C*-ALGEBRAS

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ABSTRACT

This book is addressed to those readers who are already familiar with the elements of the theory but wish to go further. While some aspects, e.g. tensor products, are summarized without proof, others are dealt with in all detail. Numerous examples have been included and I have also appended an extensive list of references.

CONTENTS

§1. *BASIC FACTS*

Let A be a complex Banach algebra, $*: A \rightarrow A$ an involution -- then the pair $(A,*)$ is said to be a C*-algebra if $\forall\ A \in A$,

$$||A^*A|| = ||A||^2.$$

N.B. It is automatic that $||A^*|| = ||A||$, thus the involution $*: A \rightarrow A$ is continuous.

1.1 LEMMA $\forall\ A \in A$,

$$||A|| = r(A^*A)^{1/2},$$

r the spectral radius.

1.2 REMARK If $(A, ||\cdot||)$ is a C*-algebra and if $||\cdot||'$ is a submultiplicative norm satisfying the C*-condition, viz.,

$$||A^*A||' = (||A||')^2 \quad (A \in A),$$

then $||\cdot||' = ||\cdot||$.

[Note: It is not assumed that $(A, ||\cdot||')$ is complete, i.e., $(A, ||\cdot||')$ is merely a pre-C*-algebra.]

1.3 EXAMPLE Given a complex Hilbert space H, denote by $B(H)$ the set of bounded linear operators on H -- then $B(H)$ is a C*-algebra. Furthermore, any $*$-subalgebra A of $B(H)$ which is closed in the norm topology is a C*-algebra. E.g.: This is the case of $A = \underline{L}_\infty(H)$, the norm closed $*$-ideal in $B(H)$ consisting of the compact operators.

1.4 EXAMPLE Take $\mathcal{H} = \underline{C}^n$ and identify $\mathcal{B}(\underline{C}^n)$ with $M_n(\underline{C})$, the algebra of n-by-n matrices over $\underline{C}$. Equip $M_n(\underline{C})$ with the induced operator norm and let the involution $*:M_n(\underline{C}) \rightarrow M_n(\underline{C})$ be "conjugate transpose" -- then with these stipulations, $M_n(\underline{C})$ is a C*-algebra. More generally, if $\underline{n} = (n_1,\ldots,n_p)$ is a p-tuple of positive integers, then

$$M_{\underline{n}}(\underline{C}) = \bigoplus_{k=1}^{p} M_{n_k}(\underline{C})$$

is a C*-algebra. Here

$$||\bigoplus_{k=1}^{p} A_k|| = \max_{1\leq k\leq p} ||A_k|| \quad (A_k \in M_{n_k}(\underline{C}))$$

or still,

$$||\bigoplus_{k=1}^{p} A_k|| = \max_{1\leq k\leq p} \lambda_k,$$

where λ_k^2 is the largest eigenvalue of $A_k^*A_k$.

[Note: Every finite dimensional C*-algebra $\mathcal{A}$ is $*$-isomorphic to an $M_{\underline{n}}(\underline{C})$ for some $\underline{n}$ and $\underline{n}$ is uniquely determined by $\mathcal{A}$ up to a permutation. If $\mathcal{B}$ is another finite dimensional C*-algebra with associated q-tuple $\underline{m} = (m_1,\ldots,m_q)$, then $\mathcal{A}$ and $\mathcal{B}$ are $*$-isomorphic iff $p = q$ and $\exists$ a permutation σ of $\{1,\ldots,p\}$ such that $m_k = n_{\sigma(k)}$ $(k = 1,\ldots,p)$.]

1.5 EXAMPLE Fix a C*-algebra $\mathcal{A}$ and let X be a compact Hausdorff space. Equip $C(X,\mathcal{A})$ with pointwise operations and define the involution by $f^*(x) = f(x)^*$ $(x \in X)$. Put

$$||f|| = \sup_{x\in X} ||f(x)||.$$

Then $C(X,A)$ is a C*-algebra.

1.6 NOTATION C*ALG is the category whose objects are the C*-algebras and whose morphisms are the *-homomorphisms.

[Note: An isomorphism is a bijective morphism.]

N.B. Let A,B be C*-algebras -- then a linear map $\Phi:A \to B$ is a *-homomorphism iff

$$\Phi(A_1A_2) = \Phi(A_1)\Phi(A_2) \ \& \ \Phi(A^*) = \Phi(A)^*.$$

1.7 LEMMA A *-homomorphism $\Phi:A \to B$ is necessarily norm decreasing, i.e., $\forall\ A \in A,\ ||\Phi(A)|| \leq ||A||$.

1.8 LEMMA An injective *-homomorphism $\Phi:A \to B$ is necessarily isometric, i.e., $\forall\ A \in A,\ ||\Phi(A)|| = ||A||$.

Suppose that $I \subset A$ is a closed ideal -- then I is a *-ideal. Equip A/I with the quotient norm, thus

$$||A + I|| = \inf_{I \in I} ||A + I||,$$

and let

$$(A + I)^* = A^* + I.$$

Then A/I is a C*-algebra and the projection $\pi:A \to A/I$ is a *-homomorphism with kernel I.

N.B. If $\Phi:A \to B$ is a *-homomorphism, then the kernel of Φ is a closed ideal in A and the image of Φ is a C*-subalgebra of B: $A/\mathrm{Ker}\ \Phi \approx \Phi(A)$.

[Note: The term "C*-subalgebra" means a norm closed subalgebra which is invariant under the *-operation.]

1.9 EXAMPLE If X is a compact Hausdorff space and if $I \subset C(X)$ is a closed ideal, then $\exists$ a unique closed subset $Y \subset X$ such that

$$I = \{f \in C(X) : f|Y = 0\}.$$

Moreover, the C*-algebra $C(X)/I$ is *-isomorphic to $C(Y)$ via the map induced by the arrow of restriction $C(X) \to C(Y)$.

A C*-algebra A is simple if it has no nontrivial closed ideals. E.g.: $\underline{L}_\infty(H)$ is simple (but $B(H)$ is not simple if H is infinite dimensional).

A C*-algebra A is unital if A has a unit 1_A; otherwise, A is nonunital.

1.10 LEMMA If A is unital, then every maximal ideal in A is closed.

A simple unital C*-algebra has no nontrivial ideals. On the other hand, a nonunital simple C*-algebra may very well have nontrivial ideals (e.g., $\underline{L}_\infty(H)$ if H is infinite dimensional).

A closed ideal I in a C*-algebra A is essential if $AI = 0 \Rightarrow A = 0$ (equivalently, $IA = 0 \Rightarrow A = 0$). In particular: A is essential in itself.

1.11 LEMMA A closed ideal $I \subset A$ is essential iff $I \cap J \neq 0$ for all nonzero closed ideals J in A.

1.12 EXAMPLE Suppose that H is a complex Hilbert space -- then $\underline{L}_\infty(H)$ is an essential ideal in $B(H)$.

A unitization of a C*-algebra A is a pair (U,i), where U is a unital C*-algebra and $i:A \rightarrow U$ is an injective *-homomorphism such that the image $i(A)$ is an essential ideal in U.

1.13 REMARK If A is unital to begin with, then the only unitization of A is A itself. Proof: Identify A and $i(A)$ and, assuming that $U \neq A$, fix $U \in U - A$ -- then $U1_A \in A$ and $U - U1_A \neq 0$. Meanwhile, $\forall\ A \in A$,

$$(U - U1_A)A = UA - U1_A A = UA - UA = 0.$$

1.14 CONSTRUCTION Given a nonunital C*-algebra A, put $A^+ = A \oplus \underline{C}$ (vector space direct sum) -- then with the operations

$$(A,\lambda)(B,\mu) = (AB + \lambda B + \mu A, \lambda\mu)$$

and

$$(A,\lambda)^* = (A^*,\bar{\lambda}),$$

A^+ acquires the structure of a unital *-algebra $(1_{A^+} = (0,1))$. Moreover, the prescription

$$||(A,\lambda)|| = \sup_{||X||\leq 1} ||AX + \lambda X||$$

is a C*-norm on A^+. Proof: It suffices to observe that

$$||(A,\lambda)^*(A,\lambda)|| = ||(A^*A + \bar{\lambda}A + \lambda A^*, \bar{\lambda}\lambda)||$$

$$= \sup_{||X||\leq 1} \{||A^*AX + \bar{\lambda}AX + \lambda A^*X + \bar{\lambda}\lambda X||\}$$

$$\geq \sup_{||X||\leq 1} \{||X^*A^*AX + \bar{\lambda}X^*AX + \lambda X^*A^*X + \bar{\lambda}\lambda X^*X||\}$$

$$= \sup_{||X||\leq 1} \{||(AX + \lambda X)^*(AX + \lambda X)||$$

$$= \sup_{||X||\leq 1} \{||AX + \lambda X||^2\}$$

$$= ||(A,\lambda)||^2.$$

Denote now by i the arrow $A \to A^+$ that sends A to (A,0) -- then the pair (A^+,i) is a unitization of A. Indeed, $i(A)$ is a closed ideal in A^+, thus one only has to check that it is essential. So suppose that $(A,\lambda)i(A) = 0$, i.e., $AB + \lambda B = 0$ $\forall\ B \in A$. Claim: $A = 0$ and $\lambda = 0$. This being obvious if $\lambda = 0$, assume that $\lambda \neq 0$: $\forall\ B \in A$,

$$AB + \lambda B = 0$$

$$\Rightarrow$$

$$(\frac{1}{\lambda} A)B + B = 0$$

$$\Rightarrow$$

$$B^*(\frac{1}{\lambda} A)^* + B^* = 0$$

$$\Rightarrow$$

$$B(\frac{1}{\lambda} A)^* + B = 0$$

$$\Rightarrow$$

$$\begin{cases} (\frac{1}{\lambda} A)(\frac{1}{\lambda} A)^* + (\frac{1}{\lambda} A)^* = 0 \\ (\frac{1}{\lambda} A)(\frac{1}{\lambda} A)^* + (\frac{1}{\lambda} A) = 0 \end{cases}$$

$$\Rightarrow$$

$$(\frac{1}{\lambda} A)^* = \frac{1}{\lambda} A.$$

Therefore $-\frac{1}{\lambda}A$ is an identity for $\mathcal{A}$. But $\mathcal{A}$ is nonunital, from which a contradiction.

[Note: The quotient $\mathcal{A}^+/i(\mathcal{A})$ is $*$-isomorphic to $\underline{C}$ $((A,\lambda) \rightarrow \lambda)$.]

1.15 EXAMPLE Let X be a noncompact locally compact Hausdorff space, $C_\infty(X)$ the algebra of complex valued continuous functions on X that vanish at infinity. Equip $C_\infty(X)$ with the sup norm and let the involution be complex conjugation -- then $C_\infty(X)$ is a nonunital C*-algebra and $C_\infty(X)^+ \approx C(X^+)$, X^+ $(= X \cup \{\infty\})$ the one point compactification of X.

[Note: Explicated, the relevant arrow

$$C_\infty(X)^+ \rightarrow C(X^+)$$

is the assignment

$$(f,\lambda) \rightarrow f + \lambda,$$

where

$$(f + \lambda)(\infty) = \lambda.]$$

Given C*-algebras $\mathcal{A}$ and $\mathcal{B}$, their direct sum $\mathcal{A} \oplus \mathcal{B}$ is the ordinary $*$-algebra direct sum with norm

$$||(A,B)|| = \max\{||A||,||B||\}.$$

This is a C*-norm. Proof:

$$\begin{aligned}||(A,B)^*(A,B)|| &= ||(A^*A,B^*B||\\ &= \max\{||A^*A||,||B^*B||\}\\ &= \max\{||A||^2,||B||^2\}\end{aligned}$$

$$= \max\{||A||,||B||\}^2$$

$$= ||(A,B)||^2.$$

N.B. $\mathcal{A} \oplus \mathcal{B}$ contains $\mathcal{A}$ and $\mathcal{B}$ as nonessential ideals and

$$\begin{cases} (\mathcal{A} \oplus \mathcal{B})/\mathcal{A} \simeq \mathcal{B} \\ (\mathcal{A} \oplus \mathcal{B})/\mathcal{B} \simeq \mathcal{A}. \end{cases}$$

In addition, $\mathcal{A} \oplus \mathcal{B}$ is unital iff $\mathcal{A}$ and $\mathcal{B}$ are unital (in which case $1_{\mathcal{A}\oplus\mathcal{B}} = (1_{\mathcal{A}},1_{\mathcal{B}})$).

1.16 REMARK Take $\mathcal{A}$ unital -- then one can form $\mathcal{A}^+$ exactly as in 1.14 and the arrow $\zeta:\mathcal{A}^+ \to \mathcal{A} \oplus \underline{C}$ that sends (A,λ) to $(A + \lambda 1_{\mathcal{A}},\lambda)$ is a unital $*$-isomorphism.

1.17 LEMMA Let $\mathcal{A},\mathcal{B}$ be C*-algebras and let $\Phi:\mathcal{A} \to \mathcal{B}$ be a $*$-homomorphism -- then Φ admits a unique extension to a unital $*$-homomorphism $\Phi^+:\mathcal{A}^+ \to \mathcal{B}^+$, viz.

$$\Phi^+(A,\lambda) = (\Phi(A),\lambda).$$

1.18 NOTATION UNC*ALG is the category whose objects are the unital C*-algebras and whose morphisms are the unital $*$-homomorphisms.

[Note: An isomorphism is a bijective morphism.]

N.B. The assignment

$$\begin{cases} A \to A^+ \\ \Phi \to \Phi^+ \end{cases}$$

is functorial, i.e., defines a functor

C*ALG → UNC*ALG.

1.19 RAPPEL Let A be a Banach algebra -- then an approximate unit per A is a norm bounded net $\{e_i : i \in I\}$ such that $\forall\, A \in A$,

$$\begin{cases} \lim_{i \in I} \|e_i A - A\| = 0 \\ \lim_{i \in I} \|A e_i - A\| = 0. \end{cases}$$

1.20 LEMMA Every C*-algebra A has an approximate unit $\{e_i : i \in I\}$ such that

$$\forall\, i, \begin{cases} e_i \geq 0 \\ \|e_i\| \leq 1 \end{cases} \text{ and } \forall\, i \leq j,\ e_i \leq e_j.$$

C*-algebras having a countable approximate unit are said to be σ-unital.

1.21 REMARK Every unital C*-algebra is σ-unital. Every separable C*-algebra is σ-unital but there are nonseparable nonunital σ-unital C*-algebras.

[Note: Not all C*-algebras are σ-unital.]

1.22 EXAMPLE Take H separable and infinite dimensional. Fix an orthonormal basis $\{e_n : n \in \underline{N}\}$ and let P_n be the orthogonal projection onto $\underline{C}e_1 + \cdots + \underline{C}e_n$ -- then the sequence $\{P_n\}$ is an approximate unit per $\underline{L}_\infty(H)$, hence $\underline{L}_\infty(H)$ is σ-unital.

[Note: $\underline{L}_\infty(H)$ is separable (but $B(H)$ is not separable).]

1.23 <u>EXAMPLE</u> Let X be a noncompact locally compact Hausdorff space — then $C_\infty(X)$ is σ-unital iff X is σ-compact.

Let $\mathcal{A}$ be a C*-algebra.

- $\mathcal{A}_{SA}$ is the collection of all selfadjoint elements in $\mathcal{A}$, i.e.,

$$\mathcal{A}_{SA} = \{A \in \mathcal{A} : A^* = A\}.$$

- $\mathcal{A}_+$ is the collection of all positive elements in $\mathcal{A}$, i.e.,

$$\mathcal{A}_+ = \{A^2 : A \in \mathcal{A}_{SA}\}$$

or still,

$$\mathcal{A}_+ = \{A^*A : A \in \mathcal{A}\}.$$

1.24 <u>LEMMA</u> The set $\mathcal{A}_+$ is a closed convex cone in $\mathcal{A}$ with the property that $\mathcal{A}_+ \cap (-\mathcal{A}_+) = \{0\}$.

Given $A, B \in \mathcal{A}_{SA}$, one writes $A \geq B$ (or $B \leq A$) iff $A - B \in \mathcal{A}_+$.

1.25 <u>LEMMA</u> If $A \geq B \geq 0$, then $||A|| \geq ||B||$.

1.26 <u>LEMMA</u> If $A \geq B \geq 0$, then $\forall X \in \mathcal{A}$,

$$X^*AX \geq X^*BX \geq 0.$$

<u>PROOF</u> Since $A - B \in \mathcal{A}_+$, $\exists C \in \mathcal{A} : A - B = C^*C$. Therefore

$$X^*AX - X^*BX = X^*(A - B)X$$

$$= X^*C^*CX$$

$$= (CX)^*CX \in A_+.$$

N.B. If A is unital, then

$$A \in A_+ \Rightarrow 0 \le A \le ||A||1_A.$$

If A is nonunital, then

$$A_+ = A \cap (A^+)_+$$

and

$$A \in A_+ \Rightarrow 0 \le A \le ||A||1_{A^+}.$$

So, in either situation, $\forall X \in A$,

$$0 \le X^*AX \le ||A||X^*X.$$

1.27 REMARK Every positive A has a unique positive square root $A^{1/2}$, thus $A = (A^{1/2})^2$.

1.28 LEMMA Given $A \in A_{SA}$, put $|A| = (A^2)^{1/2}$ and let

$$A_\pm = (|A| \pm A)/2.$$

Then

$$A_\pm \in A_+,\ A = A_+ - A_-,\ A_+A_- = 0.$$

Moreover, $A_\pm$ are the unique positive elements with these properties.

N.B. Every $A \in \mathcal{A}$ is the sum of two selfadjoint elements:

$$A = \text{Re } A + \sqrt{-1} \text{ Im } A,$$

where

$$\text{Re } A = \frac{A + A^*}{2}, \quad \text{Im } A = \frac{A - A^*}{2\sqrt{-1}},$$

and

$$||\text{Re } A||, \quad ||\text{Im } A|| \leq ||A||.$$

Therefore every $A \in \mathcal{A}$ can be written as a linear combination of four positive elements.

Suppose that $\mathcal{A}$ is unital -- then an element $U \in \mathcal{A}$ is unitary if $U^*U = UU^* = 1_{\mathcal{A}}$. If $A \in \mathcal{A}_{SA}$ and $||A|| \leq 1$, then

$$A = (U_+ + U_-)/2.$$

Here

$$U_{\pm} = A \pm \sqrt{-1}\,(1_{\mathcal{A}} - A^2)^{1/2}$$

are unitary. Therefore every $A \in \mathcal{A}$ can be written as a linear combination of four unitary elements.

1.29 REMARK If $||A|| < 1 - \frac{2}{n}$, then there are unitaries $U_1, \ldots, U_n$ such that

$$A = \frac{U_1 + \cdots + U_n}{n}.$$

Consequently, the convex hull of the set of unitary elements includes the open unit ball in $\mathcal{A}$, thus its closure is the closed unit ball in $\mathcal{A}$.

Put

$$A^1 = \{A \in A : ||A|| \leq 1\}.$$

1.30 LEMMA A C*-algebra A is unital iff A^1 has an extreme point.

1.31 EXAMPLE If A is unital, then 1_A is an extreme point of A^1.

§2. THE COMMUTATIVE CASE

A character of a commutative C*-algebra A is a nonzero homomorphism $\omega: A \rightarrow \underline{C}$ of algebras. The set of all characters of A is called the structure space of A and is denoted by $\Delta(A)$.

N.B. We have

$$\begin{cases} \Delta(A) = \emptyset & (A = \{0\}) \\ \Delta(A) \neq \emptyset & (A \neq \{0\}). \end{cases}$$

2.1 LEMMA Let $\omega \in \Delta(A)$ — then ω is necessarily bounded and, in fact, $||\omega|| = 1$. Moreover, if A is unital, then

$$1 = \omega(1_A)$$

and if A is nonunital, then

$$1 = \lim_{i \in I} \omega(e_i).$$

Given $A \in A$, define

$$\hat{A}: \Delta(A) \rightarrow \underline{C}$$

by stipulating that

$$\hat{A}(\omega) = \omega(A).$$

Equip $\Delta(A)$ with the initial topology determined by the $\hat{A}$, i.e., equip $\Delta(A)$ with the relativised weak* topology.

2.2 LEMMA $\Delta(A)$ is a locally compact Hausdorff space. Furthermore, $\Delta(A)$ is compact iff A is unital.

2.3 <u>LEMMA</u> Fix a commutative C* algebra A.

• If A is unital, then $\hat{A} \in C(\Delta(A))$ and the arrow

$$\begin{cases} A \to C(\Delta(A)) \\ A \to \hat{A} \end{cases}$$

is a unital $*$-isomorphism.

• If A is nonunital, then $\hat{A} \in C_\infty(\Delta(A))$ and the arrow

$$\begin{cases} A \to C_\infty(\Delta(A)) \\ A \to \hat{A} \end{cases}$$

is a $*$-isomorphism.

<u>N.B.</u> If $A = \{0\}$, then $\Delta(A) = \emptyset$ and there exists exactly one map $\emptyset \to \underline{C}$, namely the empty function $(\emptyset = \emptyset \times \underline{C})$, which we shall take to be $\hat{0}$.

2.4 <u>REMARK</u> It suffices to establish 2.3 in the unital case. Thus suppose that A is nonunital -- then each $\omega \in \Delta(A)$ extends to an element $\omega^+ \in \Delta(A^+)$ via the prescription $\omega^+(A,\lambda) = \omega(A) + \lambda$ and

$$\Delta(A^+) = \{\omega^+ : \omega \in \Delta(A)\} \cup \{\omega_\infty\},$$

where $\omega_\infty(A,\lambda) = \lambda$, so $\Delta(A^+)$ is homeomorphic to $\Delta(A)^+$, the one point compactification of $\Delta(A)$. But A^+ is unital, hence

$$A^+ \approx C(\Delta(A^+)) \approx C(\Delta(A)^+)$$

$$\Rightarrow$$

$$A \approx C_\infty(\Delta(A)).$$

2.5 LEMMA Fix a locally compact Hausdorff space X.

• If X is compact, then $\forall\, x \in X$, the Dirac measure $\delta_x \in \Delta(C(X))$ and the arrow

$$\begin{cases} X \to \Delta(C(X)) \\ x \to \delta_x \end{cases}$$

is a homeomorphism.

• If X is noncompact, then $\forall\, x \in X$, the Dirac measure $\delta_x \in \Delta(C_\infty(X))$ and the arrow

$$\begin{cases} X \to \Delta(C_\infty(X)) \\ x \to \delta_x \end{cases}$$

is a homeomorphism.

2.6 REMARK It suffices to establish 2.5 in the compact case. Thus suppose that X is noncompact -- then X^+ is compact, hence

$$X^+ \approx \Delta(C(X^+))$$

or still,

$$X^+ \approx \Delta(C_\infty(X)^+)$$

or still,

$$X^+ \approx \Delta(C_\infty(X))^+.$$

Therefore

$$X \approx \Delta(C_\infty(X)).$$

2.7 RAPPEL Let $\underline{C}$ and $\underline{D}$ be categories -- then a functor $F:\underline{C} \to \underline{D}$ is an

<u>equivalence</u> if there exists a functor $G:\underline{D} \to \underline{C}$ such that $G \circ F \simeq id_{\underline{C}}$ and $F \circ G \simeq id_{\underline{D}}$, the symbol $\simeq$ standing for natural isomorphism.

[Note: The term <u>coequivalence</u> is used when F is a cofunctor: $\forall\ f \in Mor(X,Y)$, $Ff \in Mor(FY,FX)$.]

<u>N.B.</u> A functor $F:\underline{C} \to \underline{D}$ is an equivalence iff it is full, faithful, and has a representative image (i.e., for any $Y \in Ob\ \underline{D}$, there exists an $X \in Ob\ \underline{C}$ such that FX is isomorphic to Y).

2.8 <u>RAPPEL</u> Categories $\underline{C}$ and $\underline{D}$ are said to be <u>equivalent</u> (<u>coequivalent</u>) provided there is an equivalence (coequivalence) $F:\underline{C} \to \underline{D}$. The object isomorphism types of equivalent (coequivalent) categories are in a one-to-one correspondence.

Let X and Y be compact Hausdorff spaces. Suppose that $\phi:X \to Y$ is a continuous function — then ϕ induces a unital $*$-homomorphism

$$\phi^*:C(Y) \to C(X),$$

viz. $\phi^*(f) = f \circ \phi$. Therefore the association C that sends X to C(X) is a cofunctor from the category of compact Hausdorff spaces and continuous functions to the category of unital commutative C*-algebras and unital $*$-homomorphisms.

Let $\mathcal{A}$ and $\mathcal{B}$ be unital commutative C*-algebras. Suppose that $\Phi:\mathcal{A} \to \mathcal{B}$ is a unital $*$-homomorphism — then Φ induces a continuous function

$$\Phi^*:\Delta(\mathcal{B}) \to \Delta(\mathcal{A}),$$

viz. $\Phi^*(\omega) = \omega \circ \Phi$. Therefore the association Δ that sends $\mathcal{A}$ to $\Delta(\mathcal{A})$ is a cofunctor from the category of unital commutative C*-algebras and unital $\star$-homomorphisms to the category of compact Hausdorff spaces and continuous functions.

2.9 THEOREM The category of compact Hausdorff spaces and continuous functions is coequivalent to the category of unital commutative C*-algebras and unital $\star$-homomorphisms.

PROOF Define

$$\Xi_X : X \to \Delta(C(X))$$

by the rule $\Xi_X(x) = \delta_x$ — then Ξ_X is a homeomorphism and there is a commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{\Xi_X} & \Delta(C(X)) \\ \phi \downarrow & & \downarrow \phi^{**} \\ Y & \xrightarrow[\Xi_Y]{} & \Delta(C(Y)). \end{array}$$

Define

$$\Xi_A : A \to C(\Delta(A))$$

by the rule $\Xi_A(A) = \hat{A}$ -- then Ξ_A is a unital $\star$-isomorphism and there is a commutative diagram

$$\begin{array}{ccc} A & \xrightarrow{\Xi_A} & C(\Delta(A)) \\ \Phi \downarrow & & \downarrow \Phi^{**} \\ B & \xrightarrow[\Xi_B]{} & C(\Delta(B)). \end{array}$$

Therefore

$$\begin{cases} \text{id} \approx \Delta \circ C \\ \text{id} \approx C \circ \Delta. \end{cases}$$

The situation for noncompact locally compact Hausdorff spaces and nonunital commutative C*-algebras is slightly more complicated. One immediate and obvious difficulty is that a continuous $\phi:X \to Y$ need not induce a map $\phi^*:C_\infty(Y) \to C_\infty(X)$. E.g.: Take $X = Y = \underline{R}$ and let

$$\phi(t) = e^{2\pi\sqrt{-1}\, t}.$$

However, the resolution turns out to be simple enough: Impose the restriction that $\phi:X \to Y$ be proper.

[Note: Let $\phi:X \to Y$ be continuous -- then ϕ is proper iff its canonical extension $\phi^+:X^+ \to Y^+$ $(\phi^+(\infty_X) = \infty_Y)$ is continuous.

2.10 <u>LEMMA</u> A proper $\phi:X \to Y$ induces a $*$-homomorphism

$$\phi^*:C_\infty(Y) \to C_\infty(X).$$

There is also a problem on the algebraic side, namely if $\mathcal{A}$ and $\mathcal{B}$ are nonunital commutative C*-algebras, then a $*$-homomorphism $\Phi:\mathcal{A} \to \mathcal{B}$ need not induce a map $\Phi^*:\Delta(\mathcal{B}) \to \Delta(\mathcal{A})$, the point being that $\omega \circ \Phi$ might very well be zero. To get around this, call Φ <u>proper</u> if for any approximate unit $\{e_i:i \in I\}$ per $\mathcal{A}$, $\{\Phi(e_i):i \in I\}$ is an approximate unit per $\mathcal{B}$ (cf. 1.20).

[Note: A surjective Φ is proper. To see this, choose an approximate unit $\{e_i:i \in I\}$ per $\mathcal{A}$ -- then $\forall\, A \in \mathcal{A}$,

$$e_iA \to A \Rightarrow \Phi(e_i)\Phi(A) \to \Phi(A).]$$

2.11 <u>LEMMA</u> A proper $\Phi:\mathcal{A} \to \mathcal{B}$ induces a continuous function

$$\Phi^*:\Delta(\mathcal{B}) \to \Delta(\mathcal{A}).$$

[$\forall\ A \in \mathcal{A}$,

$$\Phi^*(\omega)(A^*A) = \omega(\Phi(A)^*\Phi(A)) \geq 0.$$

Therefore $\Phi^*(\omega)$ is a positive linear functional, hence $\forall\ \omega \in \Delta(\mathcal{B})$,

$$\begin{aligned} ||\Phi^*(\omega)|| &= \lim_{i\in I} \Phi^*(\omega)(e_i) \\ &= \lim_{i\in I} \omega(\Phi(e_i)) \\ &= ||\omega|| \neq 0.] \end{aligned}$$

<u>N.B.</u> The ϕ^* figuring in 2.10 is proper and the Φ^* figuring in 2.11 is proper.

2.12 <u>THEOREM</u> The category of noncompact locally compact Hausdorff spaces and proper continuous functions is coequivalent to the category of nonunital commutative C*-algebras and proper *-homomorphisms.

<u>PROOF</u> Replace the commutative diagrams in 2.9 by

$$\begin{array}{ccc} X & \xrightarrow{\Xi_X} & \Delta(C_\infty(X)) \\ \phi \downarrow & & \downarrow \phi^{**} \\ Y & \xrightarrow[\Xi_Y]{} & \Delta(C_\infty(Y)) \end{array}$$

and

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{\Xi_{\mathcal{A}}} & C_\infty(\Delta(\mathcal{A})) \\ \Phi \downarrow & & \downarrow \Phi^{**} \\ \mathcal{B} & \xrightarrow[\Xi_{\mathcal{B}}]{} & C_\infty(\Delta(\mathcal{B})). \end{array}$$

§3. *CATEGORICAL CONSIDERATIONS*

We shall first review some standard terminology.

3.1 RAPPEL Let $\underline{C}$ be a category.

• A source in $\underline{C}$ is a collection of morphisms $f_i:X \to X_i$ indexed by a set I and having a common domain. An n-source is a source for which $\#(I) = n$.

• A sink in $\underline{C}$ is a collection of morphisms $f_i:X_i \to X$ indexed by a set I and having a common codomain. An n-sink is a sink for which $\#(I) = n$.

A diagram in a category $\underline{C}$ is a functor $\Delta:\underline{I} \to \underline{C}$, where $\underline{I}$ is a small category, the indexing category. To facilitate the introduction of sources and sinks associated with Δ, we shall write Δ_i for the image in Ob $\underline{C}$ of $i \in$ Ob $\underline{I}$.

3.2 LIMITS Let $\Delta:\underline{I} \to \underline{C}$ be a diagram — then a source $\{f_i:X \to \Delta_i\}$ is said to be natural if for each $\delta \in$ Mor $\underline{I}$, say $i \overset{\delta}{\to} j$, $\Delta\delta \circ f_i = f_j$. A limit of Δ is a natural source $\{\ell_i:L \to \Delta_i\}$ with the property that if $\{f_i:X \to \Delta_i\}$ is a natural source, then there exists a unique morphism $\phi:X \to L$ such that $f_i = \ell_i \circ \phi$ for all $i \in$ Ob $\underline{I}$. Limits are essentially unique. Notation: $L = \lim_{\underline{I}} \Delta$ (or $\lim \Delta$).

3.3 COLIMITS Let $\Delta:\underline{I} \to \underline{C}$ be a diagram — then a sink $\{f_i:\Delta_i \to X\}$ is said to be natural if for each $\delta \in$ Mor $\underline{I}$, say $i \overset{\delta}{\to} j$, $f_i = f_j \circ \Delta\delta$. A colimit of Δ is a natural sink $\{\ell_i:\Delta_i \to L\}$ with the property that if $\{f_i:\Delta_i \to X\}$ is a natural sink, then there exists a unique morphism $\phi:L \to X$ such that $f_i = \phi \circ \ell_i$ for all $i \in \text{Ob } \underline{I}$. Colimits are essentially unique. Notation: $L = \text{colim}_{\underline{I}}\, \Delta$ (or $\text{colim}\, \Delta$).

There are a number of basic constructions that can be viewed as a limit or colimit of a suitable diagram.

3.4 PRODUCTS Let I be a set; let $\underline{I}$ be the discrete category with $\text{Ob } \underline{I} = I$. Given a collection $\{X_i:i \in I\}$ of objects in $\underline{C}$, define a diagram $\Delta:\underline{I} \to \underline{C}$ by $\Delta_i = X_i$ $(i \in I)$ — then a limit $\{\ell_i:L \to \Delta_i\}$ of Δ is said to be a product of the X_i. Notation: $L = \prod_i X_i$ (or X^I if $X_i = X$ for all i), $\ell_i = pr_i$, the projection from $\prod_i X_i$ to X_i.

3.5 LEMMA C*ALG has products.

PROOF Let $\{A_i:i \in I\}$ be a collection of objects in C*ALG. Consider the set $\underline{A}$ of all functions $\underline{A}$ from I to $\bigcup_{i\in I} A_i$ such that $\forall\, i \in I$, $\underline{A}(i) \in A_i$ and

$$||\underline{A}|| = \sup_{i\in I} ||\underline{A}(i)|| < \infty.$$

Take the sum, product, and involution pointwise — then $\underline{A}$ is a C*-algebra and $\forall\, i \in I$, there is an arrow $pr_i:\underline{A} \to A_i$, viz.

$$pr_i(\underline{A}) = \underline{A}(i).$$

We claim that the natural source $\{\mathrm{pr}_i:\underline{A} \to A_i\}$ is the product of the A_i. For suppose that $\{\Phi_i:A \to A_i\}$ is another natural source — then $\forall$ i,

$$||\Phi_i(A)|| \leq ||A|| \quad \text{(cf. 1.7)},$$

thus the function

$$\Phi(A):I \to \bigcup_{i\in I} A_i$$

that sends i to $\Phi_i(A)$ belongs to $\underline{A}$. Moreover, the diagram

$$\begin{array}{ccc} A & = & A \\ {\scriptstyle\Phi}\downarrow & & \downarrow{\scriptstyle\Phi_i} \\ \underline{A} & \xrightarrow[\mathrm{pr}_i]{} & A_i \end{array}$$

obviously commutes, from which the claim.

[Note: $\underline{A}$ is not the cartesian product of the A_i if I is infinite.]

E.g.: Take $A_i = \underline{C}\ \forall$ i — then the product in this case is simply $\ell^\infty(I)$.

3.6 <u>COPRODUCTS</u> Let I be a set; let $\underline{I}$ be the discrete category with Ob $\underline{I}$ = I. Given a collection $\{X_i:i \in I\}$ of objects in $\underline{C}$, define a diagram $\Delta:\underline{I} \to \underline{C}$ by $\Delta_i = X_i$ $(i \in I)$ — then a colimit $\{\ell_i:\Delta_i \to L\}$ of Δ is said to be a <u>coproduct</u> of the X_i. Notation: $L = \coprod_i X_i$ (or $I\cdot X$ if $X_i = X$ for all i), $\ell_i = \mathrm{in}_i$, the <u>injection</u> from X_i to $\coprod_i X_i$.

3.7 <u>LEMMA</u> <u>C*ALG</u> has coproducts.

PROOF Let $\{A_i : i \in I\}$ be a collection of objects in C*ALG -- then their coproduct can be realized as the free product C*-algebra $*A_i$, i.e., the completion of the free *-algebra generated by the A_i w.r.t. the largest C*-norm whose restriction to each A_i is the original norm.

3.8 REMARK Let $\underline{0}$ be the category with no objects and no arrows -- then the limit of a diagram having $\underline{0}$ for its indexing category is a final object in $\underline{C}$ and the colimit of a diagram having $\underline{0}$ for its indexing category is an initial object in $\underline{C}$.

[Note: The zero C*-algebra is both a final and initial object in C*ALG.]

3.9 PULLBACKS Let $\underline{I}$ be the category $1 \bullet \xrightarrow{a} \underset{3}{\bullet} \xleftarrow{b} \bullet 2$. Given a 2-sink $X \xrightarrow{f} Z \xleftarrow{g} Y$ in $\underline{C}$, define a diagram $\Delta : \underline{I} \to \underline{C}$ by

$$\begin{cases} \Delta_1 = X \\ \Delta_2 = Y \\ \Delta_3 = Z \end{cases} \quad \& \quad \begin{cases} \Delta a = f \\ \Delta b = g. \end{cases}$$

Then a commutative diagram

$$\begin{array}{ccc} P & \xrightarrow{\eta} & Y \\ {\scriptstyle \xi}\downarrow & & \downarrow{\scriptstyle g} \\ X & \xrightarrow[f]{} & Z \end{array}$$

is said to be a pullback square if for any 2-source $X \overset{\xi'}{\leftarrow} P' \overset{\eta'}{\rightarrow} Y$ with $f \circ \xi' = g \circ \eta'$ there exists a unique morphism $\phi: P' \to P$ such that $\xi' = \xi \circ \phi$ and $\eta' = \eta \circ \phi$. The 2-source $X \overset{\xi}{\leftarrow} P \overset{\eta}{\rightarrow} Y$ is called a pullback of the 2-sink $X \overset{f}{\rightarrow} Z \overset{g}{\leftarrow} Y$. Notation: $P = X \times_Z Y$. Limits of Δ are pullback squares and conversely.

3.10 LEMMA C*ALG has pullbacks.

PROOF Given a 2-sink $A \overset{\Phi}{\rightarrow} C \overset{\Psi}{\leftarrow} B$, let

$$P = \{(A,B) \in A \oplus B : \Phi(A) = \Psi(B)\}.$$

3.11 PUSHOUTS Let $\underline{I}$ be the category $1 \bullet \overset{a}{\longleftarrow} \underset{3}{\bullet} \overset{b}{\longrightarrow} \bullet\, 2$. Given a 2-source $X \overset{f}{\leftarrow} Z \overset{g}{\rightarrow} Y$ in $\underline{C}$, define a diagram $\Delta: \underline{I} \to \underline{C}$ by

$$\begin{bmatrix} \Delta_1 = X \\ \\ \Delta_2 = Y \\ \\ \Delta_3 = Z \end{bmatrix} \quad \& \quad \begin{bmatrix} \Delta a = f \\ \\ \Delta b = g. \end{bmatrix}$$

Then a commutative diagram

$$\begin{array}{ccc} Z & \xrightarrow{g} & Y \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \eta} \\ X & \xrightarrow[\xi]{} & P \end{array}$$

is said to be a pushout square if for any 2-sink $X \overset{\xi'}{\rightarrow} P' \overset{\eta'}{\leftarrow} Y$ with

$\xi' \circ f = \eta' \circ g$ there exists a unique morphism $\phi : P \to P'$ such that $\xi' = \phi \circ \xi$ and $\eta' = \phi \circ \eta$. The 2-sink $X \xrightarrow{\xi} P \xleftarrow{\eta} Y$ is called a pushout of the 2-source $X \xleftarrow{f} Z \xrightarrow{g} Y$. Notation: $P = X \coprod_Z Y$. Colimits of Δ are pushout squares and conversely.

3.12 LEMMA C*ALG has pushouts.

PROOF Given a 2-source $\mathcal{A} \xleftarrow{\Phi} \mathcal{C} \xrightarrow{\Psi} \mathcal{B}$, let

$$\mathcal{P} = \mathcal{A} *_{\mathcal{C}} \mathcal{B},$$

the amalgamated free product.

[Note: Spelled out, $\mathcal{P}$ is the quotient of the free product C*-algebra $\mathcal{A} * \mathcal{B}$ by the closed ideal generated by the set

$$\{\Phi(C) - \Psi(C) : C \in \mathcal{C}\}.]$$

A category $\underline{C}$ is said to be complete if for each small category $\underline{I}$, every diagram $\Delta : \underline{I} \to \underline{C}$ has a limit.

3.13 CRITERION $\underline{C}$ is complete iff $\underline{C}$ has products and pullbacks.

A category $\underline{C}$ is said to be cocomplete if for each small category $\underline{I}$, every diagram $\Delta : \underline{I} \to \underline{C}$ has a colimit.

3.14 CRITERION $\underline{C}$ is cocomplete iff $\underline{C}$ has coproducts and pushouts.

What has been said above can thus be summarized as follows.

3.15 THEOREM C*ALG is both complete and cocomplete.

Let $(I,\leq)$ be an up-directed poset — then the pair $(I,\leq)$ gives rise to a small category:

$$\text{ob } \underline{I} = I,\ \text{Mor}(i,j) = \begin{cases} (i,j) \text{ if } i \leq j \\ \emptyset \text{ otherwise} \end{cases},\ \text{id}_i = (i,i),$$

composition being

$$(j,k)\circ(i,j) = (i,k) \quad (i \leq j \leq k).$$

This said, let $\underline{C}$ be a category — then by definition, a filtered colimit is the colimit of a diagram $\Delta:\underline{I} \to \underline{C}$.

3.16 LEMMA C*ALG has filtered colimits.

[On the basis of 3.15, this is clear. However, it is not difficult to proceed directly. Indeed, to specify a diagram $\Delta:\underline{I} \to$ C*ALG amounts to specifying a collection

$$\{(A_i,\Phi_{ij}):i,j \in I, i \leq j\},$$

where the A_i are C*-algebras and Φ_{ij} is a *-homomorphism from A_i to A_j with

$$\Phi_{ik} = \Phi_{jk} \circ \Phi_{ij} \text{ for } i \leq j \leq k.$$

Each Φ_{ij} is norm decreasing, so on the algebraic filtered colimit, the prescription

$$||A|| = \inf_{j>i} ||\Phi_{ij}(A)|| \quad (A \in A_i)$$

is a C*-seminorm. Dividing out the elements of seminorm 0 and completing then leads to a C*-algebra, written

$$\underrightarrow{\lim}(A_i, \Phi_{ij}),$$

which in fact is a realization of the filtered colimit.]

[Note: Put

$$A = \underrightarrow{\lim}(A_i, \Phi_{ij}).$$

Then strictly speaking, the filtered colimit is the natural sink $\{\Phi_i : A_i \rightarrow A\}$, where $\Phi_i : A_i \rightarrow A$ is the *-homomorphism defined by

$$\Phi_i(A)(i) = A, \Phi_i(A)(j) = \Phi_{ij}(A) \quad (i < j),$$

and

$$\Phi_i(A)(j) = 0 \text{ otherwise.}]$$

3.17 EXAMPLE Let $I = \underline{N}$ — then a filtered colimit of a sequence of finite dimensional C*-algebras is called an AF-algebra. E.g.: Take $A_n = M_n(\underline{C})$ and let

$$\Phi_{n,n+k} : M_n(\underline{C}) \rightarrow M_{n+k}(\underline{C})$$

be the *-homomorphism obtained by adding k rows and columns of zeros — then

$$\underrightarrow{\lim}(M_n(\underline{C}), \Phi_{n,n+k})$$

is *-isomorphic to $\underline{L}_\infty(\ell^2)$.

3.18 LEMMA Let

$$A = \underrightarrow{\lim}(A_i, \Phi_{ij}).$$

Assume: $\forall$ i, A_i is simple — then A is simple.

3.19 <u>REMARK</u> Let I be a set and let $\{A_i : i \in I\}$ be a collection of objects in <u>C*ALG</u>. Form the categorical product $\underline{A}$ as in 3.5 and denote by $\bigoplus_i A_i$ the closure in $\underline{A}$ of the algebraic direct sum -- then $\underline{A} \in \bigoplus_i A_i$ iff $\forall \, \varepsilon > 0$,

$$\#\{i : ||\underline{A}(i)|| \geq \varepsilon\} < \infty.$$

To realize $\bigoplus_i A_i$ as a filtered colimit, let $\mathcal{F}$ be the set of finite subsets of I directed by inclusion and for each $F \in \mathcal{F}$, put

$$A_F = \bigoplus_{i \in F} A_i \ (= \prod_{i \in F} A_i).$$

If $F \subset G$, define

$$\Phi_{F,G} : A_F \to A_G$$

by setting the additional coordinates equal to zero -- then

$$\varinjlim (A_F, \Phi_{F,G}) \approx \bigoplus_i A_i.$$

[Note: Take $A_i = \underline{C} \ \forall \ i$ -- then $\bigoplus_i \underline{C}$ can be identified with $c_0(I)$.]

The setting for filtered colimits is an up-directed poset I. Dually, the setting for cofiltered limits is a down-directed poset I. E.g.: If $I = \underline{N}^{OP}$, then a diagram $\Delta : \underline{I} \to \underline{C}$ is essentially a sequence

$$\cdots \to X_{n+1} \xrightarrow{f_n} X_n \to \cdots$$

of morphisms in $\underline{C}$, where

$$\Delta(n + 1 \to n) = X_{n+1} \xrightarrow{f_n} X_n.$$

3.20 LEMMA C*ALG has cofiltered limits.

Let $\underline{C}$, $\underline{D}$ be categories and let $F:\underline{C} \to \underline{D}$ be a functor.

- F is said to preserve a limit $\{\ell_i:L \to \Delta_i\}$ (colimit $\{\ell_i:\Delta_i \to L\}$) of a diagram $\Delta:\underline{I} \to \underline{C}$ if $\{F\ell_i:FL \to F\Delta_i\}$ ($\{F\ell_i:F\Delta_i \to FL\}$) is a limit (colimit) of the diagram $F \circ \Delta:\underline{I} \to \underline{D}$.

- F is said to preserve limits (colimits) over an indexing category $\underline{I}$ if F preserves all limits (colimits) of diagrams $\Delta:\underline{I} \to \underline{C}$.

- F is said to preserve limits (colimits) if F preserves limits (colimits) over all indexing categories $\underline{I}$.

3.21 ADJOINTS Given categories $\left[\begin{array}{l}\underline{C}\\ \underline{D}\end{array}\right.$, functors $\left[\begin{array}{l}F:\underline{C} \to \underline{D}\\ G:\underline{D} \to \underline{C}\end{array}\right.$ are said to be an adjoint pair if the functors

$$\left[\begin{array}{l}\text{Mor} \circ (F^{OP} \times id_{\underline{D}})\\ \\ \text{Mor} \circ (id_{\underline{C}^{OP}} \times G)\end{array}\right.$$

from $\underline{C}^{OP} \times \underline{D}$ to SET are naturally isomorphic, i.e., if it is possible to assign to each ordered pair $\left[\begin{array}{l}X \in \text{Ob } \underline{C}\\ Y \in \text{Ob } \underline{D}\end{array}\right.$ a bijective map

$$\Xi_{X,Y}:\text{Mor}(FX,Y) \to \text{Mor}(X,GY)$$

which is functorial in X and Y. When this is so, F is a <u>left adjoint</u> for G and G is a <u>right adjoint</u> for F.

Write $[\underline{I},\underline{C}]$ for the category whose objects are the diagrams $\Delta:\underline{I} \to \underline{C}$ and whose morphisms are the natural transformations $\mathrm{Nat}(\Delta,\Delta')$ from Δ to Δ'.

3.22 <u>EXAMPLE</u> Let $K:\underline{C} \to [\underline{I},\underline{C}]$ be the diagonal functor, thus $\forall\ X \in \mathrm{Ob}\ \underline{C}$,

$$(KX)(i) = X, \quad (KX)(i \xrightarrow{\delta} j) = \mathrm{id}_X$$

and $\forall\ f \in \mathrm{Mor}(X,Y)$,

$$Kf \in \mathrm{Nat}(KX,KY)$$

is the natural transformation

$$\begin{array}{ccc} (KX)(i) & \xrightarrow{\Xi_i} & (KY)(i) \\ {\scriptstyle (KX)(\delta)}\downarrow & & \downarrow{\scriptstyle (KY)(\delta)} \\ (KX)(j) & \xrightarrow[\Xi_j]{} & (KY)(j) \end{array}$$

defined by the commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle \mathrm{id}_X}\downarrow & & \downarrow{\scriptstyle \mathrm{id}_Y} \\ X & \xrightarrow[f]{} & Y \end{array} \quad .$$

Assume now that $\underline{C}$ is both complete and cocomplete — then K has a left adjoint, viz.

$$\mathrm{colim}:[\underline{I},\underline{C}] \to \underline{C},$$

and a right adjoint, viz.

$$\lim:[\underline{I},\underline{C}] \to \underline{C}.$$

3.23 REMARK If $\underline{C}$ is both complete and cocomplete, then the same holds for $[\underline{I},\underline{C}]$.

[Note: Limits and colimits in $[\underline{I},\underline{C}]$ are computed "object by object".]

3.24 THEOREM Left adjoints preserve colimits and right adjoints preserve limits.

3.25 RAPPEL Let $\underline{C}$ be a category -- then a morphism $f:X \to Y$ is said to be a monomorphism if for any pair of morphisms $A \overset{u}{\underset{v}{\rightrightarrows}} X$ such that $f \circ u = f \circ v$, there follows $u = v$.

3.26 LEMMA In $\underline{C^*ALG}$, a $*$-homomorphism $\Phi:A \to B$ is a monomorphism iff it is injective.

PROOF An injective $*$-homomorphism $\Phi:A \to B$ is trivially a monomorphism. As for the converse, consider

$$\begin{cases} \operatorname{Ker} \Phi \xrightarrow{i} A \xrightarrow{\Phi} B \\ \operatorname{Ker} \Phi \xrightarrow{0} A \xrightarrow{\Phi} B. \end{cases}$$

Then

$$\Phi \circ i = \Phi \circ 0 \Rightarrow i = 0 \Rightarrow \operatorname{Ker} \Phi = \{0\}.$$

3.27 RAPPEL Let $\underline{C}$ be a category -- then a morphism $f:X \to Y$ is said to be an epimorphism if for any pair of morphisms $Y \overset{u}{\underset{v}{\rightrightarrows}} B$ such that $u \circ f = v \circ f$, there follows u = v.

3.28 LEMMA In C*ALG, a *-homomorphism $\Phi:A \rightarrow B$ is an epimorphism iff it is surjective[†].

[†] Archiv d. Math. 20 (1969), 48-53; see also Inventiones Math. 9 (1970), 295-307.

§4. HILBERT A-MODULES

Let A be a complex Banach algebra -- then a <u>left Banach A-module</u> is a complex Banach space E equipped with a left action $(A,x) \to Ax$ such that for some constant $K > 0$,

$$||Ax|| \leq K||A|| \; ||x|| \quad (A \in A, x \in E).$$

[Note: Right Banach A-modules are defined analogously.]

<u>N.B.</u> If A is nonunital, form A^+ as in §1 (but with $||(A,\lambda)|| = ||A|| + |\lambda|$) -- then E becomes a left Banach A^+-module via the prescription

$$(A + \lambda)x = Ax + \lambda x \quad ((A,\lambda) \equiv A + \lambda).$$

[Note: We have

$$Ix = x \quad (I = 1_{A^+} = (0,1)).]$$

4.1 <u>RAPPEL</u> A <u>left approximate unit</u> per A is a norm bounded net $\{e_i : i \in I\}$ in A such that $e_i A \to A$ for all $A \in A$.

4.2 <u>THEOREM</u> Suppose that A has a left approximate unit $\{e_i : i \in I\}$ and let E be a left Banach A-module -- then the set

$$AE = \{Ax : A \in A, x \in E\}$$

is a closed linear subspace of E.

The assertion is trivial if A is unital so take A nonunital and fix $M > 0$:

$$||e_i|| \leq M \quad (i \in I).$$

4.3 <u>LEMMA</u> Let E_0 be the closed linear span of AE -- then

$$E_0 = \{x \in E: \lim_{i \in I} e_i x = x\}.$$

<u>PROOF</u> The RHS is certainly contained in the LHS. On the other hand, AE is contained in the RHS as is its linear span $[AE]$. With this in mind, take an arbitrary $x \in E_0$ and given $\varepsilon > 0$, choose $y \in [AE]: ||x - y|| < \varepsilon$. Next, choose $i_0 \in I$:

$$i \geq i_0 \Rightarrow ||e_i y - y|| < \varepsilon$$

and write

$$e_i x - x = e_i(x - y) + (y - x) + (e_i y - y).$$

Then $\forall\ i \geq i_0$,

$$\begin{aligned} ||e_i x - x|| &\leq K||e_i||\ ||x - y|| + ||y - x|| + ||e_i y - y|| \\ &\leq (KM + 2)\varepsilon. \end{aligned}$$

4.4 <u>RAPPEL</u> Let $X \in A^+$ and suppose that $||X|| < 1$ -- then $(I - X)^{-1}$ exists and there is a norm convergent expansion

$$(I - X)^{-1} = I + X + X^2 + \cdots .$$

Let $\mu = 1/M$ -- then $\forall\ i \in I$,

$$I - \frac{\mu}{1 + \mu} e_i$$

is invertible, hence the same is true of

$$(1 + \mu)I - \mu e_i$$

as well. And

$$((1 + \mu)I - \mu e_i)^{-1} = (1 + \mu)^{-1}I + A_i$$

for some $A_i \in A$.

4.5 <u>LEMMA</u> Fix $X_0 \in E_0$ -- then $\exists$ a sequence $\{e_{i_n} (\equiv e_n)\}$ in $\{e_i : i \in I\}$ such that

$$A_n^+ = ((1 + \mu)I - \mu e_n)^{-1} \ldots ((1 + \mu)I - \mu e_1)^{-1}$$

$$= (1 + \mu I)^{-n}I + A_n$$

converges in A^+ to a limit $A \in A$ and $x_n = (A_n^+)^{-1}x_0$ converges in E_0 to an element x.

Admit 4.5 for the moment -- then

$$\begin{aligned} & ||A_n^+ x_n - Ax|| \\ & \quad = ||A_n^+ x_n - Ax_n + Ax_n - Ax|| \\ & \quad \leq ||(A_n^+ - A)x_n|| + ||A(x_n - x)|| \\ & \quad \leq K||A_n^+ - A|| \ ||x_n|| + K||A|| \ ||x_n - x|| \\ & \quad \to 0 \ (n \to \infty). \end{aligned}$$

So

$$x_0 = A_n^+ x_n = \lim_{n \to \infty} A_n^+ x_n = Ax.$$

Therefore

$$x_0 \in AE.$$

Turning to the proof of 4.5, set $A_0 = 0$, $A_0^+ = I$ and choose the e_n inductively subject to

$$||x_0 - e_{n+1}x_0|| < ||(A_n^+)^{-1}||^{-1} \frac{M}{K} (1 + \mu)^{-n-1}$$

and

$$||e_{n+1}A_n - A_n|| < M(1 + \mu)^{-n-1}.$$

Since

$$A_n^+ - A_n \to 0,$$

to prove that $\{A_n^+\}$ is convergent, it suffices to prove that $\{A_n\}$ is Cauchy.

First

$$||A_{n+1} - A_n||$$

$$= ||((1 + \mu)I - \mu e_{n+1})^{-1}(\mu(1 + \mu)^{-n-1}e_{n+1} + \mu e_{n+1}A_n - \mu A_n)||.$$

But

$$||((1 + \mu)I - \mu e_{n+1})^{-1}||$$

$$\leq (1 + \mu)^{-1} \frac{1}{1 - \mu(1 + \mu)^{-1}||e_{n+1}||}$$

$$\leq \frac{1}{1 + \mu - \mu||e_{n+1}||}$$

$$\leq M,$$

$$||\mu(1 + \mu)^{-n-1}e_{n+1}|| \leq (1 + \mu)^{-n-1},$$

and

$$||\mu e_{n+1}A_n - \mu A_n|| \leq (1 + \mu)^{-n-1}.$$

Therefore

$$||A_{n+1} - A_n|| \le 2M(1 + \mu)^{-n-1}.$$

So, for $m > n$,

$$||A_n - A_m||$$

$$\le ||A_n - A_{n+1}|| + ||A_{n+1} - A_{n+2}|| + \cdots + ||A_{m-1} - A_m||$$

$$\le 2M(1 + \mu)^{-n-1}(1 + (1 + \mu)^{-1} + \cdots + (1 + \mu)^{n-m+1})$$

$$\le 2M(M + 1)(1 + \mu)^{-n-1} \to 0 \ (n \to \infty),$$

which implies that $\{A_n\}$ is Cauchy.

It remains to deal with $x_n = (A_n^+)^{-1}x_0$. For this purpose, note that

$$x_{n+1} = (A_{n+1}^+)^{-1}x_0$$

$$= (A_n^+)^{-1}((1 + \mu)I - \mu e_{n+1})x_0,$$

thus

$$||x_{n+1} - x_n|| = ||(A_n^+)^{-1}(\mu x_0 - \mu e_{n+1}x_0)||$$

$$\le \mu K||(A_n^+)^{-1}|| \ ||x_0 - e_{n+1}x_0||$$

$$\le (1 + \mu)^{-n-1}.$$

Proceeding as above, we then conclude that $\{x_n\}$ is Cauchy, thereby finishing the proof of 4.5.

4.6 EXAMPLE Let $A \in \mathcal{A}$ -- then $\overline{\mathcal{A}A}$ is a left Banach $\mathcal{A}$-module. Since $A \in \overline{\mathcal{A}A}$, it follows from 4.2 that $\exists\ B \in \mathcal{A}$, $C \in \overline{\mathcal{A}A}$ such that $A = BC$.

Maintain the assumption that $\mathcal{A}$ has a left approximate unit $\{e_i : i \in I\}$.

4.7 LEMMA Let X be a compact subset of E_0 -- then $\exists\ A \in \mathcal{A}$ and a continuous function $f:X \to E_0$ such that

$$x = Af(x) \quad \forall\ x \in X.$$

PROOF Define a left action of $\mathcal{A}$ on the Banach space $C(X,E_0)$ (sup norm) by

$$(Af)(x) = Af(x) \qquad (x \in X).$$

Then

$$||Af|| = \sup_{x\in X} ||(Af)(x)||$$

$$= \sup_{x\in X} ||Af(x)||$$

$$\leq ||A||\ ||f||.$$

Therefore $C(X,E_0)$ is a left Banach $\mathcal{A}$-module. And here

$$C(X,E_0)_0 = C(X,E_0).$$

Accordingly, thanks to 4.2, $\forall\ f_0 \in C(X,E_0)$, $\exists\ A \in \mathcal{A}$ and $f \in C(X,E_0)$:

$$f_0 = Af.$$

Conclude by applying this to the particular choice $f_0(x) = x$ $(x \in X)$.

4.8 EXAMPLE Suppose that $\{x_n\}$ is a sequence in E_0 which converges to 0. In 4.7, take $X = \{0,x_1,x_2,\ldots\}$, and put $y_n = f(x_n)$ -- then $Ay_n = x_n$, $Af(0) = 0$, and $y_n \to f(0)$. So, letting $x_n' = y_n - f(0)$, we have $Ax_n' = x_n$ and $x_n' \to 0$.

4.9 SCHOLIUM Let $\mathcal{A},\mathcal{B}$ be complex Banach algebras. Let $\Phi:\mathcal{A} \to \mathcal{B}$ be a homomorphism. Assume:

1. $\exists\ K > 0: \forall\ A \in \mathcal{A},\ ||\Phi(A)|| \leq K||A||$.
2. $\{e_i : i \in I\}$ is a left approximate unit per $\mathcal{A}$.
3. $\{\Phi(e_i) : i \in I\}$ is a left approximate unit per $\mathcal{B}$.

