ABELIAN THEORY

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CONTENTS

§1. *GROUP SCHEMES*

1: NOTATION SCH is the category of schemes, RNG is the category of commutative rings with unit.

Fix a scheme S -- then the category SCH/S of schemes over S (or of S-schemes) is the category whose objects are the morphisms $X \to S$ of schemes and whose morphisms

$$\mathrm{Mor}(X \to S,\ Y \to S)$$

are the morphisms $X \to Y$ of schemes with the property that the diagram

$$\begin{array}{ccc} X & \longrightarrow & Y \\ \downarrow & & \downarrow \\ S & = & S \end{array}$$

commutes.

[Note: Take $S = \mathrm{Spec}(\mathbb{Z})$ -- then

$$\underline{\mathrm{SCH}}/S = \underline{\mathrm{SCH}}.]$$

2: N.B. If $S = \mathrm{Spec}(A)$ (A in RNG) is an affine scheme, then the terminology is "schemes over A" (or "A-schemes") and one writes SCH/A in place of SCH/Spec(A).

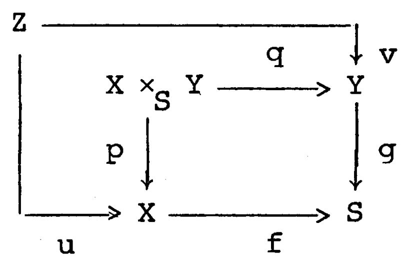
3: NOTATION Abbreviate $\mathrm{Mor}(X \to S,\ Y \to S)$ to $\mathrm{Mor}_S(X,Y)$ (or to $\mathrm{Mor}_A(X,Y)$ if $S = \mathrm{Spec}(A)$).

4: REMARK The S-scheme $\mathrm{id}_S : S \to S$ is a final object in SCH/S.

5: THEOREM SCH/S has pullbacks:

$$\begin{array}{ccc} X \times_S Y & \longrightarrow & Y \\ \downarrow & & \downarrow \\ X & \longrightarrow & S. \end{array}$$

[Note: Every diagram



$(f \circ u = g \circ v)$

admits a unique filler

$$(u,v)_S : Z \to X \times_S Y$$

such that

$$\begin{cases} p \circ (u,v)_S = u \\ q \circ (u,v)_S = v.] \end{cases}$$

6: FORMALITIES Let X,Y,Z be objects in SCH/S -- then

$$X \times_S S \approx X,$$

$$X \times_S Y \approx Y \times_S X,$$

and

$$(X \times_S Y) \times_S Z \approx X \times_S (Y \times_S Z).$$

7: REMARK If X, Y, X', Y' are objects in SCH/S and if $u: X \to X'$, $v: Y \to Y'$ are S-morphisms, then there is a unique morphism $u \times_S v$ (or just $u \times v$) rendering the diagram

$$\begin{array}{ccccc}
X & \xrightarrow{\quad u \quad} & X' & \longrightarrow & S \\
{\scriptstyle p}\uparrow & & \uparrow & & \| \\
X \times_S Y & \xrightarrow{u \times_S v} & X' \times_S Y' & & \| \\
{\scriptstyle q}\downarrow & & \downarrow & & \| \\
Y & \xrightarrow[\quad v \quad]{} & Y' & \longrightarrow & S
\end{array}$$

commutative.

[Spelled out,

$$u \times_S v = (u \circ p, v \circ q)_S.]$$

8: BASE CHANGE Let $u: S' \to S$ be a morphism in SCH.

- If $X \to S$ is an S-object, then $X \times_S S'$ is an S'-object via the projection

$$X \times_S S' \to S',$$

denoted $u^*(X)$ or $X_{(S')}$ and called the base change of X by u.

- If $X \to S$, $Y \to S$ are S-objects and if $f: (X \to S) \to (Y \to S)$ is an S-morphism, then

$$\begin{array}{ccc}
X \times_S S' & \xrightarrow{f \times_S \mathrm{id}_{S'}} & Y \times_S S' \\
\downarrow & & \downarrow \\
S' & = & S'
\end{array}$$

is a morphism of S'-objects, denoted $u^*(f)$ or $f_{(S')}$ and called <u>the base change of f by u</u>.

These considerations thus lead to a functor

$$u^*:\underline{SCH}/S \to \underline{SCH}/S'$$

called <u>the base change by u</u>.

<u>9:</u> <u>N.B.</u> If $u':S'' \to S'$ is another morphism in <u>SCH</u>, then the functors $(u \circ u')^*$ and $(u')^* \circ u$ from <u>SCH</u>/S to <u>SCH</u>/S'' are isomorphic.

<u>10:</u> LEMMA Let $u:S' \to S$ be a morphism in <u>SCH</u>. Suppose that $T' \to S'$ is an S'-object -- then T' can be viewed as an S-object T via postcomposition with u and there are canonical mutually inverse bijections

$$\mathrm{Mor}_{S'}(T', X_{(S')}) \rightleftarrows \mathrm{Mor}_S(T,X)$$

functorial in T' and X.

<u>11:</u> NOTATION Each S-scheme $X \to S$ determines a functor

$$(\underline{SCH}/S)^{OP} \to \underline{SET},$$

viz. the assignment

$$T \to \mathrm{Mor}_S(T,X) \equiv X_S(T),$$

the set of <u>T-valued points of X</u>.

[Note: In terms of category theory,

$$X_S(T) = h_{X \to S}(T \to S).]$$

<u>12:</u> LEMMA To give a morphism $(X \to S) \xrightarrow{f} (Y \to S)$ in <u>SCH</u>/S is equivalent to giving for all S-schemes T a map

$$f(T): X_S(T) \to Y_S(T)$$

which is functorial in T, i.e., for all morphisms $u: T' \to T$ of S-schemes the diagram

$$\begin{array}{ccc} X_S(T) & \xrightarrow{f(T)} & Y_S(T) \\ {\scriptstyle X_S(u)}\downarrow & & \downarrow{\scriptstyle Y_S(u)} \\ X_S(T') & \xrightarrow[f(T')]{} & Y_S(T') \end{array}$$

commutes.

<u>13:</u> DEFINITION A <u>group scheme</u> over S (or an <u>S-group</u>) is an object G of <u>SCH</u>/S and S-morphisms

$$m: G \times_S G \to G \quad (\text{"multiplication"})$$

$$e: S \to G \quad (\text{"unit"})$$

$$i: G \to G \quad (\text{"inversion"})$$

such that the diagrams

$$\begin{array}{ccc} G \times_S G \times_S G & \xrightarrow{m \times id_G} & G \times_S G \\ {\scriptstyle id_G \times m}\downarrow & & \downarrow{\scriptstyle m} \\ G \times_S G & \xrightarrow[m]{} & G \end{array}$$

$$\begin{array}{ccc} G \times_S S & \xrightarrow{(id_G, e)_S} & G \times_S G \\ \| & & \downarrow m \\ G & \xrightarrow[id_G]{} & G \end{array}$$

$$\begin{array}{ccc} G & \xrightarrow{(id_G, i)_S} & G \times_S G \\ \downarrow & & \downarrow m \\ S & \xrightarrow[e]{} & G \end{array}$$

commute.

14: REMARK To say that $(G;m,e,i)$ is a group scheme over S amounts to saying that G is a group object in SCH/S.

15: LEMMA Let G be an S-scheme -- then G gives rise to a group scheme over S iff for all S-schemes T, the set $G_S(T)$ carries the structure of a group which is functorial in T (i.e., for all S-morphisms $T' \to T$, the induced map $G_S(T) \to G_S(T')$ is a homomorphism of groups).

16: REMARK It suffices to define functorial group structures on the $G_S(A)$, where $\text{Spec}(A) \to S$ is an affine S-scheme.

[This is because morphisms of schemes can be "glued".]

17: LEMMA Let $u:S' \to S$ be a morphism in SCH. Suppose that $(G;m,e,i)$ is a group scheme over S -- then

$$(G \times_S S'; m_{(S')}, e_{(S')}, i_{(S')})$$

is a group scheme over S'.

[Note: For every S'-object $T' \to S'$,

$$(G \times_S S')_{S'}(T') = G_S(T),$$

where T is the S-object $T' \to S' \xrightarrow{u} S$.]

<u>18:</u> THEOREM If $(X,\mathcal{O}_X)$ is a locally ringed space and if A is a commutative ring with unit, then there is a functorial set-theoretic bijection

$$\mathrm{Mor}(S,\mathrm{Spec}(A)) \approx \mathrm{Mor}(A,\Gamma(X,\mathcal{O}_X)).$$

[Note: The "Mor" on the LHS is in the category of locally ringed spaces and the "Mor" on the RHS is in the category of commutative rings with unit.]

<u>19:</u> EXAMPLE Take $S = \mathrm{Spec}(\mathbb{Z})$ and let

$$\mathbb{A}^n = \mathrm{Spec}(\mathbb{Z}[t_1,\ldots,t_n]).$$

Then for every scheme X,

$$\mathrm{Mor}(X,\mathbb{A}^n) \approx \mathrm{Mor}(\mathbb{Z}[t_1,\ldots,t_n],\Gamma(X,\mathcal{O}_X))$$

$$\approx \Gamma(X,\mathcal{O}_X)^n \quad (\varphi \to (\varphi(t_1),\ldots,\varphi(t_n))).$$

Therefore $\mathbb{A}^n$ is a group object in <u>SCH</u> called <u>affine n-space</u>.

[Note: Here $\Gamma(X,\mathcal{O}_X)$ is being viewed as an additive group, hence the underlying multiplicative structure is being ignored.]

<u>20:</u> <u>N.B.</u> Given any scheme S,

$$\mathbf{A}^n_S = \mathbf{A}^n \times_{\mathbf{Z}} S \to S$$

is an S-scheme and for every morphism $S' \to S$,

$$\mathbf{A}^n_S \times_S S' \approx \mathbf{A}^n \times_{\mathbf{Z}} S \times_S S' \approx \mathbf{A}^n_{S'} .$$

<u>21:</u> NOTATION Write $\mathbf{G}_a$ in place of $\mathbf{A}^1$.

<u>22:</u> NOTATION Given A in <u>RNG</u>, denote

$$\mathbf{G}_a \times_{\mathbf{Z}} \operatorname{Spec}(A)$$

by $\mathbf{G}_a \otimes A$ or still, by $\mathbf{G}_{a,A}$.

<u>23:</u> <u>N.B.</u>

$$\begin{aligned} \mathbf{G}_{a,A} &= \operatorname{Spec}(\mathbf{Z}[t]) \times_{\mathbf{Z}} \operatorname{Spec}(A) \\ &= \operatorname{Spec}(\mathbf{Z}[t] \otimes A) = \operatorname{Spec}(A[t]). \end{aligned}$$

<u>24:</u> LEMMA $\mathbf{G}_{a,A}$ is a group object in <u>SCH</u>/A.

There are two other "canonical" examples of group objects in <u>SCH</u>/A.

- $\mathbf{G}_{m,A} = \operatorname{Spec}(A[u,v]/(uv-1))$

which assigns to an A-scheme X the multiplicative group $\Gamma(X,\mathcal{O}_X)^\times$ of invertible elements in the ring $\Gamma(X,\mathcal{O}_X)$.

- $\mathbf{GL}_{n,A} = \operatorname{Spec}(A[t_{11},\ldots,t_{nn}, \det(t_{ij})^{-1}])$

which assigns to an A-scheme X the group

$$GL_n(\Gamma(X,\mathcal{O}_X))$$

of invertible n × n-matrices with entries in the ring $\Gamma(X,\mathcal{O}_X)$.

25: DEFINITION If G and H are S-groups, then a homomorphism from G to H is a morphism $f:G \to H$ of S-schemes such that for all S-schemes T the induced map $f(T):G_S(T) \to H_S(T)$ is a group homomorphism.

26: EXAMPLE Take $S = \mathrm{Spec}(A)$ -- then

$$\det_A : GL_{n,A} \to G_{m,A}$$

is a homomorphism.

27: DEFINITION Let G be a group scheme over S -- then a subscheme (resp. an open subscheme, resp. a closed subscheme) $H \subset G$ is called an S-subgroup scheme (resp. an open S-subgroup scheme, resp. a closed S-subgroup scheme) if for every S-scheme T, $H_S(T)$ is a subgroup of $G_S(T)$.

