An Invitation to Tensor Triangulated Geometry

Wolfgang Allred

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## About these notes

These notes are the product of my writing milestone project with Professor Julia Pevtsova at the University of Washington. They are adapted from the lecture notes of an introductory course on *tensor triangulated geometry* taught by Professor Julia Pevtsova during the Winter quarter of 2024. The original document was a collaborative effort between everyone in the class, as every day a different student was assigned to transcribe and later type up that day's lecture in a shared LaTeX document. I would like to thank those students for their contributions. Since I did not attend this class myself (as I was not yet a student at Washington), this project would not have happened with without their diligent work.

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|---|---|---|
| Andrew Aguilar | Nathan Louie | Nelson Niu |
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When I set out to write up these notes, my intended audience was myself from three years in the past, i.e. an early graduate student who wants to learn some of the fundamental notions of tensor triangulated geometry and is comfortable with homological algebra. Ideally, readers should also have familiarity with either the derived categories of a ring or scheme, modular representation theory, or stable homotopy theory.

I tried to loosely follow the original class notes and to add more expository material while keeping the text as self contained as possible and with ample references. Chapters 1 through 3 cover the material from Paul Balmer's original paper [Bal05] on tensor triangulated categories while Chapters 3 through 5 cover some of the material from [Bal10]. Chapter 6 covers some material on big tensor triangulated categories.

# 0 Introduction

Mathematicians of all stripes have long toiled to classify their favorite objects of study; be they topological spaces, quasi-coherent sheaves, or modules over a $p$-group, mathematicians love to round them up and put them into neat little boxes. Sometimes they are successful after a long struggle, as in the case of finite simple groups, but sometimes it becomes clear that the damn things are just too wild, as is the case in many flavors of homotopy theory. When faced with impossibility, the mathematician's trick is not to capitulate, but to change the rules of the game. The term "up to isomorphism" is relaxed to "up to homotopy", prompting the advent of triangulated categories. Later, "homotopy" is relaxed to "stable homotopy". Ground is gained, but still the task seems intractable, even when the scope of the battle is narrowed to nice classes of objects.

Still, there are innovations to be made. When describing the structure of an algebraic object, algebraists don't often determine the behavior of individual elements, and instead study substructures. This amounts to no longer asking if $x = y$, but instead asking if $x$ and $y$ may be obtained from one and other through various operations, i.e. comparing the ideals generated by $x$ and $y$. In the context of stable homotopy theory one deals with categories that are not just triangulated but *tensor triangulated*. In this case, one asks if $x$ can by obtained from $y$ through the operations of tensoring, taking direct sums, direct summands, cones, and suspensions. Therefore, if we wish to study any kind of stable homotopy theory, we should take inspiration from the ring theorist. Instead of looking at when $x \cong y$, we should study when $x$ may be obtained from $y$ through these algebraic operations, i.e. when $x$ is contained in the thick tensor ideal generated by $y$. From a more zoomed out perspective, we are studying the lattices of certain suitable subcategories of our larger ambient category.

The first steps in this direction came from algebraic topology, primarily in work to do with Ravenel's conjectures and the work of Devinatz-Hopkins-Smith in chromatic homotopy theory; see [DHS88]. Hopkins was the first to start transplanting these results to other areas of research, specifically the derived category, but there were issues with his proof which was rescued by Neeman; see [Hop87] and [Nee92]. Later, Thomason clarified the matter definitively in [Tho97], and in doing so highlighted the importance of the tensor structure in the proof. That same year, Benson, Carlson and Rickard made analogous advances in modular representation theory. The first axiomatic approach was taken by Hovey-Palmieri-Strickland in their monograph "Axiomatic Stable Homotopy Theory", [HPS97].

Later, Balmer provided a perhaps more elementary axiomatic approach to studying stable homotopy theory in [Bal05]. In this context, we can assign a topological space to our tensor triangulated category $\mathcal{T}$ which acts as a universal support variety for $\mathcal{T}$. This space, called the Balmer spectrum of $\mathcal{T}$ and written $\mathrm{Spc}(\mathcal{T})$, classifies thick tensor ideals of $\mathcal{T}$ and provides a shared framework unifying many of the previous advances.

In the first chapter we will define tensor triangulated categories and all of the necessary accompanying structures. We will also look at the context from which tensor triangular geometry descends, namely, stable homotopy theory, derived categories of schemes and rings, and the representation theory of finite groups. This leads directly into the second chapter where we will define and establish some of the key properties of the Balmer spectrum, which will then allow us to carry out the classification of thick tensor ideals for tensor triangulated categories.

The Balmer spectrum $\mathrm{Spc}(\mathcal{T})$ is well understood in certain cases, but is in general hard to compute. One approach to this problem is to try to relate $\mathrm{Spc}(\mathcal{T})$ to more familiar algebro-geometric structures. In this vein, the third chapter focuses on the comparison map $\rho^* : \mathrm{Spc}(\mathcal{T}) \to \mathrm{Spec}(\mathrm{End}^*_{\mathcal{T}}(\mathbb{1}))$ which relates the Balmer spectrum of $\mathcal{T}$ to the spectrum of the graded endomorphism ring of the monoidal unit of $\mathcal{T}$. We will also see that $\mathrm{Spc}(\mathcal{T})$ may be given the structure of a locally ringed space, and that the comparison map is a morphism of locally ringed spaces, at least when $\mathcal{T}$ is rigid.

In the last chapter, we will see that tensor triangulated categories often live inside of stable homotopy categories as the full subcategory of compact objects. Though the Balmer spectrum cannot be defined for these large categories for set theoretic reasons, we can use the Balmer spectrum of the subcategory of compact objects to lift a notion of support to the larger ambient homotopy category.

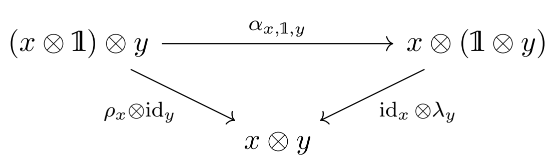
# 1 Tensor Triangulated Categories

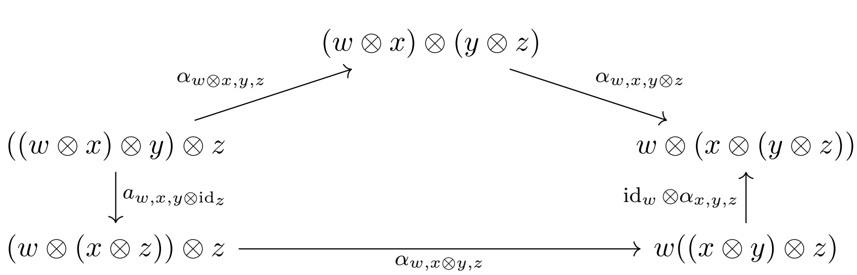
We begin our journey by introducing the basic notions from [Bal05] and developing some of the basic machinery required to study tensor triangulated geometry. Some familiarity with homological algebra and triangulated categories is required, and for this our two main references will be [Nee01] and [Wei94]. After that, we will introduce the main examples of tensor triangulated categories that we would like to study.

## 1.1 First Definitions

**Definition 1.1** (Monoidal Category)**.** A *monoidal category* is a category $\mathcal{C}$ equipped with

1. A functor
$$\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$$
$$(a, b) \mapsto a \otimes b$$
called the *tensor product*.
2. an object $\mathbb{1} \in \mathcal{C}$ called the *unit object* or *tensor unit*,
3. a natural isomorphism
$$\alpha : ((-) \otimes (-)) \otimes (-) \xrightarrow{\sim} (-) \otimes ((-) \otimes (-))$$
of the form
$$\alpha_{x,y,z} : (x \otimes y) \otimes z \to x \otimes (y \otimes z)$$
called the *associator*,
4. natural isomorphisms
$$\lambda : (\mathbb{1} \otimes (-)) \xrightarrow{\sim} (-) \quad \text{and} \quad \rho : (-) \otimes \mathbb{1} \xrightarrow{\sim} (-)$$
of the form
$$\lambda_x : \mathbb{1} \otimes x \to x \quad \text{and} \quad \rho_x : x \otimes \mathbb{1} \to x$$
such that the following diagrams commute

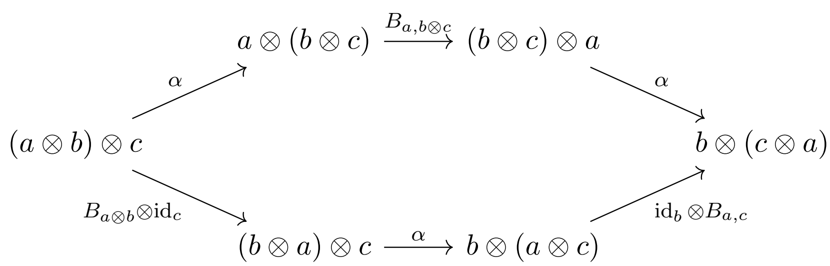


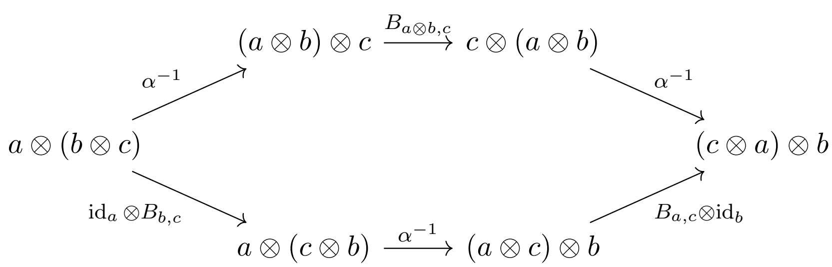



**Definition 1.2.** A *braided monoidal category* is a monoidal category $\mathcal{C}$ equipped with another natural isomorphism

$$B_{x,y} : x \otimes y \to y \otimes x$$

called the *braiding* which that the following two kinds of diagrams commute for all objects involved.




We say that $\mathcal{C}$ is a *symmetric monoidal category* or a *tensor category* if it is a braided monoidal category for which the braiding $B_{x,y} : x \otimes y \to y \otimes x$ has the property $B_{y,x} \circ B_{x,y} = \mathrm{id}_{x\otimes y}$ for all objects $x, y$ in $\mathcal{C}$.

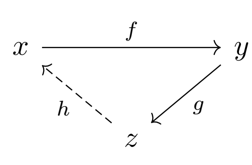
**Definition 1.3.** A *triangulated category* is a triple $(\mathcal{T}, \Sigma, \Delta)$ where $\mathcal{T}$ is an additive category equipped with an auto-equivalence

$$\Sigma : \mathcal{T} \to \mathcal{T}$$

called the *shift* or *suspension* and a class $\Delta$ of *exact* (or *distinguished*) triangles, which are triples of composable morphisms:

$$x \xrightarrow{f} y \xrightarrow{g} z \xrightarrow{h} \Sigma x$$

They are called triangles because they are sometimes represented in diagrams of the kind below, where the dotted line represents a map from $z$ to $\Sigma x$.




Some authors condense notation into the following form: $(x, y, z; f, g, h)$. We require that $\mathcal{T}$ satisfies the following axioms

(TR1) (Bookkeeping)

a. $x \xrightarrow{\text{id}} x \to 0 \to \Sigma x$ is exact for each object $x \in \mathcal{T}$.

b. Distinguished triangles are closed under isomorphisms.

c. $x \xrightarrow{f} y \xrightarrow{g} z \xrightarrow{h} \Sigma x$ is exact if and only if $y \xrightarrow{g} z \xrightarrow{h} \Sigma x \xrightarrow{-\Sigma f} \Sigma y$ is exact. This is called *rotation.*

(TR2) (Cones) For each morphism $x \xrightarrow{f} y$ there is an object cone$(f)$ (unique up to non-unique isomorphism) referred to as the *cone* of $f$ which fits into an exact triangle

$$x \xrightarrow{f} y \xrightarrow{g} \text{cone}(f) \xrightarrow{h} \Sigma x$$

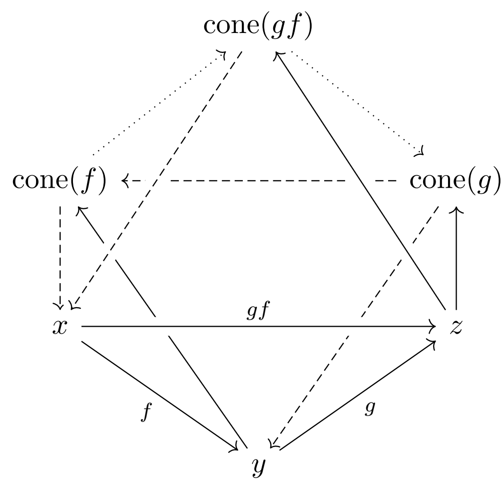
Sometimes cone$(f)$ will be used instead of cone$(f)$.

(TR3) (Extension) Given two exact triangles $x \xrightarrow{f} y \xrightarrow{g} z \xrightarrow{h} \Sigma x$ and $x' \xrightarrow{f'} y' \xrightarrow{g'} z \xrightarrow{h'} \Sigma x'$ and arrows $x \xrightarrow{\alpha} x'$ and $y' \xrightarrow{\beta} y'$ such that $\beta \circ f = f' \circ \alpha$, then there exists $z \xrightarrow{\gamma} z'$ making the diagram below commute

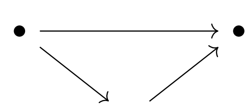
$$\begin{array}{ccccccc} x & \xrightarrow{f} & y & \xrightarrow{g} & z & \xrightarrow{h} & \Sigma x \\ \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \exists\gamma} & & \downarrow{\scriptstyle \Sigma\alpha} \\ x' & \xrightarrow{f'} & y' & \xrightarrow{g'} & z' & \xrightarrow{h'} & \Sigma x' \end{array}$$

Such a commutative diagram is called a *morphism of triangles.*

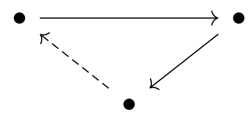
(TR4) (Octohedral) Given a composition $x \xrightarrow{f} y \xrightarrow{h} z$ there exists dotted arrows which form an octohedral diagram,




where each face is either a simplex commuting up to suspension,



or an exact triangle



For other more precise presentations of the octohedral axiom, refer to Appendix A.

**Definition 1.4** (Tensor Triangulated Category)**.** A *tensor triangulated category* (or a tt-category) is a category $\mathcal{T}$ equipped with a tensor structure $(\mathcal{T}, \otimes, \mathbb{1})$ and a triangulated structure $(\mathcal{T}, \Sigma, \Delta)$ and a natural isomorphism

$$e : \Sigma(-) \otimes y \xrightarrow{\sim} \Sigma(- \otimes -)$$

with components

$$e_{x,y} : \Sigma x \otimes y \xrightarrow{\sim} \Sigma(x \otimes y)$$

such that

1. $\otimes$ is additive and exact in both arguments (takes exact triangles to exact triangles),
2. $e$ satisfies the following commutative coherence pentagon for all objects $x, y, z$ in $\mathcal{T}$.

$$\begin{array}{ccccc} & & \Sigma(x \otimes y) \otimes z & & \\ & \nearrow^{e_{x,y} \otimes \mathrm{id}_z} & & \searrow^{e_{x \otimes y, z}} & \\ (\Sigma x \otimes y) \otimes z & & & & \Sigma((x \otimes y) \otimes z) \\ \downarrow{\scriptstyle \alpha_{\Sigma x, y, z}} & & & & \downarrow{\scriptstyle \Sigma \alpha_{x,y,z}} \\ \Sigma x \otimes (y \otimes z) & & \xrightarrow{e_{x,(y \otimes z)}} & & \Sigma(x \otimes (y \otimes z)) \end{array}$$

3. the Koszul sign rule is satisfied:

$$\begin{array}{ccc} (\Sigma^a \mathbb{1}) \otimes (\Sigma^b \mathbb{1}) & \xrightarrow{\sim} & \Sigma^{a+b} \mathbb{1} \\ \downarrow{\scriptstyle B_{\Sigma^a \mathbb{1}, \Sigma^b \mathbb{1}}} & & \downarrow{\scriptstyle (-1)^{ab}} \\ (\Sigma^b \mathbb{1}) \otimes (\Sigma^a \mathbb{1}) & \xrightarrow{\sim} & \Sigma^{a+b} \mathbb{1} \end{array}$$

If we are talking about two different tt-categories we might adorn their units and tensor products with subscripts in order to avoid confusion, e.g. $\otimes_{\mathcal{T}}$ and $\mathbb{1}_{\mathcal{T}}$.

**Definition 1.5.** Let $\mathcal{T}$ and $\mathcal{T}'$ be tt-categories. A functor $F : \mathcal{T} \to \mathcal{T}'$ is called an *exact functor* if it is equipped with functorial isomorphisms $\xi : F(\Sigma x) \to \Sigma F(x)$ such that an exact triangle $(x, y, z; f, g, h)$ in $\mathcal{T}$ is taken to an exact triangle $(F(x), F(y), F(z); F(f), F(g), \xi_x F(h))$ in $\mathcal{T}'$.

Furthermore, $F$ is called $\otimes$*-exact* if $F(\mathbb{1}_{\mathcal{T}}) = \mathbb{1}_{\mathcal{T}'}$ and if $F(x \otimes y) \cong F(x) \otimes F(y)$.

**Definition 1.6.** Given a triangulated category $(\mathcal{T}, \Sigma, \Delta)$, a *triangulated subcategory* is a pair $(\mathcal{C}, \Delta')$ such that

1. $\mathcal{C}$ is an full additive subcategory of $\mathcal{T}$ which is preserved under $\Sigma$ such that $\Sigma : \mathcal{C} \to \mathcal{C}$ is an auto-equivalence,
2. $\Delta' = \Delta \cap \mathcal{C}$, and
3. $(\mathcal{C}, \Sigma, \Delta')$ is a triangulated category.

*Remark* 1.7. Note that the above definition immediately implies that if $x \to y \to z \to \Sigma x$ is an exact triangle in $\mathcal{T}$ and any two of $x, y, z$ are in $\mathcal{C}$ then the third is in $\mathcal{C}$. As a consequence, thick $\otimes$-ideals are replete, which is to say, thick $\otimes$-ideals are closed under isomorphisms.

We are now ready for our first propositions.

**Proposition 1.8.** *For objects $a, b, c \in \mathcal{T}$ the distributive property holds:*

$$(a \oplus b) \otimes c \cong (a \otimes c) \oplus (b \otimes c)$$

*Proof.* By corollary A.8 $a \xrightarrow{i_a} a \oplus b \xrightarrow{p_b} b \xrightarrow{0} \Sigma(a)$ is an exact triangle. By applying exactness of the $- \otimes c$ functor,

$$a \otimes c \xrightarrow{i_a \otimes id_c} (a \oplus b) \otimes c \xrightarrow{p_b \otimes \mathrm{id}_c} b \otimes c \xrightarrow{0} \Sigma(a) \otimes c \tag{1}$$

is exact, as is

$$a \otimes c \xrightarrow{i_{a \otimes c}} (a \otimes c) \oplus (b \otimes c) \xrightarrow{p_{b \otimes c}} b \otimes c \xrightarrow{0} \Sigma(a \otimes c) \tag{2}$$

by another application of corollary A.8. By applying TR1(rotation), TR2, and TR3 we can get a morphism of exact triangles between the triangles on lines (1) and (2) above that is the identity on the first and third components and an isomorphism on the fourth as $\Sigma(a \otimes c) \cong \Sigma(a) \otimes c$. By proposition A.5 $(a \oplus b) \otimes c \cong (a \otimes c) \oplus (b \otimes c)$. □

**Proposition 1.9.** *For any morphism $f : a \to b$ in a tt-category $\mathcal{T}$ there is an isomorphism*

$$s \otimes \mathrm{cone}(f) \cong \mathrm{cone}(f \otimes \mathrm{id}_s)$$

*for all objects $s \in \mathcal{T}$.*

*Proof.* By TR2 the triangle $a \xrightarrow{f} b \xrightarrow{g} \mathrm{cone}(f) \xrightarrow{h} \Sigma a$ and since $\otimes$ is exact in both variables it follows that the top row of the triangle below is exact. The bottom row is also exact by TR2.

$$\begin{array}{ccccccc}
a \otimes s & \xrightarrow{f \otimes \mathrm{id}_s} & b \otimes s & \xrightarrow{g \otimes \mathrm{id}_s} & \mathrm{cone}(f) \otimes s & \xrightarrow{h \otimes \mathrm{id}_s} & \Sigma(a \otimes s) \\
\| & & \| & & \updownarrow & & \| \\
a \otimes s & \xrightarrow{f \otimes \mathrm{id}_s} & b \otimes s & \xrightarrow{g'} & \mathrm{cone}(f \otimes \mathrm{id}_s) & \xrightarrow{h'} & \Sigma(a) \otimes s
\end{array}$$

By TR3 the dotted line exists to make the diagram a morphism of triangles, and by proposition A.5 it is an isomorphism. □

*Remark* 1.10. Tensor Triangular geometry is largely inspired by the rich geometry found in the realm of ring spectra, though I would argue that it is not immediately obvious from the definitions above that one should be able to develop such a theory on tt-categories. It's also worth noting that we could have proved the proposition above if our category had an internal-hom structure, but this proof makes it clear that distributivity of the tensor product over direct sums directly comes from the triangular structure communicating well with the tensor structure, which is a sign that our tt-category should behave somewhat like a ring. With this in mind, we shall seek out other parallels between tt-categories and commutative rings.

**Definition 1.11.** If $\mathcal{T}$ is a triangulated category then $\mathcal{C} \subseteq T$ is a *thick subcategory* if it is a triangulated subcategory of $\mathcal{T}$ and $\mathcal{C}$ is closed under sums, meaning that $a, b \in \mathcal{C}$ if and only if $a \oplus b \in \mathcal{C}$.

If, in addition, $\mathcal{T}$ is a *tensor* triangulated category, and $\mathcal{C}$ is a thick subcategory of $\mathcal{T}$ that is closed under $\otimes$, i.e. $a \in \mathcal{C}$ and $b \in \mathcal{T}$ implies that $a \otimes b \in \mathcal{C}$, then we say that it is a *thick $\otimes$-ideal.* Given a collection of objects $\mathcal{S} \subseteq \mathrm{Obj}(\mathcal{T})$, write $\langle \mathcal{S} \rangle$ to denote the $\otimes$-ideal generated by $\mathcal{S}$. Equivalently, this is the smallest $\otimes$-ideal containing $\mathcal{S}$.

It is important to keep in mind that to describe a full subcategory it is sufficient to merely describe all the objects within the subcategory, which we will do regularly.

*Remark* 1.12. Though not immediately obvious from the definition, it's worth noting that if $\mathcal{I}$ is a triangulated subcategory of $\mathcal{T}$ then $a, b \in \mathcal{C}$ implies that $a \oplus b \in \mathcal{C}$, so $\mathcal{C}$ being thick only adds the requirement that direct sums of objects in $\mathcal{T}$ decompose within $\mathcal{C}$.

Additionally, note that we are not distinguishing between left, right, or two-sided ideals. This is because our monoidal structure is symmetric, though it should be said that there is no reason that we could not consider a non-commutative analogue to the definition above in the scenario that we relax the condition that $\otimes$ be symmetric. See [NVY21] for an exploration of noncommutative tensor triangulated geometry.

**Notation 1.13.** If $\mathcal{C}$ is a subcategory of a triangulated category $\mathcal{T}$, we write $\mathrm{thick}(\mathcal{C})$ be the smallest thick subcategory of $\mathcal{T}$ that contains $\mathcal{C}$. This is called the *thick closure* of C in $\mathcal{T}$.

**Proposition 1.14.** *Let $\mathcal{T}$ be a tensor triangulated category with unit $\mathbb{1}$. If $\mathrm{thick}(\mathbb{1}) = \mathcal{T}$, then all thick subcategories of $\mathcal{T}$ are $\otimes$-ideals.*

*Proof.* Let $\mathcal{I}$ be a thick subcategory of $\mathcal{T}$ and let $\mathcal{S} = \{a \in \mathcal{T} \mid a \otimes \mathcal{I} \subseteq \mathcal{I}\}$. Our aim is to show that $\mathcal{S}$ is a thick subcategory of $\mathcal{T}$.

If $x \in \mathcal{I}$ and $a, b \in \mathcal{T}$ then $x \otimes (a \oplus b) \cong (x \otimes a) \oplus (x \otimes b)$, and since $\mathcal{I}$ is a thick subcategory it follows that $a, b \in \mathcal{S}$ if and only if $a \oplus b \in \mathcal{S}$. Additionally, if $a \in \mathcal{S}$ then $\Sigma(x \otimes a) \cong x \otimes (\Sigma a) \cong (\Sigma x) \otimes a$, and since $\Sigma x \in \mathcal{I}$ it follows that $\Sigma a \in \mathcal{S}$. Finally, if $a \to b \to c \to \Sigma c$ is an exact triangle and $a, b \in \mathcal{S}$ then $a \otimes x \to b \otimes x \to c \otimes x \to \Sigma a \otimes x$ is exact as $\mathcal{T}$ is a tt-category, and since $a \otimes x, b \otimes x \in \mathcal{I}$ it follows that $c \otimes x \in \mathcal{I}$ as $\mathcal{I}$ is a triangulated subcategory, so $c \in \mathcal{S}$.

Hence, $\mathcal{S}$ is a thick subcategory of $\mathcal{T}$. Since $\mathbb{1} \in \mathcal{S}$ trivially, it follows that $\mathrm{thick}(\mathbb{1}) \subseteq \mathcal{S}$. It then follows that $\mathrm{thick}(\mathbb{1}) = \mathcal{T}$ forces $\mathcal{T} = \mathcal{S}$, making $\mathcal{I}$ a $\otimes$-ideal. □

## 1.2 Verdier Localization

Much of our thinking about tt-categories will be inspired by the structure of rings, and two of the most important ways to build new rings our of old ones are quotients and localizations. Verdier localization generalizes both ideas simultaneously.

**Definition 1.15.** Let $F : \mathcal{T}_1 \to \mathcal{T}_2$ be an exact functor of triangulated categories. The *kernel* of $F$ is defined to be the full subcategory $\mathcal{C}$ of $\mathcal{T}_1$ whose objects map to objects of $\mathcal{T}_2$ isomorphic to 0.

**Proposition 1.16.** *The kernel $\mathcal{C}$ of a triangulated functor $F : \mathcal{D} \to \mathcal{T}$ is a thick subcategory of $\mathcal{T}$.*

*Proof.* An object $x \in \mathcal{D}$ is in $\mathcal{C}$ if and only if $F(x) \cong 0$, but then $F(\Sigma x) \cong \Sigma F(x) \cong 0$, so $\Sigma F(x) \in \mathcal{C}$. Additionally, if $x \to y \to z \to \Sigma x$ is an exact triangle in $\mathcal{D}$, then $F(x) \to F(y) \to F(z) \to \Sigma F(x)$ is also an exact triangle in $\mathcal{C}$. If $x$ and $y$ are in $\mathcal{C}$, then $F(x) \to F(Y) \to F(z) \to \Sigma F(x)$ must be isomorphic to the zero triangle, making $z \in \mathcal{C}$, so $\mathcal{C}$ is a triangulated category. Now if $x \oplus y \in \mathcal{C}$ then $F(x \oplus y) = F(x) \oplus F(y)$ as $F$ is an additive functor, but since $F(x \oplus y) \cong 0$, it must be that $F(x)$ and $F(y)$ are isomorphic to 0, putting $x$ and $y$ in $\mathcal{C}$. □

**Theorem 1.17.** *Let $\mathcal{T}$ be a triangulated category and $\mathcal{C} \subseteq \mathcal{T}$ a triangulated subcategory (not necessarily thick). Then there is as universal exact functor $F_{univ} : \mathcal{T} \to \mathcal{T}\,/\,\mathcal{C}$ where $\mathcal{C} \subsetneq \ker(F)$. In other words, there exists a triangulated category $\mathcal{T}\,/\,\mathcal{C}$ and an exact functor $F_{univ} : \mathcal{T} \to \mathcal{T}\,/\,\mathcal{C}$ so that if there is another triangulated functor $G : \mathcal{T} \to \mathcal{D}$ with $\mathcal{C} \subseteq \ker(G)$, then there is a unique factorization of $G$:*

$$\mathcal{T} \xrightarrow{F_{univ}} \mathcal{T}\,/\,\mathcal{C} \to \mathcal{D}$$

*Proof.* See the proof of theorem 2.1.8 in [Nee01]. □

The quotient category $\mathcal{T}\,/\,\mathcal{C}$ is called the *Verdier quotient* of $\mathcal{T}$ by $\mathcal{C}$, and the natural map $F_{\text{univ}} : \mathcal{T} \to \mathcal{T}\,/\,\mathcal{C}$ is called the Verdier localization map. Discussing the structure of $\mathcal{T}\,/\,\mathcal{C}$ in detail is outside the scope of this text, and the interested reader should consult [Kra21] or chapter 2 of [Nee01]. For now, we simply need to know that the objects of $\mathcal{T}\,/\,\mathcal{C}$ are the objects in $\mathcal{T}$ and $F_{\text{univ}}$ is the identity on objects, and that $F_{univ}$ sends morphisms whose cone lies in $\mathcal{C}$ to isomorphisms. In fact, every morphism of $\mathcal{T}\,/\,\mathcal{C}$ may be written $gf^{-1} : X \to Y$ where $f$ and $g$ are morphisms $Z \xrightarrow{f} X$ and $Z \xrightarrow{g} Y$ such that $f$ fits into an exact triangle $Z \xrightarrow{f} X \to W \to \Sigma Z$ where $W$ is an object of $\mathcal{C}$. Recall that an exact functor $F$ takes an object $C$ to 0 if and only if it takes an exact triangle $A \xrightarrow{h} B \to C \to \Sigma C$ to a triangle isomorphic to the image of $X \xrightarrow{\text{id}_X} X \to 0 \to \Sigma X$, meaning that $F(h)$ is an isomorphism. Therefore, all morphisms that are of the same form as $f$ above must be sent to isomorphisms by $F_{univ}$, which is why we may invert $f$. To summarize:

**Fact 1.18.** Let $\mathcal{I}$ be a triangulated subcategory of $\mathcal{T}$ and set $S = \{f \in \text{Mor}(\mathcal{T}) \mid \text{cone}(f) \in \mathcal{I}\}$. Then,

$$\mathcal{T}/\mathcal{I} = \mathcal{T}[S^{-1}]$$

where $\mathcal{T}[S^{-1}]$ is the category obtained from $\mathcal{T}$ by formally inverting the morphisms in $S$.

Notice that proposition 1.16 tells us that the kernel of a triangulated functor is thick, but in theorem 1.17 we are quotienting out a triangulated subcategory $\mathcal{C}$ that may not be thick and that $\mathcal{C}$ is merely contained in the kernel of $F_{\text{univ}}$ where $F_{\text{univ}}$ is induced by quotienting out by $\mathcal{C}$. It turns out that $\ker(F_{\text{univ}}) = \widehat{\mathcal{C}}$ where $\widehat{\mathcal{C}}$ is the *thick closure* of $\mathcal{C}$, the full subcategory whose objects are direct summands of $\mathcal{C}$. Indeed, it's not hard to show that $\mathcal{C}$ is a thick subcategory if and only if $\widehat{\mathcal{C}} = \mathcal{C}$.

So far we have only discussed localization for triangulated categories, but have yet to invoke a tensor structure. Thankfully, Verdier localization behaves well with respect to tt-categories, meaning that if we are given a thick $\otimes$-ideal $\mathcal{I}$ with a $\mathcal{T}$ then the Verdier localization $\mathcal{T}/\mathcal{I}$ is a tt-category and the universal exact functor $\pi : \mathcal{T} \to \mathcal{T}/\mathcal{I}$ is an exact $\otimes$-functor.

## 1.3 Examples

Tensor Triangulated categories come in many flavors, some of which we will outline. This text will primarily focus on the three examples below, but it should be noted that tt-categories show up in many places. Examples can be found in motivic theory, stable $\mathbb{A}^1$-homotopy theory, the equivarient $KK$-theory of $C^*$-algebras, and in Fukaya categories of Calabi-Yau manifolds, to name a few.

### 1.3.1 The Derived Category

Some of the most well understood tt-categories are those from algebraic geometry and commutative algebra. The prototypical abelian category is $R$-Mod for a ring $R$, and in some sense these are the only abelian categories due to the Freyd–Mitchell theorem embedding theorem, which roughly states that if $\mathcal{A}$ is a small abelian category then $\mathcal{A}$ is equivalent to a full subcategory of left $R$-modules for some (unital, not necessarily commutative) ring $R$, so going forward, we have that $\mathcal{A}$ is just $R$-Mod for some ring $R$. From $\mathcal{A}$ we can construct $\mathrm{Ch}(\mathcal{A})$, the category of chain complexes of $\mathcal{A}$. By embedding $\mathcal{A}$ into $\mathrm{Ch}(\mathcal{A})$ we can approximate objects of $\mathcal{A}$ via resolutions of more well behaved objects. In this text, we will only care about projective and injective resolutions. Unfortunately, there is "too much noise" in $\mathrm{Ch}(\mathcal{A})$, meaning that a projective or injective resolution of $M \in \mathcal{A}$ will not be unique up to isomorphism, but is instead unique up to homotopy of chain complexes. Recall that in $\mathrm{Ch}(\mathcal{A})$, we say that two maps $f, g : C_\bullet \to D_\bullet$ are *chain homotopic* if there is a chain homotopy $h : C_\bullet \to D_{\bullet+1}$. Recall that a *chain homotopy* $h$ from $f$ to $g$ is a collection of maps $h_n : C_n \to D_{n+1}$ such that $f_n - g_n = \partial_D h_n + h_{n-1}\partial_C$. This is summed up in the following diagram:

$$\begin{array}{ccccccccc} \cdots & \longrightarrow & C_{n+1} & \xrightarrow{\partial_C} & C_n & \xrightarrow{\partial_C} & C_{n-1} & \longrightarrow & \cdots \\ & & f\Big\downarrow\Big\downarrow g & \overset{h}{\swarrow} & f\Big\downarrow\Big\downarrow g & \overset{h}{\swarrow} & f\Big\downarrow\Big\downarrow g & & \\ \cdots & \longrightarrow & D_{n+1} & \xrightarrow[\partial_D]{} & D_n & \xrightarrow[\partial_D]{} & D_{n-1} & \longrightarrow & \cdots \end{array} .$$

We say that $f$ and $g$ are *homotopic* if $f - g$ is *null-homotopic*, that is, if there is a homotopy from $f - g$ to 0, and we write $f$ $g$. It's straightforward to check that the null-homotopic maps from $C_\bullet$ to $D_\bullet$ form a subgroup of $\mathrm{Hom}_{\mathrm{Ch}(\mathcal{A})}(C_\bullet, D_\bullet)$

We care about all this because in order to make resolutions functorial, we need to pass to some sort of category in which homotopy equivalences are actually isomorphisms. Enter: $\mathrm{K}(\mathcal{A})$, the homotopy category of chain complexes over $\mathcal{A}$. The category $\mathrm{K}(\mathcal{A})$ has the same objects as $\mathrm{Ch}(\mathcal{A})$, but we define $\mathrm{Hom}_{\mathrm{K}(\mathcal{A})}(C_\bullet, D_\bullet) \coloneqq \mathrm{Hom}_{\mathrm{Ch}(\mathcal{A})}(C_\bullet, D_\bullet)/\mathrm{null}(C, D)$ where $\mathrm{null}(C, D)$ is the subgroup of null-homotopic maps. This category is naturally triangulated. The auto-equivalence $\Sigma$ is simply the *shift functor* $[1] : \mathrm{K}(\mathcal{A}) \to \mathrm{K}(\mathcal{A})$ where $C[1]_n = C_{n-1}$. Any morphism $f : C_\bullet \to D_\bullet$ may be fit into a composition

$$C_\bullet \xrightarrow{f} D_\bullet \longrightarrow \mathrm{cone}(f)_\bullet \longrightarrow C[1]_\bullet$$

where $\mathrm{cone}(f)$ is the *mapping cone* of the morphism $f$ (see [Wei94] for details). Therefore, define the class of exact triangles of $\mathrm{K}(\mathcal{A})$ to be triangles that are isomorphic in $\mathrm{K}(\mathcal{A})$ to a composition of the form above.

Much may be gleaned through studying $\mathrm{K}(\mathcal{A})$, but it is still unfortunately not entirely satisfactory. For one thing, there are chain maps which ought to be identified in $\mathrm{K}(\mathcal{A})$, but are not homotopy equivalences; see [nLa25]. Part of the problem is that we want to be able to identify an object of $\mathcal{A}$ with a resolution of that object. To do this, we need the *derived category* $\mathrm{D}(\mathcal{A})$. This is obtained from $\mathrm{K}(\mathcal{A})$ by quotienting out by the triangulated subcategory of *acyclic* complexes $\mathrm{K}_{ac}(\mathcal{A})$, i.e. complexes with zero homology. This is equivalent to localizing at the collection of *quasi-isomorphisms*, which are chain maps that induce isomorphisms on all homology groups.

There are a number of advantages to studying $\mathrm{D}(\mathcal{A})$ over $\mathrm{K}(\mathcal{A})$. One such advantage is that $\mathrm{D}(\mathcal{A})$ is the most natural setting to study derived functors. If we take $\mathcal{A}$ to be $R$-Mod or quasi-coherent sheaves over a scheme $X$, then one such functor is $-\otimes^{\mathbb{L}}_R - : R\text{-Mod} \times R\text{-Mod} \to R\text{-Mod}$, the derived tensor product. This equips $\mathrm{D}(R)$ with a monoidal structure with $R$ concentrated in degree 0 as the unit, and this in turn makes $\mathrm{D}(R)$ into a tt-category.

In addition to $\mathrm{K}(R)$ and $\mathrm{D}(R)$ there are other triangulated categories associated to $\mathcal{A}$, such as $\mathrm{K}^+(R)$, $\mathrm{K}^-(R)$, and $\mathrm{K}^b(R)$, whose objects consist of *bounded below, bounded above*, and *totally bounded* complexes respectively. Of course these have derived counterparts $\mathrm{D}^{+,-,b}(R)$ as well. Of particular interest to us is $\mathrm{D}^{\mathrm{pf}}(R)$, the derived category of *perfect complexes*, i.e. totally bounded complexes of finitely generated projective modules. The other derived categories are in some sense too large for us to easily study at present, but this is not the case for $\mathrm{D}^{\mathrm{pf}}(R)$. More on this later.

For a much more detailed account of derived categories in the context of commutative algebra, see [CFH24]. For a rapid introduction to derived categories in the context of algebraic geometry, see the first three chapters of [Huy06].