Define a left action of $\mathcal{A}$ on $\mathcal{B}$ by

$$AB = \Phi(A)B.$$

Then $\mathcal{B}$ is a left Banach $\mathcal{A}$-module and

$$\mathcal{B} = \mathcal{A}\mathcal{B}.$$

[In 4.2, take $E = \mathcal{B}$ -- then

$$\mathcal{B}_0 = \{B \in \mathcal{B} : \lim_{i\in I} \Phi(e_i)B = B\}.$$

But $\mathcal{B}_0 = \mathcal{A}\mathcal{B}$.]

Let $\mathcal{A}$ be a C*-algebra. Let E be a right $\mathcal{A}$-module -- then an $\mathcal{A}$-valued pre-inner product on E is a function $\langle\ ,\ \rangle : E \times E \to \mathcal{A}$ such that $\forall\ x,y,z \in E$, $\forall\ A \in \mathcal{A}$, $\forall\ \lambda \in \underline{C}$:

(i) $\langle x,y + z\rangle = \langle x,y\rangle + \langle x,z\rangle$;

(ii) $\langle x,\lambda y\rangle = \lambda\langle x,y\rangle$;

(iii) $\langle x,yA\rangle = \langle x,y\rangle A$;

(iv) $\langle x,y\rangle^* = \langle y,x\rangle$;

(v) $\langle x,x\rangle \geq 0$ ($\Rightarrow \langle x,x\rangle \in A_+$).

If

$$\langle x,x\rangle = 0 \Rightarrow x = 0,$$

then $\langle\ ,\ \rangle$ is called an A-valued inner product.

[Note: $\langle\ ,\ \rangle$ is "conjugate linear" in the first variable: $\langle xA,y\rangle = A^*\langle x,y\rangle$.]

A pre-Hilbert A-module is a right A-module E equipped with an A-valued pre-inner product $\langle\ ,\ \rangle$.

N.B. Tacitly E is a complex vector space with compatible scalar multiplication: $\lambda(xA) = (\lambda x)A = x(\lambda A)$.

4.10 LEMMA Suppose that E is a pre-Hilbert A-module -- then $\forall\ x,y \in E$,

$$\langle x,y\rangle^*\langle x,y\rangle \leq ||\langle x,x\rangle||\langle y,y\rangle.$$

PROOF Assume that $||\langle x,x\rangle|| = 1$ and let $A \in A$:

$$0 \leq \langle xA - y,xA - y\rangle$$

$$= A^*\langle x,x\rangle A - \langle y,x\rangle A - A^*\langle x,y\rangle + \langle y,y\rangle$$

$$\leq ||\langle x,x\rangle||A^*A - \langle y,x\rangle A - A^*\langle x,y\rangle + \langle y,y\rangle$$

$$= A^*A - \langle y,x\rangle A - A^*\langle x,y\rangle + \langle y,y\rangle.$$

Now take $A = \langle x,y\rangle$ to get

$$0 \leq \langle x,y\rangle^*\langle x,y\rangle - \langle y,x\rangle\langle x,y\rangle - \langle x,y\rangle^*\langle x,y\rangle + \langle y,y\rangle$$

or still,

$$\langle y,x\rangle\langle x,y\rangle \leq \langle y,y\rangle$$

or still,

$$\langle x,y\rangle^*\langle x,y\rangle \leq \langle y,y\rangle.$$

Put

$$||x|| = ||\langle x,x\rangle||^{1/2} \quad (x \in E).$$

Then 4.10 implies that $||.||$ is a seminorm on E:

$$\begin{cases} ||x + y|| \leq ||x|| + ||y|| \\ ||\lambda x|| \leq |\lambda| \; ||x||. \end{cases}$$

Moreover, $||.||$ is a norm if the pre-inner product is actually an inner product.

Definition: E is said to be a Hilbert A-module if the seminorm is a norm and E is complete (hence is a Banach space).

4.11 EXAMPLE Take $A = \underset{\sim}{C}$ -- then the Hilbert $\underset{\sim}{C}$-modules are the complex Hilbert spaces.

4.12 EXAMPLE Let Ξ be a hermitian vector bundle over a compact space X. Denote by $\Gamma(\Xi)$ the space of continuous sections of Ξ -- then $\Gamma(\Xi)$ is a right C(X)-module and the rule

$$\langle\sigma,\sigma'\rangle(x) = \langle\sigma(x),\sigma'(x)\rangle_x$$

equips $\Gamma(\Xi)$ with the structure of a Hilbert C(X)-module.

Let

$$N_E = \{x \in E: ||x|| = 0\}.$$

Then N_E is a sub A-module of E and the pre-inner product and seminorm drop to an inner product and norm on the quotient A-module E/N_E.

4.13 LEMMA The completion of E/N_E is a Hilbert A-module.

A Hilbert A-module E is a right Banach A-module. Proof:

$$\begin{aligned} ||xA|| &= ||\langle xA,xA\rangle||^{1/2} \\ &= ||A^*\langle x,x\rangle A||^{1/2} \\ &\leq ||A^*||^{1/2}\ ||\langle x,x\rangle||^{1/2}\ ||A||^{1/2} \\ &= ||x||\ ||A||. \end{aligned}$$

4.14 LEMMA Let E be a Hilbert A-module -- then $E = EA$.

PROOF One has only to show that EA is dense in E (cf. 4.2). But

$$\begin{aligned} &\langle x - xe_i, x - xe_i\rangle \\ &\quad = \langle x,x\rangle - e_i\langle x,x\rangle - \langle x,x\rangle e_i + e_i\langle x,x\rangle e_i \\ &\quad \to 0. \end{aligned}$$

[Note: If A is unital, then $x = x1_A$.]

Here are three examples of Hilbert A-modules which are "internal" to A.

4.15 EXAMPLE View A itself as a right A-module and put

$$<A,B> = A^*B \quad (A,B \in A).$$

Then A is a Hilbert A-module.

4.16 EXAMPLE Given $n \in \underline{N}$, let $A^n = A \oplus \cdots \oplus A$. View A^n as a right A-module in the obvious way and put

$$<A_1 \oplus \cdots \oplus A_n, B_1 \oplus \cdots \oplus B_n> = \sum_{k=1}^{n} A_k^* B_k.$$

Then A^n is a Hilbert A-module.

4.17 EXAMPLE Let H_A stand for the subset of $\prod_{k=1}^{\infty} A$ consisting of those $\underline{A}$ such that $\sum_{k=1}^{\infty} A_k^* A_k$ $(A_k = \underline{A}(k))$ converges in A. View H_A as a right A-module in the obvious way and put

$$<\underline{A},\underline{B}> = \sum_{k=1}^{\infty} A_k^* B_k.$$

Then H_A is a Hilbert A-module.

4.18 REMARK Let H_A^2 stand for the subset of $\prod_{k=1}^{\infty} A$ consisting of those $\underline{A}$ such that $\sum_{k=1}^{\infty} ||A_k||^2 < \infty$ $(A_k = \underline{A}(k))$ -- then

$$H_A^2 \subset H_A$$

and

$$H_A^2 = H_A$$

iff A is finite dimensional. E.g.: $H_{\underline{C}} = \ell^2$.

Let H be a complex Hilbert space, E a Hilbert A-module -- then their algebraic tensor product $H \otimes E$ carries an A-valued inner product given on elementary tensors by

$$<\xi \otimes x, \eta \otimes y> = <\xi, \eta><x,y>.$$

Its completion $H \underline{\otimes} E$ is therefore a Hilbert A-module (cf. 4.13).

4.19 <u>EXAMPLE</u> Suppose that H is separable and infinite dimensional -- then $H \underline{\otimes} A$ and H_A are isomorphic as Hilbert A-modules.

4.20 <u>EXAMPLE</u> Let X be a compact Hausdorff space -- then $C(X,H)$ is a Hilbert $C(X)$-module and

$$H \underline{\otimes} C(X) \approx C(X,H).$$

[Consider the map

$$H \otimes C(X) \rightarrow C(X,H)$$

that sends $\xi \otimes f$ to the function $x \rightarrow f(x)\xi$. It preserves $C(X)$-valued inner products and has a dense range.]

4.21 <u>CONSTRUCTION</u> Suppose that E and F are Hilbert A-modules -- then $E \oplus F$ is a right A-module in the obvious way and the prescription

$$<(x,y),(x',y')> = <x,x'> + <y,y'>$$

is an A-valued inner product on $E \oplus F$. Since the completeness of E and F implies that of $E \oplus F$, it follows that $E \oplus F$ is a Hilbert A-module.

One difference between Hilbert A-modules and Hilbert spaces lies in the properties of orthogonal complements. Thus let $F \subset E$ be a closed sub A-module of the Hilbert A-module E. Put

$$F^{\perp} = \{x \in E: \langle F,x\rangle = 0\}.$$

Then $F^{\perp}$ is also a closed sub A-module but in general, E is not equal to $F \oplus F^{\perp}$.

4.22 <u>EXAMPLE</u> Take $A = C[0,1] = E$ and let $F = \{g \in E: g(0) = 0\}$ -- then $F^{\perp} = \{0\}$, so $F \oplus F^{\perp} \neq E$.

Let E and F be Hilbert A-modules -- then by $\text{Hom}_A(E,F)$ we shall understand the subset of $\mathcal{B}(E,F)$ whose elements are the $T:E \to F$ which are A-linear:

$$T(xA) = (Tx)A \quad (x \in E,\ A \in A).$$

<u>N.B.</u> $\text{Hom}_A(E,F)$ is a closed subspace of $\mathcal{B}(E,F)$, hence is a Banach space.

4.23 <u>LEMMA</u> $\forall\ T \in \text{Hom}_A(E,F)$, we have

$$\langle Tx,Tx\rangle \leq ||T||^2\langle x,x\rangle \quad (x \in E).$$

Let $T \in \text{Hom}_A(E,F)$ -- then T is said to be <u>adjointable</u> if $\exists$ an operator $T^* \in \text{Hom}_A(F,E)$ such that

$$\langle Tx,y\rangle = \langle x,T^*y\rangle$$

for all $x \in E$, $y \in F$.

[Note: T^* is unique and $T^{**} = T$.]

Write $\text{Hom}_A^*(E,F)$ for the subset of $\text{Hom}_A(E,F)$ consisting of those T which are adjointable -- then $\text{Hom}_A^*(E,F)$ is a Banach space.

[Note: The containment

$$\text{Hom}_A^*(E,F) \subset \text{Hom}_A(E,F)$$

is, in general, proper (cf. infra).]

4.24 <u>EXAMPLE</u> Take $A = C[0,1] = E$ and let $F = \{g \in E: g(0) = 0\}$ (cf. 4.22). Define $T: E \oplus F \to E \oplus F$ by $T(f,g) = (g,0)$ -- then

$$T \in \text{Hom}_A(E \oplus F, E \oplus F) \text{ but } T \notin \text{Hom}_A^*(E \oplus F, E \oplus F).$$

4.25 <u>LEMMA</u> $\text{Hom}_A^*(E,E)$ is a unital C^*-algebra.

[Note: $\text{Hom}_A(E,E)$ is a unital Banach algebra.]

4.26 <u>REMARK</u> Let $T \in \text{Hom}_A^*(E,E)$ -- then $T \in \text{Hom}_A^*(E,E)_+$ iff $\forall\, x \in E$, $\langle Tx,x\rangle \geq 0$.

4.27 <u>NOTATION</u> $\underline{\text{H*MOD}_A}$ is the category whose objects are the Hilbert A-modules with

$$\text{Mor}(E,F) = \text{Hom}_A^*(E,F).$$

<u>N.B.</u> $\underline{\text{H*MOD}_A}$ is a $\star$-category in the sense that it comes equipped with an involutive, identity-on-objects, cofunctor

$$\star: \underline{\text{H*MOD}_A} \to \underline{\text{H*MOD}_A}.$$

4.28 EXAMPLE Let HILB be the category whose objects are the complex Hilbert spaces and whose morphisms are the bounded linear operators — then

$$\underline{\text{HILB}} = \underline{\text{H*MOD}}_{\underline{\mathbb{C}}}$$

and

$$*:\text{Mor}(\mathcal{H}_1,\mathcal{H}_2) = \mathcal{B}(\mathcal{H}_1,\mathcal{H}_2)$$

sends $T:\mathcal{H}_1 \to \mathcal{H}_2$ to its adjoint $T^*:\mathcal{H}_2 \to \mathcal{H}_1$.

4.29 LEMMA $\text{Hom}_A^*(E,F)$ is a Hilbert $\text{Hom}_A^*(E,E)$-module.

PROOF The right action

$$\text{Hom}_A^*(E,F) \times \text{Hom}_A^*(E,E) \to \text{Hom}_A^*(E,F)$$

is precomposition and the $\text{Hom}_A^*(E,E)$-valued inner product

$$< , >:\text{Hom}_A^*(E,F) \times \text{Hom}_A^*(E,F) \to \text{Hom}_A^*(E,E)$$

is

$$<T,S> = T^*S.$$

[Note: The induced norm on $\text{Hom}_A^*(E,F)$ is the operator norm.]

Let E be a Hilbert A-module. Given $x \in E$, define $\hat{x}:E \to A$ by

$$\hat{x}(y) = <x,y>$$

and define $L_x:A \to E$ by

$$L_x(A) = xA.$$

Then

$$\left[\begin{array}{l} \hat{x} \in \text{Hom}_A(E,A) \\ \\ L_x \in \text{Hom}_A(A,E). \end{array}\right.$$

And

$$\langle \hat{x}(y), A\rangle = \langle\langle x,y\rangle, A\rangle$$

$$= \langle x,y\rangle^* A$$

$$= \langle y,x\rangle A$$

$$= \langle y,xA\rangle$$

$$= \langle y,L_x(A)\rangle .$$

Therefore

$$(\hat{x})^* = L_x$$

=>

$$\begin{cases} \hat{x} \in \mathrm{Hom}_A^*(E,A) \\ L_x \in \mathrm{Hom}_A^*(A,E). \end{cases}$$

Put

$$\hat{E} = \mathrm{Hom}_A(E,A).$$

Then $\hat{E}$ is a right A-module:

$$(TA)x = A^*Tx.$$

4.30 <u>LEMMA</u> The arrow

$$\begin{cases} E \to \hat{E} \\ x \to \hat{x} \end{cases}$$

is an isometric conjugate linear map of right A-modules.

One then calls E <u>selfdual</u> if this arrow is surjective, thus

$$\mathrm{Hom}_A(E,A) = \mathrm{Hom}_A^*(E,A).$$

4.31 <u>EXAMPLE</u> A is selfdual iff A is unital.

4.32 <u>EXAMPLE</u> H_A is selfdual iff A is finite dimensional.

4.33 <u>LEMMA</u> Suppose that E is selfdual -- then $\forall$ Hilbert A-module F,

$$\mathrm{Hom}_A(E,F) = \mathrm{Hom}_A^*(E,F).$$

4.34 <u>REMARK</u> Suppose that A is a W*-algebra and let E be a selfdual Hilbert A-module -- then it can be shown that the unital C*-algebra $\mathrm{Hom}_A^*(E,E)$ is a W*-algebra.

Let E and F be Hilbert A-modules. Given $\begin{cases} x \in E \\ y \in F \end{cases}$, define $\Theta_{y,x}:E \to F$ by

$$\Theta_{y,x}(x') = y\langle x,x'\rangle.$$

Then

$$||\Theta_{y,x}|| \leq ||y||\ ||x||$$

and

$$\Theta_{y,x}(x'A) = y\langle x,x'A\rangle = y\langle x,x'\rangle A = \Theta_{y,x}(x')A.$$

E.g.: Take E = F = A and suppose that A is unital -- then

$$\Theta_{1_A,1_A} = \mathrm{id}_A.$$

4.35 <u>LEMMA</u> $\Theta_{y,x} \in \text{Hom}_A^*(E,F)$:

$$\Theta_{y,x}^* = \Theta_{x,y}.$$

Write $\underline{L}_\infty(E,F)$ for the closed linear subspace of $\text{Hom}_A^*(E,F)$ spanned by the $\Theta_{y,x}$.

4.36 <u>EXAMPLE</u> The image of the arrow in 4.30 is $\underline{L}_\infty(E,A)$. In fact,

$$\Theta_{A,x} = \hat{x}^*.$$

Accordingly, when E is selfdual,

$$\underline{L}_\infty(E,A) = \text{Hom}_A^*(E,A).$$

So, e.g., if A is unital, then

$$\underline{L}_\infty(A,A) = \text{Hom}_A^*(A,A),$$

but if A is nonunital, then $\text{Hom}_A^*(A,A)$ is in general much larger than $\underline{L}_\infty(A,A)$ (cf. §5).

4.37 <u>REMARK</u> If A is unital and if E is a Hilbert A-module, then

$$\underline{L}_\infty(E,A) = \text{Hom}_A^*(E,A).$$

Thus let $T \in \text{Hom}_A^*(E,A)$ and put $x = T^*(1_A)$ -- then

$$\begin{aligned} \hat{x}(y) &= \langle x,y\rangle \\ &= \langle T^*(1_A),y\rangle \\ &= \langle 1_A,Ty\rangle \\ &= 1_A^* Ty \\ &= Ty. \end{aligned}$$

Take E = F -- then

$$\Theta_{x,y}\Theta_{u,v} = \Theta_{x<y,u>,v} = \Theta_{x,v<u,y>}$$

and

$$\begin{bmatrix} T\Theta_{x,y} = \Theta_{Tx,y} \\ \Theta_{x,y}T = \Theta_{x,T^*y}. \end{bmatrix}$$

4.38 LEMMA $\underline{L}_\infty(E,E)$ is a closed ideal in $\mathrm{Hom}_A^*(E,E)$.

[Note: Therefore $\underline{L}_\infty(E,E)$ is a C*-algebra.]

More is true: $\underline{L}_\infty(E,E)$ is an essential ideal in $\mathrm{Hom}_A^*(E,E)$. To prove this, we shall need a technical preliminary.

4.39 LEMMA $\forall\ x \in E$,

$$x = \lim_{\varepsilon\to 0} x<x,x> \ (<x,x> + \varepsilon)^{-1}.$$

Bearing in mind 1.11, let $\mathcal{J} \subset \mathrm{Hom}_A^*(E,E)$ be a closed ideal such that $\mathcal{J} \cap \underline{L}_\infty(E,E) = \{0\}$. Fix $J \in \mathcal{J}$ -- then $\forall\ x \in E$, $J\Theta_{x,x} = 0$ and

$$\begin{aligned} Jx &= J(\lim_{\varepsilon\to 0} x<x,x> \ (<x,x> + \varepsilon)^{-1}) \\ &= \lim_{\varepsilon\to 0} J(x<x,x> \ (<x,x> + \varepsilon)^{-1}) \\ &= \lim_{\varepsilon\to 0} J\Theta_{x,x}(x)\,(<x,x> + \varepsilon)^{-1} \end{aligned}$$

$$= \lim_{\varepsilon \to 0} 0(\langle x,x \rangle + \varepsilon)^{-1}$$

$$= 0.$$

I.e.: $J = \{0\}$.

4.40 EXAMPLE The C*-algebra $\underline{L}_\infty(A,A)$ is $*$-isomorphic to A. To see this, define $L_A : A \to A$ by $L_A B = AB$ -- then

$$(L_A)^* = L_{A^*} \Rightarrow L_A \in \text{Hom}_A^*(A,A).$$

But

$$||L_A|| = ||A||.$$

Therefore the range of

$$L : A \to \text{Hom}_A^*(A,A)$$

is a C*-subalgebra of $\text{Hom}_A^*(A,A)$. On the other hand,

$$\Theta_{A,B}(C) = A\langle B,C \rangle = AB^*C = L_{AB^*}C,$$

from which it follows that

$$LA = \underline{L}_\infty(A,A).$$

[Note: The pair

$$(\text{Hom}_A^*(A,A), L)$$

is a unitization of A. Indeed, the image LA is $\underline{L}_\infty(A,A)$, which is an essential ideal in $\text{Hom}_A^*(A,A)$.]

4.41 REMARK Let $M_n(A)$ be the set of n-by-n matrices with entries from A --

then $M_n(A)$ is a $\star$-algebra but it is not a priori obvious that $M_n(A)$ is a C*-algebra (if $n > 1$). Here is one way to proceed. Introduce A^n per 4.16 -- then the map

$$\Theta_{A_1 \oplus \cdots \oplus A_n, B_1 \oplus \cdots \oplus B_n} \to \begin{bmatrix} A_1B_1^* & \cdots & A_1B_n^* \\ \vdots & & \vdots \\ A_nB_1^* & \cdots & A_nB_n^* \end{bmatrix}$$

implements a $\star$-isomorphism

$$\underline{L}_\infty(A^n, A^n) \to M_n(A).$$

Therefore $M_n(A)$ becomes a C*-algebra via transport of structure.

[Note: The involution $*:M_n(A) \to M_n(A)$ is

$$[A_{ij}]^* = [A_{ji}^*].]$$

§5. *MULTIPLIERS and DOUBLE CENTRALIZERS*

Given a C*-algebra A, put

$$M(A) = \text{Hom}_A^*(A,A).$$

Then $M(A)$ is a unital C*-algebra, the multiplier algebra of A. Abbreviate $\underline{L}_\infty(A,A)$ to $\underline{L}_\infty(A)$, thus $\underline{L}_\infty(A)$ is an essential ideal in $M(A)$ and there is a *-isomorphism $L:A \to \underline{L}_\infty(A)$ (cf. 4.40).

[Note: Recall that

$$\underline{L}_\infty(A) = M(A)$$

if A is unital (cf. 4.37).]

Let E be a left Banach A-module -- then according to 4.2, the set

$$AE = \{Ax:A \in A,x \in E\}$$

is a closed linear subspace of E, which can be characterized as

$$\{x \in E:\lim_{i\in I} e_i x = x\},$$

denoted by E_0 in 4.3.

N.B. E can, of course, be viewed as a left Banach $\underline{L}_\infty(A)$-module by writing

$$L_A x = Ax.$$

5.1 THEOREM Assume: $E = AE$ -- then the prescription

$$M(Ax) = (ML_A)x \quad (M \in M(A))$$

iw welldefined and serves to equip E with the structure of a left Banach $M(A)$-module.

PROOF Observe first that

$$ML_A \in \underline{L}_\infty(A),$$

so the RHS makes sense. To check that matters are welldefined, suppose that $A_1x_1 = A_2x_2$ -- then

$$\begin{aligned}(ML_{A_1})x_1 &= (M \lim_{i\in I} L_{e_iA_1})x_1 \\ &= \lim_{i\in I} (ML_{e_iA_1})x_1 \\ &= \lim_{i\in I} (ML_{e_i}L_{A_1})x_1 \\ &= \lim_{i\in I} (ML_{e_i})A_1x_1 \\ &= \lim_{i\in I} (ML_{e_i})A_2x_2 \\ &= (ML_{A_2})x_2.\end{aligned}$$

And

$$\begin{aligned}||M(Ax)|| &= ||(ML_A)x|| \\ &= ||\lim_{i\in I} (ML_{e_i})Ax|| \\ &= \lim_{i\in I} ||(ML_{e_i})Ax||.\end{aligned}$$

But

$$\begin{aligned}||(ML_{e_i})Ax|| &\le K||ML_{e_i}||\ ||Ax|| \\ &\le K||M||\ ||L_{e_i}||\ ||Ax|| \\ &\le K||M||\ ||Ax||.\end{aligned}$$

Therefore E is a left Banach $M(\mathcal{A})$-module.

Given C*-algebras $\mathcal{A}$ and $\mathcal{B}$, a $*$-homomorphism $\Phi:\mathcal{A} \rightarrow \mathcal{B}$ is said to be <u>proper</u> if for any approximate unit $\{e_i : i \in I\}$ per $\mathcal{A}$, $\{\Phi(e_i) : i \in I\}$ is an approximate unit per $\mathcal{B}$.

5.2 <u>THEOREM</u> Suppose that $\Phi:\mathcal{A} \rightarrow \mathcal{B}$ is proper -- then there is a unique unital $*$-homomorphism $\bar{\Phi}:M(\mathcal{A}) \rightarrow M(\mathcal{B})$ extending Φ_∞:

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{\Phi} & \mathcal{B} \\ {\scriptstyle L}\downarrow\wr & & \wr\downarrow{\scriptstyle L} \\ \underline{L}_\infty(\mathcal{A}) & \xrightarrow{\Phi_\infty} & \underline{L}_\infty(\mathcal{B}) \\ \downarrow & & \downarrow \\ M(\mathcal{A}) & \xrightarrow{\bar{\Phi}} & M(\mathcal{B}). \end{array}$$

<u>PROOF</u> It is a question of applying 5.1. Thus view $\mathcal{B}$ as a left Banach $\mathcal{A}$-module per 4.9 -- then $\mathcal{B} = \mathcal{A}\mathcal{B}$. This said, given $M \in M(\mathcal{A})$, define $\bar{\Phi}(M) \in M(\mathcal{B})$ by

$$\bar{\Phi}(M)(\Phi(A)B) = \Phi_\infty(ML_A)B.$$

Then

$$\bar{\Phi}|\underline{L}_\infty(\mathcal{A}) = \Phi_\infty.$$

In fact, $\forall\ A' \in \mathcal{A}$,

$$\begin{aligned} \bar{\Phi}(L_{A'})(\Phi(A)B) &= \Phi_\infty(L_{A'}L_A)B \\ &= \Phi_\infty(L_{A'A})B \end{aligned}$$

$$= L_{\Phi(A'A)} B$$

$$= L_{\Phi(A')} L_{\Phi(A)} B$$

$$= L_{\Phi(A')} (\Phi(A)B)$$

$$= \Phi_{\infty}(L_{A'})(\Phi(A)B).$$

5.3 <u>NOTATION</u> <u>PRC*ALG</u> is the category whose objects are the C*-algebras and whose morphisms are the proper *-homomorphisms.

<u>N.B.</u> The assignment

$$\begin{cases} A \rightarrow M(A) \\ \Phi \rightarrow \bar{\Phi} \end{cases}$$

is functorial, i.e., defines a functor

<u>PRC*ALG</u> → <u>UNC*ALG</u>.

Suppose that (U,i) is a unitization of A -- then (U,i) is said to be <u>maximal</u> if for every embedding $j:A \rightarrow V$ as an essential ideal of a C*-algebra V, there exists a *-homomorphism $\zeta:V \rightarrow U$ such that $\zeta \circ j = i$:

$$\begin{array}{ccc} A & = & A \\ j \downarrow & & \downarrow i \\ V & \xrightarrow[\zeta]{} & U \end{array} .$$

5.4 REMARK ζ is necessarily injective ($j(A)$ being essential) and, in fact, is unique.

[Note: If

$$\left[\begin{array}{l} (\mathcal{U}_1, i_1) \\ \\ (\mathcal{U}_2, i_2) \end{array}\right.$$

are maximal unitizations of A, then there exists a $*$-isomorphism $\Phi:\mathcal{U}_1 \to \mathcal{U}_2$ such that $\Phi \circ i_1 = i_2$.]

5.5 LEMMA The pair $(M(A), L)$ is a maximal unitization of A.

5.6 EXAMPLE Let X be a noncompact locally compact Hausdorff space and let BC(X) be the C*-algebra of complex valued bounded continuous functions on X -- then $C_\infty(X)$ sits inside BC(X) as an essential ideal, hence there is a commutative diagram

$$\begin{array}{ccc} C_\infty(X) & = & C_\infty(X) \\ {\scriptstyle j}\downarrow & & \downarrow{\scriptstyle L} \\ BC(X) & \xrightarrow[\zeta]{} & M(C_\infty(X)), \end{array}$$

where, as pointed out above, ζ is injective. But here ζ is also surjective, i.e., is a $*$-isomorphism.

Given a Hilbert A-module E, denote by $\langle E,E\rangle$ the linear span of the set $\{\langle x,y\rangle : x,y \in E\}$ -- then the closure $\overline{\langle E,E\rangle}$ of $\langle E,E\rangle$ is an ideal in A. Working with an approximate unit from $\overline{\langle E,E\rangle}$, one finds that $E\overline{\langle E,E\rangle}$ is dense in E.

Abbreviate

$$\begin{cases} \text{Hom}_A^*(E,E) \text{ to } \text{Hom}_A^*(E) \\ \underline{L}_\infty(E,E) \text{ to } \underline{L}_\infty(E). \end{cases}$$

Then $\text{Hom}_A^*(E)$ is a unital C*-algebra containing $\underline{L}_\infty(E)$ as an essential ideal.

5.7 <u>LEMMA</u> View E as a left Banach $\underline{L}_\infty(E)$-module -- then

$$\underline{L}_\infty(E)E = E.$$

<u>PROOF</u> Let $\{e_i : i \in I\}$ be an approximate unit per $\underline{L}_\infty(E)$ -- then it need only be shown that $e_i x \to x \ \forall \ x \in E$ (cf. 4.2 and 4.3). And for this, it suffices to prove that $e_i x \to x \ \forall \ x \in E\langle E,E\rangle$. So suppose that

$$x = y\langle u,v\rangle.$$

Then

$$e_i\Theta_{y,u} \to \Theta_{y,u} \text{ in } \underline{L}_\infty(E)$$

$$\Rightarrow$$

$$(e_i\Theta_{y,u})(v) \to \Theta_{y,u}(v) \text{ in } E$$

$$\Rightarrow$$

$$e_i(\Theta_{y,u}(v)) \to \Theta_{y,u}(v) \text{ in } E$$

$$\Rightarrow$$

$$e_i x \to x \text{ in } E.$$

5.8 <u>THEOREM</u> We have

$$M(\underline{L}_\infty(E)) \approx \text{Hom}_A^*(E).$$

PROOF Let

$$i: \underline{L}_\infty(E) \to \mathrm{Hom}_A^*(E)$$

be the inclusion -- then the pair

$$(\mathrm{Hom}_A^*(E), i)$$

is a unitization of $\underline{L}_\infty(E)$, which we claim is maximal. To see this, consider an embedding $j: \underline{L}_\infty(E) \to V$ as an essential ideal of a C*-algebra V. Imitating the procedure utilized in 5.1, define $\zeta: V \to \mathrm{Hom}_A^*(E)$ by

$$\zeta(v)Tx = (vj(T))x \quad (x \in E, \ T \in \underline{L}_\infty(E)).$$

And so forth... .

5.9 EXAMPLE Take $A = \underline{C}$ -- then the Hilbert $\underline{C}$-modules are the complex Hilbert spaces H, thus

$$M(\underline{L}_\infty(H)) \approx \mathrm{Hom}_{\underline{C}}^*(H) = B(H).$$

5.10 REMARK The relation

$$M(A) = \mathrm{Hom}_A^*(A)$$

is a definition. On the other hand,

$$\underline{L}_\infty(A) \approx A$$

$$\Rightarrow$$

$$M(\underline{L}_\infty(A)) \approx \mathrm{Hom}_A^*(A).$$

5.11 EXAMPLE $\forall\, n \in \underline{N}$,

$$\underline{L}_\infty(A^n) \approx M_n(A) \quad \text{(cf. 4.41)}$$

=>

$$M(\underline{L}_\infty(A^n)) \approx M(M_n(A))$$

$$\approx M_n(M(A))$$

$$\approx \text{Hom}_A^*(A^n).$$

[Note: $\forall\ n \in \underline{N}$,

$$M(A^n) \approx M(A)^n.]$$

5.12 <u>EXAMPLE</u> Suppose that H is separable and infinite dimensional -- then

$$H_A \approx H \underline{\otimes} A \qquad \text{(cf. 4.19)}$$

=>

$$\underline{L}_\infty(H_A) \approx \underline{L}_\infty(H \underline{\otimes} A)$$

$$\approx \underline{L}_\infty(H) \otimes_{\min} A,$$

the symbol $\otimes_{\min}$ standing for the minimal tensor product (cf. §6).

[Note: $\underline{L}_\infty(H)$ is nuclear, so there is only one C*-norm on $\underline{L}_\infty(H) \otimes A$.]

There is another approach to $M(A)$ based on purely algebraic tenets. Assume for the moment that A is just a complex algebra -- then a

<u>left centralizer</u>

of A

<u>right centralizer</u>

is a linear map

$$\begin{cases} L:\mathcal{A} \to \mathcal{A} \\ R:\mathcal{A} \to \mathcal{A} \end{cases}$$

such that $\forall$ A,B $\in \mathcal{A}$,

$$\begin{cases} L(AB) = L(A)B \\ R(AB) = AR(B) \end{cases}$$

and a <u>double centralizer</u> of $\mathcal{A}$ is a pair (L,R), where

$$\begin{cases} \text{L is a left centralizer} \\ \text{R is a right centralizer} \end{cases}$$

such that $\forall$ A,B $\in \mathcal{A}$,

$$AL(B) = R(A)B.$$

Write $\mathcal{DC}(\mathcal{A})$ for the set of double centralizers of $\mathcal{A}$ -- then $\mathcal{DC}(\mathcal{A})$ is a complex algebra under pointwise linear operations, multiplication being defined by

$$(L_1,R_1)(L_2,R_2) = (L_1L_2,R_2R_1).$$

Since

$$(\mathrm{id}_{\mathcal{A}},\mathrm{id}_{\mathcal{A}}) \in \mathcal{DC}(\mathcal{A}),$$

it follows that $\mathcal{DC}(\mathcal{A})$ is unital.

Given A $\in \mathcal{A}$, define

$$\begin{cases} L_A:\mathcal{A} \to \mathcal{A} \\ R_A:\mathcal{A} \to \mathcal{A} \end{cases}$$

by

$$\begin{cases} L_A(B) = AB \\ R_A(B) = BA. \end{cases}$$

Then the pair

$$(L_A, R_A) \in \mathbf{DC}(A)$$

and the map

$$\iota_A: \begin{cases} A \to \mathbf{DC}(A) \\ \\ A \to (L_A, R_A) \end{cases}$$

is a homomorphism whose kernel is called the <u>annihilator</u> of A: Ann A.

5.13 <u>LEMMA</u> ι_A is surjective iff A is unital.

<u>N.B.</u> Therefore ι_A is an isomorphism iff A is unital.

5.14 <u>LEMMA</u> $\forall$ A,B $\in A$ and $\forall$ (L,R) $\in \mathbf{DC}(A)$, we have

$$\begin{cases} L_A L(B) = AL(B) = R(A)B = L_{R(A)}B \\ RR_A(B) = R(BA) = BR(A) = R_{R(A)}B \\ LL_A(B) = L(AB) = L(A)B = L_{L(A)}B \\ R_A R(B) = R(B)A = BL(A) = R_{L(A)}B. \end{cases}$$

Consequently, $\iota_A(A)$ is an ideal in $\mathbf{DC}(A)$ and

$$\begin{cases} (L,R)(L_A,R_A) = (L_{L(A)},R_{L(A)}) \\ (L_A,R_A)(L,R) = (L_{R(A)},R_{R(A)}). \end{cases}$$

Put

$$\begin{cases} \text{Ann}_L A = \{A \in A : AB = 0 \ \forall \ B \in A\} \\ \text{Ann}_R A = \{A \in A : BA = 0 \ \forall \ B \in A\}. \end{cases}$$

Then

$$\text{Ann } A = \text{Ann}_L A \cap \text{Ann}_R A.$$

Now specialize and assume that A is a complex Banach algebra.

5.15 LEMMA Suppose that

$$\text{Ann}_L A = \{0\} \text{ and } \text{Ann}_R A = \{0\}.$$

Let $(L,R) \in \mathcal{DC}(A)$ -- then L and R are bounded:

$$L,R \in \mathcal{B}(A).$$

PROOF Let $\{A_n\}$ be a sequence which converges to 0 with $\{L(A_n)\}$ converging to B (say) -- then $\forall$ C $\in A$,

$$\begin{aligned} CB &= C(\lim_{n \to \infty} L(A_n)) \\ &= \lim_{n \to \infty} CL(A_n) \\ &= \lim_{n \to \infty} R(C)A_n \\ &= 0. \end{aligned}$$

Therefore

$$B \in \text{Ann}_R A = \{0\}.$$

So, by the closed graph theorem, L is bounded. The argument for R is analogous.

5.16 REMARK The existence of a

$$\begin{cases} \text{right approximate unit per } A \Rightarrow \text{Ann}_L A = \{0\} \\ \text{left approximate unit per } A \Rightarrow \text{Ann}_R A = \{0\}. \end{cases}$$

[Note: In particular, these conditions are met by a C*-algebra.]

Maintaining the suppositions of 5.15, place a norm on $\mathfrak{DC}(A)$ by stipulating that

$$||(L,R)|| = \max\{||L||,||R||\}.$$

Then $\mathfrak{DC}(A)$ is a unital Banach algebra and

$$\iota_A : A \to \mathfrak{DC}(A)$$

is contractive.

5.17 EXAMPLE Let G be a locally compact topological group (Hausdorff is assumed). Take $A = L^1(G)$ (left Haar measure) -- then $\forall\ f,g \in L^1(G)$,

$$\begin{cases} ||f|| = \sup\{||f*\phi||:||\phi|| \leq 1\} \\ ||g|| = \sup\{||\phi*g||:||\phi|| \leq 1\}. \end{cases}$$

Therefore

$$\begin{cases} \text{Ann}_L A = \{0\} \\ \text{Ann}_R A = \{0\}. \end{cases}$$

Given $\mu \in M(G)$, define

$$\begin{cases} L_\mu \\ R_\mu \end{cases} \in \mathcal{B}(L^1(G))$$

by

$$\begin{cases} L_\mu f = \mu * f \\ R_\mu f = f * \mu. \end{cases}$$

Then

$$\begin{cases} ||L_\mu|| = ||\mu|| \\ ||R_\mu|| = ||\mu|| \end{cases}, \quad (L_\mu, R_\mu) \in \mathfrak{DC}(L^1(G)),$$

and a classical theorem due to Wendel says that the arrow

$$\begin{cases} M(G) \to \mathfrak{DC}(L^1(G)) \\ \mu \longrightarrow (L_\mu, R_\mu) \end{cases}$$

is an isometric isomorphism.

Assume henceforth that A is a C*-algebra.

5.18 LEMMA Let $(L,R) \in \mathfrak{DC}(A)$ -- then

$$||L|| = ||R||.$$

PROOF Since

$$||AL(B)|| = ||R(A)B|| \leq ||R|| \ ||A|| \ ||B||,$$

we have

$$||L(B)|| = \sup_{||A|| \leq 1} ||AL(B)|| \leq ||R|| \ ||B||$$

$$\Rightarrow$$

$$||L|| \leq ||R||.$$

Ditto:

$$||R|| \leq ||L||.$$

[Note: $\forall\ X \in A$,

$$||X|| = \sup_{||Y||\leq 1} ||XY|| = \sup_{||Y||\leq 1} ||YX||.]$$

Define an involution

$$*:\mathfrak{DC}(A) \to \mathfrak{DC}(A)$$

by

$$(L,R)^* = (R^*,L^*),$$

where $T^*(A) = T(A^*)^*$.

5.19 THEOREM Under the multiplication, norm, and involution defined above, $\mathfrak{DC}(A)$ is a unital C^*-algebra.

PROOF To check that

$$||(L,R)^*(L,R)|| = ||(L,R)||^2,$$

note that $\forall\ A \in A$ of norm ≤ 1,

$$\begin{aligned} ||L(A)||^2 &= ||(L(A))^*L(A)|| \\ &= ||L^*(A^*)L(A)|| \\ &= ||A^*R^*(L(A))|| \\ &\leq ||A^*||\ ||R^*(L(A))|| \\ &\leq ||(R^*L)(A)|| \end{aligned}$$

$$\leq ||R^*L||$$

$$= ||(L,R)^*(L,R)||$$

=>

$$||(L,R)||^2 = ||L||^2$$

$$= \sup_{||A||\leq 1} ||L(A)||^2$$

$$\leq ||(L,R)^*(L,R)||$$

$$\leq ||(L,R)||^2.$$

It is clear that $\forall\ A \in \mathcal{A}$,

$$\begin{cases} ||(L_A,R_A)|| = ||A|| \\ (L_A,R_A)^* = (L_{A^*},R_{A^*}). \end{cases}$$

Therefore

$$\iota_{\mathcal{A}}: \begin{cases} \mathcal{A} \to \mathfrak{D}\mathfrak{C}(\mathcal{A}) \\ A \to (L_A,R_A) \end{cases}$$

is an isometric $*$-homomorphism.

5.20 <u>LEMMA</u> The ideal $\iota_{\mathcal{A}}(\mathcal{A})$ is essential in $\mathfrak{D}\mathfrak{C}(\mathcal{A})$.

PROOF If $\forall\, A \in \mathcal{A}$,

$$(L,R)\iota_{\mathcal{A}}(A) = 0 = \iota_{\mathcal{A}}(A)(L,R),$$

then

$$(L_{L(A)},R_{L(A)}) = 0 = (L_{R(A)},R_{R(A)}),$$

so

$$L(A) = 0 = R(A)$$

$$\Rightarrow$$

$$(L,R) = (0,0).$$

[Note: The quotient

$$\mathcal{C}(\mathcal{A}) = \mathfrak{DC}(\mathcal{A})/\iota_{\mathcal{A}}(\mathcal{A})$$

is called the corona algebra of $\mathcal{A}$.]

The pair $(\mathfrak{DC}(\mathcal{A}),\iota_{\mathcal{A}})$ is thus a unitization of $\mathcal{A}$, which we claim is maximal. To see this, consider an embedding $j:\mathcal{A} \to \mathcal{V}$ as an essential ideal of a C*-algebra $\mathcal{V}$ -- then the problem is to construct a *-homomorphism $\zeta:\mathcal{V} \to \mathfrak{DC}(\mathcal{A})$ such that $\zeta \circ j = \iota_{\mathcal{A}}$:

$$\begin{array}{ccc} \mathcal{A} & = & \mathcal{A} \\ \downarrow & & \downarrow \iota_{\mathcal{A}} \\ \mathcal{V} & \xrightarrow[\zeta]{} & \mathfrak{DC}(\mathcal{A}). \end{array}$$

Definition:

$$\zeta(v) = (L_v,R_v),$$

where

$$\begin{cases} L_v(A) = j^{-1}(vj(A)) \\ R_v(A) = j^{-1}(j(A)v). \end{cases}$$

The computation

$$\begin{aligned} L_v(AB) &= j^{-1}(vj(AB)) \\ &= j^{-1}(vj(A)j(B)) \\ &= j^{-1}(vj(A))B \\ &= L_v(A)B \end{aligned}$$

shows that L_v is a left centralizer of A. Analogously, R_v is a right centralizer of A. And

$$\left[\begin{array}{l} AL_v(B) = Aj^{-1}(vj(B)) = j^{-1}(j(A))j^{-1}(vj(B)) = j^{-1}(j(A)vj(B)) \\ R_v(A)B = j^{-1}(j(A)v)B = j^{-1}(j(A)v)j^{-1}(j(B)) = j^{-1}(j(A)vj(B)). \end{array}\right.$$

Therefore the pair (L_v,R_v) is a double centralizer of A. That ζ is a $*$-homomorphism is likewise immediate. Finally,

$$\zeta(j(A)) = (L_{j(A)},R_{j(A)}).$$

But

$$\left[\begin{array}{l} L_{j(A)}(B) = j^{-1}(j(A)j(B)) = j^{-1}(j(AB)) = AB = L_A(B) \\ R_{j(A)}(B) = j^{-1}(j(B)j(A)) = j^{-1}(j(BA)) = BA = R_A(B) \end{array}\right.$$

$$\Rightarrow$$

$$(L_{j(A)},R_{j(A)}) = (L_A,R_A) = \iota_A(A).$$

I.e.:

$$\zeta \circ j = \iota_A.$$

[Note: The construction of ζ uses only the fact that $j(A)$ is a closed ideal in V.]

5.21 <u>THEOREM</u> The C*-algebras $M(A)$ and $\mathbf{DC}(A)$ are *-isomorphic. Moreover, there is a commutative diagram

$$\begin{array}{ccc} A & = & A \\ L \downarrow & & \downarrow \iota_A \\ M(A) & \xrightarrow[\approx]{} & \mathbf{DC}(A). \end{array}$$

[This is because maximal unitizations are unique up to *-isomorphism.]

[Note: One can therefore realize the corona algebra of A as the quotient

$$M(A)/\mathrm{L}(A).]$$

5.22 <u>REMARK</u> Let E be a Hilbert A-module -- then according to 5.8,

$$M(\underline{\mathrm{L}}_\infty(\mathrm{E})) \approx \mathrm{Hom}_A^*(\mathrm{E}),$$

so by 5.21,

$$\mathrm{Hom}_A^*(\mathrm{E}) \approx \mathbf{DC}(\underline{\mathrm{L}}_\infty(\mathrm{E})).$$

This can be explicated, viz. define

$$\Phi:\mathrm{Hom}_A^*(\mathrm{E}) \to \mathbf{DC}(\underline{\mathrm{L}}_\infty(\mathrm{E}))$$

by assigning to $\mathrm{T} \in \mathrm{Hom}_A^*(\mathrm{E})$ the pair $(\mathrm{L}_\mathrm{T},\mathrm{R}_\mathrm{T})$, where

$$\begin{cases} \mathrm{L}_\mathrm{T}(\phi) = \mathrm{T} \circ \phi \\ \mathrm{R}_\mathrm{T}(\phi) = \phi \circ \mathrm{T} \end{cases} \qquad (\phi \in \underline{\mathrm{L}}_\infty(\mathrm{E})).$$

Then Φ is a *-isomorphism.

[Note: $\forall$ x,y,z $\in$ E,

$$\begin{cases} T \circ \Theta_{x,y}(z) = Tx\langle y,z\rangle = \Theta_{Tx,y}(z) \\ \Theta_{x,y} \circ T(z) = x\langle T^*y,z\rangle = \Theta_{x,T^*y}(z).] \end{cases}$$

Let A,B be C*-algebras -- then an extension of A by B is a C*-algebra E and a short exact sequence

$$0 \to A \overset{i}{\to} E \overset{\pi}{\to} B \to 0.$$

So: ι is injective, π is surjective, and Im ι = Ker π.

N.B. There is a commutative diagram

$$\begin{array}{ccc} A & = & A \\ \iota \downarrow & & \downarrow \iota_A \\ E & \xrightarrow[\sigma]{} & \mathbf{DC}(A) \end{array}$$

but σ need not be injective (since the closed ideal $\iota(A)$ need not be essential).

5.23 EXAMPLE The unitization extension is

$$0 \to A \overset{\iota}{\to} A^+ \overset{\pi}{\to} \underline{C} \to 0,$$

where $\iota(A) = (A,0)$ and $\pi(A,\lambda) = \lambda$.

Two extensions

$$\begin{cases} 0 \to A \xrightarrow{\iota_1} E_1 \xrightarrow{\pi_1} B \to 0 \\ 0 \to A \xrightarrow[\iota_2]{} E_2 \xrightarrow[\pi_2]{} B \to 0 \end{cases}$$

of A by B are said to be isomorphic if $\exists$ a $*$-isomorphism $\gamma: E_1 \to E_2$ rendering the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \xrightarrow{\iota_1} & E_1 & \xrightarrow{\pi_1} & B & \longrightarrow & 0 \\ & & \| & & \gamma\downarrow & & \| & & \\ 0 & \longrightarrow & A & \xrightarrow[\iota_2]{} & E_2 & \xrightarrow[\pi_2]{} & B & \longrightarrow & 0 \end{array}$$

commutative.

[Note: This notion of "isomorphic" is an equivalence relation and we write $\mathrm{Ext}(A,B)$ for the corresponding set of equivalence classes.]

Suppose that

$$0 \to A \xrightarrow{\iota} E \xrightarrow{\pi} B \to 0$$

is an extension of A by B. Postcompose $\sigma: E \to \mathbf{DC}(A)$ with the projection pr: $\mathbf{DC}(A) \to C(A)$ to get a $*$-homomorphism τ from $E/\iota(A) \approx B$ to $C(A)$, the so-called Busby invariant of the extension.

N.B. The diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \xrightarrow{\iota} & E & \xrightarrow{\pi} & B & \longrightarrow & 0 \\ & & \| & & \sigma\downarrow & & \tau\downarrow & & \\ 0 & \longrightarrow & A & \xrightarrow[\iota_A]{} & \mathbf{DC}(A) & \longrightarrow & C(A) & \longrightarrow & 0 \end{array}$$

is commutative.

5.24 LEMMA There is a pullback square

$$\begin{array}{ccc} \mathbf{DC}(A) \times_{C(A)} B & \longrightarrow & B \\ \downarrow & & \downarrow \tau \\ \mathbf{DC}(A) & \xrightarrow[\mathrm{pr}]{} & C(A), \end{array}$$

a $*$-isomorphism $\zeta:E \to \mathfrak{D}\mathcal{C}(A) \times_{C(A)} B$, and a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \xrightarrow{\iota} & E & \xrightarrow{\pi} & B & \longrightarrow & 0 \\ & & \| & & \zeta\downarrow & & \| & & \\ 0 & \longrightarrow & A & \to & \mathfrak{D}\mathcal{C}(A) \times_{C(A)} B & \to & B & \longrightarrow & 0. \end{array}$$

Two extensions

$$\begin{cases} 0 \to A \xrightarrow{\iota_1} E_1 \xrightarrow{\pi_1} B \to 0 \\ \\ 0 \to A \xrightarrow[\iota_2]{} E_2 \xrightarrow[\pi_2]{} B \to 0 \end{cases}$$

of A by B with respective Busby invariants τ_1 and τ_2 are isomorphic iff $\tau_1 = \tau_2$. Therefore the Busby invariant determines the isomorphism class of an extension, thus there is an injection

$$\text{Ext}(A,B) \to \text{Mor}(B,C(A)),$$

that, in fact, is a bijection. Proof: Let $\tau \in \text{Mor}(B,C(A))$ -- then the Busby invariant of the extension

$$0 \to A \to \mathfrak{D}\mathcal{C}(A) \times_{C(A)} B \to B \to 0$$

is τ itself.

5.25 <u>EXAMPLE</u> Take $A = C_\infty(]0,1[)$, $B = \underline{C}$ -- then up to isomorphism there are four extensions of A by B:

1. $E = C_\infty([0,1[)$
2. $E = C_\infty(]0,1])$
3. $E = C([0,1])$
4. $E = C_\infty(]0,1[) \oplus \underline{C}$.

5.26 <u>LEMMA</u> Let $\tau : \mathcal{B} \to C(A)$ be the Busby invariant of the extension

$$0 \to A \xrightarrow{\iota} \mathcal{E} \xrightarrow{\pi} \mathcal{B} \to 0.$$

Then $\tau = 0$ iff $\mathcal{E}$ is $*$-isomorphic to $A \oplus \mathcal{B}$.

5.27 <u>REMARK</u> If A is unital, then $C(A)$ is trivial and, up to isomorphism, there is only one extension of A by $\mathcal{B}$, viz.

$$0 \to A \to A \oplus \mathcal{B} \to \mathcal{B} \to 0.$$

§6. *TENSOR PRODUCTS*

A monoidal category is a category $\underline{C}$ equipped with a functor $\otimes:\underline{C} \times \underline{C} \to \underline{C}$ (the multiplication) and an object $e \in \mathrm{Ob}\,\underline{C}$ (the unit), together with natural isomorphisms R, L, and A, where

$$\begin{cases} R_X : X \otimes e \to X \\ L_X : e \otimes X \to X \end{cases}$$

and

$$A_{X,Y,Z} : X \otimes (Y \otimes Z) \to (X \otimes Y) \otimes Z,$$

subject to the following assumptions.

(MC_1) The diagram

$$\begin{array}{ccccc} X \otimes (Y \otimes (Z \otimes W)) & \xrightarrow{A} & (X \otimes Y) \otimes (Z \otimes W) & \xrightarrow{A} & ((X \otimes Y) \otimes Z) \otimes W \\ {\scriptstyle id \otimes A}\downarrow & & & & \uparrow{\scriptstyle A \otimes id} \\ X \otimes ((Y \otimes Z) \otimes W) & & \xrightarrow[A]{\qquad\qquad} & & (X \otimes (Y \otimes Z)) \otimes W \end{array}$$

commutes.

(MC_2) The diagram

$$\begin{array}{ccc} X \otimes (e \otimes Y) & \xrightarrow{A} & (X \otimes e) \otimes Y \\ {\scriptstyle id \otimes L}\downarrow & & \downarrow{\scriptstyle R \otimes id} \\ X \otimes Y & = & X \otimes Y \end{array}$$

commutes.

[Note: The "coherency" principle then asserts that "all" diagrams built up from instances of R, L, A (or their inverses), and id by repeated application of $\otimes$

necessarily commute. In particular, the diagrams

$$\begin{array}{ccc} e \otimes (X \otimes Y) & \xrightarrow{A} & (e \otimes X) \otimes Y \\ L \downarrow & & \downarrow L \otimes \mathrm{id} \\ X \otimes Y & = & X \otimes Y \end{array} \qquad \begin{array}{ccc} X \otimes (Y \otimes e) & \xrightarrow{A} & (X \otimes Y) \otimes e \\ \mathrm{id} \otimes R \downarrow & & \downarrow R \\ X \otimes Y & = & X \otimes Y \end{array}$$

commute and $L_e = R_e : e \otimes e \to e$.]

<u>N.B.</u> Technically, the categories

$$\begin{cases} \underline{C} \times (\underline{C} \times \underline{C}) \\ (\underline{C} \times \underline{C}) \times \underline{C} \end{cases}$$

are not the same so it doesn't quite make sense to say that the functors

- $\underline{C} \times (\underline{C} \times \underline{C}) \to \underline{C} \begin{cases} (X,(Y,Z)) \to X \otimes (Y \otimes Z) \\ (f,(g,h)) \to f \otimes (g \otimes h) \end{cases}$

- $(\underline{C} \times \underline{C}) \times \underline{C} \to \underline{C} \begin{cases} ((X,Y),Z) \to (X \otimes Y) \otimes Z \\ ((f,g),h) \to (f \otimes g) \otimes h \end{cases}$

are naturally isomorphic. However, there is an obvious isomorphism

$$\underline{C} \times (\underline{C} \times \underline{C}) \xrightarrow{\iota} (\underline{C} \times \underline{C}) \times \underline{C}$$

and the assumption is that $A:F \to G \circ \iota$ is a natural isomorphism, where

$$\begin{array}{ccc} \underline{C} \times (\underline{C} \times \underline{C}) & \xrightarrow{F} & \underline{C} \\ \iota \downarrow & & \\ (\underline{C} \times \underline{C}) \times \underline{C} & \xrightarrow[G]{} & \underline{C}. \end{array}$$

Accordingly,

$$\forall\ (X,(Y,Z)) \in \text{Ob}\ \underline{C} \times (\underline{C} \times \underline{C})$$

and

$$\forall\ (f,(g,h)) \in \text{Mor}\ \underline{C} \times (\underline{C} \times \underline{C}),$$

the square

$$\begin{array}{ccc} X \otimes (Y \otimes Z) & \xrightarrow{A_{X,Y,Z}} & (X \otimes Y) \otimes Z \\ {\scriptstyle f \otimes (g \otimes h)} \downarrow & & \downarrow {\scriptstyle (f \otimes g) \otimes h} \\ X' \otimes (Y' \otimes Z') & \xrightarrow[A_{X',Y',Z'}]{} & (X' \otimes Y') \otimes Z' \end{array}$$

commutes.

6.1 <u>EXAMPLE</u> Let <u>VEC</u> be the category whose objects are the vector spaces over $\underline{C}$ and whose morphisms are the linear transformations -- then <u>VEC</u> is monoidal: Take $X \otimes Y$ to be the algebraic tensor product and let e be $\underline{C}$.

[Note: If

$$\begin{cases} f:X \to X' \\ g:Y \to Y', \end{cases}$$

then

$$\otimes\ (f,g) = f \otimes g:X \otimes Y \to X' \otimes Y'$$

sends $x \otimes y$ to $f(x) \otimes g(y)$.]

6.2 <u>EXAMPLE</u> Let <u>ALG</u> be the category whose objects are the algebras over $\underline{C}$

and whose morphisms are the multiplicative linear transformations -- then ALG is monoidal: Take $A \otimes B$ to be the algebraic tensor product and let e be $\underline{C}$.

[Note: If

$$A, B \in \text{Ob } \underline{\text{ALG}},$$

then the multiplication in $A \otimes B$ on elementary tensors is given by

$$(A_1 \otimes B_1)(A_2 \otimes B_2) = A_1A_2 \otimes B_1B_2.]$$

6.3 EXAMPLE Let *ALG be the category whose objects are the $*$-algebras over $\underline{C}$ and whose morphisms are the multiplicative $*$-linear transformations -- then *ALG is monoidal: Take $A \otimes B$ to be the algebraic tensor product and let e be $\underline{C}$.

[Note: To say that $\Phi: A \to B$ is $*$-linear means that

$$\Phi(A^*) = \Phi(A)^*$$

for all $A \in A$.]

6.4 REMARK Each of these three categories also admits another monoidal structure: Take for the multiplication the direct sum $\oplus$ and take for the unit the zero object $\{0\}$.

Let H and K be complex Hilbert spaces -- then their algebraic tensor product $H \otimes K$ can be equipped with an inner product given on elementary tensors by

$$\langle x_1 \otimes y_1, x_2 \otimes y_2 \rangle = \langle x_1, x_2 \rangle \langle y_1, y_2 \rangle$$

and its completion $H \underline{\otimes} K$ is a complex Hilbert space.

N.B. If

$$\left[\begin{array}{l} A \in \mathcal{B}(\mathcal{H}_1,\mathcal{H}_2) \\ B \in \mathcal{B}(\mathcal{K}_1,\mathcal{K}_2), \end{array}\right.$$

then

$$A \otimes B:\mathcal{H}_1 \otimes \mathcal{K}_1 \to \mathcal{H}_2 \otimes \mathcal{K}_2$$

extends by continuity to a bounded linear operator

$$A \underline{\otimes} B:\mathcal{H}_1 \underline{\otimes} \mathcal{K}_1 \to \mathcal{H}_2 \underline{\otimes} \mathcal{K}_2.$$

Recall now that HILB is the category whose objects are the complex Hilbert spaces and whose morphisms are the bounded linear operators (cf. 4.28).

6.5 LEMMA HILB is a monoidal category.

PROOF Define a functor

$$\otimes:\underline{\text{HILB}} \times \underline{\text{HILB}} \to \underline{\text{HILB}}$$

by

$$\otimes(\mathcal{H},\mathcal{K}) = \mathcal{H} \underline{\otimes} \mathcal{K}$$

and

$$\otimes(\mathcal{H}_1 \xrightarrow{A} \mathcal{H}_2,\mathcal{K}_1 \xrightarrow{B} \mathcal{K}_2) = A \underline{\otimes} B$$

and let e be $\underline{C}$.

A symmetry for a monoidal category $\underline{C}$ is a natural isomorphism τ, where

$$\tau_{X,Y}:X \otimes Y \to Y \otimes X,$$

such that

$$\tau_{Y,X} \circ \tau_{X,Y}:X \otimes Y \to X \otimes Y$$

is the identity, $R_X = L_X \circ \tau_{X,e}$, and the diagram

$$\begin{array}{ccccc}
X \otimes (Y \otimes Z) & \xrightarrow{A} & (X \otimes Y) \otimes Z & \xrightarrow{\tau} & Z \otimes (X \otimes Y) \\
{\scriptstyle \mathrm{id} \otimes \tau}\downarrow & & & & \downarrow{\scriptstyle A} \\
X \otimes (Z \otimes Y) & \xrightarrow[A]{} & (X \otimes Z) \otimes Y & \xrightarrow[\tau \otimes \mathrm{id}]{} & (Z \otimes X) \otimes Y
\end{array}$$

commutes. A <u>symmetric monoidal category</u> is a monoidal category $\underline{C}$ endowed with a symmetry τ. A monoidal category can have more than one symmetry (or none at all).