28: EXAMPLE Given a positive integer n, $\underline{\mu}_{n,A}$ is the group object in SCH/A which assigns to an A-scheme X the multiplicative subgroup of $\Gamma(X,\mathcal{O}_X)^\times$ consisting of those ϕ such that $\phi^n = 1$, thus

$$\underline{\mu}_{n,A} = \mathrm{Spec}(A[t]/(t^n-1))$$

and $\underline{\mu}_{n,A}$ is a closed A-subgroup of $G_{m,A}$.

29: EXAMPLE Fix a prime number p and suppose that A has characteristic p.

Given a positive integer n, $\underline{\alpha}_{n,A}$ is the group object in $\underline{SCH}/A$ which assigns to an A-scheme X the additive subgroup of $\Gamma(X,\mathcal{O}_X)$ consisting of those ϕ such that $\phi^{p^n} = 0$, thus

$$\underline{\alpha}_{n,A} = \mathrm{Spec}(A[t]/(t^{p^n}))$$

and $\underline{\alpha}_{n,A}$ is a closed A-subgroup of $G_{a,A}$.

<u>30:</u> CONSTRUCTION Let $f:G \to H$ be a homomorphism of S-groups. Define Ker(f) by the pullback square

$$\begin{array}{ccc} \mathrm{Ker}(f) = S \times_H G & \longrightarrow & G \\ \downarrow & & \downarrow f \\ S & \xrightarrow[e]{} & H \end{array} .$$

Then for all S-schemes T,

$$\mathrm{Mor}_S(T,\mathrm{Ker}(f)) = \mathrm{Ker}(G_S(T) \xrightarrow{f(T)} H_S(T)),$$

so Ker(f) is an S-group.

<u>31:</u> EXAMPLE The kernel of $\det_A$ is $SL_{n,A}$.

<u>32:</u> <u>N.B.</u> Other kernels are $\underline{\mu}_{n,A}$ and $\underline{\alpha}_{n,A}$.

<u>33:</u> CONVENTION If P is a property of morphisms of schemes, then an S-group G has property P if this is the case of its structural morphism $G \to S$.

E.g.: The property of morphisms of schemes being quasi-compact, locally of finite type, separated, étale etc.

34: LEMMA Let

$$\begin{array}{ccc} X' & \longrightarrow & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Y' & \longrightarrow & Y \end{array}$$

be a pullback square in SCH. Suppose that f is a closed immersion -- then the same holds for f'.

35: APPLICATION Let $g:Y \to X$ be a morphism of schemes that has a section $s:X \to Y$. Assume: g is separated -- then s is a closed immersion.

[The commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{\ s\ } & Y \\ {\scriptstyle s}\downarrow & & \downarrow{\scriptstyle \Delta_{Y/X}} \\ Y & \xrightarrow[(id_Y, s \circ g)_X]{} & Y \times_X Y \end{array}$$

is a pullback square in SCH. But g is separated, hence the diagonal morphism $\Delta_{Y/X}$ is a closed immersion. Now quote the preceding lemma.]

If $G \to S$ is a group scheme over S, then the composition

$$S \xrightarrow{\ e\ } G \longrightarrow S$$

is id_S. Proof: e is an S-morphism and the diagram

$$\begin{array}{ccc} S & \xrightarrow{e} & G \\ {\scriptstyle id_S}\downarrow & & \downarrow \\ S & = & S \end{array}$$

commutes. Therefore e is a section for the structural morphism $G \to S$:

$$G \longrightarrow S \xrightarrow{e} G.$$

36: LEMMA Let $G \to S$ be a group scheme over S -- then the structural morphism $G \to S$ is separated iff $e:S \to G$ is a closed immersion.

[To see that "closed immersion" => "separated", consider the pullback square

$$\begin{array}{ccc} G & \longrightarrow & S \\ {\scriptstyle \Delta_{G/S}}\downarrow & & \downarrow{\scriptstyle e} \\ G \times_S G & \xrightarrow[m \circ (id_C \times i)]{} & G \end{array} \quad .]$$

37: LEMMA If S is a discrete scheme, then every S-group is separated.

38: APPLICATION Take $S = \text{Spec}(k)$, where k is a field -- then the structural morphism $X \to \text{Spec}(k)$ of a k-scheme X is separated.

§2. SCH/k

Fix a field k.

1: DEFINITION A k-algebra is an object in RNG and a ring homomorphism $k \to A$.

2: NOTATION ALG/k is the category whose objects are the k-algebras $k \to A$ and whose morphisms

$$(k \to A) \to (k \to B)$$

are the ring homomorphisms $A \to B$ with the property that the diagram

$$\begin{array}{ccc} A & \longrightarrow & B \\ \uparrow & & \uparrow \\ k & = & k \end{array}$$

commutes.

3: DEFINITION Let A be a k-algebra -- then A is finitely generated if there exists a surjective homomorphism $k[t_1,\ldots,t_n] \to A$ of k-algebras.

4: DEFINITION Let A be a k-algebra -- then A is finite if there exists a surjective homomorphism $k^n \to A$ of k-modules.

5: N.B. A finite k-algebra is finitely generated.

Recall now that SCH/k stands for SCH/Spec(k).

6: LEMMA The functor

$$A \to \mathrm{Spec}(A)$$

from $(\underline{ALG}/k)^{OP}$ to $\underline{SCH}/k$ is fully faithful.

7: DEFINITION Let $X \to \mathrm{Spec}(k)$ be a k-scheme — then X is locally of finite type if there exists an affine open covering $X = \bigcup_{i \in I} U_i$ such that for all i, $U_i = \mathrm{Spec}(A_i)$, where A_i is a finitely generated k-algebra.

8: DEFINITION Let $X \to \mathrm{Spec}(k)$ be a k-scheme — then X is of finite type if X is locally of finite type and quasi-compact.

9: LEMMA If a k-scheme $X \to \mathrm{Spec}(k)$ is locally of finite type and if $U \subset X$ is an open affine subset, then $\Gamma(U, \mathcal{O}_X)$ is a finitely generated k-algebra.

10: APPLICATION If A is a finitely generated k-algebra, then the k-scheme $\mathrm{Spec}(A) \to \mathrm{Spec}(k)$ is of finite type.

11: LEMMA If $X \to \mathrm{Spec}(k)$ is a k-scheme of finite type, then all subschemes of X are of finite type.

12: RAPPEL Let $(X, \mathcal{O}_X)$ be a locally ringed space. Given $x \in X$, denote the stalk of $\mathcal{O}_X$ at x by $\mathcal{O}_{X,x}$ — then $\mathcal{O}_{X,x}$ is a local ring. And:

- $\mathfrak{m}_x$ is the maximal ideal in $\mathcal{O}_{X,x}$.
- $\kappa(x) = \mathcal{O}_{X,x}/\mathfrak{m}_x$ is the residue field of $\mathcal{O}_{X,x}$.

13: CONSTRUCTION Let $(X, \mathcal{O}_X)$ be a scheme. Given $x \in X$, let $U = \mathrm{Spec}(A)$ be an affine open neighborhood of x. Denote by $\mathfrak{p}$ the prime ideal of A corresponding to x, hence $\mathcal{O}_{X,x} = \mathcal{O}_{U,x} = A_{\mathfrak{p}}$ (the localization of A at $\mathfrak{p}$) and the canonical homomorphism $A \to A_{\mathfrak{p}}$ leads to a morphism

$$\mathrm{Spec}(\mathcal{O}_{X,x}) = \mathrm{Spec}(A_{\mathfrak{p}}) \to \mathrm{Spec}(A) = U \subset X$$

of schemes (which is independent of the choice of U).

14: N.B. There is an arrow $\mathcal{O}_{X,x} \to \kappa(x)$, thus an arrow $\mathrm{Spec}(\kappa(x)) \to \mathrm{Spec}(\mathcal{O}_{X,x})$, thus an arrow

$$i_x : \mathrm{Spec}(\kappa(x)) \to X$$

whose image is x.

Let K be any field, let $f:\mathrm{Spec}(K) \to X$ be a morphism of schemes, and let x be the image of the unique point p of Spec(K). Since f is a morphism of locally ringed spaces, at the stalk level there is a homomorphism

$$\mathcal{O}_{X,x} \to \mathcal{O}_{\mathrm{Spec}(K),p} = K$$

of local rings meaning that the image of the maximal ideal $\mathfrak{m}_x \subset \mathcal{O}_{X,x}$ is contained in the maximal ideal $\{1\}$ of K, so there is an induced homomorphism

$$\iota : \kappa(x) \to K.$$

Consequently,

$$f = i_x \circ \mathrm{Spec}(\iota).$$

15: SCHOLIUM There is a bijection

$$\mathrm{Mor}(\mathrm{Spec}(K), X) \to \{(x,\iota) : x \in X,\ \iota : \kappa(x) \to K\}.$$

If $X \to \mathrm{Spec}(k)$ is a k-scheme, then for any $x \in X$, there is an arrow

$$\mathrm{Spec}(\kappa(x)) \to X,$$

from which an arrow

$$\mathrm{Spec}(\kappa(x)) \to \mathrm{Spec}(k),$$

or still, an arrow $k \to \kappa(x)$.

16: LEMMA Let $X \to \mathrm{Spec}(k)$ be a k-scheme locally of finite type -- then $x \in X$ is closed iff the field extension $\kappa(x)/k$ is finite.

17: APPLICATION Let $X \to \mathrm{Spec}(k)$ be a k-scheme locally of finite type. Assume: k is algebraically closed -- then

$$\{x \in X : x \text{ closed}\} = \{x \in X : k = \kappa(x)\}$$

$$= \mathrm{Mor}_k(\mathrm{Spec}(k), X) \equiv X(k).$$

18: DEFINITION A subset Y of a topological space X is dense in X if $\bar{Y} = X$.

19: DEFINITION A subset Y of a topological space X is very dense in X if for every closed subset $F \subset X$, $\overline{F \cap Y} = F$.

20: N.B. If Y is very dense in X, then Y is dense in X.

[Take $F = X : \overline{X \cap Y} = \bar{Y} = X$.]

21: LEMMA Let $X \to \mathrm{Spec}(k)$ be a k-scheme locally of finite type -- then

$$\{x \in X : x \text{ closed}\}$$

is very dense in X.

22: DEFINITION Let $X \to \mathrm{Spec}(k)$ be a k-scheme -- then a point $x \in X$ is called <u>k-rational</u> if the arrow $k \to \kappa(x)$ is an isomorphism.

23: N.B. Sending a k-morphism Spec(k) $\to X$ to its image sets up a bijection between the set

$$X(k) = \text{Mor}_k(\text{Spec}(k), X)$$

and the set of k-rational points of X.

24: REMARK X(k) may very well be empty.

[Consider what happens if k'/k is a proper field extension.]

Given a k-scheme $X \to \text{Spec}(k)$ and a field extension K/k, let

$$X(K) = \text{Mor}_k(\text{Spec}(K), X)$$

be the set of K-valued points of X. If $x:\text{Spec}(K) \to X$ is a K-valued point with image $x \in X$, then there are field extensions

$$k \to \kappa(x) \to K.$$

25: N.B. Spec(K) is a k-scheme, the structural morphism $\text{Spec}(K) \to \text{Spec}(k)$ being derived from the arrow of inclusion $j:k \to K$.]

Let $G = \text{Gal}(K/k)$. Given $\sigma:K \to K$ in G,

$$\text{Spec}(\sigma):\text{Spec}(K) \to \text{Spec}(K),$$

hence

$$\text{Spec}(K) \xrightarrow{\text{Spec}(\sigma)} \text{Spec}(K) \xrightarrow{x} X,$$

and we put

$$\sigma \cdot x = x \circ \text{Spec}(\sigma).$$

- $\sigma \cdot x$ is a K-valued point.

[There is a commutative diagram

$$\begin{array}{ccc} K & \xrightarrow{\sigma} & K \\ {\scriptstyle j}\uparrow & & \uparrow{\scriptstyle j} \\ k & \xrightarrow[\mathrm{id}_k]{} & k \end{array},$$

so $\sigma \circ j = j \circ \mathrm{id}_k = j$, and if $\pi: X \to \mathrm{Spec}(k)$ is the structural morphism, there is a commutative diagram

$$\begin{array}{ccc} \mathrm{Spec}(K) & \xrightarrow{x} & X \\ {\scriptstyle \mathrm{Spec}(j)}\downarrow & & \downarrow{\scriptstyle \pi} \\ \mathrm{Spec}(k) & = & \mathrm{Spec}(k) \end{array},$$

so $\pi \circ x = \mathrm{Spec}(j)$. The claim then is that the diagram

$$\begin{array}{ccc} \mathrm{Spec}(K) & \xrightarrow{x \circ \mathrm{Spec}(\sigma)} & X \\ {\scriptstyle \mathrm{Spec}(j)}\downarrow & & \downarrow{\scriptstyle \pi} \\ \mathrm{Spec}(k) & = & \mathrm{Spec}(k) \end{array}$$

commutes. But

$$\begin{aligned} \pi \circ x \circ \mathrm{Spec}(\sigma) &= \mathrm{Spec}(j) \circ \mathrm{Spec}(\sigma) \\ &= \mathrm{Spec}(\sigma \circ j) \\ &= \mathrm{Spec}(j).] \end{aligned}$$

- The operation

$$\left[\begin{array}{l} G \times X(K) \to X(K) \\ (\sigma, x) \longrightarrow \sigma \cdot x \end{array}\right.$$

is a left action of G on X(K).