### 1.3.2 The Stable Module Category

Modular representation theory is a branch of representation theory focusing on the study of linear representations of finite groups over fields $k$ of prime characteristic $p$ where $p$ divides $|G|$. In ordinary representation theory, i.e. in the case that $\mathrm{char}(k) = 0$, finite dimensional (left) $kG$-modules (representations) are classified by their characters whereby two representations are isomorphic if and only if they have the same character. Any character may be uniquely constructed from irreducible characters, of which there are only finitely many. Irreducible characters themselves correspond to simple modules, and according to Maschke's theorem, simple modules are precisely the indecomposable modules. The Krull-Schmidt theorem then tells us that all finite length modules may be decomposed into a unique-up-to-ordering finite direct sum of indecomposables. Therefore, we may classify finite length left $kG$-modules, which coincide with finitely generated $kG$-modules as $kG$ is a left-Artinian ring.

This is much more tractable than the modular case where indecomposable modules may have the same character despite being non-isomorphic. In fact, it turns out that indecomposable $kG$-modules are usually unclassifiable. To have any hope of understanding $kG$-modules, one then needs to aim for coarser classifications. Categorically, the goal is to study $kG$-Mod, the category of $kG$-modules and the full subcategory of finitely generated $kG$-modules respectively, by looking at certain informative subcategories. It turns out that a $kG$-module $M$ is projective if and only if it is injective, and there are only a finite number of projective/injective indecomposables, each of which has a unique top composition factor $S$ where $S$ is also the unique bottom composition factor. Simple modules then correspond to indecomposable projectives, where a simple $S$ is associated to its projective cover $P(S)$.

Once one understands the projective modules, it becomes convenient to study modules modulo projectives, which leads one to define $\mathsf{StMod}(kG)$, the category which has the same objects as $kG$-

Mod but the morphisms that factor through a projective have been killed. Also of prime interest is $\mathsf{stmod}(kG)$, the full subcategory of $\mathsf{StMod}(kG)$ whose objects are finitely generated $kG$-modules.

$$\underline{\mathrm{Hom}}_{kG}(M,N) := \mathrm{Hom}_{kG}(M,N)/\,\mathrm{PHom}_{kG}(M,N)$$

where $\mathrm{PHom}_{kG}(M,N)$ is the linear subspace of homomorphisms from $M$ to $N$ that factor through some projective module. Given a module $M$, let $P(M)$ and $I(M)$ denote a projective and injective envelope of $M$, respectively. To see that $\mathrm{PHom}_{kG}(M,N)$ is a linear subspace of $\mathrm{Hom}_{kG}(M,N)$, note that $M \to N$ factors through some projective if and only if it factors as $M \to P(N) \to N$, and since projectives are the same as injectives, this happens if and only if it factors $M \to I(M) \to N$.

The category $\mathsf{StMod}(kG)$ doesn't end up being abelian however. By the discussion above, morphisms $M \to N$ become equivalent to both surjective homomorphisms $M \oplus P(N) \to N$ and injective homomorphisms $M \to I(M) \oplus N$, so one cannot appropriately examine cokernels and kernels. Instead, $\mathsf{StMod}(kG)$ ends up being a tt-category.

To define the triangular structure of $\mathsf{StMod}(kG)$, let $M$ be a $kG$-module and consider an epimorphism $\alpha : P \to M$ where $P$ is projective. Define the first syzygy $\Omega(M) = \ker(\alpha)$. Dually, given a monomorphism $\beta : P \to M$ we define $\Omega^{-1}(M) = \mathrm{coker}(\beta)$. We then have short exact sequences

$$0 \to \Omega(M) \to P \to M \to 0 \qquad 0 \to M \to P \to \Omega^{-1}(M) \to 0$$

Within $\mathsf{StMod}(kG)$, $\Omega(M)$ becomes an object independent of a choice surjection $P \to M$, so $\Omega(M)$ is well defined. The same goes for $\Omega^{-1}(M)$. Additionally, due to the discussion above, any morphism $M \to N$ yields a morphism $\Omega(M) \to \Omega(N)$, which makes $\Omega$ and $\Omega^{-1}$ into endofunctors of $\mathsf{StMod}(kG)$. The short exact sequences above then yield $\Omega^{-1} \circ \Omega(M) \cong M \cong \Omega \circ \Omega^{-1}(M)$, so $\Omega$ is an auto-equivalence with inverse $\Omega^{-1}$, justifying the notation used. It makes sense to then define $\Omega^i = \Omega \circ \Omega^{i-1}$ for $i \in \mathbb{Z}$.

To define the triangles of $\mathsf{StMod}(kG)$, let $X$ and $Y$ be $kG$-modules. Given a short exact sequence $0 \to Y \to P \to \Omega^{-1}(Y) \to 0$ one can then examine the following exact sequence

$$\mathrm{Hom}_{kG}(X,P) \to \mathrm{Hom}_{kG}(X,\Omega^{-1}(Y)) \to \mathrm{Ext}^1_{kG}(X,Y) \to \mathrm{Ext}^1_{kG}(X,P)$$

Since $P \cong 0$ in $\mathsf{StMod}(kG)$, it follows that $\underline{\mathrm{Hom}}_{kG}(X,P) = \mathrm{Ext}^1_{kG}(X,P) = 0$, and therefore

$$\underline{\mathrm{Hom}}_{kG}(X,\Omega^{-1}(Y)) \cong \mathrm{Ext}^1_{kG}(X,Y)$$

Now let $0 \to L \xrightarrow{\alpha} M \xrightarrow{\beta} N \to 0$ be a short exact sequence. Viewing $\mathrm{Ext}^1_{kG}(N,L)$ as extensions of $N$ by $L$, this short exact sequence corresponds to an element of $\mathrm{Ext}^1_{kG}(N,L)$ and therefore to an element $\gamma$ of $\underline{\mathrm{Hom}}^1_{kG}(N,\Omega^{-1}(L))$. Then the short exact sequence fits into a commutative diagram with exact rows:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & L & \xrightarrow{\alpha} & M & \xrightarrow{\beta} & N & \longrightarrow & 0 \\ & & \| & & \downarrow & & \downarrow{\scriptstyle\gamma} & & \\ 0 & \longrightarrow & L & \longrightarrow & P & \longrightarrow & \Omega^{-1}(L) & \longrightarrow & 0 \end{array}$$

where $P$ is injective. It then makes sense to say that a triangle in $\mathsf{StMod}(kG)$ is exact exactly when it is isomorphic to a sequence of morphisms

$$L \xrightarrow{\alpha} M \xrightarrow{\beta} N \xrightarrow{\gamma} \Omega^{-1}L$$

It is straight forward to show that $\Omega^{-1}$ and this collection of exact triangles endows $\mathsf{StMod}(kG)$ with a triangulated structure. Without too much more effort, it can also be shown that the monoidal structure on $\mathsf{StMod}(kG)$ induced by the usual monoidal structure of $kG$-Mod makes $\mathsf{StMod}(kG)$ into a tt-category. Explicitly, given $kG$-modules $V$ and $W$, the object $V \otimes W$ is $V \otimes_k W$ as a $k$-vector space, where $kG$ acts diagonally, i.e. $g \cdot v \otimes_k w = (gv) \otimes_k (gw)$ for all $v \in V$, $w \in W$, and $g \in G$. The monoidal unit is the *trivial representation* $k$, which is a $kG$-module via the map $\epsilon : kG \to k$ where $\epsilon(\sum_{g \in G} c_g g) = \sum_{g \in G} c_g$.

More generally, this construction can be carried out by replacing $kG$ with $A$ where $A$ is a Frobenius algebra, that is, $A$ is a finite dimensional unital associative algebra over a field $k$ equipped with a bilinear form $\sigma : A \times A \to k$ that satisfies the equation $\sigma(a \cdot b, c) = \sigma(a, b \cdot c)$. Finite dimensional Hopf algebras are Frobenius algebras, and group algebras of finite groups are Hopf algebras. Even more generally, we can replace $kG$-mod with a Frobenius category $\mathcal{A}$, which is an abelian category with enough projectives and injectives and whose class of projectives and injectives coincide.

For more details, see [Hap88], [Kra21], or [BIK11].

### 1.3.3 The Stable Homotopy Category

The following discussion will be a little more informal than the last two, in large part because of the amount of machinery involved in the background. It is meant for people who have taken a first course in algebraic topology. Some of the following material loosely follows the short survey [Mal14] by Cary Malkiewich, which goes into more depth than we do here.

If we are in the mood for topology, we care about *the category of compactly generated weak Hausdorff topological spaces.* Denote this category $\mathsf{Top}$. We also care about the full subcategory of spaces that are homeomorphic to CW complexes, which we will denote $\mathsf{CW}$. Of course, $\mathsf{Top}$ category is too complicated to truly understand as continuous maps can be extremely wild, so we smooth out the noise by studying continuous maps up to homotopy. The *homotopy category of CW complexes* $\mathsf{CW}$ has the same objects as $\mathsf{CW}$ has homotopy classes of continuous maps as morphisms, so we have a functor $\mathsf{CW} \to \mathsf{CW}$ which is the identity on objects and takes continuous maps to their homotopy class.

We then define *the homotopy category of spaces* $\mathsf{HoTop}$ to have the same objects as $\mathsf{Top}$ and set morphisms from $X$ to $Y$ to be homotopy classes of maps between CW approximations of $X$ and $Y$. A map on topological spaces $f : X \to Y$ is called a *weak homotopy* equivalence if it induces isomorphisms on all homotopy groups, that is

$$f_* : \pi_n(X, x) \cong \pi_n(Y, f(y)) \quad \forall x \in X, \forall n \in \mathbb{N}$$

Whitehead showed that a weak homotopy equivalence between CW complexes which induces isomorphisms is actually a homotopy equivalence; that is to say, a $f : X \to Y$ is a weak equivalence if and only if $f$ is an isomorphism in $\mathsf{HoTop}$. Therefore, to study homotopy we need to study homotopy groups of CW complexes. Since CW complexes may be assembled out of spheres, we would like to understand the homotopy groups of the spheres, $\pi_k(S^n)$ where $S^n$ is the *n-sphere*. Unfortunately, this too turns out to be extremely difficult. While there are many patterns in the homotopy groups of the spheres, there are no general combinatorial formulas that calculate what $\pi_k(S^n)$ should be. Nevertheless, we press on.

For a based space $X$, there is the *suspension homomorphism*

$$\Sigma : \pi_k(X) \to \pi_{k+1}(\Sigma X)$$

where for $f$ a homotopy class in $\pi_k(X)$,

$$\Sigma f = f \vee \mathrm{id}_{S^1} : S^{k+1} \cong S^k \wedge S^1 \to X \wedge S^1 = \Sigma X$$

where $\Sigma X$ is the topological suspension of $X$ and $A \vee B$ denotes the smash product of the spaces $A$ and $B$. Freudenthal showed that if $X$ is an $(n-1)$-connected space for $n \geq 1$, then the map $\Sigma$ above is a bijection if $k < 2n - 1$. If we keep applying the suspension homomorphism, we get a sequence of homotopy groups:

$$\pi_k(X) \xrightarrow{\Sigma} \pi_{k+1}(\Sigma X) \xrightarrow{\Sigma} \ldots \xrightarrow{\Sigma} \pi_{k+r}(\Sigma^r X)$$

Since $\Sigma^r X$ is $(n+r-1)$-connected, the suspension homomorphism

$$\Sigma : \pi_{k+r}(\Sigma^r X) \to \pi_{k+r+1}(\Sigma^{r+1} X)$$

is an isomorphism for $k + r < 2(n+r) - 1$. In other words, for fixed $n$ and $k$, sequence above stabilizes when $r > k - 2n + 1$. For an $(n-1)$-connected space we then define *the k-th stable homotopy group*,

$$\pi_k^s(X) = \pi_{k+r}(\Sigma^r X) \quad \text{for} \quad r > k - 2n + 1$$

This definition generalizes to spaces that may not be $(n-1)$-connected:

$$\pi_k^s(X) \coloneqq \operatorname{colim} \pi_{k+r}(\Sigma^r X)$$

In particular, we are interested in the stable homotopy group of the spheres. Since $\Sigma S^n \cong S^{n+1}$ for all $n \geq 0$, the suspension map $\pi_{n+k}(S^n) \to \pi_{n+k+1}(S^{n+1})$ is an isomorphism for $n > k+1$, so we have $\pi_k^s(S^n) = \pi_{k-n}^s(S^0)$ for $n \geq k$, and therefore the stable homotopy groups of $S^n$ can be expressed in terms of the stable homotopy groups of $S^0$. Determining the behavior of the stable homotopy groups of spheres has been one of the central problems of algebraic topology since they were first defined, and has driven an enormous amount of innovation in mathematics.

Just as the homotopy category filters out the wild behavior of continuous maps, we want to pass to a new category that filters out unstable behavior in homotopy groups. One of the defining features of $\mathsf{HoTop}$ was that isomorphisms were detected by homotopy groups. Analogously, we want to pass to a category in which isomorphisms are detected by the stable homotopy groups. The right category for the job is called the *stable homotopy category*, denoted $\mathsf{SHC}$. An object of $\mathsf{SHC}$ is called a "spectrum". There are many different definitions of the category of spectra, most of which are not equivalent, each of them has an associated definition of the stable homotopy category, and these are almost always equivalent.

Earlier we saw that in calculating the stable homotopy groups of the spheres we only needed to look at $S^0$ as $\Sigma S^n \cong S^{n+1}$. Since isomorphisms in $\mathsf{SHC}$ are detected by the stable homotopy groups, it might occur to us that instead of looking at all the spheres separately in $\mathsf{SHC}$, they should all be rolled up into one construction $\Sigma^\infty S^n$. Extending this philosophy, there should be a functor $\Sigma^\infty : \mathsf{HoTop} \to \mathsf{SHC}$ which sends each suspension of $X$ to the same place. The usual reduced suspension functor should then give rise to an equivalence on $\mathsf{SHC}$:

$$\begin{array}{ccc}
\mathsf{CW} & \xrightarrow{\Sigma} & \mathsf{CW} \\
\downarrow & & \downarrow \\
\mathsf{HoTop} & \xrightarrow{\Sigma} & \mathsf{HoTop}_* \\
\downarrow{\scriptstyle \Sigma^\infty} & & \downarrow{\scriptstyle \Sigma^\infty} \\
\mathsf{SHC} & \xrightarrow{\Sigma} & \mathsf{SHC}
\end{array}$$

This means that object of $\mathsf{SHC}$ is the suspension of some other object. This should look strange, since not every space is the suspension of another space in $\mathsf{Top}$ or $\mathsf{HoTop}$. The functor $\Sigma^\infty$ has a right adjoint $\Omega^\infty : \mathsf{SHC} \to \mathsf{HoTop}$. As the notation suggests, there is a loopspace functor $\Omega : \mathsf{SHC} \to \mathsf{SHC}$ which fits into a diagram below:

$$\begin{array}{ccc} \mathsf{CW} & \xleftarrow[\Omega]{} & \mathsf{CW} \\ \updownarrow & & \uparrow \\ \mathsf{HoTop} & \xleftarrow[\Omega]{} & \mathsf{HoTop}_* \\ {\scriptstyle \Omega^\infty}\uparrow & & {\scriptstyle \Omega^\infty}\uparrow \\ \mathsf{SHC} & \xleftarrow[\Omega]{} & \mathsf{SHC} \end{array}$$

The functors $\Sigma$ and $\Omega$ are inverse equivalences. This means that every object of $\mathsf{SHC}$ is the loopspace of some other object.

Many topological constructions become algebraic in $\mathsf{SHC}$. The hom-set $\mathrm{Hom}_{\mathsf{SHC}}(X, Y)$ becomes an abelian group, and composition of morphisms becomes bilinear. Additionally, the one point base space $*$ becomes a zero object, turning $\mathsf{SHC}$ into an additive category. The wedge sum $X \vee Y$ is now the direct sum of $X$ and $Y$ in $\mathsf{SHC}$. It can also be shown that the smash product $\wedge$ induces a monoidal structure on $\mathsf{SHC}$ for which the *sphere spectrum* $\mathbb{S} := \Sigma^\infty S^0$ is the monoidal unit.

We now have almost all the ingredients for a tt-category, save for the exact triangles. In $\mathsf{Top}$ there are the classical cofiber sequences

$$A \to X \to X/A \to \Sigma A$$

and classical fiber sequences

$$\Omega B \to F \to E \to B$$

As one might expect, $\Sigma^\infty$ takes cofiber sequences to exact triangles in $\mathsf{SHC}$ and $\Omega^\infty$ takes exact triangles in $\mathsf{SHC}$ to fiber sequences in $\mathsf{Top}$. Let $\Delta$ denote the exact triangles of $\mathsf{SHC}$. With these ingredients, it can be shown with some care that $(\mathsf{SHC}, \wedge, \mathbb{S}, \Sigma, \Delta)$ is a triangulated tensor category. There is also the stable homotopy category of finite spectra, $\mathsf{shc}$, which is essentially small.

### 1.4 Rigidity

**Motivation.** Let $M, N \in \mathsf{stmod}(kG)$ for $G$ a finite group and $k$ a field. As explained in subsubsection 1.3.2, the monoidal structure of $\mathsf{stmod}(kG)$ is given by tensoring over $k$ and we make $M \otimes_k N$ a $kG$-module via the diagonal action where $g(m \otimes_k n) = (gm) \otimes_k gn$ for $g \in kG$, $m \in M$, and $n \in N$. We can similarly make $\mathrm{Hom}_k(M, N)$ into a $kG$-module via a diagonal action: for $\alpha \in \mathrm{Hom}_k(M, N)$ and $g \in kG$, define $(g \cdot \alpha)(m) = g\alpha(g^{-1}m)$.

**Definition 1.19.** For $M$ a $kG$-module, we define the *invariant* submodule

$$M^G := \{m \in M \mid g \cdot m = m \ \forall g \in G\}$$

This defines an endofunctor on $kG$-Mod

One can then verify that for any $kG$-module $M$ and $N$ there is an isomorphism

$$\mathrm{Hom}_{kG}(M, N) = \mathrm{Hom}_k(M, N)^G$$

Additionally, one can also verify that for a $kG$-module $L$ the usual hom-tensor adjunction of $k$-vector spaces,

$$\mathrm{Hom}_k(L \otimes_k M, N) \cong \mathrm{Hom}_k(L, \mathrm{Hom}_k(M, N))$$

is compatible with the $kG$-module structures. In particular, it is an isomorphism of $kG$-modules. By applying the invariant functor, we obtain

$$\mathrm{Hom}_{kG}(L \otimes_k M, N) \cong \mathrm{Hom}_{kG}(L, \mathrm{Hom}_k(M, N))$$

In other words, the functor $- \otimes_k M$ has a right adjoint $\mathrm{Hom}_k(M, -)$. More generally, we could replace $kG$ with a Hopf algebra $A$ over $k$. For more details in the group theoretic case, see lecture 1 of [BIK11].

**Exercise 1.20.** Verify the above claims.

**Definition 1.21** (Hom-Tensor Adjunction)**.** An additive symmetric monoidal category $(\mathcal{C}, \otimes, \mathbb{1})$ is called *closed* if $\mathcal{C}$ has an *internal hom* functor, that is to say, a functor

$$\underline{\mathrm{hom}}(-,-) : \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathcal{C}$$

such that for any $y \in \mathcal{C}$ the functor $(-) \otimes y$ is the left adjoint of $\underline{\mathrm{hom}}(y, -)$. The unit and counit of this adjunction are denoted

$$\eta_{x,y} : y \to \mathrm{hom}(x, x \otimes y) \quad \text{and} \quad \epsilon_{x,y} : \mathrm{hom}(x, y) \otimes x \to y$$

In this case, the *dual* of an object $x \in \mathcal{C}$ is defined to be $x^\vee \coloneqq \underline{\mathrm{hom}}(x, \mathbb{1})$.

**Example 1.22.** The functor $\mathrm{Hom}_k(-,-) : \mathsf{stmod}(kG)^{\mathrm{op}} \times \mathsf{stmod}(kG) \to \mathsf{stmod}(kG)$ is the internal hom of $\mathsf{stmod}(kG)$.

**Definition 1.23.** Let $\mathcal{C}$ be a closed monoidal category. Given objects. For objects $x, y \in \mathcal{C}$, there is a natural *evaluation map*

$$x^\vee \otimes y \to \underline{\mathrm{hom}}(x, y)$$

given by tracking what happens to the identity map on $y$ through the composition below.

$$\mathrm{Hom}_{\mathcal{C}}(y, y) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{C}}(y \otimes \mathbb{1}, y) \xrightarrow{(\mathrm{id}_y \otimes \epsilon_{x,\mathbb{1}})^*} \mathrm{Hom}_{\mathcal{C}}(y \otimes x^\vee \otimes x, y) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{C}}(y \otimes x^\vee, \underline{\mathrm{hom}}(x, y))$$

We say that an object $x$ is *rigid* if the evaluation map $x^\vee \otimes y \to \underline{\mathrm{hom}}(x, y)$ is an isomorphism for all $y$. If all objects of $\mathcal{C}$ are rigid, then we say that $\mathcal{C}$ is rigid.

**Exercise 1.24.** Show that $\mathsf{stmod}(kG)$ is rigid. *Hint:* The evaluation map $\mathrm{Hom}_k(M, k) \otimes_k N \to \mathrm{Hom}_k(M, N)$ is of the form below:

$$\phi \otimes_k n \mapsto [m \mapsto \phi(m)n]$$

for $\phi \in \mathrm{Hom}_k(M, k)$, $m \in M$, and $n \in N$.

*Remark* 1.25. Note that $\mathcal{C}$ being rigid is equivalent to the existence of a functor $D : \mathcal{C}^{\mathrm{op}} \to \mathcal{C}$ such that there is a natural isomorphism

$$\mathrm{Hom}_{\mathcal{C}}(x \otimes y, z) \cong \mathrm{Hom}_{\mathcal{C}}(x, D(y) \otimes z)$$

which is natural in all three variables. If such a functor exists then we can define the internal hom $\underline{\mathrm{hom}}(y, z) := D(y) \otimes z$, so really the functor $D$ is serving as the dual functor $(-)^{\vee}$. Additionally, if $\mathcal{C}$ is rigid, then there is a natural isomorphism $x^{\vee\vee} \cong x$. One can also show that the functor $x^{\vee} \otimes (-)$ is both a left and right adjoint to $x \otimes (-)$, and that the functor $\underline{\mathrm{hom}}$ obeys hom-tensor adjunction that $\mathrm{Hom}_{\mathcal{C}}$ does, i.e.

$$\underline{\mathrm{hom}}(x \otimes y, z) \cong \underline{\mathrm{hom}}(x, \underline{\mathrm{hom}}(y, z))$$

**Fact 1.26.** Let $\mathcal{T}$ be a rigid tt-category. If $\underline{\mathrm{hom}}(x, y) = 0$ then the only morphism between $x$ and $y$ is the trivial morphism.

*Proof.* This is a routine calculation:

$$\mathrm{Hom}_{\mathcal{T}}(x, y) \cong \mathrm{Hom}_{\mathcal{T}}(\mathbb{1} \otimes x, y) \cong \mathrm{Hom}_{\mathcal{T}}(\mathbb{1}, \underline{\mathrm{hom}}(x, y)) = \mathrm{Hom}_{\mathcal{T}}(\mathbb{1}, 0) = 0$$

□

# 2 The Spectrum of $\mathcal{T}$

In this chapter we will define the Balmer spectrum and show that it is a spectral space before then going on to state the main classification theorem for thick subcategories. Crucially, we will require that our tt-categories be *essentially small*.

**Definition 2.1.** A category $\mathcal{C}$ is *small* if it has a small set of objects and a small set of morphisms, where we use the term small to emphasize that these really are sets as opposed to proper classes. If $\mathcal{C}$ is equivalent to a small category then we say that $\mathcal{C}$ is *essentially small*.

We make this provision because the points of the Balmer spectrum are the prime $\otimes$-ideals of $\mathcal{T}$ (defined below) and we need there to be no more than a sets worth of prime $\otimes$-ideals contained in $\mathcal{T}$ to be able to define a sensible topological space on them.

### A warning to the reader

Before we begin the real content of this section, the reader should be warned. The Balmer spectrum, which will be our main object of study, behaves like and is defined similarly to the Zariski spectrum $\mathrm{Spec}(R)$ of a commutative ring $R$, and much of the theory of tt-geometry is inspired by this similarity. Indeed, the Balmer spectrum will be a *spectral space*, meaning that it is homeomorphic to the Zariski spectrum of some commutative ring. However, there is a key difference that one must pay mind to, and that is that the Balmer spectrum appears to be topologically "backwards" – by this I mean that closed points correspond to minimal primes (to be defined) in $\mathcal{T}$ and generic points correspond to maximal ideals.

The reason for this backwardness is surprisingly deep and unfortunately beyond the scope of this text to explore in its entirety, but there are authors who have written on the subject; see [BKS07] and [KP15]. For now I will simply say that it comes down to the fact that Balmer's choice of a closed basis ends up yielding the *Hochster dual* of what some consider to be the more natural choice of topology for the Balmer spectrum. This is will be expounded later in subsubsection 2.4.2.

## 2.1 Definition of Spc($\mathcal{T}$)

This section follows section 2 of [Bal05].

**Definition 2.2.** A thick $\otimes$-ideal $\mathcal{P} \subseteq \mathcal{T}$ is called *prime* if for any $a, b \in \mathrm{Obj}(\mathcal{T})$ such that $a \otimes b \in \mathcal{P}$ then either $a \in \mathcal{P}$ or $b \in \mathcal{P}$. We define $\mathrm{Spc}(\mathcal{T})$ to be the collection of all prime $\otimes$-ideals in $\mathcal{T}$. A *maximal* $\otimes$-ideal is a proper $\otimes$-ideal of $\mathcal{T}$ that is maximal with respect to inclusion.

Later on we will define $\mathrm{Spec}\,\mathcal{T}$ to be the set $\mathrm{Spc}(\mathcal{T})$ equipped with a particular ringed space structure, so we will hold the notation $\mathrm{Spec}\,\mathcal{T}$ in reserve until then. To put a ringed space structure on $\mathrm{Spc}(\mathcal{T})$, we first need to put a topology on $\mathcal{T}$.

**Definition 2.3.** For any $\mathcal{S} \subseteq \mathrm{Obj}(\mathcal{T})$, we write $\mathcal{Z}(S) := \{\mathcal{P} \in \mathrm{Spc}(\mathcal{T}) \mid \mathcal{S} \cap \mathcal{P} = \emptyset\}$.

**Proposition 2.4.** *For any tt-category $\mathcal{T}$ the following hold:*

1. $\bigcap_{i\in I} \mathcal{Z}(\mathcal{S}_i) = \mathcal{Z}(\bigcup_{i\in I} \mathcal{S}_i)$ *for a family* $\{\mathcal{S}_i\}_{i\in I}$ *where* $\mathcal{S}_i \subseteq \mathrm{Obj}(\mathcal{T})$ *for each* $i \in I$.
2. $\mathcal{Z}(\mathcal{S}_1 \oplus \mathcal{S}_2) = \mathcal{Z}(\mathcal{S}_1) \cup \mathcal{Z}(\mathcal{S}_2)$
3. $\mathcal{Z}(\mathcal{T}) = \emptyset$
4. $\mathcal{Z}(\emptyset) = \mathrm{Spc}(\mathcal{T})$

*where* $\mathcal{S}_1 \oplus \mathcal{S}_2 = \{a \oplus b \mid a \in \mathcal{S}_1, b \in \mathcal{S}_2\}$.

*Proof.*

1. $\mathcal{P} \in \bigcap_{i\in I} \mathcal{Z}(\mathcal{S}_i)$ if and only if $\mathcal{P} \cap \mathcal{S}_i = \emptyset$ for all $i \in I$ if and only if $\mathcal{P} \cap \left(\bigcup_{i\in I} \mathcal{S}_i\right) = \emptyset$ if and only if $\mathcal{P} \in \mathcal{Z}(\bigcup_{i\in I} \mathcal{S}_i)$.

2. If $\mathcal{P} \notin \mathcal{Z}(\mathcal{S}_1 \oplus \mathcal{S}_2)$ then $\exists a \oplus b \in \mathcal{P}$ where $a \in \mathcal{S}_1, b \in \mathcal{S}_2$. But as $\mathcal{P}$ is a thick $\otimes$-ideal $a, b \in \mathcal{P}$, so $\mathcal{P}$ intersects both $\mathcal{S}_1$ and $\mathcal{S}_2$ non-trivially and therefore $\mathcal{P} \notin \mathcal{Z}(\mathcal{S}_1) \cup \mathcal{Z}(\mathcal{S}_2)$. On the other hand, if $\mathcal{P}$ is not in either $\mathcal{Z}(\mathcal{S}_1)$ or $\mathcal{Z}(\mathcal{S}_2)$ then $\mathcal{P}$ must intersect both $\mathcal{S}_1$ and $\mathcal{S}_2$, so take $a \in \mathcal{P} \cap \mathcal{S}_1$ and $b \in \mathcal{P} \cap \mathcal{S}_2$. Then $a \oplus b \in \mathcal{P}$ and therefore $\mathcal{P} \cap \mathcal{S}_1 \cap \mathcal{S}_2 \neq \emptyset$, so $\mathcal{P} \notin \mathcal{Z}(\mathcal{S}_1 \oplus \mathcal{S}_2)$.

3. and 4. are obvious from the definition. □

It immediately follows that we can define a Zariski topology on $\mathrm{Spc}(\mathcal{T})$ with closed sets $\mathcal{Z}(S)$.

**Definition 2.5.** For any $a \in \mathrm{Ob}(\mathcal{T})$, we define $\mathrm{Supp}(a) = \{\mathcal{P} \in \mathrm{Spc}(\mathcal{T}) \mid a \notin \mathcal{P}\}$. For any family $\mathcal{I}$ of objects of $\mathcal{T}$, we write

$$\operatorname{supp} \mathcal{I} \coloneqq \bigcup_{a\in\mathcal{I}} \mathrm{Supp}(a)$$

*Remark* 2.6. It is the case that given an object $a \in \mathcal{T}$, $\mathcal{Z}(\{a\}) = \mathrm{supp}(a)$; however, it is not always the case that $\operatorname{supp} \mathcal{I} = \mathcal{Z}(\mathcal{I})$.

We could also define the topology using open sets of the form below:

**Definition 2.7.** Given $\mathcal{Z}(\mathcal{S})$ a closed set in $\mathrm{Spc}(\mathcal{T})$ define

$$U(\mathcal{S}) \coloneqq \mathrm{Spc}(\mathcal{T}) \setminus \mathcal{Z}(\mathcal{S}) = \{\mathcal{P} \in \mathrm{Spc}(\mathcal{T}) : \mathcal{S} \cap \mathcal{P} \neq \emptyset\}$$

**Definition 2.8.** A subset $\mathcal{S} \subseteq \mathcal{T}$ is called a $\otimes$*-multiplicative* subset if for any $a.b \in \mathcal{S}$, $a \otimes b \in \mathcal{S}$.

**Proposition 2.9.** *Let $\mathcal{T}$ be a non-zero tt-category. Then,*

1. *If $S \subseteq \mathcal{T}$ is a $\otimes$-multiplicative subset which does not contain* $0$*, then there exists $\mathcal{P} \in \mathrm{Spc}(\mathcal{T})$ such that $\mathcal{S} \cap \mathcal{P} = \emptyset$.*

2. *If $\mathcal{C} \subseteq \mathcal{T}$ is a proper thick $\otimes$-ideal then there exists a proper maximal $\otimes$-ideal $\mathcal{M} \subseteq \mathcal{T}$ such that $\mathcal{C} \subseteq \mathcal{M} \subset \mathcal{T}$.*

3. $\mathrm{Spc}(\mathcal{T}) \neq \emptyset$

*Proof.* This proposition follows easily from an application of lemma 2.10 below to the ideals generated by $0$, $\mathcal{C}$, and $\mathbb{1}$ respectively. □

**Lemma 2.10.** *Let $\mathcal{I}$ be a $\otimes$-ideal of $\mathcal{T}$ and $\mathcal{S} \subseteq \mathcal{T}$ a $\otimes$-multiplicative subset. If $\mathcal{I} \cap \mathcal{S} = \emptyset$ then there exists $\mathcal{P} \in \mathrm{Spc}(\mathcal{T})$ such that $\mathcal{I} \subseteq \mathcal{P}$ and $\mathcal{P} \cap \mathcal{S} = \emptyset$.*

*Proof.* Define $\mathcal{J} \coloneqq \{a \in \mathcal{T} : \exists s \in S, a \otimes s \in \mathcal{I}\}$. I claim that this is a thick $\otimes$-ideal containing $\mathcal{I}$. Clearly $\mathcal{I} \subseteq \mathcal{J}$ since $a \otimes x \in \mathcal{I}$ for any $a \in \mathcal{I}$ and any $x \in \mathcal{T}$.

Now we want to show that $\mathcal{J}$ is closed under taking cone, so suppose that $a \xrightarrow{f} b$ is a morphism in $\mathcal{J}$. Since $a, b \in \mathcal{J}$ there are $s, s' \in S$ such that $a \otimes s, b \otimes s \in \mathcal{I}$. In fact, we can just take $s = s'$ by replacing $s$ with $s \otimes s'$ since $a \otimes s \in \mathcal{I}$ implies that $a \otimes s \otimes s' \in \mathcal{I}$ and $s \otimes s' \in S$. By proposition 1.9 $s \otimes \text{cone}(f) \cong \text{cone}(f \otimes \text{id}_s)$. Since $\mathcal{I}$ is a thick subcategory and $a \otimes s, b \otimes s \in \mathcal{I}$ it follows that $s \otimes \text{cone}(f) \in \mathcal{I}$. Therefore, $\text{cone}(f) \in \mathcal{J}$, and by the same token we can see that $\mathcal{J}$ is closed under $\Sigma$, so $\mathcal{J}$ is a triangulated subcategory of $\mathcal{T}$.

Now suppose that $a, b \in \mathcal{J}$. It follows from corollary A.9 that $a \oplus b \in \mathcal{J}$, so $\exists s \in S$ such that $(a \oplus b) \otimes s \in \mathcal{I}$. Then, since $(a \oplus b) \otimes s \cong (a \otimes s) \oplus (b \otimes s)$ it follows that $a \otimes s, b \otimes s \in \mathcal{I}$, so $a, b \in \mathcal{J}$, so $\mathcal{J}$ is a thick triangulated subcategory. To see that $\mathcal{J}$ is a $\otimes$-ideal, let $a \in \mathcal{J}$ with $s \in \mathcal{S}$ such that $a \otimes s \in \mathcal{I}$, and since $\mathcal{I}$ is a $\otimes$-ideal we see that for any $b \in \mathcal{T}$, $(a \otimes s) \otimes b \in \mathcal{I}$, but then by associativity of $\otimes$ we have $a \otimes b \in \mathcal{J}$. Additionally, $\mathcal{J} \cap \mathcal{S} = \emptyset$ since if $a \in \mathcal{S} \cap \mathcal{J}$ then $\exists s \in \mathcal{S}$ such that $a \otimes s \in \mathcal{I}$, but then $a \otimes s \in \mathcal{I} \cap \mathcal{S}$ as $a, s \in \mathcal{S}$ implies that $a \otimes s$ as $\mathcal{S}$ is $\otimes$-multiplicatively. closed, which is a contradiction.

Now let $C$ be the collection of thick $\otimes$-ideals $\mathcal{J}$ where

(1) $\mathcal{J} \cap \mathcal{S} = \emptyset$,

(2) If $a \in \mathcal{T}$ such that there is some $s \in \mathcal{S}$ where $a \otimes s \in \mathcal{J}$, then $a \in \mathcal{J}$

(3) $\mathcal{I} \subseteq \mathcal{J}$

The paragraph above shows that $C$ is nonempty. Now let $\{J_i\}$ be an ascending chain (via containment) within $C$. Then clearly the full subcategory of $\mathcal{T}$ generated by the union of the objects within $\{J_i\}$ is also a thick $\otimes$-ideal satisfying the conditions above, so by Zorn's lemma there exists a maximal element $\mathcal{P}$ of $C$. To see that $\mathcal{P}$ is prime, suppose that $a \otimes b \in \mathcal{P}$ and $b \notin P$. Then define $(\mathcal{P} : a)$ to be the full subcategory of $\mathcal{T}$ generated by $\{x \in \mathcal{T} : x \otimes a \in \mathcal{P}\}$. This subcategory is easily shown to be a thick $\otimes$-ideal of $\mathcal{T}$ by essentially the same methods used to show that $\mathcal{J}$ above was a thick $\otimes$-ideal. If $x \in \mathcal{P}$ then $x \otimes a \in \mathcal{P}$ by definition of a $\otimes$-ideal, so $\mathcal{P} \subset (\mathcal{P} : a)$, but $b \in (\mathcal{P} : a) \setminus \mathcal{P}$, so the containment is proper. Hence $(\mathcal{P} : a) \notin C$, so one of the three properties above fails for $(\mathcal{P} : a)$. Clearly it can't be condition (3), so suppose that $x \in \mathcal{T}$ such that there is some $s \in \mathcal{S}$ where $x \otimes s \in (\mathcal{P} : a)$. Then $x \otimes s \otimes a \in \mathcal{P}$, but by condition (2) this implies that $x \otimes a \in \mathcal{P}$, and therefore $x \in (\mathcal{P} : a)$. Therefore the only condition that $(\mathcal{P} : a)$ could fail is (2), which means that $\exists s \in (\mathcal{P} : a) \cap S$, so $a \otimes s \in \mathcal{P}$. But then $a \in \mathcal{P}$ by condition (2). Hence $\mathcal{P}$ is prime. □

**Corollary 2.11.** *Maximal thick $\otimes$-ideals are prime.*

**Definition 2.12.** A thick $\otimes$-ideal $\mathcal{I} \subseteq \mathcal{T}$ is *radical* if for all $a \in \text{Obj}(\mathcal{T})$, $a \in \mathcal{I}$ whenever $a^{\otimes n} \in I$ for some $n \in \mathbb{N}$. Equivalently, $\mathcal{I}$ is equal to the *radical of* $\mathcal{I}$, denoted $\sqrt{\mathcal{I}}$ where

$$\sqrt{\mathcal{I}} \coloneqq \{a \in \mathcal{T} \mid a^{\otimes n} \in \mathcal{I} \text{ for some } n \in \mathbb{N}\}$$

The thick $\otimes$-ideal $\sqrt{\langle 0 \rangle}$ is referred to as the *nilradical*.

By definition $a$ is contained in every $\mathcal{P}$ in $\text{Spc}(\mathcal{T})$ if and only if $U(a) = \text{Spc}(\mathcal{T})$. Similarly, $a$ is contained in every $\mathcal{P} \in \text{Spc}(\mathcal{T})$ if and only if $\text{Supp}\, a = \emptyset$.