[Note: The "coherency" principle then asserts that "all" diagrams built up from instances of R, L, A, τ (or their inverses), and id by repeated application of $\otimes$ necessarily commute.]

<u>N.B.</u> Let

$$f:\underline{C} \times \underline{C} \to \underline{C} \times \underline{C}$$

be the interchange -- then f is an isomorphism and $\tau:\otimes \to \otimes \circ f$ is a natural isomorphism.

It is clear that <u>VEC</u>, <u>ALG</u>, and <u>*ALG</u> are symmetric monoidal, as is <u>HILB</u>.

6.6 <u>LEMMA</u> Let H and K be complex Hilbert spaces -- then the linear map

$$\underline{\beta}:\mathcal{B}(H) \otimes \mathcal{B}(K) \to \mathcal{B}(H \underline{\otimes} K)$$

induced by the bilinear map

$$\beta: \begin{cases} \mathcal{B}(H) \times \mathcal{B}(K) \to \mathcal{B}(H \underline{\otimes} K) \\ \\ (T,S) \to T \underline{\otimes} S \end{cases}$$

is an injective $*$-homomorphism.

From the definitions, <u>C*ALG</u> is a full subcategory of <u>*ALG</u> and while <u>*ALG</u> is symmetric monoidal, it is definitely not automatic that the same is true of <u>C*ALG</u> (the algebraic tensor product of two C*-algebras is not, in general, a C*-algebra).

Suppose that $\mathcal{A}$ and $\mathcal{B}$ are C*-algebras -- then a <u>C*-norm</u> on their algebraic tensor product $\mathcal{A} \otimes \mathcal{B}$ is a norm $||\cdot||_\alpha$ which is submultiplicative, i.e.,

$$||XY||_\alpha \leq ||X||_\alpha \ ||Y||_\alpha,$$

and satisfies the C*-condition, i.e.,

$$||X^*X||_\alpha = ||X||^2.$$

[Note: The pair $(\mathcal{A} \otimes \mathcal{B}, ||\cdot||_\alpha)$ is a pre-C*-algebra and its completion $\mathcal{A} \otimes_\alpha \mathcal{B}$ is a C*-algebra.]

Definition: A norm $||\cdot||$ on $\mathcal{A} \otimes \mathcal{B}$ is said to be a <u>cross norm</u> if $\forall$ A $\in \mathcal{A}$, $\forall$ B $\in \mathcal{B}$,

$$||A \otimes B|| = ||A|| \ ||B||.$$

6.7 <u>LEMMA</u> Every C*-norm on $\mathcal{A} \otimes \mathcal{B}$ is a cross norm.

6.8 <u>EXAMPLE</u> Given X $\in \mathcal{A} \otimes \mathcal{B}$, let

$$||X||^{\wedge} = \inf\{\Sigma ||A_i|| \ ||B_i|| : X = \Sigma A_i \otimes B_i\}.$$

Then $||\cdot||^{\wedge}$ is a submultiplicative cross norm on $\mathcal{A} \otimes \mathcal{B}$ and the completion $\mathcal{A} \hat{\otimes} \mathcal{B}$ is a Banach $*$-algebra. Still, $||\cdot||^{\wedge}$ is rarely a C*-norm.

6.9 <u>RAPPEL</u> Every C*-algebra is isometrically $*$-isomorphic to a norm closed

$*$-subalgebra of $\mathcal{B}(H)$ for some H, or in different but equivalent terminology, every C*-algebra admits a faithful $*$-representation on some complex Hilbert space (cf. 10.37).

6.10 LEMMA Suppose that

$$\begin{cases} \Phi:\mathcal{A} \to \mathcal{C} \\ \Psi:\mathcal{B} \to \mathcal{D} \end{cases}$$

are $*$-homomorphisms of C*-algebras -- then there is a unique $*$-homomorphism

$$\Phi \otimes \Psi:\mathcal{A} \otimes \mathcal{B} \to \mathcal{C} \otimes \mathcal{D}$$

of algebraic tensor products such that

$$(\Phi \otimes \Psi)(A \otimes B) = \Phi(A) \otimes \Psi(B)$$

for all $A \in \mathcal{A}$, $B \in \mathcal{B}$. And

$$\begin{cases} \Phi \text{ injective} \\ \Psi \text{ injective} \end{cases} \Rightarrow \Phi \otimes \Psi \text{ injective.}$$

Given C*-algebras $\begin{cases} \mathcal{A} \\ \mathcal{B} \end{cases}$, let

$$\begin{cases} \Phi:\mathcal{A} \to \mathcal{B}(H) \\ \Psi:\mathcal{B} \to \mathcal{B}(K) \end{cases}$$

be faithful $*$-representations -- then the composition

$$A \otimes B \xrightarrow{\Phi \otimes \Psi} B(H) \otimes B(K) \xrightarrow{\underline{\beta}} B(H \underline{\otimes} K)$$

is an injective $*$-homomorphism. One can therefore place a C*-norm on $A \otimes B$ by writing

$$||X||_{min} = ||(\underline{\beta} \circ \Phi \otimes \Psi)(X)|| \qquad (X \in A \otimes B).$$

6.11 <u>LEMMA</u> $||\cdot||_{min}$ is independent of the choice of Φ and Ψ.

[Note: If in the above Φ and Ψ are arbitrary $*$-representations, then

$$||(\underline{\beta} \circ \Phi \otimes \Psi)(X)|| \leq ||X||_{min}.]$$

One terms $||\cdot||_{min}$ the <u>minimal</u> C*-norm on $A \otimes B$. Denote its completion by $A \otimes_{min} B$ and call $A \otimes_{min} B$ the <u>minimal tensor product</u> of A and B.

6.12 <u>EXAMPLE</u> Fix a C*-algebra A. Given $X \in M_n(\underline{C}) \otimes A$, write

$$X = \sum_{i,j} E_{ij} \otimes A_{ij}.$$

Then the A_{ij} are unique and the map

$$X \to [A_{ij}]$$

defines a $*$-isomorphism

$$M_n(\underline{C}) \otimes A \to M_n(A).$$

But $M_n(A)$ is a C*-algebra (cf. 4.41), hence $M_n(\underline{C}) \otimes A$ is a C*-algebra w.r.t. the norm that it gets from $M_n(A)$. Owing to 1.2, this norm must then be $||\cdot||_{min}$, so

$$M_n(\underline{C}) \otimes A = M_n(\underline{C}) \otimes_{\min} A.$$

[Note: One can show directly that $M_n(\underline{C}) \otimes A$ is complete per $||\cdot||_{\min}$. For if $\{X_k\}$ is Cauchy and if

$$X_k = \sum_{i,j} E_{ij} \otimes A_{ij}^k,$$

then for each pair (i,j), $\{A_{ij}^k\}$ is Cauchy in A, thus

$$\lim_{k \to \infty} A_{ij}^k = A_{ij}^\infty, \text{ say.}$$

Now put

$$X_\infty = \sum_{i,j} E_{ij} \otimes A_{ij}^\infty$$

and observe that

$$||X_\infty - X_k||_{\min} = ||\sum_{i,j} E_{ij} \otimes (A_{ij}^\infty - A_{ij}^k)||_{\min}$$

$$\leq \sum_{i,j} ||A_{ij}^\infty - A_{ij}^k||$$

$$\to 0 \quad (k \to \infty).$$

Consequently matters can be turned around: The *-isomorphism

$$M_n(\underline{C}) \otimes A \to M_n(A)$$

can be used to place the structure of a C*-algebra on $M_n(A)$.]

6.13 <u>EXAMPLE</u> Suppose that X and Y are compact Hausdorff spaces -- then

$$C(X) \otimes_{\min} C(Y) \approx C(X \times Y).$$

[Note: If instead, X and Y are noncompact locally compact Hausdorff spaces, then

$$C_\infty(X) \otimes_{\min} C_\infty(Y) \approx C_\infty(X \times Y).]$$

6.14 EXAMPLE Fix a C*-algebra A and suppose that X is a compact Hausdorff space -- then

$$C(X,A) \approx C(X) \otimes_{\min} A.$$

[Note: If instead, X is a noncompact locally compact Hausdorff space, then

$$C_\infty(X,A) \approx C_\infty(X) \otimes_{\min} A.]$$

6.15 LEMMA If A and B are simple, then $A \otimes_{\min} B$ is simple.

6.16 EXAMPLE Suppose that H and K are complex Hilbert spaces -- then

$$\underline{L}_\infty(H) \otimes_{\min} \underline{L}_\infty(K)$$

is simple and

$$\underline{L}_\infty(H) \otimes_{\min} \underline{L}_\infty(K) \approx \underline{L}_\infty(H \underline{\otimes} K).$$

6.17 LEMMA Suppose that

$$\begin{cases} \Phi: A \rightarrow C \\ \Psi: B \rightarrow D \end{cases}$$

are *-homomorphisms of C*-algebras -- then

$$\Phi * \Psi: A \otimes B \rightarrow C \otimes D$$

extends by continuity to a $*$-homomorphism

$$\Phi \otimes_{\min} \Psi : A \otimes_{\min} B \to C \otimes_{\min} D.$$

6.18 <u>REMARK</u> Here

$$\left.\begin{array}{l} \Phi \text{ injective} \\ \\ \Psi \text{ injective} \end{array}\right] \Rightarrow \Phi \otimes_{\min} \Psi \text{ injective.}$$

E.g.: If A is a C*-subalgebra of C and if B is a C*-subalgebra of D, then there is an embedding

$$A \otimes_{\min} B \to C \otimes_{\min} D.$$

[Note: This is false in general if "$\otimes_{\min}$" is replaced by "$\otimes_{\max}$" (cf. infra).]

There are canonical isomorphisms

$$\left[\begin{array}{l} R_A : A \otimes_{\min} \underline{C} \ (= A \otimes \underline{C}) \to A \\ \\ L_A : \underline{C} \otimes_{\min} A \ (= \underline{C} \otimes A) \to A, \end{array}\right.$$

$$A_{A,B,C} : A \otimes_{\min} (B \otimes_{\min} C) \to (A \otimes_{\min} B) \otimes_{\min} C,$$

and

$$T_{A,B} : A \otimes_{\min} B \to B \otimes_{\min} A,$$

which are evidently natural.

6.19 SCHOLIUM Equipped with the minimal tensor product, C*ALG is a symmetric monoidal category.

[Define a functor

$$\otimes : \underline{C^*ALG} \to \underline{C^*ALG}$$

by

$$\otimes(A,B) = A \otimes_{min} B$$

and

$$\otimes(A \xrightarrow{\Phi} C, B \xrightarrow{\Psi} D) = \Phi \otimes_{min} \Psi$$

and let e be $\underline{C}$.]

6.20 THEOREM Let $||\cdot||_\alpha$ be a C*-norm on $A \otimes B$ -- then $\forall\ X \in A \otimes B$,

$$||X||_{min} \leq ||X||_\alpha.$$

[Note: This result is the origin of the term "minimal tensor product".]

6.21 LEMMA If A is nonunital, then any C*-norm $||\cdot||_\alpha$ on $A \otimes B$ can be extended to a C*-norm on $A^+ \otimes B$.

[Note: Therefore if both A and B are nonunital, then any C*-norm $||\cdot||_\alpha$ on $A \otimes B$ can be extended to a C*-norm on $A^+ \otimes B^+$.]

6.22 LEMMA If $A \otimes_\alpha B$ is simple for some C*-norm $||\cdot||_\alpha$ on $A \otimes B$, then $||\cdot||_\alpha = ||\cdot||_{min}$ and A and B are simple (cf. 6.15).

Given C*-algebras A and B, define the maximal C*-norm on $A \otimes B$ by

$$||X||_{max} = \sup_{\pi} \{||\pi(X)||\},$$

sup being taken over all $*$-representations of $A \underset{\pi}{\otimes} B$. Let $A \otimes_{max} B$ be the completion of $A \otimes B$ w.r.t. $||\cdot||_{max}$ -- then $A \otimes_{max} B$ is the <u>maximal tensor product</u> of A and B and

$$A \otimes_{max} B \simeq C^*(A \hat{\otimes} B),$$

where $C^*(A \hat{\otimes} B)$ is the enveloping C^*-algebra of $A \hat{\otimes} B$ (cf. §9), hence there is an arrow

$$A \hat{\otimes} B \to A \otimes_{max} B.$$

6.23 <u>LEMMA</u> If $\Phi: A \otimes B \to C$ is a $*$-homomorphism, then there is a unique $*$-homomorphism $\Phi_{max}: A \otimes_{max} B \to C$ which extends Φ.

6.24 <u>THEOREM</u> Let $||\cdot||_\alpha$ be a C^*-norm on $A \otimes B$ -- then $\forall\ X \in A \otimes B$,

$$||X||_\alpha \leq ||X||_{max}.$$

<u>PROOF</u> Thanks to 6.23, there is a surjective $*$-homomorphism

$$A \otimes_{max} B \to A \otimes_\alpha B,$$

so

$$||X||_\alpha \leq ||X||_{max}$$

for all $X \in A \otimes B$.

6.25 <u>REMARK</u> Equipped with the maximal tensor product, <u>C*ALG</u> is a symmetric monoidal category (cf. 6.19).

A C*-algebra A is <u>nuclear</u> if there is only one C*-norm on $A \otimes B$ for every C*-algebra B. So, if A is nuclear, then $||\cdot||_{min} = ||\cdot||_{max}$ on $A \otimes B$ and we write $A \underline{\otimes} B$ for

$$A \otimes_{min} B = A \otimes_{max} B.$$

6.26 EXAMPLE $\forall\ n \geq 1$, the C*-algebra $M_n(\underline{C})$ is nuclear (cf. 6.12).

[Note: More generally, every finite dimensional C*-algebra is nuclear (use 1.4).]

6.27 EXAMPLE If H is an infinite dimensional complex Hilbert space, then $B(H)$ is not nuclear.

[There are a number of ways to see this, none of them obvious. One method is to show that

$$B(H) \otimes_{min} B(H) \neq B(H) \otimes_{max} B(H).]$$

6.28 THEOREM Every commutative C*-algebra is nuclear.

6.29 THEOREM A filtered colimit of nuclear C*-algebras is nuclear.

6.30 EXAMPLE Every AF-algebra is nuclear (cf. 3.17).

6.31 EXAMPLE Suppose that H is an infinite dimensional complex Hilbert space -- then $\underline{L}_\infty(H)$ is nuclear.

Note: Recall that

$$M(\underline{L}_\infty(H)) \approx B(H) \quad \text{(cf. 5.9)}.$$

Since $\mathcal{B}(H)$ is not nuclear, it follows that the multiplier algebra of a nuclear C*-algebra need not be nuclear.]

6.32 LEMMA The minimal tensor product $A \otimes_{min} B$ is nuclear iff both A and B are nuclear.

PROOF If B is not nuclear and if C is a C*-algebra for which $||\cdot||_{max} \neq ||\cdot||_{min}$ on $B \otimes C$, then the surjective *-homomorphism

$$B \otimes_{max} C \to B \otimes_{min} C$$

has a nontrivial kernel, thus the same is true of the composition

$$\begin{aligned}(A \otimes_{min} B) \otimes_{max} C &\to A \otimes_{min} (B \otimes_{max} C)\\ &\to A \otimes_{min} (B \otimes_{min} C)\\ &\simeq (A \otimes_{min} B) \otimes_{min} C.\end{aligned}$$

Therefore $A \otimes_{min} B$ is not nuclear. Conversely, if A and B are nuclear, then for any C, we have

$$\begin{aligned}(A \otimes_{min} B) \otimes_{max} C &\simeq (A \otimes_{max} B) \otimes_{max} C\\ &\simeq A \otimes_{max} (B \otimes_{max} C)\\ &\simeq A \otimes_{max} (B \otimes_{min} C)\\ &\simeq A \otimes_{min} (B \otimes_{min} C)\\ &\simeq (A \otimes_{min} B) \otimes_{min} C.\end{aligned}$$

6.33 <u>EXAMPLE</u> If A is nuclear, then $\forall\ n \geq 1$, $M_n(A)$ is nuclear. In fact,

$$M_n(A) \approx M_n(\underline{C}) \underline{\otimes} A$$

$$= M_n(\underline{C}) \otimes_{\min} A \quad \text{(cf. 6.12)}.$$

6.34 <u>EXAMPLE</u> If H and K are complex Hilbert spaces, then

$$\underline{L}_\infty(H) \otimes_{\min} \underline{L}_\infty(K)$$

is nuclear and, in fact, is $*$-isomorphic to

$$\underline{L}_\infty(H \underline{\otimes} K) \quad \text{(cf. 6.16)}.$$

6.35 <u>REMARK</u> Write <u>NUCC*ALG</u> for the full subcategory of C*ALG whose objects are the nuclear C*-algebras equipped with the minimal tensor product -- then <u>NUCC*ALG</u> is a symmetric monoidal category.

A C*-algebra A is said to be <u>stable</u> if $A \approx A \otimes_{\min} \underline{L}_\infty(\ell^2)$ $(\approx \underline{L}_\infty(H_A)$ (cf. 5.12)).

6.36 <u>EXAMPLE</u> $\underline{L}_\infty(\ell^2)$ is stable:

$$\ell^2 \underline{\otimes} \ell^2 \approx \ell^2$$

=>

$$\underline{L}_\infty(\ell^2) \otimes_{\min} \underline{L}_\infty(\ell^2) \approx \underline{L}_\infty(\ell^2 \underline{\otimes} \ell^2)$$

$$\approx \underline{L}_\infty(\ell^2).$$

6.37 EXAMPLE If A is stable, then $\forall\, n \geq 1$, $M_n(A) \approx A$. Proof:

$$\begin{aligned} M_n(A) &\approx M_n(\underline{C}) \otimes_{\min} A \\ &\approx M_n(\underline{C}) \otimes_{\min} (A \otimes_{\min} \underline{L}_\infty(\ell^2)) \\ &\approx A \otimes_{\min} (M_n(\underline{C}) \otimes_{\min} \underline{L}_\infty(\ell^2)) \\ &\approx A \otimes_{\min} (\underline{L}_\infty(\underline{C}^n) \otimes_{\min} \underline{L}_\infty(\ell^2)) \\ &\approx A \otimes_{\min} \underline{L}_\infty(\underline{C}^n \underline{\otimes} \ell^2) \\ &\approx A \otimes_{\min} \underline{L}_\infty(\ell^2) \\ &\approx A. \end{aligned}$$

Two C*-algebras A and B are stably isomorphic if

$$A \otimes_{\min} \underline{L}_\infty(\ell^2) \approx B \otimes_{\min} \underline{L}_\infty(\ell^2).$$

6.38 EXAMPLE $\underline{C}$ and $\underline{L}_\infty(\ell^2)$ are stably isomorphic.

6.39 LEMMA If A is nuclear and if A and B are stably isomorphic, then B is nuclear.

PROOF For

$$\begin{aligned} A \text{ nuclear} &\Rightarrow A \otimes_{\min} \underline{L}_\infty(\ell^2) \text{ nuclear} \quad (\text{cf. } 6.32) \\ &\Rightarrow B \otimes_{\min} \underline{L}_\infty(\ell^2) \text{ nuclear} \\ &\Rightarrow B \text{ nuclear} \quad (\text{cf. } 6.32). \end{aligned}$$

It is false in general that a C*-subalgebra of a nuclear C*-algebra is nuclear. Still, there are properties of permanence.

6.40 LEMMA If A is nuclear and if $I \subset A$ is a closed ideal, then I is nuclear.

6.41 LEMMA If A is nuclear and if $I \subset A$ is a closed ideal, then A/I is nuclear.

6.42 THEOREM Suppose that $I \subset A$ is a closed ideal. Assume: I and A/I are nuclear -- then A is nuclear.

If

$$0 \to J \to B \to B/J \to 0$$

is a short exact sequence of C*-algebras and if A is a C*-algebra, then

$$0 \to A \otimes_{max} J \to A \otimes_{max} B \to A \otimes_{max} B/J \to 0$$

is again short exact. On the other hand, this need not be true if "max" is replaced by "min", leading thereby to the following definition.

A C*-algebra A is said to be exact if it has the property that $A \otimes_{min}$ -- preserves short exact sequences.

6.43 LEMMA Every nuclear C*-algebra is exact.

6.44 REMARK There are C*-algebras which are not exact and there are exact

C*-algebras which are not nuclear.

6.45 LEMMA Every C*-subalgebra of an exact C*-algebra is exact.

[Note: Thus every C*-subalgebra of a nuclear C*-algebra is exact (but not necessarily nuclear).]

The quotient of an exact C*-algebra is exact. Filtered colimits of exact C*-algebras are exact but extensions of exact C*-algebras are in general not exact.

N.B. It is a famous theorem due to Kirchberg that every separable exact C*-algebra can be embedded as a C*-subalgebra of a separable nuclear C*-algebra.

6.46 LEMMA If A and B are exact C*-algebras, then so is $A \otimes_{\min} B$.

6.47 REMARK Write EXC*ALG for the full subcategory of C*ALG whose objects are the exact C*-algebras equipped with the minimal tensor product -- then EXC*ALG is a symmetric monoidal category containing NUCC*ALG as a full subcategory.

§7. *STATES*

Let A, B be C*-algebras -- then a linear map $\Phi: A \to B$ is said to be positive if $\Phi(A_+) \subset B_+$.

7.1 LEMMA Suppose that $\Phi: A \to B$ is positive -- then $\forall\ A_1, A_2 \in A$,

$$\Phi(A_1^* A_2)^* = \Phi(A_2^* A_1).$$

[Note: Since $A = A^2$, it follows that

$$\Phi(A)^* = \Phi(A^*) \qquad (A \in A).]$$

7.2 EXAMPLE A *-homomorphism $\Phi: A \to B$ is positive:

$$\Phi(A^*A) = \Phi(A^*)\Phi(A) = \Phi(A)^*\Phi(A) \in B_+.$$

7.3 LEMMA Suppose that $\Phi: A \to B$ is positive -- then Φ is bounded.

More can be said in the unital situation.

7.4 LEMMA If A and B are unital and if $\Phi: A \to B$ is positive, then $||\Phi|| = ||\Phi(1_A)||$.

[Note: Accordingly, if Φ is in addition unital, then $||\Phi|| = 1$.]

7.5 EXAMPLE Take $A = B = M_2(\underline{C})$ and let Φ be the linear map defined by

$$\Phi\left(\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\right) = \begin{bmatrix} a_{11} & a_{22} \\ 0 & 0 \end{bmatrix}.$$

Then $||\Phi|| = ||\Phi(1_A)|| = 1$ and $\Phi(1_A) \geq 0$. Still, Φ is not positive.

7.6 LEMMA If A and B are unital and if $\Phi:A \to B$ is a unital bounded linear map such that $||\Phi|| = 1$, then Φ is positive.

Specialize now and take $B = \underline{C}$ -- then a linear functional $\omega:A \to \underline{C}$ is said to be positive if

$$A \geq 0 \Rightarrow \omega(A) \geq 0.$$

N.B. Positive linear functionals are necessarily continuous (cf. 7.3).

7.7 LEMMA Let $\omega:A \to \underline{C}$ be a positive linear functional -- then $\forall$ $A \in A$,

$$\omega(A^*) = \overline{\omega(A)}$$

and

$$|\omega(A)|^2 \leq ||\omega||\omega(A^*A).$$

7.8 LEMMA Let $\omega:A \to \underline{C}$ be a positive linear functional -- then $\forall$ $A,B \in A$,

$$|\omega(A^*B)|^2 \leq \omega(A^*A)\omega(B^*B).$$

Fix an approximate unit $\{e_i:i \in I\}$ for A per 1.20.

7.9 LEMMA Let $\omega:A \to \underline{C}$ be a positive linear functional -- then

$$||\omega|| = \lim_{i\in I} \omega(e_i).$$

In particular: If A is unital and if $\omega:A \to \underline{C}$ is positive, then $||\omega|| = \omega(1_A)$ (cf. 7.4).

[Note: This can be turned around. In other words, if $\omega:A \to \underline{C}$ is a bounded linear functional such that $||\omega|| = \omega(1_A)$, then ω is positive (cf. 7.6) $(\frac{\omega}{||\omega||}(1_A) = 1)$.]

If

$$\begin{bmatrix} \omega_1:A \to \underline{C} \\ \omega_2:A \to \underline{C} \end{bmatrix}$$

are positive linear functionals, then their sum $\omega_1 + \omega_2$ is a positive linear functional. And:

$$||\omega_1 + \omega_2|| = ||\omega_1|| + ||\omega_2||.$$

Proof:

$$\begin{aligned} ||\omega_1 + \omega_2|| &= \lim_{i\in I} (\omega_1(e_i) + \omega_2(e_i)) \\ &= \lim_{i\in I} \omega_1(e_i) + \lim_{i\in I} \omega_2(e_i) \\ &= ||\omega_1|| + ||\omega_2||. \end{aligned}$$

Suppose that A is nonunital. Given a positive linear functional $\omega:A \to \underline{C}$, define a linear functional ω^+ on A^+ by

$$\omega^+(A,\lambda) = \omega(A) + \lambda||\omega||.$$

Then ω^+ is positive. In fact,

$$\omega^+((A,\lambda)^*(A,\lambda))$$

$$= \omega^+(A^*A + \bar{\lambda}A + \lambda A^*, \bar{\lambda}\lambda)$$

$$= |\lambda|^2 \; ||\omega|| + \bar{\lambda}\omega(A) + \lambda\omega(A^*) + \omega(A^*A).$$

But

$$\bar{\lambda}\omega(A) + \lambda\omega(A^*) + 2|\lambda| \; ||\omega||^{1/2} \; \omega(A^*A)^{1/2}$$

$$\geq \bar{\lambda}\omega(A) + \lambda\omega(A^*) + 2|\lambda| \; |\omega(A)| \quad \text{(cf. 7.7)}$$

$$\geq 0.$$

Therefore

$$\omega^+((A,\lambda)^*(A,\lambda)) \geq (|\lambda| \; ||\omega||^{1/2} - \omega(A^*A)^{1/2})^2 \geq 0.$$

N.B. We have

$$||\omega^+|| = \omega^+(1_{A^+})$$

$$= \omega^+(0,1) = ||\omega||.$$

7.10 LEMMA Let $\omega: A \to \underline{C}$ be a bounded linear functional. Assume: $\forall \, A \in A$,

$$\omega(A^*) = \overline{\omega(A)}.$$

Then $\exists$ unique positive linear functionals

$$\begin{cases} \omega_+ : A \to \underline{C} \\ \omega_- : A \to \underline{C} \end{cases}$$

such that

$$\omega = \omega_+ - \omega_-$$

and

$$||\omega|| = ||\omega_+|| + ||\omega_-||.$$

7.11 REMARK Let $\omega: A \to \underline{C}$ be a bounded linear functional. Define $\omega^*: A \to \underline{C}$ by

$$\omega^*(A) = \overline{\omega(A^*)}$$

and put

$$\begin{cases} \text{Re}\ \omega = \dfrac{\omega + \omega^*}{2} \\ \\ \text{Im}\ \omega = \dfrac{\omega - \omega^*}{2\sqrt{-1}}. \end{cases}$$

Then

$$\omega = \text{Re}\ \omega + \sqrt{-1}\ \text{Im}\ \omega.$$

Since

$$\begin{cases} \text{Re}\ \omega(A^*) = \overline{\text{Re}\ \omega(A)} \\ \text{Im}\ \omega(A^*) = \overline{\text{Im}\ \omega(A)}, \end{cases}$$

it follows from 7.10 that every bounded linear functional on A can be written as a linear combination of four positive linear functionals.

A <u>state</u> on A is a positive linear functional ω of norm 1. The <u>state space</u> $S(A)$ of A is the set of states of A.

E.g.: $S(\underline{C} \oplus \underline{C})$ can be identified with $[0,1]$ and $S(M_2(\underline{C}))$ can be identified with $\underline{S}^2$.

7.12 <u>EXAMPLE</u> Fix a locally compact Hausdorff space X.

• If X is compact, then the dual $C(X)^*$ of $C(X)$ can be identified with $M(X)$, the space of complex Radon measures on X:

$$\mu \to I_\mu \ , \ I_\mu(f) = \int_X f d\mu.$$

Here

$$||\mu|| = |\mu|(X),$$

$|\mu|$ the total variation of μ. Therefore $S(C(X)) = M_1^+(X)$, the Radon probability measures on X.

• If X is noncompact, then the dual $C_\infty(X)^*$ of $C_\infty(X)$ can be identified with $M(X)$, the space of complex Radon measures on X:

$$\mu \to I_\mu, \ I_\mu(f) = \int_X f d\mu.$$

Here

$$||\mu|| = |\mu|(X),$$

$|\mu|$ the total variation of μ. Therefore $S(C_\infty(X)) = M_1^+(X)$, the Radon probability measures on X.

7.13 <u>EXAMPLE</u> Given a complex Hilbert space H, denote by $W(H)$ the set of <u>density operators</u> (i.e. the set of positive trace class operators W with $tr(W) = 1$) --

then the arrow

$$\mathcal{W}(H) \to S(\underline{L}_\infty(H))$$

that sends W to ω_W, where

$$\omega_W(T) = \mathrm{tr}(WT) \qquad (T \in \underline{L}_\infty(H))$$

is bijective.

[Note: It is clear that

$$S(\underline{L}_\infty(H)) \subset S(\mathcal{B}(H)),$$

the inclusion being proper if H is infinite dimensional.]

7.14 LEMMA $S(A)$ is a nonempty convex subset of A^*.

7.15 LEMMA $S(A)$ is weak* closed iff A is unital.

[Note: So, if A is unital, then $S(A)$ is weak* compact (Alaoglu), thus is the weak* closed convex hull of its extreme points (Krein-Milman).]

If

$$\begin{cases} \omega_1 : A \to \underline{C} \\ \omega_2 : A \to \underline{C} \end{cases}$$

are positive linear functionals, write $\omega_1 \geq \omega_2$ if $\omega_1 - \omega_2$ is positive.

Now let $\omega \in S(A)$. Denote by $[0,\omega]$ the set of all positive linear functionals $\omega' : \omega \geq \omega'$ -- then $[0,\omega]$ is a convex set and ω is said to be pure if $[0,\omega] = \{t\omega : 0 \leq t \leq 1\}$. Write $P(A)$ for the set of pure states of A.

7.16 EXAMPLE If X is a locally compact Hausdorff space, then

$$P(C(X)) = \{\delta_x : x \in X\} \quad (X \text{ compact})$$

and

$$P(C_\infty(X)) = \{\delta_x : x \in X\} \quad (X \text{ noncompact}).$$

7.17 EXAMPLE Suppose that H is a complex Hilbert space -- then

$$P(\underline{L}_\infty(H)) = \{\omega_x : ||x|| = 1\}.$$

Here

$$\omega_x(T) = \langle x, Tx \rangle$$

or still,

$$\omega_x(T) = \text{tr}(P_x T),$$

P_x the orthogonal projection onto $\underline{C}x$.

[Note: Let $\underline{P}H$ be projective Hilbert space (the quotient of the unit sphere in H by the canonical action of $\underline{U}(1)$). Give $\underline{P}H$ the quotient topology -- then $P(\underline{L}_\infty(H))$ supplied with the relativised weak* topology is homeomorphic to $\underline{P}H$.]

N.B. The $\omega_x(||x|| = 1)$ are the so-called vector states.

7.18 LEMMA If A is unital, then the extreme points of $S(A)$ are the pure states:

$$\text{ex } S(A) = P(A).$$

7.19 REMARK For any A (unital or nonunital), let $\underline{S}(A)$ stand for the set of positive linear functionals of norm ≤ 1 -- then $\underline{S}(A)$ is convex, weak* compact, and

$$\text{ex } \underline{S}(A) = \{0\} \cup P(A).$$

7.20 <u>LEMMA</u> Every multiplicative state is pure.

7.21 <u>LEMMA</u> Every pure state is multiplicative on the center of A.

7.22 <u>SCHOLIUM</u> If A is a commutative C*-algebra, then

$$P(A) = \Delta(A).$$

Suppose that A is nonunital. Given a state $\omega \in S(A)$, define as before a linear functional ω^+ on A^+ by

$$\omega^+(A,\lambda) = \omega(A) + \lambda \quad (||\omega|| = 1).$$

Then $\omega^+ \in S(A^+)$. Moreover,

$$\omega \in P(A) \Leftrightarrow \omega^+ \in P(A^+).$$

7.23 <u>THEOREM</u> If A' is a C*-subalgebra of A, then every state ω' on A' can be extended to a state ω on A.

<u>PROOF</u> It suffices to establish this when both A and A' are unital with $1_A = 1_{A'}$. So let $\omega' \in S(A')$. Owing to the Hahn-Banach theorem, $\exists$ a bounded linear functional $\omega \in A^*$ that extends ω' and is of the same norm. But

$$1 = ||\omega|| = ||\omega'|| = \omega'(1_{A'}) = \omega(1_A).$$

Therefore ω is positive (cf. 7.6), hence $\omega \in S(A)$.

7.24 <u>THEOREM</u> If A' is a C*-subalgebra of A, then every pure state ω' on A' can be extended to a pure state ω on A.

<u>PROOF</u> Let $S_{\omega'}(A)$ be the subset of $S(A)$ consisting of those states that extend ω' -- then $S_{\omega'}(A)$ is not empty (cf. 7.23). On the other hand, $S_{\omega'}(A)$ is a weak* compact face of $S(A)$, thus

$$\text{ex } S_{\omega'}(A) \neq \emptyset \quad \text{(Krein-Milman)}.$$

But

$$\text{ex } S_{\omega'}(A) \subset \text{ex } \underline{S}(H).$$

And

$$\omega \in \text{ex } S_{\omega'}(A) \Rightarrow \omega \neq 0 \Rightarrow \omega \in P(A) \quad \text{(cf. 7.19)}.$$

7.25 <u>LEMMA</u> Let $A \in A_{SA}$ -- then $\exists\ \omega \in P(A)$: $|\omega(A)| = ||A||$.

<u>PROOF</u> The C*-subalgebra $C^*(A)$ generated by A is commutative. Choose a character $\omega_A \in \Delta(C^*(A))$: $|\omega_A(A)| = ||A||$ and extend ω_A to a pure state ω on A (cf. 7.24).

Here is a corollary: If $\omega(A) = 0\ \forall\ \omega \in P(A)$, then $A = 0$. In fact, $\forall\ \omega \in P(A)$,

$$\begin{bmatrix} \omega(\text{Re } A) = 0 \\ \\ \omega(\text{Im } A) = 0 \end{bmatrix} \Rightarrow \text{Re } A = \text{Im } A = 0.$$

7.26 LEMMA Let $A \in \mathcal{A}$ -- then $A \in \mathcal{A}_{SA}$ iff $\omega(A) \in \underline{R}$ for all $\omega \in \mathcal{P}(\mathcal{A})$.

7.27 LEMMA Let $A \in \mathcal{A}$ -- then $A \in \mathcal{A}_+$ iff $\omega(A) \in \underline{R}_{\geq 0}$ for all $\omega \in \mathcal{P}(\mathcal{A})$.

A weight on $\mathcal{A}$ is a function $w:\mathcal{A}_+ \to [0,\infty]$ such that

$$\begin{cases} w(A + B) = w(A) + w(B) & (A,B \in \mathcal{A}_+) \\ w(0) = 0, \quad w(\lambda A) = \lambda w(A) & (\lambda > 0,\ A \in \mathcal{A}_+). \end{cases}$$

E.g.: The prescription $w(0) = 0$, $w(A) = \infty$ $(A \in \mathcal{A}_+, A \neq 0)$ is a weight, albeit a not very interesting one.

Every positive linear functional is, of course, a weight. More generally, any sum of positive linear functionals is a weight (in fact, any sum of weights is a weight).

7.28 EXAMPLE Let $\mathcal{H}$ be a complex Hilbert space. Fix an orthonormal basis $\{e_i : i \in I\}$ for $\mathcal{H}$ and define

$$\text{tr}:\mathcal{B}(\mathcal{H})_+ \to [0,\infty]$$

by

$$\text{tr}(A) = \sum_{i \in I} \langle e_i, Ae_i \rangle.$$

Then tr is a weight.

[Note: Recall that tr is welldefined in the sense that it is independent of the choice of orthonormal basis.]

7.29 EXAMPLE Take $X = \underline{R}^n$ -- then the Riesz representation theorem identifies the positive linear functionals on $C_c(\underline{R}^n)$ with the Radon measures and the positive linear functionals on $C_\infty(\underline{R}^n)$ with the finite Radon measures. Therefore every Radon measure μ such that $\mu(\underline{R}^n) = \infty$ determines a weight on $C_\infty(\underline{R}^n)$ which is not a positive linear functional (e.g., μ = Lebesgue measure).

[Note: Recall that a positive linear functional on $C_c(\underline{R}^n)$ is a linear functional $I:C_c(\underline{R}^n) \to \underline{C}$ such that $I(f) \geq 0$ whenever $f \geq 0$.]

Given a weight w on $\mathcal{A}$, let

$$w - \mathcal{A}_+ = \{A \in \mathcal{A}_+ : w(A) < \infty\}.$$

7.30 LEMMA If $A \geq B \geq 0$ and if $A \in w - \mathcal{A}_+$, then $B \in w - \mathcal{A}_+$.

PROOF Write

$$A = (A - B) + B.$$

Then

$$\infty > w(A) = w(A - B) + w(B).$$

Let

$$L_w = \{A \in \mathcal{A} : w(A^*A) < \infty\}.$$

[Note: In general,

$$w(A^*A) < \infty \not\Rightarrow w(AA^*) < \infty.]$$

7.31 LEMMA L_w is a left ideal.

PROOF There are two points. First, $\forall\ A,B \in \mathcal{A}$,

$$(A + B)^*(A + B) + (A - B)^*(A - B) = 2(A^*A + B^*B)$$

$$\Rightarrow$$

$$(A + B)^*(A + B) \leq 2(A^*A + B^*B).$$

Second, $\forall\ X \in \mathcal{L}_w$ & $\forall\ A \in \mathcal{A}$,

$$(AX)^*AX = X^*A^*AX$$

$$\leq ||A^*A||X^*X$$

$$= ||A||^2X^*X.$$

7.32 LEMMA The linear span $w - \mathcal{A}$ of $w - \mathcal{A}_+$ is the set of elements of the form

$$\{\sum_{i=1}^{n} Y_i^*X_i : X_i, Y_i \in \mathcal{L}_w\},$$

i.e., is

$$\mathcal{L}_w^*\mathcal{L}_w.$$

PROOF If $X,Y \in \mathcal{L}_w$, then

$$4Y^*X = \sum_{k=0}^{3} (\sqrt{-1})^k(X + (\sqrt{-1})^kY)^*(X + (\sqrt{-1})^kY),$$

which implies that

$$\mathcal{L}_w^*\mathcal{L}_w \subset w - \mathcal{A}.$$

In the other direction, if $A \in w - \mathcal{A}_+$, then $A^{1/2} \in \mathcal{L}_w$, thus

$$A = (A^{1/2})^*A^{1/2} \in \mathcal{L}_w^*\mathcal{L}_w.$$

N.B. It follows that $w - \mathcal{A}$ is a *-subalgebra of $\mathcal{A}$ with

$$(w - \mathcal{A}) \cap \mathcal{A}_+ = w - \mathcal{A}_+.$$

Given $A \in w - \mathcal{A}$, we can write

$$A = A_1 - A_2 + \sqrt{-1}\, A_3 - \sqrt{-1}\, A_4,$$

where A_1, A_2, A_3, A_4 are in $w - \mathcal{A}_+$. If

$$A = A_1' - A_2' + \sqrt{-1}\, A_3' - \sqrt{-1}\, A_4'$$

is another such decomposition, then

$$A_1 - A_2 = A_1' - A_2' \text{ and } A_3 - A_4 = A_3' - A_4'.$$

So

$$w(A_1) + w(A_2') = w(A_1') + w(A_2) \text{ and } w(A_3) + w(A_4') = w(A_3') + w(A_4)$$

$$\Rightarrow$$

$$w(A_1) - w(A_2) + \sqrt{-1}\, w(A_3) - \sqrt{-1}\, w(A_4)$$

$$= w(A_1') - w(A_2') + \sqrt{-1}\, w(A_3') - \sqrt{-1}\, w(A_4').$$

Therefore the prescription

$$w(A) \equiv w(A_1) - w(A_2) + \sqrt{-1}\, w(A_3) - \sqrt{-1}\, w(A_4)$$

unambiguously extends w from $w - \mathcal{A}_+$ to $w - \mathcal{A}$.

7.33 REMARK If $w - \mathcal{A}_+ = \mathcal{A}_+$, then $w - \mathcal{A} = \mathcal{A}$ and the extension of w to $\mathcal{A}$ is a positive linear functional, hence w is continuous.

A trace on $\mathcal{A}$ is a weight w satisfying the condition

$$w(A^*A) = w(AA^*)$$

for all $A \in \mathcal{A}$.

N.B. If $\mathcal{A}$ is commutative, then every weight is a trace.

7.34 REMARK If w is a trace, then L_w is a *-ideal, thus the same is true of $w - \mathcal{A}$ (cf. 7.32).

7.35 EXAMPLE If H is a complex Hilbert space, then

$$\mathrm{tr}:\mathcal{B}(H)_+ \to [0,\infty]$$

is a trace and

$$\mathrm{tr} - \mathcal{B}(H) = \underline{L}_1(H).$$

A tracial state on $\mathcal{A}$ is a state ω which is a trace.

N.B. If $\mathcal{A}$ is commutative, then every state is a tracial state.

7.36 EXAMPLE Take $\mathcal{A} = M_n(\underline{C})$ -- then the assignment

$$[a_{ij}] \to \frac{1}{n} \sum_{k=1}^{n} a_{kk}$$

is a tracial state on $M_n(\underline{C})$ (and there are no others).

7.37 EXAMPLE Let H be an infinite dimensional complex Hilbert space -- then $\underline{L}_\infty(H)$ does not admit a tracial state. To see this, assume the opposite and suppose that $\omega \in S(\underline{L}_\infty(H))$ is a tracial state, hence ω has the same constant value $t > 0$ on all rank one orthogonal projections (any two such being unitarily equivalent). Let $\{e_i : i \in I\}$ be an orthonormal basis for H. Given $e_{i_1}, \ldots, e_{i_n}$, let P_n be the orthogonal projection onto their closed linear span -- then

$$|\omega(P_n)| \le ||P_n|| = 1.$$

On the other hand,

$$|\omega(P_n)| = nt \Rightarrow nt \le 1 \Rightarrow n \le \frac{1}{t},$$

from which the obvious contradiction.

§8. REPRESENTATIONS OF ALGEBRAS

N.B. In what follows, the underlying scalar field is $\underline{C}$.

Let E be a linear space, $L(E)$ the linear maps from E to E -- then $L(E)$ is an algebra (multiplication being composition).

Let A be an algebra -- then a representation of A on E is a homomorphism $\pi:A \rightarrow L(E)$.

[Note: A representation $\pi:A \rightarrow L(E)$ defines a left A-module structure on E (viz. $Ax = \pi(A)x$) and conversely.]

8.1 TERMINOLOGY

- π is faithful if π is injective.
- π is trivial if $\pi(A) = 0 \ \forall \ A \in A$.
- π is algebraically irreducible if π is not trival and $\{0\}$ and E are the only π-invariant subspaces.
- π is algebraically cyclic if $\exists \ x \in E$ such that $\{\pi(A)x:A \in A\} = E$.

8.2 REMARK The definition of algebraically irreducible explicitly excludes trivial representations. If they were not excluded, then the trivial representation on a zero or one dimensional space would qualify.

8.3 LEMMA Let π be a representation of A on $E \neq 0$ -- then π is algebraically irreducible iff every nonzero vector in E is algebraically cyclic.

8.4 THEOREM Let π be an algebraically irreducible representation of A on E.

Suppose that $I \subset A$ is a nonzero ideal -- then the restriction $\pi|I$ is either trivial or an algebraically irreducible representation of I. Furthermore, every algebraically irreducible representation of I arises by restriction from some algebraically irreducible representation of A.

[Note: If $I \subset \text{Ker}\ \pi$, then π drops to an algebraically irreducible representation of A/I and every algebraically irreducible representation of A/I is obtained in this fashion.]

8.5 <u>LEMMA</u> Let π be an algebraically irreducible representation of A on E. Suppose that $A \in \text{Ann}_L A$ -- then $\pi(A) = 0$.

<u>PROOF</u> Fix $y \in E : y \neq 0$, thus $\{\pi(B)y : B \in A\} = E$. And

$$\begin{aligned} \pi(A)\pi(B)y &= \pi(AB)y \\ &= \pi(0)y = 0. \end{aligned}$$

Consequently,

$$\text{Ann}_L A \subset \text{Ker}\ \pi.$$

Since $\text{Ann}_L A$ is an ideal, it follows that the induced homomorphism

$$A/\text{Ann}_L A \rightarrow L(E)$$

is an algebraically irreducible representation of $A/\text{Ann}_L A$.

8.6 <u>THEOREM</u> Let π be an algebraically irreducible representation of A on E -- then π can be extended to an algebraically irreducible representation $\bar{\pi}$ of $\mathbb{DC}(A)$ on E. Moreover, $\bar{\pi}$ is unique.

<u>PROOF</u> Suppose that

$$\pi(X)x = \pi(Y)y \quad (X,Y \in \mathcal{A},\ x,y \in E).$$

Then $\forall\ A \in \mathcal{A}$ & $\forall\ (L,R) \in \mathfrak{DC}(\mathcal{A})$,

$$\begin{aligned}
&\pi(A)(\pi(L(X))x - \pi(L(Y))y) \\
&\quad = \pi(AL(X))x - \pi(AL(Y))y \\
&\quad = \pi(R(A)X)x - \pi(R(A)Y)y \\
&\quad = \pi(R(A))(\pi(X)x - \pi(Y)y) \\
&\quad = 0.
\end{aligned}$$

But π is irreducible, hence

$$\pi(L(X))x = \pi(L(Y))y.$$

Accordingly, if $e \in E$ and if

$$e = \left[\begin{array}{l} \pi(X)x \\ \\ \pi(Y)y, \end{array}\right.$$

then the prescription

$$\bar{\pi}((L,R))e = \left[\begin{array}{c} \pi(L(X))x \\ = \\ \pi(L(Y))y \end{array}\right.$$

makes sense and defines an algebraically irreducible representation of $\mathfrak{DC}(\mathcal{A})$ on E. Finally, $\forall\ A \in \mathcal{A}$,

$$\begin{aligned}
\bar{\pi}((L_A,R_A))e &= \pi(L_A(X))x \\
&= \pi(AX)x
\end{aligned}$$

$$= \pi(A)\pi(X)x$$

$$= \pi(A)e.$$

Given a representation π of A on E, let

$$\pi(A)' = \{T \in L(E) : T\pi(A) = \pi(A)T (A \in A)\}.$$

8.7 LEMMA Let π be an algebraically irreducible representation of A on E -- then $\pi(A)'$ is a division algebra.

[Note: In other words, $\pi(A)'$ is a unital algebra in which every nonzero element has an inverse.]

8.8 REMARK The converse is false, i.e., it may very well be the case that $\pi(A)'$ is a division algebra, yet π is not algebraically irreducible. E.g.: Let A be the algebra of all $\underline{N}$-by-$\underline{N}$ matrices which have only finitely many nonzero entries, let E be the vector space of all complex sequences, and let π be the canonical representation of A on E -- then $\pi(A)'$ can be identified with $\underline{C}$, yet the subspace of E consisting of those sequences that are finitely supported is π-invariant.

Let π be a representation of A on $E \neq 0$ -- then π is totally algebraically irreducible if $\forall\ T \in L(E)$ and every finite dimensional subspace $V \subset E$, $\exists\ A \in A$:

$$\pi(A)x = Tx \ \forall\ x \in E.$$

N.B. Evidently,

"totally algebraically irreducible" => "algebraically irreducible".

8.9 LEMMA If $\pi: A \to L(E)$ is totally algebraically irreducible, then $\pi(A)' = \underline{C}\ id_E$.

PROOF Let $T \in \pi(A)'$ and suppose that for some $x \in E$, x and Tx are linearly independent. Since π is totally algebraically irreducible, $\exists\ A \in A$:

$$\begin{cases} \pi(A)x = x \\ \pi(A)Tx = 0. \end{cases}$$

But then

$$0 = \pi(A)Tx = T\pi(A)x = Tx,$$

a contradiction, So, $\forall\ x \in E$, $\exists\ c_x \in \underline{C}: Tx = c_x x$. If $x \neq 0$, $y \neq 0$, and $c_x \neq c_y$, then $x + y$ and $T(x + y)$ would be linearly independent. This being an impossibility, the conclusion is that $\exists\ c \in \underline{C}: Tx = cx$ $(x \in E)$ or still, $T = c(id_E)$.

8.10 LEMMA If $\pi: A \to L(E)$ is algebraically irreducible and if $\pi(A)' = \underline{C}\ id_E$, then π is totally algebraically irreducible.

8.11 RAPPEL The only finite dimensional division algebra over $\underline{C}$ is $\underline{C}$ itself.

Let π be an algebraically irreducible representation of A on E. Assume: $\dim E < \infty$ -- then π is totally algebraically irreducible. Proof: $\pi(A)'$ is a finite dimensional division algebra, thus $\pi(A)' = \underline{C}\ id_E$. Now quote 8.10.

8.12 EXAMPLE If A is commutative, then every finite dimensional algebraically irreducible representation $\pi: A \to L(E)$ of A is one dimensional.

[Suppose that E has two linearly independent vectors x and y. Choose $A,B \in \mathcal{A}: \pi(A)x = x$, $\pi(A)y = 0$, $\pi(B)x = y$ -- then

$$\begin{cases} \pi(AB)x = \pi(A)\pi(B)x = \pi(A)y = 0 \\ \pi(BA)x = \pi(B)\pi(A)x = \pi(B)x = y. \end{cases}$$

But AB = BA, so we have a contradiction.]

[Note: The assumption $\dim E < \infty$ implies that π is totally algebraically irreducible and this is all that is needed. Spelled out: If $\mathcal{A}$ is commutative, then every totally algebraically irreducible representation of $\mathcal{A}$ is one dimensional.]

8.13 <u>REMARK</u> Let π be an algebraically irreducible representation of $\mathcal{A}$ on E. Assume: $\forall A \in \mathcal{A}$, $\pi(A)$ is of finite rank -- then π is totally algebraically irreducible.

Let π_1 and π_2 be representations of $\mathcal{A}$ on E_1 and E_2.

- An <u>algebraic equivalence</u> is a linear bijection $\zeta: E_1 \to E_2$ such that

$$\zeta\pi_1(A) = \pi_2(A)\zeta \quad (A \in \mathcal{A}).$$

- An <u>algebraic intertwining operator</u> is a linear map $T: E_1 \to E_2$ such that

$$T\pi_1(A) = \pi_2(A)t \quad (A \in \mathcal{A}).$$

8.14 <u>LEMMA</u> Suppose that π_1 and π_2 are algebraically irreducible representations of $\mathcal{A}$ on E_1 and E_2 -- then all nonzero algebraic intertwining operators between π_1 and π_2 are algebraic equivalences.

Let π be an algebraically irreducible representation of A on E. Fix $x \neq 0$ -- then $I = \{A \in A : \pi(A)x = 0\}$ is a modular maximal left ideal and the arrow $A \rightarrow \pi(A)x$ implements a linear bijection $\zeta : A/I \rightarrow E$ that sets up an algebraic equivalence between the canonical representation L of A on A/I and π.

[Note: To check modularity, choose $e \in A : \pi(e)x = x$ -- then $\forall\ A \in A$,

$$\pi(Ae - A)x = \pi(A)\pi(e)x - \pi(A)x = \pi(A)x - \pi(A)x = 0.$$

Therefore

$$Ae - A \in I \quad (A \in A).$$

I.e.: I is modular.]

* * * * * * *

Assume henceforth that A is a Banach algebra and that E is a Banach space -- then in this context a <u>representation</u> of A on E is a homomorphism $\pi : A \rightarrow B(E)$, where $B(E)$ is the Banach algebra whose elements are the bounded linear maps from E to E.

8.15 <u>TERMINOLOGY</u>

- π is <u>faithful</u> if π is injective.
- π is <u>trivial</u> if $\pi(A) = 0\ \forall\ A \in A$.
- π is <u>topologically irreducible</u> if π is not trivial and $\{0\}$ and E are the only closed π-invariant subspaces.
- π is <u>topologically cyclic</u> if $\exists\ x \in E$ such that $\{\pi(A)x : A \in A\}$ is dense in E.

<u>N.B.</u> It is clear that the notions "topologically irreducible" and "topologically cyclic" are weaker than their purely algebraic counterparts.

8.16 <u>LEMMA</u> Let π be a representation of A on $E \neq 0$ -- then π is topologically irreducible iff every nonzero vector in E is topologically cyclic (cf. 8.3).

8.17 <u>REMARK</u> Suppose that $I \subset A$ is a nonzero closed ideal -- then the restriction to I of a topologically irreducible representation of A is either trivial or a topologically irreducible representation of I (cf. 8.4).

[Note: It is not claimed, however, that every topologically irreducible representation of I can be extended to a topologically irreducible representation of A.]

8.18 <u>RAPPEL</u> A normed division algebra $\mathcal{D}$ is one dimensional: $\mathcal{D} \approx \underline{C}$.

8.19 <u>THEOREM</u> Let π be an algebraically irreducible representation of A on E -- then π is totally algebraically irreducible.

<u>PROOF</u> Recall first that $\pi(A)'$ is a division algebra (cf. 8.7). Accordingly, in view of 8.10, it suffices to show that $\pi(A)'$ is normed. To this end, fix a nonzero $x \in E$. Given $T \in \pi(A)'$, put

$$||T||_x = \inf\{||\pi(A)||:A \in A,\ \pi(A)x = Tx\}.$$

Since π is algebraically irreducible, the RHS is not empty (cf. 8.3) and $0 \leq ||T||_x < \infty$. Next

$$||Tx|| = ||\pi(A)x|| \leq ||\pi(A)||\ ||x||$$

=>

$$||Tx||\ ||x||^{-1} \leq ||T||_x.$$

Therefore

$$||T||_x = 0 \Rightarrow ||Tx|| = 0$$

$$\Rightarrow Tx = 0,$$

so $T = 0$ (otherwise $T^{-1}Tx = 0 \Rightarrow x = 0$). The verification that $||\cdot||_x$ is a norm is straightforward.

[Note: The commutant $\pi(A)'$ of $\pi(A)$ is computed in $L(E)$ (not $B(E)$).]

8.20 REMARK Momentarily drop the assumption that E is a Banach space (but retain the assumption that A is a Banach algebra). Consider an algebraically irreducible representation π of A on E -- then π is necessarily totally algebraically irreducible. To see this, recall that π is algebraically equivalent to the canonical representation L of A on A/I for some modular maximal left ideal $I \subset A$. But A/I is a Banach space (I being closed) and the operator $L_A : A/I \to A/I$ which sends $B + I$ to $AB + I$ is continuous (indeed, $||L_A|| \leq ||A||$). One may therefore apply 8.19.

[Note: It is thus a corollary that an algebraically irreducible representation of a commutative Banach algebra is one dimensional (cf. 8.12).]

8.21 EXAMPLE If π is an algebraically irreducible representation of A on E, then $\pi(A)' = \underline{C}\, \mathrm{id}_E$ (cf. supra) but this is false if "algebraically irreducible" is replaced by "topologically irreducible". Thus take for E a Banach space with the property that $\exists\, T \in B(E)$ which has no nontrivial closed invariant subspaces (Enflo) -- then the identity representation π of the commutative unital subalgebra A of $B(E)$ generated by T is a topologically irreducible representation. But $A \subset \pi(A)'\ldots$.

If π is a representation of A on E, then π is continuous if $\exists\, K > 0$ such that

$$||\pi(A)|| \leq K||A|| \quad (A \in \mathcal{A}).$$

[Note: If in the terminology of §4, E is a left Banach $\mathcal{A}$-module, then the associated representation π of $\mathcal{A}$ on E is continuous:

$$||\pi(A)x|| = ||Ax|| \leq K||A|| \; ||x||$$

$$\Rightarrow$$

$$||\pi(A)|| \leq K||A||.]$$

8.22 LEMMA Suppose that $\forall$ x $\in$ E, the map

$$\pi_x : \begin{cases} \mathcal{A} \to E \\ \\ A \to \pi_x(A) \quad (= \pi(A)x) \end{cases}$$

is continuous -- then π is continuous.

PROOF Consider the set

$$\{\pi_x : ||x|| \leq 1\} \subset \mathcal{B}(\mathcal{A},E).$$

Then

$$\sup_{||x|| \leq 1} ||\pi_x(A)|| \leq ||\pi(A)|| \quad (A \in \mathcal{A}).$$

So, by the uniform boundedness principle, $\exists$ K > 0:

$$\sup_{||x|| \leq 1} ||\pi_x|| \leq K.$$

And this implies that

$$||\pi(A)|| = \sup_{||x|| \leq 1} ||\pi(A)x|| = \sup_{||x|| \leq 1} ||\pi_x(A)|| \leq K||A||.$$

8.23 <u>THEOREM</u> Suppose that π is an algebraically irreducible representation of A on E -- then π is continuous.

<u>PROOF</u> In the notation of 8.22, the algebraic irreducibility of π implies that there are two possibilities: 1. $\forall$ x, π_x is continuous; 2. $\forall\ x \neq 0$, π_x is discontinuous. This said, the idea then is to assume that the second possibility obtains and from there derive a contradiction. So take E infinite dimensional and start by fixing a sequence of linearly independent elements $x_n \in E$ $(||x_n|| = 1)$. Next, choose a sequence $A_n \in A$ with the following properties:

(i) $||A_n|| < 2^{-n}$;

(ii) $\pi(A_n)x_1 = \cdots = \pi(A_n)x_{n-1} = 0$;

(iii) $||\pi(A_n)x_n|| \geq n + ||\pi(A_1)x_n + \cdots + \pi(A_{n-1})x_n||$.

That such a construction is possible will be detailed below. Proceeding, let

$$A_0 = \sum_{n=1}^{\infty} A_n.$$

Then $\forall\ k \in \underline{N}$,

$$\begin{aligned} ||\pi(A_0)x_k|| &= ||\pi(A_1)x_k + \cdots + \pi(A_k)x_k|| \\ &\geq ||\pi(A_k)x_k|| - ||\pi(A_1)x_k + \cdots + \pi(A_{k-1})x_k|| \\ &\geq k = k||x_k||. \end{aligned}$$

But $\pi(A_0) \in \mathcal{B}(E)$, from which a contradiction.