[Given $\sigma, \tau \in G$: $K \xrightarrow{\tau} K \xrightarrow{\sigma} K$, it is a question of checking that

$$(\sigma \circ \tau) \cdot x = \sigma \cdot (\tau \cdot x).$$

But the LHS equals

$$x \circ \mathrm{Spec}(\sigma \circ \tau) = x \circ \mathrm{Spec}(\tau) \circ \mathrm{Spec}(\sigma)$$

while the RHS equals

$$\tau \cdot x \circ \mathrm{Spec}(\sigma) = x \circ \mathrm{Spec}(\tau) \circ \mathrm{Spec}(\sigma).]$$

26: NOTATION Let

$$K^G = \mathrm{Inv}(G)$$

be the invariant field associated with G.

27: LEMMA The set $X(K)^G$ of fixed points in X(K) for the left action of G on X(K) coincides with the set $X(K^G)$.

28: APPLICATION If K is a Galois extension of k, then

$$X(K)^G = X(k).$$

Take $K = k^{sep}$, thus now $G = \mathrm{Gal}(k^{sep}/k)$.

29: DEFINITION Suppose given a left action $G \times S \to S$ of G on a set S -- then S is called a G-set if $\forall\, s \in S$, the G-orbit $G \cdot s$ is finite or, equivalently, the stabilizer $G_s \subset G$ is an open subgroup of G.

30: EXAMPLE Let $X \to \mathrm{Spec}(k)$ be a k-scheme locally of finite type -- then

$\forall\, x \in X(k^{sep})$, the G-orbit $G \cdot x$ of x in $X(k^{sep})$ is finite, hence $X(k^{sep})$ is a G-set.

31: DEFINITION Let $X \to \text{Spec}(k)$ be a k-scheme -- then X is étale if it is of the form

$$X = \coprod_{i \in I} \text{Spec}(K_i),$$

where I is some index set and where K_i/k is a finite separable field extension.

There is a category $\underline{\text{ÉT}}/k$ whose objects are the étale k-schemes and there is a category $\underline{\text{G-SET}}$ whose objects are the G-sets.

Define a functor

$$\Phi : \underline{\text{ÉT}}/k \to \underline{\text{G-SET}}$$

by associating with each X in $\underline{\text{ÉT}}/k$ the set $X(k^{sep})$ equipped with its left G-action.

32: LEMMA Φ is an equivalence of categories.

PROOF To construct a functor

$$\Psi : \underline{\text{G-SET}} \to \underline{\text{ÉT}}/k$$

such that

$$\Psi \circ \Phi \approx \text{id}_{\underline{\text{ÉT}}/k} \quad \text{and} \quad \Phi \circ \Psi \approx \text{id}_{\underline{\text{G-SET}}},$$

take a G-set S and write it as a union of G-orbits, say

$$S = \coprod_{i \in I} G \cdot s_i.$$

Let $K_i \supset k$ be the finite separable field extension inside k^{sep} corresponding to the open subgroup $G_{s_i} \subset G$ and assign to S the étale k-scheme $\coprod_{i \in I} \text{Spec}(K_i)$. Proceed... .

The foregoing equivalence of categories induces an equivalence between the corresponding categories of group objects:

$$\text{étale group k-schemes} \approx \text{G-groups},$$

where a G-group is a group which is a G-set, the underlying left action being by group automorphisms.

<u>33:</u> CONSTRUCTION Given a group M, let M_k be the disjoint union

$$\coprod_M \text{Spec}(k),$$

the <u>constant group k-scheme</u>, thus for any k-scheme $X \to \text{Spec}(k)$,

$$\text{Mor}_k(X, M_k)$$

is the set of locally constant maps $X \to M$ whose group structure is multiplication of functions.

[The terminology is standard but not the best since if M is nontrivial, then

$$\text{Mor}_k(X, M_k) \approx M$$

only if X is connected.]

<u>34:</u> EXAMPLE For any étale group k-scheme X,

$$X \times_k \text{Spec}(k^{\text{sep}}) \approx X(k^{\text{sep}})_k \times_k \text{Spec}(k^{\text{sep}}).$$

[Note: Here (and elsewhere),

$$\times_k = \times_{\text{Spec}(k)}.]$$

35: RAPPEL An A in RNG is reduced if it has no nilpotent elements $\neq 0$ (i.e., $\nexists\ a \neq 0 : a^n = 0\ (\exists\ n)$).

36: DEFINITION A scheme X is reduced if for any nonempty open subset $U \subset X$, the ring $\Gamma(U, \mathcal{O}_X)$ is reduced.

[Note: This is equivalent to the demand that all the local rings $\mathcal{O}_{X,x}$ $(x \in X)$ are reduced.]

37: DEFINITION Let X be a k-scheme -- then X is geometrically reduced if for every field extension $K \supset k$, the K-scheme $X \times_k \text{Spec}(K)$ is reduced.

38: LEMMA If X is a reduced k-scheme, then for every separable field extension K/k, the K-scheme $X \times_k \text{Spec}(K)$ is reduced.

39: APPLICATION Assume: k is a perfect field -- then every reduced k-scheme X is geometrically reduced.

40: THEOREM Assume: k is of characteristic zero. Suppose that X is a group k-scheme which is locally of finite type -- then X is reduced, hence is geometrically reduced.

§3. *AFFINE GROUP* k-*SCHEMES*

Fix a perfect field k.

[Recall that a field k is perfect if every field extension of k is separable (equivalently, char(k) = 0 or char(k) = p > 0 and the arrow $x \to x^p$ is surjective).]

1: DEFINITION An affine group k-scheme is a group k-scheme of the form Spec(A), where A is a k-algebra.

2: EXAMPLE

$$\underline{G}_{a,k} = \text{Spec}(k[t])$$

is an affine group k-scheme.

3: EXAMPLE

$$\underline{G}_{m,k} = \text{Spec}(k[t,t^{-1}])$$

is an affine group k-scheme.

4: EXAMPLE

$$\underline{\mu}_{n,k} = \text{Spec}(k[t]/(t^n - 1)) \qquad (n \in \mathbb{N})$$

is an affine group k-scheme.

There is a category GRP/k whose objects are the group k-schemes and whose morphisms are the morphisms $f:X \to Y$ of k-schemes such that for all k-schemes T the induced map

$$f(T):\text{Mor}_k(T,X) \to \text{Mor}_k(T,Y)$$

is a group homomorphism.

5: NOTATION

$$\underline{\text{AFF}}\text{-}\underline{\text{GRP}}/k$$

is the full subcategory of $\underline{\text{GRP}}/k$ whose objects are the affine group k-schemes.

6: NOTATION

$$\underline{\text{GRP}}\text{-}\underline{\text{ALG}}/k$$

is the category of group objects in $\underline{\text{ALG}}/k$ and

$$\underline{\text{GRP}}\text{-}(\underline{\text{ALG}}/k)^{\text{OP}}$$

is the category of group objects in $(\underline{\text{ALG}}/k)^{\text{OP}}$.

7: LEMMA The functor

$$A \rightarrow \text{Spec}(A)$$

from $(\underline{\text{ALG}}/k)^{\text{OP}}$ to $\underline{\text{SCH}}/k$ is fully faithful and restricts to an equivalence

$$\underline{\text{GRP}}\text{-}(\underline{\text{ALG}}/k)^{\text{OP}} \rightarrow \underline{\text{AFF}}\text{-}\underline{\text{GRP}}/k.$$

8: REMARK An object in $\underline{\text{GRP}}\text{-}(\underline{\text{ALG}}/k)^{\text{OP}}$ is a k-algebra A which carries the structure of a commutative Hopf algebra over k: $\exists$ k-algebra homomorphisms

$$\Delta : A \rightarrow A \otimes_k A, \quad \varepsilon : A \rightarrow k, \quad S : A \rightarrow A$$

satisfying the "usual" conditions.

9: N.B. There is another way to view matters, viz. any functor $\underline{\text{ALG}}/k \rightarrow \underline{\text{GRP}}$ which is representable by a k-algebra serves to determine an affine group k-scheme (and vice versa). From this perspective, a morphism $G \rightarrow H$ of affine group k-schemes is a natural transformation of functors, i.e., a collection of group homomorphisms

$G(A) \to H(A)$ such that if $A \to B$ is a k-algebra homomorphism, then the diagram

$$\begin{array}{ccc} G(A) & \longrightarrow & H(A) \\ \downarrow & & \downarrow \\ G(B) & \longrightarrow & H(B) \end{array}$$

commutes.

[Note: Suppose that

$$\begin{cases} G = h^X = \text{Mor}(X,-) \\ H = h^Y = \text{Mor}(Y,-). \end{cases}$$

Then from Yoneda theory,

$$\text{Mor}(G,H) \approx \text{Mor}(Y,X).]$$

<u>10:</u> EXAMPLE $k[t,t^{-1}]$ represents $G_{m,k}$ and

$$k[t_{11},\ldots,t_{nn}, \det(t_{ij})^{-1}]$$

represents $GL_{n,k}$. Given any k-algebra A, the determinant is a group homomorphism

$$GL_{n,k}(A) \to G_{m,k}(A)$$

and

$$\det_k \in \text{Mor}(GL_{n,k}, G_{m,k}).$$

[Note: There is a homomorphism

$$k[t,t^{-1}] \to k[t_{11},\ldots,t_{nn},\det(t_{ij})^{-1}]$$

of k-algebras that defines $\det_k$. E.g.: If $n = 2$, then the homomorphism in question sends t to $t_{11}t_{22} - t_{12}t_{21}$.]

11: PRODUCTS Let

$$\begin{cases} G = h^X & (X \text{ in } \underline{ALG}/k) \\ H = h^Y & (Y \text{ in } \underline{ALG}/k) \end{cases}$$

be affine group k-schemes. Consider the functor

$$G \times H : \underline{ALG}/k \to \underline{GRP}$$

defined on objects by

$$A \to G(A) \times H(A).$$

Then this functor is represented by the k-algebra $X \otimes_k Y$:

$$\begin{aligned} \mathrm{Mor}(X \otimes_k Y, A) &\approx \mathrm{Mor}(X,A) \times \mathrm{Mor}(Y,A) \\ &= G(A) \times H(A). \end{aligned}$$

12: EXAMPLE Take

$$\begin{cases} G = \mathbf{G}_{m,\mathbf{R}} \\ H = \mathbf{G}_{m,\mathbf{R}}. \end{cases}$$

Then

$$(\mathbf{G}_{m,\mathbf{R}} \times \mathbf{G}_{m,\mathbf{R}})(\mathbf{R}) = \mathbf{R}^\times \times \mathbf{R}^\times = \mathbf{C}^\times$$

and

$$(\mathbf{G}_{m,\mathbf{R}} \times \mathbf{G}_{m,\mathbf{R}})(\mathbf{C}) = \mathbf{C}^\times \times \mathbf{C}^\times.$$

Let k'/k be a field extension -- then for any k-algebra A, the tensor product $A \otimes_k k'$ is a k'-algebra, hence there is a functor

$$\underline{ALG}/k \to \underline{ALG}/k'$$

termed extension of the scalars. On the other hand, every k'-algebra B' can be regarded as a k-algebra B, from which a functor

$$\underline{ALG}/k' \to \underline{ALG}/k$$

termed restriction of the scalars.

13: LEMMA For all k-algebras A and for all k'-algebras B',

$$\mathrm{Mor}_{k'}(A \otimes_k k', B') \approx \mathrm{Mor}_k(A,B).$$

14: SCHOLIUM The functor "extension of the scalars" is a left adjoint for the functor "restriction of the scalars".