**Corollary 2.13.**

$$\begin{aligned}\bigcap_{\mathcal{P}\in\operatorname{Spc}(\mathcal{T})} \mathcal{P} &= \{a \in \mathcal{T} \mid U(a) = \operatorname{Spc}(\mathcal{T})\} \\ &= \{a \in \mathcal{T} \mid \operatorname{Supp}(a) = \emptyset\} \\ &= \{a \in \mathcal{T} \mid a^{\otimes n} = 0 \text{ for some } n \in \mathbb{N}\} \\ &= \sqrt{0}\end{aligned}$$

*Proof.* One direction is obvious, as $0 \in \mathcal{P}$ for any $\mathcal{P} \in \operatorname{Spc}(\mathcal{T})$. Now suppose that $a^{\otimes n} \neq 0$ for any $n \in \mathbb{N}$ and let $\mathcal{S} = \{a^{\otimes n}\}_{n\in\mathbb{N}}$. $\mathcal{S}$ does not contain 0, so by proposition 2.9, there exists $\mathcal{P} \in \operatorname{Spc}(\mathcal{T})$ such that $\mathcal{S} \cap \mathcal{P} = \emptyset$, but then $\operatorname{Supp}(a) \neq \emptyset$. By the contrapositive, we both containments. □

**Corollary 2.14.** *An object $a \notin \mathcal{P}$ for all $\mathcal{P} \in \operatorname{Spc}(\mathcal{T})$ if and only if $\langle a\rangle^{\otimes} = \mathcal{T}$*

Below we list some readily verified properties of sets of the form $U(a)$ and $\operatorname{supp}_{\mathcal{T}}(a)$ where $a \in \operatorname{Obj}(\mathcal{T})$.

**Proposition 2.15.** *Let $\mathcal{T}$ be a tensor-triangulated category and $a, b, c \in \operatorname{Obj}(\mathcal{T})$. The following hold:*

1. $U(0) = \operatorname{Spc}(\mathcal{T})$ *&* $\operatorname{supp}(0) = \emptyset$
2. $U(\mathbb{1}) = \emptyset$ *&* $\operatorname{supp}(\mathbb{1}) = \operatorname{Spc}(\mathcal{T})$
3. $U(a \oplus b) = U(a) \cap U(b)$ *&* $\operatorname{supp}(a \oplus b) = \operatorname{supp}(a) \cup \operatorname{supp}(b)$
4. $U(\Sigma a) = U(a)$ *&* $\operatorname{supp}(\Sigma a) = \operatorname{supp}(a)$
5. *For an exact triangle $a \to b \to c$,* $U(a) \cap U(c) \subseteq U(b)$ *&* $\operatorname{supp}(a) \cup \operatorname{supp}(c) \supseteq \operatorname{supp}(b)$
6. $U(a \otimes b) = U(a) \cup U(b)$ *&* $\operatorname{supp}(a \otimes b) = \operatorname{supp}(a) \cap \operatorname{supp}(b)$

*Remark* 2.16. The interaction between the tt-structure and the Zariski topology can be interpreted as the support remembering more than just the additive structure.

**Corollary 2.17.** *The collection $\{U(a) : a \in \mathcal{T}\}$ provides an (open) basis for the topology. The collection $\{\operatorname{supp}(a) = \operatorname{Spc}(\mathcal{T}) \setminus U(a)\}$ provides a (closed) basis for the topology.*

## 2.2 Spc($\mathcal{T}$) is a Spectral Space

This section follows the rest of section 2 of [Bal05].

So far, we have defined prime ideals and equipped $\mathcal{T}$ with a topology that is analogous to the Zariski topology found in the world of commutative rings. It turns out that there is a class of topological spaces, referred to as *spectral spaces* which exactly corresponds to the Zariski spectrums of commutative rings. The goal of this section is to show that $\operatorname{Spc}(\mathcal{T})$ is also a spectral space.

**Definition 2.18.** A topological space $X$ is called *irreducible* if it cannot be written as the union of two proper closed subsets of $X$. A topological space $X$ is called *sober* if every nonempty irreducible closed subset of $X$ is the closure of exactly one point of $X$. Given such an irreducible closed subset $Y$ of $X$ the unique point $y \in X$ such that $\bar{y} = Y$, the point $y$ is called the *generic point* of $Y$.

**Definition 2.19.** Let $X$ be a topological space and $\mathcal{K}^{\circ}(X)$ the set of quasi-compact open subsets of $X$. $X$ is *spectral* if

1. $X$ is quasi-compact and $T_0$.
2. $\mathcal{K}^\circ(X)$ is a basis for $X$.
3. $\mathcal{K}^\circ(X)$ is closed under finite intersections.
4. $X$ is sober.

Given spectral spaces $X$ and $Y$, a *spectral map* $X \xrightarrow{f} Y$ is a continuous map such that for any quasi-compact open $U \subseteq Y$ the pre-image $f^{-1}(U)$ is quasi-compact.

We will now proceed to show that $\text{Spc}(\mathcal{T})$ is also spectral.

**Proposition 2.20.** *Let $\mathcal{P} \in \text{Spc}(\mathcal{T})$. Then $\overline{P} = \{\mathcal{Q} \in \text{Spc}(\mathcal{T}) : \mathcal{Q} \subseteq \mathcal{P}\}$.*

*Proof.* Let $\mathcal{S} = \mathcal{T} \setminus \mathcal{P}$. Then,

$$\begin{aligned}\mathcal{Z}(\mathcal{S}) &= \mathcal{Z}(\mathcal{T} \setminus \mathcal{P}) \\ &= \{\mathcal{Q} : \mathcal{Q} \cap (\mathcal{T} \setminus \mathcal{P}) = \emptyset\} \\ &= \{\mathcal{Q} : \mathcal{Q} \subseteq \mathcal{P}\}\end{aligned}$$

Clearly $\mathcal{P} \in \mathcal{Z}(\mathcal{S})$ and by definition $\mathcal{Z}(\mathcal{P})$ is closed, so $\overline{P} \subseteq \mathcal{Z}(\mathcal{S})$. Now suppose that $\mathcal{P} \subseteq \mathcal{Z}(\mathcal{S}')$ where $\mathcal{S}'$ is some other collection of objects in $\mathcal{T}$. Then $\mathcal{S}' \cap \mathcal{P} = \emptyset$, so $\mathcal{S}' \subseteq \mathcal{T} \setminus \mathcal{P} = \mathcal{S}$, which implies that $\mathcal{Z}(\mathcal{S}) \subseteq \mathcal{Z}(\mathcal{S}')$, and therefore $\mathcal{Z}(\mathcal{S})$ is minimal amongst closed sets containing $\mathcal{P}$. □

*Remark* 2.21. As mentioned before, this might be confusing for those used to thinking about Spec $R$ for a commutative ring $R$, so one should take care when switching between these two contexts. In subsubsection 2.4.2 we will try to clarify this situation.

**Corollary 2.22.** *If $\mathcal{P}_1, \mathcal{P}_2 \in \text{Spc}(\mathcal{T})$ and $\overline{\mathcal{P}_1} = \overline{\mathcal{P}_2}$, then $\mathcal{P}_1 = \mathcal{P}_2$. Consequently, $\text{Spc}(\mathcal{T})$ is $T_0$.*

*Proof.* Immediate from proposition 2.20. □

**Proposition 2.23.** *There exists a minimal prime in $\mathcal{T}$, that is, there exists a prime $\otimes$-ideal $\mathcal{P} \in \text{Spc}(\mathcal{T})$ such that if $\mathcal{Q} \in \text{Spc}(\mathcal{T})$ where $\langle 0\rangle^\otimes \subseteq \mathcal{Q} \subseteq \mathcal{P}$, then either $\mathcal{Q} = \langle 0\rangle^\otimes$ or $\mathcal{Q} = \mathcal{P}$.*

*Proof.* The proof is more or less identical to the proof using Zorn's lemma used in standard ring theory. □

*Remark* 2.24. A point is called *closed* if $x = \overline{x}$. By proposition 2.20 the only closed points of $\text{Spc}(\mathcal{T})$ are exactly minimal primes. As a result, any closed set of $\text{Spc}(\mathcal{T})$ contains a closed point.

**Lemma 2.25.** *Let $a \in \mathcal{T}$ and $\mathcal{S} \subseteq \mathcal{T}$ be a collection of objects. Then the following are equivalent.*

*(a) $U(a) \subseteq U(\mathcal{S})$ (or equivalently $\mathcal{Z}(\mathcal{S}) \subseteq \text{supp}(a)$).*

*(b) There exist finitely many objects $b_1, ..., b_n \in \mathcal{S}$ such that $b_1 \otimes b_2 \otimes ... \otimes b_n \in \langle a\rangle^\otimes$.*

*Proof.* $(a) \implies (b)$. Let $\mathcal{I} = \langle a\rangle^\otimes$ and $\mathcal{S}'$ be all finite tensor products of elements in $\mathcal{S}$ and suppose that $\mathcal{I} \cap \mathcal{S}' = \emptyset$. Then by lemma 2.10 there exists $\mathcal{P} \in \text{Spc}(\mathcal{T})$ such that $\mathcal{I} \subseteq \mathcal{P}$ and $\mathcal{P} \cap \mathcal{S}' = \emptyset$. Therefore, $\mathcal{P} \in U(a)$, but $\mathcal{P} \notin U(\mathcal{S})$ as $U(\mathcal{S}) \subseteq U(\mathcal{S}')$. By the contrapositive, $U(a) \subseteq U(\mathcal{S})$ implies that $I \cap \mathcal{S}' \neq \emptyset$.

$(b) \implies (a)$. Assume $(b)$. Let $\mathcal{P} \in U(a)$. Then $a \in \mathcal{P}$ so $\langle a \rangle^{\otimes} \subseteq \mathcal{P}$. In particular, $b_1, ..., b_n \in \mathcal{P}$. Since $\mathcal{P}$ is a prime ideal, there exists some $b_j \in \mathcal{P}$, so $\mathcal{P} \cap \mathcal{S} \neq \emptyset$. Hence $\mathcal{P} \subseteq U(\mathcal{S})$. □

**Proposition 2.26.**

*(a)* $\mathrm{Spc}(\mathcal{T})$ *is quasi-compact.*

*(b)* $U(a)$ *is quasi-compact for all* $a \in \mathcal{T}$*.*

*(c) Any quasi-compact open set has the form* $U(a)$ *for some* $a \in \mathcal{T}$*.*

*Proof.* Since $U(0) = \mathrm{Spc}(\mathcal{T})$, we get $(a)$ from $(b)$.

We will first prove $(b)$. Let $a \in \mathcal{T}$ and $U(a) \subseteq \bigcup_{i \in I} U(\mathcal{S}_i)$ be an open cover where $I$ is an indexing set. Then set $\mathcal{S} \coloneqq \bigcup_{i \in I} \mathcal{S}_i$. Then $U(\mathcal{S}) = \bigcup_{i \in I} U(\mathcal{S})$. By lemma 2.25 there exists a finite subset $\mathcal{S}_1, ..., \mathcal{S}_n$ and objects $b_1, ..., b_n$ where $b_i \in \mathcal{S}$ for $1 \leq i \leq n$ such that $b_1 \otimes \ldots \otimes b_n \in \langle a \rangle^{\otimes}$. Now let $\mathcal{P} \in U(a)$. Then $\langle a \rangle^{\otimes} \subseteq \mathcal{P}$ and therefore $b_1 \otimes ... \otimes b_n \in \mathcal{P}$ and since $\mathcal{P}$ is prime there is some $b_j \in \mathcal{P}$. Thus $\mathcal{P} \cap \mathcal{S}_j \neq \emptyset$ and so $\mathcal{P} \subseteq U(S_j)$. Hence, $U(a)$ is covered by the finite subcover $U(\mathcal{S})_1, ..., U(\mathscr{S})_n$.

Now let $U(\mathcal{S})$ be a quasi-compact set. Then

$$\begin{aligned} U(\mathcal{S}) &= \bigcup_{a \in \mathcal{S}} U(a) \\ &= U\left(\bigcup_{a \in \mathcal{S}} a\right) \\ &= U(a_1) \cup \ldots \cup U(a_n) \text{by (b)} \\ &= U(a_1 \otimes \ldots \otimes a_n) \end{aligned}$$

*Remark* 2.27. Given proposition 2.26 might be tempting to think that $\mathrm{Spc}(\mathcal{T})$ is always Noetherian (as a topological space) since $\mathrm{Spc}(\mathcal{T})$ is Noetherian if and only if any $\mathcal{Z} \subseteq \mathrm{Spc}(\mathcal{T})$ can be realized as $\mathrm{supp}(a)$ for some $a \in \mathcal{T}$. However, this is not the case as Thomason showed in [Tho97] that given a quasi-separated quasi-compact scheme $X$ one has a homeomorphism $\mathrm{Spc}(\mathrm{D}^{\mathrm{pf}}(X)) \cong X$, so of course $\mathrm{Spc}(\mathrm{D}^{\mathrm{pf}}(X))$ cannot be noetherian when $X$ is not. Barthel, Heard, and Sanders give other examples of non-Noetherian Balmer spectrum in [BHS23].

□

**Corollary 2.28.** *The set of quasi-compact open subsets of* $\mathrm{Spc}(\mathcal{T})$ *form a basis for the space, and this basis is closed under intersections.*

It remains to show that $\mathrm{Spc}(\mathcal{T})$ is a sober space.

**Proposition 2.29.** *Any closed irreducible subset* $\mathcal{Z} \subseteq \mathrm{Spc}(\mathcal{T})$ *has a unique generic point, making* $\mathrm{Spc}(\mathcal{T})$ *a sober space.*

*Proof.* Our goal is to be able to find for any $\mathcal{Z}$ a $\mathcal{P} \in \operatorname{Spc}(\mathcal{T})$ such that $\mathcal{Z} = \overline{\mathcal{P}}$. Uniqueness of the generic point $\mathcal{P}$ will follow from corollary 2.22. Suggestively, denote $\mathcal{P} := \{a \in \mathcal{T} : U(a) \cap \mathcal{Z} = \emptyset\}$. We will show that $\mathcal{P}$ is a prime $\otimes$-ideal and that $\mathcal{Z} = \mathcal{P}$.

Currently $\mathcal{P}$ is merely a full subcategory of $\mathcal{T}$ and it must be upgraded to the status of prime $\otimes$-ideal. For closure of $\oplus$, let $a, b \in \mathcal{P}$. Suppose that $a \oplus b \notin \mathcal{P}$. Then

$$\emptyset = U(a \oplus b) \cap \mathcal{Z} = (U(a) \cap U(b)) \cap \mathcal{Z} = (U(a) \cap \mathcal{Z}) \cap (U(b) \cap \mathcal{Z})$$

This is a non-empty decomposition of $\mathcal{Z}$ into two disjoint and open subsets, which cannot happen as $\mathcal{Z}$ is irreducible, so $a \oplus b \in \mathcal{P}$.

Now let $a \xrightarrow{f} b$ be a morphism in $\mathcal{P}$. By TR1 $f$ fits into an exact triangle $a \xrightarrow{f} b \xrightarrow{g} \operatorname{cone}(f) \xrightarrow{h} \Sigma a$ exact, and we want to see that $c \in \mathcal{P}$. Since $\langle a \oplus b \rangle^{\otimes}$ is the smallest $\otimes$-ideal containing $a$ and $b$, it follows that $c \in \langle a \oplus b \rangle^{\otimes}$. Hence $U(a \oplus b) \subseteq U(c)$, and since $a \oplus b \in \mathcal{P}$ by the paragraph above it follows that

$$\emptyset \neq U(a \oplus b) \cap \mathcal{Z} \subseteq U(c) \cap \mathcal{Z}$$

and therefore $c \in \mathcal{P}$.

Closure under the translation functor $\Sigma$ is easy, as $U(\Sigma a) = U(a)$ by proposition 2.15, and so $a \in \mathcal{P}$ if and only if $\Sigma a \in \mathcal{P}$. We now have that $\mathcal{P}$ is a triangulated thick subcategory. Now let $a \in \mathcal{P}$ and $b \in \mathcal{T}$. Then

$$U(a \otimes b) \cap \mathcal{Z} = (U(a) \cup U(b)) \cap \mathcal{Z} = (U(a) \cap \mathcal{Z}) \cup (U(b) \cap \mathcal{Z})$$

Note that $U(a) \cap \mathcal{Z} \neq \emptyset$ as $\mathcal{P}$, so the intersection above is nonempty and therefore $a \otimes b \in \mathcal{P}$. The expression above also shows us that $\mathcal{P}$ is prime, as $U(a \otimes b) \cap \mathcal{Z} \neq \emptyset$ implies that either $U(a) \cap \mathcal{Z}$ or $U(b) \cap \mathcal{Z}$ is non-empty and therefore either $a$ or $b$ are found in $\mathcal{P}$.

It remains to show that $\mathcal{Z} = \overline{\mathcal{P}}$. By proposition 2.20 $\overline{\mathcal{P}} = \{\mathcal{Q} : \mathcal{Q} \subseteq \mathcal{P}\}$. Let $\mathcal{Q} \in \mathcal{Z}$ and $a \in \mathcal{Q}$. Since $\mathcal{Q} \in U(a)$, $U(a) \cap \mathcal{Z} \neq \emptyset$. Then $a \in \mathcal{P}$ and so $\mathcal{Q} \subseteq \mathcal{P}$ and therefore $\mathcal{Q} \in \overline{P}$ and therefore $\mathcal{Z} \subseteq \overline{\mathcal{P}}$. Since $\mathcal{Z}$ is itself closed, it suffices to show that $\mathcal{P} \in \mathcal{Z}$ to finish the proof. By corollary 2.17 we can write

$$\mathcal{Z} = \bigcap_{\mathcal{Z} \subset \operatorname{supp}(a)} \operatorname{supp}(a)$$

If $a$ is an object such that $\mathcal{Z} \subseteq \operatorname{supp}(a)$ then $a \notin \mathcal{P}$, and thus $\mathcal{P} \in \operatorname{supp} a$. As $a$ was picked generally, $\mathcal{P} \in \mathcal{Z}$, and hence $\mathcal{Z} = \overline{\mathcal{P}}$. □

**Corollary 2.30.** $\operatorname{Spc}(\mathcal{T})$ *is a spectral space.*

## 2.3 Nilpotence

**Definition 2.31.** A morphism $f : a \to b$ in $\mathcal{T}$ is called $\otimes$-*nilpotent* if there exists $n \geq 1$ such that $f^{\otimes n} : a^{\otimes n} \to b^{\otimes n}$ is zero. If $a$ is an object where $\mathrm{id}_a$ is $\otimes$-nilpotent, then we say that $a$ is $\otimes$-nilpotent.

*Remark* 2.32. Recall the nilradical of $\mathcal{T}$:

$$\sqrt{0} := \{a \in \operatorname{Obj}(\mathcal{T}) \mid a^{\otimes n} \cong 0 \text{ for some } n \in \mathbb{N}\} = \bigcap_{\mathcal{P} \in \operatorname{Spc}(\mathcal{T})} \mathcal{P}$$

In other words, $\otimes$-nilpotent objects of $\mathcal{T}$ are precisely those objects that belong to all prime $\otimes$-ideals, and are therefore the objects with empty support. Proposition 2.33 is an analog of this fact for morphisms.

**Proposition 2.33.** *A morphism $f$ is $\otimes$-nilpotent if $f$ is zero locally, i.e. $f$ is sent to the zero morphism in $\mathcal{T}/\mathcal{P}$ for all $\mathcal{P} \in \operatorname{Spc}(\mathcal{T})$.*

*Proof.* Suppose that $f : a \to b$ is a $\otimes$-nilpotent morphism.

Suppose that $f$ is zero in $\mathcal{T}/\mathcal{P}$ for all $\mathcal{P} \in \operatorname{Spc}(\mathcal{T})$. Recall that a morphism is only sent to the zero morphism under a Verdier localization if it factors through an object in the kernel. Therefore, $f$ factors through an object $c_{\mathcal{P}} \in \mathcal{P}$ for all $\mathcal{P} \in \operatorname{Spc}(\mathcal{T})$. We can then write

$$\operatorname{Spc}(\mathcal{T}) = \bigcup_{\mathcal{P} \in \operatorname{Spc}(\mathcal{T})} U(c_{\mathcal{P}})$$

Since $\operatorname{Spc}(\mathcal{T})$ is quasi-compact, there is a finite subcover

$$\operatorname{Spc}(\mathcal{T}) = \bigcup_{i=1}^{n} U_{c_i}$$

corresponding to a finite collection $\mathcal{P}_1, ..., \mathcal{P}_n \in \operatorname{Spc}(\mathcal{T})$. Then,

$$U(0) = \operatorname{Spc}(\mathcal{T}) = \bigcup_{i=1}^{n} U(c_i) = U(c_1 \otimes ... \otimes c_n)$$

Therefore, $c_1 \otimes ... \otimes c_n \in \mathcal{P}$ for all $\mathcal{P} \in \operatorname{Spc}(\mathcal{T})$ so $c_1 \otimes ... \otimes c_n$ is $\otimes$-nilpotent. By assumption $f$ factors through each $c_i$ and so $f^{\otimes n}$ factors through $c_1 \otimes ... \otimes c_n$, but the latter is $\otimes$-nilpotent and is therefore zero for some $\otimes$-power $m \geq 1$. Hence, $f^{\otimes mn} = 0$. □

*Remark* 2.34. Let $X$ be a quasi-separated and quasi-compact scheme. Thomason proved in [Tho97] the following highly nontrivial result: a morphism $f : X_\bullet \to Y_\bullet$ of complexes with quasi-coherent cohomology in $\mathrm{D}^{\mathrm{qc}}(X)$ is $\otimes$-nilpotent if and only if $f \otimes_{\mathcal{O}_x} \kappa(x) : X_* \otimes^L_{\mathcal{O}_x} \kappa(x) \to Y_* \otimes^L_{\mathcal{O}_x} \kappa(x)$ is zero in $\mathrm{D}(\kappa(x))$ for all $x \in X$. This was a crucial step in Thomason's classification of thick subcategories of $\mathrm{D}^{\mathrm{pf}}(X)$ (discussed later in subsection 3.5), and is analogous to proposition 2.33.

## 2.4 Stone Duality and Spectral Spaces

In this section I will make good on my promise to talk about why the Balmer spectrum feels backwards from the usual ring theoretic perspective. We will first take a brief digression into Stone duality before talking about Hochster duality and the specialization order on a spectral space. We will not prove anything in this section as some of the results are highly non-trivial and would take us too far out of the scope of this text. Readers in a hurry may want to just skim over 2.4.1 as it is enlightening but not strictly essential. We will however need the results stated in 2.4.2. The presentation of 2.4.1 roughly follows section 1.2 of [KP15], which references [Joh82] for results on Stone duality.

### 2.4.1 Stone Duality

There are several categorical dualities between certain categories of topological spaces and categories of lattices. Usually these dualities fall under the umbrella term of Stone duality as they generalize Stone's representation theorem for Boolean algebras. The key insight of Stone duality is that the essential data of a topological space $X$ lies in the natural lattice structure on open sets of $X$, denoted $\mathcal{O}(X)$. In fact, $\mathcal{O}(X)$ is a type of lattice known as a *frame*.

**Definition 2.35.** A frame $F$ is a poset (viewed as a category) such that,

(a) $F$ has all set indexed coproducts, which we call joins and denote $\vee$,

(b) $F$ has finite limits called meets, denoted $\wedge$,

(c) and $F$ satisfies the infinite distributive law,

$$x \wedge \left( \bigvee_{i \in I} y_i \right) = \bigvee_{i \in I} (x \wedge y_i)$$

A morphism of frames is a covariant functor between frames, so frames form a category.

*Remark* 2.36. Equivalently, we can define a frame as a complete distributive lattice in which finite meets distribute over arbitrary joins, and a morphism of frames as a morphism of lattices that preserves arbitrary joins.

**Example 2.37.** Let $X$ be a topological space and $\mathcal{O}(X)$ be the set of open sets of $X$. Then $\mathcal{O}(X)$ is a poset where $U \leq V$ when $U \subseteq V$, and it is a frame with join and meet as union and intersection of sets respectively.

There is a contravariant functor from topological spaces to the category of frames, which sends $X \mapsto \mathcal{O}(X)$. There is a less obvious functor going back from frames to topological spaces, and it is a right adjoint to $\mathcal{O}$. To construct it, let $*$ denote a singleton as a topological space. Note that $\mathcal{O}(*) = \{\emptyset, \{*\}\}$, so for clarity we will just identify $\mathcal{O}(*)$ with the frame $\{0, 1\}$ where $0 \leq 1$. We define a *point* of a frame $F$ to be a morphism $p : F \to \mathcal{O}(*)$, and denote the set of points of $F$ as $\mathrm{Pt}(F)$. This set has a natural topology where open sets are of the form

$$U(x) = \{p \in \mathrm{Pt}(F) \mid p(x) = 1\}$$

for some $x \in F$. The topological spaces in the image of the points functor are sober, and in fact any sober space may be realized this way (up to homeomorphism). Going the other way, the frames that come from $\mathcal{O}$ are called *spatial frames*, which are the frames in which any two elements are separated by some point. The intuition here is that any two open sets (represented by the elements $x, y \in F$) there is a point $p$ which belongs to one open set but not the other. This is formally captured in the definition below.

**Definition 2.38.** A frame $F$ is called *spatial* if for any two elements $x, y \in F$ there exists $p \in \mathrm{Pt}(F)$ such that $p(x) \neq p(y)$.

One can show that the adjoint pair $(\mathcal{O} \dashv \mathrm{Pt})$ defines a contravariant equivalence of categories between sober spaces and spatial frames.

In his thesis, [Hoc69], Hochster defined spectral spaces and then showed that any spectral space may be realized as the spectrum of a commutative ring. As we saw in the previous section, spectral spaces form a category with spectral maps as morphisms and spectral spaces are by definition sober. It follows that they must have a corresponding subcategory in the category of spatial frames. To characterize these frames, we restate the definition of a spectral space as follows: a topological space $X$ is spectral if it is sober and the collection of compact opens, $\mathcal{K}^\circ(X)$, is a basis for $X$ and a sublattice of $\mathcal{O}(X)$. With this in mind, we can define the frames corresponding to spectral spaces.

**Definition 2.39.** An element $x$ of a frame $F$ is called *compact* if, for any cover of $x$ there is a finite subcover of $x$, i.e. given $A \subseteq F$ we have

$$x \leq \bigvee_{a \in A} a \quad \Longrightarrow \quad \exists \text{ finite } B \subset A \text{ where } x \leq \bigvee_{b \in B} b$$

Let $K(F) = \{x \in F \mid x \text{ is compact}\}$. Then we say that $F$ is *coherent* if $K(F)$ is a sublattice of $F$ and all elements of $F$ are joins of elements of $K(F)$. A morphism of frames is called coherent if it takes compact elements to compact elements.

One can show that coherent frames are spatial and that the equivalence between spatial frames and sober spaces given by $\mathcal{O}$ and Pt restricts to an anti-equivalence between coherent frames and spectral spaces. In fact, spectral spaces are now more commonly called *coherent spaces*. Though it is a non-trivial result, one can also show that a coherent frame $F$ may be recovered from $K(F)$ as the *frame of ideals* of $K(F)$.

**Definition 2.40.** Let $L$ be a distributive lattice. We call a subset $I \subseteq L$ an *ideal* if it is closed under finite joins and if $a \in I$ and $x \leq a$ then $x \in I$, i.e. $I$ is a down set of $L$.

An ideal $P$ of $L$ is called *prime* if

$$a \wedge b \in P \quad \Longleftrightarrow \quad a \in P \text{ or } b \in P$$

When viewing a distributive lattice $L$ as a category, taking the frame of ideals of $L$, denoted $\mathrm{Id}(L)$, amounts to passing to the cocompletion of $L$. These two functors are an equivalence of categories between coherent frames and bounded distributive lattices, though this takes a little work to show. The following theorem summarizes this discussion.

**Theorem 2.41** (Stone, 1939; Joyal, 1971)**.** *The category of spectral spaces is contravariantly equivalent to the category of coherent frames, which in turn is equivalent to the category of distributive lattices.*

### 2.4.2 Hochster Duality and the Specialization Order

Hochster also showed in [Hoc69] that a spectral space $X$ may be equipped with a dual topology called the *Hochster dual* or *inverse topology.*

**Definition 2.42.** Given a spectral space $X$ with quasi-compact basis $\mathcal{K}^\circ(X)$, define $X_{\text{inv}}$ to be the topological space with the same underlying set $X$ but whose open sets are those of the form $Y = \bigcup_{i \in \Omega} Y_i$ where $X \backslash Y_i$ is quasi-compact open for all $i \in \Omega$. In other words, the set of complements of the sets in $\mathcal{K}^\circ(X)$ becomes $\mathcal{K}^\circ(X^\vee)$

This yields an involution functor on the category of spectral spaces because it can be shown that $X_{\text{inv}}$ is also spectral and that $(X_{\text{inv}})_{\text{inv}} = X$.

Under the Hochster dual $X_{\text{inv}}$, the closed and generic points of $X$ become the generic and closed points respectively of $X_{\text{inv}}$. This is a special case of a more general fact. Given a topological space $X$ and points $x, y \in X$, we say that $x$ *specializes* to $y$ if $y \in \overline{\{x\}}$.

**Definition 2.43.** Given a subset of a topological space $Y \subseteq X$, $Y$ is said to be *specialization closed* if $Y$ has the property that $y \in Y$ implies $\overline{\{y\}} \subseteq Y$ or if, equivalently, $Y$ is an arbitrary union of closed sets.

**Definition 2.44.** Every topological space is equipped with a natural quasi-order (a relation that is reflexive and transitive) called the *specialization order* where

$$x \rightsquigarrow y \quad \Longleftrightarrow \quad y \in \overline{\{x\}}$$

**Fact 2.45.** The specialization order on a topological space $X$ is a partial order if and only if $X$ is $T_0$. In particular, the specialization order on a spectral space is a partial ordering. It's worth noting that in a $T_1$ space, all singletons are closed, so the specialization order ceases to be interesting for $T_1$ spaces.

**Fact 2.46.** Let $X$ be a spectral space. Then $X_{\text{inv}}$ has the reverse specialization order of $X$.

Now we are ready to see why the Balmer spectrum looks the way it does. The prime $\otimes$-ideals of $\mathcal{T}$ form a bounded distributive lattice, and therefore we may equip $\text{Spc}(\mathcal{T})$ with the Zariski topology for lattices, lets call it $\text{Spc}_{Lat}(\mathcal{T})$ to distinguish it from the Balmer spectrum. If we were to examine the closed sets of $\text{Spc}_{Lat}(\mathcal{T})$ we would find that they are of the form $\mathcal{Z}_{Lat}(\mathcal{S}) = \{\mathcal{P} \in \text{Spc}(\mathcal{T}) \mid \mathcal{S} \subset \mathcal{P}\}$ with $\mathcal{S}$ as some collection of objects in $\mathcal{T}$. This looks like the closed sets that we are used to in commutative algebra, where the specialization order and the ideal containment order coincide. One can then show that $\text{Spc}(\mathcal{T})_{inv} = \text{Spc}_{Lat}(\mathcal{T})$, i.e. the Balmer spectrum is the Hochster dual of the Zariski spectrum on the distributive lattice of prime $\otimes$-ideals in $\mathcal{T}$. It is for this reason that some authors interpret $\text{Spc}_{Lat}(\mathcal{T})$ as the more natural topology to work with. This perspective has led some authors to generalize the content of the following sections to lattices; see [KP15] and [BKS07].

# 3 Support Data and the Main Classification

From subsection 3.2 onward this chapter largely follows section 3 of [Bal05].

At this point we now know that $\operatorname{Spc}(\mathcal{T})$ is a spectral space, so $\operatorname{Spc}(\mathcal{T})$ is homeomorphic to the spectrum of some commutative ring. This helps us understand many abstract characteristics of $\operatorname{Spc}(\mathcal{T})$, but the point of defining the Balmer spectrum is to leverage algebro-geometric intuition to study the structure of the lattice of thick $\otimes$-ideals of a given tt-category. This means that at the end of the day we would like to be able to compute what $\operatorname{Spc}(\mathcal{T})$ is for a given tt-category $\mathcal{T}$. This is a rather difficult task in general due to the abstract nature of a general tt-category. The natural remedy for this is to try to relate $\operatorname{Spc}(\mathcal{T})$ to more familiar contexts.

This will lead us to the notion of support data and morphisms of support data on a tt-category $\mathcal{T}$, which will be formally defined in subsection 3.2. A support data is a topological space $X$ along with an assignment of objects of $\mathcal{T}$ to set of closed sets of $\mathcal{T}$, which satisfies some conditions that any sensible notion of support ought to satisfy. We go on to prove in subsection 3.3 that the topological space $\operatorname{Spc}(\mathcal{T})$ and the assignment $\operatorname{supp}_{\mathcal{T}}(-)$ is the universal support datum for $\mathcal{T}$, and that this support datum classifies radical thick $\otimes$-ideals of $\mathcal{T}$. After this, we will see that all support data that classifies radical thick $\otimes$-ideals of $\mathcal{T}$ are actually isomorphic to each other.

But before all this, we will first need some intuition for what a support datum should look like.

## 3.1 Supports of Modules

Let $R$ be a commutative ring. One of the most productive ideas in commutative algebra is to relate $R$-modules to subsets of the spectrum of prime ideals. There are many ways to do this, and they are usually called *support theories*. The classical definition of the support of an $R$-module is as follows.

**Definition 3.1.** Let $M$ be an $R$-module. The *classic support* of $M$ is the set of points of $\operatorname{Spec}(R)$ at which $M$ does not vanish, i.e.

$$\operatorname{Supp}_R(M) := \{\mathfrak{p} \in \operatorname{Spec}(R) \mid M_{\mathfrak{p}} \neq 0\}$$

Let $S$ be a subset of $R$. Recall the notation

$$V(S) := \{\mathfrak{p} \in \operatorname{Spec}(R) \mid S \subseteq \mathfrak{p}\}$$

which is a closed subset of $\operatorname{Spec}(R)$. We also recall the annihilator of $M$.

$$\operatorname{Ann}_R(M) := \{r \in R \mid r \cdot M = 0\}$$

If $M$ is finitely generated and $I$ is an ideal of $R$, then a classic result from commutative algebra says that $\operatorname{Supp}_R(M/IM) = V(I + \operatorname{Ann}_R(M))$, which makes $\operatorname{Supp}_R(M)$ closed in $\operatorname{Spec}(R)$. If $M$ is not finitely generated, then $\operatorname{Supp}_R(M)$ may not be closed; however, if $R$ is Noetherian then we can at least say that $\operatorname{Supp}_R(M)$ is specialization closed. Furthermore, the classic support relates algebraic operations to topological operations.

**Proposition 3.2.** *For $R$-modules $M$, $N$, and $L$ we have the following:*

*(a)* $\operatorname{Supp}_R(R) = \operatorname{Spec}(R)$.

*(b)* $\operatorname{Supp}_R(M) = \emptyset$ *if and only if* $M = 0$

*(c)* $\operatorname{Supp}_R(M \oplus N) = \operatorname{Supp}_R(M) \cup \operatorname{Supp}_R(N)$

*(d) If* $0 \to L \to M \to N \to 0$ *is exact, then* $\operatorname{Supp}_R(M) \subseteq \operatorname{Supp}_R(N) \cup \operatorname{Supp}_R(M)$*.*

*(e)* $\operatorname{Supp}_R(M \otimes N) \subseteq \operatorname{Supp}_R(M) \cap \operatorname{Supp}_R(N)$

*If $M$ and $N$ are finitely generated, then (e) above is equality.*

This looks an awful lot like the properties of $\operatorname{supp}_{\mathcal{T}}(a)$ as seen in proposition 2.15 where $\mathcal{T}$ is a tt-category and $a \in \operatorname{Obj}(\mathcal{T})$. In fact, the definitions are analogous, as $\operatorname{supp}_{\mathcal{T}}(a)$ is defined to be the points $\mathcal{P} \in \operatorname{Spc}(\mathcal{T})$ at which $a$ is not sent to zero under Verdier localization at $\mathcal{P}$. The main difference here is that (e) of proposition 3.2 is not always equality, whereas the corresponding property is equality in the tensor triangulated case. Property (e) is important enough to have a name. It is called the *tensor formula* in the context of a monoidal category equipped with a notion of support. Keep this in mind as we will return to it in a moment.

**Example 3.3.** Here is an example of the tensor formula failing for infinitely generated modules. Consider $\mathbb{Q}$ as a $\mathbb{Z}$-module. Then $\mathbb{Q}_{p\mathbb{Z}}$ is non-trivial for any prime $p \in \mathbb{Z}$, so $\operatorname{Supp}_R(\mathbb{Q}) = \operatorname{Spec}(\mathbb{Z})$. But then note that $\mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Z}/p\mathbb{Z} \cong 0$ for any prime number $p \in \mathbb{Z}$, so

$$\operatorname{Supp}_{\mathbb{Z}}(\mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Z}/p\mathbb{Z}) = \emptyset \neq \{p\mathbb{Z}\} = \operatorname{Supp}_{\mathbb{Z}}(\mathbb{Q}) \cap \operatorname{Supp}_{\mathbb{Z}}(\mathbb{Z}/p\mathbb{Z})$$

To further explore the connection between the tensor triangulated support and the classical support of modules, we need to extend the classical support from $R$-modules to complexes of $R$-modules.

**Definition 3.4.** Let $C_\bullet \in \mathrm{D}(R)$. The classical support of $C_\bullet$ is defined as

$$\operatorname{Supp}_R(C_\bullet) \coloneqq \{\mathfrak{p} \in \operatorname{Spec}(R) \mid (C_\bullet)_{\mathfrak{p}} \text{ is not acyclic}\}$$

**Proposition 3.5.** *If $C_\bullet \in \mathrm{D}(R)$ then*

$$\begin{aligned}\operatorname{Supp}_R(C_\bullet) &= \{\mathfrak{p} \in \operatorname{Spec}(R) \mid \mathrm{H}^*((C_{\mathfrak{p}})_\bullet) \neq 0\} \\ &= \{\mathfrak{p} \in \operatorname{Spec}(R) \mid \mathrm{H}^*(C_\bullet)_{\mathfrak{p}} \neq 0\} \\ &= \bigcup_{i \in \mathbb{Z}} \operatorname{Supp}_R(\mathrm{H}^i(C_\bullet))\end{aligned}$$

Immediately, we can see that definition 3.4 extends the classical support for $R$-modules by regarding an $R$-module $M$ as a complex concentrated in degree 0. Furthermore, one can show that the classical support for complexes satisfies the properties in proposition 3.2. We are starting to see how we might be able to glean information about the Balmer spectrum of the derived category of $R$ from $\operatorname{Spec}(R)$, for we now have two notions of support with similar properties associated to complexes of $R$-modules.

Except, this is not quite true.

