

18: LEMMA If $\Gamma \subset G$ is a discrete subgroup and if $G$ is second countable, then $\Gamma$ is at most countable.

19: LEMMA If $\Gamma \subset G$ is a discrete subgroup, if $G$ is second countable and if $\mathcal{V}$ is a transversal for $G/\Gamma$, then

$$G = \bigcup_{\gamma \in \Gamma} \mathcal{V}_\gamma \quad \text{(disjoint union)},$$

$$\int_G f d\mu_G = \sum_{\gamma \in \Gamma} \int_{\mathcal{V}_\gamma} f d\mu_G$$

$$= \int_{\mathcal{V}} f^\Gamma \circ \pi d\mu_G.$$
4.

[Note: \( \forall x \in \mathbb{R}, \]

\[
(f^r \circ \pi)(x) = f^r(xr)
\]

\[
= \int_{\Gamma} f(xy) \, d\mu_{\Gamma}(y)
\]

\[
= \sum_{y \in \Gamma} f(xy).
\]

20: **RAPPEL** If \( G \) is unimodular and if \( \mu_G \) is fixed, then \( G/\Gamma \) admits an invariant measure \( \mu_{G/\Gamma} \) characterized by the condition that for all \( f \in C_c(G), \)

\[
\int_G f(x) \, d\mu_G(x) = \int_{G/\Gamma} f^r(\hat{x}) \, d\mu_{G/\Gamma}(\hat{x}) \quad (x = \hat{x} \Gamma) \quad (\text{cf. I, §4, #7}).
\]

21: **THEOREM** If \( \Gamma \subset G \) is a discrete subgroup, if \( G \) is second countable, if \( \mathcal{T} \) is a transversal for \( G/\Gamma \), if \( G \) is unimodular and if \( \mu_G \) is fixed, then

\[
\forall f \in C_c(G),
\]

\[
\int_{G/\Gamma} f^r \, d\mu_{G/\Gamma} = \int_{\mathcal{T}} f^r \circ \pi \, d\mu_G.
\]

[Simply assemble the foregoing data.]

[Note: Since the \( f^r \) (\( f \in C_c(G) \)) exhaust \( C_c(G/\Gamma) \) (cf. I, §4, #3), it follows that \( \forall \phi \in C_c(G/\Gamma), \)

\[
\int_{G/\Gamma} \phi \, d\mu_{G/\Gamma} = \int_{\mathcal{T}} \phi \circ \pi \, d\mu_G.
\]

In particular, this holds for all \( \phi \) if \( G/\Gamma \) is compact.]

22: **DEFINITION** Let \( \Gamma \) be a discrete subgroup of \( G \) -- then a Borel subset \( \mathcal{F} \subset G \) is a **fundamental domain** for \( G/\Gamma \) if it differs from a transversal by a set of Haar measure 0 (left or right).
23: EXAMPLE Take $G = \mathbb{R}$, $\Gamma = \mathbb{Z}$ -- then $[0,1]$ is a fundamental domain for $\mathbb{R}/\mathbb{Z}$.

24: N.B. What was said in #21 goes through verbatim if "transversal" is replaced by "fundamental domain".
§2. LATTICES

Let \( G \) be a second countable locally compact group, \( \Gamma \subset G \) a discrete subgroup.

1: \textbf{NOTATION} Given a finite subset \( \Delta \subset \Gamma \), let \( G_\Delta \) denote the centralizer of \( \Delta \) in \( G \).

2: \textbf{N.B.} \( G_\Delta \) is closed in \( G \).

3: \textbf{LEMMA} \( G_\Delta \Gamma \) is closed in \( G \).

\textbf{PROOF} Let \( x_n \in G_\Delta \) and \( \gamma_n \in \Gamma \) be sequences such that \( x_n \gamma_n \) converges to a limit \( x \) -- then the claim is that \( x \in G_\Delta \Gamma \). To begin with, \( \forall \gamma \in \Delta \),

\[
x^{-1} \gamma x = \lim_{n \to \infty} (\gamma_n^{-1} x_n^{-1} \gamma x \gamma_n)
\]

\[
= \lim_{n \to \infty} \gamma_n^{-1} \gamma \gamma_n.
\]

Since \( \Gamma \) is discrete, \( \exists n_0(\gamma) \):

\[
n \geq n_0(\gamma) \Rightarrow \gamma_n \gamma_n = \gamma^{-1}_n \gamma \gamma_n + 1
\]

\[
= \gamma_n + 1 \gamma^{-1}_n \in G_\Delta.
\]

But \( \Delta \) is finite, thus \( \exists n_0 \) independent of the choice of \( \gamma \) such that \( n \geq n_0 \Rightarrow \gamma_n = \gamma \gamma_0 \) (\( \gamma_n \in G_\Delta \))

\[
\Rightarrow x_n \gamma_n = x_n \gamma_n \gamma_n = z \gamma_n \gamma_0 \quad (z \in G_\Delta)
\]
2.

\[ z_n = x_n y_n^{-1} + x y_n^{-1} \quad (n \to \infty) \]

\[ x y_n^{-1} \in G_\Delta \Rightarrow x \in G_\Delta. \]

4: NOTATION Given \( \gamma \in \Gamma \), \( G_\gamma \) is its centralizer in \( G \) and \( \Gamma_\gamma \) (= \( G_\gamma \cap \Gamma \)) is its centralizer in \( \Gamma \).

5: N.B. \( G_\gamma \) is a closed subgroup of \( G \), as is \( \Gamma_\gamma \) (cf. §1, #4).

6: LEMMA \( G_\gamma \Gamma \) is closed in \( G \) (cf. #3 (take \( \Delta = \{\gamma\}\))).

7: SUBLEMMA If \( H \) is a closed subgroup of \( G \), if \( \pi:G \to G/H \) is the projection and if \( F \) is a closed subset of \( G \) that is the union of cosets \( xH \), then \( \pi(F) \) is closed in \( G/H \).

8: APPLICATION The image of \[ G_\gamma \Gamma = \bigcup_{x \in G_\gamma} x\Gamma \] in \( G/\Gamma \) is closed, hence is a locally compact Hausdorff space.

9: REMARK The projection \( \pi:G \to G/\Gamma \) is an open map (cf. §1, #7) but, in general, it is not a closed map.

[Take \( G = \mathbb{R} \), \( \Gamma = \mathbb{Z} \) and view \( \mathbb{R}/\mathbb{Z} \) as \([0,1] \) equipped with the topology in which an open basis consists of all sets \( ]a,b[ \) \((0 < a < b < 1) \) and of all sets \]a,b[ \((a < b) \) and \(]a,b[ \)\(]b,a[ \).]
[0, \alpha \cup \beta, 1] (0 < \alpha < \beta < 1) \text{ -- then}

\[ A = \left\{ \frac{3}{2}, \frac{9}{4}, \ldots, n+2^{-n}, \ldots \right\} \]

is closed in \( R \) but

\[ \pi(A) = \left\{ \frac{1}{2}, \frac{1}{4}, \ldots, \frac{1}{2^n}, \ldots \right\} \]

is not closed in \([0,1[.\]

Considered as families of subsets of \( G \), \( G_{\gamma}/\Gamma \) and \( \pi(G_{\gamma}) \) are identical: The elements of \( G_{\gamma}/\Gamma \) are the cosets \( x\Gamma \) with \( x \in G_{\gamma} \) and the elements of \( \pi(G_{\gamma}) \) are the cosets \( x\Gamma \) with \( x \in G_{\gamma} \).

10: LEMMA The identity map

\[ \{x\Gamma : x \in G_{\gamma}/\Gamma\} \to \{x\Gamma : x \in G_{\gamma}\} \]

is a homeomorphism.

[Note: That is to say, the two topologies are the same.]

11: N.B. One may then identify \( \pi(G_{\gamma}) \) with \( G_{\gamma}/\Gamma \) which is therefore closed in \( G/\Gamma \) (cf. #8).

12: NOTATION Let

\[ r : G_{\gamma}/\Gamma \to G_{\gamma}/G_{\gamma} \cap \Gamma \]

be the arrow defined by

\[ r(x\Gamma) = x(G_{\gamma} \cap \Gamma). \]

13: N.B. \( r \) is bijective.
14: THEOREM \( r \) is a homeomorphism.

This is not completely obvious and it will be best to break the proof into two parts.