Let G be an affine group k-scheme. Abusing the notation, denote still by G the associated functor

$$\underline{ALG}/k \to \underline{GRP}.$$

Then there is a functor

$$G_{k'}:\underline{ALG}/k' \to \underline{GRP},$$

namely

$$G_{k'}(A') = G(A),$$

where A is A' viewed as a k-algebra.

15: LEMMA $G_{k'}$ is an affine group k'-scheme and the assignment $G \to G_{k'}$ is functorial:

$$\underline{AFF}\text{-}\underline{GRP}/k \to \underline{AFF}\text{-}\underline{GRP}/k'.$$

[Note: Suppose that $G = h^X$ — then

$$\mathrm{Mor}_{k'}(X \otimes_k k', A') \approx \mathrm{Mor}_k(X,A)$$

$$= G(A) = G_{k'}(A').$$

Therefore $G_{k'}$ is represented by $X \otimes_k k'$:

$$G_{k'} = h^{X \otimes_k k'}.$$

Matters can also be interpreted "on the other side":

$$\begin{array}{ccc} G_{k'} = \mathrm{Spec}(X \times_k k') = \mathrm{Spec}(X) \times_k \mathrm{Spec}(k') & \longrightarrow & \mathrm{Spec}(k') \\ \downarrow & & \downarrow \\ G = \mathrm{Spec}(X) & \longrightarrow & \mathrm{Spec}(k).] \end{array}$$

<u>16:</u> DEFINITION $G_{k'}$ is said to have been obtained from G by <u>extension of the scalars</u>.

<u>17:</u> NOTATION Given an affine group k'-scheme G', let $G_{k'/k}$ be the functor

$$\underline{ALG}/k \to \underline{GRP}$$

defined by the rule

$$A \to G'(A \otimes_k k').$$

[Note: If $k' = k$, then $G_{k'/k} = G$.]

<u>18:</u> THEOREM Assume that k'/k is a finite field extension -- then $G_{k'/k}$ is an affine group k-scheme and the assignment $G' \to G_{k'/k}$ is functorial:

$$\underline{AFF}\text{-}\underline{GRP}/k' \to \underline{AFF}\text{-}\underline{GRP}/k.$$

<u>19:</u> DEFINITION $G_{k'/k}$ is said to have been obtained from G' by <u>restriction of the scalars</u>.

20: LEMMA Assume that k'/k is a finite field extension -- then for all affine group k-schemes H,

$$\mathrm{Mor}_k(H, G_{k'/k}) \approx \mathrm{Mor}_{k'}(H_{k'}, G').$$

21: SCHOLIUM The functor "restriction of the scalars" is a right adjoint for the functor "extension of the scalars".

[Accordingly, there are arrows of adjunction

$$\begin{cases} G \to (G_{k'})_{k'/k} \\ (G_{k'/k})_{k'} \to G'.] \end{cases}$$

22: NOTATION

$$\mathrm{Res}_{k'/k}: \underline{\mathrm{AFF}}\text{-}\underline{\mathrm{GRP}}/k' \to \underline{\mathrm{AFF}}\text{-}\underline{\mathrm{GRP}}/k$$

is the functor defined by setting

$$\mathrm{Res}_{k'/k}(G') = G_{k'/k}.$$

So, by definition,

$$\mathrm{Res}_{k'/k}(G')(A) = G'(A \otimes_k k').$$

And in particular:

$$\mathrm{Res}_{k'/k}(G')(k) = G'(k \otimes_k k') = G'(k').$$

23: EXAMPLE Take $G' = \underline{A}^n_{k'}$ -- then

$$\mathrm{Res}_{k'/k}(\underline{A}^n_{k'}) \approx \underline{A}^{nd}_k \quad (d = [k':k]).$$

<u>24:</u> EXAMPLE Take $k = R$, $k' = C$, $G' = G_{m,C}$, and consider

$$\mathrm{Res}_{C/R}(G_{m,C}).$$

Then

$$\mathrm{Res}_{C/R}(G_{m,C})(R) = C^{\times}$$

and

$$\mathrm{Res}_{C/R}(G_{m,C})(C) = C^{\times} \times C^{\times}.$$

[Note:

$$\mathrm{Res}_{C/R}(G_{m,C})$$

is not isomorphic to $G_{m,R}$ (its group of real points is $R^{\times}$).]

<u>25:</u> LEMMA Let k' be a finite Galois extension of k -- then

$$(\mathrm{Res}_{k'/k}(G'))_{k'} \approx \prod_{\sigma \in \mathrm{Gal}(k'/k)} \sigma G'.$$

[Note: $\forall\ \sigma \in \mathrm{Gal}(k'/k)$, there is a pullback square

$$\begin{array}{ccc} \sigma G' & \longrightarrow & \mathrm{Spec}(k') \\ \downarrow & & \downarrow \mathrm{Spec}(\sigma) \\ G' & \longrightarrow & \mathrm{Spec}(k') \end{array}$$

.]

<u>26:</u> EXAMPLE Take $k = R$, $k' = C$, $G' = G_{m,C}$ -- then

$$\begin{aligned} (\mathrm{Res}_{C/R}(G_{m,C}))_C &\approx G_{m,C} \times \sigma G_{m,C} \\ &\approx G_{m,C} \times G_{m,C}. \end{aligned}$$

Let G be an affine group k-scheme.

<u>27:</u> DEFINITION A <u>character</u> of G is an element of

$$X(G) = \mathrm{Mor}_k(G, \mathbf{G}_{m,k}).$$

Given $\chi \in X(G)$, for every k-algebra A, there is a homomorphism

$$\chi(A): G(A) \to \mathbf{G}_{m,k}(A) = A^{\times}.$$

Given $\chi_1, \chi_2 \in X(G)$, define

$$(\chi_1 + \chi_2)(A): G(A) \to \mathbf{G}_{m,k}(A) = A^{\times}$$

by the stipulation

$$(\chi_1 + \chi_2)(A)(t) = \chi_1(A)(t)\chi_2(A)(t),$$

from which a character $\chi_1 + \chi_2$ of G, hence $X(G)$ is an abelian group.

<u>28:</u> EXAMPLE Take $G = \mathbf{G}_{m,k}$ — then the characters of G are the morphisms $G \to \mathbf{G}_{m,k}$ of the form

$$t \to t^n \quad (n \in \mathbf{Z}),$$

i.e.,

$$X(G) \approx \mathbf{Z}.$$

<u>29:</u> EXAMPLE Take $G = \mathbf{G}_{m,k} \times \cdots \times \mathbf{G}_{m,k}$ (d factors) -- then the characters of G are the morphisms $G \to \mathbf{G}_{m,k}$ of the form

$$(t_1, \ldots, t_d) \to t_1^{n_1} \cdots t_d^{n_d} \quad (n_1, \ldots, n_d \in \mathbf{Z}),$$

i.e.,

$$X(G) \approx \mathbf{Z}^d.$$

30: EXAMPLE Given an abelian group M, its group algebra k[M] is canonically a k-algebra. Consider the functor $D(M):\underline{ALG}/k \to \underline{GRP}$ defined on objects by the rule

$$A \to \mathrm{Mor}(M,A^{\times}).$$

Then $\forall$ A,

$$\mathrm{Mor}(M,A^{\times}) \approx \mathrm{Mor}(k[M],A),$$

so k[M] represents D(M) which is therefore an affine group k-scheme. And

$$X(D(M)) \approx M,$$

the character of D(M) corresponding to $m \in M$ being the assignment

$$D(M)(A) = \mathrm{Mor}(M,A^{\times}) \xrightarrow{f \to f(m)} A^{\times} = G_{m,k}(A).$$

31: NOTATION Given $\chi' \in X(G')$, let $N_{k'/k}(\chi')$ stand for the rule that assigns to each k-algebra A the homomorphism

$$G_{k'/k}(A) \to G_{m,k}(A) = A^{\times}$$

defined by the composition

$$G_{k'/k}(A) \longrightarrow G'(A \otimes_k k')$$

$$G'(A \otimes_k k') \longrightarrow G_{m,k'}(A \otimes_k k') = (A \otimes_k k')^{\times}$$

$$(A \otimes_k k')^{\times} \longrightarrow A^{\times}.$$

Here the first arrow is the canonical isomorphism, the second arrow is $\chi'(A \otimes_k k')$, and the third arrow is the norm map.

32: LEMMA The arrow

$$\chi' \to N_{k'/k}(\chi')$$

is a homomorphism

$$X(G') \to X(G_{k'/k})$$

of abelian groups.

33: THEOREM The arrow

$$\chi' \to N_{k'/k}(\chi')$$

is bijective, hence defines an isomorphism

$$X(G') \to X(G_{k'/k})$$

of abelian groups.

34: APPLICATION Consider

$$\mathrm{Res}_{C/R}(G_{m,C}).$$

Then its character group is isomorphic to the character group of $G_{m,C}$, i.e., to Z. Therefore

$$\mathrm{Res}_{C/R}(G_{m,C})$$

is not isomorphic to $G_{m,R} \times G_{m,R}$.

§4. *ALGEBRAIC TORI*

Fix a field k of characteristic zero.

<u>1:</u> DEFINITION Let G be an affine group k-scheme — then G is <u>algebraic</u> if its associated representing k-algebra A is finitely generated.

<u>2:</u> REMARK It can be shown that every algebraic affine group k-scheme is isomorphic to a closed subgroup of some $GL_{n,k}$ ($\exists$ n).

<u>3:</u> CONVENTION The term <u>algebraic k-group</u> means "algebraic affine group k-scheme".

<u>4:</u> <u>N.B.</u> It is automatic that an algebraic k-group is reduced (cf. §2, #40), hence is geometrically reduced (cf. §2, #39).

<u>5:</u> LEMMA Assume that k'/k is a finite field extension — then the functor

$$\mathrm{Res}_{k'/k}:\underline{\mathrm{AFF}}\text{-}\underline{\mathrm{GRP}}/k' \to \underline{\mathrm{AFF}}\text{-}\underline{\mathrm{GRP}}/k$$

sends algebraic k'-groups to algebraic k-groups.

Given a finite field extension k'/k, let Σ be the set of k-embeddings of k' into k^{sep} and identify $k' \otimes_k k^{sep}$ with $(k^{sep})^{\Sigma}$ via the bijection which takes $x \otimes y$ to the string $(\sigma(x)y)_{\sigma\in\Sigma}$.

<u>6:</u> LEMMA Let G' be an algebraic k'-group — then

$$(G_{k'/k}) \times_k \mathrm{Spec}(k^{sep}) \approx \prod_{\sigma\in\Sigma} \sigma G',$$

where $\sigma G'$ is the algebraic k^{sep}-group defined by the pullback square

$$\begin{array}{ccc} \sigma G' & \longrightarrow & \mathrm{Spec}(k^{sep}) \\ \downarrow & & \downarrow \mathrm{Spec}(\sigma) \\ G' & \longrightarrow & \mathrm{Spec}(k') \end{array} .$$

[Note: To review, the LHS is

$$(\mathrm{Res}_{k'/k}(G'))_{k^{sep}}$$

and the Galois group $\mathrm{Gal}(k^{sep}/k)$ operates on it through the second factor. On the other hand, to each pair $(\tau,\sigma) \in \mathrm{Gal}(k^{sep}/k) \times \Sigma$, there corresponds a bijection $\sigma G' \to (\tau \circ \sigma)G'$ leading thereby to an action of $\mathrm{Gal}(k^{sep}/k)$ on

$$\prod_{\sigma\in\Sigma} \sigma G'.$$

The point then is that the identification

$$(\mathrm{Res}_{k'/k}(G'))_{k^{sep}} \approx \prod_{\sigma\in\Sigma} \sigma G'$$

respects the actions, i.e., is $\mathrm{Gal}(k^{sep}/k)$-equivariant.]

7: N.B. Consider the commutative diagram

$$\begin{array}{ccc} (\tau \circ \sigma)G' & \longrightarrow & \mathrm{Spec}(k^{sep}) \\ \downarrow & & \downarrow \mathrm{Spec}(\tau) \\ \sigma G' & \longrightarrow & \mathrm{Spec}(k^{sep}) \\ \downarrow & & \downarrow \mathrm{Spec}(\sigma) \\ G' & \longrightarrow & \mathrm{Spec}(k') \end{array} .$$

Then the "big" square is a pullback. Since this is also the case of the "small" bottom square, it follows that the "small" upper square is a pullback.

8: DEFINITION A split k-torus is an algebraic k-group T which is isomorphic to a finite product of copies of $G_{m,k}$.

9: EXAMPLE The algebraic R-group

$$\mathrm{Res}_{C/R}\,(G_{m,C})$$

is not a split R-torus (cf. §3, #24 and #34).