[Note: The existence of the A_n can be established by induction if one can prove: $\forall\ \varepsilon > 0$, $\forall\ M > 0$, $\forall\ m \in \underline{N}$, there is an $A \in A$ such that

$$||A|| < \varepsilon,\ \pi(A)x_1 = \cdots = \pi(A)x_{n-1} = 0,\ ||\pi(A)x_m|| > M.$$

To this end, let

$$\mathcal{I}_1 = \{A \in \mathcal{A} : \pi(A)x_1 = 0\}, \ldots, \mathcal{I}_m = \{A \in \mathcal{A} : \pi(A)x_m = 0\}.$$

On the basis of 8.19, $\exists\ X_m \in \mathcal{A}$:

$$\pi(X_m)x_1 = \cdots = \pi(X_m)x_{m-1} = 0,\ \pi(X_m)x_m = x_m.$$

Thus

$$X_m \in \mathcal{I}_1 \cap \cdots \cap \mathcal{I}_{m-1},\ X_m \notin \mathcal{I}_m$$

$$\Rightarrow$$

$$\mathcal{I}_1 \cap \cdots \cap \mathcal{I}_{m-1} + \mathcal{I}_m = \mathcal{A},$$

$\mathcal{I}_m$ being maximal. Therefore addition defines a continuous linear map of

$$\mathcal{I}_1 \cap \cdots \cap \mathcal{I}_{m-1} \oplus \mathcal{I}_m$$

onto $\mathcal{A}$. By the open mapping theorem, $\exists\ \delta > 0$ such that for any $C \in \mathcal{A}$ with $||C|| < \delta\varepsilon$,

$$\exists\ A \in \mathcal{I}_1 \cap \cdots \cap \mathcal{I}_{m-1},\ B \in \mathcal{I}_m$$

such that

$$C = A + B \text{ and } ||A|| < \varepsilon,\ ||B|| < \varepsilon.$$

Since the map π_{x_m} is discontinuous, one can find a $C: ||C|| < \delta\varepsilon$ and $||\pi(C)x_m|| > M$. For this choice of C, the corresponding A satisfies the required conditions.]

Let π_1 and π_2 be representations of $\mathcal{A}$ on E_1 and E_2.

- A topological equivalence is a linear homeomorphism $\zeta: E_1 \to E_2$ such that

$$\zeta\pi_1(A) = \pi_2(A)\zeta \quad (A \in A).$$

• A <u>topological intertwining operator</u> is a bounded linear map $T:E_1 \to E_2$ such that

$$T\pi_1(A) = \pi_2(A)T \quad (A \in A).$$

8.24 <u>LEMMA</u> Suppose that π_1 and π_2 are algebraically irreducible representations of A on E_1 and E_2 -- then every algebraic equivalence $\zeta:E_1 \to E_2$ is a topological equivalence.

<u>PROOF</u> In view of 18.23, π_1 and π_2 are continuous. Fix $x_1 \in E_1$ ($x_1 \neq 0$) and let

$$I = \{A \in A:\pi_1(A)x_1 = 0\}.$$

Put $x_2 = \zeta x_1$ -- then

$$I = \{A \in A:\pi_2(A)x_2 = 0\}.$$

Since the arrows

$$\begin{cases} A + I \to \pi_1(A)x_1 \\ A + I \to \pi_2(A)x_2 \end{cases}$$

are topological equivalences between

$$\begin{cases} L_1 \text{ and } \pi_1 \\ L_2 \text{ and } \pi_2, \end{cases}$$

the arrow

$$\pi_1(A)x_1 \to \pi_2(A)x_2$$

is a linear homeomorphism. But

$$\zeta(\pi_1(A)x_1) = \pi_2(A)\zeta x_1$$

$$= \pi_2(A)x_2.$$

Therefore ζ is a topological equivalence.

The radical of A is the intersection of the kernels of all the algebraically irreducible representations of A, thus is an ideal. Notation: rad A.

[Note: A priori, this is a purely algebraic notion, i.e., the representation space E of an algebraically irreducible representation π of A is merely a linear space, not a Banach space. However, as was pointed out in 8.20, one can always place a norm on E w.r.t. which E is a Banach space, the $\pi(A)$ $(A \in A)$ are bounded, and $\pi:A \to B(E)$ is continuous.]

8.25 LEMMA The radical of A is the intersection of the modular maximal left ideals in A, hence is a closed ideal.

[Note: One can replace "left" by "right".]

8.26 REMARK A modular maximal left ideal in A is closed but in general, maximal left ideals need not be closed. E.g.: Take A to be an infinite dimensional Banach space thought of as a Banach algebra with trivial multiplication $(AB = 0$ $\forall\ A,B \in A)$ and let I be any dense linear subspace of codimension 1.

[Note: If A has a right (left) approximate unit (cf. 4.1), then every maximal left (right) ideal is closed.]

N.B. If $r:A \to \underline{R}_{\geq 0}$ is the spectral radius, then

$$r|\text{rad } A \equiv 0.$$

8.27 LEMMA Let $I \subset A$ be a closed ideal -- then $\text{rad } I = I \cap \text{rad } A$.
[This is a trivial consequence of 8.4.]

If $\text{rad } A = \{0\}$, then A is said to be semisimple.

N.B. The quotient $A/\text{rad } A$ is a semisimple Banach algebra:

$$\text{rad}(A/\text{rad } A) = \{0\}.$$

[The algebraically irreducible representations of A are of the form $\pi \circ \text{pr}$, where $\text{pr}:A \to A/\text{rad } A$ is the projection and π is an algebraically irreducible representation of $A/\text{rad } A$.]

8.28 EXAMPLE Every C*-algebra is semisimple. Proof: Let $A \in \text{rad } A$ -- then

$$A^*A \in \text{rad } A \Rightarrow r(A^*A) = 0.$$

But

$$||A|| = r(A^*A)^{1/2} \quad \text{(cf. 1.1)}.$$

Therefore $A = 0$.

[Note: Not all Banach algebras are semisimple and there are plenty of instances at the extreme end, viz. those equal to their radical (hence having no algebraically irreducible representations whatsoever).]

8.29 THEOREM Let A and B be Banach algebras. Assume: A is semisimple and let $\Psi:B \to A$ be a surjective homomorphism -- then Ψ is continuous.

PROOF Suppose that Ψ is not continuous -- then $\exists$ a sequence $\{B_n\}$ in $\mathcal{B}$ such that $B_n \to 0$ and $\Psi(B_n) \to A \neq 0$. Since $\mathcal{A}$ is semisimple, $\exists$ an algebraically irreducible representation π of $\mathcal{A}$ on a Banach space E such that $\pi(A) \neq 0$ with π continuous. Because Ψ is surjective, $\pi \circ \Psi:\mathcal{B} \to \mathcal{B}(E)$ is also algebraically irreducible, thus is continuous (cf. 8.23). Therefore

$$\pi(\Psi(B_n)) = (\pi \circ \Psi)(B_n) \to (\pi \circ \Psi)(0) = 0.$$

Meanwhile, thanks to the continuity of π,

$$\pi(\Psi(B_n)) \to \pi(A) \neq 0.$$

Contradiction.

8.30 REMARK If $\mathcal{A}$ is in addition commutative, then it can be shown that any homomorphism $\Psi:\mathcal{B} \to \mathcal{A}$ is continuous.

8.31 THEOREM Any two complete norms on a semisimple Banach algebra are equivalent.

[Apply 8.29 to $\text{id}_{\mathcal{A}}:\mathcal{A} \to \mathcal{A}$.]

§9. *-REPRESENTATIONS OF *-ALGEBRAS

N.B. In what follows, the underlying scalar field is $\underline{C}$.

Let E be a Hilbert space, $\mathcal{B}(E)$ the bounded linear operators from E to E -- then $\mathcal{B}(E)$ is a C*-algebra.

Let A be a *-algebra -- then a *-representation of A on E is a *-homomorphism $\pi:A \to \mathcal{B}(E)$.

9.1 LEMMA Let π be a *-representation of A on E. Suppose that $E_0 \subset E$ is a π-invariant linear subspace of E -- then $\overline{E_0}$ and $E_0^{\perp}$ are closed π-invariant linear subspaces of E and

$$E = \overline{E_0} \oplus E_0^{\perp}.$$

[Note: Let $P_0:E \to \overline{E_0}$ be the orthogonal projection -- then

$$P_0 \in \pi(A)'.]$$

9.2 RAPPEL A subset $S \subset E$ is total if the linear span of S is dense in E.

Let π be a *-representation of A on E -- then π is nondegenerate if the set

$$AE = \{\pi(A)x:A \in A, x \in E\}$$

is total.

[Note: The trivial *-representation of A on a zero dimensional space is non-degenerate.]

E.g.: If π is topologically cyclic, then π is nondegenerate.

9.3 LEMMA Let π be a *-representation of A on E -- then π is nondegenerate iff $\forall$ nonzero $x \in E$, $\exists A \in A : \pi(A)x \neq 0$.

9.4 LEMMA Let π be a *-representation of A on E -- then π is nondegenerate iff $\forall x \in E$,

$$x \in \{\pi(A)x : A \in A\}^{-}.$$

Given a *-representation π of A on E, let E_π be the closed linear span of AE -- then E_π is π-invariant and the restriction of π to E_π is a nondegenerate *-representation of A. Write

$$E = E_\pi \oplus E_\pi^\perp.$$

Then $E_\pi^\perp$ is π-invariant and the restriction of π to $E_\pi^\perp$ is a trivial *-representation of A:

$$E_\pi^\perp = \bigcap_{A \in A} \text{Ker } \pi(A).$$

9.5 THEOREM Suppose that π is a nondegenerate *-representation of A on E -- then there is an orthogonal decomposition

$$E = \bigoplus_{i \in I} E_i,$$

where $\forall i \in I$, E_i is a closed π-invariant subspace and the restriction of π to E_i is a topologically cyclic *-representation of A.

PROOF Order the set of sets of mutually orthogonal, topologically cyclic, closed π-invariant subspaces of E by inclusion and, via Zorn, consider a maximal element

$\{E_i : i \in I\}$.

Let π_1 and π_2 be $*$-representations of A on E_1 and E_2.

- A <u>unitary equivalence</u> is a unitary operator $U:E_1 \to E_2$ such that

$$U\pi_1(A) = \pi_2(A)U \quad (A \in A).$$

9.6 REMARK Let $\zeta:E_1 \to E_2$ be a topological equivalence. Write $\zeta = U(\zeta^*\zeta)^{1/2}$ (polar decomposition) -- then $\forall\ A \in A$,

$$(\zeta^*\zeta)^{1/2}\ \pi_1(A) = \pi_1(A)(\zeta^*\zeta)^{1/2}$$

and

$$U\pi_1(A) = \pi_2(A)U.$$

Therefore

π_1,π_2 topologically equivalent

=>

π_1,π_2 unitarily equivalent.

[To begin with,

$$\begin{aligned}\zeta^*\pi_2(A) &= \zeta^*\pi_2(A^*)^* \\ &= (\pi_2(A^*)\zeta)^* \\ &= (\zeta\pi_1(A^*))^* \\ &= \pi_1(A)\zeta^*\end{aligned}$$

=>

$$\zeta^*\zeta\pi_1(A) = \zeta^*\pi_2(A)\zeta = \pi_1(A)\zeta^*\zeta$$

=>

$$(\zeta^*\zeta)^{1/2}\,\pi_1(A) = (\zeta^*\zeta)^{1/2}\,\pi_1(A).$$

And then

$$\begin{aligned} U\pi_1(A)(\zeta^*\zeta)^{1/2} &= U(\zeta^*\zeta)^{1/2}\,\pi_1(A) \\ &= \zeta\pi_1(A) \\ &= \pi_2(A)\zeta \\ &= \pi_2(A)U(\zeta^*\zeta)^{1/2}. \end{aligned}$$

But the range of $(\zeta^*\zeta)^{1/2}$ is dense, so

$$U\pi_1(A) = \pi_2(A)U.]$$

9.7 LEMMA Let π_1 and π_2 be *-representations of A on E_1 and E_2. Assume: π_2 is topologically cyclic with a topologically cyclic vector $x_2 \in E_2$ -- then π_1 is unitarily equivalent to π_2 iff π_1 is topologically cyclic with a topologically cyclic vector $x_1 \in E_1$ such that

$$\langle x_1, \pi_1(A)x_1\rangle = \langle x_2, \pi_2(A)x_2\rangle \quad (A \in A).$$

[Note: One can always arrange matters so as to ensure that $Ux_1 = x_2$.]

In §8,

$$\pi(A)' = \{T \in L(E) : T\pi(A) = \pi(A)T \quad (A \in A)\}.$$

I.e.: The commutant of π was computed in $L(E)$. However, for the purposes at hand, it is best to let

$$\pi(A)' = \{T \in B(E) : T\pi(A) = \pi(A)T \quad (A \in A)\}.$$

9.8 LEMMA Let π be a $*$-representation of A on $E \neq 0$ -- then π is topologically irreducible iff π is not trivial and $\pi(A)' = \underline{C}\, \mathrm{id}_E$.

9.9 LEMMA Let π be a $*$-representation of A on $E \neq 0$ -- then π is topologically irreducible iff π is not trivial and $\pi(A)'$ contains no nonzero orthogonal projections except for the identity map on E.

PROOF Assume that π is not trivial and that the condition on $\pi(A)'$ obtains. To get a contradiction, suppose that π is not topologically irreducible. Let $E_0 \neq E$ be a nonzero closed π-invariant subspace and let P_0 be the orthogonal projection of E onto E_0 -- then $\forall\, A \in A$,

$$P_0\pi(A)P_0 = \pi(A)P_0.$$

Therefore

$$\begin{aligned} P_0\pi(A) &= (\pi(A^*)P_0)^* \\ &= (P_0\pi(A^*)P_0)^* \\ &= P_0\pi(A)P_0 \\ &= \pi(A)P_0 \end{aligned}$$

=>

$$P_0 \in \pi(A)'.$$

Let $\pi_i:\mathcal{A} \to E_i$ $(i \in I)$ be a $*$-representation. Assume: $\forall A \in \mathcal{A}$, $\exists M_A > 0$ such that

$$\sup_{i \in I} ||\pi_i(A)|| \leq M_A.$$

Form the (Hilbert) direct sum

$$\bigoplus_{i \in I} E_i.$$

Then $\forall A \in \mathcal{A}$,

$$\bigoplus_{i \in I} \pi_i(A) \in \mathcal{B}(\bigoplus_{i \in I} E_i)$$

and the assignment

$$A \to \bigoplus_{i \in I} \pi_i(A)$$

defines a $*$-representation of $\mathcal{A}$ on $\bigoplus_{i \in I} E_i$, the (Hilbert) direct sum of the π_i.

[Note: It is clear that

$$\bigoplus_{i \in I} \pi_i$$

is nondegenerate iff $\forall i \in I$, π_i is nondegenerate.]

N.B. If π is a $*$-representation of $\mathcal{A}$ on E and if $\pi_i = \pi$ $\forall i \in I$, then the $*$-representation

$$\bigoplus_{i \in I} \pi_i$$

is denoted by $\underline{I}\pi$ ($\underline{I}$ the cardinality of I). Under the identification

$$\bigoplus_{i \in I} E \approx E \underline{\otimes} \ell^2(I),$$

$\underline{I}\pi$ becomes $\pi \otimes \mathrm{id}$.

[Note: Any $*$-representation which is topologically equivalent to a $*$-representation of this type is called a multiple or amplification of π by I.]

The definitions and results that follow can be formulated for arbitrary $*$-representations but matters simplify if we restrict to nondegenerate $*$-representations which is not an essential loss of generality.

Let π_1 and π_2 be nondegenerate $*$-representations of A on E_1 and E_2 -- then π_1 and π_2 are disjoint if no nonzero sub $*$-representation of π_1 is topologically equivalent to a nonzero sub $*$-representation of π_2.

[Note: Therefore two topologically irreducible $*$-representations of A are disjoint iff they are not topologically equivalent.]

9.10 EXAMPLE Every nontrivial nondegenerate $*$-representation of A on a finite dimensional Hilbert space is the finite direct sum of topologically irreducible sub $*$-representations (these are unique up to topological equivalence while their multiplicities are unique period). So, if π_1 and π_2 are two such, then to say π_1 and π_2 are disjoint means that the "same" topologically irreducible $*$-representation cannot appear in the decompositions of π_1 and π_2 into topologically irreducible sub $*$-representations.

Let π_1 and π_2 be nondegenerate $*$-representations of A on E_1 and E_2 -- then π_1 and π_2 are geometrically equivalent if no nonzero sub $*$-representation of π_1 is disjoint from π_2 and no nonzero sub $*$-representation of π_2 is disjoint from π_1.

9.11 <u>EXAMPLE</u> In the finite dimensional case (cf. 9.10), π_1 is geometrically equivalent to π_2 iff the "same" topologically irreducible *-representations occur in their respective decompositions into topologically irreducible components but not necessarily with the same multiplicity.

9.12 <u>LEMMA</u> Nondegenerate *-representations π_1, π_2 are geometrically equivalent iff π_1 is unitarily equivalent to a sub *-representation of a multiple of π_2 and vice versa.

[Note: Therefore a given nondegenerate *-representation is geometrically equivalent to any of its multiples.]

In particular:

"unitary equivalence" => "geometric equivalence".

9.13 <u>REMARK</u> If π_1 is topologically irreducible and π_2 is geometrically equivalent to π_1, then π_2 is unitarily equivalent to a multiple of π_1. Thus if π_2 is also topologically irreducible, then π_1 and π_2 are unitarily equivalent.

9.14 <u>LEMMA</u> Nondegenerate *-representations π_1, π_2 are geometrically equivalent iff $\exists$ a cardinal number $\underline{n}$ such that $\underline{n}\pi_1$ is unitarily equivalent to $\underline{n}\pi_2$.

[To establish sufficiency, let π be a nonzero sub *-representation of π_1 -- then π is not disjoint from $\underline{n}\pi_1$, hence is not disjoint from $\underline{n}\pi_2$, or still, is not disjoint from π_2. It remains only to reverse the roles of π_1,π_2.]

N.B. One says that π_1 is weakly equivalent to π_2 if Ker π_1 = Ker π_2. So, as a corollary to 9.14,

"geometric equivalence" => "weak equivalence".

9.15 REMARK Let Rep A be the set of all nondegenerate $\star$-representations of A -- then in Rep A there are four standard notions of equivalence:

1. unitary equivalence;
2. topological equivalence;
3. geometric equivalence;
4. weak equivalence.

All are equivalence relations and we have 1 <=> 2 => 3 => 4. Moreover, these implications are not reversible (except in certain special situations).

9.16 LEMMA Nondegenerate $\star$-representations π_1,π_2 are disjoint iff they have no geometrically equivalent nonzero sub $\star$-representations.

A nondegenerate $\star$-representation π of A on E is primary if every nonzero sub $\star$-representation of π is geometrically equivalent to π.

E.g.: If π is topologically irreducible, then π is primary (as is $\pi \oplus \pi$ which, of course, is not topologically irreducible).

[Note: Arbitrary multiples of a topologically irreducible $\star$-representation are primary.]

9.17 LEMMA Nondegenerate primary $*$-representations π_1, π_2 are either disjoint or geometrically equivalent.

PROOF Suppose that π_1, π_2 are not disjoint -- then $\exists$ nonzero sub $*$-representations π_1^0 of π_1, π_2^0 of π_2 with π_1^0 geometrically equivalent to π_2^0 (cf. 9.16). But π_1^0 is geometrically equivalent to π_1 and π_2^0 is geometrically equivalent to π_2^0. Therefore π_1 is geometrically equivalent to π_2.

* * * * * * *

Assume henceforth that A is a Banach $*$-algebra (but maintain the assumption that E is a Hilbert space).

9.18 REMARK There is no universally agreed to definition of the term "Banach $*$-algebra". Here, it simply means that A is a Banach algebra supplied with an involution. In particular: The involution is not necessarily continuous.

[Note: In my book POSITIVITY, the involution was tacitly taken to be isometric (i.e., $||A^*|| = ||A||$ for all $A \in A$) which, of course, implies its continuity. Let us also remind ourselves that this is automatic for C*-algebras.]

9.19 EXAMPLE Let A be an infinite dimensional Banach space. Fix a Hamel basis $E = \{e\}$ for A subject to $||e|| = 1 \ \forall \ e \in E$. Let $\{e_n\}$ be a sequence of distinct elements of E and put

$$e_{2n-1}^* = ne_{2n}, \ e_{2n}^* = \frac{1}{n} e_{n-1} \quad (n = 1,2,\ldots).$$

For all remaining elements of E, put $e^* = e$ and then extend $*:E \rightarrow E$ to A by conjugate linearity. Taking now the multiplication in A to be trivial ($AB = 0$ $\forall A,B \in A$) thus gives rise to a Banach $*$-algebra with a discontinuous involution.

9.20 REMARK If A is a Banach $*$-algebra, then the map

$$*: \begin{cases} \mathcal{DC}(A) \to \mathcal{DC}(A) \\ (L,R) \to (R^*,L^*), \end{cases}$$

where $T^*(A) = T(A^*)^*$, is an involution, hence $\mathcal{DC}(A)$ is a $*$-algebra. If in addition,

$$\text{Ann}_L A = \{0\} \text{ and } \text{Ann}_R A = \{0\},$$

then 5.15 (and subsequent discussion) implies that $\mathcal{DC}(A)$ is a unital Banach $*$-algebra, in which case

$$\iota_A : A \to \mathcal{DC}(A)$$

is contractive.

[Note: In the presence of the involution,

$$\text{Ann}_L A = \{0\} \Longleftrightarrow \text{Ann}_R A = \{0\}.$$

Therefore $\mathcal{DC}(A)$ is a Banach $*$-algebra if A admits a right or left approximate unit (cf. 5.16).]

9.21 LEMMA Suppose that A is semisimple -- then the involution $*: A \to A$ is continuous.

PROOF Denote a norm $||\cdot||^*$ by $||A||^* = ||A^*||$ -- then the pair $(A, ||\cdot||^*)$ is a Banach algebra. But according to 8.31, $||\cdot||$ and $||\cdot||^*$ are equivalent, from which the assertion.

9.22 REMARK The image of a left ideal under the involution is a right ideal. Therefore rad A is a closed $*$-ideal (cf. 8.25), thus $A/\text{rad } A$ is a semisimple Banach $*$-algebra and its involution is continuous w.r.t. the quotient norm.

A *$*$-representation* of A on E is a $*$-homomorphism $\pi: A \to \mathcal{B}(E)$.

N.B. If the involution is isometric, then every $*$-representation is continuous, a fact that persists in general (cf. 9.25).

[Note: A $*$-homomorphism from a Banach $*$-algebra with isometric involution to a C*-algebra is continuous (indeed, contractive).]

9.23 EXAMPLE Let H be a complex Hilbert space. Take $A = \mathcal{B}(H)$, $E = \underline{L}_2(H)$ (the $*$-ideal in $\mathcal{B}(H)$ consisting of the Hilbert-Schmidt operators) -- then the left regular representation π of $\mathcal{B}(H)$ on $\underline{L}_2(H)$ is a $*$-representation:

$$\pi(A)T = AT \quad (A \in \mathcal{B}(H),\ T \in \underline{L}_2(H)).$$

[Note:

$$\pi(A) \in \mathcal{B}(L_2(H)) \quad (||\pi(A)|| = ||A||).$$

Moreover, $\forall\ T, T' \in \underline{L}_2(H)$,

$$\begin{cases} \langle T', AT\rangle_2 = \langle A^*T', T\rangle_2 = \langle \pi(A^*)T', T\rangle_2 \\ \langle T', AT\rangle_2 = \langle T', \pi(A)T\rangle_2 = \langle \pi(A)^*T', T\rangle_2 \end{cases}$$

$$\Rightarrow$$

$$\pi(A^*) = \pi(A)^*.]$$

9.24 LEMMA Let $\mathcal{T}$ be a *-subalgebra of $\mathcal{B}(E)$ which is a Banach algebra under an auxiliary norm $||\cdot||_0$ -- then $\exists$ M > 0:

$$||T|| \leq M||T||_0 \quad (T \in \mathcal{T}).$$

9.25 THEOREM Let π be a *-representation of A on E -- then π is continuous.

PROOF The image $\pi(A)$ is a *-subalgebra of $\mathcal{B}(E)$ (hence is semisimple) and the kernel Ker π is a closed *-ideal of A:

$$\pi(\overline{\text{Ker } \pi}) \subset \text{rad } \pi(A) = \{0\} \Rightarrow \overline{\text{Ker } \pi} \subset \text{Ker } \pi.$$

Therefore $\pi(A)$ is a Banach algebra via transport of structure:

$$A/\text{Ker } \pi \approx \pi(A),$$

the auxiliary norm $||\cdot||_0$ being given by

$$||\pi(A)||_0 = \inf \{||B||:\pi(B) = \pi(A)\},$$

so

$$||\pi(A)||_0 \leq ||A||.$$

It remains only to take $\mathcal{T} = \pi(A)$ and apply 9.24:

$$||\pi(A)|| \leq M||\pi(A)||_0 \leq M||A||.$$

[Note:

$$X \in \text{rad } \pi(A) \Rightarrow X^*X \in \text{rad } \pi(A).$$

The spectrum of X^*X thus consists of $\{0\}$ alone, so the spectral radius $r(X^*X)$ computed in $\mathcal{B}(E)$ must vanish. But

$$||X||^2 = r(X^*X) = 0 \Rightarrow X = 0.]$$

9.26 <u>RAPPEL</u> In a unital *-algebra A, an element $U \in A$ is <u>unitary</u> if $U^*U = UU^* = 1_A$. In an arbitrary *-algebra A, an element $V \in A$ is <u>quasiunitary</u> if $V^*V = VV^* = V + V^*$.

[Note: If A is unital, then the map $A \to 1_A - A$ induces a bijection between the unitary elements and the quasiunitary elements.]

9.27 <u>LEMMA</u> Suppose that A is a Banach *-algebra -- then every element of A is a linear combination of quasiunitary elements.

[Note: This is a wellknown structural fact (its proof depends on Ford's famous "square root lemma").]

Let $A \in A$ -- then

$$A = \sum_{i=1}^{n} \lambda_i V_i \quad (\lambda_i \in \underline{C}),$$

where the V_i are quasiunitary.

[Note: Since 0 is quasiunitary, one can always assume that $\sum_{i=1}^{n} \lambda_i = 0$.]

Put

$$q(A) = \inf \{\sum_{i=1}^{n} |\lambda_i| : \sum_{i=1}^{n} \lambda_i = 0\}.$$

9.28 <u>LEMMA</u> $q: A \to \underline{R}_{\geq 0}$ is a submultiplicative seminorm such that $q(A^*) = q(A)$ for all $A \in A$.

9.29 <u>REMARK</u> If $A = \text{rad } A$, let $\beta(A) = 1$ but if $A \neq \text{rad } A$, let $\beta(A)$ be the norm of $*:A/\text{rad } A \to A/\text{rad } A$, i.e., let

$$\beta(A) = \sup \{\frac{||A^* + \text{rad } A||}{||A + \text{rad } A||} : A \in A - \text{rad } A\}.$$

Then it can be shown that

$$q(A) \le (1 + \beta(A))||A|| \quad (A \in A).$$

Let $I \subset A$ be a nonzero *-ideal (it is not assumed that I is closed). Suppose that $\pi:I \to B(E)$ is a *-representation -- then we claim that π can be extended to a *-representation $\bar{\pi}:A \to B(E)$. To see this, recall that on general grounds there is an orthogonal decomposition

$$E = E_\pi \oplus E_\pi^\perp,$$

where E_π is the closed linear span of IE and the restriction of π to $E_\pi^\perp$ is a trivial *-representation of I. One can certainly extend the latter to a trivial *-representation of A. So, to settle the extension question, one can assume that π is nondegenerate.

If $\sum_{i=1}^n \pi(I_i)x_i$ is a typical element in the linear span $\tilde{E}$ of IE and if $\bar{\pi}$ is an extension of π, then $\forall\, A \in A$,

$$\begin{aligned}\bar{\pi}(A)(\sum_{i=1}^n \pi(I_i)x_i) &= \sum_{i=1}^n \bar{\pi}(A)\pi(I_i)x_i \\ &= \sum_{i=1}^n \bar{\pi}(A)\bar{\pi}(I_i)x_i \\ &= \sum_{i=1}^n \bar{\pi}(AI_i)x_i\end{aligned}$$

$$= \sum_{i=1}^{n} \pi(AI_i)x_i.$$

Since $\bar{\pi}(A) \in \mathcal{B}(E)$ and since IE is total, it follows that if $\bar{\pi}$ exists, then $\bar{\pi}$ is unique.

One can also use this recipe to establish existence. For suppose that

$$\sum_{i=1}^{n} \pi(I_i)x_i = 0.$$

Then

$$||\sum_{k=1}^{n} \pi(AI_k)x_k||^2$$

$$= \sum_{i=1}^{n} \sum_{j=1}^{n} <\pi(AI_i)x_i, \pi(AI_j)x_j>$$

$$= \sum_{i=1}^{n} \sum_{j=1}^{n} <x_i, \pi(AI_i)^*\pi(AI_j)x_j>$$

$$= \sum_{i=1}^{n} \sum_{j=1}^{n} <x_i, \pi(I_i^*A^*AI_j)x_j>$$

$$= \sum_{i=1}^{n} \sum_{j=1}^{n} <\pi(I_i)x_i, \pi(A^*AI_j)x_j>$$

$$= \sum_{j=1}^{n} < \sum_{i=1}^{n} \pi(I_i)x_i, \pi(A^*AI_j)x_j>$$

$$= 0.$$

The prescription

$$\tilde{\pi}(A)(\sum_{i=1}^{n} \pi(I_i)x_i) = \sum_{i=1}^{n} \pi(AI_i)x_i$$

is thus a welldefined linear operator on $\tilde{E}$.

9.30 RAPPEL Suppose that $\tilde{E}$ is a pre-Hilbert space. Let $\tilde{T}:\tilde{E} \to \tilde{E}$ be a linear map -- then a linear map $\tilde{T}^*:\tilde{E} \to \tilde{E}$ is a formal adjoint of $\tilde{T}$ if $\forall\, x,y \in \tilde{E}$,

$$\langle \tilde{T}x, y\rangle = \langle x, \tilde{T}^* y\rangle.$$

Formal adjoints are unique and the subset

$$L_*(\tilde{E}) \subset L(\tilde{E})$$

consisting of those $\tilde{T}$ that have a formal adjoint is a unital *-algebra.

[Note: The mere existence of a formal adjoint does not imply boundedness. If, however, $\tilde{U}$ is a unitary element of $L_*(\tilde{E})$, then $\tilde{U}$ is bounded:

$$\langle \tilde{U}x, \tilde{U}y\rangle = \langle \tilde{U}^*\tilde{U}x, y\rangle = \langle x, y\rangle$$

$$\Rightarrow ||\tilde{U}x||^2 = ||x||^2 \Rightarrow ||\tilde{U}x|| = ||x|| \Rightarrow ||\tilde{U}|| = 1.$$

Incidentally, if $\tilde{E}$ is a dense linear subspace of a Hilbert space E, then the formal adjoint is the restriction to $\tilde{E}$ of the Hilbert space adjoint.]

Next, $\tilde{\pi}(A)$ has a formal adjoint, viz. $\tilde{\pi}(A^*)$. Proof:

$$\langle \sum_{j=1}^{m} \pi(J_j)y_j, \tilde{\pi}(A)(\sum_{i=1}^{n} \pi(I_i)x_i)\rangle$$

$$= \sum_{j=1}^{m} \sum_{i=1}^{n} \langle \pi(J_j)y_j, \pi(AI_i)x_i\rangle$$

$$= \sum_{j=1}^{m} \sum_{i=1}^{n} \langle y_j, \pi(J_j^*)\pi(AI_i)x_i\rangle$$

$$= \sum_{j=1}^{m} \sum_{i=1}^{n} \langle y_j, \pi(J_j^* AI_i)x_i\rangle.$$

On the other hand,

$$\langle\tilde{\pi}(A^*)\left(\sum_{j=1}^{m}\pi(J_j)y_j\right),\sum_{i=1}^{n}\pi(I_i)x_i\rangle$$

$$=\sum_{j=1}^{m}\sum_{i=1}^{n}\langle\pi(A^*J_j)y_j,\pi(I_i)x_i\rangle$$

$$=\sum_{j=1}^{m}\sum_{i=1}^{n}\langle y_j,\pi(A^*J_j)^*\pi(I_i)x_i\rangle$$

$$=\sum_{j=1}^{m}\sum_{i=1}^{n}\langle y_j,\pi(J_j^*AI_i)x_i\rangle.$$

Therefore

$$\tilde{\pi}(A)^* = \tilde{\pi}(A^*).$$

N.B. The definitions imply that $\tilde{\pi}:A \to L(\tilde{E})$ is a $*$-homomorphism.

9.31 LEMMA If $V \in A$ is quasiunitary, then

$$\mathrm{id}_{\tilde{E}} - \tilde{\pi}(V) \in L(\tilde{E})$$

is unitary.

PROOF We have

$$(\mathrm{id}_{\tilde{E}} - \tilde{\pi}(V))^*(\mathrm{id}_{\tilde{E}} - \tilde{\pi}(V))$$

$$= \mathrm{id}_{\tilde{E}} - \tilde{\pi}(V^*) - \tilde{\pi}(V) + \tilde{\pi}(V^*)\tilde{\pi}(V)$$

$$= \mathrm{id}_{\tilde{E}} + \tilde{\pi}(- V^* - V + V^*V)$$

$$= \mathrm{id}_{\tilde{E}}.$$

Ditto

$$(\mathrm{id}_{\tilde{E}} - \tilde{\pi}(V))(\mathrm{id}_{\tilde{E}} - \tilde{\pi}(V))^* = \mathrm{id}_{\tilde{E}}.$$

Therefore

$$\mathrm{id}_{\tilde{E}} - \tilde{\pi}(V) \in \mathcal{L}(\tilde{E})$$

is bounded (cf. 9.30).

Now write

$$A = \sum_{i=1}^{n} \lambda_i V_i \quad (\sum_{i=1}^{n} \lambda_i = 0).$$

Then

$$\tilde{\pi}(A) = \sum_{i=1}^{n} \lambda_i \tilde{\pi}(V_i)$$

$$= \sum_{i=1}^{n} \lambda_i \tilde{\pi}(V_i) - \sum_{i=1}^{n} \lambda_i \, \mathrm{id}_{\tilde{E}}$$

$$= \sum_{i=1}^{n} \lambda_i (\tilde{\pi}(V_i) - \mathrm{id}_{\tilde{E}}),$$

so $\tilde{\pi}(A)$ is bounded, thus can be extended by continuity to a bounded operator $\bar{\pi}(A) \in \mathcal{B}(E)$ and the resulting map $\bar{\pi}: A \to \mathcal{B}(E)$ is a nondegenerate *-representation of A which extends π.

<u>N.B.</u> If A is merely a *-algebra, then it need not be true that a *-representation $\pi: I \to \mathcal{B}(E)$ is extendible to a *-representation $\bar{\pi}: A \to \mathcal{B}(E)$.

9.32 <u>LEMMA</u> With the notation and assumptions being as above, $\bar{\pi}$ is topologically cyclic iff π is topologically cyclic and $\bar{\pi}$ is topologically irreducible iff π is topologically irreducible.

9.33 EXAMPLE Suppose that A is a C*-algebra and let $\pi: A \to \mathcal{B}(E)$ be a nondegenerate *-representation -- then $\exists$ a unique nondegenerate *-representation $\bar{\pi}$ of $\mathcal{DC}(A)$ on E such that $\forall$ $A \in A$, $\bar{\pi}(L_A, R_A) = \pi(A)$. Assume further that π is topologically irreducible. Since the same holds for $\bar{\pi}$, given any Z in the center $\mathcal{Z}(A)$ of $\mathcal{DC}(A)$, there exists a complex number $C_Z(\pi)$:

$$\bar{\pi}(Z) = C_Z(\pi)\,\mathrm{id}_E \quad (\text{cf. } 9.8).$$

[Note: Let us keep in mind that $\mathcal{DC}(A)$ is a unital C*-algebra and $\mathcal{Z}(A)$ is a unital commutative C*-algebra.]

The *-radical of A is the intersection of the kernels of all the topologically irreducible *-representations of A, thus a closed *-ideal. Notation: *-rad A.

If *-rad $A = \{0\}$, then A is said to be *-semisimple.

N.B. The quotient A/*-rad A is a *-semisimple Banach *-algebra.

9.34 THEOREM The *-radical of A is the intersection of the kernels of all the *-representations of A.

[This will emerge from the machinery developed in §10 (cf. 10.29).]

Accordingly, if A admits a faithful *-representation, then A is *-semisimple. E.g.: Every C*-algebra is *-semisimple (cf. 10.36).

9.35 EXAMPLE Consider $L^1(G)$ (cf. 5.17) -- then $L^1(G)$ is a Banach *-algebra with isometric involution but it is not a C*-algebra unless G is a singleton. Still, $L^1(G)$ is *-semisimple.

[The lift to $L^1(G)$ of the left regular representation of G on $L^2(G)$ is a faithful *-representation of $L^1(G)$.]

9.36 <u>EXAMPLE</u> Let $\underline{D} = \{z \in \underline{C}: |z| \leq 1\}$ -- then by $A(\underline{D})$ we shall understand the algebra of all continuous complex valued functions on $\underline{D}$ that are holomorphic in int $\underline{D}$. Since $A(\underline{D}) \subset C(\underline{D})$ is closed in the supremum norm, it follows that $A(\underline{D})$ is a unital commutative Banach *-algebra, the involution being given by the rule

$$f^*(z) = \overline{f(\bar{z})}.$$

Define a *-representation π of $A(\underline{D})$ on $L^2(|z| = 1)$ by

$$\pi(f)\phi = f\phi \quad \text{(pointwise product)}.$$

Then π is faithful, thus $A(\underline{D})$ is *-semisimple.

[Note: $A(\underline{D})$ is not a C*-algebra (consider $1 + \sqrt{-1}\, z$).]

9.37 <u>LEMMA</u> Let π be a *-representation of A on E -- then $\forall$ A $\in A$,

$$||\pi(A)|| \leq r(A^*A)^{1/2},$$

r the spectral radius.

9.38 <u>LEMMA</u> We have

$$\text{rad } A \subset *\text{-rad } A,$$

hence

$$A \text{ *-semisimple} \Rightarrow A \text{ semisimple}.$$

<u>PROOF</u> Let A $\in$ rad A -- then

$$A^*A \in \text{rad } A \Rightarrow r(A^*A) = 0$$

$$\Rightarrow$$

$$||\pi(A)|| = 0 \ (\forall\ \pi) \Rightarrow A \in \text{Ker}\ \pi(\forall\ \pi)$$

$$\Rightarrow$$

$$A \in *\text{-rad}\ \mathcal{A}.$$

[Note: It can happen that rad $\mathcal{A} = \{0\}$ but $*$-rad $\mathcal{A} \neq \{0\}$.]

Define $\gamma:\mathcal{A} \to \underline{R}_{\geq 0}$ by

$$\gamma(A) = \sup_{\pi} ||\pi(A)||,$$

where π ranges over the $*$-representations of $\mathcal{A}$.

[Note: $\forall\ A \in \mathcal{A}$,

$$\gamma(A) \leq r(A^*A)^{1/2} \quad (\text{cf. } 9.37).$$

But

$$r(A^*A)^{1/2} \leq ||A^*A||^{1/2}.$$

If now $*:\mathcal{A} \to \mathcal{A}$ is continuous, then $\exists\ C_A > 0: ||A^*|| \leq C_A^2\ ||A||$, so

$$\gamma(A) \leq C_A\ ||A||,$$

which proves that γ is continuous w.r.t. $||\cdot||$ (see below for the general case).]

9.39 <u>LEMMA</u> $\forall\ A \in \mathcal{A}$,

$$\gamma(A) \leq q(A).$$

<u>PROOF</u> If π is a $*$-representation of $\mathcal{A}$ on E and if $A = \sum_{i=1}^{n} \lambda_i V_i \ (\sum_{i=1}^{n} \lambda_i = 0)$, then $\text{id}_E - \pi(V_i)$ is unitary. Therefore

$$||\pi(A)x|| = ||\sum_{i=1}^{n} \lambda_i \pi(V_i)x||$$

$$= ||\sum_{i=1}^{n} \lambda_i (\pi(V_i) - \mathrm{id}_E)x||$$

$$\leq (\sum_{i=1}^{n} |\lambda_i|)\,||x||$$

$$\Rightarrow \quad ||\pi(A)|| \leq q(A) \Rightarrow \gamma(A) \leq q(A).$$

[Note: It is true (but not obvious) that $\gamma = q$.]

9.40 <u>THEOREM</u> $\forall\, A \in \mathcal{A}$,

$$\gamma(A) \leq (1 + \beta(\mathcal{A}))\,||A|| \quad \text{(cf. 9.29)}.$$

9.41 <u>REMARK</u> Here is a different approach to the continuity of γ w.r.t. $||\cdot||$: $\forall\, A \in \mathcal{A}$,

$$r(A^*A) = r(A^*A + \mathrm{rad}\ \mathcal{A})$$

$$\leq ||A^*A + \mathrm{rad}\ \mathcal{A}||$$

$$\leq ||A^* + \mathrm{rad}\ \mathcal{A}||\ ||A + \mathrm{rad}\ \mathcal{A}||$$

$$\leq \beta(\mathcal{A})\,||A + \mathrm{rad}\ \mathcal{A}||^2$$

$$\leq \beta(\mathcal{A})\,||A||^2$$

$\Rightarrow$

$$\gamma(A) \leq \beta(\mathcal{A})^{1/2}\,||A||.$$

In turn, this leads to another proof of 9.25 and also shows that

$$\gamma(A) \le m(\mathcal{A})||A||,$$

where

$$m(\mathcal{A}) = \sup\{\frac{r(A^*A)^{1/2}}{||A||} : A \in \mathcal{A} - \{0\}\}$$

$$\le \beta(\mathcal{A})^{1/2}.$$

[Note:

$$||A + \text{rad } \mathcal{A}|| \le \beta(\mathcal{A})||A^* + \text{rad } \mathcal{A}||$$

$$\le \beta(\mathcal{A})^2||A + \text{rad } \mathcal{A}||$$

=>

$$1 \le \beta(\mathcal{A}).$$

If $*:\mathcal{A} \to \mathcal{A}$ is isometric, then $*:\mathcal{A}/\text{rad } \mathcal{A} \to \mathcal{A}/\text{rad } \mathcal{A}$ is isometric, hence in this case, $\beta(\mathcal{A}) = 1$.]

It is clear that γ is a submultiplicative seminorm and

$$\gamma(A^*A) = \gamma(A)^2 \quad (A \in \mathcal{A}).$$

And

$$*\text{-rad } \mathcal{A} = \gamma^{-1}(\{0\}).$$

Therefore γ induces a C*-norm on the quotient $\mathcal{A}/*\text{-rad } \mathcal{A}$. Denote the completion of $\mathcal{A}/*\text{-rad } \mathcal{A}$ by $C^*(\mathcal{A})$, the <u>enveloping C*-algebra</u> of $\mathcal{A}$, and write $p_{\mathcal{A}}$ for the canonical *-homomorphism $\mathcal{A} \to C^*(\mathcal{A})$.

9.42 <u>EXAMPLE</u> Take $\mathcal{A} = L^1(G)$ (cf. 9.35) -- then $C^*(G) \equiv C^*(L^1(G))$ is called the <u>group C*-algebra</u> of G.

[Note: Since $L^1(G)$ is *-semisimple, it can be viewed as a dense *-subalgebra of $C^*(G)$.]

9.43 LEMMA If $\mathcal{B}$ is a C*-algebra and if $\Phi:A \to \mathcal{B}$ is a *-homomorphism, then there is a unique *-homomorphism $\bar{\Phi}:C^*(A) \to \mathcal{B}$ such that $\Phi = \bar{\Phi} \circ p_A$.

9.44 SCHOLIUM The map $\bar{\pi} \to \pi = \bar{\pi} \circ p_A$ sets up a bijection between the set of all *-representations $\bar{\pi}$ of $C^*(A)$ and the set of all *-representations π of A. This correspondence preserves the following properties: trivial, nondegenerate, topologically cyclic, topologically irreducible, unitary equivalence, geometric equivalence.

9.45 REMARK It may very well be the case that π is faithful yet $\bar{\pi}$ is not faithful.

[Note: It is also possible that π_1 and π_2 are weakly equivalent but $\bar{\pi}_1$ and $\bar{\pi}_2$ are not weakly equivalent.]

The *-representation theory of Banach *-algebras, hard one to say the least, simplifies enormously when specialized to C*-algebras. Further evidence for this is supplied by 9.48 infra, a surprise if there ever was one. Its proof depends on the two pillars of W*-algebra theory.

9.46 THEOREM Suppose that A is a nondegenerate *-subalgebra of $\mathcal{B}(E)$ -- then the weak closure of A is A'' $(= (A')')$.

[Note: In this context, to say that A is nondegenerate means that the set

$$AE = \{Ax:A \in A, x \in E\}$$

is total, i.e., A is nondegenerate in the sense used at the beginning per the identity representation of A on E.]

9.47 THEOREM Suppose that A is a $*$-subalgebra of $\mathcal{B}(E)$ and let T be an element in the weak closure of A -- then $\exists$ a net T_i in A such that $\forall$ i, $||T_i|| \leq ||T||$ and $T_i \to T$ strongly.

[Note: If T is selfadjoint, then one can take the T_i selfadjoint.]

9.48 THEOREM Let A be a C*-algebra and let $\pi:A \to \mathcal{B}(E)$ be a $*$-representation. Assume: π is topologically irreducible -- then π is algebraically irreducible.

PROOF Since A is a C*-algebra, the image $\pi(A)$ is a norm closed $*$-subalgebra of $\mathcal{B}(E)$. So, to establish algebraic irreducibility, we can replace A by $\pi(A)$, the claim being that $\forall$ $x \neq 0$, the set

$$\{Ax:A \in A\}$$

equals E (cf. 8.3). To this end, note first that A is nondegenerate and

$$A' = \underline{C}\, id_E \quad (\text{cf. } 9.8) \Rightarrow A'' = \mathcal{B}(E).$$

Therefore the weak closure of A is $\mathcal{B}(E)$. Now fix $x \neq 0$, y in E. To construct an $A \in A$ such that $Ax = y$, normalize the situation and take $||x|| = 1$, $||y|| = 1$ and for any $z \in E$, let

$$P_{z,x} = \langle x, \text{—}\rangle z \quad (||P_{z,x}|| = ||z|| \, ||x||).$$

Accordingly, $||P_{y,x}|| = 1$, so $\exists\ A_1 \in \mathcal{A}: ||A_1|| \leq 1$ and

$$||P_{y,x}x - A_1x|| \leq 2^{-1}$$

or still,

$$||y - A_1x|| \leq 2^{-1}.$$

Next, let $y_1 = y - A_1x$ -- then $||P_{y_1,x}|| \leq 2^{-1}$, so $\exists\ A_2 \in \mathcal{A}: ||A_2|| \leq 2^{-1}$ and

$$||P_{y_1,x}x - A_2x|| \leq 2^{-2}$$

or still,

$$||y - A_1x - A_2x|| \leq 2^{-2}.$$

Proceeding, $\exists\ A_n \in \mathcal{A}: ||A_n|| \leq 2^{-n}$ such that

$$||y - \sum_{i=1}^{n} A_ix|| \leq 2^{-n}.$$

Put

$$A = \sum_{n=1}^{\infty} A_n.$$

Then $A \in \mathcal{A}$ and $Ax = y$.

[Note: It is thus a corollary that every topologically irreducible *-representation of a C*-algebra is totally algebraically irreducible (cf. 8.19).]

Let $\mathcal{A}$ be a C*-algebra and let $\pi:\mathcal{A} \rightarrow \mathcal{B}(E)$ be a topologically irreducible *-representation. Suppose given

$$\begin{cases} x_1,\ldots,x_n \in E \\ y_1,\ldots,y_n \in E, \end{cases}$$

where the x_i are linearly independent.

9.49 <u>LEMMA</u> $\exists\, A \in \mathcal{A}$:

$$\pi(A)x_1 = y_1,\ldots,\pi(A)x_n = y_n.$$

9.50 <u>LEMMA</u> Assume that

$$Tx_1 = y_1,\ldots,Tx_n = y_n$$

for some selfadjoint $T:E \to E$ -- then $\exists$ a selfadjoint $A \in \mathcal{A}$:

$$\pi(A)x_1 = y_1,\ldots,\pi(A)x_n = y_n.$$

9.51 <u>LEMMA</u> Take $\mathcal{A}$ unital and assume that

$$Vx_1 = y_1,\ldots,Vx_n = y_n$$

for some unitary $V:E \to E$ -- then $\exists$ a unitary $U \in \mathcal{A}$:

$$\pi(U)x_1 = y_1,\ldots,\pi(U)x_n = y_n.$$

<u>PROOF</u> It suffices to establish this under the additional supposition that the x_i are orthonormal, hence that the y_k are also orthonormal. Let E_0 be the linear span of $x_1,\ldots,x_n,y_1,\ldots,y_n$. Extend

$$\begin{cases} x_1,\ldots,x_n \text{ to an orthonormal basis } x_1,\ldots,x_m \text{ for } E_0 \\ y_1,\ldots,y_n \text{ to an orthonormal basis } y_1,\ldots,y_m \text{ for } E_0. \end{cases}$$

Choose a unitary $V_0:E_0 \to E_0$ such that $V_0x_j = y_j$ $(j = 1,\ldots,m)$. Let $e_1,\ldots,e_m$

be an orthonormal basis for E_0:

$$V_0 e_j = \lambda_j e_j \quad (\lambda_j \in \underline{C}, |\lambda_j| = 1).$$

Write $\lambda_j = e^{\sqrt{-1}\, t_j}$ $(t_j \in \underline{R})$ and put

$$T = \sum_{j=1}^{m} t_j P_j,$$

where P_j is the orthogonal projection of E onto $\underline{C}e_j$ -- then $T:E \rightarrow E$ is selfadjoint and $Te_j = t_j e_j$. Accordingly, $\exists$ a selfadjoint $A \in \mathcal{A}$:

$$\pi(A) e_j = t_j e_j \quad (\text{cf. } 9.50).$$

Let $U = e^{\sqrt{-1}\, A}$ -- then $U \in \mathcal{A}$ is unitary and

$$\begin{aligned} \pi(U) e_j &= \pi(e^{\sqrt{-1}\, A}) e_j \\ &= e^{\sqrt{-1}\, \pi(A)} e_j \\ &= e^{\sqrt{-1}\, t_j} e_j \\ &= \lambda_j e_j \\ &= V_0 e_j. \end{aligned}$$

Therefore $\pi(U)$ equals V_0 on E_0, thus

$$\pi(U) x_i = V_0 x_i = y_i,$$

as desired.

9.52 <u>REMARK</u> Let $\mathcal{A}$ be a C*-algebra -- then every algebraically irreducible representation of A is algebraically equivalent to a topologically irreducible $*$-representation of A.

9.53 LEMMA Let A be a C*-algebra. Suppose that π_1 and π_2 are algebraically equivalent topologically irreducible $*$-representations of A on E_1 and E_2 -- then π_1 and π_2 are unitarily equivalent.

PROOF Since π_1 and π_2 are algebraically irreducible (cf. 9.48), if $\zeta:E_1 \to E_2$ is the algebraic equivalence at issue, then ζ must be a topological equivalence (cf. 8.24), so π_1 and π_2 are unitarily equivalent (cf. 9.6).

One of the objectives of the theory is the classification of all the non-degenerate $*$-representations of a given C*-algebra A, the simplest situation being when A is commutative.

Notation:

- Bor $\Delta(A)$: The σ-algebra of Borel subsets of $\Delta(A)$.
- Pro E: The lattice of orthogonal projections of E.

Suppose that

$$P:\text{Bor }\Delta(A) \to \text{Pro } E$$

is a spectral measure. Let

$$\pi_P(A) = \int_{\Delta(A)} \hat{A}(\omega)\, dP(\omega) \qquad (A \in A).$$

Then

$$||\pi_P|| = ||\hat{A}||_\infty$$

and the assignment $A \to \pi_P(A)$ is a unital $*$-representation of A on E.

[This is a simple consequence of the generalities that govern spectral integrals. In fact,

$$\pi_P(AB) = \int_{\Delta(A)} \widehat{AB}(\omega)\,dP(\omega)$$

$$= \int_{\Delta(A)} \omega(AB)\,dP(\omega)$$

$$= \int_{\Delta(A)} \omega(A)\,\omega(B)\,dP(\omega)$$

$$= \left(\int_{\Delta(A)} \omega(A)\,dP(\omega)\right)\left(\int_{\Delta(A)} \omega(B)\,dP(\omega)\right)$$

$$= \pi_P(A)\,\pi_P(B)$$

and

$$\pi_P(A)^* = \left(\int_{\Delta(A)} \hat{A}(\omega)\,dP(\omega)\right)^*$$

$$= \int_{\Delta(A)} \overline{\hat{A}(\omega)}\,dP(\omega)$$

$$= \int_{\Delta(A)} \overline{\omega(A)}\,dP(\omega)$$

$$= \int_{\Delta(A)} \omega(A^*)\,dP(\omega)$$

$$= \pi_P(A^*).]$$

<u>N.B.</u> If $\mathcal{A}$ is unital, then

$$\pi_P(1_{\mathcal{A}}) = \int_{\Delta(\mathcal{A})} \omega(1_{\mathcal{A}})\,dP(\omega) = \int_{\Delta(\mathcal{A})} 1\,dP(\omega) = \mathrm{id}_E.$$

Terminology: P is <u>regular</u> if $\forall\ S \in \mathrm{Bor}\ \Delta(\mathcal{A})$,

$$P(S) = \sup\{P(K) : K \subset S,\ K \text{ compact}\}.$$

[Note: This is "inner" regularity. It forces "outer" regularity:

$$P(S) = \inf\{P(U) : U \supset S,\ U \text{ open}\}.]$$

We then claim that π_P is nondegenerate if P is regular. Proof: There are two points.

(i) First, by regularity,

$$\mathrm{id}_E = P(\Delta(A)) = \sup_{K\subset\Delta(A)} P(K).$$

Therefore

$$\{P(K)x : K \subset \Delta(A), x \in E\}$$

is total.

(ii) Second, if $f \equiv 1$ on $K \subset \Delta(A)$ $(f \in C(\Delta(A))$ (A unital) or $f \in C_\infty(\Delta(A))$ (A nonunital)), then

$$\begin{aligned} P(K) &= \textstyle\int_{\Delta(A)} \chi_K dP \\ &= \textstyle\int_{\Delta(A)} f\chi_K dP \\ &= (\textstyle\int_{\Delta(A)} f dP)(\int_{\Delta(A)} \chi_K dP) \\ &= (\textstyle\int_{\Delta(A)} f dP) P(K) \end{aligned}$$

$\Rightarrow$

$$\mathrm{Ran}\, P(K) \subset \mathrm{Ran} \textstyle\int_{\Delta(A)} f dP.$$

Therefore

$$\{(\textstyle\int_{\Delta(A)} \hat{A} dP)x : A \in A, x \in E\}$$

is total, i.e., π_P is nondegenerate.

We thus have a map $P \to \pi_P$ from the set of regular E-valued spectral measures on $\Delta(A)$ to the set of nondegenerate *-representations of A on E.

9.54 <u>SNAG</u> The map $P \to \pi_P$ is bijective.

The details are relatively straightforward. Given $x,y \in E$, set

$$\mu_{x,y}(S) = \langle x,P(S)y\rangle \quad (S \in \text{Bor } \Delta(A)).$$

Then $\mu_{x,y}$ is a complex Radon measure on $\Delta(A)$.

If now P and Q are regular and if $\pi_P = \pi_Q$, then $P = Q$. Thus define $\nu_{x,y}$ per Q: $\forall\, A \in A$,

$$\int_{\Delta(A)} \hat{A}\, d\mu_{x,y} = \langle x,\pi_P(A)y\rangle$$

$$= \langle x,\pi_Q(A)y\rangle$$

$$= \int_{\Delta(A)} \hat{A}\, d\nu_{x,y}$$

$\Rightarrow$

$$\mu_{x,y} = \nu_{x,y} \quad (\forall\, x,y \in E)$$

$\Rightarrow$

$$P(S) = Q(S) \quad (\forall\, S \in \text{Bor } \Delta(A))$$

$\Rightarrow$

$$P = Q.$$

Therefore the map $P \to \pi_P$ is injective.

To prove surjectivity, assume initially that A is unital (so $\Delta(A)$ is compact) and let $\pi: A \to \mathcal{B}(E)$ be a nondegenerate $*$-representation (so $\pi(1_A) = \mathrm{id}_E$) -- then by the Riesz representation theorem, $\forall\, x,y \in E$, one can find a unique complex Radon measure $\mu_{x,y}$ on $\Delta(A)$ such that $\forall\, A \in A$,

$$\int_{\Delta(A)} \hat{A}\, d\mu_{x,y} = \langle x, \pi(A) y\rangle.$$

Since $\forall\, S \in \mathrm{Bor}\, \Delta(A)$,

$$\begin{aligned} |\mu_{x,y}(S)|^2 &\leq \mu_{x,x}(S)\mu_{y,y}(S) \\ &\leq \mu_{x,x}(\Delta(A))\mu_{y,y}(\Delta(A)) \\ &= ||x||^2\ ||y||^2, \end{aligned}$$

there exists a unique operator $P(S) \in \mathcal{B}(E)$ such that

$$\mu_{x,y}(S) = \langle x, P(S) y\rangle.$$

It is clear that $P(S)$ is selfadjoint and idempotent, i.e., $P(S)$ is an orthogonal projection. Moreover, the assignment

$$\left[\begin{array}{l} \mathrm{Bor}\, \Delta(A) \to \mathrm{Pro}\, E \\ \qquad\quad S \to P(S) \end{array}\right.$$

is a regular spectral measure on $\Delta(A)$. Finally, $\forall\, A \in A$,

$$\begin{aligned} \langle x, \pi_P(A) y\rangle &= \langle x, (\textstyle\int_{\Delta(A)} \hat{A}\, dP) y\rangle \\ &= \textstyle\int_{\Delta(A)} \hat{A}\, d\mu_{x,y} \\ &= \langle x, \pi(A) y\rangle, \end{aligned}$$

implying thereby that $\pi_P = \pi$.

It remains to consider a nonunital A. So let $\pi:A \to \mathcal{B}(E)$ be a nondegenerate *-representation. Extend π to A^+ by writing

$$\pi^+(A,\lambda) = \pi(A) + \lambda\, \mathrm{id}_E.$$

Then $\pi^+:A^+ \to \mathcal{B}(E)$ is a nondegenerate *-representation, thus $\exists$ a regular spectral measure

$$P^+:\mathrm{Bor}\ \Delta(A^+) \to \mathrm{Pro}\ E$$

such that $\pi_{P^+} = \pi^+$. But

$$\Delta(A^+) \approx \Delta(A)^+ \quad \text{(cf. §2)}.$$

And

$$\begin{aligned} &P^+(\{\infty\})\pi(A) \\ &= \int_{\Delta(A)^+} \chi_{\{\infty\}} \hat{A}\, dP^+ \\ &= 0 \end{aligned}$$

$\Rightarrow$

$$P^+(\{\infty\}) = 0$$

$\Rightarrow$

$$\begin{aligned} P^+(\Delta(A)) &= P^+(\Delta(A)^+ - \{\infty\}) \\ &= P^+((\Delta(A)^+ - \{\infty\}) \cup \{\infty\}) \\ &= P^+(\Delta(A)^+) \\ &= \mathrm{id}_E. \end{aligned}$$

If now $P = P^+|\Delta(A)$, then

$$P:\mathrm{Bor}\ \Delta(A) \to \mathrm{Pro}\ E$$

is a regular spectral measure such that $\pi_P = \pi$.