15: LEMMA \( r \) carries open subsets of \( G \gamma \Gamma / \Gamma \) onto open subsets of \( G \gamma / G \gamma \cap \Gamma \).

PROOF An open subset of \( G \gamma \Gamma / \Gamma \) is a subset \( \{x\Gamma : x \in X\} \), where \( X \subset G \gamma \), such that \( X \Gamma \) is open in \( G \gamma \Gamma \) viewed as a subspace of \( G \). Since

\[
X(G \gamma \cap \Gamma) = X\Gamma \cap G \gamma ,
\]

it follows that \( X(G \gamma \cap \Gamma) \) is an open subset of \( G \gamma \) in its relative topology as a subspace of \( G \), thus by the very definition of the topology on \( G \gamma / G \gamma \cap \Gamma \),

\[
r\{x\Gamma : x \in X\} = \{x(G \gamma \cap \Gamma) : x \in X\}
\]

is an open subset of \( G \gamma / G \gamma \cap \Gamma \).

16: LEMMA \( r^{-1} \) carries open subsets of \( G \gamma / G \gamma \cap \Gamma \) onto open subsets of \( G \gamma \Gamma / \Gamma \).

PROOF Let \( \{y(G \gamma \cap \Gamma) : y \in Y\} \ (Y \subset G \gamma ) \) be an open subset of \( G \gamma / G \gamma \cap \Gamma \) -- then

\( Y(G \gamma \cap \Gamma) \) is an open subset of \( G \gamma \), so

\[
\pi(Y(G \gamma \cap \Gamma)) = \{y\Gamma : y \in Y\}
\]

is open in \( G \gamma \Gamma / \Gamma \) (see the Appendix infra) or still,

\[
\{y\Gamma : y \in Y\} = r^{-1}\{y(G \gamma \cap \Gamma) : y \in Y\}
\]

is open in \( G \gamma \Gamma / \Gamma \).

17: EXAMPLE Take \( G = \mathbb{R} \), \( \Gamma = \mathbb{Z} \) and \( H = \sqrt{2} \mathbb{Z} \) -- then the argument used in
5.

#15 is applicable if the $G$ there is replaced by $H$, thus the map

$$\frac{H}{H \cap \Gamma} \to H + \Gamma/\Gamma$$

is continuous. Nevertheless, it is not a homeomorphism.

[$H \cap \Gamma$ is trivial so $H/H \cap \Gamma$ is isomorphic to $\mathbb{Z}$ and carries the discrete topology. Meanwhile, $H + \Gamma = \sqrt{2} \mathbb{Z} + \mathbb{Z}$ is dense in $\mathbb{R}$, hence

$$H + \Gamma/\Gamma = \sqrt{2} \mathbb{Z} + \mathbb{Z}/\mathbb{Z}$$

is dense in $\mathbb{R}/\mathbb{Z} \cong \mathbb{T}$. It is isomorphic to $\mathbb{Z}$ as a group but it is not discrete since every nonempty open subset of $\mathbb{T}$ intersects it in an infinite set implying thereby that none of its finite subsets are open.]

[Note: The difference here is this: $G \gamma \Gamma/\Gamma$ is locally compact but $H + \Gamma/\Gamma$ is not locally compact.]

18: DEFINITION $\Gamma$ is said to be a lattice if $G/\Gamma$ admits a finite invariant measure (cf. I, §4, #4), $\Gamma$ being termed uniform or nonuniform according to whether $G/\Gamma$ is compact or not.

19: N.B. If there is a lattice in $G$, then $G$ is necessarily unimodular (cf. I, §3, #11 and I, §4, #10).

[Note: A discrete cocompact subgroup is necessarily a uniform lattice... .]

20: EXAMPLE $\mathbb{Z}$ is a uniform lattice in $\mathbb{R}$.

21: EXAMPLE $\text{SL}(2,\mathbb{Z})$ is a nonuniform lattice in $\text{SL}(2,\mathbb{R})$.

22: THEOREM Suppose that $\Gamma \subset G$ is a uniform lattice -- then $\forall \gamma \in \Gamma$, $G_{\gamma}/\Gamma_{\gamma}$ is compact.
PROOF \( G_{\gamma} \Gamma / \Gamma \) is closed in \( G/\Gamma \), hence is compact (this being the case of \( G/\Gamma \)). On the other hand,

\[ r:G_{\gamma} \Gamma / \Gamma \to G_{\gamma}/G_{\gamma} \cap \Gamma \quad (= G_{\gamma}/\Gamma_{\gamma}) \]

is a homeomorphism (cf. #14).

[Note: Consequently, \( \Gamma_{\gamma} \subset G_{\gamma} \) is a uniform lattice and \( G_{\gamma} \) is unimodular.]

23: NOTATION \([\Gamma]\) is a set of representatives for the \( \Gamma \)-conjugacy classes in \( \Gamma \).

Put

\[ S = \bigsqcup_{\gamma \in [\Gamma]} G/\Gamma_{\gamma} \times \{\gamma\} \]

and define \( \psi:S \to G \) by the rule

\[ \psi(x\Gamma_{\gamma}, \gamma) = x\gamma x^{-1}. \]

24: N.B. \( \Gamma_{\gamma} \) is a discrete subgroup of \( G \), thus \( \Gamma_{\gamma} \) is closed in \( G \) (cf. §1, #4) and therefore the quotient \( G/\Gamma_{\gamma} \) is a locally compact Hausdorff space from which it follows that \( S \) is a locally compact Hausdorff space.

25: DEFINITION Let \( X \) and \( Y \) be locally compact Hausdorff spaces, \( f:X \to Y \) a continuous function -- then \( f \) is proper if for every compact subset \( K \) of \( Y \), the inverse image \( f^{-1}(K) \) is a compact subset of \( X \).

26: THEOREM Suppose that \( \Gamma \subset G \) is a uniform lattice -- then \( \psi \) is a proper map.
7.

27: NOTATION Given $\gamma \in \Gamma$, let

$$[\gamma]_G = \{x \gamma x^{-1} : x \in G\}.$$

28: APPLICATION In the uniform situation, for any compact subset $K \subset G$,

$$\{\gamma \in \Gamma : [\gamma]_G \cap K \neq \emptyset\}$$

is finite.

29: LEMMA A proper map $f : X \to Y$ is closed:

$S \subset X$ closed $\Rightarrow f(S) \subset Y$ closed.

30: APPLICATION In the uniform situation, $\forall \gamma \in \Gamma$, $[\gamma]_G$ is closed.

[In fact,

$$[\gamma]_G = \psi\left( \bigcup_{\gamma_0 \in [\gamma]_G \cap [\Gamma]} G/\gamma_0 \times \{\gamma_0\}\right).$$

31: N.B. Accordingly, $[\gamma]_G$ is a locally compact Hausdorff space and the canonical arrow

$$G/\gamma \to [\gamma]_G$$

is a homeomorphism.

APPENDIX

Denote by $\pi|_G$ the restriction of $\pi : G \to G/\Gamma$ to $G_\gamma$.

CRITERION Suppose that there exist nonempty open sets

$$U \subset G_\gamma, \ V \subset G_\gamma / \Gamma$$

such that the restriction of $\pi|_G$ to $U$ is an open continuous map of $U$ onto $V$ -- then $\pi|_G$ is open.
PROOF Given \( x \in G \) and an open neighborhood \( W \) of \( x \) in \( G \), it suffices to show that \( (\pi|_{G})_{\gamma}(W) \) contains an open neighborhood \( N_{x} \) of \( (\pi|_{G})_{\gamma}(x) \). So fix a point \( y \in U \) and put

\[
\tilde{U} = U \cap yx^{-1}W,
\]

an open neighborhood of \( y \) in \( U \), thus the image \( (\pi|_{G})_{\gamma}(\tilde{U}) \) is an open subset of \( G_{\gamma}/\Gamma \) or still, \( \tilde{U} \) is an open subset of \( G_{\gamma} \), hence

\[
x y^{-1} \tilde{U} = (x y^{-1} U \cap W) \Gamma
\]

is an open subset of \( G_{\gamma}/\Gamma \) and

\[
x \in x y^{-1} U \cap W.
\]

Put now

\[
N_{x} = \{ z \Gamma : z \in x y^{-1} U \cap W \}.
\]

Then \( N_{x} \) is an open subset of \( G_{\gamma}/\Gamma \) contained in

\[
(\pi|_{G})_{\gamma}(W) = \{ w \Gamma : w \in W \}
\]

to which \( (\pi|_{G})_{\gamma}(x) \) belongs.