10: LEMMA If T is a split k-torus, then X(T) is a finitely generated free abelian group.

11: THEOREM The functor

$$T \to X(T)$$

from the category of split k-tori to the category of finitely generated free abelian groups is a contravariant equivalence of categories.

12: N.B. $\forall$ k-algebra A,

$$T(A) \approx \mathrm{Mor}(X(T),A^{\times}).$$

[Note: Explicated,

$$T \approx \mathrm{Spec}(k[X(T)]) \quad \text{(cf. §3, \#30)}.$$

Therefore

$$\begin{aligned} T(A) &\approx \mathrm{Mor}(\mathrm{Spec}(A),T) \\ &\approx \mathrm{Mor}(\mathrm{Spec}(A),\mathrm{Spec}(k[X(T)])) \\ &\approx \mathrm{Mor}(k[X(T)],A) \\ &\approx \mathrm{Mor}(X(T),A^{\times}).\end{aligned}$$]

<u>13:</u> DEFINITION A <u>k-torus</u> is an algebraic k-group T such that

$$T_{k^{sep}} = T \times_k \mathrm{Spec}(k^{sep})$$

is a split k^{sep}-torus.

<u>14:</u> <u>N.B.</u> A split k-torus is a k-torus.

<u>15:</u> EXAMPLE Let k'/k be a finite field extension and take $G' = G_{m,k'}$ -- then the algebraic k-group $G_{k'/k}$ is a k-torus (cf. #6).

<u>16:</u> DEFINITION Let T be a k-torus -- then a <u>splitting field</u> for T is a finite field extension K/k such that T_K is a split K-torus.

<u>17:</u> THEOREM Every k-torus T admits a splitting field which is minimal (i.e., contained in any other splitting field) and Galois.

<u>18:</u> NOTATION Given a k-scheme X and a Galois extension K/k, the Galois group Gal(K/k) operates on

$$X_K = X \times_k \mathrm{Spec}(K)$$

via the second term, hence $\sigma \to 1 \otimes \sigma$.

[Note: $1 \otimes \sigma$ is a k-automorphism of X_K.]

<u>19:</u> NOTATION Given k-schemes X,Y and a Galois extension K/k, the Galois group Gal(K/k) operates on $\mathrm{Mor}_K(X_K, Y_K)$ by the prescription

$$\sigma f = (1 \otimes \sigma) f (1 \otimes \sigma)^{-1}.$$

[Note: If $f \in \mathrm{Mor}_K(X_K, Y_K)$, then the condition $\sigma f = f$ for all $\sigma \in \mathrm{Gal}(K/k)$

is equivalent to the condition that f is the lift of a k-morphism $\phi:X \to Y$, i.e., $f = \phi \otimes 1$.]

<u>20:</u> LEMMA Let K/k be a Galois extension and let G = Gal(K/k) -- then for any k-algebra A and for any k-scheme X,

$$X(A \otimes_k K)^G = X(A).$$

[Note: This generalizes §2, #28 to which it reduces if A = k.]

<u>21:</u> DEFINITION Let G be a finite group -- then a <u>G-module</u> is an abelian group M supplied with a homomorphism $G \to \mathrm{Aut}(M)$.

<u>22:</u> <u>N.B.</u> A G-module is the same thing as a $\mathbb{Z}[G]$-module (in the usual sense when $\mathbb{Z}[G]$ is viewed as a ring).

<u>23:</u> DEFINITION Let G be a finite group -- then a <u>G-lattice</u> is a $\mathbb{Z}$-free G-module M of finite rank.

<u>24:</u> LEMMA If T is a k-torus split by a finite Galois extension K/k, then

$$X(T_K) = \mathrm{Mor}_K(T_K, G_{m,K})$$

is a Gal(K/k)-lattice.

<u>25:</u> THEOREM Fix a finite Galois extension K/k -- then the functor

$$T \to X(T_K)$$

from the category of k-tori split by K/k to the category of Gal(K/k)-lattices is a contravariant equivalence of categories.

<u>26:</u> <u>N.B.</u> Suppose that T is a k-torus split by a finite Galois extension

K/k. Form $K[X(T_K)]$, thus operationally, $\forall\ \sigma \in Gal(K/k)$,

$$\sigma(\sum_i a_i \chi_i) = \sum_i \sigma(a_i)\sigma(\chi_i) \quad (a_i \in K,\ \chi_i \in X(T_K)).$$

Pass now to the invariants

$$K[X(T_K)] \quad (G = Gal(K/k)).$$

Then

$$T \approx Spec(K[X(T_K)]^G).$$

And

$$\begin{aligned} T(A \otimes_k K)^G &= T(A) \\ &\approx Mor(Spec(A), T) \\ &\approx Mor(Spec(A), Spec(K[X(T_K)]^G)) \\ &\approx Mor_k(K[X(T_K)]^G, A) \\ &\approx Mor_K(K[X(T_K)], A \otimes_k K)^G \\ &\approx Mor_Z(X(T_K), (A \otimes_k K)^\times)^G \\ &\approx Mor_{Z[G]}(X(T_K), (A \otimes_k K)^\times). \end{aligned}$$

[Note: Let $T = Res_{K/k}(G_{m,K})$ -- then on the one hand,

$$Mor_{Z[G]}(Z[G],\ (A \otimes_k K)^\times) \approx (A \otimes_k K)^\times,$$

while on the other,

$$\begin{aligned} Res_{K/k}(G_{m,K})(A) &= (A \otimes_k K)^\times \\ &\approx Mor_{Z[G]}(X(T_K), (A \otimes_k K)^\times). \end{aligned}$$

Therefore

$$X(T_K) \approx Z[G].]$$

Take $k = R$, $K = C$, and let σ be the nontrivial element of $Gal(C/R)$ -- then every R-torus T gives rise to a Z-free module of finite rank supplied with an involution corresponding to σ. And conversely... .

There are three "basic" R-tori.

1. $T = G_{m,R}$. In this case,

$$X(T_C) = X(G_{m,C}) \approx Z$$

and the Galois action is trivial.

2. $T = Res_{C/R}(G_{m,C})$. In this case,

$$X(T_C) \approx X(G_{m,C} \times G_{m,C}) \qquad \text{(cf. §3, \#26)}$$

$$\approx Z \times Z$$

and the Galois action swaps coordinates.

3. $T = SO_2$. In this case,

$$X((SO_2)_C) \approx X(G_{m,C})$$
$$\approx Z$$

and the Galois action is multiplication by -1.

[Note:

$$SO_2 : \underline{ALG}/R \to \underline{GRP}$$

is the functor defined by the rule

$$SO_2(A) = \{\begin{vmatrix} a & b \\ -b & a \end{vmatrix} : a,b \in A \ \& \ a^2 + b^2 = 1\}.$$

Then SO_2 is an algebraic R-group such that

$$(SO_2)_C \approx G_{m,C},$$

so SO_2 is an R-torus and $SO_2(R)$ can be identified with S (= $\{z \in C: z\bar{z} = 1\}$).

27: THEOREM Every R-torus is isomorphic to a finite product of copies of the three basic tori described above.

Here is the procedure. Fix a Z-free module M of finite rank and an involution $\iota: M \to M$ -- then M can be decomposed as a direct sum

$$M_+ \oplus M_{sw} \oplus M_- ,$$

where $\iota = 1$ on M_+, ι is a sum of 2-dimensional swaps on M_{sw} (or still, $M_{sw} = \oplus Z[Gal(C/R)]$), and $\iota = -1$ on M_-.

28: SCHOLIUM If T is an R-torus, then there exist unique nonnegative integers a,b,c such that

$$T(R) \approx (R^\times)^a \times (C^\times)^b \times S^c.$$

29: REMARK The classification of C-tori is trivial: Any such is a finite product of the $G_{m,C}$.

30: RAPPEL Let K/k be a finite Galois extension and let A be a k-algebra -- then there is a norm map

$$(A \otimes_k K)^\times \to A^\times \ (\approx (A \otimes_k k)^\times).$$

31: CONSTRUCTION Let K/k be a finite Galois extension -- then there is a norm map

$$N_{K/k}: Res_{K/k}(G_{m,K}) \to G_{m,k}.$$

[For any k-algebra A,

$$\begin{aligned}&\text{Res}_{K/k}(G_{m,K})(A)\\&= G_{m,K}(A \otimes_k K)\\&= (A \otimes_k K)^\times \to A^\times = G_{m,k}(A).\end{aligned}$$

[Note: $N_{K/k}$ is not to be confused with the arrow of adjunction

$$G_{m,k} \to \text{Res}_{K/k}(G_{m,K}).]$$

<u>32:</u> <u>N.B.</u>

$$N_{K/k} \in X(\text{Res}_{K/k}(G_{m,K})).$$

<u>33:</u> NOTATION Let $\text{Res}^{(1)}_{K/k}(G_{m,K})$ be the kernel of $N_{K/k}$.

<u>34:</u> LEMMA $\text{Res}^{(1)}_{K/k}(G_{m,K})$ is a k-torus and there is a short exact sequence

$$1 \to \text{Res}^{(1)}_{K/k}(G_{m,K}) \to \text{Res}_{K/k}(G_{m,K}) \to G_{m,k} \to 1.$$

<u>35:</u> EXAMPLE Take k = R, K = C -- then

$$\text{Res}^{(1)}_{C/R}(G_{m,C}) \approx SO_2$$

and there is a short exact sequence

$$1 \to SO_2 \to \text{Res}_{C/R}(G_{m,C}) \to G_{m,R} \to 1.$$

[Note: On R-points, this becomes

$$1 \to S \to C^\times \to R^\times \to 1.]$$

36: DEFINITION Let T be a k-torus -- then T is k-anisotropic if $X(T) = \{0\}$.

37: EXAMPLE SO_2 is R-anisotropic.

38: THEOREM Every k-torus T has a unique maximal k-split subtorus T_s and a unique maximal k-anisotropic subtorus T_a. The intersection $T_s \cap T_a$ is finite and $T_s \cdot T_a = T$.

39: LEMMA $\mathrm{Res}^{(1)}_{K/k}(G_{m,K})$ is k-anisotropic.

PROOF Setting $G = \mathrm{Gal}(K/k)$, under the functoriality of #25, the norm map

$$N_{K/k}:\mathrm{Res}_{K/k}(G_{m,K}) \to G_{m,k}$$

corresponds to the homomorphism $Z \to Z[G]$ of G-modules that sends n to $n(\sum_G \sigma)$, the quotient $Z[G]/Z(\sum_G \sigma)$ being $X(T_K)$, where

$$T = \mathrm{Res}^{(1)}_{K/k}(G_{m,K}).$$

And

$$Z[G]^G = Z(\sum_G \sigma).$$

40: N.B. $\mathrm{Res}^{(1)}_{K/k}(G_{m,K})$ is the maximal k-anisotropic subtorus of $\mathrm{Res}_{K/k}(G_{m,K})$.

41: DEFINITION Let G,H be algebraic k-groups -- then a homomorphism $\phi:G \to H$ is an isogengy if it is surjective with a finite kernel.

<u>42:</u> DEFINITION Let G,H be algebraic k-groups -- then G,H are said to be <u>isogeneous</u> if there is an isogengy between them.

<u>43:</u> THEOREM Two k-tori T', T'' per #25 are isogeneous iff the Q[Gal(k/k)]-modules

$$\begin{cases} X(T'_K) \otimes_Z Q \\ X(T''_K) \otimes_Z Q \end{cases}$$

are isomorphic.

§5. *THE LLC*

1: N.B. The term "LLC" means "local Langlands correspondence" (cf. #26).

Let K be a non-archimedean local field -- then the image of $rec_K:K^\times \to G_K^{ab}$ is W_K^{ab} and the induced map $K^\times \to W_K^{ab}$ is an isomorphism of topological groups.

2: SCHOLIUM There is a bijective correspondence between the characters of W_K and the characters of $K^\times$:

$$\text{Mor}(W_K, C^\times) \approx \text{Mor}(K^\times, C^\times).$$

[Note: "Character" means continuous homomorphism. So, if $\chi:W_K \to C^\times$ is a character, then χ must be trivial on W_K^* ($C^\times$ being abelian), hence by continuity, trivial on $\overline{W_K^*}$, thus χ factors through $W_K/\overline{W_K^*} = W_K^{ab}$.]

Let T be a K-torus -- then T is isomorphic to a closed subgroup of some $GL_{n,K}$ ($\exists$ n). But $GL_{n,K}(K)$ is a locally compact topological group, thus T(K) is a locally compact topological group (which, moreover, is abelian).