9.55 EXAMPLE Let A be a commutative C*-algebra. Suppose that $\mu \in M_1^+ (\Delta(A))$ (cf. 7.12). Take $E = L^2(\Delta(A),\mu)$ and define $\pi_\mu(A)$ by

$$(\pi_\mu(A)f)(\omega) = \hat{A}(\omega)f(\omega) \quad (f \in E).$$

Then π_μ is a nondegenerate *-representation of A on E and its associated spectral measure P_μ is the prescription

$$P_\mu(S)f = \chi_S f \quad (S \in \text{Bor } \Delta(A)).$$

Let

$$P:\text{Bor } \Delta(A) \to \text{Pro } E$$

be a regular spectral measure -- then the support of P, denoted spt P, is the set of all $\omega \in \Delta(A)$ such that $P(U) \neq 0$ $\forall$ open neighborhood of ω.

[Note: The support of P is a closed subset of $\Delta(A)$.]

N.B. If $\pi_P = \pi$, then spt P is called the spectrum of π.

9.56 LEMMA Suppose that $\pi:A \to \mathcal{B}(E)$ is a nondegenerate *-representation of A -- then Ker π consists of those $A \in A$ such that $\hat{A}$ vanishes on the spectrum of π.

[Note: π is faithful iff its spectrum is all of $\Delta(A)$.]

9.57 REMARK The machinery assembled for the proof of 9.54 and its consequences provides a direct route to the spectral theorem for normal operators.

§10. GNS

Let $\mathcal{A}$ be a Banach *-algebra -- then a linear functional $\omega:\mathcal{A} \to \underline{C}$ is positive if $\forall\ A \in \mathcal{A}$, $\omega(A^*A) \geq 0$.

10.1 LEMMA Let $\omega:\mathcal{A} \to \underline{C}$ be a positive linear functional -- then $\forall\ A,B \in \mathcal{A}$,

$$\omega(A^*B) = \overline{\omega(B^*A)}$$

and

$$|\omega(A^*B)|^2 \leq \omega(A^*A)\omega(B^*B).$$

N.B. Suppose that $\mathcal{A}$ is unital -- then $\forall\ A \in \mathcal{A}$,

$$\omega(A^*) = \overline{\omega(A)}$$

and

$$|\omega(A)|^2 \leq \omega(1_{\mathcal{A}})\omega(A^*A).$$

[Note: Therefore

$$\omega(1_{\mathcal{A}}) = 0 \Rightarrow \omega \equiv 0.]$$

10.2 EXAMPLE There are Banach *-algebras that have no nonzero positive linear functionals. Thus take any unital Banach algebra $\mathcal{A} \neq \{0\}$ and form the cartesian product $\mathcal{A} \times \mathcal{A}$. Introduce operations and norm by $(A_1,B_1) + (A_2,B_2) = (A_1 + A_2, B_1 + B_2)$, $\lambda(A,B) = (\lambda A,\bar{\lambda}B)$, $(A_1,B_1)\cdot(A_2,B_2) = (A_1A_2,B_2B_1)$, $(A,B)^* = (B,A)$, and $||(A,B)|| = \max(||A||,||B||)$ -- then $\mathcal{A} \times \mathcal{A}$ is a Banach *-algebra with unit

$$1_{\mathcal{A} \times \mathcal{A}} = (1_{\mathcal{A}},1_{\mathcal{A}}).$$

Since

$$(1_A,-1_A)^*(1_A,-1_A) = -(1_A,1_A),$$

it follows that every positive linear functional on $A \times A$ must vanish at $(1_A,1_A)$, hence from the above, must vanish identically.

If A is a C*-algebra, then positive linear functionals are continuous (cf. 7.3) but if A is just a Banach *-algebra, this need not be true.

10.3 EXAMPLE Let A be the Banach space $C[0,1]$, take the multiplication to be trivial ($fg = 0 \ \forall \ f,g$) and set $f^* = \bar{f}$ -- then A is a Banach *-algebra and every linear functional $\omega: A \to \underline{C}$ is positive. On the other hand, A is infinite dimensional, thus admits a discontinuous linear functional.

Let $\omega: A \to \underline{C}$ be a positive linear functional. Given $B \in A$, define $\omega^B: A \to \underline{C}$ by

$$\omega^B(A) = \omega(B^*AB) \qquad (A \in A).$$

10.4 LEMMA We have

$$|\omega^B(A)| \le \omega(B^*B)\gamma(A) \le \omega(B^*B)m(A)||A||.$$

[Looking ahead, the computation

$$\begin{aligned} |\omega^B(A)|^2 &= |\omega(B^*AB)|^2 \\ &\le \omega(B^*B)\omega(B^*A^*AB) \\ &= \omega(B^*B)\omega^B(A^*A) \end{aligned}$$

shows that ω^B satisfies condition H with

$$||\omega^B||_H \leq \omega(B^*B).$$

Therefore ω^B is representable (cf. 10.10), hence (cf. 10.12)

$$|\omega^B(A)| \leq ||\omega^B||_H \gamma(A)$$

$$\leq \omega(B^*B)\gamma(A).]$$

Let $\omega: A \to \underline{C}$ be a positive linear functional. Given $B, C \in A$, define $\omega^{B,C}: A \to \underline{C}$ by

$$\omega^{B,C}(A) = \omega(B^*AC) \qquad (A \in A).$$

10.5 <u>LEMMA</u> We have

$$|\omega^{B,C}(A)| \leq \omega(B^*B)^{1/2}\omega(C^*C)^{1/2}\gamma(A)$$

$$\leq \omega(B^*B)^{1/2}\omega(C^*C)^{1/2}m(A)||A||.$$

<u>PROOF</u> In fact,

$$|\omega^{B,C}(A)|^2 = |\omega(B^*AC)|^2$$

$$\leq \omega(B^*B)\omega(C^*A^*AC) \qquad \text{(cf. 10.1)}$$

$$= \omega(B^*B)\omega^C(A^*A)$$

=>

$$|\omega^{B,C}(A)| \leq \omega(B^*B)^{1/2}\omega^C(A^*A)^{1/2}$$

$$\leq \omega(B^*B)^{1/2}\omega(C^*C)^{1/2}\gamma(A^*A)^{1/2}$$

$$= \omega(B^*B)^{1/2}\omega(C^*C)^{1/2}(\gamma(A)^2)^{1/2}$$

$$= \omega(B^*B)^{1/2}\omega(C^*C)^{1/2}\gamma(A)$$

$$\leq \omega(B^*B)^{1/2}\omega(C^*C)^{1/2}m(A)\,||A||.$$

N.B. Recall from 9.41 that

$$\gamma(A) \leq m(A)\,||A|| \qquad (A \in A).$$

10.6 THEOREM Suppose that A has a left approximate unit (cf. 4.1) -- then all positive linear functionals on A are continuous.

PROOF Let $\omega: A \to \underline{C}$ be a positive linear functional.

Step 1: $\omega|{*}\text{-rad}\, A \equiv 0$. Thus let $A \in {*}\text{-rad}\, A$ and using 4.6, write $A = B^*C$, where $B \in A$, $C \in \overline{AA} \subset {*}\text{-rad}\, A$. Repeat and write $C^* = E^*D^*$, where $E^* \in A$, $D^* \in \overline{AC^*} \subset {*}\text{-rad}\, A$, so $C = DE$, where $D \in \overline{AC^{**}} \subset {*}\text{-rad}\, A$, $E \in A$. Therefore $A = B^*DE$ and

$$\begin{aligned}
|\omega(A)|^2 &= |\omega(B^*DE)|^2 \\
&= |\omega^{B,E}(D)|^2 \\
&\leq \omega(B^*B)^{1/2}\omega(E^*E)^{1/2}\gamma(D) \\
&= \omega(B^*B)^{1/2}\omega(E^*E)^{1/2}0 = 0.
\end{aligned}$$

Step 2: Since ω drops to $A/{*}\text{-rad}\, A$, it can be assumed that A is $*$-semisimple, hence semisimple (cf. 9.38). In particular: The involution $*: A \to A$ is continuous (cf. 9.21).

Step 3: Let $A_n \in A$ be a sequence in A such that $A_n \to 0$. Claim: $\omega(A_n) \to 0$ (=> ω is continuous). To see this, use 4.8 to first write $A_n = A^*B_n^*$, where $B_n^* \to 0$. But then, thanks to the continuity of the involution, $B_n \to 0$, thus by a second application of 4.8, we can write $B_n = B^*C_n^*$, where $C_n^* \to 0$, so $A_n = A^*C_nB$ and $C_n \to 0$.

Therefore

$$|\omega(A_n)|^2 = |\omega(A^*C_nB)|^2$$
$$= |\omega^{A,B}(C_n)|^2$$
$$\leq \omega(A^*A)\omega(B^*B)m(A)^2||C_n||^2$$
$$\to 0 \quad (n \to \infty).$$

10.7 LEMMA Suppose that A is unital -- then all positive linear functionals on A are continuous. Moreover, if $\omega:A \to \underline{C}$ is one such, then

$$||\omega|| \leq \omega(1_A)m(A).$$

PROOF $\forall\ A \neq 0$,

$$\frac{|\omega(A)|}{||A||} = \frac{|\omega(1_A^* A 1_A)|}{||A||}$$
$$= \frac{|\omega^{1_A,1_A}(A)|}{||A||}$$
$$\leq \omega(1_A)^{1/2}\omega(1_A)^{1/2}m(A)$$
$$= \omega(1_A)m(A)$$

=>

$$||\omega|| \leq \omega(1_A)m(A).$$

[Note: If $*:A \to A$ is isometric, then $\beta(A) = 1$ (cf. 9.41) and

$$m(A) \leq \beta(A)^{1/2} = 1$$

=>

$$||\omega|| \leq \omega(1_A)$$

which can be improved to

$$||\omega|| = \omega(1_A)$$

when A is a C*-algebra (cf. 7.4).]

10.8 EXAMPLE It is not always true that $||\omega|| \leq \omega(1_A)$. Thus let $A = \mathcal{B}(\underline{C}^2)$, where $\underline{C}^2$ has the norm

$$\left|\left|\begin{bmatrix} z_1 \\ z_2 \end{bmatrix}\right|\right| = |z_1| + t|z_2| \quad (t > 2).$$

Represent the elements of A as 2-by-2 complex matrices $[\lambda_{ij}]$ and put $[\lambda_{ij}]^* = [\bar{\lambda}_{ji}]$ -- then A is a Banach *-algebra with a continuous (but not isometric) involution. Define $\omega: A \to \underline{C}$ by $\omega([\lambda_{ij}]) = \sum_{i,j} \lambda_{ij}$ -- then ω is a positive linear functional on A such that $\omega(1_A) = 2$. If

$$A = \begin{bmatrix} 0 & t^2 \\ 1 & 0 \end{bmatrix},$$

then

$$\left|\left| A \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}\right|\right| = t \left|\left|\begin{bmatrix} z_1 \\ z_2 \end{bmatrix}\right|\right|$$

$\Rightarrow$

$$||A|| = t$$

$\Rightarrow$

$$||\omega|| \geq \frac{\omega(A)}{||A||} = \frac{t^2 + 1}{t} > t > 2 = \omega(1_A).$$

10.9 REMARK Write A_+ for the set of all finite sums of the form $\sum_{i=1}^{n} A_i^* A_i$ -- then the linear span of A_+ is A^2 (a.k.a. the linear span of the A^*B). Proof:

$$A^*B = \frac{1}{4} \sum_{k=0}^{3} (\sqrt{-1})^{-k} (A + (\sqrt{-1})^k B)^* (A + (\sqrt{-1})^k B).$$

If A^2 is not closed or is closed but not of finite codimension, then one can use a Hamel basis for A to construct a discontinuous linear functional ω that vanishes on A^2. Such an ω is necessarily positive.

Note: [It therefore follows that a necessary condition for the continuity of all positive linear functionals on a Banach $*$-algebra A is that A^2 be closed of finite codimension.]

Let $\omega: A \to \underline{C}$ be a positive linear functional.

- ω is said to be representable if $\exists$ a topologically cyclic $*$-representation π of A on E with a topologically cyclic vector $x \in E$ such that

$$\omega(A) = \langle x, \pi(A)x \rangle \quad (A \in A).$$

- ω is said to satisfy condition H if

$$||\omega||_H = \sup\{|\omega(A)|^2 : \omega(A^*A) \leq 1\} < \infty.$$

10.10 <u>THEOREM</u> Let $\omega : \mathcal{A} \to \underline{C}$ be a positive linear functional -- then ω is representable iff ω satisfies condtion H.

<u>N.B.</u> The equivalences in 10.10 are of central importance for the theory. One direction is immediate, viz.:

10.11 <u>LEMMA</u> Suppose that $\omega : \mathcal{A} \to \underline{C}$ is representable -- then ω satisfies condition H.

<u>PROOF</u> By definition,

$$\omega(A) = \langle x, \pi(A)x \rangle \quad (A \in \mathcal{A}),$$

where $x \in E$ is topologically cyclic. Therefore

$$\begin{aligned} |\omega(A)|^2 &= |\langle x, \pi(A)x \rangle|^2 \\ &\leq (||x|| \ ||\pi(A)x||)^2 \\ &= ||x||^2 \langle \pi(A)x, \pi(A)x \rangle \\ &= ||x||^2 \langle x, \pi(A^*A)x \rangle \\ &= ||x||^2 \omega(A^*A) \end{aligned}$$

=>

$$||\omega||_H \leq ||x||^2.$$

I.e.: ω satisfies condition H.

[Note: Since $x \in E$ is topologically cyclic, we have

$$||x||^2 = ||\omega||_H.$$

In fact,

$$||x||^2 = \sup\{|<x,y>|^2 : ||y|| \le 1\}$$

$$= \sup\{|<x,\pi(A)x>|^2 : ||\pi(A)x|| \le 1\}$$

$$= \sup\{|\omega(A)|^2 : \omega(A^*A) \le 1\}$$

$$= ||\omega||_H.]$$

10.12 REMARK In view of 9.25, a representable ω is necessarily continuous. [Note: This can be pinned down in that

$$|\omega(A)|^2 \le ||x||^4 \; ||\pi(A^*A)||$$

$$\le ||x||^4 \; \gamma(A^*A)$$

$$= ||x||^4 \; \gamma(A)^2$$

=>

$$|\omega(A)| \le ||x||^2 \; \gamma(A)$$

$$= ||\omega||_H \; \gamma(A)$$

$$\le ||\omega||_H \; m(A) \; ||A||$$

=>

$$||\omega|| \le ||\omega||_H \; m(A).]$$

10.13 LEMMA Suppose that $\omega: A \to \underline{C}$ is representable -- then ω is hermitian: $\forall \; A \in A$,

$$\omega(A^*) = \overline{\omega(A)}.$$

PROOF In fact,

$$\begin{aligned}\omega(A^*) &= \langle x,\pi(A^*)x\rangle \\ &= \langle x,\pi(A)^*x\rangle \\ &= \langle \pi(A)x,x\rangle \\ &= \overline{\langle x,\pi(A)x\rangle} \\ &= \overline{\omega(A)}.\end{aligned}$$

The proof that

"condition H" => "representable"

is a special case of the Kolmogorov construction. However, proceeding to the details, we shall first look for conditions on a Banach *-algebra that are sufficient to ensure that all its positive linear functionals satisfy condition H.

10.14 LEMMA If A is unital, then every positive linear functional $\omega:A \to \underline{C}$ satisfies condition H and $||\omega||_H = \omega(1_A)$.

PROOF To begin with,

$$1_A^* = 1_A 1_A^* = 1_A^{**} 1_A^* = (1_A 1_A^*)^* = 1_A^{**} = 1_A.$$

Accordingly,

$$\begin{aligned}\omega(A^*) &= \omega(A^* 1_A) \\ &= \overline{\omega(1_A^* A)} \qquad \text{(cf. 10.1)} \\ &= \overline{\omega(A)}.\end{aligned}$$

Therefore

$$
\begin{aligned}
|\omega(A)|^2 &= |\overline{\omega(A)}|^2 \\
&= |\omega(A^*)|^2 \\
&= |\omega(A^* 1_A)|^2 \\
&\leq \omega(A^*A)\,\omega(1_A^* 1_A) \qquad \text{(cf. 10.1)} \\
&= \omega(A^*A)\,\omega(1_A)
\end{aligned}
$$

=>

$$||\omega||_H \leq \omega(1_A).$$

On the other hand,

$$
\begin{aligned}
\omega(1_A)^2 &\leq ||\omega||_H\,\omega(1_A^* 1_A) \\
&= ||\omega||_H\,\omega(1_A)
\end{aligned}
$$

=>

$$\omega(1_A) \leq ||\omega||_H.$$

[Note: If $\omega(1_A) = 0$, then ω is the zero functional and matters are trivial.]

10.15 LEMMA If A is a C*-algebra, then every positive linear functional $\omega: A \to \underline{C}$ satisfies condition H and $||\omega||_H = ||\omega||$.

PROOF Work with an approximate unit $\{e_i : i \in I\}$ per 1.20: $\forall\, A \in A$,

$$|\omega(A)|^2 = \lim_{i \in I} |\omega(e_i A)|^2$$

$$= \lim_{i\in I} |\omega(e_i^*A)|^2$$

$$\leq \liminf_{i\in I} \omega(e_i^*e_i)\omega(A^*A) \qquad \text{(cf. 10.1)}$$

$$= \liminf_{i\in I} \omega(e_i^2)\omega(A^*A)$$

$$\leq ||\omega||\omega(A^*A)$$

=>

$$||\omega||_H \leq ||\omega||.$$

On the other hand,

$$\omega\left(\frac{e_i^*}{||\omega||^{1/2}} \frac{e_i}{||\omega||^{1/2}}\right) \leq 1$$

=>

$$||\omega||_H \geq \omega\left(\frac{e_i}{||\omega||^{1/2}}\right)^2 = \frac{\omega(e_i)^2}{||\omega||}$$

=>

$$||\omega||_H \geq \lim_{i\in I} \frac{\omega(e_i)^2}{||\omega||}$$

$$= \frac{||\omega||^2}{||\omega||} \qquad \text{(cf. 7.9)}$$

$$= ||\omega||.$$

The preceding lemmas are special cases of the following result.

10.16 <u>THEOREM</u> Suppose that A has a left approximate unit (cf. 4.1) — then every positive linear functional $\omega:A \to \underset{\sim}{C}$ satisfies condition H.

<u>PROOF</u> There is no loss of generality in taking A $*$-semisimple (see the proof of 10.6), so the involution $*:A \to A$ is continuous (cf. 9.21). If now $\{e_i : i \in I\}$ is a left approximate unit per A and if $M > 0 : ||e_i|| \le M$ $(i \in I)$, then arguing as in 10.15 (bearing in mind that ω is continuous (cf. 10.6)), $\forall$ A $\in A$, we have

$$\begin{aligned}|\omega(A)|^2 &= \lim_{i\in I} |\omega(e_iA)|^2 \\ &= \lim_{i\in I} |\omega((e_i^*)^*A)|^2 \\ &\le \liminf_{i\in I} \omega(e_ie_i^*)\omega(A^*A) \\ &\le \liminf_{i\in I} ||e_ie_i^*||\ ||\omega||\omega(A^*A) \\ &\le \liminf_{i\in I} ||e_i||\ ||e_i^*||\ ||\omega||\omega(A^*A) \\ &\le \liminf_{i\in I} ||e_i||^2\ \beta(A)\ ||\omega||\omega(A^*A) \\ &\le M^2\beta(A)\ ||\omega||\omega(A^*A)\end{aligned}$$

$\Rightarrow$

$$||\omega||_H \le M^2\beta(A).$$

[Note: Here $\beta(A)$ is the norm of the involution $*:A \to A$ (cf. 9.29).]

Returning to 10.10, assume that ω satisfies condition H and put

$$A^\omega = A/A_\omega,$$

where $\mathcal{A}_\omega$ is the left ideal

$$\{A \in \mathcal{A} : \omega(A^*A) = 0\} \quad (\text{cf. } 10.1).$$

Given $\begin{cases} A \in \mathcal{A} \\ B \in \mathcal{A} \end{cases}$, write $\begin{cases} A^\omega \\ B^\omega \end{cases}$ in place of $\begin{cases} A + \mathcal{A}_\omega \\ B + \mathcal{A}_\omega \end{cases}$ -- then the prescription

$$\begin{cases} \langle\ ,\ \rangle_\omega : \mathcal{A}^\omega \times \mathcal{A}^\omega \to \underline{C} \\ \langle A^\omega, B^\omega\rangle_\omega = \omega(A^*B) \end{cases}$$

equips $\mathcal{A}^\omega$ with the structure of a pre-Hilbert space ($\Rightarrow ||A^\omega||_\omega = \omega(A^*A)^{1/2}$). Define π^ω by

$$\pi^\omega(A)B^\omega = (AB)^\omega.$$

Then

$$\pi^\omega : \mathcal{A} \to \mathcal{L}_*(\mathcal{A}^\omega)$$

is a $*$-homomorphism.

<u>N.B.</u> $\pi^\omega(A)$ has a formal adjoint, viz. $\pi^\omega(A^*)$. Proof:

$$\begin{aligned} \langle\pi^\omega(A)B^\omega, C^\omega\rangle_\omega &= \langle(AB)^\omega, C^\omega\rangle_\omega \\ &= \omega((AB)^*C) \\ &= \omega(B^*A^*C) \\ &= \langle B^\omega, (A^*C)^\omega\rangle_\omega \\ &= \langle B^\omega, \pi^\omega(A^*)C\rangle_\omega \end{aligned}$$

$\Rightarrow$

$$\pi^\omega(A)^* = \pi^\omega(A^*).$$

10.17 LEMMA $\forall A \in \mathcal{A}$, $\pi^{\omega}(A)$ is bounded.

PROOF This is because $\pi^{\omega}(A)$ can be written as a finite linear combination of unitary elements of $L_{*}(A^{\omega})$ (cf. 9.31 and subsequent discussion).

[Note: It is thus a corollary that $\forall A \in \mathcal{A}$,

$$\sup\{\omega(B^*A^*AB) : B \in \mathcal{A}, \omega(B^*B) \leq 1\} < \infty].$$

Let E^{ω} be the Hilbert space completion of A^{ω} -- then π^{ω} extends by continuity to a $*$-representation of $\mathcal{A}$ on E^{ω}, denoted still by π^{ω}. Since ω satisfies condition H, it vanishes on $\mathcal{A}_{\omega}$, hence induces a linear functional on A^{ω} which is continuous w.r.t. $||\cdot||_{\omega}$, thus extends to E^{ω} with the same bound, namely $||\omega||_H^{1/2}$: $\forall A \in \mathcal{A}$,

$$|\omega(A)| \leq ||\omega||_H^{1/2}\omega(A^*A)^{1/2} = ||\omega||_H^{1/2}||A^{\omega}||_{\omega}.$$

Owing to the Riesz representation theorem, $\exists$ a unique vector $x_{\omega} \in E^{\omega}$ such that

$$\omega(A) = \langle x_{\omega}, A^{\omega}\rangle_{\omega}.$$

Here

$$||x_{\omega}||_{\omega}^2 = ||\omega||_H.$$

10.18 LEMMA $\forall A \in \mathcal{A}$,

$$\pi^{\omega}(A)x_{\omega} = A^{\omega}.$$

PROOF $\forall B \in \mathcal{A}$,

$$\begin{aligned}&\langle \pi^{\omega}(A)x_{\omega} - A^{\omega}, B^{\omega}\rangle_{\omega}\\ &\quad = \langle x_{\omega}, \pi^{\omega}(A)^*B^{\omega}\rangle_{\omega} - \langle A^{\omega}, B^{\omega}\rangle_{\omega}\end{aligned}$$

$$= \langle x_\omega, \pi^\omega(A^*)B^\omega\rangle_\omega - \omega(A^*B)$$

$$= \langle x_\omega, (A^*B)^\omega\rangle_\omega - \omega(A^*B)$$

$$= \omega(A^*B) - \omega(A^*B)$$

$$= 0.$$

To summarize: π^ω is a topologically cyclic *-representation of $\mathcal{A}$ on E^ω with topologically cyclic vector $x_\omega \in E^\omega$ such that

$$\omega(A) = \langle x_\omega, \pi^\omega(A)x_\omega\rangle_\omega \quad (A \in \mathcal{A}).$$

Therefore ω is representable, which completes the proof of 10.10.

[Note: π^ω is called the GNS representation attached to ω.]

10.19 EXAMPLE Take $\mathcal{A}$ unital -- then $\forall\ A \in \mathcal{A}$,

$$\pi^\omega(A)1_A^\omega = A^\omega,$$

so 1_A^ω is topologically cyclic. And

$$\langle 1_A^\omega, \pi^\omega(A)1_A^\omega\rangle_\omega$$

$$= \langle 1_A^\omega, A^\omega\rangle_\omega$$

$$= \omega((1_A^\omega)^*A)$$

$$= \omega(A)$$

$$\Rightarrow$$

$$x_\omega = 1_A^\omega.$$

10.20 LEMMA Suppose that π is a topologically cyclic *-representation of A on E with topologically cyclic vector $x \in E$ -- then π is unitarily equivalent to π^{ω} for some ω satisfying condition H.

PROOF Define $\omega: A \to \underline{C}$ by

$$\omega(A) = \langle x, \pi(A)x \rangle \qquad (A \in A).$$

Then ω is representable, hence satisfies condition H (cf. 10.11), so

$$\omega(A) = \langle x_{\omega}, \pi^{\omega}(A)x_{\omega} \rangle_{\omega} \qquad (A \in A).$$

Now quote 9.7.

[Note: The trivial *-representation on the zero dimensional Hilbert space "is" $\pi^{\omega=0}$.]

10.21 LEMMA Suppose that π is a nondegenerate *-representation of A on E -- then $\exists$ a set Ω of representable positive linear functionals ω on A such that π is unitarily equivalent to $\bigoplus_{\omega \in \Omega} \pi^{\omega}$ and $\forall$ $A \in A$,

$$||\pi(A)|| = \sup_{\omega \in \Omega} ||\pi^{\omega}(A)||.$$

[This is an immediate consequence of 9.5 and 10.20.]

Suppose that $\omega: A \to \underline{C}$ is a positive linear functional which satisfies condition H -- then ω is said to be a state if $||\omega||_H = 1$.

[Note: This terminology is consistent with that used for C*-algebras (cf. 10.15).]

If $\omega \neq 0$ satisfies condition H, then $\forall$ $t > 0$, $t\omega$ satisfies condition H:

$$||t\omega||_H = t||\omega||_H.$$

Also $\forall$ A $\in$ $\mathcal{A}$,

$$\omega(A) = \langle x^{\omega}, \pi^{\omega}(A) x_{\omega} \rangle_{\omega}$$

$$\Rightarrow$$

$$t\omega(A) = \langle \sqrt{t}\, x^{\omega}, \pi^{\omega}(A) \sqrt{t}\, x_{\omega} \rangle.$$

And $\pi^{t\omega}$ is unitarily equivalent to π^{ω} via the arrow $E^{t\omega} \to E^{\omega}$ that sends $A^{t\omega}$ to $\sqrt{t}\, A^{\omega}$ $(A \in \mathcal{A})$.

10.22 THEOREM Every nontrivial topologically cyclic *-representation of $\mathcal{A}$ is unitarily equivalent to π^{ω} for some state ω (cf. 10.20).

PROOF If $\omega \neq 0$, then

$$\frac{\omega}{||\omega||_H}$$

is a state.

If

$$\begin{cases} \omega_1 : \mathcal{A} \to \underline{C} \\ \omega_2 : \mathcal{A} \to \underline{C} \end{cases}$$

are positive linear functionals satisfying condition H, write $\omega_1 \geq \omega_2$ if $\omega_1 - \omega_2$ is positive.

10.23 LEMMA If $\omega, \omega' : \mathcal{A} \to \underline{C}$ satisfy condition H and if $\omega \geq \omega'$, then $\exists$ $T \in \pi^{\omega}(A)'$ $(0 \leq T \leq I)$ such that

$$\omega'(A) = \langle x_{\omega}, \pi^{\omega}(A) T x_{\omega} \rangle_{\omega} \quad (A \in \mathcal{A}).$$

PROOF Noting that

$$\omega \geq \omega' \Rightarrow \mathcal{A}_{\omega} \subset \mathcal{A}_{\omega'},$$

put

$$\langle A^{\omega}, B^{\omega} \rangle_{\omega'} = \omega'(A^*B).$$

Then

$$|\omega'(A^*B)|^2 \leq \omega'(A^*A)\,\omega'(B^*B)$$

$$\leq \omega(A^*A)\,\omega(B^*B)$$

$$= ||A^{\omega}||_{\omega}^2 \; ||B^{\omega}||_{\omega}^2.$$

Therefore $\langle \; , \; \rangle_{\omega'}$ can be extended to $E^{\omega} \times E^{\omega}$. Fix $T \in \mathcal{B}(E^{\omega})$:

$$\omega'(A^*B) = \langle A^{\omega}, TB^{\omega} \rangle_{\omega}.$$

Then

$$\omega \geq \omega' \geq 0 \Rightarrow 0 \leq T \leq I.$$

And $\forall$ A,B,C,

$$\langle A^{\omega}, \pi^{\omega}(C)TB^{\omega} \rangle_{\omega}$$

$$= \langle \pi^{\omega}(C^*)A^{\omega}, TB^{\omega} \rangle_{\omega}$$

$$= \langle (C^*A)^{\omega}, TB^{\omega} \rangle_{\omega}$$

$$= \omega'(A^*CB)$$

$$= \langle A^{\omega}, T(CB)^{\omega} \rangle_{\omega}$$

$$= \langle A^{\omega}, T\pi^{\omega}(C)B^{\omega} \rangle_{\omega}$$

=>

$$T \in \pi^{\omega}(A)'.$$

Finally, choose a sequence $\{A_n\}$ in A such that $A_n^{\omega} \to x_{\omega}$:

$$\begin{aligned}
|\omega'(AA_n - A)|^2 &\le ||\omega'||_H \, \omega'((AA_n - A)^*(AA_n - A)) \\
&\le ||\omega'||_H \, \omega((AA_n - A)^*(AA_n - A)) \\
&= ||\omega'||_H \, ||(AA_n - A)^{\omega}||_{\omega}^2 \\
&= ||\omega'||_H \, ||\pi^{\omega}(A)(A^{\omega} - x_{\omega})||_{\omega}^2 \quad \text{(cf. 10.18)} \\
&\le ||\omega'||_H \, ||\pi^{\omega}(A)||^2 \, ||A_n^{\omega} - x_{\omega}||_{\omega}^2 \\
&\to 0 \ (n \to \infty)
\end{aligned}$$

=>

$$\begin{aligned}
\omega'(A) &= \lim_{n \to \infty} \omega'(AA_n) \\
&= \lim_{n \to \infty} \omega'((A^*)^*A_n) \\
&= \lim_{n \to \infty} \langle (A^*)^{\omega}, TA_n^{\omega} \rangle_{\omega} \\
&= \langle (A^*)^{\omega}, Tx_{\omega} \rangle_{\omega} \\
&= \langle \pi^{\omega}(A^*)x_{\omega}, Tx_{\omega} \rangle_{\omega} \quad \text{(cf. 10.18)} \\
&= \langle x_{\omega}, \pi^{\omega}(A)Tx_{\omega} \rangle_{\omega}.
\end{aligned}$$

[Note:

$$T \in \pi^{\omega}(A)' \Rightarrow T^{1/2} \in \pi^{\omega}(A)'$$

$\Rightarrow$

$$\begin{aligned}\omega'(A) &= \langle x_{\omega}, \pi^{\omega}(A) T x_{\omega}\rangle_{\omega} \\ &= \langle x_{\omega}, \pi^{\omega}(A) T^{1/2} T^{1/2} x_{\omega}\rangle_{\omega} \\ &= \langle T^{1/2} x_{\omega}, \pi^{\omega}(A) T^{1/2} x_{\omega}\rangle_{\omega}.]\end{aligned}$$

Suppose that $\omega \neq 0$ satisfies condition H -- then ω is said to be pure if

$$\omega \geq \omega \Rightarrow \omega' = t\omega \quad (\exists\ t \geq 0).$$

10.24 LEMMA If π^{ω} is topologically irreducible, then ω is pure.

PROOF Assuming that $\omega \geq \omega'$, produce $T \in \pi_{\omega}(A)'$ per 10.23:

$$0 \leq T \leq I \Rightarrow T = tI \ (0 \leq t \leq 1) \quad \text{(cf. 9.8)}.$$

So, $\forall\ A \in A$,

$$\begin{aligned}\omega'(A) &= \langle x_{\omega}, \pi^{\omega}(A)(tI) x_{\omega}\rangle_{\omega} \\ &= t\langle x_{\omega}, \pi^{\omega}(A) x_{\omega}\rangle_{\omega} \\ &= t\omega(A).\end{aligned}$$

10.25 LEMMA If ω is pure, then π^{ω} is topologically irreducible.

PROOF Let $P \in \pi^{\omega}(A)'$ be a nonzero orthogonal projection. Define $\omega': A \to \underline{C}$ by

$$\omega'(A) = \langle P x_{\omega}, \pi^{\omega}(A) P x_{\omega}\rangle_{\omega} \quad (A \in A).$$

Then

$$|\omega'(A)|^2 = |\langle Px_\omega, \pi^\omega(A)Px_\omega\rangle_\omega|^2$$

$$\leq ||Px_\omega||_\omega^2 \; ||\pi^\omega(A)Px_\omega||_\omega^2$$

$$= ||Px_\omega||^2 \; \langle Px_\omega, \pi^\omega(A^*A)Px_\omega\rangle_\omega$$

$$= ||Px_\omega||^2 \omega'(A^*A).$$

Therefore ω' satisfies condition H. And

$$\omega(A^*A) = ||\pi^\omega(A)x_\omega||_\omega^2$$

$$\geq ||P\pi^\omega(A)x_\omega||_\omega^2$$

$$= ||\pi^\omega(A)Px_\omega||_\omega^2$$

$$= \omega'(A^*A)$$

$$\Rightarrow$$

$$\omega \geq \omega'.$$

But ω is pure and ω' is nonzero, hence $\omega' = t\omega$ ($\exists\ t > 0$). So, $\forall\ A \in \mathcal{A}$,

$$0 = \omega'(A^*A) - t\omega(A^*A)$$

$$= \langle A^\omega, (P - tI)A^\omega\rangle_\omega.$$

Since A^ω is dense in E^ω, it follows that $P = tI \Rightarrow t = 1$, thus π^ω is topologically irreducible (cf. 9.9).

10.26 THEOREM Suppose that $\omega \neq 0$ satisfies condtion H -- then π^ω is topologically irreducible iff ω is pure.

PROOF Combine 10.24 and 10.25.

10.27 THEOREM Every topologically irreducible *-representation of A is unitarily equivalent to π^{ω} for some pure state ω (cf. 10.22).

PROOF If ω is pure, then

$$\frac{\omega}{||\omega||_H}$$

is a pure state. Proof:

$$\frac{\omega}{||\omega||_H} \geq \omega' \Rightarrow \omega = ||\omega||_H \omega'$$

$$\Rightarrow ||\omega||_H \omega' = t\omega$$

$$\Rightarrow \omega' = t\left(\frac{\omega}{||\omega||_H}\right).$$

10.28 EXAMPLE Take $A = \underline{L}_{\infty}(H)$ (H a complex Hilbert space) -- then the pure states are the ω_x ($||x|| = 1$), where

$$\omega_x(T) = \langle x, Tx \rangle \quad \text{(cf. 7.17)}.$$

Since the identity representation π_{∞} of $\underline{L}_{\infty}(H)$ on H is a topologically irreducible *-representation (cf. 9.8 $(\pi_{\infty}(\underline{L}_{\infty}(H))' = \underline{C}\ \mathrm{id}_H)$), it follows that $\forall$ x, π_{∞} is unitarily equivalent to π^{ω_x}. On the other hand, an arbitrary topologically irreducible *-representation π of $\underline{L}_{\infty}(H)$ is unitarily equivalent to some π^{ω_x} (cf. 10.27). Therefore π is unitarily equivalent to π_{∞}.

[Note: Every nondegenerate $*$-representation of $\underline{L}_{\infty}(H)$ is unitarily equivalent to a direct sum of copies of π_{∞}.]

10.29 THEOREM The $*$-radical of A is the intersection of the kernels of all the $*$-representations of A (cf. 9.34).

The proof requires some ancillary considerations. Thus given a nondegenerate $*$-representation π of A, let

$$\sigma(A) = ||\pi(A)||$$

and for any ω satisfying condition H, put

$$\sigma^{\omega}(A) = ||\pi^{\omega}(A)||.$$

10.30 LEMMA $\exists$ a set Ω_{π} of pure states with the property that $\forall$ $A \in A$,

$$\sigma(A) = \sup_{\omega\in\Omega_{\pi}} \sigma^{\omega}(A).$$

Grant this temporarily -- then

$$10.30 \Rightarrow 10.29.$$

For in the first place, it is obvious that

$$\bigcap_{\pi} \text{Ker } \pi \subset *\text{-rad } A,$$

where $\bigcap_{\pi}$ is taken over all the $*$-representations π of A. Conversely, let $A \in *\text{-rad } A$ -- then A is annihilated by all the π^{ω} (ω pure). In particular: Given π,

$$\forall\ \omega \in \Omega_{\pi},\ \sigma^{\omega}(A) = 0 \Rightarrow \sigma(A) = 0 \Rightarrow A \in \text{Ker } \pi.$$

Therefore

$$\bigcap_{\pi} \text{Ker } \pi = *\text{-rad } A.$$

Proceeding:

• Write $\underline{S}(A)$ for the set of positive linear functionals ω on A that satisfy condition H subject to $||\omega||_H \leq 1$ and write $\underline{S}(A,\sigma)$ for the subset of $\underline{S}(A)$ consisting of those ω such that $\sigma^\omega \leq \sigma$.

• Write $P(A)$ for the set of pure states ω on A and write $P(A,\sigma)$ for the subset of $P(A)$ consisting of those ω such that $\sigma^\omega \leq \sigma$.

N.B. $\underline{S}(A)$ and $\underline{S}(A,\sigma)$ are convex sets.

[Note: If ω_1,ω_2 both satisfy condition H, then so does $\omega_1 + \omega_2$ and

$$||\omega_1 + \omega_2||_H \leq ||\omega_1||_H + ||\omega_2||_H.$$

Therefore $\underline{S}(A)$ is convex. Suppose further that $\omega_1,\omega_2 \in \underline{S}(A,\sigma)$ and let $0 \leq \lambda \leq 1$ -- then

$$\begin{aligned}\sigma^{\lambda\omega_1 + (1-\lambda)\omega_2} &\leq \sup\{\sigma^{\lambda\omega_1},\sigma^{(1-\lambda)\omega_2}\}\\ &= \sup\{\sigma^{\omega_1},\sigma^{\omega_2}\}\\ &\leq \sigma.]\end{aligned}$$

10.31 LEMMA Suppose that $\omega:A \to \underline{C}$ satisfies condition H -- then ω is a pure state iff ω is a nonzero extreme point of $\underline{S}(A)$ (cf. 7.19).

10.32 LEMMA $P(A,\sigma)$ is the set of nonzero extreme points of $\underline{S}(A,\sigma)$ and $P(A,\sigma) \cup \{0\}$ is the set of all extreme points of $\underline{S}(A,\sigma)$.

Equip $\underline{S}(A,\sigma)$ with the topology of pointwise convergence -- then the image of

$\underline{S}(A,\sigma)$ in $\prod_{A \in A} \sigma(A)\underline{D}$ (product topology) under the natural embedding

$$\omega \rightarrow \{\omega(A) : A \in A\}$$

is closed, hence $\underline{S}(A,\sigma)$ is compact.

[Note: Recall that $\forall\ A \in A$,

$$\omega(A) = <x_\omega, \pi^\omega(A) x_\omega>_\omega$$

$$\Rightarrow$$

$$\begin{aligned}|\omega(A)| &\leq ||\pi^\omega(A)||\ ||x_\omega||_\omega^2 \\ &= \sigma^\omega(A)\,||\omega||_H \\ &\leq \sigma^\omega(A) \leq \sigma(A).]\end{aligned}$$

10.33 <u>LEMMA</u> The closed convex hull of $P(A,\sigma) \cup \{0\}$ is $\underline{S}(A,\sigma)$.

<u>PROOF</u> Apply the Krein-Milman theorem.

Let us pass now to the proof of 10.30 -- then $\exists$ a set Ω of representable positive linear functionals ω on A such that $\forall\ A \in A$,

$$\sigma(A) = \sup_{\omega \in \Omega} \sigma^\omega(A) \quad \text{(cf. 10.21)}$$

and we claim that

$$\sigma(A) = \sup_{\omega \in \Omega_\pi} \sigma^\omega(A),$$

where $\Omega_\pi = P(A,\sigma)$.

To this end, fix $A \in A$ and $\varepsilon > 0$ -- then it suffices to produce $\omega \in P(A,\sigma)$ such that

$$\sigma^\omega(A) > \sigma(A) - \varepsilon.$$

Choose $\omega_\varepsilon \in \Omega$:

$$\sigma^{\omega_\varepsilon}(A) > \sigma(A) - \varepsilon.$$

Because $\sigma^{t\omega_\varepsilon} = \sigma^{\omega_\varepsilon}$ $(t > 0)$, we can assume that ω_ε is a state, hence $\omega_\varepsilon \in \underline{S}(A,\sigma)$. Using 10.33, choose a net ω_i $(i \in I)$ that converges to ω_ε, where each ω_i is a convex combination of elements from $P(A,\sigma) \cup \{0\}$ -- then $\exists\ i_0 \in I$:

$$\sigma^{\omega_0}(A) > \sigma(A) - \varepsilon \ (\omega_0 \equiv \omega_{i_0}).$$

Let $\omega_1,\ldots,\omega_n$ be the elements of $P(A,\sigma)$ which occur with nonzero coefficients in the expression of ω_0 as a convex combination per the above. Since

$$\pi_{\omega_1 + \cdots + \omega_n}$$

is unitarily equivalent to a sub $*$-representation of

$$\pi_{\omega_1} \oplus \cdots \oplus \pi_{\omega_n}$$

with

$$\sigma^{\omega_1 + \cdots + \omega_n} \le \sup\{\sigma^{\omega_1},\ldots,\sigma^{\omega_n}\},$$

there is an index $k \in \{1,\ldots,n\}$:

$$\sigma^{\omega_k}(A) > \sigma(A) - \varepsilon.$$

Therefore

$$\sigma = \sup_{P(A,\sigma)} \sigma^{\omega}$$

as claimed.

10.34 <u>REMARK</u> It is false in general that a nondegenerate $*$-representation π

decomposes into a direct sum of topologically irreducible *-representations. However, on the basis of the preceding discussion, $\forall\ A \in A$,

$$||\pi(A)|| = ||\bigoplus_{\omega\in\Omega_\pi} \pi^\omega(A)||.$$

Set

$$\pi_{UN} = \bigoplus_{\omega} \pi^\omega,$$

where ω ranges over those positive linear functionals that satisfy condition H (meaningful since $\forall\ A \in A$, $||\pi^\omega(A)|| \leq \gamma(A)$) -- then π_{UN} is a nondegenerate *-representation of A. It is "universal" in the sense that every nondegenerate *-representation of A is unitarily equivalent to a sub *-representation of a multiple of π_{UN}.

N.B. We have

$$*\text{-rad } A = \text{Ker } \pi_{UN}$$

and $\forall\ A \in A$,

$$\gamma(A) = ||\pi_{UN}(A)||.$$

Therefore the extension $\bar{\pi}_{UN}$ of π_{UN} to a *-representation of $C^*(A)$ is isometric (cf. 9.44), so the image

$$\bar{\pi}_{UN}(C^*(A))$$

is a norm closed *-subalgebra of $E_{UN} = \bigoplus_{\omega} E^\omega$.

[Note: Suppose that A is *-semisimple:

$$*\text{-rad } A = \{0\}.$$

Then π_{UN} is a faithful *-representation of $\mathcal{A}$.]

10.35 RAPPEL If $\mathcal{A}$ is a C*-algebra, then every positive linear functional $\omega:\mathcal{A} \to \underline{C}$ satisfies condition H (cf. 10.15).

10.36 LEMMA Suppose that $\mathcal{A}$ is a C*-algebra and let $A \in \mathcal{A}$ be nonzero -- then $\exists$ a topologically irreducible *-representation π of $\mathcal{A}$ such that

$$||\pi(A)|| = ||A||,$$

hence $\mathcal{A}$ is *-semisimple.

PROOF Choose $\omega \in P(\mathcal{A})$:

$$\begin{aligned} \omega((A^*A)^2) &= ||(A^*A)^2|| \quad \text{(cf. 7.25)} \\ &= ||A||^4. \end{aligned}$$

Then

$$\begin{aligned} ||A||^2 &= \omega((A^*A)^2)^{1/2} \\ &= \omega((A^*A)(A^*A))^{1/2} \\ &= ||(A^*A)^{\omega}||_{\omega} \\ &= ||\pi^{\omega}(A^*)A^{\omega}||_{\omega} \\ &= ||\pi^{\omega}(A)^*A^{\omega}||_{\omega} \\ &\leq ||\pi^{\omega}(A)^*|| \; ||A^{\omega}||_{\omega} \\ &= ||\pi^{\omega}(A)|| \; ||A^{\omega}||_{\omega}. \end{aligned}$$

But

$$||A^{\omega}||_{\omega} = \omega(A^*A)^{1/2}$$

$$\leq ||\omega||^{1/2}\ ||A^*A||^{1/2}$$

$$\leq ||A||.$$

Therefore

$$||A|| \leq ||\pi^{\omega}(A)||.$$

On the other hand,

$$||\pi^{\omega}(A)|| \leq r(A^*A)^{1/2} \quad \text{(cf. 9.37)}$$

$$= ||A|| \quad \text{(cf. 1.1)}.$$

So

$$||\pi^{\omega}(A)|| = ||A||,$$

thus it remains only to recall that π^{ω} is a topologically irreducible $*$-representation of A (cf. 10.26).

Put

$$\pi_{AT} = \bigoplus_{\omega \in P(A)} \pi^{\omega}.$$

Then 10.36 implies that π_{AT} is a faithful $*$-representation of A on $E_{AT} = \bigoplus_{\omega \in P(A)} E^{\omega}$.

10.37 SCHOLIUM Every C*-algebra is isometrically $*$-isomorphic to a norm closed $*$-subalgebra of the bounded linear operators on some complex Hilbert space.

[Note: Every separable C*-algebra is isometrically $*$-isomorphic to a norm closed $*$-subalgebra of the bounded linear operators on some separable complex Hilbert space.]

If $I \subset A$ is a nonzero $*$-ideal (not necessarily closed), then every nondegenerate $*$-representation $\pi:I \to \mathcal{B}(E)$ can be extended to a nondegenerate $*$-representation $\bar{\pi}:A \to \mathcal{B}(E)$ (see the discussion leading up to 9.32).

[Note: Recall that

$$\pi \text{ topologically irreducible} \Rightarrow \bar{\pi} \text{ topologically irreducible.}]$$

Suppose now that $A' \subset A$ is a C*-subalgebra -- then a $*$-representation $\pi:A \to \mathcal{B}(E)$ is said to be an <u>extension</u> of a $*$-representation $\pi':A' \to \mathcal{B}(E')$ if $\exists$ a closed subspace $X \subset E$ which is invariant under $\pi|A'$ and has the property that the sub representation

$$\pi|A':A' \to \mathcal{B}(X)$$

is unitarily equivalent to π'.

10.38 <u>LEMMA</u> Every topologically irreducible $*$-representation $\pi':A' \to \mathcal{B}(E')$ has a topologically irreducible extension to A.

<u>PROOF</u> Take $\pi = \pi^{\omega'}$, where ω' is pure (cf. 10.27). Using 7.24, extend ω' to a pure state ω on A and let X be the closure of

$$\{\pi^{\omega}(A')x_{\omega}:A' \in A'\}$$

in E^{ω} -- then X is invariant under $\pi^{\omega}|A'$ and if x'_{ω} is the orthogonal projection of x_{ω} onto X, we have

$$\pi^{\omega}(A')x_{\omega} = \pi^{\omega}(A')x'_{\omega} \qquad (A' \in A'),$$

so x'_{ω} is topologically cyclic for the sub representation of $\pi^{\omega}|A'$ on X.

Finally, $\forall$ A' $\in$ A',

$$<x'_{\omega},\pi^{\omega}(A')x'_{\omega}>_{\omega}$$

$$= <x_{\omega},\pi^{\omega}(A')x_{\omega}>_{\omega}$$

$$= \omega(A') = \omega'(A')$$

$$= <x_{\omega'},\pi^{\omega'}(A')x_{\omega'}>_{\omega'}.$$

Therefore $\pi^{\omega}|A'$ on X is unitarily equivalent to $\pi^{\omega'}$ (cf. 9.7).

[Note: The same kind of argument shows that every topologically cyclic $*$-representation $\pi':A' \to B(E')$ has a topologically cyclic extension to A, thus every nondegenerate $*$-representation $\pi':A' \to B(E')$ has a nondegenerate extension to A (cf. 9.5).]

10.39 LEMMA Suppose that $A' \subset A$ is a commutative C*-subalgebra -- then $\forall$ $\omega' \in \Delta(A')$, $\exists$ a topologically irreducible $*$-representation $\pi:A \to B(E)$ and a nonzero vector $x \in E$ such that $\forall$ A' $\in$ A',

$$\pi(A')x = \omega'(A')x.$$

[This is a special case of 10.38.]

10.40 REMARK The analog of 10.38 for Banach $*$-algebras is false in general. [Consider an A whose only $*$-representations are trivial.]

Let H be an infinite dimensional complex Hilbert space -- then $B(H)$ is a unital

C*-algebra but its representation theory is far more complicated than that of $\underline{L}_\infty(H)$ (cf. 10.28).

10.41 DICHOTOMY PRINCIPLE Suppose that π is a topologically irreducible *-representation of $B(H)$ -- then either

$$\pi(\underline{L}_\infty(H)) = \{0\}$$

or

π is unitarily equivalent to the identity representation of $B(H)$ on H.

[The point is that if ω is a pure state on $B(H)$, then either $\omega|\underline{L}_\infty(H) = \{0\}$ or $\omega = \omega_x$ ($\exists$ x:$||x|| = 1$).]

10.42 REMARK Every nondegenerate *-representation of $B(H)$ is unitarily equivalent to one of the form

$$\pi_0 \oplus (\oplus \pi_i),$$

where π_0 is nondegenerate and vanishes on $\underline{L}_\infty(H)$ and π_i is unitarily equivalent to the identity representation of $B(H)$ on H.

10.43 LEMMA If ω_1, ω_2 are pure states on $B(H)$, then π^{ω_1} is unitarily equivalent to π^{ω_2} iff $\exists$ a unitary $U:H \rightarrow H$ such that $\forall$ $A \in B(H)$,

$$\omega_1(A) = \omega_2(U^{-1}AU).$$

PROOF If there is a $U \in U(H)$ with the stated property, then $\forall$ $A \in B(H)$,

$$<x_{\omega_1}, \pi^{\omega_1}(A) x_{\omega_1}>_{\omega_1}$$

$$= <x_{\omega_2}, \pi^{\omega_2}(U^{-1}AU) x_{\omega_2}>_{\omega_2}$$

$$= <\pi^{\omega_2}(U) x_{\omega_2}, \pi^{\omega_2}(A) \pi^{\omega_2}(U) x_{\omega_2}>_{\omega_2}.$$

Therefore π^{ω_1} and π^{ω_2} are unitarily equivalent (cf. 9.7). Conversely, suppose that π^{ω_1} and π^{ω_2} are unitarily equivalent and let $W:E^{\omega_1} \to E^{\omega_2}$ be a unitary operator such that

$$W\pi^{\omega_1}(A) = \pi^{\omega_2}(A)W \quad (A \in \mathcal{B}(\mathcal{H})).$$

Choose a unitary $V:E^{\omega_2} \to E^{\omega_2}:Vx_{\omega_2} = Wx_{\omega_1}$ -- then $\exists\ U \in \mathcal{U}(\mathcal{H})$:

$$\pi^{\omega_2}(U) x_{\omega_2} = Wx_{\omega_1} \quad \text{(cf. 9.51)}.$$

So, $\forall\ A \in \mathcal{B}(\mathcal{H})$,

$$\omega_1(A) = <x_{\omega_1}, \pi^{\omega_1}(A) x_{\omega_1}>_{\omega_1}$$

$$= <x_{\omega_1}, W^{-1}\pi^{\omega_2}(A) Wx_{\omega_1}>_{\omega_1}$$

$$= <Wx_{\omega_1}, \pi^{\omega_2}(A) Wx_{\omega_1}>_{\omega_2}$$

$$= <\pi^{\omega_2}(U) x_{\omega_2}, \pi^{\omega_2}(A) \pi^{\omega_2}(U) x_{\omega_2}>_{\omega_2}$$

$$= \langle x_{\omega_2}, \pi^{\omega_2}(U^{-1}AU)x_{\omega_2}\rangle_{\omega_2}$$

$$= \omega_2(U^{-1}AU).$$

10.44 <u>EXAMPLE</u> If H is a separable infinite dimensional complex Hilbert space, then there are $2^{\underline{c}}$ unitary equivalence classes of topologically irreducible $*$-representations of $\mathcal{B}(H)$.

[This is a counting argument.

1. The cardinality of $\mathcal{B}(H)$ is $\underline{c}$.

2. The cardinality of $\mathcal{P}(\mathcal{B}(H))$ is $2^{\underline{c}}$.

3. The cardinality of $\mathcal{U}(H)$ is $\underline{c}$.

Now let κ be the cardinality of the set of unitary equivalence classes of topologically irreducible $*$-representations of $\mathcal{B}(H)$. Stipulate that pure states ω_1, ω_2 are equivalent (denoted $\omega_1 \sim \omega_2$) iff $\exists$ a unitary $U: H \to H$ such that $\forall$ $A \in \mathcal{B}(H)$,

$$\omega_1(A) = \omega_2(U^{-1}AU).$$

Then in view of 10.43,

$$\kappa = \#(\mathcal{P}(\mathcal{B}(H))/\sim).$$

But each equivalence class of pure states has at least one and at most $\underline{c}$ members. Therefore

$$\kappa \leq \#(\mathcal{P}(H)) = 2^{\underline{c}} \leq \kappa\underline{c} = \max(\kappa, \underline{c}).$$

Since $\underline{c} < 2^{\underline{c}}$, it follows that $\kappa = 2^{\underline{c}}$.]

§11. *STRUCTURE THEORY*

Given a C*-algebra A, denote by $\hat{A}$ the set of unitary equivalence classes $[\pi]$ of topologically irreducible *-representations π of A -- then $\hat{A}$ is called the structure space of A.

E.g.: If A is commutative, then

$$\hat{A} \longleftrightarrow \Delta(A).$$

11.1 EXAMPLE Let H be a complex Hilbert space. Take $A = \underline{L}_\infty(H)$ -- then $\#(\hat{A}) = 1$ (cf. 10.28).

11.2 DICHOTOMY PRINCIPLE Let $\pi : A \to B(E)$ be a topologically irreducible *-representation -- then either

$$\pi(A) \supset \underline{L}_\infty(E)$$

or

$$\pi(A) \cap \underline{L}_\infty(E) = \{0\}.$$

11.3 EXAMPLE Let

$$\begin{cases} \pi_1 : A \to B(E_1) \\ \pi_2 : A \to B(E_2) \end{cases}$$

be topologically irreducible *-representations of A such that $\text{Ker } \pi_1 = \text{Ker } \pi_2$. Assume:

$$\pi_1(A) \cap \underline{L}_\infty(E_1) \neq \{0\}.$$

Then π_1 and π_2 are unitarily equivalent.

[Note: Therefore a topologically irreducible *-representation π of A is determined by its kernel to within unitary equivalence provided $\pi(A)$ contains a nonzero compact operator. But all bets are off if $\pi(A) \cap \underline{L}_\infty(E) = \{0\}$ (cf. 11.11).]

11.4 <u>LEMMA</u> If $\#(\hat{A}) = 1$, then π is faithful ($[\pi] \in \hat{A}$) and A is simple.

<u>PROOF</u> $\forall$ A $\neq$ 0,

$$||\pi(A)|| = ||A|| > 0 \quad \text{(cf. 10.36)}.$$

Therefore Ker $\pi = \{0\}$. If $I \subset A$ is a proper closed ideal, then $I = \{0\}$. This is because A/I, being a C*-algebra, admits a topologically irreducible *-representation the lift of which to A is unitarily equivalent to π, so $I \subset$ Ker $\pi = \{0\}$.

A C*-algebra A is said to be <u>elementary</u> if A is *-isomorphic to $\underline{L}_\infty(H)$ for some complex Hilbert space H.

11.5 <u>LEMMA</u> Let $\pi: A \to \mathcal{B}(E)$ be a *-representation. Assume: π is nondegenerate and $\pi(A) \subset \underline{L}_\infty(E)$ -- then π is discretely decomposable, i.e., there is an orthogonal decomposition

$$E = \bigoplus_{i \in I} E_i,$$

where each E_i is a closed π-invariant subspace of E on which π acts irreducibly.

[Note: To be completely precise, $\forall$ i $\in$ I, the assignment

$$\pi_i : \begin{cases} A \to \mathcal{B}(E_i) \\ \\ A \to \pi(A)|E_i \end{cases}$$

is a topologically irreducible $*$-representation of A on E_i.]

11.6 <u>THEOREM</u> Suppose that A is $*$-isomorphic to a C*-subalgebra of an elementary C*-algebra -- then A is $*$-isomorphic to a (C*) direct sum $\bigoplus_i A_i$ (cf. 3.19) of elementary C*-algebras A_i.

If A is elementary, then $\#(\hat{A}) = 1$ (cf. 11.1) and this can be reversed provided A is separable.

11.7 <u>THEOREM</u> Suppose that A is separable and $\#(\hat{A}) = 1$ -- then A is elementary.

<u>PROOF</u> The nontrivial argument is lengthy and best broken up into pieces.

<u>Step 1</u>: Take π per 11.4, say $\pi: A \to \mathcal{B}(E)$ -- then E is separable. Thus fix $x \neq 0$ in E and let $D \subset A$ be a countable dense subset of A -- then $\pi(D)x$ is dense in $\pi(A)x$, which is dense in E.

<u>Step 2</u>: Let $A' \subset A$ be a maximal commutative C*-subalgebra -- then $\Delta(A')$ is countable. In fact, $\forall\ \omega' \in \Delta(A')$, $\exists$ a unit vector $x(\omega') \in E: \forall\ A' \in A'$,

$$\pi(A')x(\omega') = \omega'(A')x(\omega') \quad \text{(cf. 10.39)}.$$

Given $\omega_1' \neq \omega_2'$, $\exists\ A' \in A_{SA}'$:

$$\omega_1'(A') \neq \omega_2'(A').$$

Therefore

$$\omega_2'(A')\langle x(\omega_1'), x(\omega_2')\rangle$$

$$= \langle x(\omega_1'), \omega_2'(A')x(\omega_2')\rangle$$

$$= \langle x(\omega_1'), \pi(A')x(\omega_2')\rangle$$

$$= \langle \pi(A')x(\omega_1'), x(\omega_2') \rangle$$

$$= \langle \omega_1'(A')x(\omega_1'), x(\omega_2') \rangle$$

$$= \omega_1'(A')\langle x(\omega_1'), x(\omega_2') \rangle$$

$\Rightarrow$

$$\langle x(\omega_1'), x(\omega_2') \rangle = 0.$$

So if $\Delta(A')$ was uncountable, then E would have uncountably many mutually orthogonal unit vectors contradicting its separability.