There is a commutative diagram

\[
\begin{array}{ccc}
G_{\gamma} & \xrightarrow{\pi|_{G}} & G_{\gamma}/\Gamma \\
\| & & \| \\
G_{\gamma} & \xrightarrow{\pi_{\gamma}} & G_{\gamma}/G_{\gamma} \cap \Gamma \\
\end{array}
\]

and \( G_{\gamma}/\Gamma \) is a locally compact Hausdorff space (cf. \#8), thus is a Baire space.
LEMMA $\pi|_{G_\gamma}$ is an open map.

PROOF The quotient $G_\gamma/G_\gamma \cap \Gamma$ is second countable, hence $\sigma$-compact, hence

$$G_\gamma/G_\gamma \cap \Gamma = \bigcup_{n=1}^{\infty} K_n,$$

where $K_1, K_2, \ldots$ are compact. In view of #15,

$$r^{-1}: G_\gamma/G_\gamma \cap \Gamma \to G_\gamma \Gamma/\Gamma$$

is continuous and one-to-one, so $\forall n$ the restriction of $r^{-1}$ to $K_n$ is a homeomorphism of $K_n$ onto $L_n \equiv r^{-1}(K_n)$:

$$G_\gamma \Gamma/\Gamma = \bigcup_{n=1}^{\infty} L_n,$$

a countable union of compacta. Being Baire, it therefore follows that $\exists n \in \mathbb{N}$ and a nonempty open subset $V$ of $G_\gamma \Gamma/\Gamma$ such that $V \subset r^{-1}(L_n)$. Put

$$U = (\pi|_{G_\gamma})^{-1}(V).$$

Then $U \subset G_\gamma$ is nonempty and open and the restriction of $\pi_\gamma$ to $U$ is an open continuous map of $U$ onto $r(V)$ or still, the restriction of $\pi|_{G_\gamma}$ to $U$ is an open continuous map of $U$ onto $V$. 
§3. UNIFORMLY INTEGRABLE FUNCTIONS

Let $G$ be a unimodular locally compact group and, generically, let $U$ be a compact symmetric neighborhood of the identity in $G$.

1: NOTATION Given a continuous function $f$ on $G$, put

$$f_U(y) = \sup_{x,z \in U} |f(xyz)| \quad (y \in G).$$

2: LEMMA $f_U \in C(G)$, i.e., is a continuous function on $G$.

3: DEFINITION A continuous function $f$ on $G$ is said to be uniformly integrable if there exists a $U$ such that $f_U \in L^1(G)$.

4: N.B. Since $|f| \leq f_U$, it is clear that if $f$ is uniformly integrable, then $f$ is integrable: $f \in L^1(G)$.

5: NOTATION Write $C_{UN}(G)$ for the set of continuous functions on $G$ that are uniformly integrable.

6: LEMMA

$$C_c(G) \subset C_{UN}(G) \subset C_0(G).$$

[Note: As usual, $C_c(G)$ is the set of continuous functions on $G$ that are compactly supported and $C_0(G)$ is the set of continuous functions on $G$ that vanish at infinity.]

7: LEMMA

$$C_{UN}(G) \subset L^2(G).$$
2.

[Integrable functions in $C_0(G)$ are square integrable.

8: EXAMPLE Take $G = \mathbb{R}$ -- then $f(x) = e^{-x^2}$ is uniformly integrable.

9: LEMA If $f, g \in C_{\text{un}} (G)$, then $f * g \in C_{\text{un}} (G)$.

[Working with a common $\mathcal{H}$,

$$
(f * g)_{\mathcal{H}} (y) = \sup_{x,z \in \mathcal{H}} \left| \int_G f(u)g(g^{-1}(xyz))d\mu_G(u) \right|
$$

$$
= \sup_{x,z \in \mathcal{H}} \left| \int_G f(xu)g(g^{-1}(yz))d\mu_G(u) \right|
$$

$$
\leq \sup_{x,z \in \mathcal{H}} \int_G |f(xu)g(g^{-1}(yz))|d\mu_G(u)
$$

$$
\leq \int_G f_{\mathcal{H}}(u)g_{\mathcal{H}}(u^{-1}y)d\mu_G(u)
$$

$$
= (f_{\mathcal{H}} * g_{\mathcal{H}}) (y),
$$

which suffices.

[Note: The convolution $f * g$ is continuous.]

Let $H \subset G$ be a closed subgroup and assume that $H$ is unimodular and cocompact.

10: NOTATION $L^2(G/H)$ is the Hilbert space associated with $\mu_{G/H}$ (the invariant measure on $G/H$ per I, §4, #5).

11: NOTATION $L_{G/H}$ is the left translation representation of $G$ on $L^2(G/H)$.

12: THEOREM Let $f \in C_{\text{un}} (G)$ -- then

$$
L_{G/H}(f) = \int_G f(x)L_{G/H}(x)d\mu_G(x)
$$
is an integral operator on $L^2(G/H)$ with continuous kernel

$$K_f(x,y) = \int_H f(xh^{-1})d\mu_H(h).$$

Since

$$C(G/H \times G/H) \subset L^2(G/H \times G/H),$$

it follows that $\forall f \in C_{\text{c}}(G)$, $L_{G/H}(f)$ is Hilbert-Schmidt, hence is compact.

13: Sublemma Let $U$ be a unitary representation of $G$ on a Hilbert space $H$ with the property that $\forall f \in C_{\text{c}}(G)$, the operator

$$U(f) = \int_G f(x)U(x)d\mu_G(x)$$

is compact -- then $U$ is discretely decomposable, a given irreducible unitary representation of $G$ occurring at most a finite number of times in the orthogonal decomposition of $U$.

[Note: If $G$ is a Lie group, then one can replace $C_{\text{c}}(G)$ by $C_{\text{c}}^\infty(G)$.]

14: N.B. If $G$ is second countable, then $H$ is separable.

15: Application Take $H = L^2(G/H)$, $U = L_{G/H}$ -- then there exist non-negative integers $m(\Pi,L_{G/H})$ ($\Pi \in \hat{G}$) such that

$$L_{G/H} = \bigoplus_{\Pi \in \hat{G}} m(\Pi,L_{G/H})\Pi.$$
§4. THE SELBERG TRACE FORMULA

Let $G$ be a second countable locally compact group, $\Gamma \subset G$ a discrete subgroup. Assume: $\Gamma$ is a uniform lattice -- then $G/\Gamma$ is cocompact and $G$ is necessarily unimodular (cf. §2, #19).

Working with $L^2(G/\Gamma)$, there is an orthogonal decomposition

$$L^2_G/\Gamma = \bigoplus_{\gamma \in \Gamma} m(\pi, L_G/\Gamma) \pi$$

(cf. §3, #15),

the multiplicities $m(\pi, L_G/\Gamma)$ being certain nonnegative integers.

1: RAPPEL $\forall f \in C^\infty(G)$, $L_G/\Gamma (f)$ is an integral operator on $L^2(G/\Gamma)$ with continuous kernel

$$K_f(x,y) = \sum_{\gamma \in \Gamma} f(x\gamma^{-1}y)$$

(cf. §3, #12).

[Note: This implies that $L_G/\Gamma (f)$ is Hilbert-Schmidt.]

2: CONVENTION Fix a Haar measure $\mu_G$ on $G$, take the counting measure on $\Gamma$, and normalize the invariant measure $\mu_{G/\Gamma}$ on $G/\Gamma$ by the stipulation

$$\int_G = \int_{G/\Gamma} \int_\Gamma (\int_{G/\Gamma} \Sigma_\Gamma) .$$

If $f = g * g^*$ $(g \in C^\infty(G))$, then $f \in C^\infty(G)$ (cf. §3, #9),

$$L_G/\Gamma(f) = L^\Gamma_G(g) L^\Gamma_G(g)^*$$

is trace class and (cf. B, II, §2, #8)

$$\text{tr}(L_G/\Gamma(f)) = \int_{G/\Gamma} K_f(\mathbf{x}, \mathbf{x}) d\mu_{G/\Gamma}(\mathbf{x}) \quad (\mathbf{x} = x\Gamma).$$
3. REMARK The assumption that $f = g * g^*$ ($g \in C_{\text{un}}(G)$) is not restrictive.