There are multiple problems with the statement above. First, the Balmer spectrum can only be defined for essentially small tensor triangulated categories, and $\mathrm{D}(R)$ is not essentially small, meaning that we actually do not yet have a notion of tensor triangulated support for arbitrary complexes, though we will return to this in section 6. Another issue is that, as we will see in subsection 3.2, we want a support theory to assign objects to closed subsets and to satisfy the tensor formula, and neither of these properties hold for $\operatorname{Supp}_R$ on arbitrary complexes. For example, from example 3.3 and proposition 3.5 we can see that if a complex $C_\bullet$ has infinitely generated cohomology, then the tensor formula may not hold.

It turns out that there is a more natural notion of support in the derived setting. This was first defined by Foxby in [Fox79], and he called it *small support*, but we will call it the *homological support* in the context of complexes.

**Definition 3.6.** Let $R$ be a commutative ring and $\mathfrak{p} \in \operatorname{Spec}(R)$ a prime ideal. We denote the residue field of $R_\mathfrak{p}$ to be $\kappa(\mathfrak{p}) \coloneqq R_\mathfrak{p}/\mathfrak{p}_\mathfrak{p} \cong (R/\mathfrak{p})_\mathfrak{p}$.

**Definition 3.7.** We define the *homological support* of a complex $C_\bullet$ of $R$-modules to be

$$\operatorname{supp}^{\mathrm{h}}(C_\bullet) = \{\mathfrak{p} \in \operatorname{Spec}(R) \mid C_\bullet \otimes^L_R \kappa(\mathfrak{p}) \neq 0\}$$

*Remark* 3.8. The classical support is often referred to as the *big support*, for reasons that will become clear imminently.

**Proposition 3.9.** *For any complex $C_\bullet$ of $R$-modules,*

$$\operatorname{supp}^{\mathrm{h}}(C_\bullet) \subseteq \operatorname{Supp}_R(C_\bullet)$$

*with equality if $\mathrm{H}_i(C_\bullet)$ is finitely generated for all $i \in \mathbb{Z}$.*

*Proof.* Here is a proof in the case that $C_\bullet$ is just an $R$-module $M$, i.e. a complex concentrated in degree zero. The proof for arbitrary complexes is similar; see chapter 15.1 of [CFH24] for details.

Recall that $M_\mathfrak{p} = M \otimes_R R_\mathfrak{p}$. There is an isomorphism

$$M \otimes_R \kappa(\mathfrak{p}) = M \otimes_R R_\mathfrak{p}/\mathfrak{p}_\mathfrak{p} \cong M_\mathfrak{p}/\mathfrak{p}_\mathfrak{p} M_\mathfrak{p}$$

If $\mathfrak{p} \notin \operatorname{Supp}_R(M)$, i.e. if $M_\mathfrak{p} = 0$, then clearly $M \otimes_R \kappa(\mathfrak{p}) \cong 0$ by the isomorphism above, so in this case $\mathfrak{p} \notin \operatorname{supp}^{\mathrm{h}}(M)$. This proves the inclusion.

To see equality in the finitely generated case, suppose that $M$ is finitely generated and that $M_\mathfrak{p} \neq 0$. Then, $M_\mathfrak{p}$ is a non-trivial finitely generated $R_\mathfrak{p}$ module, and as $R_\mathfrak{p}$ is a local ring, it follows from Nakayama's lemma that $\mathfrak{p}_\mathfrak{p} M_\mathfrak{p} \neq M_\mathfrak{p}$. Therefore, $M \otimes_R \kappa(\mathfrak{p}) \cong M_\mathfrak{p}/\mathfrak{p}_\mathfrak{p} M_\mathfrak{p} \neq 0$, so $\mathfrak{p} \in \operatorname{supp}^{\mathrm{h}}(M)$. □

**Proposition 3.10.** *For complexes $A_\bullet$, $B_\bullet$, and $C_\bullet$ we have the following:*

*(a)* $\operatorname{supp}^{\mathrm{h}}(C_\bullet) = \emptyset$ *if and only if $C_\bullet$ is acyclic.*

*(b)* $\operatorname{supp}^{\mathrm{h}}(C[1]_\bullet) = \operatorname{supp}^{\mathrm{h}}(C_\bullet)$

*(c)* $\operatorname{supp}^{\mathrm{h}}(B_\bullet \oplus C_\bullet) = \operatorname{supp}^{\mathrm{h}}(B_\bullet) \cup \operatorname{supp}^{\mathrm{h}}(C_\bullet)$

*(d)* *If $A_\bullet \to B_\bullet \to C_\bullet$ is an exact triangle, then* $\operatorname{supp}^{\mathrm{h}}(B_\bullet) \subseteq \operatorname{supp}^{\mathrm{h}}(A_\bullet) \cup \operatorname{supp}^{\mathrm{h}}(C_\bullet)$.

*(e)* $\operatorname{supp}^{\mathrm{h}}(B_\bullet \otimes^L_R C_\bullet) = \operatorname{supp}^{\mathrm{h}}(B_\bullet) \cap \operatorname{supp}^{\mathrm{h}}(C_\bullet)$

*Proof.* See chapter 15.1 of [CFH24] for proofs of these facts. □

**Proposition 3.11.** *For perfect complexes, homological support and classical support coincide, i.e.*

$$\mathrm{supp}^{\mathrm{h}}(C_\bullet) = \mathrm{Supp}_R(C_\bullet) \quad \forall C_\bullet \in \mathrm{D}^{pf}(R)$$

*Proof.* See lemma 3.3 of [Tho97]. □

Proposition 3.10 shows that homological support satisfies the properties that we would like out of a support theory, and proposition 3.11 shows us that homological support and classical support are actually the same for $\mathrm{D}^{\mathrm{pf}}(R)$. Since perfect complexes are bounded and have finitely generated homology, it follows from proposition 3.5 that $\mathrm{supp}^{\mathrm{h}}(C_\bullet)$ is a finite union of closed subsets of $\mathrm{Spec}(R)$ for any $C_\bullet \in \mathrm{D}^{\mathrm{pf}}(R)$, making $\mathrm{supp}^{\mathrm{h}}(C_\bullet)$ closed in $\mathrm{Spec}(R)$. This will be important what follows in subsection 3.2.

### 3.2 Support data

As stated in 3.1, we now have two notions of support for $\mathrm{D}^{\mathrm{pf}}(R)$ obeying similar properties: one coming from the tt-structure on $\mathrm{D}^{\mathrm{pf}}(R)$ (see definition 2.5), and one coming from the tensor triangulated structure of $\mathrm{Spc}(\mathrm{D}^{\mathrm{pf}}(R))$. This begs the question: can we relate these two notions of support?

**Definition 3.12.** A *support datum* on a tt-category $\mathcal{T}$ is a pair $(X, \sigma)$ where $X$ is a topological space and $\sigma$ is an assignment which associates to any object $a \in \mathcal{T}$ a closed subset $\sigma(a) \subseteq X$ such that:

(SD1) $\sigma(0) = \emptyset$ and $\sigma(\mathbb{1}) = X$,

(SD2) and $\sigma(a \oplus b) = \sigma(a) \cup \sigma(b)$.

(SD3) $\sigma(\Sigma a) = \sigma(a)$,

(SD4) $\sigma(b) \subseteq \sigma(a) \cup \sigma(c)$ whenever $a \to b \to c \to \Sigma a$ is exact

(SD5) $\sigma(a \otimes b) = \sigma(a) \cap \sigma(b)$,

A morphism $f : (X, \sigma) \to (Y, \tau)$ of support data on $\mathcal{T}$ is a continuous map $f : X \to Y$ such that $\sigma(a) = f^{-1}(\tau(a))$ for all objects $a \in \mathcal{T}$. A morphisms of support data is an isomorphism if and only if $f$ is a homeomorphism.

**Example 3.13.** Let $R$ be a commutative ring. From proposition 3.10, the pair $(\mathrm{Spec}(R), \mathrm{supp}^{\mathrm{h}})$ is a support datum for $\mathrm{D}^{\mathrm{pf}}(R)$.

**Lemma 3.14.** *Let $(X, \sigma)$ be a support datum for $\mathcal{T}$. For any choice of $Y \subset X$, the full subcategory of $\mathcal{T}$ consisting of objects $\{a \in \mathcal{T} : \sigma(a) \subseteq Y\}$ is a thick $\otimes$-ideal in $\mathcal{T}$.*

*Proof.* Denote $\mathcal{C}$ as the full subcategory generated by the objects $\{a \in \mathcal{T} : \sigma(a) \subseteq Y\}$ for some fixed $Y \subseteq X$. If $a \to b \to c \to \Sigma a$ is an exact triangle in $\mathcal{T}$ with any two of $a$, $b$, or $c$ in $\mathcal{C}$ then the third must be in $\mathcal{C}$ via application of (SD 3),(SD 4), and rotation of the triangle, so $\mathcal{C}$ is triangulated. If $x \subseteq a \oplus b$ and $x \in \mathcal{C}$ then $a, b \in \mathcal{C}$ via (SD 2). Finally, if $a \in \mathcal{C}$ and $b \in \mathcal{T}$ then $a \otimes b \in \mathcal{C}$ by (SD 5). □

**Lemma 3.15.** *Let $X$ be a set and let $f_1, f_2 : X \to \mathrm{Spc}(\mathcal{T})$ be two maps such that $f_1^{-1}(\mathrm{Supp}(a)) = f_2^{-1}(\mathrm{Supp}(a))$ for all $a \in \mathcal{T}$. Then $f_1 = f_2$.*

*Proof.* Let $x \in X$. For any $a \in \mathcal{T}$ our assumption on $f_1$ and $f_2$ means $f_1(x) \in \mathrm{Supp}(a) \iff f_2(x) \in \mathrm{Supp}(a)$. It follows that

$$\bigcap_{f_1(x) \in \mathrm{Supp}(a)} \mathrm{Supp}(a) = \bigcap_{f_2(x) \in \mathrm{Supp}(a)} \mathrm{Supp}(a)$$

which is a closed set. But then the set above is equal to $\overline{\{f_1(x)\}}$ by corollary 2.17. But then $\overline{\{f_1(x)\}} = \overline{\{f_2(x)\}}$, and since $\mathrm{Spc}(\mathcal{T})$ is a sober space, it must be that $f_1(x) = f_2(x)$. □

**Theorem 3.16.** *Let $\mathcal{T}$ be a tt-category. The support datum $(\mathrm{Spc}(\mathcal{T}), \mathrm{Supp})$ is the final support datum on $\mathcal{T}$, meaning that $(\mathrm{Spc}(\mathcal{T}), \mathrm{Supp})$ is a support datum on $\mathcal{T}$ and for any support datum $(X, \sigma)$ on $\mathcal{T}$ there exists a unique support map $f : (X, \sigma) \to (\mathrm{Spc}(\mathcal{T}), \mathrm{Supp})$ where for $x \in X$*

$$f(x) = \{a \in \mathcal{T} : x \notin \sigma(a)\}$$

*Proof.* From proposition 2.15 it follows that $(\mathrm{Spc}(\mathcal{T}), \mathrm{supp})$ is a support datum for $\mathcal{T}$ as $\mathrm{supp}(a) = \mathrm{Spc}(\mathcal{T}) \backslash U(a)$. Let $(X, \sigma)$ be a support datum on $\mathcal{T}$ and let $f$ be the map in the theorem statement. We must check that $f$ is a support data morphism. Since $f(x) = \{a \in \mathcal{T} : \sigma(a) \subseteq X \setminus \{x\}\}$ it follows from lemma 3.14 that $f(x)$ is indeed a thick $\otimes$-ideal. To see that it is prime, let $a \otimes b \in f(x)$. Then $x \notin \sigma(a \otimes b) = \sigma(a) \cap \sigma(b)$ and so $x \notin \sigma(a)$ or $x \notin \sigma(b)$, so $a \in f(x)$ or $b \in f(x)$. Hence, $f(x)$ is prime.

It remains to show that $f$ is continuous and that it is a morphism of support data. To see that it is a morphism of support data, we need to check that $f^{-1}(\mathrm{supp}(a)) = \sigma(a)$ for any $a \in \mathrm{Obj}(\mathcal{T})$.

$$\begin{aligned} f^{-1}(\mathrm{supp}(a)) :=& \{x \in X \mid f(x) \in \mathrm{supp}(a)\} \\ =& \{x \in X : a \notin f(x)\} \\ =& \{x \in X : a \notin \{b \in \mathcal{T} \mid x \notin \sigma(b)\}\} \\ =& \{x \in X : x \in \sigma(a)\} \\ =& \sigma(a) \end{aligned}$$

Since sets of the form $\mathrm{supp}(a)$ form a closed basis for $\mathrm{Spc}(\mathcal{T})$, and $\sigma(a)$ is closed in $X$, we can also see from the expression above that the inverse image of a closed basis set is closed, making $f$ continuous. □

*Remark* 3.17. Notice that in the proof of theorem 3.16 we absolutely needed support data to satisfy the tensor formula $\sigma(a \otimes b) = \sigma(a) \cap \sigma(b)$, for if this were not the case we would not be able to conclude that $f(x)$ is a prime ideal.

Now we want to show that the universal support datum is functorial. Going forward, if we wish to distinguish between different tt-categories we will differentiate supports by the notations $\mathrm{supp}_{\mathcal{T}}(a)$ for the tt-category $\mathcal{T}$.

**Proposition 3.18.** *The spectrum is functorial, meaning that given a $\otimes$-exact $F : \mathcal{T} \to \mathcal{K}$, the map*

$$\mathrm{Spc}\, F : \mathrm{Spc}(\mathcal{K}) \to \mathrm{Spc}(\mathcal{T})$$

*given by $\mathcal{Q} \mapsto F^{-1}(\mathcal{Q})$ is well-defined and continuous. Furthermore, for all objects $a \in \mathcal{T}$,*

$$(\mathrm{Spc}\, F)^{-1}(\mathrm{supp}_{\mathcal{T}}(a)) = \mathrm{supp}_{\mathcal{K}}(F(a))$$

*This assignment defines a contravariant functor* $\mathrm{Spc}(-)$ *from essentially small tensor triangulated categories to the category of topological spaces.*

*Proof.* Immediate from the construction. $\square$

**Corollary 3.19.** *If two* $\otimes$*-exact functors* $F_1, F_2 : \mathcal{T} \to \mathcal{K}$ *have the property that* $\langle F_1(a)\rangle = \langle F_2(a)\rangle$ *in* $\mathcal{L}$ *for any object* $a \in \mathcal{T}$*, then* $\mathrm{Spc}\, F_1 = \mathrm{Spc}\, F_2$*. In particular, agreement of* $F_1$ *and* $F_2$ *on objects implies that they have the same map on spectra.*

*Proof.* If $\mathcal{Q} \in \mathrm{Spc}(\mathcal{K})$ and $a \in \mathrm{Spc}(\mathcal{T})$ then

$$a \in F_i^{-1}(\mathcal{Q}) \iff \langle F_i(a)\rangle \subseteq \mathcal{Q}$$

Therefore, $\langle F_1(a)\rangle = \langle F_2(a)\rangle$ implies that $F_1^{-1}(\mathcal{Q}) = F_2^{-1}(\mathcal{Q})$ making $\mathrm{Spc}(F_1) = \mathrm{Spc}(F_2)$. $\square$

**Corollary 3.20.** *Suppose that a* $\otimes$*-exact functor* $F : \mathcal{T} \to \mathcal{K}$ *is essentially surjective. Then* $\mathrm{Spc}(F)$ *is a homeomorphism onto its image.*

*Proof.* Any thick $\otimes$-ideal is replete, i.e. closed under isomorphism, so $\langle F(F^{-1}(\mathcal{Q}))\rangle = \mathcal{Q}$ as $F$ is essentially surjective. Therefore if $F^{-1}(\mathcal{Q}_1) = F^{-1}(Q_2)$ then $\mathcal{Q}_1 = \mathcal{Q}_2$, so $\mathrm{Spc}(F)$ is injective. The rest of the proof proceeds similarly to the proof of the analogous statement for commutative rings. $\square$

**Theorem 3.21.** *If* $\mathcal{I}$ *is a thick* $\otimes$*-ideal of* $\mathcal{T}$ *and* $\pi : \mathcal{T} \to \mathcal{T}/\mathcal{I}$ *is the canonical localization functor, then*

$$\mathrm{Spc}\,\pi : \mathrm{Spc}(\mathcal{T}/\mathcal{I}) \hookrightarrow \mathrm{Spc}(\mathcal{T})$$

*is a homeomorphism on its image,* $\{\mathcal{P} \in \mathrm{Spc}(\mathcal{T}) \mid \mathcal{I} \subseteq \mathcal{P}\}$*.*

*Proof.* The functor $\pi$ is essentially surjective by construction so the map $\mathrm{Spc}(\pi)$ is a homeomorphism onto its image by corollary 3.20. It remains to see that the image of $\mathrm{Spc}(\pi)$ is $V := \{\mathcal{P} \in \mathrm{Spc}(\mathcal{T}) \mid \mathcal{I} \subseteq \mathcal{P}\}$, the proof of which proceeds exactly as in the case of a commutative ring. $\square$

**Example 3.22.** Lets try our hand at calculating the spectrum of $\mathrm{D}^{\mathrm{pf}}(\mathbb{Z})$ from scratch. Recall that $\mathrm{D}^{\mathrm{pf}}(\mathbb{Z})$ is the category of chain complexes over $\mathbb{Z}$ that are quasi-isomorphic to bounded complexes of finitely generated projective modules. In order to figure out what the spectrum of this tt-category is, we should examine the prime $\otimes$-ideals and try to make a guess at what a classifying support datum might be, and if we're lucky we'll get it in one try.

From example 3.13 we know that $(\mathrm{Spec}(\mathbb{Z}), \mathrm{supp}^{\mathrm{h}})$ is a support datum for $\mathrm{D}^{\mathrm{pf}}(\mathbb{Z})$, and theorem 3.16 tells us that there is a unique morphism of support data

$$f : (\mathrm{Spec}(\mathbb{Z}), \mathrm{supp}^{\mathrm{h}}) \xrightarrow{x \mapsto \{a \in \mathrm{D}^{\mathrm{pf}}(\mathbb{Z}) \mid x \notin \mathrm{supp}^{\mathrm{h}}(a)\}} (\mathrm{Spc}\,\mathrm{D}^{\mathrm{pf}}(\mathbb{Z}), \mathrm{supp})$$

Maybe if we can understand this map, then we can extract an explicit picture of $\mathrm{Spc}(\mathrm{D}^{\mathrm{pf}}(\mathbb{Z}))$. As a first observation, recall that $\mathbb{Z}$ is a hereditary ring, meaning that all submodules of projective modules over $\mathbb{Z}$ are also projective. Take $C_\bullet \in \mathrm{D}^{\mathrm{pf}}(\mathbb{Z})$, which is quasi-isomorphic to a bounded complex of finitely generated projective modules.

$$\ldots \to 0 \to P_n \xrightarrow{d_n} P_n \xrightarrow{d_{n-1}} \ldots \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \to 0 \to \ldots$$

Since $\mathbb{Z}$ is hereditary, $\mathrm{im}(d_i) \subseteq P_{i-1}$ is itself projective. We then have that $P_i \cong \mathrm{im}(d_i) \oplus \ker(d_i)$ for each $i$. It follows that $C_\bullet$ is isomorphic to

$$\ldots \to 0 \to \ker(d_n) \oplus \mathrm{im}(d_n) \to \ker(d_{n-1}) \oplus \mathrm{im}(d_{n-1}) \to \ldots \to \ker(d_1) \oplus \mathrm{im}(d_1) \to \ker(d_0) \to 0 \to \ldots$$

We then have a quasi-isomorphism coming from the quotient map $\ker(d_i) \to \ker(d_i)/\,\mathrm{im}(d_{i+1}) = \mathrm{H}_i(C_\bullet)$ with zero differentials. Therefore, in the derived category $\mathrm{D}^{\mathrm{pf}}(\mathbb{Z})$, $C_\bullet$ is isomorphic to

$$\bigoplus_{i \in \mathbb{Z}} \mathrm{H}_i(C_\bullet)[i] \tag{3}$$

Since $\sigma(a \oplus b) = \sigma(a) \cup \alpha(b)$ and $\sigma(\Sigma a) = \sigma(a)$ for a support datum $(X, \sigma)$ of a tt-category with objects $a$ and $b$, it follows that if this is a support datum for $\mathrm{D}^{\mathrm{pf}}(\mathbb{Z})$ then $\sigma(C_\bullet)$ is just the union

$$\bigcup_{i \in \mathbb{Z}} \sigma(\mathrm{H}_i(C_\bullet))$$

where $\mathrm{H}_i(C_\bullet)$ is taken to be a chain complex that is everywhere 0 except for a single term $\mathrm{H}_i(C_\bullet)$. Since $\mathrm{H}_i(C_\bullet)$ is finitely generated in this case, we may reduce to the case of determining $\sigma$ on $M$ a finitely generated module taken to be a complex concentrated in degree zero. Because of the classification of finitely generated $\mathbb{Z}$-modules, it follows that

$$\sigma(M) = \bigcup_{j \in I} \sigma\left(\frac{\mathbb{Z}}{q_j^{n_j}\mathbb{Z}}\right)$$

where $\{q_j\}_{j \in I}$ is a finite set of prime numbers coming from the canonical decomposition of $M$ indexed by some set $I$. Hence, we may further reduce to determining $\sigma$ on $\frac{\mathbb{Z}}{q^n\mathbb{Z}}$ with $q$ a prime. Recall that for integers $m, n$ the expression below holds

$$\frac{\mathbb{Z}}{m\mathbb{Z}} \otimes_{\mathbb{Z}} \frac{\mathbb{Z}}{n\mathbb{Z}} = \frac{\mathbb{Z}}{\gcd(m,n)\mathbb{Z}} \tag{4}$$

Now we turn back the specific case of $\mathrm{supp}^{\mathrm{h}}$ and the map $f : \mathrm{Spec}(\mathbb{Z}) \to \mathrm{Spc}(\mathrm{D}^{\mathrm{pf}}(\mathbb{Z}))$ from earlier. Since $\mathrm{supp}^{\mathrm{h}}(C_\bullet) = \{q\mathbb{Z} \in \mathrm{Spec}(\mathbb{Z}) \mid C_\bullet \otimes^L_{\mathbb{Z}} \frac{\mathbb{Z}}{q\mathbb{Z}} \neq 0\}$ then have by the discussion above that for $p\mathbb{Z} \in \mathrm{Spec}(\mathbb{Z})$

$$\begin{aligned} f(p\mathbb{Z}) &= \{C_\bullet \in \mathrm{D}^{\mathrm{pf}}(\mathbb{Z}) \mid C_\bullet \notin \mathrm{supp}^{\mathrm{h}}(p\mathbb{Z})\} \\ &= \{C_\bullet \in \mathrm{D}^{\mathrm{pf}}(\mathbb{Z}) \mid C_\bullet \otimes^L_{\mathbb{Z}} \tfrac{\mathbb{Z}}{p\mathbb{Z}} = 0\} \end{aligned}$$

By equation (4) and the decomposition in equation (3), it follows that $C_\bullet$ is in the collection above if and only if $C_\bullet \in \mathrm{thick}(\mathbb{Z}/q\mathbb{Z} \mid q \neq p)$, so we have

$$f(p\mathbb{Z}) = \mathrm{thick}(\mathbb{Z}/q\mathbb{Z} \mid q \neq p)$$

Note that by theorem 3.16 it must be that $\mathrm{thick}(\mathbb{Z}/q\mathbb{Z} \mid q \neq p)$ must be a prime $\otimes$-ideal. This also immediately guarantees injectivity of $f$. At this point, we should be suspicious that $(\mathrm{Spec}(\mathbb{Z}), \mathrm{supp}^{\mathrm{h}})$ is actually isomorphic as a support datum to $(\mathrm{Spc}(\mathrm{D}^{\mathrm{pf}}(\mathbb{Z})), \mathrm{Supp})$, but we don't yet have surjectivity.

Let $\mathcal{P}$ be a prime $\otimes$-ideal of $\mathrm{D}^{\mathrm{pf}}(\mathbb{Z})$. Since $0 \in \mathcal{P}$ we can see from equation (3) that if we set $m = p$ and $n = q$ for primes $q$ and $p$ then at most one $\mathbb{Z}/p\mathbb{Z}$ is not contained in $\mathcal{P}$. Of course this is the case for $f(p\mathbb{Z})$ as we saw just a moment ago, but now we want to show that all prime $\otimes$-ideals $\mathcal{P}$ are of this form. Let $\mathcal{I}$ be a thick $\otimes$-ideal in $\mathrm{D}^{\mathrm{pf}}(\mathbb{Z})$ and let $\mathbb{Z}/p\mathbb{Z} \in \mathcal{I}$. Then note that

$$\mathrm{cone}(\mathbb{Z}/p\mathbb{Z} \xrightarrow{m \mapsto p^{n-1}m} \mathbb{Z}/p^n\mathbb{Z}) \cong \mathbb{Z}/p\mathbb{Z}$$

and therefore $\mathbb{Z}/p^n\mathbb{Z} \in \mathcal{I}$ since $\mathcal{I}$ is closed under cones. It can be similarly shown that if $\mathbb{Z}/p^n\mathbb{Z} \in \mathcal{I}$ for some $n$, then $\mathbb{Z}/p\mathbb{Z} \in \mathcal{I}$. Hence, if $\mathcal{I}$ contains any $\mathbb{Z}/p^n\mathbb{Z}$ for some $n$ then it contains $\mathbb{Z}/p^n\mathbb{Z}$ for all $n$. Using these two observations, the decomposition of objects in $\mathrm{D}^{\mathrm{pf}}(\mathbb{Z})$ shown above, classification of finitely generated $\mathbb{Z}$-algebras, and proposition 1.14, it's straightforward to check that if $\mathcal{P} \in \mathrm{Spc}(\mathcal{T})$ does not contain $\mathbb{Z}/p\mathbb{Z}$ then $\mathrm{thick}(\mathbb{Z}/q\mathbb{Z} \mid q \neq p) = \mathcal{P}$. This then means that there $f$ is a continuous bijection, and it's straightforward to see that $f$ must also be a closed map. Therefore, $f$ is a homeomorphism between $\mathrm{Spec}(\mathbb{Z})$ and $\mathrm{Spc}(\mathrm{D}^{\mathrm{pf}}(\mathbb{Z}))$.

*Remark* 3.23. Let's reflect on the previous example. As foreshadowed in section 2, the closed points of $\mathrm{Spc}(\mathrm{D}^{\mathrm{pf}}(\mathbb{Z}))$ are the minimal primes of $\mathrm{D}^{\mathrm{pf}}(\mathbb{Z})$ and the generic points are the maximal primes, and we saw that the universal morphism $f$ is a homeomorphism that associates a prime $p\mathbb{Z} \in \mathrm{Spec}(\mathbb{Z})$ to the prime $\otimes$-ideal $\mathrm{thick}(\mathbb{Z}/q\mathbb{Z} \mid q \neq p)$. Since $f$ is a homeomorphism of spectral spaces, it is an isomorphism between the specialization orders of the two spaces. The interesting thing here is that $f$ reverses the containment order. It is worth figuring out why this happens. This is not unique to this example, so going forward let us instead speak of a more general ring $R$ instead of $\mathbb{Z}$.

In algebraic contexts it is highly common to think of the prime ideals ordered with respect to containment, which coincides with the specialization order on $\mathrm{Spec}(R)$ under the usual Zariski topology. In the geometry of $\mathrm{Spec}(R)$, the specialization of a point $x$ to a point $y$ corresponds to the geometry of $x$ containing the geometry of $y$, while algebraically the ideal corresponding to $y$ contains the ideal corresponding to $x$. By examining the definition of the universal morphism $f : \mathrm{Spec}(R) \to \mathrm{Spc}(\mathrm{D}^{\mathrm{pf}}(R))$, we see that a prime ideal $\mathfrak{p} \in \mathrm{Spec}(R)$ is send to the prime $\otimes$-ideal $\{C^\bullet \mid C^\bullet \otimes_R k(\mathfrak{p}) = 0\}$, i.e. a point in $\mathrm{Spec}(R)$ is sent to the collection of perfect complexes which vanish at $\mathfrak{p}$.

When thinking classically this is perfectly sensible since the containment of ideals corresponds to the reverse containment of subvarieties through the Nullstellensatz, so really, the specialization order

Let's reflect on the previous example. As foreshadowed in section 2, the closed points of $\mathrm{Spc}(\mathrm{D}^{\mathrm{pf}}(\mathbb{Z}))$ are the minimal primes of $\mathrm{D}^{\mathrm{pf}}(\mathbb{Z})$ and the generic points are the maximal primes. We also see a sort of containment order reversing phenomenon in the universal map $f : \mathrm{Spec}(\mathbb{Z}) \to \mathrm{Spc}(\mathrm{D}^{\mathrm{pf}}(\mathbb{Z}))$. This universal morphism happened to be an isomorphism in this case, and in fact it was a homeomorphism of spectral spaces. In particular, $f$ is an isomorphism between the specialization orders of the two spaces. In algebraic contexts it is common to think of the containment order on prime ideals, hence the terminology of maximal and minimal primes. The containment order of prime ideals coincides with the specialization order on prime ideals when they are interpreted as points in the geometric object $\mathrm{Spec}(R)$. But under the image of $f$, the containment order is reversed while the specialization order is preserved. This is because $f$ sends a point $p$ to the collection of complexes vanishing at that point $p$.

It is worth examining why this happens. Fundamentally, it goes back to order reversing phenomena found in nearly all algebro-geometric correspondences, such as Hilbert's Nullstellensatz.

Concretely, on the left hand side we have a point $p$ which is associated to the ideal of functions vanishing at that point. The map $f$ sends this to $f(p) = \{C_\bullet \in \mathrm{D}^{\mathrm{pf}}(\mathbb{Z}) \mid C_\bullet \notin \mathrm{supp}^{\mathrm{h}}(p)\}$, i.e. the complexes that vanish

## 3.3 The Main Classification Theorem

Now we want to classify thick $\otimes$-ideals using topological data. In classical algebraic geometry we have Hilbert's Nullstellensatz which gives a correspondence between radical ideals of a coordinate ring and the subvarieties of the variety associated to that ring. The main classification theorem can be viewed in a similar light, but first we need some definitions.

**Definition 3.24.** Given a topological space $X$, a subset $Y \subseteq X$ is a *Thomason subset* if $Y = \bigcup_{i \in I} Y_i$ such that $X \setminus Y_i$ is open and quasi-compact for any $i \in I$.

**Proposition 3.25.** *If $X$ is a spectral space, then the Thomason subsets of $X$ are precisely the open sets of the Hochster dual $X_{\mathrm{inv}}$.*

*Remark* 3.26. Since $\operatorname{supp} \mathcal{S} = \bigcup_{a \in \mathcal{S}} \operatorname{supp}(a)$ for any subset of objects $\mathcal{S} \subseteq \mathcal{T}$ and $U(a) = \operatorname{Spc}(\mathcal{T}) \setminus \operatorname{supp}(a)$ is open and quasi-compact for any $a \in \mathcal{T}$ (proposition 2.26), it follows that $\operatorname{supp} \mathcal{S}$ is a Thomason subset for any collection of objects $\mathcal{S}$.

*Remark* 3.27. If $X$ is a Noetherian topological space, then a Thomason subset is precisely one that is specialization closed (definition 2.43) as all open subsets of a Noetherian topological space are compact.

**Definition 3.28.** Given $Y \subseteq \operatorname{Spc}(\mathcal{T})$, we define the *objects supported at $Y$* to be the set

$$C_Y \coloneqq \{a \in \mathcal{T} \mid \operatorname{supp}(a) \subseteq Y\}$$

The main classification theorem is as follows:

**Theorem 3.29** ([Bal05])**.** *There exists a 1-to-1 order reversing correspondence*

$$\Phi : \{\textit{radical } \otimes\textit{-ideals in } \mathcal{T}\} \to \{\textit{Thomason subsets of } \operatorname{Spc}(\mathcal{T})\}$$

*where $\Phi$ is given by $\Phi(\mathcal{I}) \coloneqq \operatorname{supp} \mathcal{I}$ and its inverse $\Psi$ is given by $\Psi(Y) \coloneqq C_Y$.*

To prove this we first need some lemmas. Those familiar with commutative algebra have probably anticipated the one below.

**Lemma 3.30.** *For a thick $\otimes$-ideal $\mathcal{I}$, we have that*

$$\sqrt{\mathcal{I}} = \bigcap_{\substack{\mathcal{I} \subseteq \mathcal{P} \\ \mathcal{P} \in \operatorname{Spc}(\mathcal{T})}} \mathcal{P}$$

*Proof.* It is a standard fact in commutative algebra that for any ideal $I$ of a commutative ring $R$ once has

$$\sqrt{I} = \bigcap_{\substack{I \subseteq \mathfrak{p} \\ P \in \operatorname{Spec}(R)}} \mathfrak{p}$$

A proof of this fact may be found in nearly any textbook on basic commutative algebra. The proof of lemma 3.30 follows, mutatis mutandis, in the same manner as in the ring theoretic case. See corollary 2.12 of [Eis95]. ☐

**Lemma 3.31.** *For a thick $\otimes$-ideal $\mathcal{I}$, we have*

$$\operatorname{supp}\mathcal{I} = \{\mathcal{P} \mid \mathcal{I} \nsubseteq \mathcal{P}\}$$

*Proof.* Let $\mathcal{I}$ be a thick $\otimes$-ideal. Then,

$$\begin{aligned}
\operatorname{supp}\mathcal{I} &= \bigcup_{a\in\mathcal{I}} \operatorname{supp}(a) \\
&= \bigcup_{a\in\mathcal{I}} \{\mathcal{P} \in \operatorname{Spc}(\mathcal{T}) \mid a \notin \mathcal{P}\} \\
&= \{\mathcal{P} \in \operatorname{Spc}(\mathcal{T}) \mid a \notin \mathcal{P} \text{ for some } a \in \mathcal{I}\} \\
&= \{\mathcal{P} \in \operatorname{Spc}(\mathcal{T}) \mid \mathcal{I} \not\subset \mathcal{P}\}
\end{aligned}$$

□

**Lemma 3.32.** *Given $Y \subseteq \operatorname{Spc}(\mathcal{T})$, we have*

$$C_Y = \bigcap_{\mathcal{P}\notin Y} \mathcal{P}$$

*which is radical.*

*Proof.* Suppose that $a \in C_Y$. Then $\mathcal{P} \notin Y$ means that $a \in \mathcal{P}$ by definition of $C_Y$, so $a \in \cap_{\mathcal{P}\notin Y} \mathcal{P}$. On the other hand, if $a \in \cap_{\mathcal{P}\notin Y}$ then $U(a) \subseteq \operatorname{Spc}(\mathcal{T}) \setminus Y$, so $\operatorname{supp}(a) \subseteq Y$ and therefore $a \in C_Y$, so we have equality. Note that $C_Y$ is a thick $\otimes$-ideal by *lemma* 3.14. □

*Proof of theorem 3.29.*

The maps $\Phi$ and $\Psi$ in the statement of theorem 3.29 are well defined by lemma 3.32 and *remark* 3.26. Let $\mathcal{I}$ be a radical $\otimes$-ideal. Then we know that

$$\Psi \circ \Phi(\mathcal{I}) = C_{\operatorname{supp}\mathcal{I}} = \bigcap_{\mathcal{P}\in\operatorname{Supp}\mathcal{I}} \mathcal{P}$$

The condition $\mathcal{P} \in \operatorname{supp}\mathcal{I}$ is equivalent to $\mathcal{P} \notin \operatorname{supp}(a)$ for any $a \in \mathcal{I}$, which is in turn equivalent to $a \in \mathcal{P}$ for all $a \in \mathcal{I}$, i.e. $\mathcal{I} \subseteq \mathcal{P}$. As $\mathcal{I}$ is radical, it follows that

$$C_{\operatorname{supp}\mathcal{I}} = \bigcap_{\mathcal{I}\subseteq\mathcal{P}} \mathcal{P} = \mathcal{I}$$

Now consider a Thomason subset $Y$. We have

$$\Phi \circ \Psi(Y) = \operatorname{supp}(C_Y) = \{\mathcal{P} \in \operatorname{Spc}(\mathcal{T}) \mid C_Y \not\subset \mathcal{P}\}$$

Now $C_Y$ not being contained in $\mathcal{P}$ means precisely that there exists $a \in C_Y$ not contained in $\mathcal{P}$. Since $a \notin \mathcal{P}$ means by definition that $\mathcal{P} \in \operatorname{supp}(a)$ and $a \in C_Y$ if and only if $\operatorname{supp}(a) \subseteq Y$, it follows that $\mathcal{P} \not\subset C_Y$ if and only if $\mathcal{P} \in \operatorname{supp}(a) \subseteq Y$. Hence, $\operatorname{supp}(C_Y) = Y$. □

## 3.4 Rigidity and Radicality

Here we are going to take a brief detour to discuss rigidity and radicality and how these two properties interact with each other and with $\mathrm{Spc}(\mathcal{T})$.

There are many important examples of tt-categories in which all thick $\otimes$-ideals are radical, and it turns out that this is a very desirable property for a tt-category to have. This makes the utility of the following proposition evident.

**Proposition 3.33** (Radicality)**.** *The following are equivalent:*

1. *Any thick $\otimes$-ideal of $\mathcal{T}$ is radical.*
2. *For all objects $a \in \mathcal{T}$, $a \in \langle a \otimes a \rangle$.*

*Proof.* Clearly, if all thick $\otimes$-ideals of $\mathcal{T}$ are radical, then $\langle a \otimes a \rangle$ is radical and so $a \in \langle a \otimes a \rangle$.

On the other hand, suppose that $a^{\otimes n} \in \mathcal{I}$ for some $n$, where $\mathcal{I}$ is a thick $\otimes$-ideal. We wish to show that $a \in \mathcal{I}$. The $n = 2$ case holds by assumption, so assume that cases $n$ and below hold. If $a^{\otimes n+1} \in \mathcal{I}$ and $n = 2k$ for some $k \in \mathbb{N}$ then $n+2 = 2k+2$, so $a^{\otimes k+1} \in \langle a^{\otimes 2k+2} \rangle \subseteq \mathcal{I}$. If $n = 2k+1$, then $n+1 = 2k+2$, so similarly $a^{\otimes k+1} \in \mathcal{I}$. As $k+1 \subseteq n$, it follows from the inductive hypothesis that $a \in \mathcal{I}$. □

**Example 3.34.** proposition 3.33 may be applied to $\mathsf{stmod}(kG)$. To see this, take $M$ in $kG$-mod. There is a split embedding

$$M \hookrightarrow M \otimes_k \mathrm{End}_k(M)$$

given my $m \mapsto m \otimes \mathrm{id}_M$. Its retraction is the evaluation map $m \otimes \phi \to \phi(m)$, so $M$ is a direct summand of $M \otimes_k \mathrm{End}_k(M)$. If $M$ is finite dimensional then it is a standard linear algebra fact that $\mathrm{End}_k(M) \cong M^\vee \otimes_k M$ where $M^\vee = \mathrm{Hom}_k(M, k)$. Then $M$ is a direct summand of $M \otimes_k \mathrm{End}_k(M) \cong M \otimes_k M^\vee \otimes_k M$, but then $M \in \langle M \otimes M \rangle$. It follows that all thick tensor ideals of $\mathsf{stmod}(kG)$ are radical.