3: N.B. For the record,

$$G_{m,K}(K) = K^\times = GL_{1,K}(K).$$

4: EXAMPLE Let L/K be a finite extension and consider $T = \text{Res}_{L/K}(G_{m,L})$ -- then $T(K) = L^\times$.

Roughly speaking, the objective now is to describe $\text{Mor}(T(K), C^\times)$ in terms of data attached to W_K but to even state the result requires some preparation.

<u>5:</u> <u>N.B.</u> The case when $T = G_{m,K}$ is local class field theory... .

<u>6:</u> EXAMPLE Suppose that T is K-split:

$$T \approx G_{m,K} \times \cdots \times G_{m,K} \quad \text{(d factors)}.$$

Then

$$\prod_{i=1}^{d} \text{Mor}(W_K, C^\times) \approx \prod_{i=1}^{d} \text{Mor}(K^\times, C^\times)$$

$$\approx \text{Mor}(\prod_{i=1}^{d} K^\times, C^\times)$$

$$\approx \text{Mor}(T(K), C^\times).$$

Given a K-torus T, put

$$\begin{cases} X^*(T) = \text{Mor}_{K^{sep}}(T_{K^{sep}}, G_{m,K^{sep}}) \\ X_*(T) = \text{Mor}_{K^{sep}}(G_{m,K^{sep}}, T_{K^{sep}}). \end{cases}$$

<u>7:</u> LEMMA Canonically,

$$X_*(T) \otimes_Z C^\times \approx \text{Mor}(X^*(T), C^\times).$$

PROOF Bearing in mind that

$$\text{Mor}_{K^{sep}}(G_{m,K^{sep}}, G_{m,K^{sep}}) \approx Z,$$

define a pairing

$$X^*(T) \times X_*(T) \xrightarrow{\langle\, ,\, \rangle} Z$$

by sending (χ^*,χ_*) to $\chi^* \circ \chi_* \in Z$. This done, given $\chi_* \otimes z$, assign to it the homomorphism

$$\chi^* \to z^{\langle \chi^*,\chi_* \rangle}.$$

8: NOTATION Given a K-torus T, put

$$\hat{T} = \mathrm{Spec}(C[X_*(T)]).$$

9: LEMMA $\hat{T}$ is a split C-torus such that

$$\begin{cases} X^*(\hat{T}) \equiv \mathrm{Mor}_C(\hat{T},G_{m,C}) \approx X_*(T) \\ X_*(\hat{T}) \equiv \mathrm{Mor}_C(G_{m,C},\hat{T}) \approx X^*(T). \end{cases}$$

Therefore

$$\begin{aligned} \mathrm{Mor}(X_*(T),C^\times) &\approx \mathrm{Mor}(X^*(\hat{T}),C^\times) \\ &\approx X_*(\hat{T}) \otimes_Z C^\times \\ &\approx X^*(T) \otimes_Z C^\times. \end{aligned}$$

10: LEMMA

$$\hat{T}(C) \approx X^*(T) \otimes_Z C^\times.$$

PROOF In fact,

$$\begin{aligned} \hat{T}(C) &\approx \mathrm{Mor}(X^*(\hat{T}),C^\times) \qquad \text{(cf. §4, \#12)} \\ &\approx \mathrm{Mor}(X_*(T),C^\times) \\ &\approx X^*(T) \otimes_Z C^\times. \end{aligned}$$

11: DEFINITION $\hat{T}$ is the complex dual torus of T.

12: EXAMPLE Under the assumptions of #6,

$$\begin{aligned}\hat{T}(C) &\approx X^*(T) \otimes_Z C^\times \\ &\approx Z^d \otimes_Z C \approx (C^\times)^d.\end{aligned}$$

Therefore

$$\begin{aligned}\mathrm{Mor}(W_K,\hat{T}(C)) &\approx \mathrm{Mor}(W_K,(C^\times)^d) \\ &\approx \prod_{i=1}^{d} \mathrm{Mor}(W_K,C^\times) \\ &\approx \mathrm{Mor}(T(K),C^\times).\end{aligned}$$

13: RAPPEL If G is a group and if A is a G-module, then

$$H^1(G,A) = \frac{Z^1(G,A)}{B^1(G,A)}.$$

- $Z^1(G,A)$ (the 1-cocycles) consists of those maps $f:G \to A$ such that $\forall\ \sigma,\tau \in G$,

$$f(\sigma\tau) = f(\sigma) + \sigma(f(\tau)).$$

- $B^1(G,A)$ (the 1-coboundaries) consists of those maps $f:G \to A$ for which $\exists$ an $a \in A$ such that $\forall\ \sigma \in G$,

$$f(\sigma) = \sigma a - a.$$

[Note:

$$H^1(G,A) = \mathrm{Mor}(G,A)$$

if the action is trivial.]

14: NOTATION If G is a topological group and if A is a topological G-module, then

$$\mathrm{Mor}_c(G,A)$$

is the group of continuous group homomorphisms from G to A. Analogously,

$$\begin{bmatrix} Z_c^1(G,A) = \text{"continuous 1-cocycles"} \\ B_c^1(G,A) = \text{"continuous 1-coboundaries"} \end{bmatrix}$$

and

$$H_c^1(G,A) = \frac{Z_c^1(G,A)}{B_c^1(G,A)}.$$

Let T be a K-torus -- then G_K (= $\mathrm{Gal}(K^{sep}/K)$) operates on $X^*(G)$, thus $W_K \subset G_K$ operates on $X^*(G)$ by restriction. Therefore $\hat{T}(C)$ is a W_K-module, so it makes sense to form

$$H_c^1(W_K,\hat{T}(C)).$$

15: NOTATION $\underline{\mathrm{TOR}}_K$ is the category of K-tori.

16: LEMMA The assignment

$$T \rightarrow H_c^1(W_K,\hat{T}(C))$$

defines a functor

$$\underline{\mathrm{TOR}}_K^{OP} \rightarrow \underline{\mathrm{AB}}.$$

[Note: Suppose that $T_1 \to T_2$ -- then

$$(T_1)_{K^{sep}} \to (T_2)_{K^{sep}}$$

=>

$$X^*(T_2) \to X^*(T_1)$$

=>

$$\hat{T}_2(C) \to \hat{T}_1(C)$$

=>

$$H_c^1(W_K,\hat{T}_2(C)) \to H_c^1(W_K,\hat{T}_1(C)).]$$

<u>17:</u> LEMMA The assignment

$$T \to \text{Mor}_c(T(K),C^\times)$$

defines a functor

$$\underline{\text{TOR}}_K^{\text{OP}} \to \underline{\text{AB}}.$$

<u>18:</u> THEOREM The functors

$$T \to H_c^1(W_K,\hat{T}(C))$$

and

$$T \to \text{Mor}_c(T(K),C^\times)$$

are naturally isomorphic.

<u>19:</u> SCHOLIUM There exist isomorphisms

$$\iota_T:H_c^1(W_K,\hat{T}(C)) \to \text{Mor}_c(T(K),C^\times)$$

such that if $T_1 \to T_2$, then the diagram

$$\begin{array}{ccc} H_c^1(W_K,\hat{T}_1(C)) & \xrightarrow{\iota_{T_1}} & \mathrm{Mor}_c(T_1(K),C^\times) \\ \uparrow & & \uparrow \\ H_c^1(W_K,\hat{T}_2(C)) & \xrightarrow{\iota_{T_2}} & \mathrm{Mor}_c(T_2(K),C^\times) \end{array}$$

commutes.

20: EXAMPLE Under the assumptions of #12, the action of G_K is trivial, hence the action of W_K is trivial. Therefore

$$\begin{aligned} H_c^1(W_K,\hat{T}(C)) &= \mathrm{Mor}_c(W_K,\hat{T}(C)) \\ &\approx \mathrm{Mor}_c(T(K),C^\times). \end{aligned}$$

[Note: The earlier use of the symbol Mor tacitly incorporated "continuity".]

There is a special case that can be dealt with directly, viz. when L/K is a finite Galois extension and

$$T = \mathrm{Res}_{L/K}(G_{m,L}).$$

The discussion requires some elementary cohomological generalities which have been collected in the Appendix below.

21: RAPPEL W_L is a normal subgroup of W_K of finite index:

$$W_K/W_L \approx G_K/G_L \approx \mathrm{Gal}(L/K).$$

Proceeding,

$$T_{K^{\mathrm{sep}}} \approx \prod_{\sigma\in \mathrm{Gal}(L/K)} {}^{\sigma}G_{m,L} \quad \text{(cf. \#6)},$$

so

$$X^*(T) \approx Z[W_K/W_L],$$

where

$$Z[W_K/W_L] \approx \mathrm{Ind}_{W_L}^{W_K} Z$$

$$\equiv Z[W_K] \otimes_{Z[W_L]} Z.$$

It therefore follows that

$$\hat{T}(C) \approx X^*(T) \otimes_Z C^\times$$

$$\approx Z[W_K] \otimes_{Z[W_L]} Z \otimes_Z C^\times$$

$$\approx Z[W_K] \otimes_{Z[W_L]} C^\times$$

$$\equiv \mathrm{Ind}_{W_L}^{W_K} C^\times.$$

Consequently

$$H^1(W_K, \hat{T}(C)) \approx H^1(W_K, \mathrm{Ind}_{W_L}^{W_K} C^\times)$$

$$\approx H^1(W_L, C^\times) \quad \text{(Shapiro's lemma)}$$

$$\approx \mathrm{Mor}(W_L, C^\times)$$

$$\approx \mathrm{Mor}(L^\times, C^\times)$$

$$\approx \mathrm{Mor}(T(K), C^\times),$$

which completes the proof modulo "continuity details" that we shall not stop to sort out.

22: DEFINITION The L-group of T is the semidirect product

$$^{L}T = \hat{T}(C) \times\!| \; W_K.$$

Because of this, it will be best to first recall "semidirect product theory".

23: RAPPEL If G is a group and if A is a G-module, then there is a canonical extension of G by A, namely

$$0 \to A \xrightarrow{i} A \times\!| \; G \xrightarrow{\pi} G \to 1,$$

where $A \times\!| \; G$ is the semidirect product.

24: DEFINITION A splitting of the extension

$$0 \to A \xrightarrow{i} A \times\!| \; G \xrightarrow{\pi} G \to 1$$

is a homomorphism $s:G \to A \times\!| \; G$ such that $\pi \circ s = id_G$.

25: FACT The splittings of the extension

$$0 \to A \xrightarrow{i} A \times\!| \; G \xrightarrow{\pi} G \to 1$$

determine and are determined by the elements of $Z^1(G,A)$.

Two splittings s_1, s_2 are said to be equivalent if there is an element $a \in A$ such that

$$s_1(\sigma) = i(a)s_2(\sigma)i(a)^{-1} \quad (\sigma \in G).$$

If

$$\begin{bmatrix} f_1 \longleftrightarrow s_1 \\ f_2 \longleftrightarrow s_2 \end{bmatrix}$$

are the 1-cocycles corresponding to $\left[\begin{array}{c} s_1 \\ s_2 \end{array}\right.$, then their difference $f_2 - f_1$ is a 1-coboundary.

26: SCHOLIUM The equivalence classes of splittings of the extension

$$0 \to A \xrightarrow{i} A \rtimes G \xrightarrow{\pi} G \to 1$$

are in a bijective correspondence with the elements of $H^1(G,A)$.

Return now to the extension

$$0 \to \hat{T}(C) \to \underset{\displaystyle {}^{L}T}{\underset{||}{\hat{T}(C) \rtimes W_K}} \to W_K \to 1$$

but to reflect the underlying topologies, work with continuous splittings and call them admissible homomorphisms. Introducing the obvious notion of equivalence, denote by $\Phi_K(T)$ the set of equivalence classes of admissible homomorphisms, hence

$$\Phi_K(T) \approx H^1_c(W_K, \hat{T}(C)).$$

On the other hand, denote by $A_K(T)$ the group of characters of $T(K)$, i.e.,

$$A_K(T) \approx \mathrm{Mor}_c(T(K), C^\times).$$

27: THEOREM There is a canonical isomorphism

$$\Phi_K(T) \to A_K(T).$$

[This statement is just a rephrasing of #18 and is the LLC for tori.]

28: HEURISTICES To each admissible homomorphism of W_K into ${}^{L}T$, it is possible to associate an irreducible automorphic representation of $T(K)$ (a.k.a. a character of $T(K)$) and all such arise in this fashion.