For if $f = g * h^*$ ($g, h \in C_{\text{un}}(G)$), put

$$T(g,h) = g * h^*$$

and using the same letter for the diagonal, note that

$$T(g,h) = \frac{1}{4}(T(g+h) - T(g-h) - \sqrt{-1} T(g - \sqrt{-1} h) + \sqrt{-1} T(g + \sqrt{-1} h)).$$

Let $\chi_G/\Gamma$ be the characteristic function of $G/\Gamma$, i.e., the function $\equiv 1$.

Choose $\alpha \in C_c(G): a^\Gamma = \chi_G/\Gamma$ (cf. I, §4, #3), thus $\forall x \in G$,

$$\alpha^\Gamma(x\Gamma) = \sum_{y \in \Gamma} \alpha(xy) = 1.$$ 

One can then write

$$\sum_{\Pi \in G} m(\Pi, L_{G/\Gamma}) \text{tr}(\Pi(f))$$

$$= \text{tr}(L_{G/\Gamma}(f))$$

$$= \int_{G/\Gamma} K_f(x, x) d\mu_{G/\Gamma}(x)$$

$$= \int_T K_f(x\Gamma, x\Gamma) d\mu_G(x) \quad (\text{cf. } \S 1, \#21)$$

$$= \int_T \left( \sum_{y \in \Gamma} \alpha(xy) \right) K_f(x\Gamma, x\Gamma) d\mu_G(x)$$

$$= \int_T \sum_{y \in \Gamma} \alpha(xy) K_f(x\Gamma, x\Gamma) d\mu_G(x)$$

$$= \int_T (\alpha K_f)^\Gamma \circ \pi(x) d\mu_G(x)$$

$$= \int_G \alpha(x) K_f(x, x) d\mu_G(x)$$
\[ = \int_G \alpha(x) \sum_{\gamma \in \Gamma} f(x\gamma x^{-1}) d\mu_G(x) \]
\[ = \sum_{\gamma \in \Gamma} \int_G \alpha(x)f(x\gamma x^{-1}) d\mu_G(x). \]

4: NOTATION For any \( \gamma \in \Gamma, \)

\[
\begin{align*}
G_{\gamma} &= \text{centralizer of } \gamma \text{ in } G \\
\Gamma_{\gamma} &= \text{centralizer of } \gamma \text{ in } \Gamma.
\end{align*}
\]

5: RAPPEL \( \Gamma_{\gamma} \) is a uniform lattice in \( G_{\gamma} \) (cf. §2, §22).

[Note: Consequently, \( G_{\gamma} \) is unimodular.]

6: NOTATION For any \( \gamma \in \Gamma, \)

\[
\begin{align*}
[\gamma]_{\Gamma} &= \text{conjugacy class of } \gamma \text{ in } \Gamma \\
[\gamma]_{G} &= \text{conjugacy class of } \gamma \text{ in } G.
\end{align*}
\]

7: RAPPEL There are canonical bijections

\[
\begin{align*}
\Gamma/\Gamma_{\gamma} &\leftrightarrow [\gamma]_{\Gamma} \\
G/G_{\gamma} &\leftrightarrow [\gamma]_{G}.
\end{align*}
\]

Returning to the computation, break the sum over \( \Gamma \) into conjugacy classes in \( \Gamma \), the contribution from

\[ [\gamma]_{\Gamma} = \{ \delta \gamma \delta^{-1} : \delta \in \Gamma/\Gamma_{\gamma} \} \]
being

\[ \sum_{\delta \in \Gamma/\Gamma_Y} J_G (x, \delta^{-1}) f(x, \delta^{-1}) d\mu_G(x) \]

\[ = \sum_{\delta \in \Gamma/\Gamma_Y} J_G (x, \delta^{-1}) f(x, \delta^{-1}) d\mu_G(x) \]

\[ = J_G (\sum_{\delta \in \Gamma/\Gamma_Y} \alpha(x, \delta^{-1})) f(x, \delta^{-1}) d\mu_G(x). \]

8: CONVENTION

Supplementing the agreements in #2, fix a Haar measure \( \mu_G \) on \( G \), take the counting measure on \( \Gamma_Y \), and normalize the invariant measure \( \mu_{G/\Gamma_Y} \) on \( G/\Gamma_Y \) by the stipulation

\[ \int_{G_Y} = \int_{G/\Gamma_Y} \int_{\Gamma_Y} = \int_{G/\Gamma_Y} \sum_{\Gamma_Y} . \]

Next, fix \( \mu_{G/\Gamma_Y} \) via

\[ \int_{G_Y} = \int_{G/\Gamma_Y} \int_{G_Y} \cdot \]

Finally, make the identification

\[ G/\Gamma_Y \cong (G/\Gamma_Y) \backslash (G/\Gamma_Y) \]

and put

\[ \int_{G/\Gamma_Y} = \int_{G/\Gamma_Y} \int_{G_Y/\Gamma_Y} \cdot \]

Moving on,

\[ \int_G (\sum_{\delta \in \Gamma/\Gamma_Y} \alpha(x, \delta^{-1})) f(x, \delta^{-1}) d\mu_G(x) \]
5.

\[ \int_{G/G_Y} \int_{G_Y} \int_{T_Y} \cdots \]

\[ = \int_{G/G_Y} \int_{G_Y} \int_{T_Y} \cdots. \]

But

\[ \int_{T_Y} \int_{\Gamma_T} \alpha(x\eta^{-1}) \]

is \( \equiv 1 \), leaving

\[ \int_{G/G_Y} \int_{G_Y} \int_{T_Y} \cdots = \int_{G/\Gamma_Y} \cdots. \]

Summary:

\[ \text{tr}(L_{G/\Gamma}(f)) = \sum_{\gamma \in [\Gamma]} \int_{G/\Gamma_\gamma} f(x\gamma x^{-1})d\mu_{G/\Gamma_\gamma}(\hat{x}), \]

the sum being taken over a set of representatives for the \( \Gamma \)-conjugacy classes in \( \Gamma \) (cf. §2, #23).

9: N.B.

\[ \int_{G/\Gamma_\gamma} f(x\gamma x^{-1})d\mu_{G/\Gamma_\gamma}(\hat{x}) \]

\[ = \int_{G/G_Y} \left( \int_{G_Y} \int_{T_Y} f(x\gamma x^{-1})d\mu_{G_Y/\Gamma_Y}(\hat{x}) \right) d\mu_{G/G_Y}(\hat{x}) \]

\[ = \int_{G/G_Y} \left( \int_{G_Y} \int_{T_Y} f(x\gamma x^{-1})d\mu_{G_Y/\Gamma_Y}(\hat{x}) \right) d\mu_{G/G_Y}(\hat{x}) \]

\[ = \int_{G/G_Y} f(x\gamma x^{-1}) \int_{G_Y} d\mu_{G_Y/\Gamma_Y} d\mu_{G/G_Y}(\hat{x}) \]

\[ = \text{vol}(G_Y/\Gamma_\gamma) \int_{G/G_Y} f(x\gamma x^{-1})d\mu_{G/G_Y}(\hat{x}). \]

10: DEFINITION Given \( f \in C_{\text{un}}(G) \ast C_{\text{un}}(G) \), the Selberg trace formula is
the relation
\[
\sum_{\Pi \in \mathcal{G}} m(\Pi, L_{G/\Gamma}) \text{tr}(\Pi(f)) = \sum_{\gamma \in \Gamma} \text{vol}(G/\Gamma \gamma) \int_{G/\Gamma \gamma} f(x \gamma x^{-1}) d\mu_{G/\Gamma \gamma}(x),
\]
their common value being
\[
\text{tr}(L_{G/\Gamma}(f)).
\]

11: REMARK Suppose that $G$ is a Lie group -- then
\[
C_c^\infty(G) \ast C_c^\infty(G) = C_c^\infty(G) \quad \text{(Dixmier-Malliavin)}.
\]
Since
\[
C_c^\infty(G) \subseteq C_{\text{un}}(G),
\]
it follows that the Selberg trace formula is valid for all $f \in C_c^\infty(G)$.

Let $G$ be a second countable locally compact group, $\Gamma \subset G$ a uniform lattice.