The example above may be generalized to any rigid tt-category as shown in the proposition below.

**Proposition 3.35.** *Let $\mathcal{T}$ be rigid. Then $x$ is a direct summand of $x \otimes x^\vee \otimes x$ for all objects $x$ of $\mathcal{T}$. Consequently, all $\otimes$-ideals $\mathcal{C}$ of $\mathcal{T}$ are self-dual, i.e. $\mathcal{I} = \mathcal{I}^\vee := \{a^\vee \mid a \in \mathcal{I}\}$. Additionally, all thick $\otimes$-ideals are radical.*

*Proof.* By remark 1.25, we have that $\underline{\mathrm{hom}}(x, x) \cong x \otimes x^\vee$ for any $x \in \mathcal{T}$. Additionally, by the unit-counit equations of hom-tensor adjunction, the composition below is equal to the identity on $x$.

$$x \cong \mathbb{1} \otimes x \xrightarrow{\eta_{x,1\otimes x}} \underline{\mathrm{hom}}(x, \mathbb{1} \otimes x) \otimes x \xrightarrow{\epsilon_{x,1\otimes x}} \mathbb{1} \otimes x \cong x$$

which is isomorphic to the composition

$$x \to \underline{\mathrm{hom}}(x, x) \otimes x \cong x \otimes x^\vee \otimes x \to x$$

Therefore, $x$ is a direct summand of $x \otimes x^\vee \otimes x \in \langle x \otimes x \rangle$ and therefore $x \in \langle x \otimes x \rangle$ as this is a thick $\otimes$-ideal. By proposition 3.33 all thick $\otimes$-ideals of $\mathcal{T}$ are radical.

Let $x \in \mathcal{I}$. By applying the functor $(-)^\vee$, argument above also yields that $x^\vee$ is a direct summand of $x^\vee \otimes x^{\vee\vee} \otimes x^\vee \cong= x^\vee \otimes x \otimes x^\vee$, so $x^\vee \in \mathcal{I}$ making $\mathcal{I}$ self dual. □

**Proposition 3.36.** *Let $a$ and $b$ be objects in a rigid tt-category $\mathcal{T}$. Then,*

- $\mathrm{supp}(a^\vee) = \mathrm{supp}(a)$*, and*
- $\mathrm{supp}(\underline{\mathrm{hom}}(a,b)) = \mathrm{supp}(a) \cap \mathrm{supp}(b)$

*Proof.* From proposition 3.35 we have that $\langle a \rangle = \langle a^\vee \rangle$. Therefore, given $\mathcal{P} \in \mathrm{Spc}(\mathcal{T})$, $a \in \mathcal{P}$ if and only if $a^\vee \in \mathcal{P}$. From the definition of support, $\mathrm{supp}(a) = \mathrm{supp}(a^\vee)$. For (b), recall that $\underline{\mathrm{hom}}(a,b) \cong a^\vee \otimes b$ for all objects $a$ and $b$ in $\mathcal{T}$ when $\mathcal{T}$ is rigid. Therefore,

$$\mathrm{supp}(\underline{\mathrm{hom}}(a,b)) = \mathrm{supp}(a^\vee \otimes b) = \mathrm{supp}(a^\vee) \cap \mathrm{supp}(b) = \mathrm{supp}(a) \cap \mathrm{supp}(b)$$

□

**Corollary 3.37.** *Let $\mathcal{T}$ be a rigid tt-category. If $a$ and $b$ are objects of $\mathcal{T}$ with disjoint supports, then* $\mathrm{Hom}_{\mathcal{T}}(a,b) = 0$.

*Proof.* If $\mathrm{supp}(a) \cap \mathrm{supp}(b) = \emptyset$ then $\mathrm{supp}(\underline{\mathrm{hom}}(a,b)) = \emptyset$ by proposition 3.36. This then forces $\underline{\mathrm{hom}}(a,b) = 0$, which in turn implies that $\mathrm{Hom}_{\mathcal{T}}(a,b) = 0$ by fact 1.26. □

*Remark* 3.38. It should be noted that, in the context of a general tt-category, if we are given a morphism $f : a \to b$ with $\mathrm{supp}(a) \cap \mathrm{supp}(b) = \emptyset$, we have that $f$ is zero locally, i.e. $f$ is sent to zero in $\mathcal{T}/\mathcal{P}$ for any $\mathcal{P} \in \mathrm{Spc}(\mathcal{T})$; however, when $f$ is zero locally we can only conclude that $f^{\otimes n} = 0$ for some $n \in \mathbb{N}$ (see proposition 2.33). This is one reason why being rigid is desirable, as it allows much finer control of $\mathcal{T}$ from the perspective of the Balmer spectrum.

### 3.5 Classifying Support Data

Here we loosely follow section 5 of [Bal05], which is where classifying support data are first defined, but the results given here are the stronger versions proved in [BKS07]. See remark 3.43.

**Definition 3.39.** A support datum $(X, \delta)$ for a tt-category $\mathcal{T}$ is a *classifying support datum* if the following two conditions hold:

(a) The topological space $X$ is spectral.

(b) We have a bijection

$$\theta : \{Y \subset X \mid Y \text{ is a Thomason subset}\} \xrightarrow{\sim} \{\mathcal{J} \subset \mathcal{T} \mid \mathcal{J} \text{ radical thick } \otimes\text{-ideal}\}$$

defined by $Y \mapsto \{a \in \mathcal{T} \mid \sigma(a) \subset Y\}$, with inverse $\mathcal{J} \mapsto \sigma(\mathcal{J}) = \bigcup_{a \in \mathcal{J}} \sigma(a)$.

*Remark* 3.40. In this light, theorem 3.29 is just saying that $(\mathrm{Spc}(\mathcal{T}), \mathrm{supp})$ is a classifying support datum.

**Theorem 3.41.** *Let $f : (X, \sigma) \to (X', \sigma')$ be a morphism of support datum for $\mathcal{T}$. If both support data are classifying, then the map $f : X \to X'$ is a homeomorphism.*

*Proof.* Let $(X, \sigma)$ and $(X', \sigma')$ be as in the statement of theorem and let $Y \subseteq X$ and $Y' \subseteq X'$ be Thomason subsets such that

$$\{b \in \mathcal{T} \mid \sigma(b) \subseteq Y\} = \mathcal{I} = \{b \in \mathcal{T} \mid \sigma'(b) \subseteq Y'\}$$

where $\mathcal{I}$ is a radical thick $\otimes$-ideal by assumption. Since $X$ and $X'$ are classifying,

$$Y = \bigcup_{b\in\mathcal{I}} \sigma(b) \quad \text{and} \quad Y' = \bigcup_{b\in\mathcal{I}} \sigma'(b)$$

Therefore,

$$f^{-1}(Y') = \bigcup_{b\in\mathcal{I}} f^{-1}(\sigma'(b)) = \bigcup_{b\in\mathcal{I}} \sigma(b) = Y$$

Therefore, the assignment $Y' \mapsto f^{-1}(Y')$ is a bijection between Thomason subsets of $X$ and $X'$. In particular, it an order preserving bijection since $f$ is a morphism of support data. Since Thomason subsets are precisely the open subsets of the inverse topology, it follows that $f : X_{\mathrm{inv}} \to X'_{\mathrm{inv}}$ is an order preserving continuous map. Since sobriety is exactly the property allows the lattice of open sets of a spaces to determine it up to homeomorphism, it follows that $f$ is a homeomorphism on the dual topologies. It follows from theorem B.13 that $f$ is an order isomorphism on the original topologies and is therefore a homeomorphism on the original topologies. □

**Corollary 3.42.** *A support datum $(X, \sigma)$ on $\mathcal{T}$ is classifying if and only if the canonical morphism $(X, \sigma) \to (\operatorname{Spc}\mathcal{T}, \operatorname{supp})$ is an isomorphism.*

*Remark* 3.43. Corollary 3.42 was originally proved by Balmer in [Bal05] under the assumption that $X$ was Noetherian. Under his original definition, a classifying space had to be Noetherian. The more general version given above is due to Buan-Krause-Solberg (Cor. 6.2 of [BKS07]) who generalized support datum to the context of lattices and frames.

The main classification was first established in stable homotopy theory by Devinatz-Hopkins-Smith in [DHS88], and Hopkins realized that the result could be carried over into the realm of algebra where $\mathcal{T} = \mathrm{D}^{\mathrm{pf}}(R)$ for a commutative ring $R$; see [Hop87]. His result was a correspondence between $\otimes$-ideals and specialization closed subsets, but his proof necessitated that $R$ be Noetherian — a hypothesis that he forgot to include. Neeman pointed out in [Nee92] that the proof was incorrect without the assumption that $R$ be noetherian, and rescued the proof in this case. It was Thomason [Tho97] who proved the more general correspondence between subsets of the kind found in definition 3.24 (i.e. open subsets of the Hochster dual) and the $\otimes$-ideals of $\mathrm{D}^{\mathrm{pf}}(X)$ for $X$ a quasi-separated and quasi-compact scheme.

**Theorem 3.44** (Thomason, Balmer)**.** *Let $X$ be a quasi-compact and quasi-separated scheme. Then the pair $(X, \operatorname{supp}^{\mathrm{h}})$ is a classifying support datum, and there is a homeomorphism $X \cong \operatorname{Spc}(\mathrm{D}^{pf}(X))$ given by the canonical morphism of support data $(X, \operatorname{supp}^{\mathrm{h}}) \to (\operatorname{Spc}(\mathrm{D}^{pf}(X)), \operatorname{supp}_{\mathrm{D}^{pf}(X)})$.*

It didn't stop there though, as soon Benson-Carlson-Rickard [BCR97] carried out the classification in modular representation theory of finite groups, and later Friedlander-Pevtsova [FP07] generalized this to finite group schemes. We will look at this in closer detail in subsection 4.2.

# 4 Central Rings and the Structure Sheaf of $\mathrm{Spc}(\mathcal{T})$

In the previous chapter, we looked at support data as a way to get a handle on the structure of a tt-category $\mathcal{T}$. This is a good start, but one downside with this approach is that if you are looking at some particular $\mathcal{T}$ it can be difficult to figure out what a classifying support data might be if you don't already have a good guess based on prior knowledge of $\mathcal{T}$. In light of this, we need other methods of analyzing $\mathrm{Spc}(\mathcal{T})$.

We know that $\mathrm{Spc}(\mathcal{T})$ is a spectral space, and that all spectral spaces arise as the Zariski spectrum of some commutative ring. We might then ask if there is a ring naturally associated to $\mathcal{T}$ whose Zariski spectrum may be related to $\mathrm{Spc}(\mathcal{T})$. The most obvious choice for such a ring is the endomorphism ring of the monoidal unit, which we will denote $R_{\mathcal{T}} := \mathrm{End}_{\mathcal{T}}(\mathbb{1})$. This makes $\mathrm{Hom}_{\mathcal{T}}(a, b)$ into an $R_{\mathcal{T}}$-module for all $a, b \in \mathrm{Obj}(\mathcal{T})$. We will see that given an $f \in R_{\mathcal{T}}$, the thick $\otimes$-ideal $\langle \mathrm{cone}(f) \rangle$ coincides with the subcategory of objects that are nilpotent with respect to $f$. We will then be able to construct localizations $S^{-1}\mathcal{T}$ of $\mathcal{T}$ with respect to a multiplicatively closed subset $S$ of $R_{\mathcal{T}}$. Additionally, we will equip $\mathrm{Spc}(\mathcal{T})$ with a sheaf of rings.

Much of the content of this chapter of comes from [Bal10], with the exception of subsection 4.1 which comes from [BS01]. To maintain focus, we have not stated everything in the full generality of Balmer's original results in this presentation of the material.

## 4.1 Idempotent Completion

The content of this section is mostly taken from [BS01].

**Definition 4.1.** Let $e : a \to a$ be a morphism in a category $\mathcal{C}$ for which $e^2 = e$. Such a morphism is called *idempotent*. An idempotent *splits* if there exists an object $b$ and morphisms $f : a \to b$ and $g : b \to a$ such that $g \circ f = e$ and $f \circ g = \mathrm{id}_b$. If $\mathcal{C}$ is additive then it follows that $b$ is a direct summand of $a$. If all idempotents of $\mathcal{C}$ split then we say that $\mathcal{C}$ is *idempotent complete*.

For geometric reasons, we would like localization to preserve idempotent completeness, but unfortunately this does not always happen. Counter examples are rather involved to describe, so we will omit them. However, we can always pass to the idempotent completion.

**Definition 4.2.** Given an additive category $\mathcal{K}$, an *idempotent completion* $\mathcal{K}^e$ (sometimes also referred to as its *Karoubian envelope*), is an idempotent complete category with a fully faithful additive functor $e : \mathcal{K} \to \mathcal{K}^e$ such that every object of $\mathcal{K}^e$ is a retract of an object in $\mathcal{K}$.

It turns out that $\mathcal{K}$ is unique up to equivalence, which is why we call it *the* idempotent completion; see 5.1.4.9 of [Lur09]. We can construct it by letting $\mathcal{K}^e$ have objects $(a, e)$ where $a$ is an object of $\mathcal{K}$ and $e : a \to a$ is an idempotent in $\mathcal{K}$, and morphisms $f : (a, e) \to (a', e')$ where $f$ is a morphism in $\mathcal{K}$ such that the diagram below commutes:

$$\begin{array}{ccc} a & \xrightarrow{e} & a \\ {\scriptstyle f}\downarrow & \searrow^{f} & \downarrow{\scriptstyle f} \\ a' & \xrightarrow{e'} & a' \end{array}$$

The identity on $(a, e)$ is $e$, while the functor $i : \mathcal{K} \to \mathcal{K}^e$ sends $a \mapsto (a, \mathrm{id}_a)$. Clearly $\mathcal{K}$ is a full subcategory of $\mathcal{K}^e$, and it is relatively straightforward to check that this category is in fact idempotent complete, since if $(a,e) \xrightarrow{\epsilon} (a,e)$ is an idempotent, it follows that $\epsilon \circ e = \epsilon = \epsilon^2 = e \circ \epsilon$ within $\mathcal{K}$, so $(a,e)$ splits via

$$(a,e) \xrightarrow{\epsilon} (a,\epsilon) \xrightarrow{\epsilon} (a,e)$$

*Remark* 4.3. Another equivalent way to define the idempotent completion of a category $\mathcal{C}$ is to define $\mathcal{C}^e$ as the full subcategory of $\widehat{\mathcal{C}}$ consisting of retracts of representable functors.

**Exercise 4.4.** Given a ring $R$, Show that the idempotent completion of the category of free $R$-modules is equivalent to the category of projective $R$-modules.

The following theorem is due to Balmer and Schlichting in [BS01]. We will not prove this, as it will take us too far afield for the moment. However, we will take a moment to describe the triangulated structure of $\mathcal{K}^e$ in construction 4.6.

**Theorem 4.5.** *If $\mathcal{K}$ is a triangulated category, then $\mathcal{K}^e$ inherits its (tensor-)triangulated structure and $i : \mathcal{K} \to \mathcal{K}$ is exact.*

**Construction 4.6.** Let $(\mathcal{K}, \Sigma, \Delta)$ be a triangulated category, and $\mathcal{K}^e$ its idempotent completion. We take the suspension functor to be $\widetilde{\Sigma} : \mathcal{K}^e \to \mathcal{K}^e$ where $\widetilde{\Sigma}(a,e) = (\Sigma a, \Sigma e)$, so clearly the embedding $i : \mathcal{K} \to \mathcal{K}^e$ has the property $i \circ \Sigma = \Sigma \circ i$. We define the exact triangles $\Delta'$ of $\mathcal{K}^e$ to be those that are a direct factor of an exact triangle of $\mathcal{K}$, that is, a triangle $\Delta_1$ in $\mathcal{K}^e$ is exact when there exists a triangle $\Delta_2$ in $\mathcal{K}$ such that $\Delta_1 \oplus \Delta_2$ is isomorphic to an exact triangle in $cK$.

*Remark* 4.7. A consequence of the construction above is that a triangle is exact in $\mathcal{K}$ if and only if it is exact in $\mathcal{K}^e$.

Similarly, it is not difficult (but somewhat tedious) to show that the idempotent completion of a symmetric monoidal category is also a symmetric monoidal category such that the map $\mathcal{K} \to \mathcal{K}^e$ is a monoidal functor, so combining all these results one can see that the embedding of a tt-category $\mathcal{T}$ into its monoidal completion is a $\otimes$-exact functor.

**Exercise 4.8.** Verify the above claim using theorem 4.5

Fortunately, it follows from [Bal05] that Spc does not see the difference between a triangulated category $\mathcal{T}$ and its completion $\mathcal{T}^e$.

**Theorem 4.9.** *Let $\mathcal{T}$ be a tt-category and let $\mathcal{T} \subset \mathcal{K}$ be a full tensor-triangulated subcategory with the same unit and which is cofinal, i.e. for any object $a \in \mathcal{K}$ there exists $a' \in \mathcal{K}$ such that $a \oplus a' \in \mathcal{T}$. Then the map $Q \mapsto Q \cap \mathcal{T}$ defines a homeomorphism* $\mathrm{Spc}(\mathcal{K}) \to \mathrm{Spc}(\mathcal{T})$.

*In particular, the embedding $i : \mathcal{T} \to \mathcal{T}^e$ induces a homeomorphism*

$$\mathrm{Spc}(i) : \mathrm{Spc}(\mathcal{T}^e) \to \mathrm{Spc}(\mathcal{T})$$

*Proof.* We will identify $\mathcal{T}$ with its essential image. The map $\mathrm{Spc}(\mathcal{K}) \to \mathrm{Spc}(\mathcal{T})$ is simply $\mathrm{Spc}(i)$ where $i : \mathcal{T} \to \mathcal{K}$ is the inclusion functor. We first need to prove that for all $a \in \mathcal{K}$ we have $a \oplus \Sigma a \in \mathcal{T}$. By assumption, there is an $a' \in \mathcal{K}$ such that $a \oplus a' \in \mathcal{T}$. We then take the direct sum of the exact triangles $a' \to 0 \to \Sigma(a') \to \Sigma(a')$, $a \to a \to 0 \to \Sigma a$, and $0 \to \Sigma a \to \Sigma a \to 0$ to obtain

$$(a \otimes a') \to a \oplus \Sigma a \to \Sigma(a \oplus a') \to \Sigma(a \oplus a')$$

Since direct sums of exact triangles are exact, it follows that the triangle above is exact, and since two entries are in $\mathcal{T}$, so is the third, i.e. $a \oplus \Sigma(a) \in \mathcal{T}$. This then shows that if $\mathcal{P} \in \mathrm{Spc}(\mathcal{T})$ then

$$\{a \in \mathcal{K} \mid a \oplus \Sigma(a) \in \mathcal{P}\} = \{a \in \mathcal{K} \mid \exists a' \in \mathcal{K} \text{ s.t. } a \oplus a' \in \mathcal{P}\} =: \widetilde{P}$$

We want to show that $\widetilde{\mathcal{P}} \in \mathrm{Spc}(\mathcal{K})$. It is straightforward to check that $\widetilde{\mathcal{P}}$ is as thick $\otimes$-ideal, so we leave it as an exercise. it remains to check that it is prime. Suppose that $a \otimes b \in \widetilde{\mathcal{P}}$ and that $a \notin \widetilde{\mathcal{P}}$. Define $x = a \oplus \Sigma a$. Then $a \in \mathcal{T} \setminus \mathcal{P}$ and $x \otimes b \cong (a \otimes b) \oplus \Sigma(a \otimes b) \in \mathcal{P}$. This then implies that $(x \otimes b) \oplus (x \otimes \Sigma b) \cong x \otimes (b \otimes \Sigma b) \in \mathcal{P}$ so $b \oplus \Sigma b \in \mathcal{P}$ since $\mathcal{P}$ is prime and cannot contain $x$. Therefore, $b \in \widetilde{\mathcal{P}}$, so $\widetilde{\mathcal{P}}$ is in fact prime. Additionally, since $\mathcal{P}$ is thick we have $\widetilde{\mathcal{P}} \cap \mathcal{T} = \mathcal{P}$ and therefore we have a right inverse to $\mathrm{Spc}(i)$.

On the other hand, if $\mathcal{Q} \in \mathrm{Spc}(\mathcal{K})$ then its straightforward to check that $\mathcal{Q} = \widetilde{\mathcal{P}}$ where $\mathcal{P} = \mathcal{Q} \cap \mathcal{T}$. Suppose $a \in \mathcal{Q}$. Then $a \oplus \Sigma a \in \mathcal{T}$, but then $a \oplus \Sigma a \in \mathcal{Q} \cap \mathcal{T}$, so $a \in \mathcal{P}$, so $a \in \widetilde{\mathcal{P}}$. On the other hand, if $a \in \widetilde{\mathcal{P}}$ then $a \oplus \Sigma a \in \mathcal{P}$ by definition, and since $\mathcal{P} = \mathcal{Q} \cap \mathcal{T}$, it follows that $a \oplus \Sigma a \in \mathcal{Q}$, so $a \in \mathcal{Q}$ by thickness of $\mathcal{Q}$.

We conclude that $\mathrm{Spc}(i)$ is a continuous bijection with inverse $\mathcal{P} \mapsto \widetilde{\mathcal{P}}$. The above argument showed that for $a \in \mathcal{K}$ we have

$$a \in \mathcal{Q} \iff a \oplus \Sigma a \in \mathcal{Q} \iff a \oplus \Sigma a \in \mathcal{P}$$

where $\mathcal{P} = \mathcal{Q} \cap \mathcal{T}$ and $\mathcal{Q} = \widetilde{\mathcal{P}}$. Therefore,

$$\mathrm{Spc}(i)(\mathrm{supp}_{\mathcal{K}}(a)) = \mathrm{supp}_{\mathcal{K}}(a \oplus \Sigma a)$$

which is closed, so $\mathrm{Spc}(i)$ is a closed map and is therefore a homeomorphism. □

There are several reasons to want a tt-category to be idempotent complete. One such reason is corollary 4.11 of the theorem below.

**Theorem 4.10.** *Let $\mathcal{T}$ be a rigid idempotent complete tt-category. Then if $Y_1$ and $Y_2$ are disjoint thomason subsets of $\mathrm{Spc}(\mathcal{T})$, then $C_{Y_1 \cup Y_2}$ coincides with the full subcategory*

$$C_{Y_1} \oplus C_{Y_2} \coloneqq \{a \in \mathrm{Obj}(\mathcal{T}) \mid a \cong a_1 \oplus a_1, a_i \in C_{Y_i}\}$$

*Proof.* Let $Y_1$ and $Y_2$ be disjoint Thomason subsets of $\mathrm{Spc}(\mathcal{T})$. Clearly, $C_{Y_1} \oplus C_{Y_2} \subseteq C_{Y_1 \cup Y_2}$. If we can prove that $C_{Y_1} \oplus C_{Y_2}$ is a thick $\otimes$-ideal (and therefore a radical thick $\otimes$-ideal as $\mathcal{T}$ is rigid), it then follows from the main classification theorem of thick $\otimes$-ideals that there is a Thomason subset $Y$ such that $C_Y = C_{Y_1} \oplus C_{Y_2}$. Since $C_Y$ is the smallest thick $\otimes$-ideal containing both $C_{Y_1}$ and $C_{Y_2}$, it follows from the classification that $Y$ is the smallest Thomason subset containing $Y_1$ and $Y_2$, i.e. $Y = Y_1 \cup Y_2$. The claim follows.

It remains to show that $C_{Y_1} \oplus C_{Y_2}$ is as thick $\otimes$-ideal. It's easy to see that $C_{Y_1} \oplus C_{Y_2}$ is a $\otimes$-ideal. We now want to see that it is triangulated. Let the composition of morphisms below be distinguished,

$$a_1 \oplus a_2 \xrightarrow{f} b_1 \oplus b_2 \xrightarrow{g} c \xrightarrow{h} \Sigma(a_1 \oplus a_2)$$

where $\mathrm{supp}(a_i), \mathrm{supp}(b_i) \subset Y_i$ for $i = 1, 2$. Since $\mathrm{supp}(a_i) \cap \mathrm{supp}(b_j) \subseteq Y_i \cap Y_j = \emptyset$ when $i \neq j$, it follows from proposition 3.36 that $\mathrm{Hom}_{\mathcal{T}}(a_i, b_j) = \emptyset$ when $i \neq j$. Therefore, $f = \begin{pmatrix} f_1 & 0 \\ 0 & f_2 \end{pmatrix}$ where $f_1 \in \mathrm{Hom}_{\mathcal{T}}(a_1, b_1)$ and $f_2 \in \mathrm{Hom}_{\mathcal{T}}(a_2, b_2)$. It follows that $c \cong \mathrm{cone}(f_1) \oplus \mathrm{cone}(f_2)$, so $c \in C_{Y_1} \oplus C_{Y_2}$. Additionally, $\Sigma$ commutes with direct sums, so $C_{Y_1} \oplus C_{Y_2}$ is a triangulated subcategory.

Now we must show that $C_{Y_1} \oplus C_{Y_2}$ is closed under direct summands. Suppose that $x \oplus y$ is an object of some full subcategory $\mathcal{C}$, and observe that $x = \mathrm{im}(e)$ where $e$ is the idempotent obtained from the composition $x \oplus y \xrightarrow{\pi} x \xrightarrow{\iota} x \oplus y$ where $\pi$ is the projection onto $x$ and $\iota$ is the inclusion of

$x$ into $x \oplus y$. Therefore, if we wish to show that $\mathcal{C}$ is closed under direct summands, it suffices to show that all images of idempotents in $\mathcal{C}$ are contained in $\mathcal{C}$.

Let $a \cong a_1 \oplus a_2 \in C_{Y_1} \oplus C_{Y_2}$ where $a_1 \in C_{Y_1}$ and $a_2 \in C_{Y_2}$, and let $e \in \mathrm{End}_{\mathcal{T}}(a)$ be an idempotent. Since $\mathrm{supp}(a_1) \cap \mathrm{supp}(a_2) = \emptyset$, we have $\mathrm{Hom}_{\mathcal{T}}(a_1, a_2) = \mathrm{Hom}_{\mathcal{T}}(a_2, a_1) = 0$. It follows that $e = \left(\begin{smallmatrix} e_1 & 0 \\ 0 & e_2 \end{smallmatrix}\right)$ where $e_1$ and $e_2$ are idempotents for $a_1$ and $a_2$ respectively. Since $\mathcal{T}$ is idempotent complete, it follows that $a_i \cong \mathrm{im}(e_i) \oplus \ker(e_i)$ for $i = 1, 2$. In particular, this means that

$$\mathrm{supp}(a_i) = \mathrm{supp}(\mathrm{im}(e_i)) \cup \mathrm{supp}(\ker(e_i)) \quad \Longrightarrow \quad \mathrm{supp}(\mathrm{im}(e_i)) \subseteq \mathrm{supp}(a_i) \subseteq Y_i$$

Therefore, $\mathrm{im}(e_i) \in C_{Y_i}$, so $\mathrm{im}(e) \cong \mathrm{im}(e_1) \oplus \mathrm{im}(e_2) \in C_{Y_1} \oplus C_{Y_2}$. Therefore, $C_{Y_1} \oplus C_{Y_2}$ contains the images of idempotents within itself and is therefore closed under direct summands, making it thick. □

**Corollary 4.11.** *Let $\mathcal{T}$ be a rigid idempotent complete tt-category. If the support of an object can be decomposed as $\mathrm{supp}(a) = Y_1 \cup Y_2$ where $Y_1$ and $Y_2$ are disjoint and quasi-compact open in $\mathrm{Spc}(\mathcal{T})_{\mathrm{inv}}$, then $a \cong a_1 \oplus a_2$ with $\mathrm{supp}(a_1) = Y_1$ and $\mathrm{supp}(a_2) = Y_2$.*

## 4.2 The Structure Sheaf of Spc($\mathcal{T}$)

For the remainder of the chapter, assume that $\mathcal{T}$ is essentially small.

Let $\mathcal{T}$ be a tt-category and let $U$ be a quasi-compact subset $U \subseteq \mathrm{Spc}(\mathcal{T})$ and $Z_U := \mathrm{Spc}(\mathcal{T}) \setminus U$. Recall that the subcategory $C_{Z_U}$ of objects supported on $Z$ is a thick $\otimes$-ideal by lemma 3.14, so we may take the Verdier localization $\mathcal{T}/C_{Z_U}$ which we informally think of as the category of objects supported away from $U$. We define $\mathcal{T}(U)$ to be the idempotent completion of the objects supported away from $U$, i.e. $\mathcal{T}(U) := (\mathcal{T}/C_{Z_U})^e$, and denote the composition $\mathcal{T} \to \mathcal{T}/C_{Z_U} \to \mathcal{T}(U)$ as $\mathrm{res}_U$. The functor $\mathrm{res}_U$ sends $\mathbb{1}$ to $\mathbb{1}_{\mathcal{T}(U)}$ which we will just denote $\mathbb{1}_U$. Note that $\mathrm{res}_U$ is also the localization $S^{-1}\mathcal{T}$ where

$$S = \{s : a \to b \mid \mathrm{supp}(\mathrm{cone}(s)) \subset Z_U\}$$

**Proposition 4.12.** *Given $\mathcal{T}$ and $U$ as above, the functor $\mathrm{res}_U$ is an exact $\otimes$-functor.*

*Proof.* The functor $\mathrm{res}_U$ is the composition of idempotent completion and Verdier localization at a thick $\otimes$-ideal, both of which are $\otimes$-exact functors. □

**Definition 4.13.** Define a presheaf of commutative rings on the poset category of quasi-compact opens $U \subseteq \mathrm{Spc}(\mathcal{T})$ by

$$\mathcal{F}(U) = \mathrm{End}_{\mathcal{T}(U)}(\mathbb{1}_U)$$

Now set $\mathcal{O}_{\mathcal{T}}$ to be the sheafification of $\mathcal{F}$ and call it the *structure sheaf* of $\mathcal{C}$. We write $\mathrm{Spec}(\mathcal{T}) = (\mathrm{Spc}(\mathcal{T}), \mathcal{O}_{\mathcal{T}})$ when we wish to refer to the Balmer spectrum as a ringed space.

We will soon show that $\mathcal{O}_{\mathcal{T}}$ is a sheaf of commutative rings (definition 4.19). Now observe that if $\mathcal{P} \in \mathrm{Spec}(\mathcal{T})$ then there is an isomorphism

$$\mathcal{O}_{\mathcal{T},\mathcal{P}} \cong \mathrm{End}_{\mathcal{T}/\mathcal{P}}(\mathbb{1}_{\mathcal{T}/\mathcal{P}})$$

This follows from the lemma below:

**Lemma 4.14.** *Let $\mathcal{T}$ be a tt-category and $\mathcal{P} \in \operatorname{Spec}(\mathcal{T})$. Then, there is an isomorphism*

$$\varinjlim_{\mathcal{P}\in U} \operatorname{Hom}_{\mathcal{T}(U)}(x,y) \xrightarrow{\sim} \operatorname{Hom}_{\mathcal{T}/\mathcal{P}}(x,y) \quad \forall x,y \in \mathcal{T}$$

*where $U$ varies through open subsets of* $\operatorname{Spc}(\mathcal{T})$ *containing $\mathcal{P}$.*

*Proof.* Note that since quasi-compact opens form a basis for $\operatorname{Spc}(\mathcal{T})$, is suffices to let $U$ vary over quasi-compact opens. Let $a, b \in \mathcal{T}$, $\mathcal{P} \in \operatorname{Spc}(\mathcal{T})$, and $U$ be a quasi-compact open set containing $\mathcal{P}$. Since idempotent completion is fully faithful, we have that $\operatorname{Hom}_{\mathcal{T}(U)}(a,b) = \operatorname{Hom}_{\mathcal{T}/C_{Z_U}}(a,b)$. Recall that morphisms in the localization $\mathcal{T}/C_{Z_U}$ are equivalence classes of fractions $a \xleftarrow{s} x \to b$ where $\operatorname{cone}(s) \in C_{Z_U}$, i.e. $\operatorname{supp}(\operatorname{cone}(s)) \subseteq Z_U$. But this is equivalent to $\operatorname{supp}(\operatorname{cone}(s)) \cap U = \emptyset$. Since $\mathcal{P} \in U$, this means $\mathcal{P} \notin \operatorname{supp}(\operatorname{cone}(s))$, i.e., $\operatorname{cone}(s) \in \mathcal{P}$. The collection of fractions $a \xleftarrow{s} x \to b$ where $\operatorname{cone}(s) \in \mathcal{P}$ is exactly $\operatorname{Hom}_{\mathcal{T}/\mathcal{P}}(a,b)$, so the claim follows. □

As previously stated, we will show in section 5.3 that $\operatorname{Spec}(\mathcal{T})$ is a locally ringed space, and by lemma 4.14 we just need to show that $\operatorname{End}_{\mathcal{T}/\mathcal{P}}(\mathbb{1}_{\mathcal{T}/\mathcal{P}})$ is a local ring.

Before going on, let's examine some of our running examples.

### Algebraic Geometry

Let $X$ be a quasi-separated and quasi-compact scheme. The universal homeomorphism of theorem 3.44

$$X \to \operatorname{Spec}(\mathrm{D}^{\mathrm{pf}}(X))$$

may be enhanced to an isomorphism of locally ringed spaces. Note that this immediately shows that $\operatorname{Spec}(\mathrm{D}^{\mathrm{pf}}(X))$ is a scheme. This is known as *Balmer's reconstruction theorem* since this shows that $X$ may be recovered from the derived category of perfect complexes on $X$, not just as a topological space, but as a scheme; see [Bal10]. This may seem surprising at first since there are non-isomorphic varieties with equivalent derived categories (notable examples can be found in Fourier-Mukai pairs coming from mirror symmetry). The fundamental difference here is that these categories are equivalent merely on a triangulated level, but not on the tensor triangulated level. Informally, this is like having rings which are isomorphic as additive groups, but not as rings.

### Modular Representation Theory

The following content summarizes material from [BIK11].

Let $G$ be a finite group and $k$ a field of positive characteristic dividing the order of $G$. We define the *group cohomology* of $G$ to be $H^*(G,k) \coloneqq \operatorname{Ext}^*_{kG}(k,k)$. It turns out that $\mathrm{H}^*(G,k)$ is a finitely generated graded-commutative $k$-algebra. Since $kG$-mod is highly non-commutative and sometimes hard to understand, it would be nice if we could cook up a good functor from $kG$-mod to $\mathrm{H}^*(G,k)$-mod, the latter of which is graded-commutative and therefore easier to understand. To this end, let $M \in kG$-mod. We can then realize $\operatorname{Ext}^*_{kG}(M,M)$ as an $\mathrm{H}^*(G,k)$-algebra via a graded-ring map $f_M : \operatorname{Ext}^*_{kG}(k,k) \to \operatorname{Ext}^*_{kG}(M,M)$. Explicitly, take $\eta$, an $n$-extension of $k$ by $k$, and send it to $\eta \otimes_k M \in \operatorname{Ext}^*_{kG}(M,M)$. We then define the *cohomological support variety* of $M$ to be

$$\mathcal{V}_G(M) \coloneqq \{\mathfrak{p} \in \operatorname{Spec}^{\mathrm{h}}(\mathrm{H}^*(G,k)) \mid \ker(f_M) \subseteq \mathfrak{p}\}$$

where $\operatorname{Spec}^{\mathrm{h}}(R)$ denotes the homogeneous prime spectrum of a ring $R$. We then have the following facts. See chapter 5 of [Ben98].

**Fact 4.15.** Let $M$ and $N$ be $kG$-modules.

(a) $\mathcal{V}_G(M) = \emptyset$ if and only if $M$ is projective.

(b) $\mathcal{V}_G(\oplus_{\alpha\in I} M_\alpha) = \bigcup_{\alpha\in I} \mathcal{V}_G(M_\alpha)$ for a finite indexing set $I$.

(c) $\mathcal{V}_G(M \otimes_k N) = \mathcal{V}_G(M) \cap \mathcal{V}_G(N)$

This suggests a support theory for $kG$-mod, but we'd like to get a support theory of a more homotopic flavor. First notice that $\mathcal{V}_G(M) = \emptyset$ for projective $M$ and recall that earlier we defined $\mathsf{stmod}(kG)$ as the category obtained by quotienting out morphisms that factor through a projective. Rickard showed that there is another way to construct the stable module category in [Ric89], seen below.

**Theorem 4.16.** *Let $A$ be a self injective $k$-algebra, that is, let $A$ be injective as a module over itself. The essential image of the embedding*

$$\mathrm{K^b}(\text{proj-}\,A)) \hookrightarrow \mathrm{D^b}(A)$$

*of the homotopy category of bounded complexes over the category of projective $A$-modules into the derived category of bounded complexes over $A$ is a thick subcategory. There is then a tensor triangulated equivalence of categories*

$$\mathsf{stmod}(A) \to \mathrm{D^b}(A)/\,\mathrm{K^b}(\mathrm{proj}-A)$$

*Proof.* See proposition 4.4.18 of [Kra21] for a concise proof. □

This leads to a support theory both for $\mathrm{D^b}(kG)$ and for $\mathsf{stmod}(kG)$.

**Theorem 4.17.** *Let $k$ be a field of positive characteristic and $G$ a finite group (scheme over $k$). Consider the graded-commutative cohomology ring $\mathrm{H}^\bullet(G,k)$. Then, for $\mathcal{T} = \mathrm{D^b}(kG\text{-mod})$, the comparison map $\rho^\bullet_\mathcal{T}$ of definition 5.9 induces*

$$\mathrm{Spec}(\mathrm{D^b}(kG\text{-}mod)) \cong \mathrm{Spec^h}(\mathrm{H}^*(G,k))$$

*Using theorem 4.16, this restricts to an isomorphism $\mathrm{Spec}(\mathsf{stmod}(kG)) \cong \mathrm{Proj}(\mathrm{H}^*(G,k))$ where the latter is the support variety $\mathcal{V}_G(k)$.*

We will revisit the theorem above in example 5.37.

### The Stable Homotopy Category

The Balmer spectrum for the stable homotopy category is a locally ringed space, but not a scheme. See the section on the stable homotopy category in [Bal10].