It remains to consider the archimedean case: $\mathbf{C}$ or $\mathbf{R}$.

- If T is a $\mathbf{C}$-torus, then T is isomorphic to a finite product

$$G_{m,\mathbf{C}} \times \cdots \times G_{m,\mathbf{C}}$$

and

$$T(\mathbf{C}) \approx \mathrm{Mor}(X^*(T), \mathbf{C}^\times)$$

$$\approx X_*(T) \otimes_{\mathbf{Z}} \mathbf{C}^\times.$$

Furthermore, $W_{\mathbf{C}} = \mathbf{C}^\times$ and the claim is that

$$H^1_c(W_{\mathbf{C}}, \hat{T}(\mathbf{C})) \equiv \mathrm{Mor}_c(\mathbf{C}^\times, \hat{T}(\mathbf{C}))$$

is isomorphic to

$$\mathrm{Mor}_c(T(\mathbf{C}), \mathbf{C}^\times).$$

But

$$\mathrm{Mor}_c(\mathbf{C}^\times, \hat{T}(\mathbf{C}))$$

$$\approx \mathrm{Mor}_c(\mathbf{C}^\times, X^*(T) \otimes_{\mathbf{Z}} \mathbf{C}^\times)$$

$$\approx \mathrm{Mor}_c(\mathbf{C}^\times, \mathrm{Mor}(X_*(T), \mathbf{C}^\times))$$

$$\approx \mathrm{Mor}_c(X_*(T) \otimes_{\mathbf{Z}} \mathbf{C}^\times, \mathbf{C}^\times)$$

$$\approx \mathrm{Mor}_c(T(\mathbf{C}), \mathbf{C}^\times).$$

- If T is an $\mathbf{R}$-torus, then T is isomorphic to a finite product

$$(G_{m,\mathbf{R}})^a \times (\mathrm{Res}_{\mathbf{C}/\mathbf{R}}(G_{m,\mathbf{C}}))^b \times (SO_2)^c$$

and it is enough to look at the three irreducible possibilities.

1. $T = G_{m,R}$. The point here is that $W_R^{ab} \approx R^\times \equiv T(R)$.

2. $T = \mathrm{Res}_{C/R}(G_{m,C})$. One can imitate the argument used above for its non-archimedean analog.

3. $T = SO_2$. The initial observation is that $X(T) = Z$ with action $n \to -n$, so $\hat{T}(C) = C^\times$ with action $z \to \frac{1}{z}$. And ... .

APPENDIX

Let G be a group (written multiplicatively).

1: DEFINITION A left (right) G-module is an abelian group A equipped with a left (right) action of G, i.e., with a homomorphism $G \to \mathrm{Aut}(A)$.

2: N.B. Spelled out, to say that A is a left G-module means that there is a map

$$\begin{cases} G \times A \to A \\ (\sigma, a) \to \sigma a \end{cases}$$

such that

$$\tau(\sigma a) = (\tau\sigma)a, \quad 1a = a,$$

thus A is first of all a left G-set. To say that A is a left G-module then means in addition that

$$\sigma(a + b) = \sigma a + \sigma b.$$

[Note: For the most part, the formalities are worked out from the left, the agreement being that

"left G-module" = "G-module".]

3: NOTATION The group ring $Z[G]$ is the ring whose additive group is the free abelian group with basis G and whose multiplication is determined by the multiplication in G and the distributive law.

A typical element of $Z[G]$ is

$$\sum_{\sigma \in G} m_\sigma \sigma,$$

where $m_\sigma \in Z$ and $m_\sigma = 0$ for all but finitely many σ.

4: N.B. A G-module is the same thing as a $Z[G]$-module.

5: LEMMA Given a ring R, there is a canonical bijection

$$\mathrm{Mor}(Z[G],R) \approx \mathrm{Mor}(G,R^\times).$$

6: CONSTRUCTION Given a G-set X, form the free abelian group $Z[X]$ generated by X and extend the action of G on X to a Z-linear action of G on $Z[X]$ -- then the resulting G-module is called a permutation module.

7: EXAMPLE Let H be a subgroup of G and take $X = G/H$ (here G operates on G/H by left translation), from which $Z[G/H]$.

8: DEFINITION A G-module homomorphism is a $Z[G]$-module homomorphism.

9: NOTATION $\underline{\mathrm{MOD}}_G$ is the category of G-modules.

10: NOTATION Given A,B in $\underline{\mathrm{MOD}}_G$, write $\mathrm{Hom}_G(A,B)$ in place of $\mathrm{Mor}(A,B)$.

11: LEMMA Let $A,B \in \underline{\mathrm{MOD}}_G$ — then $A \otimes_Z B$ carries the G-module structure defined by $\sigma(a \otimes a') = \sigma a \otimes \sigma a'$ and $\mathrm{Hom}_Z(A,B)$ carries the G-module structure defined by $(\sigma\phi)(a) = \sigma\phi(\sigma^{-1}a)$.

<u>12:</u> LEMMA If G' is a subgroup of G, then there is a homomorphism $Z[G'] \to Z[G]$ of rings and a functor

$$\mathrm{Res}_{G'}^{G}:\underline{\mathrm{MOD}}_G \to \underline{\mathrm{MOD}}_{G'}$$

of restriction.

<u>13:</u> DEFINITION Let G' be a subgroup of G — then the <u>functor of induction</u>

$$\mathrm{Ind}_{G'}^{G}:\underline{\mathrm{MOD}}_{G'} \to \underline{\mathrm{MOD}}_G$$

sends A' to

$$Z[G] \otimes_{Z[G']} A'.$$

[Note: $Z[G]$ is a right $Z[G']$-module and A' is a left $Z[G']$-module. Therefore the tensor product

$$Z[G] \otimes_{Z[G']} A'$$

is an abelian group. And it becomes a left G-module under the operation $\sigma(r \otimes a') = \sigma r \otimes a'$.]

<u>14:</u> EXAMPLE Let H be a subgroup of G. Suppose that H operates trivially on Z — then

$$Z[G/H] \approx \mathrm{Ind}_H^G Z.$$

<u>15:</u> FROBENIUS RECIPROCITY $\forall$ A in $\underline{\mathrm{MOD}}_G$, $\forall$ A' in $\underline{\mathrm{MOD}}_{G'}$,

$$\mathrm{Hom}_{G'}(A',\mathrm{Res}_{G'}^{G}A) \approx \mathrm{Hom}_G(\mathrm{Ind}_{G'}^{G}A',A).$$

16: REMARK $\forall$ A in $\underline{MOD}_G$,

$$\mathrm{Ind}_{G'}^{G} \circ \mathrm{Res}_{G'}^{G} A \approx Z[G/G'] \otimes_{Z[G]} A.$$

[G operates on the right hand side diagonally: $\sigma(r \otimes a) = \sigma r \otimes \sigma a$.]

17: LEMMA There is an arrow of inclusion

$$Z[G] \otimes_{Z[G']} A' \to \mathrm{Hom}_{G'}(Z[G], A')$$

which is an isomorphism if $[G:G'] < \infty$.

18: NOTATION Given a G-module A, put

$$A^G = \{a \in A : \sigma a = a \ \forall \ \sigma \in G\}.$$

[Note: A^G is a subgroup of A, termed the invariants in A.]

19: LEMMA $A^G = \mathrm{Hom}_G(Z,A)$ (trivial G-action on Z).

[Note: By comparison,

$$A = \mathrm{Hom}_G(Z[G],A).]$$

20: LEMMA $\mathrm{Hom}_Z(A,B)^G = \mathrm{Hom}_G(A,B)$.

$\underline{MOD}_G$ is an abelian category. As such, it has enough injectives (i.e., every G-module can be embedded in an injective G-module).

21: DEFINITION The group cohomology functor $H^q(G,-):\underline{MOD}_G \to \underline{AB}$ is the right derived functor of $(-)^G$.

[Note: Recall the procedure: To compute $H^q(G,A)$, choose an injective resolution

$$0 \to A \to I^0 \to I^1 \to \dots .$$

Then $H^*(G,A)$ is the cohomology of the complex $(I)^G$. In particular: $H^0(G,A) = A^G$.]

<u>22:</u> LEMMA $H^q(G,A)$ is independent of the choice of injective resolutions.

<u>23:</u> LEMMA $H^q(G,A)$ is a covariant functor of A.

<u>24:</u> LEMMA If

$$0 \to A \to B \to C \to 0$$

is a short exact sequence of G-modules, then there is a functorial long exact sequence

$$\begin{aligned} 0 &\to H^0(G,A) \to H^0(G,B) \to H^0(G,C) \\ &\to H^1(G,A) \to H^1(G,B) \to H^1(G,C) \to H^2(G,A) \to \cdots \\ \cdots &\to H^q(G,A) \to H^q(G,B) \to H^q(G,C) \to H^{q+1}(G,A) \to \cdots \end{aligned}$$

in cohomology.

<u>25:</u> <u>N.B.</u> If $G = \{1\}$ is the trivial group, then

$$H^0(G,A) = A, \; H^q(G,A) = 0 \quad (q > 0).$$

[Note: Another point is that for any G, every injective G-module A is cohomologically acyclic:

$$\forall\, q > 0,\; H^q(G,A) = 0.]$$

<u>26:</u> THEOREM (SHAPIRO'S LEMMA) If $[G:G'] < \infty$, then $\forall\, q$,

$$H^q(G',A') \approx H^q(G,\mathrm{Ind}_{G'}^{G}A').$$

27: EXAMPLE Take $A' = Z[G']$ -- then

$$H^q(G',Z[G']) \approx H^q(G,Z[G] \otimes_{Z[G']} Z[G'])$$

$$\approx H^q(G,Z[G]).$$

28: EXAMPLE Take $G' = \{1\}$ (so G is finite) -- then $Z[G'] = Z$ and

$$H^q(\{1\},Z) \approx H^q(G,Z[G]).$$

But the LHS vanishes if $q > 0$, thus the same is true of the RHS. However this fails if G is infinite. E.g.: Take for G the infinite cyclic group: $H^1(G,Z[G]) \approx Z$.

[Note: If G is finite, then $H^0(G,Z[G]) \approx Z$ while if G is infinite, then $H^0(G,Z[G]) = 0$.]

29: EXAMPLE Take $A' = Z$ -- then

$$H^q(G',Z) \approx H^q(G,\mathrm{Ind}_{G'}^{G} Z)$$

$$\approx H^q(G,Z[G/G']).$$

§6. TAMAGAWA MEASURES

Suppose given a Q-torus T of dimension d — then one can introduce

$$\begin{array}{ccc} T(Q) \subset T(R), & T(Q) \subset & T(Q_p) \\ & & \cup \\ & & T(Z_p) \end{array}$$

and

$$T(Q) \subset T(A).$$

1: EXAMPLE Take $T = G_{m,Q}$ -- then the above data becomes

$$\begin{array}{ccc} Q^\times \subset R^\times, & Q^\times \subset & Q_p^\times \\ & & \cup \\ & & Z_p^\times \end{array}$$

and

$$Q^\times \subset A^\times = I.$$

2: LEMMA $T(Q)$ is a discrete subgroup of $T(A)$.

3: RAPPEL $I^1 = \text{Ker}|\cdot|_A$, where for $x \in I$,

$$|x|_A = \prod_{p \leq \infty} |x_p|_p.$$

And the quotient $I^1/Q^\times$ is a compact Hausdorff space.

Each $\chi \in X(T)$ generates continuous homomorphisms

$$\begin{cases} \chi_p : T(Q_p) \rightarrow Q_p^\times \xrightarrow{|\cdot|_p} R_{>0}^\times \\ \chi_\infty : T(R) \rightarrow R^\times \xrightarrow{|\cdot|_\infty} R_{>0}^\times \end{cases}$$

from which an arrow

$$\begin{cases} \chi_A : T(A) \to R_{>0}^{\times} \\ \quad x \to \prod_{p \le \infty} \chi_p(x_p). \end{cases}$$

4: NOTATION

$$T^1(A) = \bigcap_{\chi \in X(T)} \operatorname{Ker} \chi_A.$$

5: N.B. The infinite intersection can be replaced by a finite intersection since if $\chi_1, \ldots, \chi_d$ is a basis for $X(T)$, then

$$T^1(A) = \bigcap_{i=1}^{d} \operatorname{Ker}(\chi_i)_A.$$

6: THEOREM The quotient $T^1(A)/T(Q)$ is a compact Hausdorff space.