12: LEMMA Let $\chi: G \to \Gamma$ be a unitary character -- then the multiplicity of $\chi$ in $L^2(G/\Gamma)$ is 1 if $\chi(\Gamma) = \{1\}$ and is 0 otherwise.

Now take $G$ abelian and identify $\hat{G}$ with the unitary character group of $G: \mathbb{C} \leftrightarrow \chi$, the Fourier transform being defined by
\[
\text{tr}(\Pi(f)) = \hat{f}(\chi) = \int_G f(x) \chi(x) d\mu_G(x).
\]

13: NOTATION Let
\[
\Gamma^\perp = \{ \chi \in \hat{G} : \chi(\gamma) = 1 \forall \gamma \in \Gamma \}.
\]
14: N.B. Therefore

\[
\begin{align*}
    &\chi \in \Gamma^1 \Rightarrow m(\chi, L_{G/\Gamma}) = 1 \\
    &\chi \in \Gamma^1 \Rightarrow m(\chi, L_{G/\Gamma}) = 0.
\end{align*}
\]

The Selberg trace formula thus simplifies:

- Matters on the "spectral side" reduce to
  \[ \sum_{\gamma \in \Gamma^1} \hat{f}(\chi). \]

- Matters on the "geometric side" reduce to
  \[ \text{vol}(G/\Gamma) \sum_{\gamma \in \Gamma} f(\gamma). \]

15: DEFINITION The relation

\[ \sum_{\chi \in \Gamma^1} \hat{f}(\chi) = \text{vol}(G/\Gamma) \sum_{\gamma \in \Gamma} f(\gamma) \]

is the Poisson summation formula (cf. A, III, §4, #7) (in that situation

\[ \text{vol}(G/\Gamma) = \frac{|G|}{|\Gamma|}. \]
§5. FUNCTIONS OF REGULAR GROWTH

Let $G$ be a second countable locally compact group, $\Gamma \subset G$ a uniform lattice. While $C^1_{\text{un}}(G)$ is theoretically convenient, there is a larger class of functions that can be fed into the Selberg trace formula.

1. DEFINITION Let $\phi \in C(G) \cap L^1(G)$ be nonnegative -- then $\phi$ is said to be of regular growth if there is a compact symmetric neighborhood $\mathcal{U}$ of the identity in $G$ and a positive constant $C$ (depending on $\phi$ and $\mathcal{U}$) such that $\forall y \in G$,

$$\phi(y) \leq C \int_{\mathcal{U}} \phi(xy) \, d\mu_G(x).$$

2. N.B. In terms of the characteristic function $\chi_{\mathcal{U}}$ of $\mathcal{U}$, $\forall y \in G$,

$$(\chi_{\mathcal{U}} \ast \phi)(y) = \int_G \chi_{\mathcal{U}}(x) \phi(x^{-1}y) \, d\mu_G(x)$$

$$= \int_{\mathcal{U}} \phi(x^{-1}y) \, d\mu_G(x)$$

$$= \int_{\mathcal{U}^{-1}} \phi(xy) \, d\mu_G(x)$$

$$= \int_{\mathcal{U}} \phi(xy) \, d\mu_G(x).$$

3. EXAMPLE Take $G = \mathbb{R}^n$ and fix a real number $r > 0$ such that

$$\int_{\mathbb{R}^n} \frac{1}{(1 + ||Y||)^r} \, dY < \infty.$$ 

Given $\mathcal{U}$, fix a real number $N > 0$ such that $\forall x \in \mathcal{U}$,

$$(1 + ||Y||)^{-r} \leq N(1 + ||X + Y||)^{-r}.$$
Then

\[ \frac{1}{\text{vol}(\mathcal{U})} \int_{\mathcal{U}} \frac{dx}{(1 + ||x + y||)^r} \]

\[ \geq \frac{1}{\text{vol}(\mathcal{U})} \int_{\mathcal{U}} \frac{dx}{(1 + ||y||)^r} \]

\[ = (1 + ||y||)^{-r}. \]

Therefore

\[ \phi(y) = (1 + ||y||)^{-r} \]

is of regular growth.

**4. EXAMPLE** Let \( G \) be a connected semisimple Lie group with finite center and fix a real number \( r > 0 \) such that

\[ f_G \left| \frac{d}{dy} \right|^2(y) (1 + \sigma(y))^{-r} \leq \infty. \]

Given \( \mathcal{U} \), fix a real number \( M > 0 \) such that \( \forall x \in \mathcal{U}, \)

\[ \left| \frac{d}{dy} \right|^2(y) \leq M \left| \frac{d}{dy} \right|^2(xy) \]

and fix a real number \( N > 0 \) such that \( \forall x \in \mathcal{U}, \)

\[ (1 + \sigma(y))^{-r} \leq N(1 + \sigma(xy))^{-r}. \]

Then

\[ \frac{MN}{\text{vol}(\mathcal{U})} \int_{\mathcal{U}} \left| \frac{d}{dy} \right|^2(xy) (1 + \sigma(xy))^{-r} \mu_G(x) \]

\[ \geq \frac{1}{\text{vol}(\mathcal{U})} \int_{\mathcal{U}} \left| \frac{d}{dy} \right|^2(y) (1 + \sigma(y))^{-r} \mu_G(x) \]

\[ = \left| \frac{d}{dy} \right|^2(y) (1 + \sigma(y))^{-r}. \]
Therefore

\[ \phi(y) = \frac{2}{(1 + \sigma(y))} \]

is of regular growth.

5: **DEFINITION** Let \( f \) be a continuous function on \( G \) -- then \( f \) is **admissible** if there exists a function \( \phi \) of regular growth such that \( \forall y \in G \),

\[ |f(y)| \leq \phi(y) \leq C(\chi_{\mathbb{A}} * \phi)(y). \]

[Note: Admissible functions are integrable.]

6: **EXAMPLE** The rapidly decreasing functions on \( \mathbb{R}^n \) are admissible (cf. #3).

7: **LEMMA** If \( f \in \mathcal{C}_{\mathcal{U}N}(G) \), then \( f \) is admissible.

**PROOF** \( \forall y \in G \), \( |f(y)| \leq f_\mathbb{U}(y) \). And

\[ f_\mathbb{U}(y) = \sup_{u,z \in \mathbb{U}} |f(uyz)| \]

\[ \leq \sup_{u,z \in \mathbb{U}} |f(uxyz)| \quad (x \in \mathbb{U}) \]

\[ \Rightarrow \]

\[ f_\mathbb{U}(y) = \frac{\text{vol}(\mathbb{U})}{\text{vol}(\mathbb{U})} f_\mathbb{U}(y) \]

\[ \leq \frac{1}{\text{vol}(\mathbb{U})} \int_{\mathbb{U}} f_\mathbb{U}(y) \, d\mu_G(x) \]

\[ \leq \frac{1}{\text{vol}(\mathbb{U})} \int_{\mathbb{U}} \sup_{x,z \in \mathbb{U}} |f(uxyz)| \, d\mu_G(x) \]

\[ = \frac{1}{\text{vol}(\mathbb{U})} \int_{\mathbb{U}} f_\mathbb{U}(xy) \, d\mu_G(x). \]
Therefore $f_\|\|$ is of regular growth, hence $f$ is admissible.

8: LEMMA Suppose that $|f| \leq |g|$, where $g$ is admissible, say $|g| \leq \psi$ -- then $f$ is admissible (clear) as is $f \ast f$.