Step 3: $\Delta(A')$ is a countable locally compact Hausdorff space, hence by the Baire category theorem, has at least one isolated point ω_0'. On the other hand,

$$A' \approx C_\infty(\Delta(A')),$$

so there is a projection P in A' ($P = P^* = P^2$) such that $\omega_0'(P) = 1$ and $\omega'(P) = 0$ for $\omega' \neq \omega_0'$. Moreover, every element $A' \in A'$ decomposes as

$$A' = \lambda P + B',$$

where $\lambda \in \underline{C}$ and $B'P = PB' = 0$.

Step 4: Let $A \in A$ — then

$$\begin{aligned} A'(PAP) &= (\lambda P + B')\ (PAP) \\ &= \lambda PAP \\ &= \lambda PAP + PAPB' \\ &= PAP\ (\lambda P + B') \\ &= (PAP)A'. \end{aligned}$$

But A' is maximal:

$$PAP \in A' \Rightarrow PAP \subset A'.$$

Step 5: Since π is faithful, $\pi(P) \neq 0$. Therefore Ran $\pi(P)$ is a nonzero closed linear subspace of E which is invariant under the commutative *-algebra PAP. Denote by π_P the associated *-representation

$$PAP \rightarrow \pi(PAP)|\text{Ran } \pi(P) \quad (A \in A).$$

Then π_P is topologically irreducible. Proof: Let $x,y \in$ Ran $\pi(P)$ with $x \neq 0$ and choose a net $\{A_i : i \in I\}$ in A:

$$\pi(A_i)x \rightarrow y \quad (\text{cf. } 8.16)$$

$$\Rightarrow$$

$$\begin{aligned}\pi(PA_iP)x &= \pi(P)\pi(A_i)\pi(P)x \\ &= \pi(P)\pi(A_i)x \\ &\rightarrow \pi(P)y = y.\end{aligned}$$

That π_P is topologically irreducible follows upon citing 8.16 once again.

Step 6: Due to the topological irreducibility of π_P, the $\pi_P(PAP)$ $(A \in A)$ are scalar operators (cf. 9.8). In turn, this forces Ran $\pi(P)$ to be one dimensional, i.e., $\pi(P)$ is rank 1. Accordingly,

$$\pi(A) \cap \underline{L}_\infty(E) \neq \{0\}$$

$$\Rightarrow$$

$$\pi(A) \supset \underline{L}_\infty(E) \quad (\text{cf. } 11.2).$$

Step 7: The inverse image $\pi^{-1}(\underline{L}_\infty(E))$ is a nonzero closed ideal in A, so, as A is simple (cf. 11.4),

$$\pi^{-1}(\underline{L}_\infty(E)) = A.$$

Therefore

$$\pi : A \to \underline{L}_\infty(E)$$

is a $*$-isomorphism or still, A is elementary.

11.8 REMARK Consult Akemann-Weaver[†] for a discussion of the situation when A is nonseparable (but $\#(\hat{A}) = 1$).

11.9 RAPPEL A primitive ideal of A is an ideal which is the kernel of a topologically irreducible $*$-representation of A.

Write Prim A for the set of primitive ideals of A and equip it with the hull-kernel topology -- then Prim A is T_0.

The obvious arrow

$$\begin{cases} \hat{A} \to \text{Prim } A \\ [\pi] \to \text{Ker } \pi \end{cases}$$

is surjective (but, in general, is not injective). Therefore the hull-kernel topology on Prim A can be pulled back to $\hat{A}$ to give what is called the regional topology on $\hat{A}$.

[Note: A subset $S \subset \hat{A}$ is open in the regional topology iff it is of the form $\{[\pi] \in \hat{A} : \text{Ker } \pi \in O\}$ for some subset $O \subset$ Prim A which is open in the hull-kernel topology.]

N.B. In general, $\hat{A}$ need not be T_0 but if it is T_0, it need not be T_1 but if

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[†]Proc. Natl. Acad. Sci. USA, 101 (2004), 7522-7525.

it is T_1, it need not be T_2 (cf. infra).

11.10 <u>LEMMA</u> The following conditions are equivalent: (i) $\hat{A}$ is T_0; (ii) Two topologically irreducible $*$-representations of A with the same kernel are unitarily equivalent; (iii) The canonical map $\hat{A} \to \text{Prim } A$ is a homeomorphism.

[This is a simple deduction from the definitions.]

11.11 <u>EXAMPLE</u> Suppose that A is simple -- then $\text{Prim } A = \{0\}$. So, if $\hat{A}$ has more than one element, then $\hat{A}$ will not be T_0.

[Note: There are simple A for which $\hat{A}$ is uncountable ("Glimm algebras").]

11.12 <u>EXAMPLE</u> Let H be an infinite dimensional complex Hilbert space. Take $A = \underline{L}_\infty(H)^+$ -- then $\#(\hat{A}) = 2$, say $\hat{A} = \{\pi_1, \pi_2\}$. Here $\text{Ker } \pi_1 = \{0\}$, $\text{Ker } \pi_2 = \underline{L}_\infty(H)$, so $\hat{A}$ is T_0. But $\hat{A}$ is not T_1: $[\pi_1]$ is a dense open point ($[\pi_2]$ is a closed point).

11.13 <u>EXAMPLE</u> Let

$$A = \{f \in C([0,1], M_2(\underline{C})) : f(0) = \begin{bmatrix} \lambda & 0 \\ 0 & \mu \end{bmatrix} \ (\exists\ \lambda, \mu \in \underline{C})\}.$$

Then

$$\hat{A} =]0,1] \cup \{\pi_1, \pi_2\},$$

where

$$t \longleftrightarrow f(t) \quad (0 < t \leq 1)$$

and

$$\pi_1(f) = f(0)_{11}, \ \pi_2(f) = f(0)_{22}.$$

Topologically, $]0,1]$ has its usual topology and sets of the form $\{\pi_1\} \cup]0,\varepsilon[$, $\{\pi_2\} \cup]0,\varepsilon[$ are also open. Therefore $\hat{A}$ is T_1. Still, it is not T_2:

$$\frac{1}{n} \to 0 \ \& \ \left[\begin{array}{l} \frac{1}{n} \to \pi_1 \\ \\ \frac{1}{n} \to \pi_2. \end{array}\right.$$

11.14 <u>LEMMA</u> Let $S \subset \hat{A}$ -- then

$$[\pi] \in \bar{S} \iff \bigcap_{[S]\in S} \text{Ker } S \subset \text{Ker } \pi.$$

E.g.: If $S = \{[S]\}$ and Ker $S = \{0\}$, then $\bar{S} = \hat{A}$.

11.15 <u>THEOREM</u> Suppose that A is separable -- then for a given $[\pi] \in \hat{A}$, the following conditions are equivalent: (i) $[\pi]$ is closed in $\hat{A}$; (ii) $\pi(A) = \underline{L}_\infty(E)$.

<u>PROOF</u> Assume (i) -- then

$$\{[\pi'] \in \hat{A}: \text{Ker } \pi \subset \text{Ker } \pi'\}$$

is a one element set (cf. 11.14), so the C*-algebra $\pi(A)$ is elementary (cf. 11.7), hence $\exists$ a $*$-isomorphism $\Phi:\pi(A) \to \underline{L}_\infty(\mathcal{H})$ ($\mathcal{H}$ a complex Hilbert space). But the identity representation of $\pi(A)$ on E is topologically irreducible, thus $\exists$ a unitary operator $U:\mathcal{H} \to E$ such that $\forall$ A $\in A$,

$$U\Phi(\pi(A))U^{-1} = \pi(A).$$

I.e.:

$$U\underline{L}_\infty(\mathcal{H})U^{-1} = \pi(A)$$

=>

$$\underline{L}_\infty(E) = \pi(A).$$

Assume (ii) and consider a $[\pi_0] \in \overline{[\pi]}$, thus Ker $\pi \subset$ Ker π_0 (cf. 11.14), so there is a topologically irreducible $*$-representation π' of $\underline{L}_\infty(E)$ on E' such that $\pi_0 = \pi' \circ \pi$. Bearing in mind 10.28, fix a unitary operator $U:E \to E'$ with the property that $\forall\, A \in A$,

$$U\pi(A)U^{-1} = \pi'(\pi(A)) \quad (\equiv \pi_0(A)).$$

Then obviously

$$[\pi_0] = [\pi],$$

which establishes that $[\pi]$ is closed in $\hat{A}$.

[Note: The proof of the implication (ii) => (i) does not use the assumption that A is separable.]

A C*-algebra A is said to be <u>liminal</u> if for every topologically irreducible $*$-representation $\pi:A \to B(E)$, we have $\pi(A) = \underline{L}_\infty(E)$.

11.16 <u>EXAMPLE</u> Every commutative C*-algebra is liminal.

11.17 <u>EXAMPLE</u> Every finite dimensional C*-algebra is liminal.

11.18 <u>EXAMPLE</u> Every elementary C*-algebra is liminal.

<u>N.B.</u> If H is an infinite dimensional complex Hilbert space, then $B(H)$ is not

liminal (just consider the identity representation of $\mathcal{B}(H)$ on H).

11.19 <u>LEMMA</u> Suppose that A is liminal -- then its C*-subalgebras are liminal (in particular, its closed ideals are liminal).

[One has only to apply 10.38 (restrictions of compact operators are compact).]

11.20 <u>LEMMA</u> Suppose that A is liminal -- then $\forall$ closed ideal $I \subset A$, the quotient A/I is liminal.

If A is unital and liminal, then its topologically irreducible $*$-representations are necessarily finite dimensional ($\forall\ \pi, \pi(1_A) = \mathrm{id}_E$). This said, let H be an infinite dimensional complex Hilbert space -- then $\underline{L}_\infty(H)^+$ is not liminal (consider $\pi(A,\lambda) = A + \lambda \mathrm{id}_H$). Still, $\underline{L}_\infty(H)$ is a liminal closed ideal of $\underline{L}_\infty(H)^+$ and the quotient $\underline{L}_\infty(H)^+/\underline{L}_\infty(H) \approx \underline{C}$ is liminal as well.

11.21 <u>LEMMA</u> If A is liminal, then $\hat{A}$ is T_1, the converse being valid if in addition A is separable (cf. 11.15).

11.22 <u>EXAMPLE</u> Suppose that A is $*$-isomorphic to a C*-subalgebra of an elementary C*-algebra -- then A is liminal (cf. 11.19), hence $\hat{A}$ is T_1 and, in fact, $\hat{A}$ is discrete.

A C*-algebra A is said to be <u>postliminal</u> if for every topologically irreducible $*$-representation $\pi:A \rightarrow \mathcal{B}(E)$, we have $\pi(A) \supset \underline{L}_\infty(E)$.

Trivially,

$$\text{"liminal"} \Rightarrow \text{"postliminal"}.$$

11.23 EXAMPLE Let H be an infinite dimensional complex Hilbert space -- then $\underline{L}_\infty(H)^+$ is postliminal (but not liminal).

11.24 LEMMA Suppose that A is postliminal -- then its C*-subalgebras are postliminal (in particular, its closed ideals are postliminal).

11.25 LEMMA Suppose that A is postliminal -- then $\forall$ closed ideal $I \subset A$, the quotient A/I is postliminal.

11.26 LEMMA Let $I \subset A$ be a closed ideal. Assume: I and A/I are postliminal -- then A is postliminal.

[Note: If I and A/I are liminal, then A is postliminal (but, as observed above (and will be seen again below), A need not be liminal).]

11.27 EXAMPLE Take $H = \ell^2$ with its usual orthonormal basis $\{e_n\}$ and let S be the unilateral shift characterized by $Se_n = e_{n+1}$ -- then the Toeplitz algebra T is the C*-subalgebra of $B(H)$ generated by S. It is wellknown that T properly contains $\underline{L}_\infty(H)$ and $T/\underline{L}_\infty(H) \approx C(\underline{T})$. Since $\underline{L}_\infty(H)$ and $C(\underline{T})$ are liminal, hence postliminal, it follows from 11.26 that T is postliminal. Nevertheless, T is not liminal: The identity representation is topologically irreducible and T properly contains $\underline{L}_\infty(H)$.

N.B. One consequence of 11.25 and 11.26 is this: Suppose that A is non-unital -- then A is postliminal iff A^+ is postliminal.

11.28 LEMMA Suppose that A is postliminal. Let

$$\begin{cases} \pi_1 : A \to \mathcal{B}(E_1) \\ \pi_2 : A \to \mathcal{B}(E_2) \end{cases}$$

be topologically irreducible *-representations of A such that $\text{Ker}\ \pi_1 = \text{Ker}\ \pi_2$ -- then

$$[\pi_1] = [\pi_2] \quad \text{(cf. 11.3).}$$

Therefore $\hat{A}$ is T_0 and the canonical map $\hat{A} \to \text{Prim}\ A$ is a homeomorphism (cf. 11.10).

[Note: $\hat{A}$ is T_1 if A is liminal (cf. 11.21).]

11.29 REMARK It is a fact that if A is separable and $\hat{A}$ is T_0, then A is postliminal.

[Note: This is definitely not obvious.]

11.30 LEMMA Suppose that A is simple and postliminal -- then A is elementary.

PROOF Let $\pi : A \to \mathcal{B}(E)$ be a topologically irreducible *-representation -- then $\pi(A) \supset \underline{L}_\infty(E)$. But $\pi^{-1}(\underline{L}_\infty(E))$ is a closed ideal, thus $A = \pi^{-1}(\underline{L}_\infty(E))$. At the same time, π is faithful. Therefore $\pi : A \to \underline{L}_\infty(E)$ is a *-isomorphism, so A is elementary.

An elementary C*-algebra is unital iff it is finite dimensional. Combining this with 11.30, we conclude that an infinite dimensional unital simple C*-algebra is not postliminal.

11.31 EXAMPLE Let H be a separable infinite dimensional complex Hilbert space — then the quotient $\mathcal{B}(H)/\underline{L}_\infty(H)$ is not postliminal, hence $\mathcal{B}(H)$ is not postliminal either (cf. 11.25).

[Note: For the record, Prim $\mathcal{B}(H) = \{0, \underline{L}_\infty(H)\}$ (cf. 10.41), while

$$\#(\mathcal{B}(H)^\wedge) = 2^{\underline{c}} \quad \text{(cf. 10.44).]}$$

11.32 THEOREM Suppose that A is postliminal -- then every primary *-representation of A is geometrically equivalent to a topologically irreducible *-representation of A or still, is unitarily equivalent to a multiple of a topologically irreducible *-representation of A.

11.33 LEMMA Let A and B be C*-algebras and suppose that A is postliminal. Fix a C*-norm $||\cdot||_\alpha$ on $A \otimes B$ -- then every topologically irreducible *-representation ζ of $A \otimes_\alpha B$ is unitarily equivalent to one of the form $\pi \underline{\otimes} \zeta$, where $[\pi] \in \hat{A}$ and $[\zeta] \in \hat{B}$.

PROOF On elementary general grounds, there are nondegenerate *-representations

$$\left[\begin{array}{l} \zeta_A : A \rightarrow \mathcal{B}(E) \\ \\ \zeta_B : B \rightarrow \mathcal{B}(E) \end{array}\right. \qquad (\zeta : A \otimes_\alpha B \rightarrow \mathcal{B}(E))$$

such that $\forall\, A \in \mathcal{A}$, $\forall\, B \in \mathcal{B}$,

$$\zeta(A \otimes B) = \begin{cases} \zeta_{\mathcal{A}}(A)\zeta_{\mathcal{B}}(B) \\ \\ \zeta_{\mathcal{B}}(B)\zeta_{\mathcal{A}}(A). \end{cases}$$

Both $\zeta_{\mathcal{A}}$ and $\zeta_{\mathcal{B}}$ are primary. But $\mathcal{A}$ is also postliminal, so $\exists$ a topologically irreducible $*$-representation π of $\mathcal{A}$ such that $\zeta_{\mathcal{A}}$ is unitarily equivalent to $\underline{I}\pi \simeq \pi \otimes \mathrm{id}$ (cf. 11.32). And, under this equivalence, $\zeta_{\mathcal{B}}$ takes the form $\mathrm{id} \otimes \eta$, where η is a topologically irreducible $*$-representation of $\mathcal{B}$.

11.34 THEOREM Suppose that $\mathcal{A}$ is postliminal — then $\mathcal{A}$ is nuclear.

PROOF Let $\mathcal{B}$ be a C*-algebra and let $X \in \mathcal{A} \otimes \mathcal{B}$ $(X \neq 0)$. Given a C*-norm $||\cdot||_{\alpha}$ on $\mathcal{A} \otimes \mathcal{B}$, choose a topologically irreducible $*$-representation ζ of $\mathcal{A} \otimes_{\alpha} \mathcal{B}$ such that

$$||X||_{\alpha} = ||\zeta(X)|| \quad \text{(cf. 10.36).}$$

Then

$$||X||_{\alpha} = ||(\pi \underline{\otimes} \eta)(X)|| \quad \text{(cf. 11.33).}$$

But

$$||(\pi \underline{\otimes} \eta)(X)|| \leq ||X||_{\min} \quad \text{(cf. 6.11)}$$

$\Rightarrow$

$$||X||_{\alpha} \leq ||X||_{\min}$$

$\Rightarrow$

$$||X||_{\max} \leq ||X||_{\min}.$$

Therefore $\mathcal{A}$ is nuclear.

11.35 REMARK It can be shown that

$$\mathcal{A},\mathcal{B} \text{ postliminal} \Rightarrow \mathcal{A} \underset{\sim}{\otimes} \mathcal{B} \text{ postliminal.}$$

11.36 EXAMPLE Let $\mathcal{H}$ be an infinite dimensional complex Hilbert space -- then $\mathcal{B}(\mathcal{H})$ is not postliminal.

[In fact, $\mathcal{B}(\mathcal{H})$ is not nuclear (cf. 6.27).]

[Note: If $\mathcal{H}$ is not separable, then for each cardinal $\kappa \leq \dim \mathcal{H}$ there is a closed ideal $\mathcal{I}_\kappa \subset \mathcal{B}(\mathcal{H})$ containing $\underline{L}_\infty(\mathcal{H})$, hence $\mathcal{B}(\mathcal{H})/\underline{L}_\infty(\mathcal{H})$ is not simple.]

11.37 LEMMA Fix $A \in \mathcal{A}$ -- then the function

$$[\pi] \rightarrow ||\pi(A)||$$

is lower semicontinuous on $\hat{\mathcal{A}}$.

PROOF Fix $\varepsilon > 0$. Given a topologically irreducible *-representation $\pi:\mathcal{A} \rightarrow \mathcal{B}(E)$, choose unit vectors $x,y \in E$:

$$|\langle x,\pi(A)y\rangle| > ||\pi(A)|| - \frac{\varepsilon}{2}.$$

Then $\exists$ a neighborhood U of $[\pi]$ such that $\forall$ $[\pi'] \in U$, there are unit vectors x',y' in E' for which

$$|\langle x',\pi'(A)y'\rangle - \langle x,\pi(A)y\rangle| < \frac{\varepsilon}{2},$$

thus

$$|\langle x',\pi'(A)y'\rangle| > ||\pi(A)|| - \varepsilon$$

$$\Rightarrow$$

$$||\pi'(A)|| > ||\pi(A)|| - \varepsilon.$$

Suppose now that $\{[\pi_i]:i \in I\}$ is a net in $\hat{A}$:

$$[\pi_i] \to [\pi].$$

Then $[\pi_i]$ is eventually in U, so

$$\liminf_{i \in I} ||\pi_i(A)|| \geq ||\pi(A)|| - \varepsilon$$

or still,

$$\liminf_{i \in I} ||\pi_i(A)|| \geq ||\pi(A)|| \quad (\varepsilon \to 0).$$

11.38 <u>REMARK</u> In general, the function

$$[\pi] \to ||\pi(A)||$$

is not continuous on $\hat{A}$ but it will be if $\hat{A}$ is T_2 (see the next lemma) (a compact subset of a Hausdorff space is closed).

[Note: The continuity of the function

$$[\pi] \to ||\pi(A)||$$

$\forall\, A \in A$ is equivalent to the condition that $\hat{A}$ be T_2.]

11.39 <u>LEMMA</u> Fix $A \in A$ and $r > 0$ -- then

$$S_r(A) = \{[\pi] \in \hat{A}: ||\pi(A)|| \geq r\}$$

is a compact subset of $\hat{A}$.

<u>PROOF</u> Let $\{S_i:i \in I\}$ be a decreasing net of relatively closed nonempty subsets of $S_r(A)$ -- then it will be enough to prove that $\bigcap_{i \in I} S_i \neq \emptyset$. To this end, let

$$I_i = \bigcap_{[\pi] \in S_i} \text{Ker } \pi.$$

Claim:

$$||A + I_i|| \geq r.$$

In fact,

$$||A + I_i|| = \sup_{[\pi]\in S_i} ||A + \text{Ker } \pi||.$$

But

$$||A + \text{Ker } \pi|| = \inf_{B\in\text{Ker } \pi} ||A + B||.$$

And $\forall$ B $\in$ Ker π,

$$\begin{aligned} r \leq ||\pi(A)|| &= ||\pi(A + B)|| \\ &\leq ||\pi|| \; ||A + B|| \\ &\leq ||A + B||. \end{aligned}$$

Continuing, put

$$I = \overline{(\bigcup_{i\in I} I_i)},$$

so

$$||A + I|| \geq r.$$

Since A/I is a C*-algebra, $\exists$ a topologically irreducible $*$-representation π of A:

$$I \subset \text{Ker } \pi \ \& \ ||\pi(A + I)|| = ||A + I|| \quad \text{(cf. 10.36)}.$$

Therefore

$$[\pi] \in S_r(A).$$

But $\forall$ i $\in$ I,

$$I_i \subset \text{Ker } \pi$$

=>

$$[\pi] \in \overline{S_i} \text{ (cf. 11.14)}$$

=>

$$[\pi] \in S_i \quad (S_i = \overline{S}_i \cap S_r(A))$$

=>

$$\bigcap_{i \in I} S_i \neq \emptyset.$$

11.40 <u>THEOREM</u> $\hat{A}$ is locally compact.

<u>PROOF</u> Fix $[\pi_0] \in \hat{A}$ -- then the claim is that $[\pi_0]$ has a basis of compact neighborhoods. Thus let U be an open neighborhood of $[\pi_0]$. Since $S = \hat{A} - U$ is closed, $\exists$ A $\in$ A:

$$\pi_0(A) \neq 0 \text{ and } S(A) = 0 \ \forall \ [S] \in S \quad \text{(cf. 11.14).}$$

Choose $r > 0 : r < ||\pi_0(A)||$ -- then

$$\{[\pi] \in \hat{A} : ||\pi(A)|| > r\}$$

is open (cf. 11.37), so

$$\{[\pi] \in \hat{A} : ||\pi(A)|| \geq r\}$$

is a compact neighborhood of $[\pi_0]$ (cf. 11.39) which is contained in U.

11.41 <u>REMARK</u> If A is unital, then $\hat{A}$ is compact. Proof:

$$\{[\pi] \in \hat{A} : ||\pi(1_A)|| \geq 1\}$$

is a compact subset of $\hat{A}$. But

$$||\pi(1_A)|| = ||id_E|| \quad (\pi : A \to B(E))$$

$$= 1.$$

[Note: The converse is false: If H is an infinite dimensional complex Hilbert space and if $A = \underline{L}_\infty(H)$, then $\#(\hat{A}) = 1$ (cf. 11.1), yet $\mathrm{id}_H \notin \underline{L}_\infty(H)$.]

N.B. The preceding considerations imply that Prim A is locally compact, Prim A being compact if A is unital.

Using the notation of 9.33, each Z in the center $\mathcal{Z}(A)$ of $\mathfrak{DC}(A)$ determines a bounded continuous complex valued function

$$\chi_Z : \hat{A} \rightarrow \underline{C}$$

via the prescription

$$\chi_Z([\pi]) = C_Z(\pi).$$

If instead, we hold $[\pi]$ fixed and let Z vary, then the assignment

$$Z \rightarrow \chi_Z([\pi])$$

defines a character $\omega_{[\pi]}$ of $\mathcal{Z}(A)$ (note that

$$\omega_{[\pi]}(1_{\mathfrak{DC}(A)}) = 1).$$

In summary:

$$\left[\begin{array}{l} \chi_Z \in BC(\hat{A}) \\ \\ \omega_{[\pi]} \in \Delta(\mathcal{Z}(A)). \end{array}\right.$$

11.42 RAPPEL An element $Z \in \mathcal{Z}(A)$ is a pair (ζ,ζ) such that $\forall$ A,B $\in A$,

$$\zeta(A)B = \zeta(AB) = A\zeta(B).$$

11.43 <u>LEMMA</u> $\forall\ [\pi] \in \hat{A}$,

$$\operatorname{Ker} \omega_{[\pi]} = \{Z \in \mathcal{Z}(A) : \zeta(A) \subset \operatorname{Ker} \pi\}.$$

[One has only to recall that by construction (cf. 9.32),

$$\bar{\pi}(Z)\left(\sum_{i=1}^{n} \pi(A_i)x_i\right) = \sum_{i=1}^{n} \pi(\zeta(A_i))x_i.]$$

It follows that $\omega_{[\pi]}$ depends only on Ker π, so there is a continuous function

$$\phi : \operatorname{Prim} A \rightarrow \Delta(\mathcal{Z}(A))$$

such that $\forall\ \pi$,

$$\phi(\operatorname{Ker} \pi) = \omega_{[\pi]}.$$

11.44 <u>THEOREM</u> The map

$$\left[\begin{array}{l} \mathcal{Z}(A) \rightarrow BC(\operatorname{Prim} A) \\ \quad Z \rightarrow \hat{Z} \circ \phi \end{array}\right.$$

is a $*$-isomorphism.

[Note: We have

$$\begin{aligned} (\hat{Z} \circ \phi)(\operatorname{Ker} \pi) &= \hat{Z}(\omega_{[\pi]}) \\ &= \omega_{[\pi]}(Z) \\ &= \chi_Z([\pi]).] \end{aligned}$$

The only issue is surjectivity and for that we'll need a couple of lemmas, the first of which is standard fare.

11.45 <u>LEMMA</u> Let $I_k \subset A$ $(k = 0,1,\ldots,n)$ be closed ideals. Suppose that

$$A \in I_0 + I_1 + \cdots + I_n.$$

Then $\forall\ \varepsilon > 0$, $\exists\ A_k \in I_k$:

$$A = A_0 + A_1 + \cdots + A_n \text{ and } ||A_k|| \leq (2 + \varepsilon)||A||.$$

<u>PROOF</u> Proceed by induction, the statement being trivial if $n = 0$. To pass from n to $n + 1$, choose

$$B \in I_0 + I_1 + \cdots + I_n$$

such that $A - B \in I_{n+1}$. Since

$$(I_0 + \cdots + I_{n+1})/I_{n+1}$$

$$\approx (I_0 + \cdots + I_n)/(I_0 + \cdots + I_n) \cap I_{n+1},$$

one can assume that

$$||B|| \leq (1 + \varepsilon')||A||,$$

where $\varepsilon' > 0$ will be specified below. Let ε'' be another positive parameter which will also be specified below -- then the induction hypothesis applied to the pair (B,ε'') gives rise to a decomposition

$$B = A_0 + A_1 + \cdots + A_n \quad (A_k \in I_k)$$

with

$$||A_k|| \leq (2 + \varepsilon'')||B||.$$

Put

$$A_{n+1} = A - B.$$

Then

$$A = B + (A - B)$$

$$= A_0 + A_1 + \cdots + A_{n+1}$$

$\Rightarrow$

$$||A_0|| \leq (2 + \varepsilon'')||B|| \leq (2 + \varepsilon'')(1 + \varepsilon')||A||$$

$$||A_1|| \leq (2 + \varepsilon'')||B|| \leq (2 + \varepsilon'')(1 + \varepsilon')||A||$$

$$\vdots \qquad\qquad \vdots$$

$$||A_{n+1}|| \leq ||A|| + ||B|| \leq (2 + \varepsilon')||A||.$$

Now take $\varepsilon', \varepsilon''$ small enough to force

$$2\varepsilon' + \varepsilon'' + \varepsilon''\varepsilon' \leq \varepsilon.$$

N.B. Take $\varepsilon = 1$ to get the estimate

$$||A_k|| \leq 3||A||.$$

To simplify the writing, let P stand for a generic element of Prim $\mathcal{A}$ and let $pr_P : \mathcal{A} \rightarrow \mathcal{A}/P$ be the quotient map -- then

$$\bigcap_P \text{Ker } pr_P = \{0\}.$$

11.46 LEMMA Fix $\varepsilon > 0$ and $A \in \mathcal{A}$. Let $f \in BC(\text{Prim } \mathcal{A})$ -- then $\exists\ B_\varepsilon \in \mathcal{A}$ such that $\forall\ P \in \text{Prim } \mathcal{A}$,

$$||pr_P(B_\varepsilon) - f(P)pr_P(A)|| < \varepsilon.$$

PROOF Assume for sake of argument that $f:\text{Prim } \mathcal{A} \rightarrow [0,1]$. Fix n and define open sets

$$\mathcal{O}_k = \{P \in \text{Prim } A : \frac{k-1}{n} < f(P) < \frac{k+1}{n}\} \quad (k = 0,1,\ldots,n).$$

Obviously,

$$\text{Prim } A = \bigcup_{k=0}^{n} \mathcal{O}_k$$

and each $P \in \text{Prim } A$ belongs to at most two of the $\mathcal{O}_k$. Let

$$\mathcal{I}_k = \cap \{P \in \text{Prim } A : P \not\in \mathcal{O}_k\}.$$

Then

$$P \in \mathcal{O}_k \iff \mathcal{I}_k \not\subset P$$

and

$$\mathcal{I}_0 + \mathcal{I}_1 + \cdots + \mathcal{I}_n = A.$$

By 11.45, $\exists\, A_k \in \mathcal{I}_k$:

$$A = A_0 + A_1 + \cdots + A_n \text{ and } ||A_k|| \le 3||A||.$$

Let

$$B_n = \sum_{k=0}^{n} \frac{k}{n} A_k.$$

Then $\forall\, P \in \text{Prim } A$,

$$||\text{pr}_P(B_n) - f(P)\text{pr}_P(A)||$$

$$= ||\sum_{k=0}^{n} (\frac{k}{n}\text{pr}_P(A_k) - f(P)\text{pr}_P(A_k))||$$

$$= ||\sum_{k=0}^{n} (\frac{k}{n} - f(P))\text{pr}_P(A_k)||$$

$$\le \frac{6}{n}\, ||A||.$$

Choose n > > 0:

$$\frac{6}{n} ||A|| < \varepsilon$$

and put

$$B_\varepsilon = B_n.$$

<u>N.B.</u> If B'_ε also has the stated property, then

$$||B_\varepsilon - B'_\varepsilon|| < 2\varepsilon.$$

Proof:

$$||B_\varepsilon - B'_\varepsilon|| = \sup_{[\pi]\in\hat{A}} ||\pi(B_\varepsilon - B'_\varepsilon)||$$

$$= \sup_{P\in \text{Prim } A} ||\text{pr}_P(B_\varepsilon) - \text{pr}_P(B'_\varepsilon)||$$

$$\leq \sup_{P\in \text{Prim } A} (||\text{pr}_P(B_\varepsilon) - f(P)\text{pr}_P(A)||$$

$$+ ||f(P)\text{pr}_P(A) - \text{pr}_P(B'_\varepsilon)||)$$

$$< 2\varepsilon.$$

The sequence $\{B_{2^{-n}}\}$ generated per 11.46 is therefore Cauchy, hence converges to an element $T(f,A) \in A$, and $\forall$ $P \in$ Prim A,

$$\text{pr}_P(T(f,A)) = f(P)\text{pr}_P(A),$$

an equation that characterizes $T(f,A)$ (since $\bigcap_P \text{Ker pr}_P = \{0\}$).

Let

$$\zeta_f(A) = T(f,A) \quad (A \in A).$$

Then $\zeta_f : A \to A$ is linear.

[Note:

$$\zeta_f = \zeta_g \Rightarrow f = g.$$

Proof: $\forall\ P \in \text{Prim}\ A$,

$$\text{pr}_P(T(f,A)) = \text{pr}_P(T(g,A))$$

$$\Rightarrow$$

$$f(P)\text{pr}_P(A) = g(P)\text{pr}_P(A)$$

$$\Rightarrow$$

$$f(P) = g(P) \quad (\exists\ A : \text{pr}_P(A) \neq 0).]$$

11.47 LEMMA $\forall\ A,B \in A$,

$$\zeta_f(A)B = \zeta_f(AB) = A\zeta_f(B).$$

PROOF $\forall\ P \in \text{Prim}\ A$,

$$\begin{aligned} \text{pr}_P(T(f,A)B) &= \text{pr}_P(T(f,A))\text{pr}_P(B) \\ &= f(P)\text{pr}_P(A)\text{pr}_P(B) \\ &= f(P)\text{pr}_P(AB) \\ &= \text{pr}_P(T(f,AB)). \end{aligned}$$

ETC.

Put

$$Z_f = (\zeta_f, \zeta_f).$$

Then

$$Z_f \in \mathfrak{Z}(A) \quad \text{(cf. 11.42)}$$

and we claim that

$$\hat{Z}_f \circ \phi = f,$$

thereby establishing surjectivity in 11.44.

First, $\forall\ \pi$ & $\forall\ A$,

$$\begin{aligned}\pi(\zeta_f(A)) &= \bar{\pi}(Z_f)\pi(A)\\ &= \chi_Z([\pi])\pi(A)\\ &= (\hat{Z}_f \circ \phi)(\operatorname{Ker}\pi)\pi(A).\end{aligned}$$

But

$$\begin{aligned}&\pi(\zeta_{\hat{Z}_f \circ \phi}(A))\\ &= \pi(\zeta_{\hat{Z}_f \circ \phi}(A) - (\hat{Z}_f \circ \phi)(\operatorname{Ker}\pi)A + (\hat{Z}_f \circ \phi)(\operatorname{Ker}\pi)A)\\ &= \pi(\zeta_{\hat{Z}_f \circ \Phi}(A) - (\hat{Z}_f \circ \phi)(\operatorname{Ker}\pi)A) + \pi((\hat{Z}_f \circ \phi)(\operatorname{Ker}\pi)A)\\ &= \pi(T(\hat{Z}_f \circ \phi, A) - (\hat{Z}_f \circ \phi)(\operatorname{Ker}\pi)A) + \pi((\hat{Z}_f \circ \phi)(\operatorname{Ker}\pi)A)\\ &= \pi((\hat{Z}_f \circ \phi)(\operatorname{Ker}\pi)A)\\ &= (\hat{Z}_f \circ \phi)(\operatorname{Ker}\pi)\pi(A).\end{aligned}$$

So $\forall\ \pi$ & $\forall\ A$,

$$\pi(\zeta_f(A)) = \pi(\zeta_{\hat{Z}_f \circ \phi}(A))$$

=>

$$\zeta_f(A) = \zeta_{\hat{Z}_f \circ \phi}(A)$$

=>

$$\zeta_f = \zeta_{\hat{Z}_f} \circ \phi$$

=>

$$f = \hat{Z}_f \circ \phi.$$

11.48 REMARK One can work with $\hat{A}$ rather than Prim A provided $\hat{A}$ is T_0 (cf. 11.10), in which case

$$\mathcal{Z}(A) \simeq BC(\hat{A}).$$

11.49 LEMMA The map

$$\text{Prim } A \to \text{Prim } \mathfrak{DC}(A)$$

that sends

$$\text{Ker } \pi \text{ to Ker } \bar{\pi}$$

is a continuous injection with a dense range.

[The closure of the image of Prim A in Prim $\mathfrak{DC}(A)$ consists of those Q:

$$Q \supset \bigcap_{\pi} \text{Ker } \bar{\pi}.]$$

Since $\mathfrak{DC}(A)$ is a unital C*-algebra, Prim $\mathfrak{DC}(A)$ is compact. And, as will be seen momentarily, one can assign to each

$$f \in C(\text{Prim } \mathfrak{DC}(A))$$

an element

$$\Phi(f) \in BC(\text{Prim } A)$$

with the property that

$$\Phi(f)(\text{Ker } \pi) = f(\text{Ker } \bar{\pi}).$$

11.50 THEOREM The map

$$\begin{cases} C(\text{Prim } \mathcal{DC}(A)) \to BC(\text{Prim } A) \\ f \to \Phi(f) \end{cases}$$

is a $*$-isomorphism.

PROOF Injectivity is implied by 11.49, leaving surjectivity. To deal with it, note that the arrow

$$\begin{cases} \text{Prim } A \to \text{Prim } \mathcal{Z}(A) \\ \text{Ker } \pi \to \text{Ker } \bar{\pi}|\mathcal{Z}(A) \end{cases}$$

factors as

$$\text{Prim } A \to \text{Prim } \mathcal{DC}(A) \to \text{Prim } \mathcal{Z}(A)$$

from which an induced map

$$C(\text{Prim } \mathcal{Z}(A)) \to C(\text{Prim } \mathcal{DC}(A)) \overset{\Phi}{\to} BC(\text{Prim } A).$$

But

$$C(\text{Prim } \mathcal{Z}(A)) \approx C(\Delta(\mathcal{Z}(A)))$$

$$\approx \mathcal{Z}(A),$$

so from 11.44, the arrow

$$C(\text{Prim } \mathcal{Z}(A)) \to BC(\text{Prim } A)$$

is bijective, hence Φ is surjective.

11.51 <u>RAPPEL</u> Let X be a toplogical space -- then a <u>Stone-Cech compactification</u> of X is a compact Hausdorff space βX and a continuous map $\beta_X:X \to \beta X$ such that for every compact Hausdorff space Y and every continuous function $f:X \to Y$ there is a unique continuous function $f':\beta X \to Y$ with $f = f' \circ \beta_X$.

[Note: It is not assumed that X is Hausdorff. Still, βX always exists (cf. 11.53) and is essentially unique. Incidentally, the image of X in βX is dense and is all of βX if X is compact.]

11.52 <u>REMARK</u> Let <u>TOP</u> be the category of topological spaces and continuous functions and let $\underline{CPT_2}$ be the full subcategory of <u>TOP</u> whose objects are the compact Hausdorff spaces -- then the Stone-Cech compactification determines a functor

$$\beta:\underline{TOP} \to \underline{CPT_2}.$$

Indeed, if X,Y are topological spaces and if $f:X \to Y$ is a continuous function then there is a commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle \beta_X}\downarrow & & \downarrow{\scriptstyle \beta_Y} \\ \beta X & \xrightarrow[\beta f]{} & \beta Y \end{array} \quad ,$$

βf being the unique filler for

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle \beta_X}\downarrow & & \downarrow{\scriptstyle \beta_Y} \\ \beta X & \cdots\!> & \beta Y \end{array} \quad .$$

On the other hand, there is a forgetful functor

$$\underline{\text{CPT}_2} \rightarrow \underline{\text{TOP}}$$

and β is its left adjoint, so β preserves colimits (cf. 3.24). E.g.: If $\{X_i : i \in I\}$ is a collection of compact Hausdorff spaces, then its coproduct in $\underline{\text{CPT}_2}$ is

$$\beta(\coprod_i X_i).$$

11.53 LEMMA Let X be a topological space. Define $\varepsilon : X \rightarrow \text{Prim } BC(X)$ by $\varepsilon(x) = \text{Ker } \varepsilon_x$, where ε_x is evaluation at x — then the pair

$$(\text{Prim } BC(X), \varepsilon)$$

is "the" Stone-Cech compactification of X.

[Note: BC(X) is a unital commutative C*-algebra, hence Prim BC(X) is a compact Hausdorff space.]

E.g.: Tracing through the various identifications, we have

$$\begin{aligned} \beta \text{ Prim } A &\approx \text{Prim } BC(\text{Prim } A) \\ &\approx \text{Prim } \mathcal{Z}(A) \quad (\text{cf. } 11.44). \end{aligned}$$

§12. $\mathcal{W}^*$-ALGEBRAS

Let $\mathcal{H}$ be a complex Hilbert space -- then a $*$-subalgebra $\mathcal{A} \subset \mathcal{B}(\mathcal{H})$ is nondegenerate if the linear span of the set

$$\mathcal{A}\mathcal{H} = \{Ax : A \in \mathcal{A}, x \in \mathcal{H}\}$$

is dense in $\mathcal{H}$, i.e., if $\mathcal{A}\mathcal{H}$ is total.

[Note: A unital $*$-subalgebra $\mathcal{A} \subset \mathcal{B}(\mathcal{H})$ is automatically nondegenerate.]

12.1 REMARK If $\mathcal{A} \subset \mathcal{B}(\mathcal{H})$ is a C*-subalgebra, then $\mathcal{H}$ is a left Hilbert $\mathcal{A}$-module ($||Ax|| \leq ||A|| \ ||x||$), hence in this situation, $\mathcal{A}\mathcal{H}$ is a closed linear subspace of $\mathcal{H}$ (cf. 4.2), thus $\mathcal{H} = \mathcal{A}\mathcal{H}$ if $\mathcal{A}$ is nondegenerate.

12.2 RAPPEL The arrow

$$\mathcal{B}(\mathcal{H}) \to \underline{L}_1(\mathcal{H})^*$$

that sends A to λ_A ($A \in \mathcal{B}(\mathcal{H})$), where

$$\lambda_A(T) = \text{tr}(AT) \quad (T \in \underline{L}_1(\mathcal{H})),$$

is an isometric isomorphism, thus $\mathcal{B}(\mathcal{H})$ can be equipped with the weak* topology arising from this identification.

[Note: Accordingly, the weak* topology on $\mathcal{B}(\mathcal{H})$ is generated by the seminorms

$$||A||_T = |\text{tr}(TA)| \quad (T \in \underline{L}_1(\mathcal{H})).]$$

12.3 THEOREM Suppose that $\mathcal{A}$ is a nondegenerate $*$-subalgebra of $\mathcal{B}(\mathcal{H})$ -- then $\mathcal{A}$ is dense in $\mathcal{A}''$ per the weak, the strong, and the weak* topologies.

So, as a corollary, if $A \subset B(H)$ is a nondegenerate $*$-subalgebra, then the following conditions are equivalent:

1. $A = A''$;
2. A is weakly closed;
3. A is strongly closed;
4. A is weak* closed.

N.B. Therefore A is necessarily unital.

A von Neumann algebra is a $*$-subalgebra $A \subset B(H)$ such that $A = A''$.

E.g.: A' is a von Neumann algebra. In fact, $(A')'' = A''' = A'$.

12.4 REMARK A von Neumann algebra A is weakly closed, hence norm closed, so A is a unital C*-algebra.

[Note: Suppose that A is a weakly closed C*-subalgebra of $B(H)$. Let

$$H_0 = \bigcap_{A \in A} \text{Ker } A.$$

Then $H_0^\perp$ is A-invariant and $A|H_0^\perp$ is a weakly closed nondegenerate $*$-subalgebra of $B(H_0^\perp)$, hence is a von Neumann algebra.]

12.5 EXAMPLE $B(H)$ is a von Neumann algebra. On the other hand, $\underline{L}_\infty(H)$ is not a von Neumann algebra if H is infinite dimensional. To see this, fix an orthonormal basis $\{e_i : i \in I\}$ for H. Write P_i for the orthogonal projection onto $\underline{C}e_i$ and given a finite subset $F \subset I$, put

$$P_F = \sum_{i \in F} P_i.$$

Then the net $\{P_F\}$ is strongly convergent to id_H. But $\mathrm{id}_H \notin \underline{L}_\infty(H)$.

12.6 LEMMA If S is a subset of $B(H)$ which is closed under the $*$-operation, then S'' is the smallest von Neumann algebra containing S (the von Neumann algebra generated by S).

12.7 RAPPEL Suppose that $\{A_i : i \in I\}$ is a bounded increasing net of self-adjoint operators on H -- then

$$\sup_{i \in I} A_i \in B(H)_{SA}$$

exists, call it A. So, $\forall$ i, $A_i \le A$ and if $B \in B(H)_{SA}$ has the property that $\forall$ i, $A_i \le B$, then $A \le B$.

[Note: We have

1. $A_i \to A$ weakly;

2. $A_i \to A$ strongly;

3. $A_i \to A$ weak*.]

If $A \subset B(H)$ is a von Neumann algebra and if $\{A_i : i \in I\} \subset A_{SA}$ is a bounded increasing net, then it is clear that

$$\sup_{i \in I} A_i \in A_{SA}.$$

Conversely:

12.8 THEOREM Let $A \subset B(H)$ be a unital C*-algebra. Assume: $\forall$ bounded increasing net $\{A_i : i \in I\} \subset A_{SA}$,

$$\sup_{i \in I} A_i \in A_{SA}.$$

Then A is a von Neumann algebra.

A C*-algebra A is monotone complete if every bounded increasing net $\{A_i : i \in I\}$ in A_{SA} has a supremum in A_{SA}.

E.g.: Every von Neumann algebra is monotone complete.

12.9 LEMMA Suppose that A is monotone complete -- then A is unital.

PROOF Let $\{e_i : i \in I\}$ be an approximate unit per A (cf. 1.20). Put

$$e = \sup_{i \in I} e_i$$

and let $\pi : A \to \mathcal{B}(E)$ be a faithful *-representation of A (cf. 10.37) -- then, due to the nondegeneracy of π, $\pi(e_i) \to \text{id}_E$ strongly. But $\forall\ i \in I$, $\pi(e_i) \leq \pi(e)$, thus $\text{id}_E \leq \pi(e)$, so $\pi(e)$ is invertible in $\mathcal{B}(E)$ or still, is invertible in $\pi(A) + \underline{C}\ \text{id}_E$. Accordingly, $\forall\ A \in A$, $\forall\ c \in \underline{C}$,

$$\pi(e)\pi(e)^{-1}(\pi(A) + c\ \text{id}_E) = \pi(A) + c\ \text{id}_E.$$

Write

$$\pi(e)^{-1} = \pi(A_e) + c_e\ \text{id}_E$$

and take $A = 0$, $c = 1$ to get

$$\pi(e)(\pi(A_e) + c_e\ \text{id}_E)\text{id}_E = \text{id}_E,$$

i.e.,

$$\pi(eA_e + c_e e) = \mathrm{id}_E$$

$$\Rightarrow$$

$$\mathrm{id}_E \in \pi(A).$$

Therefore A is unital.

12.10 REMARK Let A be a unital commutative C*-algebra -- then A is monotone complete iff $\Delta(A)$ is a compact extremely disconnected Hausdorff space.

[Note: The term "extremely disconnected" means that the closure of every open set is open.]

A W*-algebra is a C*-algebra A which is *-isomorphic to a von Neumann algebra.

N.B. A W*-algebra is unital and monotone complete.

12.11 REMARK Let A be a unital commutative C*-algebra -- then A is a W*-algebra iff there exists a locally compact Hausdorff space X equipped with a positive Radon measure μ such that A is isometrically *-isomorphic to the algebra $L^\infty(X,\mu)$ of essentially bounded μ-measurable functions on X.

[Note: The pair (X,μ) is not unique.]

If A and B are monotone complete C*-algebras, then a positive linear map $\Phi: A \to B$ is said to be normal if for every bounded increasing net $\{A_i : i \in I\} \subset A_{SA}$, we have

$$\Phi(\sup_{i\in I} A_i) = \sup_{i\in I} \Phi(A_i).$$

12.12 <u>LEMMA</u> A $*$-isomorphism between monotone complete C*-algebras is normal.

Take $\mathcal{B} = \underline{C}$ -- then it makes sense to consider normal positive linear functionals on $\mathcal{A}$, in particular rormal states on $\mathcal{A}:S_{\underline{n}}(\mathcal{A}) \subset S(\mathcal{A})$.

12.13 <u>LEMMA</u> Suppose that $\mathcal{A}$ is a von Neumann algebra. Let $\omega:\mathcal{A} \rightarrow \underline{C}$ be a positive linear functional -- then ω is normal iff ω is weak* continuous.

12.14 <u>THEOREM</u> Suppose that $\mathcal{A} \subset \mathcal{B}(\mathcal{H})$ is a von Neumann algebra. Let $\omega \in S(\mathcal{A})$ -- then ω is normal iff $\exists$ a density operator $W \in \mathcal{W}(\mathcal{H})$ such that $\forall\ A \in \mathcal{A}$,

$$\omega(A) = \mathrm{tr}(WA).$$

[Note: Recall that a <u>density operator</u> is a positive trace class operator W with $\mathrm{tr}(W) = 1$ (cf. 7.13).]

<u>N.B.</u> It is thus immediate that the normal states separate the points of $\mathcal{A}$, i.e., $\forall\ A \neq 0,\ \exists\ \omega \in S_{\underline{n}}(\mathcal{A}):\omega(A) \neq 0$.

[Note: Consequently, $\forall\ A \neq 0,\ \exists\ \omega \in S_{\underline{n}}(\mathcal{A}):\pi^{\omega}(A) \neq 0$.]

Suppose that

$$\left[\begin{array}{l} \mathcal{A} \subset \mathcal{B}(\mathcal{H}) \\ \\ \mathcal{B} \subset \mathcal{B}(\mathcal{K}) \end{array}\right.$$

are van Neumann algebras.

12.15 <u>LEMMA</u> Let $\Phi:\mathcal{A} \rightarrow \mathcal{B}$ be a positive linear map -- then Φ is rormal iff Φ is weak* continuous.

12.16 <u>THEOREM</u> Let $\Phi:A \to B$ be a *-homomorphism. Assume: Φ is normal -- then Ker Φ is weak* closed and Ran Φ is weak* closed.

[Note: It follows that Ran Φ is a von Neumann algebra if Φ is unital.]

12.17 <u>EXAMPLE</u> Let $\omega \in S(A)$ and consider its GNS representation π^{ω} -- then $\pi^{\omega}:A \to B(E^{\omega})$ is a unital *-homomorphism. Moreover,

$$\omega \text{ normal} \Rightarrow \pi^{\omega} \text{ normal},$$

hence $\pi^{\omega}(A) \subset B(E^{\omega})$ is a von Neumann algebra.

A projection in the center of A is called a <u>central projection</u>.

12.18 <u>LEMMA</u> Suppose that $I \subset A$ is a weak* closed ideal -- then $\exists$ a unique central projection P such that $I = PA$ $(= AP)$ and $\forall$ $A \in A$,

$$PA = P(PA) = P(AP) = (PA)P = AP.$$

[Note: We have

$$A = PA \oplus P^{\perp}A.]$$

12.19 <u>REMARK</u> In the context of 12.16, one can thus say that there exists a unique central projection P such that Ker $\Phi = PA$ and Φ is a *-isomorphism of $P^{\perp}A$ onto Ran Φ.

Suppose that A is a W*-algebra -- then A is monotone complete and the normal states separate the points of A. Conversely, as we shall now see, these properties are characteristic.

[Note: If $A_0 \subset B(H_0)$ is a von Neumann algebra and if $\Phi:A \rightarrow A_0$ is a $*$-isomorphism, then Φ is normal (cf. 12.12). So, $\forall\ \omega_0 \in S_{\underline{n}}(A_0)$, $\omega_0 \circ \Phi \in S_{\underline{n}}(A)$. But $S_{\underline{n}}(A_0)$ separates the points of A_0. Therefore $S_{\underline{n}}(A)$ separates the points of A.]

12.20 LEMMA Suppose that A is monotone complete. Let ω be a normal positive linear functional on A -- then for any bounded increasing net $\{A_i : i \in I\} \subset A_{SA}$, $\pi^\omega(A_i)$ converges strongly to $\pi^\omega(A)$ $(A = \sup_{i \in I} A_i)$.

PROOF Let $U \in A$ be unitary -- then

$$UAU^{-1} = \sup_{i \in I} UA_iU^{-1}$$

=>

$$<\pi^\omega(U)x_\omega, \pi^\omega(A)\pi^\omega(U)x_\omega>_\omega$$

$$= <x_\omega, \pi^\omega(U^{-1}AU)x_\omega>_\omega$$

$$= \omega(U^{-1}AU)$$

$$= \sup_{i \in I} \omega(U^{-1}A_iU)$$

$$= \sup_{i \in I} <x_\omega, \pi^\omega(U^{-1}A_iU)x_\omega>_\omega$$

$$= \sup_{i \in I} <\pi^\omega(U)x_\omega, \pi^\omega(A_i)\pi^\omega(U)x_\omega>_\omega$$

=>

$$\lim_{i \in I} ||(\pi^\omega(A) - \pi^\omega(A_i))^{1/2}\pi^\omega(U)x_\omega||_\omega^2$$

$$= 0.$$

Since the finite linear combinations of unitary elements exhaust A and since $\pi^{\omega}(A)x_{\omega}$ is dense in E^{ω}, it follows that

$$(\pi^{\omega}(A) - \pi^{\omega}(A_i))^{1/2}$$

converges strongly to zero, which implies that $\pi^{\omega}(A_i)$ converges strongly to $\pi^{\omega}(A)$.

12.21 <u>THEOREM</u> Let A be a C*-algebra. Assume: A is monotone complete and the normal states separate the points of A -- then A is a W*-algebra.

<u>PROOF</u> Let

$$\pi_{NOR} = \bigoplus_{\omega \in S_{\underline{n}}(A)} \pi^{\omega}.$$

Then π_{NOR} is a faithful $*$-representation of A on

$$E_{NOR} = \bigoplus_{\omega \in S_{\underline{n}}(A)} E^{\omega}.$$

So

$$\pi_{NOR}: A \to \pi_{NOR}(A)$$

is a $*$-isomorphism, thus to prove that A is a W*-algebra, it suffices to prove that

$$\pi_{NOR}(A) \subset \mathcal{B}(E_{NOR})$$

is a von Neumann algebra and for this, we shall appeal to 12.8 ($\pi_{NOR}(A)$ is unital (cf. 12.9)). Let $\{A_i : i \in I\} \subset A_{SA}$ be a bounded increasing net and put $A = \sup_{i \in I} A_i$ -- then $\forall\ \omega \in S_{\underline{n}}(A)$, $\pi^{\omega}(A_i)$ converges strongly to $\pi^{\omega}(A)$ (cf. 12.20), hence $\pi_{NOR}(A_i)$ converges strongly to $\pi_{NOR}(A)$. Meanwhile

$$\pi_{NOR}(A_i) \to \sup_{i \in I} \pi_{NOR}(A_i)$$

strongly. Therefore

$$\sup_{i\in I} \pi_{NOR}(A_i) = \pi_{NOR}(A) \in \pi_{NOR}(A).$$

I.e.: $\pi_{NOR}(A)$ is monotone complete.

12.22 REMARK There are examples of monotone complete C*-algebras A: $S_{\underline{n}}(A) = \{0\}$. Such an A cannot be a W*-algebra.

The predual of a von Neumann algebra A is the set of all weak* continuous linear functionals on A. Notation: A_*.

So, e.g.,

$$\mathcal{B}(H)_* \approx \underline{L}_1(H).$$

12.23 LEMMA Let $\omega: A \to \underline{C}$ be a weak* continuous linear functional. Assume: $\forall\ A \in A$,

$$\omega(A^*) = \overline{\omega(A)}.$$

Then $\exists$ unique weak* continuous positive linear functionals

$$\begin{cases} \omega_+ : A \to \underline{C} \\ \omega_- : A \to \underline{C} \end{cases}$$

such that

$$\omega = \omega_+ - \omega_-$$

and

$$||\omega|| = ||\omega_+|| + ||\omega_-||.$$

[Note: It is a corollary that every element of A_* can be written as a linear combination of four weak* continuous positive linear functionals (cf. 7.11).]

12.24 LEMMA A_* is a norm closed subspace of A^*.

Therefore A_* is a Banach space.

12.25 THEOREM Let A be a von Neumann algebra — then the arrow $\Gamma:A \rightarrow (A_*)^*$ defined by the rule

$$\Gamma(A)(\omega) = \omega(A) \qquad (A \in A, \ \omega \in A_*)$$

is an isometric isomorphism.

[Note: Γ is also a homeomorphism if A and $(A_*)^*$ are endowed with their respective weak* topologies, thus the closed unit ball A^1 of A is weak* compact.]

Let X be a complex Banach space — then a complex Banach space Y is called a predual of X if X is isometrically isomorphic to Y^*.

[Note: If X is reflexive, then $X \approx (X^*)^*$, thus the dual X^* is a predual.]

E.g.: Take $X = A$, $Y = A_*$.

12.26 LEMMA Let A be a C*-algebra — then up to isometric isomorphism, A admits at most one predual.

12.27 EXAMPLE In general, preduals are not unique: Take $H = \ell^1$ and let

$$\begin{cases} Y_1 = c \\ Y_2 = c_0 \end{cases}$$

— then c is not isometrically isomorphic to c_0, yet c^* and c_0^* are both isometrically isomorphic to ℓ^1.

12.28 THEOREM Let A be a C*-algebra. Suppose that A has a predual V -- then A is a W*-algebra.

Because the proof is somewhat involved, it will be convenient to proceed via a series of lemmas, the goal being to finesse the matter by an application of 12.21.

So let A be a C*-algebra with a predual V -- then by definition, there is an isometric isomorphism $\phi:A \to V^*$. Use ϕ to transfer the weak* topology on V^* to A and call it the V^*-topology. This done, given $v \in V$, define $\omega_v:A \to \underline{C}$ by

$$\omega_v(A) = \langle v,\phi(A)\rangle \quad (A \in A).$$

Then the set

$$\{\omega_v:v \in V\}$$

is the subset of A^* consisting of those linear functionals that are continuous per the V^*-topology.

[Note: To say that $A_i \to A$ in the V^*-topology means that $\forall\ v \in V$,

$$\omega_v(A_i) \to \omega_v(A).]$$

12.29 LEMMA A is unital.

PROOF The closed unit ball A^1 of A is compact in the V^*-topology (Alaoglu), hence has an extreme point (Krein-Milman). Therefore A is unital (cf. 1.30).

12.30 RAPPEL (Krein-Smulian) Let E be a complex Banach space; let E* be its dual and let B* be the closed unit ball in E* -- then a convex subset $S \subset E^*$ is weak* closed iff each of the sets $S \cap rB^*$ $(r > 0)$ is weak* closed.

[Note: Here is an application. Suppose that $\omega:E^* \to \underline{C}$ is a linear functional -- then ω is weak* continuous iff the restriction $\omega|B^*$ is weak* continuous. Proof: $\text{Ker}\ \omega \cap B^*$ is weak* closed, thus $\text{Ker}\ \omega$ is weak* closed, which implies that ω is weak* continuous.]