*Remark* 4.18. In algebraic geometry, fields are essential as they play the role of points for schemes, so it is natural to want to develop a notion of "tensor triangulated fields" or tt-fields. One reason for this comes from the desire for a purely tensor triangulated analog of homological support. Recall that the homological support of a complex $C_\bullet$ of $R$-modules is

$$\mathrm{supp^h}(C_\bullet) \coloneqq \{\mathfrak{p} \in \mathrm{Spec}(R) \mid C_\bullet \otimes^L_R \kappa(\mathfrak{p}) \neq 0\}$$

Homological support communicates directly with the monoidal structure of $\mathrm{D^{pf}}(R)$, and this makes it easier to work with than classical support. Finding and axiomatizing the correct notion of a tt-field is still an open problem in tensor triangulated geometry. Morally, the class of tt-fields should include $\mathrm{D}^{\mathrm{pf}}(\mathrm{Spec}(k))$ for all fields $k$, and they should satisfy various intuitive notions; for instance, $\mathrm{Spc}(\mathcal{T})$ should be a singleton. But it turns out to be a more complicated situation that it seems to be from the outset, and one needs to ask for additional restrictions to get anything approaching an appropriate notion. This direction of inquiry is explored in [BKS19].

## 4.3 The Central Ring

The goal of this section is lay the foundations for relating the spectrum of $\mathrm{End}_{\mathcal{T}}(\mathbb{1})$ to $\mathrm{Spec}(\mathcal{T})$. To this end, we define an action of $\mathrm{End}_{\mathcal{T}}(\mathbb{1})$ on the morphisms of $\mathcal{T}$ and then try to figure out how this action affects objects. The most obvious objects to investigate are the cones of certain well behaved morphisms. The most important results in this direction are proposition 4.33 and proposition 4.34 at the end of the section, where it is shown that if $f : x \to y$ is a morphism between invertible objects, then $\langle \mathrm{cone}(f) \rangle = \{a \in \mathrm{Obj}(\mathcal{T}) \mid f^{\otimes n} \otimes \mathrm{id}_a = 0 \text{ for some } n \geq 1\}$. This allows us to construct localizations of $\mathcal{T}$ by multiplicatively closed subsets of $\mathrm{End}_{\mathcal{T}}(\mathbb{1})$, which will be crucial in our effort to mine information about $\mathrm{Spec}(\mathcal{T})$ from the spectrum of $\mathrm{End}_{\mathcal{T}}(\mathbb{1})$.

We follow [Bal10] for this section.

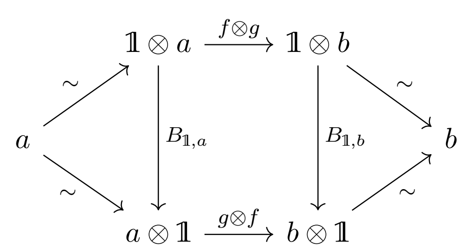
**Definition 4.19.** Let $\mathcal{T}$ be a tt-category. The *central ring* of $\mathcal{T}$ is defined to be

$$R_{\mathcal{T}} \coloneqq \mathrm{End}_{\mathcal{T}}(\mathbb{1})$$

It is not immediately obvious, but this ring is actually commutative due to the symmetric monoidal structure of $\mathcal{T}$. This fact can be seen as a corollary to the following proposition.

**Proposition 4.20.** *For all $a, b \in \mathcal{T}$, the group $\mathrm{Hom}_{\mathcal{T}}(a, b)$ is a left $R_{\mathcal{T}}$-module via $(f, g) \mapsto f \otimes g$ for $f \in R_{\mathcal{T}}$ and $g \in \mathrm{Hom}_{\mathcal{T}}(a, b)$ where we are identifying $\mathbb{1} \otimes a$ with $a$ and $\mathbb{1} \otimes b$ with $b$. This left action coincides with the right action $(g, f) \mapsto g \otimes f$ defined analogously. We simply denote this action $f \cdot g$. Given this structure, composition $\mathrm{Hom}_{\mathcal{T}}(b, c) \times \mathrm{Hom}_{\mathcal{T}}(a, b) \to \mathrm{Hom}_{\mathcal{T}}(a, b)$ becomes $R_{\mathcal{T}}$-bilinear.*

*Proof.* First we want to check that the left and right actions coincide, i.e. $f \otimes g = g \otimes f$. Consider the commutative diagram below:




where $B_{x,y}$ is the monoidal braiding of $\mathcal{T}$. The diagram commutes directly due to the axioms of a symmetric monoidal category. The rest of the proof is straightforward. □

**Definition 4.21.** If $u$ is an object of $\mathcal{T}$ such that there exists $v \in \mathcal{T}$ such that $u \otimes v \cong \mathbb{1}$ then we say that $u$ is *invertible*.

*Remark* 4.22. If $u \in \mathcal{T}$ is invertible, then so is $\Sigma^i(u)$ for all $i \in \mathbb{Z}$. Additionally, note that if $u \otimes v \cong \mathbb{1}$ in $\mathcal{T}$ and $\mathcal{T}$ is rigid, then $v \cong u^\vee$ necessarily.

*Remark* 4.23. Let $u$ be an invertible object of $\mathcal{T}$. Then for any $a, b \in \mathcal{T}$ there is an isomorphism

$$\mathrm{Hom}_{\mathcal{T}}(a,b) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{T}}(a\otimes u, b\otimes u) \quad \text{where} \quad f \mapsto f \otimes \mathrm{id}_u$$

induced by the functor $(-)\otimes u$. This is because $u$ is invertible if and only if $(-)\otimes u$ is an auto-equivalence. If we set $a$ and $b$ to be $\mathbb{1}$ and the invertible object to be $u \otimes u$ for $u$ invertible, then we get an isomorphism

$$R_{\mathcal{T}} = \mathrm{Hom}_{\mathcal{T}}(\mathbb{1},\mathbb{1}) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{T}}(u\otimes u, u\otimes u)$$
$$\epsilon \longmapsto \epsilon \cdot \mathrm{id}_{u\otimes u}$$

In particular, this means that there is an $\epsilon \in R_{\mathcal{T}}$ such that $\epsilon \cdot \mathrm{id}_{u\otimes u} = B_{u,u}$. In other words, the braiding $B_{u,u}$ on $u \otimes u$ is encoded in the $R_{\mathcal{T}}$-module structure of $\mathrm{Hom}_{\mathcal{T}}(u\otimes u, u\otimes u)$.

**Definition 4.24.** An object $x \in \mathcal{T}$ is said to have *central switch* if there exists a unit $\epsilon \in R_{\mathcal{T}}$ such that $\epsilon \cdot \mathrm{id}_{x\otimes x} = B_{x,x}$. In other words, $x$ has central switch if there is an element of the central ring that induces the braiding morphism on $x \otimes x$.

*Remark* 4.25. As we saw in remark 4.23, units of $\mathcal{T}$ have central switch. Additionally, if $x$ has central switch, then so does $x^{\otimes n}$ for all $n \in \mathbb{N}$. This property is desirable since it allows us to more freely manipulate compositions of morphisms, especially when we want to show that certain morphisms compose to zero.

**Proposition 4.26.** *Let $x$ be an object with central switch and let $f : a \to x$ and $g : b \to x$ be morphisms. Then there is an isomorphism $\tau : b \otimes a \to a \otimes b$ such that $g \otimes f = (f \otimes g) \circ \tau$. Similarly, for $f' : x \to a'$ and $g' : x \to b'$ there exists an isomorphism $\tau' : a' \otimes b' \to b' \otimes a'$ such that $g' \otimes f' = \tau' \circ (f' \otimes g')$.*

*Proof.* From the remark above we can find $\epsilon \in R_{\mathcal{T}}$ such that $B_{x,x} = \epsilon \cdot \mathrm{id}_{x\otimes x}$. Then the diagram below commutes

$$\begin{array}{ccc} b\otimes a & \xrightarrow{g\otimes f} & x\otimes x \\ \downarrow{\scriptstyle B_{b,a}} & & \downarrow{\scriptstyle B_{x,x}=\epsilon\cdot\mathrm{id}_{x\otimes x}} \\ a\otimes b & \xrightarrow{f\otimes g} & x\otimes x \end{array}$$

Since $\epsilon \cdot \epsilon = \mathrm{id}_{\mathbb{1}}$ we can multiply the two vertical maps by $\epsilon$ and set $\tau = \epsilon \cdot B_{x,x}$ to get the result. The second half of the claim is symmetric. ☐

**Lemma 4.27.** *Let $f : x \to y$ be a morphism in $\mathcal{T}$ and let the triangle below be exact:*

$$a \xrightarrow{k} b \xrightarrow{l} c \xrightarrow{m} \Sigma a$$

*Suppose that the diagram below commutes:*

$$\begin{array}{ccccccc} x\otimes a & \xrightarrow{\mathrm{id}_x\otimes k} & x\otimes b & \xrightarrow{\mathrm{id}_x\otimes l} & x\otimes c & \xrightarrow{\mathrm{id}_x\otimes m} & \Sigma(x\otimes a) \\ \downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle f\otimes\mathrm{id}_b} & & \downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle 0} \\ y\otimes a & \xrightarrow[\mathrm{id}_y\otimes k]{} & y\otimes b & \xrightarrow[\mathrm{id}_y\otimes l]{} & y\otimes c & \xrightarrow[\mathrm{id}_y\otimes m]{} & \Sigma(y\otimes a) \end{array}$$

*Then the morphism $f \otimes f \otimes \mathrm{id}_b : x\otimes x\otimes b \to y \otimes y \otimes b$ is zero.*

*Proof.* Note that since $(f\otimes \mathrm{id}_b)\circ(\mathrm{id}_x \otimes k) = 0$ by commutativity of the diagram, there is a morphism $h : x \otimes c \to y \otimes b$ and a factorization $f \otimes \mathrm{id}_b = h \circ (\mathrm{id}_x \otimes l)$ by lemma A.10. Hence,

$$f \otimes f \otimes \mathrm{id}_b = f \otimes (h \circ (\mathrm{id}_x \otimes l)) = (\mathrm{id}_y \otimes h) \circ (f \otimes \mathrm{id}_x \otimes l)$$

But then note that $f \otimes l = (\mathrm{id}_y \otimes l) \otimes (f \otimes \mathrm{id}_b) = 0$. Then this means that $0 = \mathrm{id}_x \otimes f \otimes l$, but then

$$f \otimes \mathrm{id}_x \otimes l = (B_{x,y} \otimes \mathrm{id}_c) \circ (\mathrm{id}_x \otimes f \otimes l) \circ (B_{x,x} \otimes \mathrm{id}_b) = (B_{x,y} \otimes \mathrm{id}_c) \circ 0 \circ (B_{x,x} \otimes \mathrm{id}_b) = 0$$

Putting the two equations above together, $f \otimes f \otimes \mathrm{id}_b = 0$. □

**Proposition 4.28.** *Let $f : x \to y$ be a morphism. Then the objects $a \in \mathcal{T}$ for which there exists $n \geq 1$ with $f^{\otimes n} \otimes \mathrm{id}_a = 0$ form a thick $\otimes$-ideal of $\mathcal{T}$.*

*Proof.* Define $\mathcal{J} \coloneqq \{a \in \mathrm{Obj}(\mathcal{T}) \mid \exists n \geq 1 \text{ s.t. } f^{\otimes n} \otimes \mathrm{id}_a = 0\}$. Now let $a, b \in \mathcal{J}$, so there exists $n \geq 1$ such that $f^{\otimes n} \otimes \mathrm{id}_a = f^{\otimes n} \otimes \mathrm{id}_b = 0$. Let $g : a \to b$ be a morphism. Then apply lemma 4.27 to the morphism $f^{\otimes n} : x^{\otimes n} \to y^{\otimes n}$ and the exact triangle below (under the appropriate rotation)

$$a \xrightarrow{g} b \xrightarrow{g_1} \mathrm{cone}(g) \xrightarrow{g_2} \Sigma a$$

We then have $f^{\otimes n+1} \otimes \mathrm{id}_{\mathrm{cone}(g)} = 0$, so $\mathrm{cone}(g) \in \mathcal{J}$. The fact that $\mathcal{J}$ is closed under $\Sigma$ follows easily since $\Sigma$ is an auto-equivalence, so $\mathcal{J}$ is a triangulated subcategory of $\mathcal{T}$. That $\mathcal{J}$ is thick and a $\otimes$-ideal follows almost immediately from the definition of $\mathcal{J}$. □

*Remark* 4.29. Recall that $x \otimes \mathrm{cone}(f) \cong \mathrm{cone}(f \otimes \mathrm{id}_x)$ from proposition 1.9.

**Lemma 4.30.** *Let $f : x \to y$ be a morphism where $x$ and $y$ both have central switch. Then $f \otimes f \otimes \mathrm{id}_{\mathrm{cone}(f)} \cong 0$.*

*Proof.* Let $f$ be as in the statement above, so we have an exact triangle

$$x \xrightarrow{f} y \xrightarrow{f_1} \mathrm{cone}(f) \xrightarrow{f_2} \Sigma x$$

This exact triangle fits into the diagram below. Note that the top and bottom rows of the diagram are just the triangle above after being tensored with $x$ and $y$ respectively. It is straightforward to check that the diagram commutes, so this is a morphism of triangles.

$$\begin{array}{ccccccc}
x \otimes x & \xrightarrow{\mathrm{id}_x \otimes f} & x \otimes y & \xrightarrow{\mathrm{id}_x \otimes f_1} & x \otimes \mathrm{cone}(f) & \xrightarrow{\mathrm{id}_x \otimes f_2} & \Sigma(x \otimes x) \\
{\scriptstyle f\otimes \mathrm{id}_x}\downarrow & & {\scriptstyle f\otimes \mathrm{id}_y}\downarrow & & {\scriptstyle f\otimes \mathrm{id}_{\mathrm{cone}(f)}}\downarrow & & {\scriptstyle \Sigma(f\otimes \mathrm{id}_x)}\downarrow \\
y \otimes x & \xrightarrow[\mathrm{id}_y \otimes f]{} & y \otimes y & \xrightarrow[\mathrm{id}_y \otimes f_1]{} & y \otimes \mathrm{cone}(f) & \xrightarrow[\mathrm{id}_y \otimes f_2]{} & \Sigma(y \otimes x)
\end{array}$$

Since $y$ has central switch, we may apply proposition 4.26 to $f \otimes \mathrm{id}_y$ and find an isomorphism $\tau : y \otimes x \to x \otimes y$ such that $(f \otimes \mathrm{id}_y) \circ \tau = \mathrm{id}_y \otimes f$. Then,

$$(\mathrm{id}_y \otimes f_1) \circ (f \otimes \mathrm{id}_y) \circ \tau = (\mathrm{id}_y \otimes f_1) \circ (\mathrm{id}_y \otimes f) = 0$$

and since $\tau$ is an isomorphism, it follows that the middle diagonal morphism is zero. The third square's diagonal is also zero for similar reasons, this time using the second statement of proposition 4.26 and the fact that $x$ has central switch. It follows that the diagram below is commutative:

$$\begin{array}{ccccccc} x \otimes x & \xrightarrow{\mathrm{id}_x \otimes f} & x \otimes y & \xrightarrow{\mathrm{id}_x \otimes f_1} & x \otimes \mathrm{cone}(f) & \xrightarrow{\mathrm{id}_x \otimes f_2} & \Sigma(x \otimes x) \\ {\scriptstyle 0}\downarrow & & {\scriptstyle 0}\downarrow & & {\scriptstyle f \otimes \mathrm{id}_{\mathrm{cone}(f)}}\downarrow & & {\scriptstyle 0}\downarrow \\ y \otimes x & \xrightarrow[\mathrm{id}_y \otimes f]{} & y \otimes y & \xrightarrow[\mathrm{id}_y \otimes f_1]{} & y \otimes \mathrm{cone}(f) & \xrightarrow[\mathrm{id}_y \otimes f_2]{} & \Sigma(y \otimes x) \end{array}$$

By rotating the diagram it follows from lemma 4.27 that $f \otimes f \otimes \mathrm{id}_{\mathrm{cone}(f)} = 0$. □

*Remark* 4.31. The case that we most care about applying lemma 4.30 to is when $x = \mathbb{1}$ and $y = \Sigma^d \mathbb{1}$ as we shall soon see. We are proving things in slightly more generality than we need since it comes at no extra cost and these more general results are useful for studying other rings associated to $\mathcal{T}$; and indeed there is merit to studying these other rings. See remark 4.45.

**Lemma 4.32.** *Let $\mathcal{E}$ be a collection of objects in $\mathcal{T}$.*

*(a) If $a \in \langle \mathcal{E} \rangle$ and $b \in \mathrm{Obj}(\mathcal{T})$, then $a \otimes b \in \langle \mathcal{E} \otimes b \rangle$.*

*(b) If $\langle \mathcal{E} \rangle = \mathcal{T}$, then for every $n \geq 1$ we have $\langle \{x^{\otimes n} \mid x \in \mathcal{E}\} \rangle = \mathcal{T}$ as well.*

*Proof.*

(a) Let $a \in \mathcal{E}$. Similar to the middle paragraph of proposition 1.14, we have that $\{x \in \mathrm{Obj}(\mathcal{T}) \mid x \otimes b \in \langle \mathcal{E} \otimes b \rangle\}$ is a thick $\otimes$-ideal. This thick tensor-ideal contains $\mathcal{E}$, and so it contains $\langle \mathcal{E} \rangle$, and therefore it contain $a$. Hence, $a \otimes b \in \langle \mathcal{E} \otimes b \rangle$.

(b) Suppose that $\langle \mathcal{E} \rangle = \mathcal{T}$, it follows that $\mathbb{1} \in \langle x_1, ..., x_m \rangle$ for some $x_1, ..., x_m \in \mathcal{E}$, so we may assume that $\mathcal{E}$ is finite. Denote $\mathcal{E}^{\otimes r} = \{y_1 \otimes y_r \mid y_1, ..., y_r \in \mathcal{E}\}$. From (a) we have that, for each $x_i$,

$$x_i \cong \mathbb{1} \otimes x_i \in \langle \mathcal{E} \otimes x_i \rangle \subseteq \langle \mathcal{E}^{\otimes 2} \rangle$$

But then $\mathbb{1} \in \langle \mathcal{E}^{\otimes r} \rangle$ for each $r \geq 1$. Since $\mathcal{E}$ is finite, for any $n \in \mathbb{N}$ there exists $r$ large enough such that $\mathcal{E}^{\otimes r} \subseteq \langle \{x^{\otimes n} \mid x \in \mathcal{E}\} \rangle$. But then $\mathbb{1} \in \langle x^{\otimes n} \mid x \in \mathcal{E} \rangle$. Therefore, $\mathcal{T} = \langle \{x^{\otimes n} \mid x \in \mathcal{E}\} \rangle$ for all $n \geq 1$.

□

**Proposition 4.33.** *Let $f : x \to y$ be a morphism in $\mathcal{T}$ such that $\langle x, y \rangle = \mathcal{T}$. If $a \in \mathrm{Obj}(\mathcal{T})$ such that $f \otimes \mathrm{id}_a = 0$ then $a \in \langle \mathrm{cone}(f) \rangle$.*

*Proof.* Let the morphism $f : x \to y$ and the object $a \in \mathrm{Obj}(\mathcal{T})$ be as in the proposition statement. Take the exact triangle $x \xrightarrow{f} y \xrightarrow{f_1} \mathrm{cone}(f) \xrightarrow{f_2} \Sigma x$ and tensor it with $a$ to get the exact triangle below:

$$x \otimes a \xrightarrow[=0]{f \otimes \mathrm{id}_a} y \otimes a \xrightarrow{f_1 \otimes \mathrm{id}_a} \mathrm{cone}(f) \otimes a \xrightarrow{f_2 \otimes \mathrm{id}_a} \Sigma x \otimes a$$

Since the first map is 0, it follows that $a \otimes \mathrm{cone}(f) \cong (y \otimes a) \oplus (\Sigma x \otimes a)$, so $x \otimes a$ and $y \otimes a$ lie in $\langle \mathrm{cone}(f) \rangle$ as the latter is a thick $\otimes$-ideal. Additionally, since $\mathbb{1} \in \langle x, y \rangle$, it follows from part (a) of lemma 4.32 that $a \cong a \in \langle x \otimes a, y \otimes a\rangle$. Therefore,

$$a \in \langle x \otimes a, y \otimes a\rangle \subseteq \langle \mathrm{cone}(f)\rangle$$

□

**Proposition 4.34.** *Let $f : x \to y$ be a morphism of $\mathcal{T}$ such that $\langle x, y\rangle = \mathcal{T}$, and $x$ and $y$ both have central switch, e.g. $x$ and $y$ are invertible. Then $\langle \mathrm{cone}(f)\rangle$ coincides with the subcategory of objects of $a \in \mathcal{T}$ for which $f^{\otimes n} \otimes \mathrm{id}_a = 0$ for some $n \geq 1$.*

*Proof.* Let $\mathcal{J} = \{a \in \mathcal{T} \mid \exists n \geq 1 \text{ such that } f^{\otimes n} \otimes \mathrm{id}_a = 0\}$ as in proposition 4.28. We need to show that $\mathcal{J} = \langle \mathrm{cone}(f)\rangle$. Note that by proposition 4.28 $\mathcal{J}$ is a thick triangulated $\otimes$-ideal, and by lemma 4.30 we have $\langle \mathrm{cone}(f)\rangle \subseteq \mathcal{J}$. It remains to show that $\mathcal{J} \subseteq \langle \mathrm{cone}(f)\rangle$.

Let $a \in \mathcal{J}$, so $f^{\otimes n} \otimes \mathrm{id}_a = 0$ for some $n$. Since $\langle x, y\rangle = \mathcal{T}$ it follows from (b) of lemma 4.32 that $\langle x^{\otimes n}, y^{\otimes n}\rangle = \mathcal{T}$. Then we have that $a \in \langle \mathrm{cone}(f^{\otimes n})\rangle$ by taking $f^{\otimes n}$ to be the $f$ in the statement of proposition 4.33. If we show that $\langle \mathrm{cone}(f^{\otimes n})\rangle \subseteq \langle \mathrm{cone}(f)\rangle$ then the result holds.

We will proceed via induction on $n$. The base case is a tautology, so suppose that the result holds up to $n \geq 1$, i.e. that $\mathrm{cone}(f^{\otimes n}) \in \langle \mathrm{cone}(f)\rangle$. Now decompose $f^{\otimes(n+1)} = (f \otimes \mathrm{id}_{y^{\otimes n}}) \circ (\mathrm{id}_x \otimes f^{\otimes n})$ and apply TR4 to this composition to yield that $\mathrm{cone}(f^{\otimes(n+1)}) \in \langle \mathrm{cone}(f \otimes \mathrm{id}_{y^{\otimes n}}), \mathrm{cone}(\mathrm{id}_x \otimes f^{\otimes n})\rangle$. By proposition 1.9 there are isomorphisms $\mathrm{cone}(f \otimes \mathrm{id}_{y^{\otimes n}}) \cong y^{\otimes n} \otimes \mathrm{cone}(f)$ and $\mathrm{cone}(\mathrm{id}_x \otimes f^{\otimes n}) \cong x \otimes \mathrm{cone}(f^{\otimes n})$. Then, we apply the inductive hypothesis and obtain

$$\mathrm{cone}(f^{\otimes(n+1)}) \in \langle \mathrm{cone}(f \otimes \mathrm{id}_{y^{\otimes n}}), \mathrm{cone}(\mathrm{id}_x \otimes f^{\otimes n})\rangle = \langle y^{\otimes n} \otimes \mathrm{cone}(f), x \otimes \mathrm{cone}(f^{\otimes n})\rangle \subseteq \langle \mathrm{cone}(f)\rangle$$

thereby finishing the proof. □

**Proposition 4.35.** *Let $f : x \to y$ be a morphism, and assume the same hypotheses as proposition 4.34. Then*

$$\langle \mathrm{cone}(f)^{\otimes n}\rangle = \langle \mathrm{cone}(f^{\otimes n})\rangle = \langle \mathrm{cone}(f)\rangle \quad \forall n \in \mathbb{N}$$

*Proof.* It follows directly from proposition 4.34 that $\langle \mathrm{cone}(f)\rangle = \langle \mathrm{cone}(f^{\otimes n})\rangle$, and clearly $\mathrm{cone}(f)^{\otimes n} \in \langle \mathrm{cone}(f)\rangle$, so the claim will be proved if $\mathrm{cone}(f) \in \langle \mathrm{cone}(f)^{\otimes n}\rangle$.

We will induct on $n$. If we apply TR4 to the composition below,

$$(x \otimes x) \otimes \mathrm{cone}(f)^{\otimes n} \xrightarrow{f \otimes \mathrm{id} \otimes \mathrm{id}} (y \otimes x) \otimes \mathrm{cone}(f)^{\otimes n} \xrightarrow{\mathrm{id} \otimes f \otimes \mathrm{id}} (y \otimes y) \otimes \mathrm{cone}(f)^{\otimes n}$$

Since $\mathrm{cone}(f) \otimes x \otimes \mathrm{cone}(f)^{\otimes n}$ and $y \otimes \mathrm{cone}(f) \otimes \mathrm{cone}(f)^{\otimes n}$ are both contained in $\langle \mathrm{cone}(f)^{\otimes(n+1)}\rangle$, it must be that

$$\mathrm{cone}(f \otimes f \otimes \mathrm{id}_{\mathrm{cone}(f)^{\otimes n}}) \in \langle \mathrm{cone}(f)^{\otimes(n+1)}\rangle$$

But $f \otimes f \otimes \mathrm{id}_{\mathrm{cone}(f)^{\otimes n}} = 0$ by lemma 4.30. Since the cone of the zero morphism $a \xrightarrow{0} b$ for any objects $a$ $b$ is the direct sum $\Sigma(a) \oplus b$, we then have

$$\mathrm{cone}(f \otimes f \otimes \mathrm{id}_{\mathrm{cone}(f)^{\otimes n}}) \cong \Sigma(x^{\otimes 2} \otimes \mathrm{cone}(f)^{\otimes n}) \oplus (y^{\otimes 2} \otimes \mathrm{cone}(f)^{\otimes n})$$

Since thick $\otimes$-ideals are closed under summands,

$$\langle x^{\otimes 2} \otimes \mathrm{cone}(f)^{\otimes n}, y^{\otimes 2} \otimes \mathrm{cone}(f)^{\otimes n}\rangle \in \langle \mathrm{cone}(f)^{\otimes(n+1)}\rangle$$

It then follows from parts (a) and (b) of lemma 4.32 that

$$\mathrm{cone}(f)^{\otimes n} \in \langle \mathrm{cone}(f)^{\otimes(n+1)}\rangle$$

□

### 4.4 Localization and the Graded Central Ring

The central ring $R_{\mathcal{T}}$ can capture a lot of information about $\mathcal{T}$ in certain cases, but in many cases it is more useful to look at the *graded central ring*, of which the central ring is just the degree 0 part.

**Definition 4.36.** Let $\mathcal{T}$ be a tt-category. Given any two objects $a, b \in \mathcal{T}$ we use the notation below:

$$\operatorname{Hom}^n_{\mathcal{T}}(a,b) \coloneqq \operatorname{Hom}_{\mathcal{T}}(a, \Sigma^n b)$$

We then define the abelian group of $\mathbb{Z}$-graded homomorphisms

$$\operatorname{Hom}^*_{\mathcal{T}}(a,b) \coloneqq \bigoplus_{n\in\mathbb{Z}} \operatorname{Hom}_{\mathcal{T}}(a, \Sigma^n b)$$

There is an obvious composition

$$\operatorname{Hom}^j_{\mathcal{T}}(b,c) \times \operatorname{Hom}^i_{\mathcal{T}}(a,b) \longrightarrow \operatorname{Hom}^{i+j}_{\mathcal{T}}(a,c)$$

where

$$(g,f) \longmapsto g \diamond f \coloneqq a \xrightarrow{(\Sigma^j g)\circ f} \Sigma^{i+j} c$$

which of course is just the usual composition when $i = j = 0$.

**Definition 4.37.** The *graded central ring* is

$$R^*_{\mathcal{T}} \coloneqq \operatorname{Hom}^*_{\mathcal{T}}(\mathbb{1}, \mathbb{1})$$

When there is no confusion about what category we are in, we will just write $R$ or $R^*$.

Note that $R^0 = R$, and similar to $R$, there is a graded left action of $R^*$ on $\operatorname{Hom}^*_{\mathcal{T}}(a,b)$ given by

$$R^n \times \operatorname{Hom}^m_{\mathcal{T}}(a,b) \to \operatorname{Hom}^{n+m}_{\mathcal{T}}(a,b)$$

$$(r,f) \mapsto [r \cdot f \coloneqq r \otimes f : a \cong \mathbb{1} \otimes a \longrightarrow \Sigma^n \mathbb{1} \otimes \Sigma^m b \cong \Sigma^{n+m} b]$$

There is also a graded right action defined analogously. This action agrees with the left action up to a sign, making $R^*$ into a graded-commutative algebra and every $\operatorname{Hom}^*_{\mathcal{T}}(a,b)$ into a left and right graded $R^*$-module via the tensor product. Composition

$$\operatorname{Hom}^j_{\mathcal{T}}(b,c) \times \operatorname{Hom}^i_{\mathcal{T}}(a,b) \to \operatorname{Hom}^{i+j}_{\mathcal{T}}(a,c)$$

is $R^*$-bilinear up to a sign, i.e. for $r \in R^i$ and $f \in \operatorname{Hom}^j_{\mathcal{T}}(a,b)$

$$r \cdot f = (-1)^{i+j} f \cdot r$$

The graded-commutative nature of $R^*$ comes from the Koszul sign rule of definition 1.4.

**Construction 4.38.** Let $\mathcal{T}$ be a tt-category. Let $S$ be a homogeneous multiplicatively closed (abbreviated to m.c.) central subset of $R^*$. Then we can localize $R^*$ to acquire the graded-commutative algebra $S^{-1}R^*$. Since $\operatorname{Hom}^*_{\mathcal{T}}(a,b)$ is an $R^*$-module, we also get $S^{-1}\operatorname{Hom}^*_{\mathcal{T}}(a,b)$ as an $S^{-1}R^*$-module. Denote $(S^{-1}\operatorname{Hom}_{\mathcal{T}}(a,b))^0$ as the degree 0 component. There are then well defined $S^{-1}R^*$-homomorphisms

$$(S^{-1}\operatorname{Hom}^*_{\mathcal{T}}(b,c))^0 \times (S^{-1}\operatorname{Hom}^*_{\mathcal{T}}(a,b))^0 \longrightarrow (S^{-1}\operatorname{Hom}^*_{\mathcal{T}}(a,c))^0$$

We can then generate a new category $S^{-1}\mathcal{T}$ which has the same objects as $\mathcal{T}$ and has morphisms

$$\mathrm{Hom}_{S^{-1}\mathcal{T}}(a,b) \coloneqq (S^{-1}\,\mathrm{Hom}^*_{\mathcal{T}}(a,b))^0$$

The canonical morphism of $R^*$-modules $\mathrm{Hom}_{\mathcal{T}}(a,b) \to (S^{-1}\,\mathrm{Hom}^*(a,b))^0$ yields a functor

$$q_S : \mathcal{T} \to S^{-1}\mathcal{T}$$

We want to show that $q_S$ in construction 4.38 is actually a Verdier localization, so to that end we prove two preliminary results.

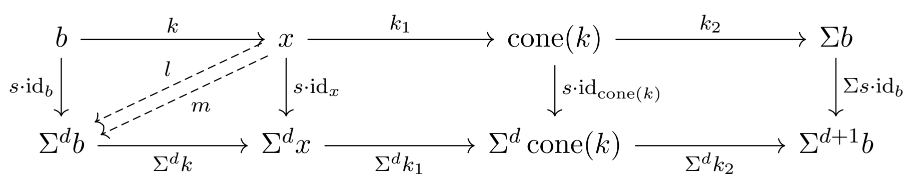
**Lemma 4.39.** *With notation as in construction 4.38, define* $\mathcal{J} := \langle \mathrm{cone}(s) \mid s \in S\rangle$. *Then,*

$$\mathcal{J} = \{c \in \mathcal{T} \mid \exists s \in S \text{ such that } s \cdot \mathrm{id}_c = 0\}$$

*Proof.* This follows immediately from proposition 4.34. ☐

**Lemma 4.40.** *With notation as in construction 4.38 and lemma 4.39, a morphism $k : b \to x$ in $\mathcal{T}$ has its cone in $\mathcal{J}$ if and only if there exists $s \in S$ of degree $d$ for some $d \in \mathbb{Z}$ and $l, m \in \mathrm{Hom}^d_{\mathcal{T}}(x,b)$ such that $l \circ k = s \cdot \mathrm{id}_b \in \mathrm{Hom}^d_{\mathcal{T}}(b,b)$ and $k \diamond m = s \cdot \mathrm{id}_x \in \mathrm{Hom}^d_{\mathcal{T}}(x,x)$.*

*Proof.* To set this up, consider the commutative diagram below. The top row is the exact triangle coming from $k : b \to x$ and the bottom row is just the $d$-th shift of the top triangle. The diagram commutes because the column maps are just the action of $s \in R^d_{\mathcal{T}}$ on the identities.




Suppose that $\mathrm{cone}(k) \in \mathcal{J}$. By lemma 4.39, we may pick $s \in S$ such that $s \cdot \mathrm{id}_{\mathrm{cone}(k)} = 0$. Then the diagonals of the middle and right square vanish by commutativity. By rotation of the triangle and application of lemma A.10, $l$ and $m$ exist as in the diagram such that $l \circ k = s \cdot \mathrm{id}_b$ and $\Sigma^d(k) \circ m = s \cdot \mathrm{id}_x$.

Now suppose that $l$ and $m$ satisfying our conditions exist. Then,

$$(s \cdot \mathrm{id}_{\mathrm{cone}(k)}) \circ k_1 = \Sigma^d(k_1) \circ (s \cdot \mathrm{id}_x) = \Sigma^d(k_1) \circ \Sigma^d(k) \circ m = \Sigma^d(k_1 \circ k) \circ m = 0$$

Analogously, $\Sigma^d(k_2) \circ (s \cdot \mathrm{id}_{\mathrm{cone}(k)}) = 0$. By lemma 4.27, $(s^2) \cdot \mathrm{id}_{\mathrm{cone}(f)} = 0$, and since $S$ is m.c., we have that $s^2 \in S$ so $\mathrm{cone}(k) \in \mathcal{J}$ by lemma 4.39 ☐

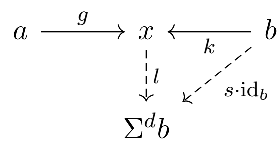
**Theorem 4.41.** *Keep the notation as above and let $q : \mathcal{T} \to \mathcal{T}/\mathcal{J}$ be the canonical Verdier localization. There is an equivalence $\alpha : S^{-1}\mathcal{T} \xrightarrow{\sim} \mathcal{T}/\mathcal{J}$ such that $\alpha \circ q_S = q$. This endows $S^{-1}\mathcal{T}$ with the structure of a tt-category such that $q_S : \mathcal{T} \to S^{-1}\mathcal{T}$ is a morphism of tt-categories, i.e. a $\otimes$-exact functor.*

Readers who are not yet comfortable with Verdier localization may want to just skim the next proof.

*Proof.* Define $\alpha : S^{-1}\mathcal{T} \to \mathcal{T}/\mathcal{J}$ to be the identity on objects, which we can do since $S^{-1}\mathcal{T}$ and $\mathcal{T}/\mathcal{J}$ share the same collection of objects as $\mathcal{T}$ by definition. Let $a, b$ be objects of $\mathcal{T}$. Note that $\text{cone}(s \cdot \text{id}_b) \cong b \otimes \text{cone}(s) \in \mathcal{J}$. This forces $q(s \cdot \text{id}_b)$ to be an isomorphism in $\mathcal{T}/\mathcal{J}$, as the cone of a morphism is in the kernel of a Verdier localization if and only if the image of the morphism is an isomorphism. We now define a natural morphism

$$\begin{aligned} \alpha_{a,b} : (S^{-1}\operatorname{Hom}^*_{\mathcal{T}}(a,b))^0 &\longrightarrow \operatorname{Hom}_{\mathcal{T}/\mathcal{J}}(a,b) \\ \tfrac{f}{s} &\longmapsto q(s\cdot \text{id}_b)^{-1} \circ f \end{aligned}$$

The goal is to show that $\alpha_{a,b}$ is an isomorphism. Take a morphism $k^{-1}g$ in $\operatorname{Hom}_{\mathcal{T}/\mathcal{J}}(a,b)$ represented by $a \xrightarrow{g} x \xleftarrow{k} b$ where $\text{cone}(k) \in \mathcal{J}$. Then lemma 4.40 yields a morphism $l \in \operatorname{Hom}^d_{\mathcal{T}}(x,b)$ such that $l \circ k = s \cdot \text{id}_b$ for some $s \in S$. Then, $k^{-1}g = \alpha_{a,b}(lg/s)$ as shown in the diagram below:




Hence, $\alpha_{a,b}$ is surjective. For injectivity, suppose that $q(s \cdot \text{id}_b)^{-1} \circ f = 0$. Then since $q(s \cdot \text{id}_b)$ is an isomorphism, it must be that $f$ is the zero map. □

**Corollary 4.42.** *If $\mathfrak{p}^* \in \operatorname{Spec}^{\text{h}}(R^*_{\mathcal{T}})$ then the localization $\mathcal{T}_{\mathfrak{p}^*} := S^{-1}_{\mathfrak{p}^*}\mathcal{T}$ where $S_{\mathfrak{p}^*} = \{s \in R^{even}_{\mathcal{T}} \mid s \notin \mathfrak{p}^*\}$ has graded central ring $\operatorname{End}^*_{\mathcal{T}_{\mathfrak{p}^*}}(\mathbb{1}) \cong (R^*_{\mathcal{T}})_{\mathfrak{p}^*}$.*

*Remark* 4.43. So far we have been localizing in a graded manner, but that doesn't have to be the case. This is because if $R^*$ is a graded ring and $M^*$ a graded $R^*$-module and $S \subseteq R^0$ an m.c. subset, then $(S^{-1}M^*)^0 = S^{-1}(M^0)$.

**Corollary 4.44.** *Let $S \subseteq R_{\mathcal{T}}$ be a m.c. subset. Then the Verdier localization $S^{-1}\mathcal{T}$ is equivalent to $\mathcal{T}/\mathcal{J}$ where $\mathcal{J} = \langle \text{cone}(s) \mid s \in S\rangle$. In particular, if $\mathfrak{p} \in \operatorname{Spec}(R_{\mathcal{T}})$ then $\mathcal{T}_{\mathfrak{p}} := S^{-1}_{\mathfrak{p}}\mathcal{T}$ where $S_{\mathfrak{p}} := R_{\mathcal{T}} \setminus \mathfrak{p}$ has central ring isomorphic to $(R_{\mathcal{T}})_{\mathfrak{p}}$.*

*Remark* 4.45. The presentation of the material in sections 4.4 and 4.3 are not as general as was originally stated in [Bal10]. Balmer constructs the central graded ring $R^*_{\mathcal{T},u}$ with respect to an invertible object $u \in \mathcal{T}$ as below:

$$R^*_{\mathcal{T},u} := \bigoplus_{i\in\mathbb{Z}} \operatorname{Hom}^i_{\mathcal{T}}(\mathbb{1}, u^{\otimes i})$$

This can be useful, but for readability and brevity I decided to stick to the case $u = \Sigma(\mathbb{1})$ since we will only need the case when $u = \mathbb{1}$.