[For]

$$|f \ast f| \leq |f| \ast |f|$$
$$\leq |g| \ast |g|$$
$$\leq \psi \ast |g|.$$ 

And $\psi \ast |g|$ is of regular growth:

$$\psi \ast |g| \leq (C_{\|\|} \ast \psi) \ast |g|$$

$$= C(\chi_{\|\|} \ast (\psi \ast |g|)).$$ 

The condition of admissible is then met by

$$\phi = \psi \ast |g|.$$ 

[Note: If $f_1, f_2 \in C(G) \cap L^1(G)$ and if $f_1$ is admissible, then $f_1 \ast f_2$ is admissible. Proof:

$$|f_1 \ast f_2| \leq |f_1| \ast |f_2|$$
$$\leq \phi_1 \ast |f_2|$$
$$\leq C(\chi_{\|\|} \ast (\phi_1 \ast |f_2|)).]$$

9: DEFINITION A series of functions $f_1, f_2, \ldots$ on a locally compact Hausdorff space $X$ is locally dominantly absolutely convergent (ldac) if for every compact set $K \subset X$ there exists a positive constant $M_K$ such that $\forall k \in K$,
5.

$$\sum_n |f_n(k)| < M_K.$$ 

10: CRITERION Let \(f \in C(G) \cap L^1(G)\). Assume: The operator \(L_{G/\Gamma}(f)\) is trace class and the series

$$\sum_{\gamma \in \Gamma} f(x\gamma^{-1})$$

is ldac on \(G \times G\) to a separately continuous function — then the Selberg trace formula obtains:

$$\text{tr}(L_{G/\Gamma}(f)) = \sum_{\gamma \in \Gamma} \text{vol}(G/\Gamma) \int_{G/\Gamma} f(x\gamma^{-1}) \, \text{d}\mu_{G/\Gamma}(\hat{x}),$$

the sum on the right hand side being absolutely convergent.

[First of all,

$$\text{tr}(L_{G/\Gamma}(f)) = \int_{G/\Gamma} \sum_{\gamma \in \Gamma} f(x\gamma^{-1}) \, \text{d}\mu_{G/\Gamma}(\hat{x}) \quad (\text{cf. B, II, §2, §8}).$$

Proceeding, fix a compact set \(K \subset G\): \(K\Gamma = G\) (cf. §11 infra) and choose \(M_K > 0\):

$$k, \ell \in K \Rightarrow \sum_{\gamma \in \Gamma} |f(\gamma \ell^{-1})| < M_K.$$ 

Here, of course, the ldac condition is per \(K \times K^{-1} \subset G \times G\). Given \(x, y \in G\), \(\exists \gamma_x, \gamma_y \in \Gamma: x\gamma_x, y\gamma_y \in K\), so

$$\sum_{\gamma \in \Gamma} |f(x\gamma^{-1})| = \sum_{\gamma \in \Gamma} |f(x\gamma x\gamma^{-1} y^{-1})| < M_{K'},$$

from which

$$M_K \, \text{vol}(G/\Gamma) \geq \int_{G/\Gamma} \sum_{\gamma \in \Gamma} |f(x\gamma^{-1})| \, \text{d}\mu_{G/\Gamma}(\hat{x}).$$

Now interchange sum and integral, the ensuing formal manipulation being justified by Fubini.]

11: SUBLEMMA There exists a compact set \(K \subset G\) such that \(K\Gamma = G\).
[Let $U$ be an open neighborhood of $e$ such that $\bar{U}$ is compact -- then the collection $\{\pi(xU) : x \in G\}$ is an open covering of $G/\Gamma$, thus there is a finite sub-collection

$$\pi(x_1U), \pi(x_2U), \ldots, \pi(x_nU)$$

that covers $G/\Gamma$ and one may take

$$K = x_1\bar{U} \cup x_2\bar{U} \cup \ldots \cup x_n\bar{U}.$$ 

Indeed,

$$G/\Gamma = \{k\Gamma : k \in K\},$$

so given $x \in G$,

$$x\Gamma = k\Gamma \ (\exists \ k) \Rightarrow x = k\gamma \ (\exists \ \gamma) \Rightarrow x \in k\Gamma.$$ 

[Note: It can be shown that $K$ contains a transversal $\mathcal{T}$ which is therefore relatively compact.]

Suppose that $f$ is admissible -- then $\forall x, y \in G$,

$$|f(x\gamma^{-1})| \leq \phi(x\gamma^{-1})$$

$$\leq C \int_{\Gamma} \phi(ux\gamma^{-1})d\mu_G(u).$$

12: LEMMA Fix $x, y \in G$ -- then $\forall \gamma_1, \gamma_2 \in \Gamma$,

$$\text{rlb}(x\gamma_1y^{-1}) \cap \text{rlb}(x\gamma_2y^{-1}) \neq \emptyset$$

iff

$$\gamma_2\gamma_1^{-1} \in x^{-1}u^{-1}\text{rlb}.$$ 

[In one direction,
Since $\Gamma$ is discrete, the compact set $x^{-1} \cup Ux$ contains a finite number $N$ of elements of $\Gamma$. So, for fixed $x, y$, not more than $N$ of the $Uxy^{-1}$ can intersect $Ux \cup y^{-1}$.

13: N.B. Consider the case when $N = 1$. Since it is always true that $e \in x^{-1} \cup Ux$, in this situation the $Uxy^{-1}$ are disjoint, hence

$$\sum_{\gamma \in \Gamma} \int \phi(uxy^{-1}) \, d\mu_G(u)$$

$$\leq \int_G \phi \, d\mu_G < \infty.$$
\[ n \mu(n \times n) = \mu(\bigcup_{i=1}^{n} \bigcup_{j=1}^{n} \bigcup_{k=1}^{n} X_i \cap X_j \cap X_k) + \cdots + \mu(\bigcap_{i=1}^{n} X_i). \]

15: **Lemma** Fix \( x, y \in G \) -- then
\[ \sum_{\gamma \in \Gamma} |f(x\gamma^{-1}y)| \leq N \int_{G} f \, d\mu < \infty. \]

16: **N.B.** More is true: The series
\[ \sum_{\gamma \in \Gamma} f(x\gamma^{-1}y) \]

is ldac on \( G \times G \) to a continuous function.

[The point is that the preceding estimate is uniform in \( x \) and \( y \) if these variables are confined to compacta \( K_x \) and \( K_y \).]

[Note: Consequently,
\( f \) admissible \( \Rightarrow \) \( L_{G/\Gamma}(f) \) Hilbert-Schmidt.]

17: **Theorem** If \( f \) is admissible and if \( L_{G/\Gamma}(f) \) is trace class, then the Selberg trace formula obtains (cf. \#10).

18: **N.B.**
\( f \) admissible \( \Rightarrow f \ast f \) admissible (cf. \#8).

Therefore
\[ L_{G/\Gamma}(f \ast f) = L_{G/\Gamma}(f)L_{G/\Gamma}(f) \]

is trace class and the foregoing is applicable.
Specialize now to the case when $G$ is a connected semisimple Lie group with finite center.

19: RAPPEL $C^1(G)$ is the $L^1$-Schwartz space of $G$. It is closed under convolution and contains $C_c^\infty(G)$ as a dense subspace.

Let $f \in C^1(G)$ and take $r > 0$ per #4 -- then there exists a constant $C > 0$ such that

$$|f(y)| \leq C |y|^{-2r} (y \in G).$$

Therefore $f$ is admissible.

20: LEMMA $L_{G/\Gamma}(f)$ is trace class.

[Using the theory of the parametrix, write

$$f = g \ast \mu \ast f \ast \nu,$$

where $g \in C^1(G)$ (a certain derivative of $f$), $\mu \in C_c^\infty(G)$, $\nu \in C_c^\infty(G)$, so

$$L_{G/\Gamma}(f) = L_{G/\Gamma}(g)L_{G/\Gamma}(\mu) + L_{G/\Gamma}(f)L_{G/\Gamma}(\nu).$$

The functions $f, g, \mu, \nu$ are admissible, hence the operators $L_{G/\Gamma}(f), L_{G/\Gamma}(g), L_{G/\Gamma}(\mu), L_{G/\Gamma}(\nu)$ are Hilbert-Schmidt.]

21: SCHOLIUM $\forall f \in C^1(G)$, the Selberg trace formula obtains.
22: N.B. The assignment
\[ f \rightarrow \text{tr}(L_{G/\Gamma}(f)) \]
is continuous in the topology of \( C^1(G) \).

[Note: Analogously, the assignment
\[ f \rightarrow \text{tr}(L_{G/\Gamma}(f)) \]
is continuous in the topology of \( C_c^\infty(G) \), i.e., is a distribution on G.]