12.31 LEMMA A_{SA} is closed in the $\mathcal{U}^*$-topology.

PROOF It is enough to prove that

$$A_{SA}^1 = A_{SA} \cap A^1$$

is closed in the $\mathcal{U}^*$-topology (Krein-Smulian). So let $\{A_i : i \in I\}$ be a $\mathcal{U}^*$-convergent net in A_{SA}^1 and write the limit as $X + \sqrt{-1}\ Y$ $(X,Y \in A_{SA})$, the claim being that $Y = 0$. To establish this, note that $\forall\ n \in \underline{N}$, $\{A_i + \sqrt{-1}\ n1_A\}$ is $\mathcal{U}^*$-convergent to $X + \sqrt{-1}$ $(n1_A + Y)$. And then

$$(1 + n^2)^{1/2} \geq ||A_i + \sqrt{-1}\ n1_A||$$

$\Rightarrow$

$$\begin{aligned}(1 + n^2)^{1/2} &\geq \liminf_{i \in I} ||A_i + \sqrt{-1}\ n1_A|| \\ &\geq ||X + \sqrt{-1}\ (n1_A + Y)|| \\ &\geq ||n1_A + Y||.\end{aligned}$$

If Y is not zero, one can assume that its spectrum contains some $r > 0$ (otherwise work with $\{-A_i : i \in I\}$), thus $\forall\ n \in \underline{N}$,

$$r + n \leq ||n1_A + Y|| \leq (1 + n^2)^{1/2}$$

or still,

$$r^2 + 2rn + n^2 \leq 1 + n^2,$$

an impossibility. Therefore $Y = 0$, as claimed.

12.32 <u>LEMMA</u> $\mathcal{A}_+$ is closed in the $\mathcal{V}$*-topology.

<u>PROOF</u> It is enough to prove that

$$\mathcal{A}_+^1 = \mathcal{A}_+ \cap \mathcal{A}^1$$

is closed in the $\mathcal{V}$*-topology (Krein-Smulian). But

$$\mathcal{A}_+^1 = \frac{1}{2}(\mathcal{A}_{SA}^1 + 1_{\mathcal{A}}).$$

12.33 <u>LEMMA</u> $\mathcal{A}$ is monotone complete.

<u>PROOF</u> Let $\{A_i : i \in I\}$ be a bounded increasing net in $\mathcal{A}_{SA}$. Because $\mathcal{A}_{SA}^1$ is compact in the $\mathcal{V}$*-topology, there is a subnet $\{A_j : j \in J\}$ which is convergent to an element $A \in \mathcal{A}_{SA}$. But $\forall A_i$, A_j is $\geq A_i$ eventually, hence $A \geq A_i$ ($\mathcal{A}_+$ being closed in the $\mathcal{V}$*-topology (cf. 12.32)). On the other hand, if $B \in \mathcal{A}_{SA}$ and if $B \geq A_i$ for all i, then $B \geq A_j$ for all j, so $B \geq A$. Therefore

$$A = \sup_{i \in I} A_i,$$

which proves that $\mathcal{A}$ is monotone complete.

Bearing in mind 12.21, to finish the proof of 12.28, we have to show that the normal states separate the points of $\mathcal{A}$. And for this, some additional preparation is required.

12.34 RAPPEL (Urysohn) Let X be a topological space. Suppose that $\{x_i\}$ is a net in X -- then $\lim x_i = x$ iff every subnet $\{x_j\}$ has a subnet $\{x_k\}$ such that $\lim x_k = x$.

[If x_i does not converge to x, then $\exists$ a neighborhood U of x with the following property: $\forall\ i,\ \exists\ j \geq i : x_j \notin U$. But the subnet $\{x_j\}$ has a subnet $\{x_k\}$ such that the x_k are eventually in U.]

12.35 LEMMA The involution $*:A \to A$ is V^*-continuous.

PROOF The V^*-topology is the initial topology per the linear functionals

$$A \to \omega_v(A) \qquad (v \in V).$$

So, to conclude that the involution $*:A \to A$ is V^*-continuous, it suffices to prove that $\forall\ v \in V$, the arrow

$$A \to \omega_v(A^*)$$

is V^*-continuous and for this, it can be assumed that $||A|| \leq 1$ (cf. 12.30). Accordingly, fix $v \in V$ and suppose that $A_i \to 0$ in the V^*-topology -- then the contention is that $\omega_v(A_i^*) \to 0$. Consider an arbitrary subnet $\{\omega_v(A_j^*)\}$. Since

$$||A_j^*|| = ||A_j|| \leq 1,$$

it follows from the V^*-compactness of A^1 that the net $\{A_j^*\}$ has a V^*-convergent subnet $\{A_k^*\}$:

$$A_k^* \to B.$$

Claim: $B = 0$. To see this, note that

$$\begin{cases} A_k + A_k^* \to 0 + B \in \mathcal{A}_{SA} \\ \dfrac{A_k - A_k^*}{\sqrt{-1}} \to -\dfrac{B}{\sqrt{-1}} \in \mathcal{A}_{SA}, \end{cases}$$

$\mathcal{A}_{SA}$ being closed in the $\mathcal{V}^*$-topology (cf. 12.31). But

$$B^* = B, \ \left(-\frac{B}{\sqrt{-1}} \right)^* = -\frac{B}{\sqrt{-1}}$$

$$\Rightarrow$$

$$\left(-\frac{B}{\sqrt{-1}} \right)^* = \frac{-B^*}{-\sqrt{-1}} = \frac{B}{\sqrt{-1}} = -\frac{B}{\sqrt{-1}}$$

$$\Rightarrow$$

$$B = 0.$$

Therefore

$$\omega_v(A_k^*) \to \omega_v(0) = 0.$$

Now apply 12.34 to get

$$\omega_v(A_i^*) \to 0.$$

12.36 <u>LEMMA</u> If ω_v is positive, then ω_v is normal.

<u>PROOF</u> In the notation of 12.33,

$$A \geq A_i \Rightarrow \omega_v(A) \geq \omega_v(A_i)$$

$$\Rightarrow \omega_v(A) \geq \sup_{i \in I} \omega_v(A_i).$$

And

$$\omega_v(A_j) \leq \sup_{i \in I} \omega_v(A_i)$$

=>

$$\omega_v(A) = \lim_{j \in J} \omega_v(A_j) \leq \sup_{i \in I} \omega_v(A_i).$$

[Note: Recall that $A_j \to A$ in the $\mathcal{U}$*-topology and $\omega_v: \mathcal{A} \to \underline{C}$ is continuous in the $\mathcal{U}$*-topology.]

12.37 RAPPEL (Hahn-Banach) Let E be a real Hausdorff LCTVS. Let $S \subset E$ be a closed convex cone -- then $\forall\, x \in E - S$, $\exists$ a continuous linear functional $\theta: E \to \underline{R}$ such that $\theta(x) < 0$ and $\theta | S \geq 0$.

E.g.: Take $E = \mathcal{A}_{SA}$, $S = \mathcal{A}_+$ and work with the $\mathcal{U}$*-topology -- then $\forall\, A \in \mathcal{A}_{SA} - \mathcal{A}_+$, $\exists$ a $\mathcal{U}$*-continuous linear functional $\theta: \mathcal{A}_{SA} \to \underline{R}$ such that $\theta(A) < 0$ and $\theta | \mathcal{A}_+ \geq 0$.

N.B. Extend θ to a linear functional ω on all of $\mathcal{A}$ by writing

$$\omega(X + \sqrt{-1}\, Y) = \theta(X) + \sqrt{-1}\, \theta(Y) \qquad (X, Y \in \mathcal{A}_{SA}).$$

Then ω is $\mathcal{U}$*-continuous (cf. 12.35) and, by construction, is positive, hence normal (cf. 12.36).

12.38 LEMMA Let $A \in \mathcal{A}$ and assume that $\omega_v(A) = 0$ for all $\mathcal{U}$*-continuous positive linear functionals ω_v on $\mathcal{A}$ -- then $A = 0$.

PROOF Write

$$A = \text{Re}\, A + \sqrt{-1}\, \text{Im}\, A.$$

Then $\forall\, \omega_v$,

$$\begin{cases} \omega_v(\text{Re}\, A) = 0 \\ \omega_v(\text{Im}\, A) = 0 \end{cases}$$

and from this we want to conclude that A = 0, which will be the case if

$$\begin{cases} \mathrm{Re}\, A = 0 \\ \mathrm{Im}\, A = 0. \end{cases}$$

Consider Re A:

$$\mathrm{Re}\, A = (\mathrm{Re}\, A)_+ - (\mathrm{Re}\, A)_-.$$

Suppose that $\mathrm{Re}\, A \notin \mathcal{A}_+$ (=> $(\mathrm{Re}\, A)_- \neq 0$) -- then $\exists\ \omega_V$:

$$\omega_V(\mathrm{Re}\, A) < 0 \quad \text{(cf. supra)}.$$

As this can't be, it follows that $(\mathrm{Re}\, A)_- = 0$. Analogous considerations apply to Im A, thus $(\mathrm{Im}\, A)_- = 0$. Therefore

$$A = (\mathrm{Re}\, A)_+ + \sqrt{-1}\ (\mathrm{Im}\, A)_+$$

and $\forall\ \omega_V$,

$$\begin{cases} \omega_V((\mathrm{Re}\, A)_+) = 0 \\ \omega_V((\mathrm{Im}\, A)_+) = 0. \end{cases}$$

Consider $(\mathrm{Re}\, A)_+$. If $(\mathrm{Re}\, A)_+ \neq 0$, then

$$-\ (\mathrm{Re}\, A)_+ \in \mathcal{A}_{SA} - \mathcal{A}_+ \quad \text{(Cf. 1.24)},$$

so $\exists\ \omega_V$:

$$\omega_V(-\ (\mathrm{Re}\, A)_+) < 0 \quad \text{(cf. supra)},$$

a contradiction, hence $(\mathrm{Re}\, A)_+ = 0$. Similarly, $(\mathrm{Im}\, A)_+ = 0$. Therefore A = 0.

The upshot, then, is that the normal states separate the points of $\mathcal{A}$, which completes the proof of 12.28.

12.39 REMARK Write A_* for the subspace of A^* spanned by the normal positive linear functionals -- then

$$A_* = \{\omega_v : v \in V\}.$$

Suppose that A is a von Neumann algebra.

- Write $\underline{\text{Pro}}(A)$ for the set of all projections P in A.
- Write $F_{\underline{n}}(A)$ for the set of all norm closed faces F in $S_{\underline{n}}(A)$.
- Write $I_{\underline{L}}(A)$ for the set of all weak* closed left ideals I in A.

Equip each of these entities with their natural ordering.

12.40 THEOREM

- There is an order preserving bijection

$$\Phi:\underline{\text{Pro}}(A) \to F_{\underline{n}}(A).$$

- There is an order reversing bijection

$$\Psi:\underline{\text{Pro}}(A) \to I_{\underline{L}}(A).$$

- There is an order reversing bijection

$$\Theta:F_{\underline{n}}(A) \to I_{\underline{L}}(A).$$

[The relevant definitions are as follows.

$\underline{\Phi}$: Let

$$\Phi(P) = \{\omega \in S_{\underline{n}}(A):\omega(P) = 1\}.$$

Then $\Phi^{-1}(F) = P$, where P is the smallest projection such that $\omega(P) = 1$ for all $\omega \in F$.

$\underline{\Psi}$: Let

$$\Psi(P) = \{A \in A : AP = 0\}.$$

Then $\Psi^{-1}(I) = P^{\perp}$, where P is the unique projection such that $I = AP$.

$\underline{\Theta}$: Take $\Theta = \Psi \circ \Phi^{-1}$ -- then

$$\left[\begin{array}{l} \Theta(F) = \{A \in A : \omega(A^*A) = 0 \ \forall \ \omega \in F\} \\ \\ \Theta^{-1}(I) = \{\omega \in S_{\underline{n}}(A) : \omega(A^*A) = 0 \ \forall \ A \in I\}. \end{array}\right]$$

Given $P \in \underline{\text{Pro}}(A)$, let

$$F_P = \Phi(P),$$

thus

$$F_P = \{\omega \in S_{\underline{n}}(A) : \omega(P) = 1\}.$$

12.41 LEMMA Every γ in the convex hull of $F_P \cup F_{P^{\perp}}$ can be written as a unique convex combination

$$\gamma = \lambda\sigma + (1 - \lambda)\tau,$$

where $\sigma \in F_P$, $\tau \in F_{P^{\perp}}$.

Let $F_P \subset S_{\underline{n}}(A)$ be a norm closed face -- then F_P is said to be a split face if the convex hull of $F_P \cup F_{P^{\perp}}$ is all of $S_{\underline{n}}(A)$.

12.42 LEMMA F_P is a split face iff P is a central projection.

[Note: Suppose that P is a central projection -- then

$$\omega \in F_P \iff \forall\, A \in \mathcal{A},\ \omega(PA) = \omega(A).]$$

Let $\omega \in S_{\underline{n}}(\mathcal{A})$ and form ω^B as in 10.4:

$$\omega^B(A) = \omega(B^*AB) \qquad (A \in \mathcal{A})$$

or still,

$$\omega^B(A) = \langle B^\omega, \pi^\omega(A) B^\omega \rangle_\omega.$$

If $\omega(B^*B) \neq 0$, then

$$\frac{\omega^B(A)}{\omega(B^*B)} = \left\langle \frac{B^\omega}{\omega(B^*B)^{1/2}}, \pi^\omega(A) \frac{B^\omega}{\omega(B^*B)^{1/2}} \right\rangle_\omega.$$

But

$$||B^\omega||_\omega = \omega(B^*B)^{1/2} \qquad (\text{cf. } \S 10).$$

Therefore

$$\omega_B \equiv \frac{\omega^B}{\omega(B^*B)}$$

is a vector state which, moreover, is normal (cf. 12.17).

12.43 <u>LEMMA</u> Let $F_P \subset S_{\underline{n}}(\mathcal{A})$ be a split face. Fix $\omega \in F_P$ and suppose that $\omega(B^*B) \neq 0$ -- then $\omega_B \in F_P$.

<u>PROOF</u> We have

$$F_P = \{\omega \in S_{\underline{n}}(\mathcal{A}) : \omega(P) = 1\}.$$

Since F_P is a split face, P is central, so

$$\omega(B^*B)\omega_B(P^\perp) = \omega^B(P^\perp)$$

$$= \omega(B^*P^\perp B)$$

$$= \omega((B^*B)P^\perp)$$

$$\leq \omega(B^*B)^{1/2}\omega(P^\perp)^{1/2} \quad \text{(cf. 7.8)}.$$

But

$$1 = \omega(1_A) = \omega(P + P^\perp)$$

$$= \omega(P) + \omega(P^\perp)$$

$$= 1 + \omega(P^\perp)$$

$$\Rightarrow \omega(P^\perp) = 0 \Rightarrow \omega_B(P^\perp) = 0 \Rightarrow \omega_B(P) = 1 \Rightarrow \omega \in F_P.$$

§13. THE DOUBLE DUAL

Given a C*-algebra A, let

$$\bar{\pi} = \bigoplus_{\omega \in S(A)} \pi^{\omega}.$$

Then $\bar{\pi}$ is faithful. Moreover, the image $\bar{A} = \bar{\pi}(A)$ is a nondegenerate $*$-subalgebra of $\mathcal{B}(\bar{E})$ $(\bar{E} = \bigoplus_{\omega \in S(A)} E^{\omega})$. Therefore $\bar{A}$ is dense in $\bar{A}''$ per the weak, the strong, and the weak* topologies (cf. 12.3).

13.1 <u>LEMMA</u> Each $\omega \in S(A)$ has a unique extension to an element $\bar{\omega} \in S_{\underline{n}}(\bar{A}'')$: $\omega = \bar{\omega} \circ \bar{\pi}$.

<u>PROOF</u> Uniqueness follows from 12.13. As for existence, view $x_{\omega} \in E^{\omega}$ as an element $\bar{x}_{\omega}$ of $\bar{E}$ and let $\bar{\omega}$ be the restriction to $\bar{A}''$ of the vector state $\omega_{\bar{x}_{\omega}}$ -- then $\forall\ A \in A$,

$$\begin{aligned}
\omega(A) &= \langle x_{\omega}, \pi^{\omega}(A) x_{\omega} \rangle_{\omega} \\
&= \langle \bar{x}_{\omega}, \bar{\pi}(A) \bar{x}_{\omega} \rangle \\
&= \omega_{\bar{x}_{\omega}}(\bar{\pi}(A)) \\
&= (\bar{\omega} \circ \bar{\pi})(A).
\end{aligned}$$

<u>N.B.</u> The procedure is exhaustive in that every element of $S_{\underline{n}}(\bar{A}'')$ arises in this way.

13.2 <u>REMARK</u> On $\bar{A}''$, the weak and the weak* topologies coincide.

[Every normal state on $\bar{A}''$ is a vector state.]

13.3 <u>THEOREM</u> The map

$$\begin{cases} S(A) \to S_{\underline{n}}(\bar{A}'') \\ \omega \longrightarrow \bar{\omega} \end{cases}$$

is an affine isomorphism and extends to an isometric isomorphism

$$\begin{cases} A^* \to (\bar{A}'')_* \\ \omega \longrightarrow \bar{\omega}\ . \end{cases}$$

<u>PROOF</u> The only thing that has to be checked is the fact that

$$||\bar{\omega}|| = ||\omega|| \qquad (\omega \in A^*).$$

However, according to 9.47, the closed unit ball $\bar{A}^1$ $(= \bar{\pi}(A^1))$ is weakly dense in the closed unit ball of $\bar{A}''$. But $\bar{\omega}$ is weakly continuous (cf. 13.2), so

$$\begin{aligned} ||\bar{\omega}|| &= \sup_{A\in A^1} |(\bar{\omega} \circ \bar{\pi})(A)| \\ &= \sup_{A\in A^1} |\omega(A)| \\ &= ||\omega||. \end{aligned}$$

13.4 <u>REMARK</u> The dual of the arrow

$$A^* \to (\bar{A}'')_*$$

is an isometric isomorphism

$$((\bar{A}'')_*)^* \to A^{**}.$$

Therefore $(\bar{A}'')_*$ is a predual of A^{**}. As it will be shown below that A^{**} is a C*-algebra (cf. 13.20), this means that A^{**} is actually a W*-algebra (cf. 12.28).

There is an arrow

$$\bar{A}'' \overset{\Gamma}{\rightarrow} ((\bar{A}'')_*)^*,$$

viz.

$$\Gamma(\bar{A})(\bar{\omega}) = \bar{\omega}(\bar{A}) \quad \text{(cf. 12.25)}.$$

Denote by Δ the composite

$$\bar{A}'' \overset{\Gamma}{\rightarrow} ((\bar{A}'')_*)^* \rightarrow A^{**}.$$

Then $\forall\ \bar{A} \in \bar{A}''$, $\Delta(\bar{A})$ is that element of A^{**} which sends ω to $\bar{\omega}(\bar{A})$ and by construction, Δ is an isometric isomorphism.

<u>N.B.</u> The diagram

$$\begin{array}{ccc} \bar{A}'' & \xrightarrow{\Delta} & A^{**} \\ \uparrow & & \uparrow \\ \bar{A} & \underset{\bar{\pi}}{\longleftarrow} & A \end{array}$$

commutes. For let $A \in A$ -- then on the one hand, $\hat{A}(\omega) = \omega(A)$, while on the other,

$$\Delta(\bar{\pi}(A))(\omega) = \bar{\omega}(\bar{\pi}(A)) = (\bar{\omega} \circ \bar{\pi})(A) = \omega(A).$$

To proceed further, it will be convenient to introduce some formalities.

So let A be a Banach algebra.

- Given $A \in A$, define linear maps $A \rightarrow A$ by

$$\begin{cases} L_A(B) = AB \\ R_A(B) = BA. \end{cases}$$

Then

$$\begin{cases} L_A^*: A^* \to A^* \\ R_A^*: A^* \to A^*. \end{cases}$$

- Given $\omega \in A^*$, define

$$\begin{cases} \omega_A \in A^* \\ {}_A\omega \in A^* \end{cases}$$

by

$$\begin{cases} \omega_A = (L_A^*)(\omega) \\ {}_A\omega = (R_A^*)(\omega). \end{cases}$$

- Given $f \in A^{**}$, define

$$\begin{cases} f^\omega \in A^* \\ \omega_f \in A^* \end{cases}$$

by

$$\begin{cases} f^\omega(A) = f(\omega_A) \\ \omega_f(A) = f({}_A\omega). \end{cases}$$

13.5 <u>ARENS PRODUCT</u> Given $f,g \in A^{**}$, define

$$\begin{cases} f \,{}_L\!\times g \in A^{**} \\ f \times_R g \in A^{**} \end{cases}$$

by

$$\begin{cases} (f\ {}_L\times g)(\omega) = f({}_g\omega) \\ (f \times{}_R\ g)(\omega) = g(\omega_f). \end{cases}$$

13.6 <u>LEMMA</u> We have

$$\begin{cases} ||\omega_A|| \leq ||\omega||\ ||A|| \\ ||{}_A\omega|| \leq ||\omega||\ ||A||. \end{cases}$$

13.7 <u>LEMMA</u> We have

$$(\omega_A)_B = \omega_{AB},\ ({}_A\omega)_B = {}_A(\omega_B),\ {}_A({}_B\omega) = {}_{AB}\omega.$$

13.8 <u>LEMMA</u> We have

$$\begin{cases} ||{}_f\omega|| \leq ||\omega||\ ||f|| \\ ||\omega_f|| \leq ||\omega||\ ||f||. \end{cases}$$

13.9 <u>LEMMA</u> We have

$$\begin{cases} ({}_f\omega)_A = {}_f(\omega_A) \\ {}_A(\omega_f) = ({}_A\omega)_f. \end{cases}$$

13.10 <u>LEMMA</u> We have

$$\begin{cases} ||f \ {}_L\times g|| \leq ||f|| \ ||g|| \\ ||f \times_R g|| \leq ||f|| \ ||g||. \end{cases}$$

13.11 <u>LEMMA</u> We have

$$\begin{cases} f \ {}_L\times g^{\ \omega} = f^{(g^{\omega})} \\ {}^{\omega}f \times_R g = {}^{({}^{\omega}f)}g. \end{cases}$$

Now bring in the canonical injection

$$\begin{cases} A \rightarrow A^{**} \\ A \rightarrow \hat{A}. \end{cases}$$

13.12 <u>LEMMA</u> We have

$$\begin{cases} \hat{A}^{\omega} = A^{\omega} \\ \omega_{\hat{A}} = \omega_A. \end{cases}$$

13.13 <u>LEMMA</u> We have

$$\begin{cases} \hat{A} \ {}_L\times f = \hat{A} \times_R f = (L_A^{**})(f) \\ f \ {}_L\times \hat{A} = f \times_R \hat{A} = (R_A^{**})(f). \end{cases}$$

13.14 <u>THEOREM</u> Either Arens product makes A^{**} into a Banach algebra and the arrow $A \to A^{**}$ is an injective homomorphism w.r.t. both:

$$\begin{cases} \hat{A}\ {}_L\!\times \hat{B} = \widehat{(AB)} \\ \hat{A} \times_R \hat{B} = \widehat{(AB)}. \end{cases}$$

[Note: If A is unital, then $\hat{1}_A$ is a unit for either Arens product.]

Definition: A is <u>Arens regular</u> if the two products ${}_L\!\times$, $\times_R$ coincide (in which case we simply write $f \times g$).

13.15 <u>EXAMPLE</u> Take G per 5.17 -- then $L^1(G)$ is Arens regular iff G is finite.

13.16 <u>EXAMPLE</u> Take $A = c_0$ -- then $c_0^* \approx \ell^1$ and $c_0^{**} \approx \ell^\infty$. Here ${}_L\!\times = \times_R$ and is just the elementwise multiplication on ℓ^∞.

Suppose in addition that A is a Banach $*$-algebra. Assume: The involution $*: A \to A$ is continuous.

- Given $\omega \in A^*$, define $\omega^*: A \to \underline{C}$ by

$$\omega^*(A) = \overline{\omega(A^*)}.$$

Then $\omega^* \in A^*$, the map $\omega \to \omega^*$ is a linear involution on A^*, and

$$\begin{cases} (\omega_A)^* = {}_{A^*}(\omega^*) \\ ({}_A\omega)^* = (\omega^*)_{A^*}. \end{cases}$$

• Given $f \in A^{**}$, define $f^*: A^* \to \underline{C}$ by

$$f^*(\omega) = \overline{f(\omega^*)}.$$

Then $f^* \in A^{**}$, the map $f \to f^*$ is a linear involution on A^{**}, and

$$\begin{cases} ({}_f\omega)^* = (\omega^*)_{f^*} \\ (\omega_f)^* = {}_{f^*}(\omega^*). \end{cases}$$

13.17 <u>EXAMPLE</u> Take A to be a C*-algebra -- then

$$\Delta: \bar{A}'' \to A^{**}$$

is $*$-linear: $\forall \bar{A} \in \bar{A}''$,

$$\Delta(\bar{A}^*) = \Delta(\bar{A})^*.$$

In fact, $\forall \omega \in A^*$,

$$\Delta(\bar{A}^*)(\omega) = \bar{\omega}(\bar{A}^*),$$

while

$$\Delta(\bar{A})^*(\omega) = \overline{\Delta(\bar{A})(\omega^*)}$$

$$= \overline{\overline{(\omega^*)}(\bar{A})}$$

$$= \overline{(\bar{\omega})^*(\bar{A})}$$

$$= \bar{\omega}(\bar{A}^*).$$

13.18 <u>LEMMA</u> We have

$$\left[\begin{array}{l} (f\ {}_L\!\times g)^* = g^* \times_R f^* \\ \\ (f \times_R g)^* = g^*\ {}_L\!\times f^*. \end{array}\right.$$

Consequently, if A is Arens regular, then A^{**} is a Banach $*$-algebra.

13.19 THEOREM Suppose that A is a C*-algebra -- then A is Arens regular.

PROOF Given $\bar{x},\bar{y} \in \bar{E}$, define $\omega_{\bar{x},\bar{y}}$ by

$$\omega_{\bar{x},\bar{y}}(T) = \langle\bar{x},T\bar{y}\rangle \quad (T \in \mathcal{B}(\bar{E})).$$

Then

$$\omega_{\bar{x},\bar{y}} \circ \bar{\pi} \in A^*$$

and $\forall\ f \in A^{**}$, the expression

$$f(\omega_{\bar{x},\bar{y}} \circ \bar{\pi})$$

is conjugate linear in $\bar{x}$, linear in $\bar{y}$, and

$$\begin{aligned} |f(\omega_{\bar{x},\bar{y}} \circ \bar{\pi})| &\leq ||f||\ ||\omega_{\bar{x},\bar{y}} \circ \bar{\pi}|| \\ &\leq ||f||\ ||\bar{x}||\ ||\bar{y}||, \end{aligned}$$

so $\exists$ a unique operator

$$\Omega_f \in \mathcal{B}(\bar{E})$$

such that

$$f(\omega_{\bar{x},\bar{y}} \circ \bar{\pi}) = \langle\bar{x},\Omega_f\bar{y}\rangle.$$

The map

$$\Omega : \left| \begin{array}{l} \mathcal{A}^{**} \to \mathcal{B}(\bar{E}) \\ \\ f \longrightarrow \Omega_f \end{array} \right.$$

is norm preserving ($||\Omega_f|| = ||f||$) and $\forall\, A \in \mathcal{A}$, $\Omega(\hat{A}) = \bar{\pi}(A)$, i.e., there is a commutative diagram

$$\begin{array}{ccc} \mathcal{A} & \longrightarrow & \mathcal{A}^{**} \\ \bar{\pi} \downarrow & & \downarrow \Omega \\ \mathcal{B}(\bar{E}) & = & \mathcal{B}(\bar{E}). \end{array}$$

Lastly:

- $\langle \bar{x}, \Omega_{f \,{}_L\!\times g}\, \bar{y} \rangle$

$$= f \,{}_L\!\times g\, (\omega_{\bar{x},\bar{y}} \circ \bar{\pi})$$

$$= f({}_g(\omega_{\bar{x},\bar{y}} \circ \bar{\pi}))$$

$$= f(\omega_{\bar{x},\Omega_g \bar{y}} \circ \bar{\pi})$$

$$= \langle \bar{x}, \Omega_f \Omega_g \bar{y} \rangle.$$

- $\langle \bar{x}, \Omega_{f \times_R g}\, \bar{y} \rangle$

$$= f \times_R g(\omega_{\bar{x},\bar{y}} \circ \bar{\pi})$$

$$= g((\omega_{\bar{x},\bar{y}} \circ \bar{\pi})_f)$$

$$= g(\omega_{\Omega_f^*\bar{x},\bar{y}} \circ \bar{\pi})$$

$$= \langle \bar{x}, \Omega_f \Omega_g \bar{y} \rangle$$

=>

$$f \,{}_L\!\times g = f \times_R g.$$

[Note: It is clear that the span of the $\omega_{\bar{x},\bar{y}} \circ \bar{\pi}$ is all of A^* but more is true: Every $\omega \in A^*$ "is" an $\omega_{\bar{x},\bar{y}} \circ \bar{\pi}$.]

<u>N.B.</u> We have

$$\Omega_{f \times g} = \Omega_f \Omega_g$$

and

$$\Omega_f^* = \Omega_{f^*}.$$

[To check the second point, write

$$\langle \bar{x}, \Omega_{f^*} \bar{y} \rangle = f^*(\omega_{\bar{x},\bar{y}} \circ \bar{\pi})$$

$$= \overline{f((\omega_{\bar{x},\bar{y}} \circ \bar{\pi})^*)}$$

$$= \overline{f(\omega_{\bar{y},\bar{x}} \circ \bar{\pi})}$$

$$= \overline{\langle \bar{y}, \Omega_f \bar{x} \rangle}$$

$$= \langle \Omega_f \bar{x}, \bar{y} \rangle$$

$$= \langle \bar{x}, \Omega_f^* \bar{y} \rangle.]$$

13.20 <u>LEMMA</u> Suppose that A is a C*-algebra -- then A^{**} is a C*-algebra.

<u>PROOF</u> $\forall f \in A^{**}$,

$$||f^* \times f|| = ||\Omega_{f^* \times f}||$$

$$= ||\Omega_{f^*} \Omega_f||$$

$$= ||\Omega_f^* \Omega_f||$$

$$= ||\Omega_f||^2$$

$$= ||f||^2.$$

Maintaining the supposition that A is a C*-algebra, note that $\Omega(A^{**}) = \bar{A}''$ and consider the composite $\Delta \circ \Omega$:

$$A^{**} \overset{\Omega}{\rightarrow} \bar{A}'' \overset{\Delta}{\rightarrow} A^{**}.$$

Then $\forall f \in A^{**}$,

$$(\Delta \circ \Omega)(f) = \Delta(\Omega_f)$$

and

$$\Delta(\Omega_f)(\omega_{\bar{x},\bar{y}} \circ \bar{\pi})$$

$$= \omega_{\bar{x},\bar{y}}(\Omega_f)$$

$$= \langle \bar{x}, \Omega_f \bar{y} \rangle$$

$$= f(\omega_{\bar{x},\bar{y}} \circ \bar{\pi}).$$

Therefore

$$\Delta \circ \Omega = \text{id}_{A^{**}} .$$

13.21 LEMMA Δ is a $*$-isomorphism.

PROOF We already know that Δ is an isometric isomorphism which, moreover, is $*$-linear (cf. 13.17), thus one has only to show that

$$\Delta(\bar{A}\bar{B}) = \Delta(\bar{A}) \times \Delta(\bar{B}) \quad (\bar{A},\bar{B} \in \bar{A}'').$$

But

$$\begin{aligned}\Delta(\bar{A}\bar{B}) &= \Delta(\Omega_f\Omega_g) \qquad (\bar{A} = \Omega_f, \ \bar{B} = \Omega_g)\\ &= \Delta(\Omega_{f \times g})\\ &= f \times g\\ &= \Delta(\Omega_f) \times \Delta(\Omega_g)\\ &= \Delta(\bar{A}) \times \Delta(\bar{B}).\end{aligned}$$

N.B. Therefore Δ is normal (cf. 12.12).

[Note: Recall that A^{**} is a W^*-algebra (cf. 13.4), hence is monotone complete.]

13.22 EXAMPLE Let H be a complex Hilbert space -- then

$$\begin{cases} \underline{L}_\infty(H)^* \approx \underline{L}_1(H) \\ \underline{L}_1(H)^* \approx \mathcal{B}(H). \end{cases}$$

And the isometric isomorphism $\mathcal{B}(H) \to \underline{L}_\infty(H)^{**}$ arising therefrom is a $*$-isomorphism, where $\underline{L}_\infty(H)^{**}$ carries the Arens product.

13.23 <u>LEMMA</u> Let π be a nondegenerate $*$-representation of A on E -- then there is a unique normal $*$-homomorphism π'' of $\bar{A}''$ onto $\pi(A)''$ such that $\pi'' \circ \bar{\pi} = \pi$:

$$\begin{array}{ccc} A & \xrightarrow{\bar{\pi}} & \bar{A}'' \\ \pi \downarrow & & \downarrow \pi'' \\ \pi(A)'' & = & \pi(A)''. \end{array}$$

<u>PROOF</u> Take for π'' the composite

$$\begin{array}{l} \bar{A}'' \xrightarrow{\Delta} A^{**} \xrightarrow{\pi^{**}} (\pi(A)'')^{**} \\ \qquad\qquad\qquad \downarrow \text{inc}^* \\ \qquad\qquad\qquad ((\pi(A)'')_*)^* \\ \qquad\qquad\qquad \downarrow \Gamma^{-1} \\ \qquad\qquad\qquad \pi(A)''. \end{array}$$

Here

$$(\pi(A)'')_* \xrightarrow{\text{inc}} (\pi(A)'')^*$$

=>

$$(\pi(A)'')^{**} \xrightarrow{\text{inc}^*} ((\pi(A)'')_*)^*.$$

And

$$X \in \pi(A)'' \Rightarrow \hat{X} \in (\pi(A)'')^{**}$$

=>

$$\Gamma^{-1} \circ \text{inc}^*(\hat{X}) = X.$$

There remains the claim that $\pi'' \circ \bar{\pi} = \pi$. So let $A \in \mathcal{A}$:

$$\left[\begin{array}{l} \Delta(\bar{\pi}(A)) = \hat{A} \\ \pi^{**}(\hat{A}) = \hat{A} \circ \pi^* = \widehat{\pi(A)} \\ \Gamma^{-1} \circ \text{inc}^*(\widehat{\pi(A)}) = \pi(A). \end{array}\right.$$

[Note: π'' is necessarily weak* continuous (cf. 12.15).]

Now specialize and assume further that $\mathcal{A} \subset \mathcal{B}(\mathcal{H})$ is a von Neumann algebra. Let $\pi:\mathcal{A} \to \mathcal{B}(\mathcal{H})$ be the identity map -- then $\exists$ a unique central projection P in $\bar{A}''$ such that Ker $\pi'' = P\bar{A}''$ and π'' is a $*$-isomorphism of $P^{\perp}\bar{A}''$ onto $\mathcal{A}$ (cf. 12.19).

<u>N.B.</u> $\forall\, A \in \mathcal{A}$,

$$\pi''(\bar{\pi}(A)) = \pi(A) = A,$$

so

$$\begin{aligned} \pi''(P^{\perp}\bar{\pi}(A)) &= \pi''(P^{\perp})\pi''(\bar{\pi}(A)) \\ &= 1_{\mathcal{A}}A \\ &= A. \end{aligned}$$

Therefore

$$P^{\perp}\bar{A} = \bar{P}^{\perp}\bar{A}''$$

and π'' is a $*$-isomorphism of $P^{\perp}\bar{A}$ onto $\mathcal{A}$.

Definition: $S(A)$ is the convex direct sum of convex subsets $S_1, S_2 \subset S(A)$ if each $\omega \in S(A)$ admits a unique decomposition

$$\omega = \lambda\omega_1 + (1 - \lambda)\omega_2 \quad (\omega_1 \in S_1, \omega_2 \in S_2, \ 0 \le \lambda \le 1).$$

Notation:

$$S(A) = S_1 \underset{\text{cvx}}{\oplus} S_2.$$

A norm closed face $F \subset S(A)$ is said to be a split face if there is a face $F^{\perp} \subset S(A)$ such that $S(A)$ is the convex direct sum of F and $F^{\perp}$:

$$S(A) = F \underset{\text{cvx}}{\oplus} F^{\perp}.$$

[Note: $F^{\perp}$ is norm closed and is uniquely determined by F.]

13.24 LEMMA $S_{\underline{n}}(A)$ is a split face of $S(A)$.

PROOF Let

$$F_{P^{\perp}} \subset S_{\underline{n}}(\bar{A}'')$$

be the split face corresponding to $P^{\perp}$ per 12.40, thus

$$F_{P^{\perp}} = \{\bar{\omega} \in S_{\underline{n}}(\bar{A}''): \bar{\omega}(P^{\perp}) = 1\}.$$

Taking into account the identification

$$S(A) \longleftrightarrow S_{\underline{n}}(\bar{A}'') \qquad \text{(cf. 13.3)},$$

let

$$F \longleftrightarrow F_{P^{\perp}},$$

the contention being that

$$F = S_{\underline{n}}(A).$$

Thus let $\omega \in S_{\underline{n}}(A)$ and consider $\omega \circ \pi'' \in S_{\underline{n}}(\bar{A}'')$ -- then

$$\omega \circ \pi'' \circ \bar{\pi} = \omega \circ \pi = \omega$$

and

$$\omega \circ \pi''(P^{\perp}) = \omega(1_A) = 1.$$

Therefore

$$S_{\underline{n}}(A) \subset F.$$

As for the other direction, let $\omega \in F$, so $\omega = \bar{\omega} \circ \bar{\pi}$ $(\bar{\omega} \in F_{P^{\perp}})$. To verify that $\omega \in S_{\underline{n}}(A)$, let $\{A_i : i \in I\} \subset A_{SA}$ be a bounded increasing net and put $A = \sup_{i \in I} A_i$ -- then

$$P^{\perp}\bar{\pi}(A_i) \uparrow P^{\perp}\bar{\pi}(A)$$

=>

$$\bar{\omega}(P^{\perp}\bar{\pi}(A_i)) \uparrow \bar{\omega}(P^{\perp}\bar{\pi}(A))$$

=>

$$\bar{\omega}(\bar{\pi}(A_i)) \uparrow \bar{\omega}(\bar{\pi}(A)) \quad \text{(cf. 12.42)}$$

=>

$$\omega(A_i) \uparrow \omega(A)$$

=>

$$\omega \in S_{\underline{n}}(A).$$

Therefore

$$F \subset S_{\underline{n}}(A).$$

Consequently,

$$S(A) = S_{\underline{n}}(A) \underset{\text{cvx}}{\oplus} S_{\underline{n}}(A)^{\perp}.$$

N.B. The elements of $S_{\underline{n}}(A)^{\perp}$ are said to be singular.

E.g.: A pure state is either normal or singular.

13.25 REMARK We have

$$S_{\underline{n}}(\bar{A}'') = F_{P^{\perp}} \underset{\text{cvx}}{\oplus} F_P$$

and

$$\begin{cases} F_{P^{\perp}} \longleftrightarrow S_{\underline{n}}(A) \\ F_P \longleftrightarrow S_{\underline{n}}(A)^{\perp}. \end{cases}$$

13.26 LEMMA Fix $\omega_0 \in S(A)$ -- then ω_0 is singular iff there is no nonzero weak* continuous positive linear functional ω on A such that $\omega \leq \omega_0$.

PROOF Write $\omega_0 = \lambda\sigma + (1 - \lambda)\tau$ $(\sigma \in S_{\underline{n}}(A),\ \tau \in S_{\underline{n}}(A)^{\perp})$. If ω_0 is not singular, then $\lambda\sigma$ is a nonzero weak* continuous positive linear functional on A such that $\lambda\sigma \leq \omega_0$. Suppose, conversely, that there is such an ω. Introduce

$$\begin{cases} \bar{\omega}_0 \\ \bar{\omega} \end{cases} \text{by} \begin{cases} \bar{\omega}_0 \circ \bar{\pi} = \omega_0 \\ \bar{\omega} \circ \bar{\pi} = \omega \end{cases} \qquad \text{(cf. 13.3).}$$

Since $\omega > 0$ is weak* continuous,

$$\bar{\omega}(P^{\perp}) = ||\bar{\omega}|| > 0,$$

hence

$$\bar{\omega}_0(P^{\perp}) \geq \bar{\omega}(P^{\perp}) > 0$$

$$\Rightarrow$$

$$\bar{\omega}_0 \notin F_P.$$

Therefore ω_0 is not singular.

13.27 EXAMPLE Suppose that H is an infinite dimensional complex Hilbert space. Let $\omega \in S(\mathcal{B}(H))$ -- then ω is singular iff $\omega|\underline{L}_\infty(H) = 0$.

13.28 REMARK Let $\omega \in \mathcal{B}(H)^*$ -- then ω is weak* continuous iff $||\omega|| = ||\omega|\underline{L}_\infty(H)||$.

§14. *FOLIA*

Suppose that A is a C*-algebra. Let π be a nondegenerate *-representation of A on E. Take $\pi"$ as in 13.23 (so $\pi" \circ \bar{\pi} = \pi$) -- then $\exists$ a unique central projection $P(\pi)$ in $\bar{A}"$ such that $\text{Ker } \pi" = P(\pi)\bar{A}"$ (cf. 12.19). Now put

$$C(\pi) = P(\pi)^{\perp}$$

and call $C(\pi)$ the central cover of π.

N.B. $\pi"$ is a *-isomorphism of $C(\pi)\bar{A}"$ onto $\pi(A)"$ (cf. 12.19).

14.1 LEMMA Let π_1 and π_2 be nondegenerate *-representations of A on E_1 and E_2 -- then

$$C(\pi_1) = C(\pi_2) \iff \text{Ker } \pi_1" = \text{Ker } \pi_2".$$

[This is trivial:

$$\begin{cases} \text{Ker } \pi_1" = C(\pi_1)^{\perp}\bar{A}" \\ \text{Ker } \pi_2" = C(\pi_2)^{\perp}\bar{A}".] \end{cases}$$

14.2 RAPPEL Suppose that A is a *-algebra. Let π_1 and π_2 be nondegenerate *-representations of A on E_1 and E_2 -- then π_1 and π_2 are geometrically equivalent iff $\exists$ a *-isomorphism

$$\Phi:\pi_1(A)" \rightarrow \pi_2(A)"$$

such that $\forall$ A $\in$ A,

$$\Phi(\pi_1(A)) = \pi_2(A).$$

14.3 <u>LEMMA</u> Let π_1 and π_2 be nondegenerate $*$-representations of A on E_1 and E_2 -- then π_1 and π_2 are geometrically equivalent iff $C(\pi_1) = C(\pi_2)$.

<u>PROOF</u> Suppose first that π_1 and π_2 are geometrically equivalent and take Φ as in 14.2 -- then Φ is normal (cf. 12.12), hence is weak* continuous (cf. 12.15), and $\forall\ A \in A$,

$$\Phi(\pi_1(A)) = \pi_2(A)$$

or still,

$$\Phi(\pi_1''(\bar{\pi}(A))) = \pi_2''(\bar{\pi}(A)).$$

But $\bar{A} = \bar{\pi}(A)$ is dense in $\bar{A}''$ per the weak* topology, so

$$\Phi(\pi_1''(\bar{A})) = \pi_2''(\bar{A}) \quad (\bar{A} \in \bar{A}'').$$

Therefore

$$\text{Ker } \pi_1'' = \text{Ker } \pi_2'',$$

i.e.,

$$C(\pi_1) = C(\pi_2).$$

Conversely,

$$C(\pi_1) = C(\pi_2) \Rightarrow \text{Ker } \pi_1'' = \text{Ker } \pi_2'',$$

thus the prescription

$$\Phi(\pi_1''(\bar{A})) = \pi_2''(\bar{A}) \quad (\bar{A} \in \bar{A}'')$$

makes sense.

14.4 <u>SCHOLIUM</u> Let Rep A be the set of all nondegenerate $*$-representations of A (cf. 9.15) and let $C(A)$ be the set of all central projections in $\bar{A}''$ -- then

$$\left[\begin{array}{l} \text{Rep } A/\sim \;\longleftrightarrow\; C(A) \\ \quad [\pi] \;\longleftrightarrow\; C(\pi). \end{array}\right.$$

E.g.: $C(\pi) = 0$ corresponds to $\pi : A \to \{0\}$.

Given $\pi_1, \pi_2 \in \text{Rep } A$, write $\pi_1 \le \pi_2$ if π_1 is geometrically equivalent to a sub $*$-representation of π_2.

14.5 LEMMA We have $\pi_1 \le \pi_2$ iff $C(\pi_1) \le C(\pi_2)$.

E.g.: $\forall\, \pi \in \text{Rep } A$, $\pi \le \bar{\pi}$.

14.6 REMARK π_1 and π_2 are disjoint iff $C(\pi_1)C(\pi_2) = 0$.

Definition: A folium $\mathcal{F}$ is a norm closed convex subset of $S(A)$ which is "invariant" in the sense that if $\omega \in \mathcal{F}$ and if $\omega(B^*B) \ne 0$, then $\omega_B \in \mathcal{F}$.

[Note: Here

$$\omega_B(A) = \frac{\omega^B(A)}{\omega(B^*B)} = \frac{\omega(B^*AB)}{\omega(B^*B)} .]$$

Given a nondegenerate $*$-representation $\pi : A \to B(E)$, put

$$A_\pi = \pi(A)''$$

and let

$$\mathcal{F}(\pi) = \{\omega \circ \pi : \omega \in S_{\underline{n}}(A_\pi)\}.$$

14.7 LEMMA $\mathcal{F}(\pi)$ is a folium.

[To check invariance, suppose that

$$(\omega \circ \pi)(B^*B) \neq 0$$

and then write

$$\frac{\omega(\pi(B^*AB))}{\omega(\pi(B^*B))} = \frac{\omega(\pi(B)^*\pi(A)\pi(B))}{\omega(\pi(B)^*\pi(B))}$$

$$= \omega_{\pi(B)}(\pi(A))$$

$$= (\omega_{\pi(B)} \circ \pi)(A).$$

But $\omega_{\pi(B)} \in S_{\underline{n}}(A_\pi)$ (see the discussion prefacing 12.43).]

14.8 LEMMA $\forall\ \pi \in \text{Rep}\ A$,

$$\text{Ker}\ \pi = \bigcap_{\omega \in S_{\underline{n}}(A_\pi)} \text{Ker}\ \omega \circ \pi.$$

[The normal states separate the points of A_π.]

14.9 THEOREM Let π_1 and π_2 be nondegenerate $*$-representations of A in E_1 and E_2 -- then π_1 and π_2 are geometrically equivalent iff $F(\pi_1) = F(\pi_2)$.

PROOF Suppose first that π_1 and π_2 are geometrically equivalent and take Φ per 14.2. Since Φ and Φ^{-1} are weak* continuous, the arrow

$$\begin{cases} S_{\underline{n}}(A_{\pi_2}) \to S_{\underline{n}}(A_{\pi_1}) \\ \omega_2 \to \omega_2 \circ \Phi \end{cases}$$

is bijective, thus $F(\pi_1) = F(\pi_2)$. Turning to the converse,

$$F(\pi_1) = F(\pi_2) \Rightarrow \text{Ker } \pi_1 = \text{Ker } \pi_2 \quad \text{(cf. 14.8)},$$

from which a *-isomorphism

$$\begin{cases} \phi:\pi_1(A) \to \pi_2(A) \\ \phi(\pi_1(A)) = \pi_2(A) \quad (A \in A). \end{cases}$$

Next, $\forall\ \omega_2 \in S_{\underline{n}}(A_{\pi_2})$,

$$\omega_2 \circ \pi_2 \in F(\pi_2) = F(\pi_1)$$

$$\Rightarrow$$

$$\omega_2 \circ \pi_2 = \omega_1 \circ \pi_1 \quad (\exists\ \omega_1 \in S_{\underline{n}}(A_{\pi_1}))$$

$$\Rightarrow$$

$$(\omega_2 \circ \phi)(\pi_1(A)) = \omega_2(\pi_2(A)) = \omega_1(\pi_1(A)).$$

Therefore $\omega_2 \circ \phi$ $(= \omega_1)$ is weak* continuous. But every weak* continuous linear functional on $\pi_2(A)$ is a linear combination of (restrictions) of elements of $S_{\underline{n}}(A_{\pi_2})$. Accordingly, from the very definition of the weak* topology as an initial topology, ϕ (and its inverse) must be weak* continuous, so $\exists$ a weak* continuous *-isomorphism $\Phi:A_{\pi_1} \to A_{\pi_2}$ such that $\Phi|A = \phi$. Now quote 14.2 to conclude that π_1 and π_2 are geometrically equivalent.

The following generality was tacitly used above.

14.10 <u>LEMMA</u> Let H and K be complex Hilbert spaces. Suppose that $A \subset B(H)$ is a C*-subalgebra and $\phi:A \to B(K)$ is a linear map. Assume: ϕ is weak* continuous --

then ϕ extends uniquely to a weak* continuous linear map $\Phi: A'' \to B(K)$. Moreover, if ϕ is a $*$-homomorphism (hence $\phi(A)$ is a C*-subalgebra of $B(K)$), then Φ is a $*$-homomorphism and $\Phi(A'') = \phi(A)''$.

[Note: In particular, every weak* continuous linear functional $\omega: A \to \underline{C}$ extends uniquely to a weak* continuous linear functional $\bar{\omega}: A'' \to \underline{C}$.]

14.11 <u>LEMMA</u> We have $\pi_1 \leq \pi_2$ iff $F(\pi_1) \subset F(\pi_2)$.

E.g.: $\forall\ \pi \in \text{Rep } A,\ F(\pi) \subset F(\bar{\pi})\ \ (= S(A))$.

14.12 <u>REMARK</u> π_1 and π_2 are disjoint iff $F(\pi_1) \cap F(\pi_2) = \emptyset$.

Given $\omega \in S(A)$, let

$$F(\omega) = F(\pi^{\omega}).$$

Then

$$\omega \in F(\omega).$$

Proof: $\forall\ A \in A$,

$$\omega(A) = \langle x_{\omega}, \pi^{\omega}(A) x_{\omega} \rangle_{\omega}.$$

On the other hand, the orthogonal projection P_{ω} of E^{ω} onto $\underline{C}x_{\omega}$ is a density operator and the assignment

$$A \to \text{tr}(P_{\omega}\pi^{\omega}(A)) \equiv \omega(A)$$

is an element of $F(\pi^{\omega})$.

<u>N.B.</u> $F(\omega)$ is the smallest folium containing ω.

14.13 LEMMA If F is a folium in $S(A)$, then $\exists$ a $\pi \in \text{Rep } A$, determined up to geometric equivalence, such that $F(\pi) = F$.

[One has only to take for π the direct sum of the π^{ω} ($\omega \in F$).]

The folia in $S(A)$ are thus in a one-to-one correspondence with the geometric equivalence classes in Rep A.

[Note: Conventionally, the empty folium corresponds to $\pi: A \rightarrow \{0\}$.]

§15. *C*-CATEGORIES*

Given a category $\underset{\sim}{C}$, denote by Ob $\underset{\sim}{C}$ its class of objects and by Mor $\underset{\sim}{C}$ its class of morphisms. If $X,Y \in \text{Ob } \underset{\sim}{C}$ is an ordered pair of objects, then Mor(X,Y) is the set of morphisms (or arrows) from X to Y. An element $f \in \text{Mor}(X,Y)$ is said to have <u>domain</u> X and <u>codomain</u> Y. One writes $f:X \rightarrow Y$ or $X \overset{f}{\rightarrow} Y$.

We shall now impose a series of conditions which in total lead to the notion of C*-category.

1. $\forall\ X,Y \in \text{Ob } \underset{\sim}{C}$, Mor(X,Y) is a complex vector space and composition

$$\text{Mor}(X,Y) \times \text{Mor}(Y,Z) \rightarrow \text{Mor}(X,Z),$$

denoted by $(f,g) \rightarrow g \circ f$, is bilinear.

2. $\forall\ X,Y \in \text{Ob } \underset{\sim}{C}$, Mor(X,Y) is a Banach space and

$$\forall \left[\begin{array}{l} f \in \text{Mor}(X,Y) \\ \\ g \in \text{Mor}(Y,Z) \end{array} \right., \quad ||g \circ f|| \leq ||g||\ ||f||.$$

3. $\exists$ an involutive, identity on objects, cofunctor

$$*:\underset{\sim}{C} \rightarrow \underset{\sim}{C}.$$

Spelled out (in superscript notation):

$$\forall\ X \in \text{Ob } \underset{\sim}{C},\ X^* = X$$

and

$$\forall\ X,Y \in \text{Ob } \underset{\sim}{C},\ *:\text{Mor}(X,Y) \rightarrow \text{Mor}(Y,X)$$

subject to

$$(af + bg)^* = \bar{a}f^* + \bar{b}g^* \quad (a,b \in \underset{\sim}{C}).$$

In addition,

$$\begin{cases} f^{**} = f \\ (g \circ f)^* = f^* \circ g^*. \end{cases}$$

4. $\forall$ X,Y $\in$ Ob $\underline{C}$ & f $\in$ Mor(X,Y),

$$||f||^2 = ||f^* \circ f||$$

and

$$f^* \circ f \in \text{Mor}(X,X)_+.$$

Summing up: $\underline{C}$ is said to be a <u>C*-category</u> if conditions 1,2,3,4 are satisfied.

<u>N.B.</u> $\forall$ X $\in$ Ob $\underline{C}$, Mor(X,X) is a unital C*-algebra.

[Note: Every unital C*-algebra A can be viewed as a C*-cateogory with one object.]

15.1 <u>EXAMPLE</u> Take $\underline{C}$ = <u>HILB</u> (cf. 4.28) — then $\underline{C}$ is a C*-category.

15.2 <u>EXAMPLE</u> Let A be a C*-algebra and take $\underline{C} = \underline{\text{H*MOD}}_A$ (cf. 4.27) — then $\underline{C}$ is a C*-category (use 4.26).

15.3 <u>EXAMPLE</u> Let A be a unital C*-algebra — then by End A we shall understand the C*-category whose objects are the unital $\star$-homomorphisms $\Phi: A \to A$ and whose arrows $\Phi \to \Psi$ are the intertwiners, i.e.,

$$\text{Mor}(\Phi,\Psi) = \{T \in A : T\Phi(A) = \Psi(A)T \ \forall \ A \in A\}.$$

Here, the composition of arrows, when defined, is given by the product in A and $1_A \in \text{Mor}(\Phi,\Phi)$ is 1_Φ. As for

$$\star : \text{End } A \to \text{End } A,$$

take it to be the identity on objects and then define

$$*:\mathrm{Mor}(\Phi,\Psi) \to \mathrm{Mor}(\Psi,\Phi)$$

by sending T to T*.

15.4 EXAMPLE Given a C*-algebra A, there is a C*-category whose objects are the elements π of Rep A (cf. 9.15) and whose morphisms $\pi_1 \to \pi_2$ are the topological intertwining operators, i.e.,

$$\mathrm{Mor}(\pi_1,\pi_2) = \{T \in \mathcal{B}(E_1,E_2) : T\pi_1(A) = \pi_2(A)T \ \forall\ A \in A\}.$$

[Note: $\mathrm{Mor}(\pi_1,\pi_2)$ is a nonempty closed subspace of $\mathcal{B}(E_1,E_2)$ which, moreover, is trivial iff π_1 and π_2 are disjoint.]

15.5 EXAMPLE Let A be a unital C*-algebra -- then by Mat A we shall understand the category whose objects are the natural numbers and whose morphisms $n \to m$ are the n-by-m matrices with entries in A (cf. 4.41). Here, composition of

$$\begin{cases} A \in \mathrm{Mor}(n,m) \\ B \in \mathrm{Mor}(m,p) \end{cases}$$

is the prescription

$$B \circ A = AB,$$

where AB is the usual multiplication of matrices, and id_n is the unit diagonal n-by-n matrix, i.e., $id_n = \mathrm{diag}\ 1_A$. As for

$$*:\mathrm{Mat}\ A \to \mathrm{Mat}\ A,$$

take it to be the identity on objects and then define

$$*:\mathrm{Mor}(n,m) \to \mathrm{Mor}(m,n)$$

by sending $[A_{ij}]$ to $[A^*_{ji}]$.

15.6 <u>REMARK</u> The technical requirement that

$$f^* \circ f \in \text{Mor}(X,X)_+$$

is not an automatic consequence of the other conditions. To see this, consider the category with two objects X and Y, where

$$\begin{cases} \text{Mor}(X,X) = \text{Mor}(Y,Y) = \underline{C} \\ \text{Mor}(X,Y) = \text{Mor}(Y,X) = \underline{C}, \end{cases}$$

and composition is multiplication of complex numbers. Take the norm of $z \in \underline{C}$ to be $|z|$ and define $*$ by

$$X^* = X,\ Y^* = Y$$

and

$$z^* = \begin{cases} \bar{z} \text{ if } z \in \text{Mor}(X,X) \text{ or } \text{Mor}(Y,Y) \\ -\bar{z} \text{ if } z \in \text{Mor}(X,Y) \text{ or } \text{Mor}(Y,X). \end{cases}$$

Then $\forall\ z \in \text{Mor}(X,Y)$,

$$z^* \circ z = (-\bar{z})(z) = -|z|^2 \notin \text{Mor}(X,X)_+.$$

Let $\underline{C}$ and $\underline{D}$ be C*-categories -- then a functor $F:\underline{C} \to \underline{D}$ is said to be a <u>C*-functor</u> if $\forall\ X,Y \in \text{Ob } \underline{C}$,

$$F:\text{Mor}(X,Y) \to \text{Mor}(FX,FY)$$

is linear and $\forall\ f \in \text{Mor}(X,Y)$,

$$F(f^*) = (Ff)^*.$$

N.B. $\forall\ X \in \text{Ob}\ \underline{C}$, the map

$$\text{Mor}(X,X) \to \text{Mor}(FX,FX)$$

is a unital $*$-homomorphism.

15.7 LEMMA Suppose that $F:\underline{C} \to \underline{D}$ is a C*-functor -- then $\forall\ f \in \text{Mor}(X,Y)$,

$$||Ff|| \leq ||f||.$$

PROOF By hypothesis, $\exists\ A \in \text{Mor}(X,X)$ such that

$$f^* \circ f = A^* \circ A.$$

But

$$||F(A^* \circ A)|| \leq ||A^* \circ A|| \quad \text{(cf. 1.7)}.$$

Therefore

$$||F(f^* \circ f)|| \leq ||f^* \circ f||$$

=>

$$||(Ff)^* \circ Ff|| \leq ||f^* \circ f||$$

=>

$$||Ff||^2 \leq ||f||^2.$$

Accordingly, if $F:\underline{C} \to \underline{D}$ is a C*-functor, then the linear maps

$$\text{Mor}(X,Y) \to \text{Mor}(FX,FY)$$

are continuous.

15.8 LEMMA Suppose that $F:\underline{C} \to \underline{D}$ is a C*-functor. Assume: F is faithful -- then $\forall\ f \in \text{Mor}(X,Y)$,

$$||Ff|| = ||f||.$$

PROOF $\forall X \in \text{Ob } \underline{C}$, the map

$$\text{Mor}(X,X) \rightarrow \text{Mor}(FX,FX)$$

is injective (F being faithful), hence $\forall A \in \text{Mor}(X,X)$,

$$||F(A^* \circ A)|| = ||A^* \circ A|| \quad (\text{cf. } 1.8).$$

Now repeat the argument of 15.7.