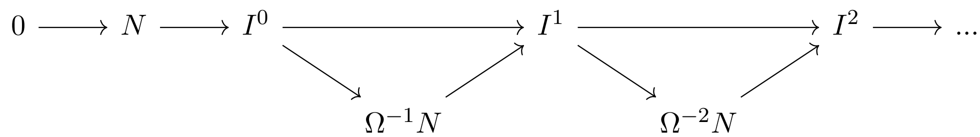
**Example 4.46.** Consider $\mathcal{T} = \mathrm{D}^{\text{pf}}(R)$ for a commutative ring $R$. Since the monoidal unit of $\mathrm{D}^{\text{pf}}(R)$ is a complex with $R$ concentrated in degree 0, it is clear that $R_{\mathrm{D}^{\text{pf}}(R)} \cong R$. The action of $f \in R_{\mathrm{D}^{\text{pf}}(R)}$ on $\operatorname{Hom}_{\mathrm{D}^{\text{pf}}(R)}(C_\bullet, D_\bullet)$ is simply to multiply the chain maps by $r$. Since $\operatorname{Hom}_{\mathrm{D}^{\text{pf}}(R)}(R, R[i]) = 0$ for non-zero $i \in \mathbb{Z}$, we can see that $R^*_{\mathrm{D}^{\text{pf}}(R)} = R_{\mathrm{D}^{\text{pf}}(R)}$.

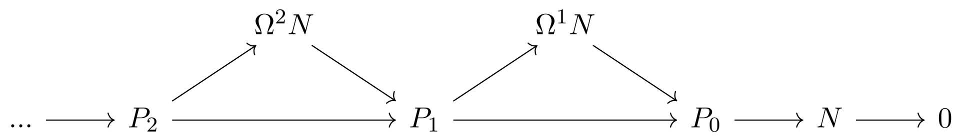
The previous example is a case where the central ring and the central graded ring coincide since the higher and lower degree components vanish. In the next section we will explore a case where the central graded ring is very rich in structure, as it recovers the notion of Tate cohomology.

## 4.5 Tate cohomology

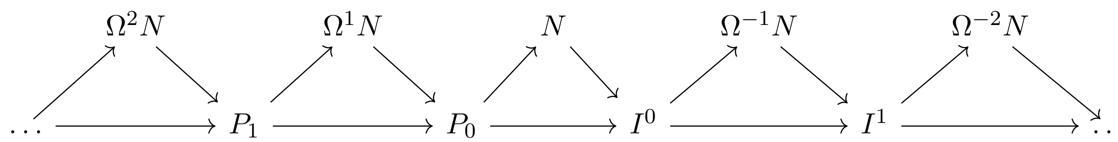
We will use [Kra21] as a reference for this section.

Let $A$ be a finite dimensional Hopf algebra over a field $k$. Readers unfamiliar with Hopf algebras should just think of $A$ as a group ring $kG$ of a finite group $G$. Let $N$ be a representation of $A$. Since injectives and projectives coincide in this context, the only difference between injective and projective resolutions is whether or not they are indexed homologically or cohomologically. Let $P_\bullet$ be a projective resolution of $N$ and $I^\bullet$ an injective resolution of $N$. In 1.3.2 we defined the syzygies $\Omega^n N$ as seen in the diagrams below.




We can then splice the resolutions together to get a complex




**Definition 4.47.** The complex constructed above is called the *Tate resolution* (or sometimes *complete resolution*) of $N$. We will denote it $tN$

*Remark* 4.48. In definition 4.47 we use the definite article "the" even though the construction is not unique; rather, it is unique up to homotopy. Another way to look at $tN$ is to take projective and injective resolutions $pN$ and $iN$ and to use the quasi-isomorphisms $pN \xrightarrow{j} N$ and $N \xrightarrow{i} iN$ to construct the exact triangle below:

$$pN \xrightarrow{ij} iN \to tN \to \Sigma pN$$

So $tN$ is the cone of the composition $ij$ in the homotopy category of chain complexes in $\mathrm{Rep}_A$.

**Definition 4.49.** Let $M, N \in \mathrm{Rep}_A$. First, reindex the projective part of the Tate resolution cohomologically. We then define the $n$-th *Tate extension* as below

$$\widehat{\mathrm{Ext}}^n(M,N) := \widehat{\mathrm{H}}^n(\mathrm{Hom}_A(M, tN))$$

and *Tate Cohomology*

$$\widehat{\mathrm{H}}^n(A,M) := \widehat{\mathrm{Ext}}^n_A(k,M)$$

If we look at the Tate resolution we can see that it's not clear where we should put our zero index. However, because of the definition of $tN$ we can immediately see that $\widehat{\mathrm{Ext}}^n_A(M,N)$ should agree with good old fashioned $\mathrm{Ext}^n_A(M,N)$ for $n > 0$. Therefore, we stipulate that $I^0$ occupies the zeroth index of $tN$ in the resolution constructed above. We saw in 1.3.2 that $\mathrm{Ext}^1_A(M,N) \cong \underline{\mathrm{Hom}}_A(M,\Omega^{-1}N)$. This turns out to be the $n = 1$ case of a more general relationship. Let $n \geq 0$ and let $0 \to \Omega^{-n+1}N \to I \to \Omega^{-n}N \to 0$ be a short exact sequence from which arise the exact sequences

$$\mathrm{Hom}_A(M,I) \to \mathrm{Hom}_A(M,\Omega^{-n}) \to \mathrm{Ext}^1_A(M,\Omega^{-n+1}N) \to \mathrm{Ext}^1_A(M,I)$$

and

$$\mathrm{Ext}^{i-1}_A(M,I) \to \mathrm{Ext}^{i-1}_A(M,\Omega^{-n},N) \to \mathrm{Ext}^i_A(M,\Omega^{-n+1}N) \to \mathrm{Ext}^i_A(M,I)$$

for all $i > 0$. When we pass to the stable module category, we have that $\underline{\mathrm{Hom}}_A(M,I) \cong \mathrm{Ext}^1_A(M,I) \cong 0$ as $I$ is both injective and projective, so

$$\underline{\mathrm{Hom}}_A(M,\Omega^n N) \cong \mathrm{Ext}^1_A(M,\Omega^{n-1}N) \quad \text{and} \quad \mathrm{Ext}^{i-1}_A(M,\Omega^n N) \cong \mathrm{Ext}^i_A(M,\Omega^{n-1}N)$$

for all $i > 0$. Hence, for $n > 0$ there are isomorphisms,

$$\underline{\mathrm{Hom}}_A(M,\Omega^{-n}N) \cong \mathrm{Ext}^1_A(M,\Omega^{-n+1}N) \cong \mathrm{Ext}^2_A(M,\Omega^{-n+2}N) \cong \ldots \cong \mathrm{Ext}^n_A(M,N)$$

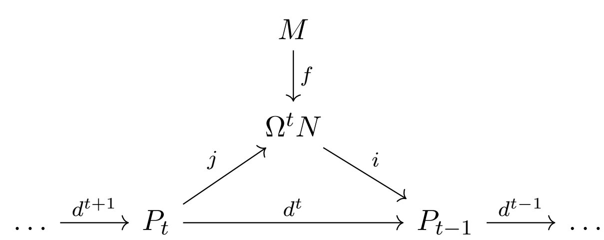
We generalize this in the following proposition.

**Proposition 4.50.** *For $M, N \in \mathrm{Rep}_A$ there are isomorphisms*

$$\widehat{\mathrm{Ext}}^n_A(M,N) \cong \underline{\mathrm{Hom}}_A(M,\Omega^{-n}N)$$

*and $\widehat{\mathrm{Ext}}^n_A(M,N) \cong \mathrm{Ext}^n_A(M,N)$ for $n > 0$.*

*Proof.* We already proved the claim for $n \geq 0$, so it remains to check $n \leq -1$. To avoid confusion, I want to clarify that for the remainder of the proof we will be in negative cohomological indices $n < -1$ which correspond to the homologically indexed part of $tN$, so really just looking at the projective resolution $pN$. Set $t = -n$. Let $f \in \mathrm{Hom}_A(M,\Omega^t N)$. We then have the diagram below:




The map $i$ induces $i_* : \mathrm{Hom}_A(M,\Omega^{-t}N) \to \mathrm{Hom}_A(M,P_{t-1})$, and since $\mathrm{im}\, d_t = \ker d^{t-1} = \Omega^t N$, we see that $d^{t-1}_* \circ i_* = 0$ and therefore we have a composition below:

$$\alpha : \mathrm{Hom}_A(M,\Omega^t N) \to \ker d^t_* \to \frac{\ker d^t_*}{\mathrm{im}\, d^{t+1}_*} = \mathrm{H}^n \,\mathrm{Hom}_A(M,tN) = \widehat{\mathrm{Ext}}^n_A(M,tN)$$

where $d^t_*$ is the map induced on $\mathrm{Hom}_A(M,P_t)$. If $\alpha$ is surjective and $\ker(\alpha) = \mathrm{PHom}_A(M,\Omega^t N)$ then the claim will have been proved.

Let $g \in \mathrm{Hom}_A(M, P_{t-1})$ such that $d^{t-1} \circ g = 0$. Then $\mathrm{im}(g) \subseteq \mathrm{im}(d^t) \cong \Omega^t N$, so we can factor $g$ through $\Omega^t N$. This proves the surjectivity of $\alpha$. Now suppose that $\alpha(f) = 0$, i.e. $i \circ f \in \mathrm{im}(d^t_*)$. By definition, this means that $i \circ f$ factors through a morphism $M \xrightarrow{h} P_t$. By commutativity of the diagram,

$$i \circ f = d^t \circ h = i \circ j \circ h$$

Since $i$ is injective it follows that $f = j \circ h$, and since $P_t$ is projective we have $f \in \mathrm{PHom}_A(M, \Omega^t N)$. On the other hand, suppose that $f \in \mathrm{PHom}_A(M, \Omega^t)$. Then $f$ factors through a map $P' \xrightarrow{\phi} \Omega^t N$ where $P'$ is projective, but then $j : P_t \to \Omega^t N$ is a surjection, so $\phi$ factors through $j$, which forces $f$ to also factor through $j$. Then $f \in \ker(\alpha)$. Hence, $\alpha$ is an isomorphism. □

**Corollary 4.51.** *For a finite dimensional Hopf algebra $A$ over a field $k$ we have*

$$\mathrm{Hom}^*_{\mathsf{StMod}(A)}(M, N) := \bigoplus_{n \in \mathbb{Z}} \underline{\mathrm{Hom}}_A(M, \Omega^{-n} N) = \widehat{\mathrm{Ext}}^*_A(M, N)$$

In the previous two sections we were exploring the action of $R^*_{\mathcal{T}}$ on $\mathrm{Hom}_{\mathcal{T}}(A, B)$ in a tt-category $\mathcal{T}$. In $\mathrm{Rep}_A$ the monoidal unit is the one dimensional representation $k$, and so in $\mathcal{T} = \mathsf{stmod}(A)$ we have

$$R^*_{\mathsf{stmod}(A)} := \mathrm{End}^*_{\mathcal{T}}(\mathbb{1}) = \widehat{\mathrm{Ext}}^*_A(k, k) = \widehat{\mathrm{H}}^*(A, k)$$

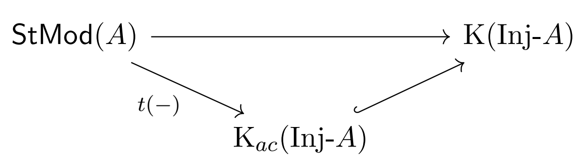
In the previous two sections we saw an action of $R^*_{\mathcal{T}}$ on $\mathrm{Hom}^*_{\mathcal{T}}(M, N)$:

$$R^i_{\mathcal{T}} \times \mathrm{Hom}^j_{\mathcal{T}}(M, N) \to \mathrm{Hom}^{i+j}_{\mathcal{T}}(M, N)$$

and using the isomorphisms constructed above we can see this action explicitly within $\mathsf{stmod}(A)$:

$$\widehat{\mathrm{H}}^n(A, k) \times \widehat{\mathrm{Ext}}^m_A(M, N) \cong \underline{\mathrm{Hom}}_A(k, \Omega^{-n} k) \times \underline{\mathrm{Hom}}_A(M, \Omega^{-n} N) \to \underline{\mathrm{Hom}}_A(M, \Omega^{-(m+n)} N)$$

If we take $M \in \mathrm{Rep}_A$, the Tate resolution $tM$ is of course a chain complex, so it lies in $\mathrm{Ch}(A)$, but more specifically it lies in $\mathrm{Ch}(\mathrm{Inj}\text{-}A)$, the category of injective chain complexes. As remarked earlier, the assignment $M \to tM$ is not functorial in $\mathrm{Ch}(\mathrm{Inj}\text{-}A)$, but it is functorial when we pass to $\mathrm{K}(\mathrm{Inj}\text{-}A)$ since injective resolutions are equivalent up to homotopy. Therefore we can regard $t(-)$ as a functor $\mathsf{StMod}(A) \to \mathrm{K}(\mathrm{Inj}\text{-}A)$, and since $tM$ is acyclic it actually factors as below:




But we can always recover $M$ from $tM$ by taking the zero-th syzygy. This induces an equivalence of tt-categories between $\mathsf{StMod}(A)$ and $\mathrm{K}_{ac}(\mathrm{Inj}\text{-}A)$. See 4.4.18 of [Kra21].

# 5 Local tt-categories and the comparison map

In this chapter we will continue to follow [Bal10] for sections 5.1, 5.2, 5.3, and 5.4. We will assume that $\mathcal{T}$ is essentially small for the entire chapter.

The previous chapter we realized $\mathrm{Spec}(\mathcal{T}) = (\mathrm{Spc}(\mathcal{T}), \mathcal{O}_{\mathcal{T}})$ as a ringed space and built up some machinery around the central ring $R_{\mathcal{T}}$ and the graded central ring $\mathbb{R}^{\bullet}_{\mathcal{T}}$. In this chapter we will show that $\mathrm{Spec}(\mathcal{T})$ is not just ringed, but locally ringed. This is done by defining the notion of a *local tt-category* topologically, and then showing that the central rings of local tt-categories are in fact local rings. We will then employ the machinery developed in section 4.4 to develop the comparison map $\mathrm{Spc}(\mathcal{T}) \to \mathrm{Spec}(R_{\mathcal{T}})$, which is a morphism of spectral spaces relating Balmer spectra to Zariski spectra.

## 5.1 Local tt-categories

**Definition 5.1.** A tt-category is called *local* if $\mathrm{Spc}(\mathcal{T})$ is a local topological space, that is, if $\mathrm{Spc}(\mathcal{T}) = \bigcup_{\alpha \in \Omega} U_\alpha$ is an open cover of $\mathrm{Spc}(\mathcal{T})$ then $U_\alpha = \mathrm{Spc}(\mathcal{T})$ for some $\alpha \in \Omega$.

**Proposition 5.2.** *The following are equivalent:*

*(a) $\mathcal{T}$ is local as defined in definition 5.1.*

*(b) The space $\mathrm{Spc}(\mathcal{T})$ has a unique closed point.*

*(c) The category $\mathcal{T}$ has a unique minimal prime.*

*(d) The ideal $\sqrt{0} \subset \mathcal{T}$ of $\otimes$-nilpotent objects is the minimal prime of $\mathcal{T}$.*

*(e) For any objects $a, b \in \mathcal{T}$, if $a \otimes b = 0$ then $a$ or $b$ is $\otimes$-nilpotent.*

*Moreover, if $\mathcal{T}$ is rigid, then the above is further equivalent to:*

*(f) if $a \otimes b = 0$ then $a = 0$ or $b = 0$.*

*(g) $\langle 0 \rangle$ is the unique closed point.*

*Proof.*

(a) $\Longrightarrow$ (b). Argue by the contrapositive. If $\mathcal{P}$ and $\mathcal{Q}$ were distinct closed points then the cover $\mathrm{Spc}(\mathcal{T}) = (\mathrm{Spc}(\mathcal{T}) \setminus \mathcal{P}) \cup (\mathrm{Spc}(\mathcal{T}) \setminus \mathcal{Q})$ is an open cover of $\mathrm{Spc}(\mathcal{T})$ that cannot be reduced to a single constituent open set as in definition 5.1, so $\mathcal{T}$ is not local.

(b) $\Longrightarrow$ (a). Any closed set of $\mathrm{Spc}(\mathcal{T})$ contains a closed point by remark 2.24. If there is only one closed point $\mathcal{M}$ of $\mathrm{Spc}(\mathcal{T})$, then any open subset of $U \subset \mathrm{Spc}(\mathcal{T})$ containing $\mathcal{M}$ must have empty complement, i.e. $U = \mathrm{Spc}(\mathcal{T})$

(b) $\Longleftrightarrow$ (c). By proposition 2.20, the closure of $\mathcal{P} \in \mathrm{Spc}(\mathcal{T})$ is $\overline{\{\mathcal{P}\}} = \{\mathcal{Q} \in \mathrm{Spc}(\mathcal{T}) \mid \mathcal{Q} \subseteq \mathcal{P}\}$, so the closed points of $\mathrm{Spc}(\mathcal{T})$ are exactly the minimal primes of $\mathcal{T}$.

(c) $\Longleftrightarrow$ (d). This follows immediately from corollary 2.13, which is exactly the statement that

$$\bigcap_{\mathcal{P} \in \mathrm{Spc}(\mathcal{T})} \mathcal{P} = \sqrt{0}$$

(d) $\Longleftrightarrow$ (e). This is immediate from the fact that $a \in \sqrt{0}$ if and only if $a^{\otimes n} = 0$ for some $n$.

If $\mathcal{T}$ is rigid then all $\otimes$-ideals are radical by proposition 3.35. Consequently, $\sqrt{0} = \langle 0 \rangle$. $\square$

*Remark* 5.3. Observe that proposition 5.2 tells us that $\mathcal{T}/\mathcal{P}$ is indeed a local tt-category since the nilradical becomes prime and is therefore the unique closed point of $\mathrm{Spc}(\mathcal{T}/\mathcal{P})$.

Proposition 5.2 may seem backwards since in algebra we usually associate the term 'local' with the property of having a unique maximal ideal, but here it corresponds to having a unique minimal prime $\otimes$-ideal. According to intuition from traditional ring theory, this looks more like being an integral domain. This phenomenon is once again related to the fact that the Balmer's spectrum is topologically the Hochster dual of the topology coming from the distributive lattice of thick $\otimes$-ideals of $\mathcal{T}$. The following example should dispel any doubt that definition 5.1 is the correct definition.

**Example 5.4.** If $R$ is a commutative ring then $\mathrm{D}^{\mathrm{pf}}(R)$ is a local tt-category if and only if $R$ is a local ring. To see this, recall the universal morphism of support data from theorem 3.44, which yields the homeomorphism below:

$$\begin{array}{ccc} \mathrm{Spec}(R) & \xrightarrow{\sim} & \mathrm{Spc}(\mathrm{D}^{\mathrm{pf}}(R)) \\ \mathfrak{p} & \longmapsto & \mathcal{P}(\mathfrak{p}) \coloneqq \{M_\bullet \in \mathrm{D}^{\mathrm{pf}}(a) \mid (M_\bullet)_{\mathfrak{p}} \simeq 0\} \end{array}$$

The critical observation is that the map above is inclusion reversing, so $\mathfrak{p} \subset \mathfrak{q} \iff \mathcal{P}(\mathfrak{p}) \supset \mathcal{P}(\mathfrak{q})$. Hence, the maximal primes of $R$ correspond to minimal prime $\otimes$-ideals of $\mathrm{D}^{\mathrm{pf}}(R)$. Therefore, $R$ is local if and only if $\mathrm{Spc}(\mathrm{D}^{\mathrm{pf}}(R))$ has a unique closed point, which is the case if and only if $\mathrm{D}^{\mathrm{pf}}(R)$ has a unique minimal prime $\otimes$-ideal.

In fact, the example above is indicative of the more general result below:

**Theorem 5.5.** *Let $\mathcal{T}$ be a local tt-category. Then,*

1. *$R^*_{\mathcal{T}}$ is a local graded ring, and*
2. *$R_{\mathcal{T}}$ is a local ring.*

*Proof.* Define an ideal

$$\mathfrak{m} \coloneqq \langle \text{homogeneous non-invertible elements of } R^*_{\mathcal{T}} \rangle$$

We must prove that this ideal is maximal, so let $f, g \in R^d_{\mathcal{T}}$. Note that if $f$ or $g$ are both non-invertible, i.e. non-isomorphisms, then $f \cdot g$ must be non-invertible. To show that $\mathfrak{m}$ is closed under addition, is suffices to show that if $f+g$ is invertible then either $f$ or $g$ is invertible, so assume that $f+g$ is invertible. Define

$$\phi \coloneqq (f+g) \cdot (\mathrm{id}_{\mathrm{cone}(f)} \otimes \mathrm{id}_{\mathrm{cone}(g)}) \in \mathrm{End}^*_{\mathcal{T}}(\mathrm{cone}(f) \otimes \mathrm{cone}(g))$$

Since $f+g$ is an isomorphism, $\phi$ is clearly invertible. Note that since $f$ and $g$ have invertible objects ($\mathbb{1}$ and $\Sigma^d \mathbb{1}$ respectively) as their source and target it follows from lemma 4.30 that $f \otimes f \otimes \mathrm{id}_{\mathrm{cone}(f)} = g \otimes g \otimes \mathrm{id}_{\mathrm{cone}(g)} = 0$. Then, by expanding out $(f+g)^3$ we can see that $(f+g)^3 \cdot (\mathrm{id}_{\mathrm{cone}(f)} \otimes \mathrm{id}_{\mathrm{cone}(g)}) = 0$, so in fact $\phi^3 = 0$. Since $\phi$ is both invertible and nilpotent in the ring $\mathrm{End}^*_{\mathcal{T}}(\mathrm{cone}(f) \otimes \mathrm{cone}(g))$ it follows that this ring is the 0 ring, so $\mathrm{cone}(f) \otimes \mathrm{cone}(g) = 0$. From proposition 5.2 it follows that either $\mathrm{cone}(f)$ or $\mathrm{cone}(g)$ is $\otimes$-nilpotent. Assume that this is the case for $f$ without loss of generality. By proposition 4.35

$$\langle 0 \rangle = \langle \mathrm{cone}(f)^{\otimes n} \rangle = \langle \mathrm{cone}(f) \rangle$$

Hence $\mathrm{cone}(f) = 0$ and therefore $f$ is an isomorphism, i.e. invertible. □

**Proposition 5.6.** *For $\mathcal{P} \in \mathrm{Spc}(\mathcal{T})$, the stalks of $\mathcal{F}_\mathcal{T}$ and $\mathcal{O}_\mathcal{T}$ are naturally isomorphic to $R_{\mathcal{T}/\mathcal{P}}$.*

*Proof.* The result follows immediately from lemma 4.14 by setting $x = y = \mathbb{1}$. □

**Corollary 5.7.** *If $\mathcal{T}$ is a tt-category then $\mathrm{Spec}(\mathcal{T})$ is a locally ringed space.*

*Proof.* Note that for any $\mathcal{P} \in \mathrm{Spc}(\mathcal{T})$ the tt-category $\mathcal{T}/\mathcal{P}$ is local by remark 5.3. The result then follows from proposition 5.6 and theorem 5.5. □

**Example 5.8.** The converse to theorem 5.5 is not true. For example, set $\mathcal{T} = \mathrm{D}^{\mathrm{pf}}(X)$ where $X = \mathbb{P}^n_k$ with $k$ a field. Then $R_\mathcal{T} = \Gamma(X, \mathcal{O}_X) = k$ which is obviously local, while $\mathrm{D}^{\mathrm{pf}}(X)$ is not local as a tt-category since $\mathrm{Spc}(\mathrm{D}^{\mathrm{pf}}(X)) \cong X$.

### 5.2 The comparison map

In 3.2 we learned about methods for understanding $\mathrm{Spc}(\mathcal{T})$ by mapping continuously into $\mathrm{Spc}(\mathcal{T})$ using support data. Now we will look at a way to study $\mathrm{Spc}(\mathcal{T})$ through continuous maps out of $\mathrm{Spc}(\mathcal{T})$.

**Definition 5.9.** Given a tt-category $\mathcal{T}$ we define a map $\rho^*$ from $\mathrm{Spc}(\mathcal{T})$ to the homogeneous prime spectrum of $\mathbb{R}^*_\mathcal{T}$, denoted $\mathrm{Spec}^{\mathrm{h}}(R^*_\mathcal{T})$, where

$$\rho^*_\mathcal{T}(\mathcal{P}) := \langle f \text{ homogeneous } \in R^*_\mathcal{T} \mid \mathrm{cone}(f) \notin \mathcal{P} \rangle$$

We call $\rho^*_\mathcal{T}$ the *comparison map.*

For each $\mathcal{P} \in \mathrm{Spc}(\mathcal{T})$ there is a Verdier localization $\mathcal{T} \to \mathcal{T}/\mathcal{P}$ which induces a homomorphism $R^*_\mathcal{T} \to R^*_{\mathcal{T}/\mathcal{P}}$ on their central graded endomorphism rings. We now want to investigate the comparison map $\rho^*$ and how it relates to localizations of tt-categories.

**Theorem 5.10.** *Let $\mathcal{T}$ be a tt-category and let $\rho^* : \mathrm{Spc}(\mathcal{T}) \to \mathrm{Spec}^{\mathrm{h}}(\mathcal{T})$ be the map in definition 5.9. Given $\mathcal{P} \in \mathrm{Spc}(\mathcal{T})$, let $q : \mathcal{T} \to \mathcal{T}/\mathcal{P}$ be the associated localization functor and $q_* : R^*_\mathcal{T} \to R^*_{\mathcal{T}/\mathcal{P}}$ the associated ring map.*

*(a) $\rho^*(\mathcal{P})$ is a homogeneous prime ideal in $R^*_\mathcal{T}$.*

*(b) If $\mathfrak{m}$ is the unique homogeneous maximal ideal in $R^*_{\mathcal{T}/\mathcal{P}}$ then $\rho^*(\mathcal{P}) = q_*^{-1}(\mathfrak{m})$.*

*(c) The map $\rho^*$ is continuous. That is, for $s \in R^*_\mathcal{T}$, the preimage of the principal open $D(s) \subset \mathrm{Spec}^{\mathrm{h}}(R^*_\mathcal{T})$ is the open set $U(\mathrm{cone}(s)) \subset \mathrm{Spc}(\mathcal{T})$.*

*(d) $\rho^*$ defines a natural transformation between the contravariant functors $\mathrm{Spc}(-) \to \mathrm{Spec}^{\mathrm{h}}(R^*_-)$.*

*Proof.* The statement of (b) implies (a), so it suffices to prove (a). Since the functor $q : \mathcal{T} \to \mathcal{T}/\mathcal{P}$ is a ⊗-exact functor, it must be that $q(\mathrm{cone}(f)) \cong \mathrm{cone}(q_* f)$ for any $f \in R^d_{\mathcal{T}/\mathcal{P}}$. Then,

$$\begin{aligned} \mathrm{cone}(f) \in \mathcal{P} &\iff q(\mathrm{cone}(f)) \cong 0 \\ &\iff \mathrm{cone}(q_* f) \cong 0 \\ &\iff q_* f \text{ invertible} \\ &\iff q_* f \notin \mathfrak{m} \end{aligned}$$

Therefore,

$$q_*^{-1}(\mathfrak{m}) = \{f \in R_{\mathcal{T}}^* \mid q_* f \in \mathfrak{m}\} = \langle f \in R_{\mathcal{T}}^{hom} \mid \operatorname{cone}(f) \notin \mathcal{P} \rangle = \rho^*(\mathcal{P})$$

which is the statement of (b).

Let $s \in R_{\mathcal{T}}^{hom}$ and $D(s)$ be the principle open set of $s$, i.e. $D(s) \coloneqq \{\mathfrak{p} \in \operatorname{Spec}^{\mathrm{h}}(R_{\mathcal{T}^*}) \mid s \notin \mathfrak{p}\}$. Then,

$$\begin{aligned}
(\rho^*)^{-1}(D(s)) &= \{\mathcal{P} \in \operatorname{Spc}(\mathcal{T}) \mid s \notin \rho^*(\mathcal{P})\} \\
&= \{\mathcal{P} \in \operatorname{Spc}(\mathcal{T}) \mid s \notin \langle f \mid \operatorname{cone}(f) \notin \mathcal{P} \rangle\} \\
&= \{\mathcal{P} \in \operatorname{Spc}(\mathcal{T}) \mid \operatorname{cone}(s) \in \mathcal{P}\} \\
&= U(\operatorname{cone}(s))
\end{aligned}$$

which is the claim of (c).

For (d), let $F : \mathcal{T} \to \mathcal{T}'$ be a $\otimes$-exact functor between tt-categories. There is an induced ring homomorphism $R_{\mathcal{T}}^* \to R_{\mathcal{T}'}^*$ induced by $F$ where $f \mapsto F(f)$, so we will also call this ring homomorphism $F$. Denote $\rho_{\mathcal{T}}^*$ and $\rho_{\mathcal{T}'}^*$ to be the comparison maps for $\mathcal{T}$ and $\mathcal{T}'$ respectively. The claim comes down to showing that the diagram below commutes:

$$\begin{array}{ccc}
\operatorname{Spc}(\mathcal{T}') & \xrightarrow{\operatorname{Spc} F} & \operatorname{Spc}(\mathcal{T}) \\
{\scriptstyle \rho_{\mathcal{T}'}^*}\downarrow & & \downarrow{\scriptstyle \rho_{\mathcal{T}}^*} \\
\operatorname{Spec}^{\mathrm{h}}(R_{\mathcal{T}'}^*) & \xrightarrow[\operatorname{Spec}^{\mathrm{h}} F]{} & \operatorname{Spec}^{\mathrm{h}}(R_{\mathcal{T}}^*)
\end{array}$$

Recall from proposition 3.18 that $(\operatorname{Spc} F)(\mathcal{Q}') = F^{-1}(\mathcal{Q}')$ for $\mathcal{Q}' \in \operatorname{Spc}(\mathcal{T}')$. Then,

$$\begin{aligned}
f \in \rho_{\mathcal{T}}^*(\operatorname{Spc} F(\mathcal{Q}')) &\iff \operatorname{cone}(f) \notin (\operatorname{Spc} F)(\mathcal{Q}') = F^{-1}(\mathcal{Q}') \\
&\iff F(\operatorname{cone}(f)) \cong \operatorname{cone}(F(f)) \notin \mathcal{Q}' \\
&\iff F(f) \in \rho_{\mathcal{T}'}^*(\mathcal{Q}') \\
&\iff f \in F^{-1}(\rho_{\mathcal{T}'}^*(\mathcal{Q}')) = (\operatorname{Spec}^{\mathrm{h}} F)(\rho_{\mathcal{T}'}^*(\mathcal{Q}'))
\end{aligned}$$

Hence, commutativity of the diagram holds. □

*Remark* 5.11. Part (c) and (d) of theorem 5.10 tell us that $\rho_{\mathcal{T}}^*$ is a morphism of spectral spaces, so $\rho_{\mathcal{T}}^*$ is actually a natural transformation between functors into the category of spectral spaces. In light of Stone duality, the comparison map is equivalent to a morphism from the lattice of homogeneous prime ideals of $R_{\mathcal{T}}^*$ to the lattice of prime $\otimes$-ideals of $\mathcal{T}$.

The following theorem shows that localization in the central endomorphism ring communicates with localization of the larger tt-category.

**Theorem 5.12.** *Let $S \subseteq R_{\mathcal{T}}^{hom}$ be a central multiplicatively closed subset and let $q : \mathcal{T} \to S^{-1}\mathcal{T}$ be* *the corresponding localization functor. Then the diagram*

$$\begin{array}{ccc} \operatorname{Spc}(S^{-1}\mathcal{T}) & \xhookrightarrow{\operatorname{Spc} q} & \operatorname{Spc}\mathcal{T} \\ \downarrow{\scriptstyle \rho^*_{S^{-1}\mathcal{T}}} & & \downarrow{\scriptstyle \rho^*_{\mathcal{T}}} \\ \operatorname{Spec}^{\mathrm{h}}(S^{-1}R^*_{\mathcal{T}}) = \operatorname{Spec}^{\mathrm{h}}(R^*_{S^{-1}\mathcal{T}}) & \hookrightarrow & \operatorname{Spec}^{\mathrm{h}}(R^*_{\mathcal{T}}) \end{array}$$

*commutes and is cartesian, that is,* $\operatorname{Spc}(S^{-1}\mathcal{T}) \cong \{\mathcal{P} \in \operatorname{Spc}(\mathcal{T}) \mid S \cap \rho^*_{\mathcal{T}}(\mathcal{P}) = \emptyset\}$

*Proof.* The diagram commutes due to the naturality of $\rho^*$. From theorem 4.41, we know that $S^{-1}\mathcal{T} \cong \mathcal{T}/\mathcal{J}$ where $\mathcal{J} = \langle \operatorname{cone}(s) \mid s \in S\rangle$. By corollary 3.20, $\operatorname{Spc}(q)$ is a homeomorphism onto its image, i.e. $\operatorname{Spc}(S^{-1}\mathcal{T}) \cong \{\mathcal{P} \in \operatorname{Spc}(\mathcal{T}) \mid \mathcal{J} \subseteq \mathcal{P}\}$. By definition, $\mathcal{J} \subset \mathcal{P}$ if and only if $\operatorname{cone}(s) \in \mathcal{P}$ for all $s \in S$, so $s \notin \rho^*_{\mathcal{T}}(\mathcal{P})$ for all $s \in S$ by the definition of $\rho^*_{\mathcal{T}}$. □

*Remark* 5.13. Let $\mathfrak{p}^* \in \operatorname{Spec}^{\mathrm{h}}(R^*_{\mathcal{T}})$ and set $S = R^*_{\mathcal{T}} \setminus \mathfrak{p}^*$. If $\mathfrak{m}$ is the maximal ideal of $S^{-1}R^*_{\mathcal{T}}$ then theorem 5.12 shows that if $\mathfrak{m} \in \operatorname{im}(\rho^*_{S^{-1}\mathcal{T}})$ then $\mathfrak{p}^* \in \operatorname{im}(\rho^*_{\mathcal{T}})$.

We have been focusing on the graded central ring $R^*_{\mathcal{T}}$ for a little while now, but now we can return to $R_{\mathcal{T}}$, which is just the 0-th graded component of $R^*_{\mathcal{T}}$.

*Remark* 5.14. There is a natural transformation $(-)^0 : \operatorname{Spec}^{\mathrm{h}}(-) \to \operatorname{Spec}(-^0)$ where $\operatorname{Spec}(-^0)$ takes the spectrum of the 0-th component of a graded ring. In other words, given a graded ring $A^*$ there is a continuous map $\operatorname{Spec}^{\mathrm{h}}(A^*) \to \operatorname{Spec}(A^0)$, where $\mathfrak{p}^* \mapsto \mathfrak{p} = \mathfrak{p}^* \cap R^0$, that is natural with respect to graded ring homomorphisms. This continuous map is surjective. To see this, first assume that $A^0$ is local with maximal ideal $\mathfrak{m}$. Then if $\mathfrak{q}^*$ is a homogeneous prime containing $\mathfrak{m} \cdot A^*$ then $\mathfrak{m} = \mathfrak{q}^* \cap A^0$ by maximality of $\mathfrak{m}$. In the non-local case, take $\mathfrak{p} \in \operatorname{Spec}(A^0)$ and set $S = A^0 \setminus \mathfrak{p}$. Note that $S$ is a homogeneous multiplicatively closed subset of $A^*$, so we can localize to $S^{-1}A^0 = A^0_{\mathfrak{p}}$ and $S^{-1}A^*$. By the universal property of localization there is a map $A^0_{\mathfrak{p}} \to S^{-1}A^*$, and in fact $A^0_{\mathfrak{p}} = (S^{-1}A^*)^0$, so we reduce to the local case from earlier.

**Definition 5.15.** Let $\mathcal{T}$ be a tt-category and $\mathcal{P} \in \operatorname{Spc}(\mathcal{T})$. Then define $\rho_{\mathcal{T}} : \operatorname{Spc}(\mathcal{T}) \to \operatorname{Spec}(R_{\mathcal{T}})$ as

$$\rho_{\mathcal{T}}(\mathcal{P}) = \{\mathcal{P} \in \operatorname{Spc}(\mathcal{T}) \mid \operatorname{cone}(s) \notin \mathcal{P}\}$$

**Corollary 5.16.** *Let $\mathcal{T}$ be a tt-category. Then,*

*(a) $\rho_{\mathcal{T}}(\mathcal{P})$ is well defined, i.e. $\rho_{\mathcal{T}}(\mathcal{P})$ is a prime ideal of $R_{\mathcal{T}}$.*

*(b) $\rho_{\mathcal{T}}$ is continuous and natural in $\mathcal{T}$. Additionally, the diagram below commutes.*

$$\begin{array}{ccc} \operatorname{Spc}(\mathcal{T}) & \xrightarrow{\rho^*_{\mathcal{T}}} & \operatorname{Spec}^{\mathrm{h}}(R^*_{\mathcal{T}}) \\ & {\scriptstyle \rho_{\mathcal{T}}} \searrow & \downarrow{\scriptstyle (-)^0} \\ & & \operatorname{Spec}(R_{\mathcal{T}}) \end{array}$$

(c) *Let $S \subset R_{\mathcal{T}}$ be a m.c. set and $q : \mathcal{T} \to S^{-1}\mathcal{T}$ the corresponding localization. Then the diagram below commutes and* $\operatorname{Spc}(S^{-1}\mathcal{T}) \cong \{\mathcal{P} \in \operatorname{Spc}(\mathcal{T}) \mid S \cap \rho_{\mathcal{T}}(\rho) = \emptyset\}$

$$\begin{array}{ccc} \operatorname{Spc}(S^{-1}\mathcal{T}) & \xhookrightarrow{\operatorname{Spc} q} & \operatorname{Spc}\mathcal{T} \\ \downarrow{\scriptstyle \rho_{S^{-1}\mathcal{T}}} & & \downarrow{\scriptstyle \rho_{\mathcal{T}}} \\ \operatorname{Spec}(S^{-1}R_{\mathcal{T}}) = \operatorname{Spec}(R_{S^{-1}\mathcal{T}}) & \hookrightarrow & \operatorname{Spec}(R_{\mathcal{T}}) \end{array}$$

*Proof.* Follows from discussion in remark 5.14, as well as theorem 5.12 and theorem 5.10. □

Now that we know that $\rho^*$ can help get a handle on $\operatorname{Spc}(\mathcal{T})$, there are some natural questions to ask:

1. Is $\rho^*$ a morphism of locally ringed spaces?
2. When is $\rho^*$ surjective?
3. When is $\rho^*$ injective?
4. What does $\rho^*$ do to the support of an object?