APPENDIX

By way of reconciliation, consider the case when \( G \) is finite and use the notation of A, III, §3 and §4 -- then given \( f \in C(G), \phi \in C(G/\Gamma) \), we have
\[ (L_{G/\Gamma}(f)\phi)(x) = \sum_{y \in G} K_f(x,y)\phi(y), \]
where in this context
\[ K_f(x,y) = \frac{1}{|\Gamma|} \sum_{\gamma \in \Gamma} f(x\gamma y^{-1}). \]

Here
\[
\begin{align*}
\mu_G &= \text{counting measure on } G \\
\mu_\Gamma &= \text{counting measure on } \Gamma.
\end{align*}
\]

Write
\[ G = \bigcup_{k=1}^n x_k \Gamma. \]

Then for any \( f \in C(G) \),
\[ \int_G f d\mu_G = \sum_{x \in G} f(x) \]
11.

\[ = \int_{G/\Gamma} f^\Gamma d\mu_{G/\Gamma} \]

\[ = \sum_{k=1}^{n} \sum_{\gamma \in \Gamma} f(x_k \gamma), \]

so \( \mu_{G/\Gamma} \) is counting measure on \( G/\Gamma \).

Now explicate matters:

\[
(L_{G/\Gamma}(f)\phi)(x) = \sum_{y \in G} K_f(x, y) \phi(y)
\]

\[ = \sum_{k=1}^{n} \sum_{\gamma \in \Gamma} K_f(x, x_k \gamma) \phi(x_k \gamma) \]

\[ = \sum_{k=1}^{n} \sum_{\gamma \in \Gamma} K_f(x, x_k) \phi(x_k) \]

\[ = \sum_{k=1}^{n} |\Gamma| \cdot K_f(x, x_k) \phi(x_k) \]

\[ = \sum_{k=1}^{n} |\Gamma| \cdot \frac{1}{|\Gamma|} \sum_{\gamma \in \Gamma} f(x \gamma x_k^{-1}) \phi(x_k) \]

\[ = \sum_{k=1}^{n} \sum_{\gamma \in \Gamma} f(x \gamma x_k^{-1}) \phi(x_k) \]

which establishes that \( L_{G/\Gamma}(f) \) is an integral operator on \( C(G/\Gamma) \) with kernel

\[ \sum_{\gamma \in \Gamma} f(x \gamma x^{-1}) \]

this being the "\( K_f \)" of §4, #1.

There is more to be said. Thus given \( f \in C(G) \), we have

\[
\text{tr}(L_{G/\Gamma}(f)) = \sum_{x \in G} \frac{1}{|\Gamma|} \sum_{\gamma \in \Gamma} f(x \gamma x^{-1}) \quad (\text{cf. A, III, §3, #8})
\]
\[ n \prod_{i=1}^{n} 0(f, \gamma_i) \text{ (cf. A, III, \$4, \#2)}. \]

Here

\[ [\Gamma] = \{\gamma_1, \ldots, \gamma_n\} \]

while

\[ 0(f, \gamma_i) = \sum_{x \in G} f(xy_i^{-1}) \]

\[ = |G_{\gamma_i}| \sum_{x \in G/G_{\gamma_i}} f(xy_i^{-1}). \]

Therefore

\[ \text{vol}(G_{\gamma_i} / \Gamma_{\gamma_i}) = \frac{|G_{\gamma_i}|}{|\Gamma_{\gamma_i}|} \]

\[ = [G_{\gamma_i} : \Gamma_{\gamma_i}]. \]

N.B. The Haar measures \( \mu_G \) (or \( \mu_G^{\gamma} \)) and \( \mu_{\Gamma} \) (or \( \mu_{\Gamma}^{\gamma} \)) are counting measures, hence the invariant measure \( \mu_{G/\Gamma} \) (or \( \mu_{G/\Gamma}^{\gamma} \)) is counting measure, hence the invariant measure \( \mu_{G/G_{\gamma}} \) per

\[ \int_G = \int_{G/G_{\gamma}} \int_{G_{\gamma}} \]

is counting measure, its total volume being

\[ [G:G_{\gamma}] = \frac{|G|}{|G_{\gamma}|}. \]

Finally, the invariant measure \( \mu_{G/\Gamma_{\gamma}} \) per

\[ \int_{G/\Gamma_{\gamma}} = \int_{G/G_{\gamma}} \int_{G_{\gamma}/\Gamma_{\gamma}} \]
is counting measure and
\[
\text{vol}(G/\Gamma_Y) = \text{vol}(G/G_Y)\text{vol}(G_Y/\Gamma_Y),
\]
i.e.,
\[
[G:\Gamma_Y] = [G:G_Y][G_Y:\Gamma_Y],
\]
i.e.,
\[
\frac{|G|}{|\Gamma_Y|} = \frac{|G|}{|G_Y|}\frac{|G_Y:\Gamma_Y|}{|\Gamma_Y|}.
\]
=>
\[
\frac{|G_Y|}{|\Gamma_Y|} = [G_Y:\Gamma_Y].
\]

Matters are thus consistent, so the bottom line is that the global trace formula of A, III, §4, #6 is in this context the Selberg trace formula.
§6. DISCRETE SERIES

Let \( G \) be a unimodular locally compact group.

1: DEFINITION Let \( \Pi \) be an irreducible unitary representation of \( G \) on a Hilbert space \( \mathcal{H}(\Pi) \) — then \( \Pi \) is called \textit{square integrable} if \( \exists \ v \neq 0 \) in \( \mathcal{H}(\Pi) \) such that the coefficient

\[
x \to \langle \Pi(x)v, v \rangle
\]

is square integrable on \( G \).

2: THEOREM If \( \Pi \) is square integrable, then for all \( v_1, v_2 \in \mathcal{H}(\Pi) \), the coefficient

\[
x \to \langle \Pi(x)v_1, v_2 \rangle
\]

lies in \( L^2(G) \) and there exists a unique positive real number \( d_{\Pi} \) (depending on the normalization of the Haar measure on \( G \) but independent of \( v_1, v_2 \)) such that

\[
\int_G |\langle \Pi(x)v_1, v_2 \rangle|^2 \, d\mu_G(x) = \frac{1}{d_{\Pi}} |v_1||^2 |v_2||^2.
\]

3: DEFINITION \( d_{\Pi} \) is called the \textit{formal dimension} of \( \Pi \).

[Note: If \( G \) is compact, then every irreducible unitary representation of \( G \) is square integrable and \( d_{\Pi} \) is the dimension of \( \Pi \) in the usual sense provided \( \int_G \, d\mu_G = 1 \).

4: NOTATION \( \hat{G}_d \) is the subset of \( \hat{G} \) comprised of the square integrable representations and is called the \textit{discrete series} for \( G \).

[Note: \( \hat{G}_d \) may very well be empty (e.g., take \( G = \mathbb{R} \)).]
5: REMARK If \( \hat{G}_d \) is not empty, then the center of \( G \) is compact (the converse being false).

6: N.B. The elements of \( \hat{G}_d \) are precisely those irreducible unitary representations of \( G \) which occur as irreducible subrepresentations of the left translation representation of \( G \) on \( L^2(G) \).

7: NOTATION Given a \( \Pi \in \hat{G}_d \), let

\[ \phi_{\Pi,\cdot}(x) = \langle \Pi(x) \cdot, \cdot \rangle \quad (x \in G) \]

stand for a generic coefficient.

8: THEOREM Suppose that \( \Pi \) is square integrable -- then \( \forall \, v_1, v_2, \forall \, w_1, w_2 \) in \( V(\Pi) \),

\[
\int_G \phi_{v_1,v_2}(x) \overline{\phi_{w_1,w_2}(x)} d\mu_G(x)
\]

\[ = \frac{1}{\|\Pi\|} \langle v_1, w_1 \rangle \overline{\langle v_2, w_2 \rangle}. \]

9: APPLICATION

\[ \phi_{v_1,v_2} \ast \phi_{w_1,w_2} = \frac{1}{\|\Pi\|} \langle v_1, w_2 \rangle \phi_{w_1,v_2}. \]

[Note: If \( v_1 = v_2 = w_1 = w_2 \) is a unit vector, call it \( v \) and abbreviate \( \phi_{v,v} \) to \( \phi \), then

\[ ||\phi||_2^2 = \frac{1}{\|\Pi\|} \text{ and } \phi \ast \phi = \frac{1}{\|\Pi\|} \phi. \]
10: **Definition** Let \( \Pi \) be an irreducible unitary representation of \( G \) on a Hilbert space \( V(\Pi) \). Then \( \Pi \) is integrable if there exists a vector \( v \neq 0 \) in \( V(\Pi) \) such that the coefficient

\[
   x \rightarrow \langle \Pi(x)v, v \rangle
\]

is integrable on \( G \).