7: CONSTRUCTION Let Ω_T denote the collection of all left invariant d-forms on T, thus Ω_T is a 1-dimensional vector space over Q. Choose a nonzero element $\omega \in \Omega_T$ -- then ω determines a left invariant differential form of top degree on the $T(Q_p)$ and $T(R)$, which in turn determines a Haar measure $\mu_{Q_p,\omega}$ on the $T(Q_p)$ and a Haar measure $\mu_{R,\omega}$ on $T(R)$.

The product

$$\prod_p \mu_{Q_p,\omega}(T(Z_p))$$

may or may not converge.

8: DEFINITION A sequence $\Lambda = \{\Lambda_p\}$ of positive real numbers is said to be a system of convergence coefficients if the product

$$\prod_p \Lambda_p \mu_{Q_p,\omega}(T(Z_p))$$

is convergent.

9: N.B. Convergence coefficients always exist, e.g.,

$$\Lambda_p = \frac{1}{\mu_{Q_p,\omega}(T(Z_p))}.$$

10: LEMMA If the sequence $\Lambda = \{\Lambda_p\}$ is a system of convergence coefficients, then

$$\mu_{\omega,\Lambda} \equiv \prod_p \Lambda_p \mu_{Q_p,\omega} \times \mu_{R,\omega}$$

is a Haar measure on $T(A)$.

11: N.B. Let λ be a nonzero rational number -- then

$$\mu_{Q_p,\lambda\omega} = |\lambda|_p \mu_{Q_p,\omega}, \quad \mu_{R,\lambda\omega} = |\lambda|_\infty \mu_{R,\omega}.$$

Therefore

$$\begin{aligned} \mu_{\lambda\omega,\Lambda} &\equiv \prod_p \Lambda_p \mu_{Q_p,\lambda\omega} \times \mu_{R,\lambda\omega} \\ &= (\prod_p |\lambda|_p) \prod_p \Lambda_p \mu_{Q_p,\omega} \times |\lambda|_\infty \mu_{R,\omega} \\ &= \prod_{p\leq\infty} |\lambda|_p \prod_\rho \Lambda_p \mu_{Q_p,\omega} \times \mu_{R,\omega} \\ &= \mu_{\omega,\Lambda}. \end{aligned}$$

And this means that the Haar measure $\mu_{\omega,\lambda}$ is independent of the choice of the rational density ω.

Let $K \supset Q$ be a Galois extension relative to which T splits -- then

$$X(T_K) = \mathrm{Mor}_K(T_K, G_{m,K})$$

is a Gal(K/Q) lattice. Call Π the representation thereby determined and denote its character by χ_Π. Let

$$L(s,\chi_\Pi,K/Q) = \prod_p L_p(s,\chi_\Pi,K/Q)$$

be the associated Artin L-function and denote by S the set of primes that ramify in K plus the "prime at infinity".

<u>12:</u> LEMMA $\forall\, p \notin S$,

$$\mu_{Q_p,\omega}(T(Z_p)) = L_p(1,\chi_\Pi,K/Q)^{-1}.$$

<u>13:</u> SCHOLIUM The sequence $\Lambda = \{\Lambda_p\}$ defined by the prescription

$$\Lambda_p = L_p(1,\chi_\Pi,K/Q) \text{ if } p \notin S$$

and

$$\Lambda_p = 1 \text{ if } p \in S$$

is a system of convergence coefficients termed <u>canonical</u>.

<u>14:</u> LEMMA The Haar measure $\mu_{\omega,\Lambda}$ on T(A) corresponding to a canonical system of convergence coefficients is independent of the choice of K, denote it by μ_T.

<u>15:</u> DEFINITION μ_T is the <u>Tamagawa measure</u> on T(A).

Owing to Brauer theory, there is a decomposition of the character χ_Π of Π as a finite sum

$$\chi_\Pi = d\chi_0 + \sum_{j=1}^{M} m_j\chi_j,$$

where χ_0 is the principal character of Gal(K/Q) ($\chi_0(\sigma) = 1$ for all $\sigma \in \mathrm{Gal}(K/Q)$), the m_j are positive integers, and the χ_j are irreducible characters of Gal(K/Q). Standard properties of Artin L-functions then imply that

$$L(s,\chi_\Pi,K/Q) = \zeta(s)^d \prod_{j=1}^{M} L(s,\chi_j,K/Q)^{m_j}.$$

<u>16:</u> FACT

$$L(1,\chi_j,K/Q)^{m_j} \neq 0 \quad (1 \leq j \leq M).$$

Therefore

$$\lim_{s \to 1} (s-1)^d L(s,\chi_\Pi,K/Q) = \prod_{j=1}^{M} L(1,\chi_j,K/Q)^{m_j}$$

$$\neq 0.$$

<u>17:</u> LEMMA The limit on the left is positive and independent of the choice of K, denote it by ρ_T.

<u>18:</u> DEFINITION ρ_T is the <u>residue</u> of T.

Define a map

$$T:T(A) \to (R^\times_{>0})^d$$

by the rule

$$T(x) = ((\chi_1)_A(x), \ldots, (\chi_d)_A(x)).$$

Then the kernel of T is $T^1(A)$, hence T drops to an isomorphism

$$T^1 : T(A)/T^1(A) \to (R^\times_{>0})^d.$$

<u>19:</u> DEFINITION The <u>standard measure</u> on $T(A)/T^1(A)$ is the pullback via T^1 of the product measure

$$\prod_{i=1}^{d} \frac{dt_i}{t_i}$$

on $(R^\times_{>0})^d$.

Consider now the formalism

$$d(T(A)) = d(T(A)/T^1(A))d(T^1(A)/T(Q))d(T(Q))$$

in which:

- $d(T(A))$ is the Tamagawa measure on $T(A)$ multiplied by $\frac{1}{\rho_T}$.
- $d(T(A)/T^1(A))$ is the standard measure on $T(A)/T^1(A)$.
- $d(T(Q))$ is the counting measure on $T(Q)$.

<u>20:</u> DEFINITION The <u>Tamagawa number</u> $\tau(T)$ is the volume

$$\tau(T) = \int_{T^1(A)/T(Q)} 1$$

of the compact Hausdorff space $T^1(A)/T(Q)$ per the invariant measure

$$d(T^1(A)/T(Q))$$

such that

$$\frac{\mu_T}{\rho_T} = d(T(A)T^1(A))d(T^1(A)/T(Q))d(T(Q)).$$

21: N.B. To be completely precise, the integral formula

$$\int_{T(A)} = \int_{T(A)/T^1(A)} \int_{T^1(A)}$$

fixes the invariant measure on $T^1(A)$ and from there the integral formula

$$\int_{T^1(A)} = \int_{T^1(A)/T(Q)} \int_{T(Q)}$$

fixes the invariant measure on $T^1(A)/T(Q)$, its volume then being the Tamagawa number $\tau(T)$.

[Note: If T is Q-anisotropic, then $T(A) = T^1(A)$.]

22: EXAMPLE Take $T = G_{m,Q}$ and $\omega = \frac{dx}{x}$ -- then

$$\mathrm{vol}_{\frac{dx}{|x|_p}} (Z_p^\times) = \frac{p-1}{p} = 1 - \frac{1}{p}$$

and the canonical convergence coefficients are the

$$(1 - \frac{1}{p})^{-1}.$$

Here $d = 1$ and

$$\lim_{s \to 1} (s-1)\zeta(s) = 1 \Rightarrow \rho_T = 1.$$

Working through the definitions, one concludes that $\tau(T) = 1$ or still,

$$\mathrm{vol}(I^1/Q^\times) = 1.$$

23: REMARK Take $T = \mathrm{Res}_{K/Q}(G_{m,K})$ -- then it turns out that $\tau(T)$ is the Tamagawa number of $G_{m,K}$ computed relative to K (and not relative to Q...). From this, it follows that $\tau(T) = 1$, matters hinging on the "famous formula"

$$\lim_{s \to 1} (s-1)\zeta_K(s) = \frac{2^{r_1}(2\pi)^{r_2}}{w_K |d_K|^{1/2}} h_K R_K.$$

24: LEMMA Let F be an integrable function on $(\mathbf{R}^{\times}_{>0})^d$ -- then

$$\tau(T) = \frac{\frac{1}{\rho_T} \int_{T(A)/T(Q)} F(T(x)) d\mu_T(x)}{\int_{(\mathbf{R}^{\times}_{>0})^d} F(t_1,\ldots,t_d) \frac{dt_1}{t_1} \cdots \frac{dt_d}{t_d}}.$$

25: EXAMPLE Take $T = G_{m,Q}$ -- then

$$\tau(T) = \frac{\int_{I/Q^{\times}} F(|x|_A) d\mu_T(x)}{\int_0^{\infty} \frac{F(t)}{t}\, dt},$$

ρ_T being 1 in this case. To see that $\tau(T) = 1$, make the calculation by choosing

$$F(t) = 2te^{-\pi t^2}.$$

[Note: Recall that

$$\prod_p Z_p^{\times} \times \mathbf{R}^{\times}_{>0}$$

is a fundamental domain for $I/Q^{\times}$.]

26: NOTATION Put

$$H^1(Q,T) = H^1(\mathrm{Gal}(Q^{\mathrm{sep}}/Q),\ T(Q^{\mathrm{sep}}))$$

and for $p \leq \infty$,

$$H^1(Q_p,T) = H^1(Gal(Q_p^{sep}/Q_p),\ T(Q_p^{sep})).$$

27: LEMMA There is a canonical arrow

$$H^1(Q,T) \to H^1(Q_p,T).$$

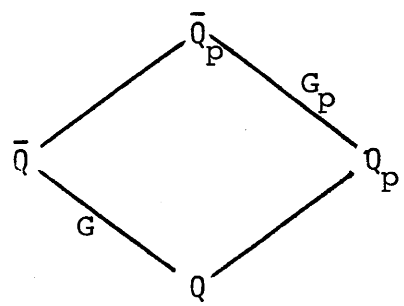
PROOF Put

$$G = Gal(\bar{Q}/Q) \quad (\bar{Q} = Q^{sep})$$

and

$$G_p = Gal(\bar{Q}_p/Q_p) \quad (\bar{Q}_p = Q_p^{sep}).$$

Then schematically



1. There is an arrow of restriction

$$\rho: G_p \to G$$

and a morphism $T(Q) \to T(\bar{Q}_p)$ of G_p-modules, $T(Q)$ being viewed as a G_p-module via ρ.

2. The canonical arrow

$$H^1(Q,T) \to H^1(Q_p,T)$$

is then the result of composing the map

$$H^1(G,T(Q)) \to H^1(G_p,T(Q))$$

with the map

$$H^1(G_p, T(Q)) \to H^1(G_p, T(\bar{Q}_p)).$$

28: NOTATION Put

$$Ш(T) = \text{Ker}(H^1(Q,T) \to \coprod_{p\leq\infty} H^1(Q_p,T)).$$

29: DEFINITION Ш(T) is the Tate-Shafarevich group of T.

30: THEOREM Ш(T) is a finite group.

31: EXAMPLE If K is a finite extension of Q, then

$$H^1(Q, \text{Res}_{K/Q}(G_{m,K})) = 1.$$

Therefore in this case

$$\#(Ш(T)) = 1.$$

32: REMARK By comparison,

$$H^1(Q, \text{Res}^{(1)}_{K/Q}(G_{m,K})) \approx Q^\times / N_{K/Q}(K^\times).$$

[Consider the short exact sequence

$$1 \to \text{Res}^{(1)}_{K/Q}(G_{m,K}) \to \text{Res}_{K/Q}(G_{m,K}) \xrightarrow{N_{k/Q}} G_{m,Q} \to 1.]$$

33: NOTATION Put

$$Ч(T) = \text{CoKer}(H^1(Q,T) \to \coprod_{p\leq\infty} H^1(Q_p,T)).$$

34: THEOREM Ч(T) is a finite group.

35: MAIN THEOREM The Tamagawa number $\tau(T)$ is given by the formula

$$\tau(T) = \frac{\#(Ч(T))}{\#(Ш(T))}.$$

36: EXAMPLE If K is a finite extension of Q, then

$$H^1(Q_p, \mathrm{Res}_{K/Q}(G_{m,K})) = 1.$$

Therefore in this case

$$\#(Ч(T)) = 1.$$

It follows from the main theorem that $\tau(T)$ is a positive rational number. Still, there are examples of finite abelian extensions $K \supset Q$ such that

$$\tau(\mathrm{Res}^{(1)}_{K/Q} G_{m,K})$$

is not a positive integer.