Let $\underline{C}$ be a C*-category -- then a representation of $\underline{C}$ is a C*-functor

$$\pi:\underline{C} \rightarrow \underline{\text{HILB}}.$$

15.9 THEOREM Fix $X \in \text{Ob } \underline{C}$ and let $\omega \in S(\text{Mor}(X,X))$ -- then there is a representation $\pi^\omega:\underline{C} \rightarrow \underline{\text{HILB}}$ and an element $x_\omega \in \pi^\omega X$ of norm 1 such that

$$\omega(f) = \langle x_\omega, \pi^\omega(f) x_\omega \rangle$$

for all $f \in \text{Mor}(X,X)$.

[This is a straightforward extension of the standard GNS construction.]

15.10 THEOREM Suppose that $\underline{C}$ is small -- then $\underline{C}$ admits a faithful representation $\pi:\underline{C} \rightarrow \underline{\text{HILB}}$.

PROOF Fix $X \in \text{Ob } \underline{C}$ and let $\underline{C}_X$ be the full subcategory of $\underline{C}$ consisting of those $Y \in \text{Ob } \underline{C}$ such that $\text{Mor}(X,Y) \neq \{0\}$. Given $\omega \in S(\text{Mor}(X,X))$, choose $\pi^\omega:\underline{C}_X \rightarrow \underline{\text{HILB}}$ per 15.9 and set

$$\pi^X = \bigoplus_\omega \pi^\omega,$$

where $\oplus_\omega$ is taken over $S(\text{Mor}(X,X))$. Claim: π^X is faithful. For let $g \in \text{Mor}(Y,Z)$ and choose $f \in \text{Mor}(X,Y): ||f|| = 1$ -- then $\exists\ A \in \text{Mor}(X,X)$ such that

$$(g \circ f)^* \circ (g \circ f) = A^* \circ A,$$

thus

$$||A||^2 = ||g \circ f||^2.$$

But $\exists\ \omega \in S(\text{Mor}(X,X))$:

$$\omega(A^* \circ A) = ||A||^2 \quad (\text{cf. } 7.25),$$

so

$$\omega((g \circ f)^* \circ (g \circ f)) = ||g \circ f||^2,$$

from which

$$||\pi^X(g)|| \geq ||g||.$$

Therefore π^X is faithful. Now put

$$\pi = \bigoplus_{X \in \text{Ob}\,\underline{C}} \pi^X.$$

Then $\pi:\underline{C} \to \underline{\text{HILB}}$ is faithful.

15.11 RAPPEL Let $\underline{C}$, $\underline{D}$ be categories and let

$$\begin{cases} F:\underline{C} \to \underline{D} \\ G:\underline{C} \to \underline{D} \end{cases}$$

be functors -- then a natural transformation Ξ from F to G is a function that assigns to each $X \in \text{Ob}\,\underline{C}$ an element $\Xi_X \in \text{Mor}(FX,GX)$ such that $\forall\ f \in \text{Mor}(X,Y)$ the square

$$\begin{array}{ccc} FX & \xrightarrow{\Xi_X} & GX \\ {\scriptstyle Ff}\downarrow & & \downarrow{\scriptstyle Gf} \\ FY & \xrightarrow[\Xi_Y]{} & GY \end{array}$$

commutes.

Let $\underline{C}$, $\underline{D}$ be C*-categories and let

$$\begin{cases} F:\underline{C} \to \underline{D} \\ G:\underline{C} \to \underline{D} \end{cases}$$

be C*-functors. Given a natural transformation $\Xi \in \text{Nat}(F,G)$, put

$$||\Xi|| = \sup_{X \in \text{Ob } \underline{C}} ||\Xi_X||$$

and call Ξ <u>bounded</u> if

$$||\Xi|| < \infty.$$

15.12 <u>REMARK</u> A natural transformation $\Xi:F \to G$ need not be bounded. Thus let $\underline{C} = \underline{D}$ be the C*-category whose objects are the positive integers 1,2,... with $\text{Mor}(n,m) = \underline{C}$, composition being induced by multiplication in $\underline{C}$ with involution complex conjugation. Take $F = \text{id}_{\underline{C}}$ and define $\Xi:\text{id}_{\underline{C}} \to \text{id}_{\underline{C}}$ by specifying that $\Xi_n:n \to n$ sends z to nz -- then Ξ is not bounded.

15.13 <u>LEMMA</u> Let $\underline{C}$, $\underline{D}$ be C*-categories -- then the category $[\underline{C},\underline{D}]^*$ whose objects are the C*-functors $F:\underline{C} \to \underline{D}$ and whose morphisms are the bounded natural transformations $\Xi:F \to G$ is a C*-category.

[To define

$$*:[\underline{C},\underline{D}]^* \to [\underline{C},\underline{D}]^*,$$

take it to be the identity on objects and given $\Xi:F \to G$, specify $\Xi^*:G \to F$ in the obvious way, viz.

$$\Xi_X \in \mathrm{Mor}(FX,GX) \Rightarrow \Xi_X^* \in \mathrm{Mor}(GX,FX).$$

Then $\forall\ f \in \mathrm{Mor}(X,Y)$, the square

$$\begin{array}{ccc} GX & \xrightarrow{\Xi_X^*} & FX \\ {\scriptstyle Gf}\downarrow & & \downarrow{\scriptstyle Ff} \\ GY & \xrightarrow[\Xi_Y^*]{} & FY \end{array}$$

commutes. Indeed,

$$\begin{aligned} Ff \circ \Xi_X^* &= F(f^{**}) \circ \Xi_X^* \\ &= F(f^*)^* \circ \Xi_X^* \\ &= (\Xi_X \circ F(f^*))^* \\ &= (G(f^*) \circ \Xi_Y)^* \\ &= \Xi_Y^* \circ G(f^*)^* \\ &= \Xi_Y^* \circ G(f^{**}) \\ &= \Xi_Y^* \circ Gf. \end{aligned}$$

Moreover, $\Xi^* \in \mathrm{Mor}(G,F)$, i.e., is bounded:

$$||\Xi_X^*||^2 = ||\Xi_X \circ \Xi_X^*||$$

$$\leq ||\Xi_X||\ ||\Xi_X^*||$$

=>

$$||\Xi_X^*|| \leq ||\Xi_X|| \leq ||\Xi|| < \infty.]$$

[Note: Strictly speaking, [C,D]* is a metacategory, not a category.]

E.g.: The objects of [C,HILB]* are the representations of C.

§16. *THE CATEGORY OF CATEGORIES*

Let $i:A \to Y$, $p:X \to B$ be morphisms in a category $\underline{C}$ — then i is said to have the *left lifting property with respect to p* (LLP w.r.t. p) and p is said to have the *right lifting property with respect to i* (RLP w.r.t. i) if for all $u:A \to X$, $v:Y \to B$ such that $p \circ u = v \circ i$, there is a $w:Y \to X$ such that $w \circ i = u$, $p \circ w = v$.

Schematically: The commutative diagram

$$\begin{array}{ccc} A & \xrightarrow{u} & X \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle p} \\ Y & \xrightarrow[v]{} & B \end{array}$$

admits a filler $w:Y \to X$.

Consider a category $\underline{C}$ equipped with three composition closed classes of morphisms termed *weak equivalences*, *cofibrations*, and *fibrations*, each containing the isomorphisms of $\underline{C}$. Agreeing to call a morphism which is both a weak equivalence and a cofibration (fibration) an *acyclic cofibration* (*acyclic fibration*), $\underline{C}$ is said to be a *model category* provided that the following axioms are satisfied.

(MC-1) $\underline{C}$ is finitely complete and finitely cocomplete.

(MC-2) Given composable morphisms f,g, if any two of $f, g, g \circ f$ are weak equivalences, so is the third.

(MC-3) Every retract of a weak equivalence, cofibration, or fibration is again a weak equivalence, cofibration, or fibration.

[Note: To say that $f:X \to Y$ is a *retract* of $g:W \to Z$ means that there exist morphisms $i:X \to W$, $r:W \to X$, $j:Y \to Z$, $s:Z \to Y$ with $g \circ i = j \circ f$, $f \circ r = s \circ g$, $r \circ i = \mathrm{id}_X$, $s \circ j = \mathrm{id}_Y$, thus there is a commutative diagram

$$\begin{array}{ccccc} X & \xrightarrow{i} & W & \xrightarrow{r} & X \\ {\scriptstyle f}\downarrow & & {\scriptstyle g}\downarrow & & {\scriptstyle f}\downarrow \\ Y & \xrightarrow[j]{} & Z & \xrightarrow[s]{} & Y. \end{array}$$

Fact: A retract of an isomorphism is an isomorphism.]

(MC-4) Every cofibration has the LLP w.r.t. every acyclic fibration and every fibration has the RLP w.r.t. every acyclic cofibration.

(MC-5) Every morphism can be written as the composite of a cofibration and an acyclic fibration and the composite of an acyclic cofibration and a fibration.

N.B. For a systematic introduction to model category theory (with numerous examples), see Chapter 12 of my book TOPICS IN TOPOLOGY AND HOMOTOPY THEORY.

16.1 REMARK A model category $\underline{C}$ has an initial object (denoted $\emptyset$) and a final object (denoted $*$). An object X in $\underline{C}$ is said to be cofibrant if $\emptyset \to X$ is a cofibration and fibrant if $X \to *$ is a fibration.

16.2 NOTATION CAT is the category whose objects are the small categories and whose morphisms are the functors.

Definition: Given small categories $\underline{C}$, $\underline{D}$, a functor $F:\underline{C} \to \underline{D}$ is a cofibration if the map

$$\begin{cases} \text{Ob } \underline{C} \to \text{Ob } \underline{D} \\ X \to FX \end{cases}$$

is injective.

Definition: Given small categories $\underline{C}$, $\underline{D}$, a functor $F:\underline{C} \to \underline{D}$ is a fibration if $\forall$ X $\in$ Ob $\underline{C}$ and $\forall$ isomorphism $\psi:FX \to Y$ in $\underline{D}$, $\exists$ an isomorphism $\phi:X \to X'$ in $\underline{C}$ such that $F\phi = \psi$.

16.3 THEOREM CAT is a model category if weak equivalence = equivalence, the cofibrations and fibrations being as above.

The first step is the verification of MC-1 which, being of independent interest, will be isolated.

16.4 THEOREM CAT is finitely complete and finitely cocomplete.

16.5 RAPPEL The following conditions on a category $\underline{C}$ are equivalent.

(1) $\underline{C}$ is finitely complete.

(2) $\underline{C}$ has finite products and equalizers.

(3) $\underline{C}$ has finite products and pullbacks.

(4) $\underline{C}$ has a final object and pullbacks.

Let $\underline{1}$ be the category with one object and one arrow — then $\underline{1}$ is a final object in CAT.

Finite Products Given objects $\underline{C}$, $\underline{D}$ in CAT, their (binary) product is the category $\underline{C} \times \underline{D}$ defined by

$$\mathrm{Ob}(\underline{C} \times \underline{D}) = \mathrm{Ob}\ \underline{C} \times \mathrm{Ob}\ \underline{D},$$

$$\mathrm{Mor}((X,Y),(X',Y')) = \mathrm{Mor}(X,X') \times \mathrm{Mor}(Y,Y')$$

$$\mathrm{id}_{(X,Y)} = \mathrm{id}_X \times \mathrm{id}_Y,$$

with composition

$$(f',g') \circ (f,g) = (f' \circ f, g' \circ g).$$

[Note: If a category has a final object and (binary) products, then it has finite products.]

Equalizers Given objects C, D in CAT and morphisms F, G:C → D in CAT, their equalizer eq(F,G) is the inclusion inc of the subcategory of C on which F, G coincide:

$$\mathrm{eq}(F,G) \xrightarrow{\mathrm{inc}} \underline{C} \overset{F}{\underset{G}{\rightrightarrows}} \underline{D},$$

where

$$\begin{cases} \mathrm{Ob}\ \mathrm{eq}(F,G) = \{X \in \mathrm{Ob}\ \underline{C} : FX = GX\} \\ \mathrm{Mor}\ \mathrm{eq}(F,G) = \{f \in \mathrm{Mor}\ \underline{C} : Ff = Gf\}. \end{cases}$$

Pullbacks Suppose that $\underline{A} \xrightarrow{T} \underline{C} \xleftarrow{S} \underline{B}$ is a 2-sink in CAT. Form the product

$$\underline{A} \xleftarrow{\mathrm{pr}_A} \underline{A} \times \underline{B} \xrightarrow{\mathrm{pr}_B} \underline{B}$$

and note that

$$\underline{A} \times \underline{B} \overset{T \circ \mathrm{pr}_A}{\underset{S \circ \mathrm{pr}_B}{\rightrightarrows}} \underline{C}.$$

Let

$$\underline{A} \times_{\underline{C}} \underline{B} = \mathrm{eq}(T \circ \mathrm{pr}_A, S \circ \mathrm{pr}_B).$$

$$\begin{array}{ccc}
\underline{A} \times_{\underline{C}} \underline{B} & \xrightarrow{\ pr_B \circ \text{inc}\ } & \underline{B} \\
{\scriptstyle pr_A \circ \text{inc}} \downarrow & & \downarrow {\scriptstyle S} \\
\underline{A} & \xrightarrow[T]{\qquad} & \underline{C}
\end{array}$$

is a pullback square. I.e.: The 2-source

$$\underline{A} \xleftarrow{\ pr_A \circ \text{inc}\ } \underline{A} \times_{\underline{C}} \underline{B} \xrightarrow{\ pr_B \circ \text{inc}\ } \underline{B}$$

is a pullback of the 2-sink $\underline{A} \overset{T}{\rightarrow} \underline{C} \overset{S}{\leftarrow} \underline{B}$.

[Note: In $\underline{SET}$, there is a pullback square

$$\begin{array}{ccc}
\text{Ob } \underline{A} \times_{\text{Ob } \underline{C}} \text{Ob } \underline{B} & \longrightarrow & \text{Ob } \underline{B} \\
\downarrow & & \downarrow {\scriptstyle S} \\
\text{Ob } \underline{A} & \xrightarrow[T]{\qquad} & \text{Ob } \underline{C}.]
\end{array}$$

16.6 RAPPEL The following conditions on a category $\underline{C}$ are equivalent.

(1) $\underline{C}$ is finitely cocomplete.

(2) $\underline{C}$ has finite coproducts and coequalizers.

(3) $\underline{C}$ has finite coproducts and pushouts.

(4) $\underline{C}$ has an initial object and pushouts.

Let $\underline{0}$ be the category with no objects and no arrows -- then $\underline{0}$ is an initial object in $\underline{CAT}$.

<u>Finite Coproducts</u> Given objects $\underline{C}$, $\underline{D}$ in $\underline{CAT}$, their (binary) <u>coproduct</u> is the category $\underline{C} \coprod \underline{D}$ defined by

$$\begin{cases} \text{Ob}(\underline{C} \coprod \underline{D}) = \text{Ob}\,\underline{C} \coprod \text{Ob}\,\underline{D} \\ \text{Mor}(\underline{C} \coprod \underline{D}) = \text{Mor}\,\underline{C} \coprod \text{Mor}\,\underline{D}, \end{cases}$$

the coproducts on the RHS being taken in $\underline{SET}$ with the obvious composition of morphisms.

[Note: If a category has an initial object and (binary) coproducts, then it has finite coproducts.]

<u>Coequalizers</u> Given objects $\underline{C}$, $\underline{D}$ in $\underline{CAT}$ and morphisms $F, G:\underline{C} \rightarrow \underline{D}$ in $\underline{CAT}$, consider the smallest equivalence relation on Ob $\underline{D}$ w.r.t. which FX and GX are equivalent for all $X \in \text{Ob}\,\underline{C}$ and let $S_{F,G}$ be the set of pairs (Ff,Gf), where the domain and codomain are equivalent. Denote by ~ the principal congruence on $\underline{D}$ generated by this data and form the quotient $\underline{D}/\simeq$ (cf.[†]) -- then $\underline{D} \xrightarrow{\text{pro}} \underline{D}/\simeq$ is a coequalizer of F,G:

$$\underline{C} \underset{G}{\overset{F}{\rightrightarrows}} \underline{D} \xrightarrow{\text{pro}} \text{coeq}(F,G).$$

<u>Pushouts</u> Suppose that $\underline{A} \overset{T}{\leftarrow} \underline{C} \overset{S}{\rightarrow} \underline{B}$ is a 2-source in $\underline{CAT}$. Form the coproduct

$$\underline{A} \xrightarrow{\text{in}_A} \underline{A} \coprod \underline{B} \xleftarrow{\text{in}_B} \underline{B}$$

and note that

[†] Theory Appl. Categ. $\underline{5}$ (1999), 266-280.

$$\underline{C} \overset{in_A \circ T}{\underset{in_B \circ S}{\rightrightarrows}} \underline{A} \coprod \underline{B}.$$

Let

$$A \coprod_{\underline{C}} B = \text{coeq}(in_A \circ T,\ in_B \circ S).$$

Then the commutative diagram

$$\begin{array}{ccc} \underline{C} & \xrightarrow{S} & \underline{B} \\ {\scriptstyle T}\downarrow & & \downarrow{\scriptstyle pro \circ in_B} \\ \underline{A} & \xrightarrow[pro \circ in_A]{} & \underline{A} \coprod_{\underline{C}} \underline{B} \end{array}$$

is a pushout square. I.e.: The 2-sink

$$\underline{A} \xrightarrow{pro \circ in_A} \underline{A} \coprod_{\underline{C}} \underline{B} \xleftarrow{pro \circ in_B} \underline{B}$$

is a pushout of the 2-source $\underline{A} \overset{T}{\leftarrow} \underline{C} \overset{S}{\rightarrow} \underline{B}$.

[Note: In $\underline{SET}$, there is a pushout square

$$\begin{array}{ccc} \text{Ob } \underline{C} & \xrightarrow{S} & \text{Ob } \underline{B} \\ {\scriptstyle T}\downarrow & & \downarrow \\ \text{Ob } \underline{A} & \longrightarrow & \text{Ob } \underline{A} \coprod_{\text{Ob } \underline{C}} \text{Ob } \underline{B}.] \end{array}$$

There remains the verification of MC-2, MC-3, MC-4, and MC-5.

16.7 LEMMA If $F:\underline{C} \to \underline{D}$ and $G:\underline{D} \to \underline{E}$ are equivalences, then $G \circ F:\underline{C} \to \underline{E}$ is an equivalence.

16.8 LEMMA Suppose that $F:\underline{C} \to \underline{D}$ and $G:\underline{D} \to \underline{E}$ are functors. Assume: F and $G \circ F$ are equivalences -- then G is an equivalence.

PROOF Choose $F':\underline{D} \to \underline{C}$ such that

$$\begin{cases} F \circ F' \approx \text{id}_{\underline{D}} \\ F' \circ F \approx \text{id}_{\underline{C}}. \end{cases}$$

Choose $H:\underline{E} \to \underline{C}$ such that

$$\begin{cases} G \circ F \circ H \approx \text{id}_{\underline{E}} \\ H \circ G \circ F \approx \text{id}_{\underline{C}}. \end{cases}$$

Let $G' = F \circ H$ — then $G \circ G' \approx \text{id}_{\underline{E}}$ and

$$\begin{aligned} G' \circ G = F \circ H \circ G &= F \circ H \circ G \circ \text{id}_{\underline{D}} \\ &\approx F \circ H \circ G \circ F \circ F' \\ &\approx F \circ \text{id}_{\underline{C}} \circ F' \\ &= F \circ F' \approx \text{id}_{\underline{D}}. \end{aligned}$$

16.9 LEMMA Suppose that $F:\underline{C} \to \underline{D}$ and $G:\underline{D} \to \underline{E}$ are functors. Assume: G and

$G \circ F$ are equivalences -- then F is an equivalence.

Therefore MC-2 is satisfied.

16.10 LEMMA A retract of an equivalence is an equivalence.

PROOF Consider a commutative diagram

$$\begin{array}{ccccc} \underline{C} & \xrightarrow{i} & \underline{K} & \xrightarrow{r} & \underline{C} \\ F\big\downarrow & & \Lambda\big\downarrow & & F\big\downarrow \\ \underline{D} & \xrightarrow[j]{} & \underline{L} & \xrightarrow[s]{} & \underline{D}, \end{array}$$

where $r \circ i = \mathrm{id}_{\underline{C}}$, $s \circ j = \mathrm{id}_{\underline{D}}$, and Λ is an equivalence -- then the claim is that F is an equivalence. Thus fix $\Lambda' : \underline{L} \to \underline{K}$ such that

$$\begin{cases} \Lambda \circ \Lambda' \approx \mathrm{id}_{\underline{L}} \\ \Lambda' \circ \Lambda \approx \mathrm{id}_{\underline{K}}. \end{cases}$$

Then

$$\begin{aligned} r \circ \Lambda' \circ j \circ F &= r \circ \Lambda' \circ \Lambda \circ i \\ &\approx r \circ \mathrm{id}_{\underline{K}} \circ i \\ &= r \circ i = \mathrm{id}_{\underline{C}} \end{aligned}$$

and

$$\begin{aligned} F \circ r \circ \Lambda' \circ j &= s \circ \Lambda \circ \Lambda' \circ j \\ &\approx s \circ \mathrm{id}_{\underline{L}} \circ j \\ &= s \circ j = \mathrm{id}_{\underline{D}}. \end{aligned}$$

16.11 <u>LEMMA</u> A retract of a cofibration is a cofibration.

<u>PROOF</u> Consider a commutative diagram

$$\begin{array}{ccccc} \underline{C} & \xrightarrow{i} & \underline{K} & \xrightarrow{r} & \underline{C} \\ F\downarrow & & \Lambda\downarrow & & F\downarrow \\ \underline{D} & \xrightarrow[j]{} & \underline{L} & \xrightarrow[s]{} & \underline{D}, \end{array}$$

where $r \circ i = \text{id}_{\underline{C}}$, $s \circ j = \text{id}_{\underline{D}}$, and Λ is injective on objects -- then the claim is that F is injective on objects. So suppose that

$$FX = FY \quad (X,Y \in \text{Ob } \underline{C}).$$

Then

$$\begin{aligned} jFX = jFY &\Rightarrow \Lambda iX = \Lambda iY \\ &\Rightarrow iX = iY \\ &\Rightarrow riX = riY \Rightarrow X = Y. \end{aligned}$$

<u>N.B.</u> Let $\underline{I}$ denote the category with objects a,b and arrows id_a, id_b, $a \overset{\alpha}{\to} b$, $b \overset{\beta}{\to} a$, where $\alpha \circ \beta = \text{id}_b$, $\beta \circ \alpha = \text{id}_a$ -- then $F:\underline{C} \to \underline{D}$ is a fibration iff every commutative diagram

$$\begin{array}{ccc} \underline{1} & \xrightarrow{\mu} & \underline{C} \\ \pi\downarrow & & \downarrow F \\ \underline{I} & \xrightarrow[\nu]{} & \underline{D} \end{array} \qquad (\pi(*) = a)$$

admits a filler $\rho:\underline{I} \to \underline{C}$, i.e.,

$$\begin{cases} \rho \circ \pi = \mu \\ F \circ \rho = \nu. \end{cases}$$

16.12 LEMMA A retract of a fibration is a fibration.

PROOF Consider a commutative diagram

$$\begin{array}{ccccccc}
\underline{1} & \xrightarrow{\mu} & \underline{C} & \xrightarrow{i} & \underline{K} & \xrightarrow{r} & \underline{C} \\
{\scriptstyle \pi}\downarrow & & {\scriptstyle F}\downarrow & & {\scriptstyle \Lambda}\downarrow & & {\scriptstyle F}\downarrow \\
\underline{I} & \xrightarrow[\nu]{} & \underline{D} & \xrightarrow[j]{} & \underline{L} & \xrightarrow[s]{} & \underline{D},
\end{array}$$

where $r \circ i = \mathrm{id}_{\underline{C}}$, $s \circ j = \mathrm{id}_{\underline{D}}$, and Λ is a fibration -- then $\exists\ \lambda:\underline{I} \rightarrow \underline{K}$ such that

$$\begin{cases} \lambda \circ \pi = i \circ \mu \\ \Lambda \circ \lambda = j \circ \nu, \end{cases}$$

so if $\rho = r \circ \lambda:\underline{I} \rightarrow \underline{C}$, we have

$$\begin{cases} \rho \circ \pi = r \circ \lambda \circ \pi = r \circ i \circ \mu = \mathrm{id}_{\underline{C}} \circ \mu = \mu \\ F \circ \rho = F \circ r \circ \lambda = s \circ \Lambda \circ \lambda = s \circ j \circ \nu = \mathrm{id}_{\underline{D}} \circ \nu = \nu. \end{cases}$$

Therefore MC-3 is satisfied.

16.13 LEMMA Every cofibration has the LLP w.r.t. every acyclic fibration.

PROOF Consider the commutative diagram

$$\begin{array}{ccc}
\underline{C} & \xrightarrow{U} & \underline{K} \\
{\scriptstyle F}\downarrow & & \downarrow{\scriptstyle \Lambda} \\
\underline{D} & \xrightarrow[V]{} & \underline{L},
\end{array}$$

where F is a cofibration and Λ is an acyclic fibration -- then the claim is that $\exists\ W:\underline{D} \rightarrow \underline{K}$ such that $W \circ F = U$, $\Lambda \circ W = V$. Since Λ is an equivalence, it has a representative image, hence, being in addition a fibration, it is surjective on objects. Accordingly, define W on objects by first demanding that $WFX = UX$ $(X \in \text{Ob } \underline{C})$ (F is injective on objects, thus this makes sense). Next, given $Y \in \text{Ob } \underline{C}$, choose $A \in \text{Ob } \underline{K}$ such that $\Lambda A = VY$ and put $WY = A$, all the while maintaining the relation $WFX = UX$ (possible, as $VFX = \Lambda UX$). Turning to morphisms, there is an arrow

$$\text{Mor}(Y,Y') \rightarrow \text{Mor}(VY,VY').$$

On the other hand,

$$\text{Mor}(WY,WY') \approx \text{Mor}(\Lambda WY,\Lambda WY') = \text{Mor}(VY,VY').$$

So the data at hand does indeed give rise to a functor $W:\underline{D} \rightarrow \underline{K}$ with the chosen object map such that $W \circ F = U$, $\Lambda \circ W = V$.

16.14 LEMMA Every fibration has the RLP w.r.t. every acyclic cofibration.

PROOF Consider the commutative diagram

$$\begin{array}{ccc} \underline{C} & \xrightarrow{U} & \underline{K} \\ {\scriptstyle F}\downarrow & & \downarrow{\scriptstyle \Lambda} \\ \underline{D} & \xrightarrow[V]{} & \underline{L}, \end{array}$$

where F is an acyclic cofibration and Λ is a fibration -- then the claim is that $\exists$ $W:\underline{D} \rightarrow \underline{K}$ such that $W \circ F = U$, $\Lambda \circ W = V$. The initial step is to construct $F':\underline{D} \rightarrow \underline{C}$ subject to

$$\begin{cases} F' \circ F = \text{id}_{\underline{C}} \\ F \circ F' \approx \text{id}_{\underline{D}}, \end{cases}$$

which can be done by the usual procedure, viz. given $Y \in \text{Ob } \underline{D}$, choose an object $F'Y \in \text{Ob } \underline{C}$ and an isomorphism $FF'Y \to Y$, where if $Y = FX$, we take $F'FX = X$ (permissible, F being injective on objects). As regards the natural isomorphism $\Xi':F \circ F' \to id_{\underline{D}}$, matters can be arranged so that $\forall\, X \in \text{Ob } \underline{C}$,

$$\Xi'_{FX}:FF'FX \to id_{FX}$$

is id_{FX}. With this preparation, we shall start by defining W on objects, observing first that $\forall\, Y \in \text{Ob } \underline{D}$,

$$\Lambda UF'Y = VFF'Y.$$

But

$$\Xi'_Y:FF'Y \to Y$$

$$\Rightarrow$$

$$V\Xi'_Y:VFF'Y \to VY,$$

thus, since Λ is a fibration, $\exists$ an object $WY \in \text{Ob } \underline{K}$ and an isomorphism $\zeta_Y:UF'Y \to WY$ with

$$\Lambda\zeta_Y = V\Xi'_Y \quad (\Lambda WY = VY).$$

We can further assume that

$$\zeta_{FX} = id_{UX} \quad (WFX = UX).$$

Passing to morphisms, let $g \in \text{Mor}(Y,Y')$ and define $Wg \in \text{Mor}(WY,WY')$ by

$$Wg = \zeta_{Y'} \circ UF'g \circ \zeta_Y^{-1}.$$

Then $W:\underline{D} \to \underline{K}$ is a functor with the desired properties.

Therefore MC-4 is satisfied.

16.15 LEMMA Every morphism can be written as the composite of a cofibration and an acyclic fibration.

PROOF Suppose that $F:\underline{C} \to \underline{D}$ is a morphism in $\underline{CAT}$. Let $\underline{D}'$ be the category with

$$\text{Ob } \underline{D}' = \text{Ob } \underline{C} \coprod \text{Ob } \underline{D}$$

and for

$$\begin{cases} X,X' \in \text{Ob } \underline{C} \\ Y,Y' \in \text{Ob } \underline{D}, \end{cases}$$

viewed as objects in $\underline{D}'$, let

$$\text{Mor}(X,X') = \text{Mor}(FX,FX'),\ \text{Mor}(X,Y') = \text{Mor}(FX,Y')$$

$$\text{Mor}(Y,X') = \text{Mor}(Y,FX'),\ \text{Mor}(Y,Y') = \text{Mor}(Y,Y').$$

Define a functor $U:\underline{C} \to \underline{D}'$ by

$$\begin{cases} UX = X & (X \in \text{Ob } \underline{C}) \\ Uf = Ff & (f \in \text{Mor}(X,X')). \end{cases}$$

Then U is injective on objects, hence is a cofibration. Define a functor $V:\underline{D}' \to \underline{D}$ by

$$\begin{cases} VX = FX & (X \in \text{Ob } \underline{C}) \\ VY = Y & (Y \in \text{Ob } \underline{D}) \end{cases}$$

and on each of the four possibilities for morphisms, take V to be the identity, thus V is fully faithful and surjective on objects, so V is an acyclic fibration. And from the definitions, $F = V \circ U$.

16.16 LEMMA Every morphism can be written as the composite of an acyclic cofibration and a fibration.

PROOF Suppose that $F:\underline{C} \to \underline{D}$ is a morphism in $\underline{CAT}$. Let $\underline{C}'$ be the category whose objects are the triples (X,Ξ,Y), where $X \in \text{Ob } \underline{C}$, $Y \in \text{Ob } \underline{D}$, and $\Xi:FX \to Y$ is an isomorphism. Put

$$\text{Mor}((X,\Xi,Y),(X',\Xi',Y')) = \text{Mor}(X,X').$$

Define a functor $U:\underline{C} \to \underline{C}'$ by

$$\left[\begin{array}{l} UX = (X,\text{id}_{FX},FX) \\ Uf = f \quad (f:X \to X'). \end{array}\right.$$

Then it is clear that U is an acyclic cofibration. Define a functor $V:\underline{C}' \to \underline{D}$ by

$$\left[\begin{array}{l} V(X,\Xi,Y) = Y \\ Vf = \Xi' \circ Ff \circ \Xi^{-1}. \end{array}\right.$$

In this connection, note that

$$Vf:V(X,\Xi,Y) \to V(X',\Xi',Y'),$$

i.e.,

$$Vf:Y \to Y'.$$

Meanwhile, $\Xi' \circ Ff \circ \Xi^{-1}$ is the composition

$$\begin{array}{ccccc} Y & \xrightarrow{\Xi^{-1}} & FX & & \\ & & \downarrow{\scriptstyle Ff} & & \\ & & FX' & \xrightarrow{\Xi'} & Y'. \end{array}$$

So

$$(V \circ U)(f) = \text{id}_{FX'} \circ Ff \circ \text{id}_{FX}^{-1} = Ff.$$

To verify that V is a fibration, let

$$\psi : V(X,\Xi,Y) \to Y'$$

be an isomorphism -- then we want to produce an isomorphism

$$\phi : (X,\Xi,Y) \to (X',\Xi',Y')$$

such that $V\phi = \psi$. To this end, take

$$X' = X, \ \Xi' = \psi \circ \Xi,$$

and let

$$\phi = \mathrm{id}_X \in \mathrm{Mor}((X,\Xi,Y),(X,\psi \circ \Xi,Y')).$$

Then

$$V\phi = \psi \circ \Xi \circ \mathrm{id}_{FX} \circ \Xi^{-1}$$

$$= \psi \circ \Xi \circ \Xi^{-1} = \psi \circ \mathrm{id}_Y = \psi.$$

Therefore MC-5 is satisfied.

16.17 REMARK In CAT, all objects are both cofibrant and fibrant.

In addition to the categories $\underline{0}$ and $\underline{1}$, let $\underline{2}$ be the category with two objects and one arrow not the identity, let $\underline{d2}$ be the discrete category with two objects, and let $\underline{p2}$ be the category with two objects and two parallel arrows -- then the canonical functors

$$\begin{cases} u : \underline{0} \to \underline{1} \\ v : \underline{d2} \to \underline{2} \\ w : \underline{p2} \to \underline{2} \end{cases}$$

are cofibrations, hence every acyclic fibration has the RLP w.r.t. each of them (cf. MC-4), a property that turns out to be characteristic.

16.18 <u>LEMMA</u> Let $F:\underline{C} \to \underline{D}$ be a morphism in <u>CAT</u>.

(u) F has the RLP w.r.t. u iff F is surjective on objects.

(v) F has the RLP w.r.t. v iff F is full.

(w) F has the RLP w.r.t. w iff F is faithful.

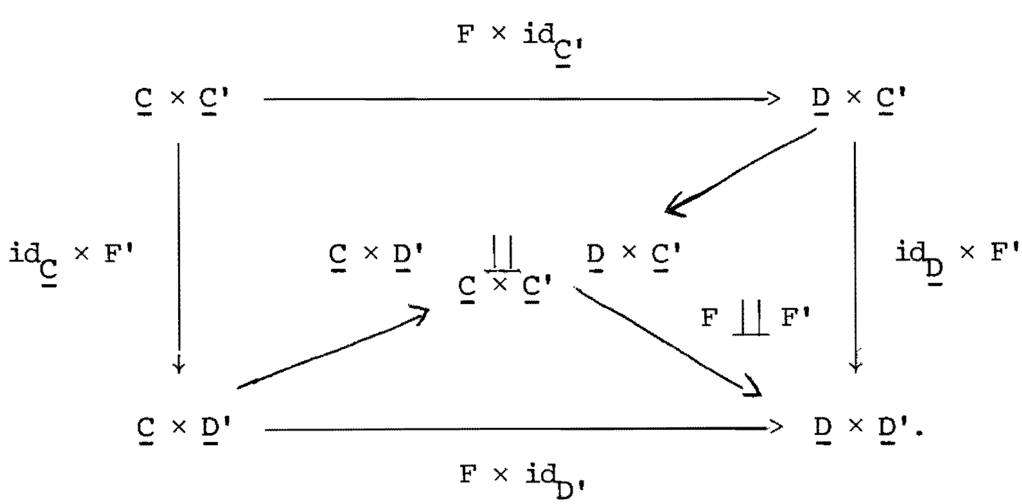
Consequently, if $F:\underline{C} \to \underline{D}$ has the RLP w.r.t. u,v,w, then F is an acyclic fibration. Proof: F is surjective on objects and fully faithful.

[Note: By comparison, recall that $F:\underline{C} \to \underline{D}$ is a fibration iff F has the RLP w.r.t. $\pi:\underline{1} \to \underline{I}$ (which is an acyclic cofibration).]

16.19 <u>LEMMA</u> Let

$$\begin{cases} F:\underline{C} \to \underline{D} \\ F':\underline{C}' \to \underline{D}' \end{cases}$$

be cofibrations. Consider the diagram



Then $F \coprod F'$ is a cofibration.

[Note: Working in $\underline{SET}$, suppose that $X \subset Y$, $X' \subset Y'$ — then

$$X \times X' = (X \times Y') \cap (Y \times X')$$

and the diagram

$$\begin{array}{ccc} X \times X' & \longrightarrow & Y \times X' \\ \downarrow & & \downarrow \\ X \times Y' & \longrightarrow & (X \times Y') \cup (Y \times X') \end{array}$$

is a pushout square, thus trivially the arrow

$$(X \times Y') \cup (Y \times X') \to Y \times Y'$$

is one-to-one.]

N.B. If in addition, either F or F' is an equivalence, then so is $F \coprod F'$.

16.20 RAPPEL A category $\underline{C}$ with finite products is said to be cartesian closed provided that each of the functors $— \times Y: \underline{C} \to \underline{C}$ has a right adjoint $Z \to Z^Y$, so

$$\text{Mor}(X \times Y, Z) \approx \text{Mor}(X, Z^Y).$$

The object Z^Y is called an exponential object. The evaluation morphism $ev_{Y,Z}$ is the morphism $Z^Y \times Y \to Z$ such that for every arrow $\phi: X \times Y \to Z$ there is a unique arrow $\lambda\phi: X \to Z^Y$ such that $\phi = ev_{Y,Z} \circ (\lambda\phi \times id_Y)$.

[Note: Each $Y \in \text{Ob}\,\underline{C}$ determines a functor $F:\underline{C} \to \underline{C}$ defined on objects by $FZ = Z^Y$ and on morphisms $Z \xrightarrow{f} X$ by

$$Ff = \lambda(f \circ ev_{Y,Z}),$$

so

$$Ff:Z^Y \to X^Y.$$

On the other hand, each $X \in \text{Ob } \underline{C}$ determines a functor $G:\underline{C}^{OP} \to \underline{C}$ defined on objects by $GY = X^Y$ and on morphisms $Z \overset{g}{\to} Y$ by

$$Gg = \lambda(ev_{Y,X} \circ id_{X^Y} \times g),$$

so

$$Gg:X^Y \to X^Z.]$$

<u>Functor Categories</u> Given small categories $\underline{C},\underline{D}$, $[\underline{C},\underline{D}]$ is the small category whose objects are the functors $F:\underline{C} \to \underline{D}$ and whose morphisms are the natural transformations $\text{Nat}(F,G)$ from F to G.

16.21 <u>LEMMA</u> <u>CAT</u> is cartesian closed:

$$\text{Mor}(\underline{C} \times \underline{D},\underline{E}) \approx \text{Mor}(\underline{C},\underline{E}^{\underline{D}}),$$

where $\underline{E}^{\underline{D}} = [\underline{D},\underline{E}]$.

16.22 <u>REMARK</u> The product operation

$$\times : \underline{CAT} \times \underline{CAT} \to \underline{CAT}$$

equips <u>CAT</u> with the structure of a symmetric monoidal category (here, $e = \underline{1}$).

16.23 <u>LEMMA</u> Let $F:\underline{C} \to \underline{D}$ be a cofibration and let $\Lambda:\underline{K} \to \underline{L}$ be a fibration.

Consider the diagram

$$\begin{array}{ccccc}
\underline{K}^{\underline{D}} & & \longrightarrow & & \underline{L}^{\underline{D}} \\
\big\downarrow & \overset{\Gamma}{\searrow} & & \nearrow & \big\downarrow \\
 & & \underline{K}^{\underline{C}} \times_{\underline{L}^{\underline{C}}} \underline{L}^{\underline{D}} & & \\
\big\downarrow & \swarrow & & & \big\downarrow \\
\underline{K}^{\underline{C}} & & \longrightarrow & & \underline{L}^{\underline{C}}
\end{array}$$

Then Γ is a fibration.

PROOF One has merely to show that every commutative diagram of the form

$$\begin{array}{ccc}
\underline{1} & \xrightarrow{\ \mu\ } & \underline{K}^{\underline{D}} \\
\big\downarrow & & \big\downarrow \Gamma \\
\underline{I} & \xrightarrow[\ \nu\ ]{} & \underline{K}^{\underline{C}} \times_{\underline{L}^{\underline{C}}} \underline{L}^{\underline{D}}
\end{array} \qquad (\pi(*) = a)$$

admits a filler $\rho:\underline{I} \to \underline{K}^{\underline{D}}$, i.e.,

$$\left[\begin{array}{l} \rho \circ \pi = \mu \\ \Gamma \circ \rho = \nu \end{array}\right.$$

But this lifting problem is equivalent to a lifting problem for the diagram

$$\begin{array}{ccc}
\underline{C} \times \underline{I} \coprod_{\underline{C} \times \underline{1}} \underline{D} \times \underline{1} & \longrightarrow & \underline{K} \\
F \coprod \pi \big\downarrow & & \big\downarrow \Lambda \\
\underline{D} \times \underline{I} & \longrightarrow & \underline{L}.
\end{array}$$

Since π is an acyclic cofibration, the same holds for $F \coprod \pi$. Therefore Λ has the RLP w.r.t. $F \coprod \pi$ (cf. MC-4), from which the assertion.

N.B. If in addition, either F or Λ is an equivalence, then so is Γ.

16.24 NOTATION GRD is the full subcategory of CAT whose objects are the groupoids, i.e., the small categories in which every morphism is invertible.

16.25 REMARK GRD is a model category if the cofibrations, fibrations, and weak equivalences are defined per CAT.

16.26 RAPPEL Let iso:CAT $\rightarrow$ GRD be the functor that sends $\underline{C}$ to iso $\underline{C}$, the groupoid whose objects are those of $\underline{C}$ and whose morphisms are the invertible morphisms -- then iso is a right adjoint for the inclusion ι:GRD $\rightarrow$ CAT. Let π_1:CAT $\rightarrow$ GRD be the functor that sends $\underline{C}$ to $\pi_1(\underline{C})$, the fundamental groupoid of $\underline{C}$ (a.k.a. the localization of $\underline{C}$ at Mor $\underline{C}$) -- then π_1 is a left adjoint for the inclusion ι:GRD $\rightarrow$ CAT.

16.27 NOTATION SISET is the category of simplicial sets.

16.28 RAPPEL There is a functor

$$c:\underline{SISET} \rightarrow \underline{CAT}$$

that assigns to each simplicial set X its categorical realization cX and there is a functor

$$ner:\underline{CAT} \rightarrow \underline{SISET}$$

§17. THE UNITARY MODEL STRUCTURE

In this § we shall take up the C*-analogs of the purely categorical results that were obtained in §16.

17.1 NOTATION C*CAT is the category whose objects are the small C*-categories and whose morphisms are the C*-functors.

N.B. 0 is an initial object in C*CAT and 1 is a final object in C*CAT.

17.2 THEOREM C*CAT is finitely complete and finitely cocomplete.

[Note: The inclusion

$$\underline{\text{UNC*ALG}} \rightarrow \underline{\text{C*CAT}}$$

preserves finite limits (obvious) but does not preserve finite colimits (as can be seen by considering binary coproducts).]

Let $\underline{C},\underline{D}$ be small C*-categories -- then their algebraic tensor product $\underline{C} \otimes \underline{D}$ is the category defined by

$$\text{Ob } \underline{C} \otimes \underline{D} = \text{Ob } \underline{C} \times \text{Ob } \underline{D}$$

and

$$\text{Mor}((X,Y),(X',Y')) = \text{Mor}(X,X') \otimes_{\underline{C}} \text{Mor}(Y,Y')$$

equipped with the involution

$$(\sum_k z_k(f_k \otimes g_k))^* = \sum_k \bar{z}_k(f_k^* \otimes g_k^*).$$

Consider the diagram

$$\begin{array}{ccccc} \underline{K}^{\underline{D}} & & \longrightarrow & & \underline{L}^{\underline{D}} \\ \downarrow & \overset{\Gamma}{\searrow} & & \nearrow & \downarrow \\ & & \underline{K}^{\underline{C}} \times_{\underline{L}^{\underline{C}}} \underline{L}^{\underline{D}} & & \\ & \swarrow & & & \\ \underline{K}^{\underline{C}} & & \longrightarrow & & \underline{L}^{\underline{C}} \end{array}$$

Then Γ is a fibration.

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admits a filler $\rho:\underline{I} \to \underline{K}^{\underline{D}}$, i.e.,

$$\left[\begin{array}{l} \rho \circ \pi = \mu \\ \\ \Gamma \circ \rho = \nu \end{array}\right.$$

But this lifting problem is equivalent to a lifting problem for the diagram

$$\begin{array}{ccc} \underline{C} \times \underline{I} \coprod_{\underline{C} \times \underline{1}} \underline{D} \times \underline{1} & \longrightarrow & \underline{K} \\ F \coprod \pi \downarrow & & \downarrow \Lambda \\ \underline{D} \times \underline{I} & \longrightarrow & \underline{L}. \end{array}$$

Since π is an acyclic cofibration, the same holds for $F \coprod \pi$. Therefore Λ has the RLP w.r.t. $F \coprod \pi$ (cf. MC-4), from which the assertion.

N.B. If in addition, either F or Λ is an equivalence, then so is Γ.

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$$c:\text{SISET} \to \text{CAT}$$

that assigns to each simplicial set X its categorical realization cX and there is a functor

$$\text{ner}:\text{CAT} \to \text{SISET}$$

that assigns to each small category $\underline{C}$ its nerve ner $\underline{C}$.

Fact: c is a left adjoint for ner.

Let $\Pi = \pi_1 \circ c$ -- then

$$\Pi:\underline{\text{SISET}} \to \underline{\text{GRD}}$$

is a functor that sends a simplicial set X to its fundamental groupoid ΠX.

16.29 LEMMA The functor

$$\iota \circ \Pi:\underline{\text{SISET}} \to \underline{\text{CAT}}$$

is a left adjoint for the functor

$$\text{ner} \circ \iota \circ \text{iso}:\underline{\text{CAT}} \to \underline{\text{SISET}}.$$

PROOF $\forall$ X & $\forall$ $\underline{C}$, we have

$$\begin{aligned}
\text{Mor}((\iota \circ \Pi)(X),\underline{C}) &= \text{Mor}(\iota(\Pi X),\underline{C}) \\
&= \text{Mor}(\iota \circ \pi_1(cX),\underline{C}) \\
&\approx \text{Mor}(\pi_1(cX),\text{iso }\underline{C}) \\
&\approx \text{Mor}(cX,\iota(\text{iso }\underline{C})) \\
&\approx \text{Mor}(X,\text{ner }\iota(\text{iso }\underline{C})).
\end{aligned}$$

Take SISET in its canonical model category structure -- then it can be shown that $\iota \circ \Pi$ preserves cofibrations and acyclic cofibrations while ner $\circ$ ι $\circ$ iso preserves fibrations and acyclic fibrations.

§17. *THE UNITARY MODEL STRUCTURE*

In this § we shall take up the C*-analogs of the purely categorical results that were obtained in §16.

17.1 NOTATION C*CAT is the category whose objects are the small C*-categories and whose morphisms are the C*-functors.

N.B. $\underline{0}$ is an initial object in C*CAT and $\underline{1}$ is a final object in C*CAT.

17.2 THEOREM C*CAT is finitely complete and finitely cocomplete.

[Note: The inclusion

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preserves finite limits (obvious) but does not preserve finite colimits (as can be seen by considering binary coproducts).]

Let $\underline{C},\underline{D}$ be small C*-categories -- then their algebraic tensor product $\underline{C} \otimes \underline{D}$ is the category defined by

$$\text{Ob } \underline{C} \otimes \underline{D} = \text{Ob } \underline{C} \times \text{Ob } \underline{D}$$

and

$$\text{Mor}((X,Y),(X',Y')) = \text{Mor}(X,X') \otimes_{\underline{C}} \text{Mor}(Y,Y')$$

equipped with the involution

$$(\sum_k z_k(f_k \otimes g_k))^* = \sum_k \bar{z}_k(f_k^* \otimes g_k^*).$$

This said, there are small C*-categories

$$\begin{cases} \underline{C} \otimes_{\min} \underline{D} \\ \underline{C} \otimes_{\max} \underline{D} \end{cases}$$

which reduce to the usual minimal and maximal tensor products of C*-algebras (details left to the reader).

N.B. The canonical functors

$$\begin{cases} \underline{C} \otimes \underline{D} \longrightarrow \underline{C} \otimes_{\min} \underline{D} \\ \underline{C} \otimes \underline{D} \longrightarrow \underline{C} \otimes_{\max} \underline{D} \end{cases}$$

are faithful.

17.3 LEMMA C*CAT is a symmetric monoidal category per

$$\otimes_{\max}: \underline{C^*CAT} \times \underline{C^*CAT} \rightarrow \underline{C^*CAT},$$

the unit e being the complex numbers (viewed as a C*-category).

17.4 REMARK The functor $— \otimes_{\max} \underline{D}$ admits a right adjoint, viz.

$$\underline{E} \rightarrow [\underline{D},\underline{E}]^*,$$

thus

$$\text{Mor}(\underline{C} \otimes_{\max} \underline{D},\underline{E}) \approx \text{Mor}(\underline{C},[\underline{D},\underline{E}]^*)$$

or still,

$$[\underline{C} \otimes_{\max} \underline{D},\underline{E}]^* \approx [\underline{C},[\underline{D},\underline{E}]^*]^*.$$

In any C*-category, an arrow $f:X \to Y$ is <u>unitary</u> if $f^*f = id_X$ and $ff^* = id_Y$.

Definition: Let $\underline{C},\underline{D}$ be objects in <u>C*CAT</u> -- then a C*-functor $F:\underline{C} \to \underline{D}$ is a <u>unitary equivalence</u> if $\exists$ a C*-functor $G:\underline{D} \to \underline{C}$ and natural isomorphisms

$$\begin{cases} G \circ F \overset{\mu}{\to} id_{\underline{C}} \\ F \circ G \overset{\nu}{\to} id_{\underline{D}} \end{cases}$$

such that

$$\begin{cases} \forall\ X \in \text{Ob } \underline{C},\ \mu_X \in \text{Mor}(GFX,X) \text{ is unitary} \\ \forall\ Y \in \text{Ob } \underline{C},\ \nu_Y \in \text{Mor}(FGY,Y) \text{ is unitary.} \end{cases}$$

[Note: An isomorphism $\underline{C} \to \underline{D}$ is necessarily unitary.]

17.5 <u>LEMMA</u> A functor $F:\underline{C} \to \underline{D}$ is a unitary equivalence iff it is fully faithful and $\forall\ Y \in \text{Ob } \underline{D}$, $\exists\ X \in \text{Ob } \underline{C}$ and a unitary isomorphism $FX \to Y$.

Definition: Given small C*-categories $\underline{C},\underline{D}$, a functor $F:\underline{C} \to \underline{D}$ is a <u>cofibration</u> if the map

$$\begin{cases} \text{Ob } \underline{C} \to \text{Ob } \underline{D} \\ \quad X \to FX \end{cases}$$

is injective on objects.

Definition: Given small C*-categories $\underline{C},\underline{D}$, a functor $F:\underline{C} \to \underline{D}$ is a <u>fibration</u> if $\forall\ X \in \text{Ob } \underline{C}$ and $\forall$ unitary isomorphism $\psi:FX \to Y$ in $\underline{D}$, $\exists$ a unitary isomorphism $\phi:X \to X'$ in $\underline{C}$ such that $F\phi = \psi$.

17.6 THEOREM C*CAT is a model category if weak equivalence = unitary equivalence, the cofibrations and fibrations being as above.

[The proof is similar to but not identical with that of 16.3.]

Let $\underline{G}$ be a small groupoid, i.e., let $\underline{G} \in \text{Ob } \underline{\text{GRD}}$ -- then by fr $\underline{G}$ we shall understand the category whose objects are those of $\underline{G}$ but

$$\text{Mor}_{\text{fr } \underline{G}} (X,Y)$$

is the free complex vector space generated by $\text{Mor}_{\underline{G}} (X,Y)$, thus the elements of

$$\text{Mor}_{\text{fr } \underline{G}} (X,Y)$$

are the formal finite linear combinations

$$\sum_{i=1}^{n} c_i \phi_i \quad (c_i \in \underline{C},\ \phi_i \in \text{Mor}_{\underline{G}} (X,Y))$$

with composition law

$$\left(\sum_{i=1}^{n} c_i \phi_i\right) \circ \left(\sum_{j=1}^{m} d_j \psi_j\right) = \sum_{i,j=1}^{n,m} (c_i d_j) \phi_i \circ \psi_j.$$

17.7 LEMMA The prescription

$$\left(\sum_{i=1}^{n} c_i \phi_i\right)^* = \sum_{i=1}^{n} \bar{c}_i \phi_i^{-1}$$

generates an involutive, identity on objects, cofunctor

$$*:\text{fr } \underline{G} \rightarrow \text{fr } \underline{G}.$$

[Note: $\forall\ \phi \in \text{Mor}_{\underline{G}} (X,Y),\ \phi^* = \phi^{-1}$.]

A representation of fr $\underline{G}$ is a $*$-preserving linear functor π:fr $\underline{G} \to \underline{\text{HILB}}$.

[Note: In particular, the elements of $\pi(\text{Mor}_{\underline{G}}(X,Y))$ are unitary operators from πX to πY.]

Given $f \in \text{Mor}_{\text{fr }\underline{G}}(X,Y)$, let

$$||f||_{\max} = \sup_{\pi} ||\pi(f)||,$$

where the sup is taken over the representations π of fr $\underline{G}$ -- then $||f||_{\max} < \infty$.

Proof: $\forall\ \pi$,

$$||\pi(f)|| = ||\pi(\sum_{i=1}^{n} c_i\phi_i)||$$

$$\leq \sum_{i=1}^{n} |c_i|\ ||\pi(\phi_i)|| = \sum_{i=1}^{n} |c_i| < \infty.$$

It is therefore clear that fr $\underline{G}$ is a pre-C*-category, hence its completion is a C*-category, call it $C^*_{\max}(\underline{G})$.

17.8 EXAMPLE Take $\underline{G} = \underline{I}$ as in §16 and form $C^*_{\max}(\underline{I})$ (= fr $\underline{I}$ here) -- then $a \xrightarrow{\alpha} b$ is unitary and for every small C*-category $\underline{C}$, the C*-functors $C^*_{\max}(\underline{I}) \to \underline{C}$ are in a one-to-one correspondence with the unitary elements of Mor $\underline{C}$.

17.9 LEMMA The association $\underline{G} \to C^*_{\max}(\underline{G})$ defines a functor

$$C^*_{\max}:\underline{\text{GRD}} \to \underline{\text{C*CAT}}.$$

PROOF Let $\underline{G},\underline{H}$ be small groupoids and let $F:\underline{G} \to \underline{H}$ be a functor -- then F induces in the evident manner a functor fr F:fr $\underline{G} \to$ fr $\underline{H}$ (on morphisms

$$\text{fr } F\left(\sum_{i=1}^{n} c_i\phi_i\right) = \sum_{i=1}^{n} c_i F\phi_i\right).$$

Accordingly, one has only to show that $\forall\ X,Y \in \text{Ob } \underline{G}$,

$$\text{fr } F:\text{Mor}_{\text{fr } \underline{G}}(X,Y) \to \text{Mor}_{\text{fr } \underline{H}}(FX,FY)$$

is continuous. But for any representation π of fr $\underline{H}$, $\pi \circ$ fr F is a representation of fr $\underline{G}$, so $\forall\ f \in \text{Mor}_{\text{fr } \underline{G}}(X,Y)$,

$$||(\pi \circ \text{fr } F)f|| \leq ||f||_{max}$$

$$\Rightarrow$$

$$||\text{fr } F(f)||_{max} \leq ||f||_{max}.$$

<u>N.B.</u> C^*_{max} takes equivalences to unitary equivalences.

Let uni:<u>C*CAT</u> $\to$ <u>GRD</u> be the functor that sends $\underline{C}$ to uni $\underline{C}$, the groupoid whose objects are those of $\underline{C}$ and whose morphisms are the unitary morphisms -- then uni is a right adjoint for C^*_{max}:

$$\text{Mor}(C^*_{max}(\underline{G}),\underline{C}) \approx \text{Mor}(\underline{G},\text{uni } \underline{C}).$$

Indeed, to proceed from the LHS to the RHS send

$$F:C^*_{max}(\underline{G}) \to \underline{C}$$

to the composition

$$\underline{G} \xrightarrow{\rho} \text{uni } C^*_{max}(\underline{G}) \xrightarrow{\text{uni } F} \text{uni } \underline{C}.$$

17.10 <u>LEMMA</u> We have

$$\text{uni}[C^*_{max}(\underline{G}),\underline{C}]^* \approx [\underline{G},\text{uni } \underline{C}].$$

<u>PROOF</u> The bijection on objects is the gist of the preceding observations. Suppose now that $F,F':C^*_{max}(\underline{G}) \to \underline{C}$ are C^*-functors and let $\Xi:F \to F'$ be a unitary natural isomorphism, so

$$\forall\ X \in \text{Ob } C^*_{max}(\underline{G}) = \text{Ob } \underline{G},$$

$\exists$ a unitary arrow $\Xi_X:FX \to FX'$ in $\underline{C}$ and $\forall$ $f:X \to Y$ in Mor $C^*_{max}(\underline{G})$, there is a commutative diagram

$$\begin{array}{ccc} FX & \xrightarrow{\Xi_X} & F'X \\ {\scriptstyle Ff}\downarrow & & \downarrow{\scriptstyle F'f} \\ FY & \xrightarrow[\Xi_Y]{} & F'Y. \end{array}$$

It is thus immediate that the data generates a natural isomorphism $F\rho \to F'\rho$.

17.11 <u>EXAMPLE</u> Let $\underline{G}_1,\underline{G}_2$ be small groupoids and let $\underline{C}$ be a small C^*-category -- then there is a string of isomorphisms of categories:

$$\text{uni}[C^*_{max}(\underline{G}_1 \times \underline{G}_2),\underline{C}]^*$$

$$\approx [\underline{G}_1 \times \underline{G}_2,\text{uni } \underline{C}]$$

$$\approx [\underline{G}_1,[\underline{G}_2,\text{uni } \underline{C}]]$$

$$\approx [\underline{G}_1,\text{uni}[C^*_{max}(\underline{G}_2),\underline{C}]^*]$$

$$\approx \text{uni}[C^*_{max}(\underline{G}_1),[C^*_{max}(\underline{G}_2),\underline{C}]^*]^*$$

$$\approx \text{uni}[C^*_{max}(\underline{G}_1) \otimes_{max} C^*_{max}(\underline{G}_2),\underline{C}]^*.$$

[Note: It follows that

$$C^*_{max}(\underline{G}_1 \times \underline{G}_2) \approx C^*_{max}(\underline{G}_1) \otimes_{max} C^*_{max}(\underline{G}_2).]$$

Let

$$\Pi_{max} = C^*_{max} \circ \Pi.$$

17.12 LEMMA The functor

$$\Pi_{max}:\underline{\text{SISET}} \to \underline{\text{C*CAT}}$$

is a left adjoint for the functor

$$\text{ner} \circ \iota \circ \text{uni}:\underline{\text{C*CAT}} \to \underline{\text{SISET}}.$$

PROOF $\forall$ X & $\forall$ $\underline{C}$, we have (cf. 16.29)

$$\begin{aligned} \text{Mor}(\Pi_{max}(X),\underline{C}) &= \text{Mor}(C^*_{max}(\Pi X),\underline{C}) \\ &\approx \text{Mor}(\Pi X,\text{uni } \underline{C}) \\ &= \text{Mor}(\pi_1(cX),\text{uni } \underline{C}) \\ &\approx \text{Mor}(cX,\iota(\text{uni } \underline{C})) \\ &\approx \text{Mor}(X,\text{ner } \iota(\text{uni } \underline{C})). \end{aligned}$$

Take SISET in its canonical model category structure — then it can be shown that Π_{max} preserves cofibrations and acyclic cofibrations while ner $\circ$ ι $\circ$ uni preserves fibrations and acyclic fibrations.

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