11: **N.B.** The coefficient

\[
   x \rightarrow \langle \Pi(x)v, v \rangle
\]

is bounded and \( L^1 \), hence is \( L^2 \). Therefore

"\( \Pi \) integrable" \( \Rightarrow \) "\( \Pi \) square integrable"

but the converse is false.

12: **Theorem** If \( \Pi \) is integrable, then there exists a dense subspace \( V(\Pi)^\sim \) of \( V(\Pi) \) such that for all \( v_1, v_2 \) in \( V(\Pi)^\sim \) the coefficient

\[
   x \rightarrow \langle \Pi(x)v_1, v_2 \rangle
\]

lies in \( L^1(G) \).

[Note: If \( \phi_{v,v} \in L^1(G) \), then one can take

\[
   V(\Pi)^\sim = \Pi(C_c(G))v.\]

Take \( G \) second countable and assume that \( \Pi \in \hat{G} \) is integrable, say \( \phi_{v,v} \in L^1(G) \). Then for all \( f \in C_c(G) \),

\[
   \phi_{\Pi(f)v, \Pi(f)v} \in L^1(G) \quad (\text{cf. #12}).
\]

Put \( v_0 = \Pi(f)v \), normalized by \( ||v_0|| = 1 \), and let

\[
   \phi_0 = d_\Pi \phi_{v_0,v_0}.
\]
13: N.B.

\[ \phi_0 \star \phi_0 = d_\Pi \phi v_0, v_0 \star d_\Pi \phi v_0, v_0 \]

\[ = d^2_\Pi \phi v_0, v_0 \star \phi v_0, v_0 \]

\[ = d^2_\Pi \frac{1}{d_\Pi} \phi v_0, v_0 \quad \text{(cf. #9)} \]

\[ = d_\Pi \phi v_0, v_0 = \phi_0. \]

It is also clear that \( \phi^*_0 = \phi_0 \) and \( \Pi(\phi_0)v_0 = v_0. \)

14: NOTATION If \( \Pi \) is an irreducible unitary representation of \( G \) and if \( \pi \) is a unitary representation of \( G \), then

\[ I_G(\Pi, \pi) \]

is the set of intertwining operators between \( \Pi \) and \( \pi \).

15: LEMMA For any unitary representation \( \pi \) of \( G \), \( \pi(\phi_0) \) is the orthogonal projection onto

\[ \{Tv_0: T \in I_G(\Pi, \pi) \}. \]

[Note: It's \( \pi(\phi_0) \), not \( \pi(\phi_0) \)... .]

Suppose that \( \Gamma \subset G \) is a uniform lattice and take \( \pi = I_{G/\Gamma}. \)

16: APPLICATION

\[ I_{G/\Gamma}(\phi_0) \]

is trace class and

\[ \text{tr}(I_{G/\Gamma}(\phi_0)) = \dim I_G(\Pi, I_{G/\Gamma}) = m(\Pi, I_{G/\Gamma}). \]
THEOREM 17: The series
\[ \sum_{\gamma \in \Gamma} \phi_0(\gamma \gamma^{-1}) \]
is ldac on \( G \times G \) to a separately continuous function.

PROOF Let \( K \subset G \) be compact and let
\[ n(K) = |\Gamma \cap K^{-1} \text{spt}(f) \cdot \text{spt}(f)^{-1} \cdot K| \]
Then the arrow
\[ \text{spt}(f)^{-1} \cdot K \rightarrow G / \Gamma \]
is at most \( n(K) \)-to-1 and \( \forall x \in K \),
\[ \sum_{\gamma \in \Gamma} |\phi_0(\gamma \gamma^{-1})| = \prod_{\gamma \in \Gamma} |\langle \gamma \gamma^{-1} \cdot \nu_0, \nu_0 \rangle| \]
\[ = \prod_{\gamma \in \Gamma} |\langle \gamma \gamma^{-1} \cdot \nu_0, \nu \rangle| \]
\[ = \prod_{\gamma \in \Gamma} |\langle \nu, \gamma \gamma^{-1} \cdot \nu_0 \rangle| \]
\[ = \prod_{\gamma \in \Gamma} |\int_G f(z) \langle \gamma \gamma^{-1} \cdot \nu_0, \nu \rangle \, d\mu_G(z) | \]
\[ = \prod_{\gamma \in \Gamma} |\int_G \tilde{f}(z) \langle z \gamma \gamma^{-1} \cdot \nu_0, \nu \rangle \, d\mu_G(z) | \]
\[ = \prod_{\gamma \in \Gamma} |\int_G \tilde{f}(z) \langle \gamma \gamma^{-1} \cdot \nu_0, \nu \rangle \, d\mu_G(z) | \]
\[ \leq \prod_{\gamma \in \Gamma} \int_G |\tilde{f}(z)| \langle \gamma \gamma^{-1} \cdot \nu_0, \nu \rangle \, d\mu_G(z) \]
6.

\[ d_\Pi \sum_{\gamma \in \Gamma} f_G |f(z^{-1})| |\langle \Pi(z\gamma y^{-1})v_0, v \rangle| d\mu_G(z) \]

\[ \leq d_\Pi \|f\|_\infty \sum_{\gamma \in \Gamma} f_{G/\Gamma} |\langle \Pi(z\gamma y^{-1})v_0, v \rangle| d\mu_{G/\Gamma}(z) \]

\[ \leq d_\Pi \|f\|_\infty n(K) \int_{\text{supp}(f)} |\langle \Pi(z\gamma y^{-1})v_0, v \rangle| d\mu_G(z) \]

\[ = d_\Pi \|f\|_\infty n(K) \int_{\text{supp}(f)} |\langle \Pi(z\gamma y^{-1})v_0, v \rangle| d\mu_G(z) \]

\[ = d_\Pi \|f\|_\infty n(K) \int_{\text{supp}(f)} |\phi_{v_0,v}|_1. \]

And

\[ |\phi_{v_0,v}|_1 \leq f_G \int f(y) \langle \Pi(xy)v, v \rangle d\mu_G(x) d\mu_G(y) \]

\[ \leq |f|_1 |\phi_{v,v}|_1 < \infty, \]

thereby settling the ldac condition (and then some (no restriction on "y")), leaving the claim of separate continuity which can be left to the reader.

The operator \( L_{G/\Gamma}(\phi_0) \) is trace class (cf. #16). So, in view of what has been said above, the criterion of \( \S 5, \#10 \) is applicable.

18: SCHOLIUM

\[ m(\Pi, L_{G/\Gamma}) = \sum_{\gamma \in \Gamma} \text{vol}(G/\Gamma \gamma) \int_{G/G \gamma} \frac{\phi_0(xyx^{-1})}{\phi_0(y)} d\mu_{G/G \gamma}(x), \]

the sum on the right hand side being absolutely convergent.
19: REMARK There are circumstances in which the integral
\[ \int_{G/G} \phi_0(xy^{-1}) d\mu_{G/G}(x) \]
vanishes for all \( \gamma \) except \( \gamma = e \), hence then
\[ m(\Pi, L_{G/\Gamma}) = \text{vol}(G/\Gamma) \phi_0(e) = \text{vol}(G/\Gamma) <\Pi(e)v_0, v_0> \]
\[ = \text{vol}(G/\Gamma) d_\Pi <v_0, v_0> \]
\[ = \text{vol}(G/\Gamma) d_\Pi. \]
Therefore \( m(\Pi, L_{G/\Gamma}) \) is positive, so \( \Pi \) definitely occurs in \( L_{G/\Gamma} \).

[Note: To run a reality check, take \( G \) finite, \( \Gamma = \{e\} \) -- then \( \text{vol}(G/\Gamma) = \text{vol}(G) = 1 \) and \( \forall \Pi \in \hat{G}, \)
\[ m(\Pi, L_{G/\Gamma}) = d_\Pi \) (cf. A, II, §5, #8 and A, III, §3, #15).]

20: N.B. The situation envisioned in #19 is realized if \( G \) is a connected semisimple Lie group with finite center and if \( \Gamma \) has no elements of finite order other than the identity.

21: LEMMA If \( G \) is a Lie group and if \( f \in C_c^\infty(G) \), then the series
\[ \sum_{\gamma \in \Gamma} \phi_0(xy^{-1}) \]
is a \( C^\infty \) function of \( x, y \).