Three out of four of these questions will be explored in the following subsections. Interestingly, most results about $\rho$ being injective are actually cases when $\rho$ is bijective as it turns out that it is much harder to determine when $\rho$ is simply just injective. This should not be too surprising since $\operatorname{Spec}^{\mathrm{h}}(R^*_{\mathcal{T}})$ is really just capturing a small piece of information internal to the generally *much* larger structure of $\mathcal{T}$.

## 5.3 The comparison map as a locally ringed morphism

For the duration of 5.3 we will require that all our tt-categories be rigid. We require this to ensure that all thick ⊗-ideals are radical.

Let's return to the structure sheaf of $\operatorname{Spc}(\mathcal{T})$. Recall that in definition 4.13 we assigned to each quasi-compact open set $U \subset \operatorname{Spc}(\mathcal{T})$ a new tt-category $\mathcal{T}(U)$ which was defined as below:

$$\mathcal{T}(U) \coloneqq (\mathcal{T}/C_{Z_U})^e$$

where $Z_U = \operatorname{Spc}(\mathcal{T}) \setminus U$, $C_{Z_U}$ is the thick ⊗-ideal $\{a \in \mathcal{T} \mid \operatorname{supp}(a) \subset \mathcal{T}\}$, and $(-)^e$ denotes the idempotent completion. Although the notation $R_{\mathcal{T}}$ had not yet been introduced in this text, we defined the presheaf of rings below:

$$\mathcal{F}(U) \coloneqq R_{\mathcal{T}(U)}$$

Note that since $\operatorname{Spc}(\mathcal{T})$ is a spectral space, it has a basis of quasi-compact open sets, so it suffices to define the presheaf on quasi-compact open sets. The restriction maps come from the fact that any morphism $\mathcal{T} \to \mathcal{K}$ of tt-categories induces a ring morphism $R_{\mathcal{T}} \to R_{\mathcal{K}}$. This presheaf is in terms of the central endomorphism rings, but there is no reason that we couldn't have done the same construction for the central graded endomorphism ring. In this vein we define the presheaf

$$\mathcal{F}^*(U) \coloneqq R^*_{\mathcal{T}(U)}$$

If we wish to keep track of the specific tt-category that we are talking about, we will write $\mathcal{F}_{\mathcal{T}}$ or $\mathcal{F}^*_{\mathcal{T}}$.

**Definition 5.17.** Given a tt-category $\mathcal{T}$ define $\mathcal{O}^*_{\mathcal{T}}$ to be the sheafification of $\mathcal{F}^*_{\mathcal{T}}$. Further define the ringed space $\operatorname{Spec}^*(\mathcal{T}) := (\operatorname{Spc}(\mathcal{T}), \mathcal{O}^*_{\mathcal{T}})$.

**Theorem 5.18.** *For $\mathcal{P} \in \operatorname{Spc}(\mathcal{T})$, the stalks of $\mathcal{F}^*_{\mathcal{T}}$ and $\mathcal{O}^*_{\mathcal{T}}$ are naturally isomorphic to $R^*_{\mathcal{T}}$. Furthermore, $\operatorname{Spec}^*(\mathcal{T})$ is a locally ringed space.*

*Proof.* The proof is the same as the one leading up to corollary 5.7 except we just change our sheaf to $\mathcal{O}^*_{\mathcal{T}}$ and we set $x = \mathbb{1}$ and $y = \Sigma^d \mathbb{1}$ for all $d \in \mathbb{Z}$ when applying lemma 4.14. □

Our goal now is to show that the continuous comparison maps $\rho_{\mathcal{T}}$ and $\rho^*_{\mathcal{T}}$ may be enhanced into maps of locally ringed spaces. Recall that a morphism of ringed spaces $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ consists of a continuous map $f : X \to Y$ along with ring homomorphisms $f^\# : \mathcal{O}_Y(U) \to \mathcal{O}_X(f^{-1}(U))$ for all open $U \subseteq Y$ that is compatible with the restriction ring morphisms.

Going forward, we will only talk about the graded case, but all of the analogous results hold in the ungraded case.

**Lemma 5.19.** *Let $\mathcal{T}$ be rigid and denote $X = \operatorname{Spec}^{\mathrm{h}}(R^*_{\mathcal{T}})$. For every even degree $s \in R^{hom}_{\mathcal{T}}$ there is a natural isomorphism between the sections of the sheaf $\mathcal{O}^*_X$ over the principal open set $D(s) \subset \operatorname{Spec}^{\mathrm{h}}(R^*_{\mathcal{T}})$ and the sections of $\mathcal{F}^*_{\mathcal{T}}$ over $(\rho^*_{\mathcal{T}})^{-1}(D(s)) = U(\operatorname{cone}(s))$. In particular, they are isomorphic to $R^*_{\mathcal{T}}[1/s]$.*

*Proof.* Denote $U(s) := U(\operatorname{cone}(s)) \subset \operatorname{Spc}(\mathcal{T})$ to condense notation. By definition, $\mathcal{O}^*_X(R^*_{\mathcal{T}}) = R^*_{\mathcal{T}}[1/s]$. Recall from theorem 5.10 that $(\rho^*_{\mathcal{T}})^{-1}(D(s)) = U(s)$. By definition $\mathcal{F}^*_{\mathcal{T}}(U(s))$ is the central graded ring of $\mathcal{T}/C_{Z_{U(s)}}$, so it is this ring that we want to analyze.

By definition, $Z_{U(s)} = \operatorname{supp}(s)$. Since $\mathcal{T}$ is rigid, all thick $\otimes$-ideals are radical, so from the main classification theorem we have $C_{\operatorname{supp}(\operatorname{cone}(s))} = \langle \operatorname{cone}(s) \rangle$. If we then set $S = \{s^i \mid i > 0\}$ then we get the expression below from theorem 4.41:

$$\begin{aligned} \mathcal{F}^*_{\mathcal{T}}(U(s)) &= R^*_{\mathcal{T}/\langle \operatorname{cone}(s) \rangle} \\ &\cong R^*_{S^{-1}\mathcal{T}} \\ &= S^{-1} R^*_{\mathcal{T}} \\ &= R^*_{\mathcal{T}}[1/s] \end{aligned}$$

□

**Construction 5.20.** We now want to use the previous lemma to construct a morphism of locally ringed spaces. For every even degree $s \in R^*_{\mathcal{T}}$, let $U(s) := U(\operatorname{cone}(s))$ which we know is equal to $\rho^*_{\mathcal{T}}(D(s)) \subset \operatorname{Spc}(\mathcal{T})$. Furthermore, let $(X, \mathcal{O}^*_X)$ be the affine scheme $\operatorname{Spec}^{\mathrm{h}}(R^*_{\mathcal{T}})$. Then define,

$$r_{D(s)} : \mathcal{O}^*_X(D(s)) \to \mathcal{O}^*_{\mathcal{T}}(U(s))$$

as the composition $\mathcal{O}^*_X(D(s)) \xrightarrow{\sim} \mathcal{F}^*_{\mathcal{T}}(U(s)) \to \mathcal{O}^*_{\mathcal{T}}(U(s))$ where the first map is the isomorphism of the previous lemma and the second is the canonical sheafification morphism. It is straightforward to check that this map is compatible with the restriction morphisms. By lemma 5.19, this defines a morphism of ringed spaces

$$(\rho^*_{\mathcal{T}}, r) : \operatorname{Spec}^*(\mathcal{T}) \to \operatorname{Spec}^{\mathrm{h}}(R^*_{\mathcal{T}})$$

It remains to check that this is a morphism of *locally* ringed spaces.

**Theorem 5.21.** *Let $\mathcal{T}$ be rigid. Then, the comparison map $(\rho^*_{\mathcal{T}}, r) : \mathrm{Spec}(\mathcal{T}) \to \mathrm{Spec}^{\mathrm{h}}(R^*_{\mathcal{T}})$ is a morphism of locally ringed spaces with the following properties:*

*(a) For a prime $\mathcal{P} \in \mathrm{Spc}(\mathcal{T})$ let $\mathfrak{p} = \rho_{\mathcal{T}}(\mathcal{P}) \in \mathrm{Spec}(R_{\mathcal{T}})$ and $\mathfrak{p}^* = \rho^*_{\mathcal{T}}(\mathcal{P}) \in \mathrm{Spec}^{\mathrm{h}}(R^*_{\mathcal{T}})$. Then the induced homeomorphisms on stalks are the natural ones, $(R_{\mathcal{T}})_{\mathfrak{p}} \to R_{\mathcal{T}/\mathcal{P}}$ and $(R^*_{\mathcal{T}})_{\mathfrak{p}^*} \to R^*_{\mathcal{T}/\mathcal{P}}$.*

*(b) If $\rho_{\mathcal{T}}$ or $\rho^*_{\mathcal{T}}$ is a homeomorphism on the underlying spaces, it is automatically an isomorphism of ringed spaces.*

*Proof.* For readability we will write $\rho$ instead of $\rho^*_{\mathcal{T}}$ and $X$ instead of $\mathrm{Spec}^{\mathrm{h}}(R^*_{\mathcal{T}})$.

Part (b) follows from (a) and lemma 5.19 since $\rho$ being a homeomorphism implies that $\mathcal{F}^*_{\mathcal{T}}$ and $\mathcal{O}^*_X$ are the same on principal open sets, so the sheafification $\mathcal{O}^*_{\mathcal{T}}$ must then agree with $\mathcal{O}^*_X$ since the latter was a sheaf to begin with.

We continue on to part (a). Let $\mathcal{P} \in \mathrm{Spc}(\mathcal{T})$ and recall from theorem 5.10 that under the ring map $R^*_{\mathcal{T}} \to R^*_{\mathcal{T}/\mathcal{P}}$ induced by the functor $\mathcal{T} \to \mathcal{T}/\mathcal{P}$ the preimage of the maximal homogeneous ideal of $R_{\mathcal{T}/\mathcal{P}}$ is $\rho(\mathcal{P})$. Denote this homogeneous prime ideal $\mathfrak{p}$. Hence, the morphism $\phi : (R^*_{\mathcal{T}})_{\mathfrak{p}} \to R^*_{\mathcal{T}/\mathcal{P}}$ is a local ring map. Now we want to show that $\phi$ is actually the map on induced on stalks by $\rho : \mathrm{Spec}^*(\mathcal{T}) \to X$. This follows from lemma 4.14 in a straightforward manner once we trace through the definitions of the rings involved. □

## 5.4 Surjectivity criteria

Lets examine two criteria for $R^*_{\mathcal{T}}$ which ensure that the comparison map is surjective.

**Definition 5.22.** A ring $R$ is called *coherent* if every finitely generated ideal is finitely presented, and a graded ring is coherent if every finitely generated homogeneous ideal is finitely presented. An $R$-module is called coherent if it is finitely presented and if its submodules are finitely presented.

**Fact 5.23.** All Noetherian modules and rings are coherent.

**Notation 5.24.** Let $a \in \mathrm{Obj}(\mathcal{T})$ and $f \in R^d_{\mathcal{T}}$. To make our notation more compact for the remainder of the section, we will use $a/\!\!/f$ to denote the cone of the morphism $a \xrightarrow{f} \Sigma^d a$. Here we are abusing notation as this is really the cone of the composition $a \cong a \otimes \mathbb{1} \xrightarrow{\mathrm{id}_a \otimes f} a \otimes \Sigma^d \mathbb{1} \cong \Sigma^d a$. Given $f_1, ..., f_n \in R^*_{\mathcal{T}}$ we write

$$\mathbb{1}/\!\!/\langle f_1, ..., f_n\rangle \coloneqq (\mathbb{1}/\!\!/f_1) \otimes ... \otimes (\mathbb{1}/\!\!/f_n) = \mathrm{cone}(f_1) \otimes ... \otimes \mathrm{cone}(f_n)$$

.

**Lemma 5.25.** *Let $\mathcal{T}$ be a tensor triangulated category for which $R^*_{\mathcal{T}}$ is a local ring, and let $\mathfrak{m}$ be the unique maximal ideal of $R^*_{\mathcal{T}}$. Then the following are equivalent:*

*(a) $\mathfrak{m}$ is in the image of the comparison map, i.e. there is some $\mathcal{P} \in \mathrm{Spc}(\mathcal{T})$ such that $\rho^*_{\mathcal{T}}(\mathcal{P}) = \mathfrak{m}$.*

*(b) For all homogeneous $f_1, ..., f_n \in \mathfrak{m}$, one has $\mathbb{1}/\!\!/\langle f_1, ..., f_n\rangle \neq 0$.*

*Proof.* Suppose that $\mathfrak{m} = \rho^*_{\mathcal{T}}(\mathcal{P})$ for some $\mathcal{P} \in \mathrm{Spc}(\mathcal{T})$. For the sake of contradiction, further suppose that we may find homogeneous $f_1, ..., f_n \in \mathfrak{m}$ such that $\mathbb{1}/\!\!/\langle f_1, ..., f_n\rangle = 0$. Then $\mathbb{1}/\!\!/f_i \in \mathcal{P}$

for some $f_i$ since $\mathcal{P}$ is prime. By definition of the comparison map, $f_i \notin \rho^*_{\mathcal{T}}(\mathcal{P}) = \mathfrak{m}$. Therefore, we have a contradiction.

For the other direction, let $\mathcal{S} = \{\mathbb{1}/\!\!/\langle f_1, ..., f_n\rangle \mid f_i \in \mathfrak{m} \text{ homogeneous}\} \cup \{\mathbb{1}\}$ and suppose that $\mathcal{S}$ does not contain zero. This set is a multiplicative set, so by proposition 2.9 there exists $\mathcal{P} \in \mathrm{Spc}(\mathcal{T})$ such that $\mathcal{P} \cap \mathcal{S} = \emptyset$. For all homogeneous $f \in \mathfrak{m}$ we have $\mathbb{1}/\!\!/f \notin \mathcal{P}$ and therefore $f \in \rho^*_{\mathcal{T}}(\mathcal{P})$, hence $\mathfrak{m} \subseteq \rho^*_{\mathcal{T}}(\mathcal{P})$, and by maximality we have equality. ☐

*Remark* 5.26. The analogous theorem for the ungraded case of lemma 5.25 holds, i.e. for a local tt-category $\mathcal{T}$ with central ring $(R_{\mathcal{T}}, \mathfrak{m})$, it is the case that $\mathfrak{m} \in \mathrm{im}(\rho_{\mathcal{T}})$ if and only if $\mathbb{1}/\!\!/\langle f_1, ..., f_n\rangle \neq 0$ for all $f_1, ..., f_n \in \mathfrak{m}$.

**Proposition 5.27.** *Assume that $(R^*_{\mathcal{T}}, \mathfrak{m})$ is a graded local coherent (resp. Noetherian) ring. Let $f \in \mathfrak{m}$ be homogeneous and $a, x \in \mathcal{T}$ be objects. If $\mathrm{Hom}^*_{\mathcal{T}}(x, a)$ is non-zero and coherent as a graded (left) module over $R^*_{\mathcal{T}}$ then $\mathrm{Hom}^*_{\mathcal{T}}(x, a/\!\!/f)$ is non-zero and coherent (resp. Noetherian).*

*Proof.* Suppose $f$ is of degree $d$. Then the triangle below is exact

$$\mathbb{1} \xrightarrow{f} \Sigma^d\mathbb{1} \xrightarrow{f_1} \mathbb{1}/\!\!/f \xrightarrow{f_2} \Sigma\mathbb{1}$$

so by applying the functor $-\otimes a$ the triangle below is exact.

$$a \xrightarrow{f\otimes \mathrm{id}_a} \Sigma^d a \xrightarrow{f_1} \mathbb{1}/\!\!/f \otimes \mathrm{id}_a \xrightarrow{f_2} \Sigma a$$

There is then a long exact sequence coming from the homological functor $\mathrm{Hom}_{\mathcal{T}}(x, a)$ which may be wound into the periodic exact sequence below:

$$\begin{array}{ccc} \mathrm{Hom}^*_{\mathcal{T}}(x,a) & \xrightarrow[(d)]{\cdot f} & \mathrm{Hom}^*_{\mathcal{T}}(x,a) \\ {}_{(1)}\nwarrow & & \swarrow \\ & \mathrm{Hom}^*_{\mathcal{T}}(x,a/\!\!/f) & \end{array}$$

where labels $(d)$ and $(1)$ denote the degrees of the morphisms. Suppose that the lower term in the triangle above is zero then $f$ must act surjectively. Since $\mathrm{Hom}^*_{\mathcal{T}}(x, a)$ is coherent as a $R^*_{\mathcal{T}}$-module, it is finitely generated. Then by the graded version of NAK it would then follow that $\mathrm{Hom}^*_{\mathcal{T}}(x, a) = 0$, but this is a contradiction, and therefore the bottom term cannot be 0. From the triangle above we can extract the short exact sequence below:

$$0 \to \ker(\cdot f) \to \mathrm{Hom}^*_{\mathcal{T}}(x, a) \to \mathrm{coker}(\cdot f) \to 0$$

Note that the left and right modules in the short exact sequence are coherent, so it follows that $\mathrm{Hom}^*_{\mathcal{T}}(x, a)$ is coherent. ☐

**Theorem 5.28.** *If $R^*_{\mathcal{T}}$ is coherent, then the comparison map $\rho^*_{\mathcal{T}} : \mathrm{Spc}(\mathcal{T}) \to \mathrm{Spec}^{\mathrm{h}}(R^*_{\mathcal{T}})$ is surjective.*

*Proof.* Let $\mathfrak{p}^\bullet \in \mathrm{Spec}^{\mathrm{h}}(R^*_{\mathcal{T}})$ and denote $S_{\mathfrak{p}^*} := R^*_{\mathcal{T}} \setminus \mathfrak{p}^*$. From corollary 4.42 we have that the central graded ring of $S^{-1}_{\mathfrak{p}^*}\mathcal{T}$ is a local ring. Write $\mathfrak{m}$ for the maximal ideal of $R_{S^{-1}_{\mathfrak{p}^*}\mathcal{T}}$. Recall from remark 5.13 that $\mathfrak{p}^*$ being in the image of $\rho^*_{\mathcal{T}}$ follows from $\mathfrak{m}$ being in the image of $\rho^*_{S^{-1}_{\mathfrak{p}^*}\mathcal{T}}$. Therefore, we reduce to the case in which $R^*_{\mathcal{T}}$ is a local ring. By lemma 5.25, it suffices to show that $\mathbb{1}/\!\!/\langle f_1, ..., f_n\rangle \neq 0$ for all homogeneous $f_1, ..., f_n \in \mathfrak{m}$.

By setting $x = \mathbb{1}$, proposition 5.27 tells us $\mathrm{Hom}^*_{\mathcal{T}}(\mathbb{1}, \mathbb{1}/\!\!/\langle f_1, ..., f_n\rangle) \neq 0$ for all homogeneous $f_1, ..., f_n \in \mathfrak{m}$ by induction on $n$. □

**Definition 5.29.** A tt-category $\mathcal{T}$ is called *connective* if $\mathrm{Hom}_{\mathcal{T}}(\mathbb{1}, \Sigma^d\mathbb{1}) = 0$ for all $d > 0$.

**Lemma 5.30.** *Let $\mathcal{T}$ be connective and $f_1, ..., f_n \in R_{\mathcal{T}}$. Let $c = \mathbb{1}/\!\!/\langle f_1, ..., f_n\rangle$. Then,*

*(a)* $\mathrm{Hom}_{\mathcal{T}}(\mathbb{1}, \Sigma^i(c)) = 0$ *for all* $i > 0$.

*(b) There is a natural isomorphism* $R_{\mathcal{T}}/\langle f_1, ..., f_n\rangle \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{T}}(\mathbb{1}, c)$

*Proof.* Predictably, we proceed by induction. For $n = 0$ we have $c = \mathbb{1}/\!\!/\langle 0\rangle = \mathbb{1}$. In this case, (a) is just the property of connectivity and (b) is trivial. Now we proceed with induction, so let $n \geq 1$ and set $d = \mathbb{1}/\!\!/\langle f_1, ..., f_{n-1}\rangle$. Consider the exact triangle below:

$$d \xrightarrow{f_n \cdot \mathrm{id}_d} d \longrightarrow c \longrightarrow \Sigma d$$

which is just the exact triangle obtained by tensoring $d$ with the exact triangle coming from the cone of $f_n$. We then apply the cohomological functor $\mathrm{Hom}_{\mathcal{T}}(\mathbb{1}, \Sigma^i d)$ to get the long exact sequence

$$\ldots \to \mathrm{Hom}_{\mathcal{T}}(\mathbb{1}, \Sigma^i d) \xrightarrow{\cdot f_n} \mathrm{Hom}_{\mathcal{T}}(\mathbb{1}, \Sigma^i d) \to \mathrm{Hom}_{\mathcal{T}}(\mathbb{1}, \Sigma^i c) \to \mathrm{Hom}_{\mathcal{T}}(\mathbb{1}, \Sigma^{i+1} d) \to \ldots$$

If we apply the inductive hypothesis at $i > 0$, then $\mathrm{Hom}_{\mathcal{T}}(\mathbb{1}, \Sigma^i d)$ vanishes, which forces $\mathrm{Hom}_{\mathcal{T}}(\mathbb{1}, \Sigma^i c) = 0$, so (a) must be true. Application of the inductive hypothesis at $i = 0$ yields (b). □

**Corollary 5.31.** *If $\mathcal{T}$ is a connective tt-category and $I \subseteq R_{\mathcal{T}}$ is a proper ideal, then for all $f_1, ..., f_n \in I$ the product $\mathbb{1}/\!\!/\langle f_1, \ldots f_n\rangle$ is nonzero.*

*Proof.* Let $f_1, ..., f_n \in I$. By lemma 5.30 we have the surjective composition of maps

$$\mathrm{Hom}_{\mathcal{T}}(\mathbb{1}, \mathbb{1}/\!\!/\langle f_1, ..., f_n\rangle) \xrightarrow{\sim} R_{\mathcal{T}}/\langle f_1, ..., f_n\rangle \to R_{\mathcal{T}}/I$$

Since $I$ is a proper ideal, $R/I$ is nonzero. □

**Corollary 5.32.** *Let $\mathcal{T}$ be a connective tt-category such that $(R_{\mathcal{T}}, \mathfrak{m})$ is a local ring. Then there exists $\mathcal{P} \in \mathrm{Spc}(\mathcal{T})$ such that $\rho_{\mathcal{T}}(\mathcal{P}) = \mathfrak{m}$.*

*Proof.* Condition (b) of remark 5.26 is satisfied by corollary 5.31 when we take $\mathfrak{m}$ to be $I$. □

**Theorem 5.33.** *Let $\mathcal{T}$ be a connective tt-category. Then $\rho_{\mathcal{T}} : \mathrm{Spc}(\mathcal{T}) \to \mathrm{Spec}(R_{\mathcal{T}})$ is surjective.*

*Proof.* If we can show that $\mathcal{T}_{\mathfrak{p}}$ is connective for any $\mathfrak{p} \in \mathrm{Spec}(R_{\mathcal{T}})$ then the result follows from corollary 5.32 and remark 5.13. But this is almost immediate from corollary 4.42 since we are localizing at the multiplicatively closed set $R_{\mathcal{T}} \setminus \mathfrak{p}$ which lives only in degree 0.

$$R^*_{\mathcal{T}_{\mathfrak{p}}} \cong (R^*_{\mathcal{T}})_{\mathfrak{p}} = (R_{\mathcal{T}})_{\mathfrak{p}}$$

□

**Example 5.34.** The stable homotopy category of finite spectra is connective.

**Example 5.35.** Let $R$ be a commutative ring. In example 4.46 we saw that $\mathrm{D}^{\mathrm{pf}}(R)$ is connective, so we know that the comparison map is surjective. This should not be surprising since we already knew that $\mathrm{Spec}(R) \cong \mathrm{Spc}(\mathrm{D}^{\mathrm{pf}}(R))$ due to the universal morphism of support data of theorem 3.44,

$$u : \mathrm{Spec}(R) \to \mathrm{Spc}(\mathrm{D}^{\mathrm{pf}}(R))$$

Explicitly, it sends

$$\mathfrak{p} \longmapsto \{C_\bullet \mid C_{\mathfrak{p}\bullet} \cong 0\}$$

Recall that the central ring of $\mathrm{D}^{\mathrm{pf}}(R)$ is isomorphic, essentially by definition, to $R$. Given this, we might hope that the comparison map $\rho : \mathrm{Spc}(\mathrm{D}^{\mathrm{pf}}(R)) \to \mathrm{Spec}(R)$ is an inverse to $u$, and indeed this is the case. Recall that $\rho$ acts via

$$\mathcal{P} \longmapsto \{f \in R \mid \mathbb{1}/\!\!/ f \notin \mathcal{P}\}$$

Since the monoidal unit of $\mathrm{D}^{\mathrm{pf}}(R)$ is a complex consisting of $R$ concentrated in degree zero, the cone of $\mathbb{1} \xrightarrow{f} \mathbb{1}$ is simply the complex

$$\ldots \to 0 \to R \xrightarrow{f} R \to 0 \to \ldots$$

If we localize this complex at $\mathfrak{p} \in \mathrm{Spec}(R)$ the resulting complex is acyclic if and only if $R_{\mathfrak{p}} \xrightarrow{\frac{f}{1}} R_{\mathfrak{p}}$ is an isomorphism, i.e. $f \notin \mathfrak{p}$. Hence,

$$\rho(u(\mathfrak{p})) = \{f \in R \mid \mathbb{1}/\!\!/ f \notin u(\mathfrak{p})\} = \{f \in A \mid (\mathbb{1}/\!\!/ f)_{\mathfrak{p}} \ncong 0\} = \mathfrak{p}$$

Since $u$ is surjective, we also know that $u(\rho(\mathcal{P})) = \mathcal{P}$, so $u$ and $\rho$ are mutually inverse.

The comparison map may not be surjective for $\mathrm{D}^{\mathrm{pf}}(X)$ where $X$ is not affine. This happens because there aren't enough global sections. This happens for example when $X = \mathbb{P}^n_k$ for a field $k$. See remark 8.2 of [Bal10].

**Example 5.36.** Here is another example where the comparison map may not be surjective. Let $(A, \mathfrak{m})$ be a regular local ring and let $U = \mathrm{Spec}(A) \backslash \{\mathfrak{m}\}$ be its punctured spectrum. Set $\mathcal{T} = \mathrm{D}^{\mathrm{pf}}(U)$. It follows from Hartog's lemma that $\mathrm{H}^0(U, \mathcal{O}_U) \cong \mathrm{H}(\mathrm{Spec}(R), \mathcal{O}_{\mathrm{Spec}(R)}) \cong A$ when $\dim(R) \geq 2$. In this case, $R_{\mathcal{T}} \cong A$ so $\mathrm{Spec}(R_{\mathcal{T}}) \cong \mathrm{Spec}(A)$ while $\mathrm{Spec}(\mathcal{T}) \cong U$. Therefore $\rho_{\mathcal{T}}$ cannot be surjective as $\mathrm{Spec}(A)$ is strictly larger than $U$.

**Example 5.37.** Let $G$ be a finite group over a field $k$. More generally we could instead consider $G$ a finite group scheme, or equivalently, a finite dimensional cocommutative Hopf algebra $A$. Let $\mathcal{T} = \mathrm{D}^{\mathrm{b}}(kG\text{-mod}) \cong \mathrm{K}(\mathrm{Inj}\text{-}kG)^c$. Here the superscript $(-)^c$ denotes the compact objects of the associated category. This will be defined in the next section. We have in this case,

$$R^*_{\mathcal{T}} = \mathrm{H}^*(G, k) = \bigoplus_{d \geq 0} \mathrm{H}^d(G, k)$$

It is a theorem of Golod, Venkov, and Evens that the group cohomology $\mathrm{H}^*(G, k)$ is a Noetherian ring when $G$ is a finite group. Friedlander and Suslin proved this in the case of $G$ a finite group scheme in [FS97]. It is an open question whether this is true of Hopf algebra cohomology in the non-cocommutative case. Since the group cohomology is noetherian, it follows from theorem 5.28 that the comparison map is an epimorphism $\mathrm{Spec}(\mathrm{D}^{\mathrm{b}}(kG\text{-mod})) \cong \mathrm{Spec}^{\mathrm{h}}(\mathrm{H}^\bullet(G, k))$. In fact, this map is an isomorphism, as shown in proposition 8.5 of [Bal10]. Using theorem 4.16, this restricts to an isomorphism $\mathrm{Spec}(\mathsf{stmod}(kG)) \cong \mathrm{Proj}(\mathrm{H}^\bullet(G, k))$. Recall that the latter is just the support variety $\mathcal{V}_G(k)$ detailed in section 4.2. For details, see [BCR97], [FP07], and [Bal10].

### 5.5 Cohomological Support

This section is loosely based on ideas that can be found in section 2 of [Lau22].

Let $\mathcal{T}$ be an essentially small tensor triangulated category and let $a \in \text{Obj}(\mathcal{T})$. For legibility we often denote $R^*_{\mathcal{T}}$ as $R$ for the rest of this section unless otherwise stated. At the end of subsection 5.2 we asked what the comparison map $\rho = \rho^*_{\mathcal{T}} : \text{Spc}(\mathcal{T}) \to \text{Spec}^{\text{h}}(R)$ does to the closed sets $\text{Supp}_{\mathcal{T}}(a) \subseteq \text{Spc}(\mathcal{T})$. In general, this question is difficult to answer. If we are lucky

**Proposition 5.38.** **??** *If $\rho : \text{Spc}(\mathcal{T}) \to \text{Spec}^{\text{h}}(R^*_{\mathcal{T}})$ induces a bijection on Thomason subsets, then $\rho$ is a homeomorphism.*

**Definition 5.39.** If a comparison map $\rho$ induces a bijection on Thomason subsets as in **??**, then we say that $\rho$ classifies Thomason subsets.

*Proof.* This proof is very similar to what happens in theorem 3.41.

As $\rho$ is a spectral map of spectral spaces, and the Thomason subsets are precisely the open sets of the inverse topology, it follows that $\rho$ induces an isomorphism on the lattice of open sets for the inverse topologies. When a continuous map induces an isomorphism on the lattice of open sets between two sober spaces, it must be that the map was a homeomorphism. Therefore, $\rho$ is a homeomorphism on the inverse topologies of $\text{Spc}(\mathcal{T})$ and $\text{Spec}^{\text{h}}(R)$. It follows that $\rho$ is a homeomorphism on the original topologies. □

The situation above is a best case scenario, and we would like to have some more general results. One might hope that $\text{Spec}^{\text{h}}(R)$ can be realized as a support datum of some kind with $\rho^*_{\mathcal{T}}$ acting as a morphism of support data. From this angle, the obvious thing to do is to try to relate $\text{supp}(a)$ to $\text{Supp}_R(M)$ where $M$ is some kind of $R$-module depending on $a$. Luckily, there is a natural way to do this.

**Notation 5.40.** Given an object $x \in \mathcal{T}$ we will write

$$M_a := \text{End}^*_{\mathcal{T}}(a)$$

to denote the graded endomorphism ring of $a$.

We saw in subsection 4.4 that $M_a$ is an $R$-module, and indeed $M_a$ is an $R$-algebra through a ring map

$$\begin{aligned} R &\longrightarrow M_a \\ f &\longmapsto (a \xrightarrow{f \cdot \text{id}_a} \Sigma^d a) \end{aligned}$$

where $f$ is a homogeneous element of degree $d$. Recall the definition of the support of an $R$-module:

$$\text{Supp}_R(M) := \{\mathfrak{p} \in \text{Spec}(R) \mid M_{\mathfrak{p}} \neq 0\}$$

Let $\mathfrak{p} \in \text{Spec}^{\text{h}}(R)$ and let $q_{\mathfrak{p}} : \mathcal{T} \to \mathcal{T}_{\mathfrak{p}}$ be the canonical localization functor. Recall from theorem 4.41 and corollary 4.42 that for any objects $a, b \in \mathcal{T}$ we have

$$\text{Hom}_{\mathcal{T}_{\mathfrak{p}}}(a_{\mathfrak{p}}, b_{\mathfrak{p}}) \cong (\text{Hom}^*_{\mathcal{T}}(a, b)_{\mathfrak{p}})^0 = \text{Hom}_{\mathcal{T}/\mathcal{J}}(a, b)$$

where $\mathcal{J} = \langle \mathbb{1} /\!\!/ s \mid s \notin \mathfrak{p} \rangle$. Tracing through the definition we have that

$$\mathfrak{p} \in \text{Supp}_R(M_a) \iff (M_a)_{\mathfrak{p}} \neq 0 \iff a_{\mathfrak{p}} \neq 0 \qquad (\star)$$

This then leads us to the following proposition.

**Proposition 5.41.** *If $f \in R^{hom}$ then*

$$V(f) = \mathrm{Supp}_R(M_{\mathbb{1}/\!\!/f})$$

*Proof.* Let $\mathfrak{p} \in V(f)$. Let $q_{\mathfrak{p}} : \mathcal{T} \to \mathcal{T}_{\mathfrak{p}}$ be the localization functor. Then,

$$\begin{aligned} \mathfrak{p} \in \mathrm{Supp}_R(M_{\mathbb{1}/\!\!/f}) &\iff q_{\mathfrak{p}}(\mathbb{1}/\!\!/f) \neq 0 \\ &\iff q_{\mathfrak{p}}(f) \text{ is not invertible in } \mathcal{T}_{\mathfrak{p}} \\ &\iff f \text{ is a not unit in } R^*_{\mathcal{T}_{\mathfrak{p}}} = (R^*_{\mathcal{T}})_{\mathfrak{p}} \\ &\iff f \in \mathfrak{p} \\ &\iff \mathfrak{p} \in V(f) \end{aligned}$$

□

With this result in mind, and because the notations $\mathrm{Supp}_{\mathcal{T}}(-)$ and $\mathrm{Supp}_R(-)$ are easy to confuse at a glance, we will introduce the following notation/definition.

**Definition 5.42.** Given a tt-category $\mathcal{T}$, we define the *cohomological support* of $\mathcal{T}$:

$$\mathcal{V}(a) \coloneqq \mathrm{Supp}_R(M_a)$$

We now want to examine the following questions:

1. Is $(\mathrm{Spec}^{\mathrm{h}}(R), \mathcal{V}(-))$ a support datum for $\mathcal{T}$?
2. Is it the case that $\rho(\mathrm{supp}_{\mathcal{T}}(a)) = \mathrm{Supp}_R(M_a)$?

Properties (SD1) through (SD4) of definition 3.12 are satisfied more or less immediately due to properties of the ring theoretic support of a module and equation $(\star)$ above. Unfortunately, this is as far as we can get without additional assumptions. Let $S$ be a commutative ring and let $M, N$ be $S$-modules. Recall from subsection 3.1 that if $M$ and $N$ are finitely generated then it the case that $\mathrm{supp}_S(M) = V(\mathrm{Ann}_S(M))$ (so $\mathrm{supp}_S(M)$ is closed in $\mathrm{Spec}(S)$) and $\mathrm{supp}_S(M \otimes_S N) = \mathrm{Supp}_S(M) \cap \mathrm{Supp}_S(N)$. However, neither of these relations necessarily hold when $M$ and $N$ are not finitely generated, so we cannot expect $\mathcal{V}$ to give us a support datum in general. We can summarize this as follows:

**Proposition 5.43.** $(\mathrm{Spec}^{\mathrm{h}}(R), \mathcal{V})$ *forms a support datum for $\mathcal{T}$ if and only if $\mathcal{V}(a)$ is closed in* $\mathrm{Spec}^{\mathrm{h}}(R)$ *and $\mathcal{V}(a \otimes b) = \mathcal{V}(a) \cap \mathcal{V}(b)$ for all objects $a, b \in \mathcal{T}$.*

Now the question is, what conditions are sufficient to enforce that $(\mathrm{Spec}^{\mathrm{h}}(R), \mathcal{V})$ is a support datum? Recall from commutative algebra that for a Noetherian commutative ring $S$, an $S$-module $M$ is Noetherian if and only if $M$ is finitely generated. It follows that $M_a$ is Noetherian for all $a \in \mathcal{T}$ if and only if $R$ is Noetherian and $M_a$ is finitely generated as an $R$-module for all objects $a$ of $\mathcal{T}$. If this holds, it follows that $\mathcal{V}(a) = \mathrm{supp}_R(M_a)$ is a closed subset for all $a \in \mathcal{T}$. This leads us to the following definition:

**Definition 5.44.** If $M_a$ is noetherian for all objects $a \in \mathcal{T}$, we say that $\mathcal{T}$ is *end-finite*.

*Remark* 5.45. The terminology of definition 5.44 comes from [Lau22].

For the next lemma, recall notation 5.24, where for an object $a$ and morphisms $f_1, ..., f_n \in R^*_{\mathcal{T}}$ we write

$$a /\!\!/ \langle f_1, ..., f_n \rangle \coloneqq a \otimes \operatorname{cone}(f_1) \otimes ... \otimes \operatorname{cone}(f_n)$$

**Lemma 5.46.** *Given an object $a \in \mathcal{T}$,*

$$\rho(\operatorname{supp}_{\mathcal{T}}(a)) \subseteq \mathcal{V}(a)$$

*Equality holds if for all homogeneous $f_1, ..., f_n \in \mathfrak{p}$ one has $a /\!\!/ \langle f_1, ..., f_n \rangle \neq 0$.*

*Proof.* Let $\mathcal{P} \in \operatorname{supp}_{\mathcal{T}}(a)$, so $a \notin \mathcal{P}$. Denote $\mathfrak{p} \coloneqq \rho(\mathcal{P})$, so $\mathfrak{p} = \langle f \in R^{\text{hom}} \mid \mathbb{1} /\!\!/ f \notin \mathcal{P} \rangle$. Recall that $\mathcal{T}_{\mathfrak{p}} \cong \mathcal{T}/\mathcal{J}$ where $\mathcal{J} = \langle \mathbb{1} /\!\!/ f \mid f \notin \mathfrak{p} \rangle$. Unraveling definitions, we see that

$$\mathcal{J} = \langle \mathbb{1} /\!\!/ f \mid f \in R^{\text{hom}}, \mathbb{1} /\!\!/ f \in \mathcal{P} \rangle \subseteq \mathcal{P}$$

Therefore, the Verdier quotient $q : \mathcal{T} \to \mathcal{T}/\mathcal{P}$ factors through the map $q_{\mathfrak{p}} : \mathcal{T} \to \mathcal{T}_{\mathfrak{p}}$. It follows that if $q(a) \neq 0$ then $q_{\mathfrak{p}}(a) \neq 0$, so by $(\star)$ we have $\rho(\mathcal{P}) = \mathfrak{p} \in \mathcal{V}(a)$. Hence, $\rho(\operatorname{supp}_{\mathcal{T}}(a)) \subseteq \mathcal{V}(a)$.

To show equality, for all $\mathfrak{p} \in \mathcal{V}(a) = \operatorname{supp}_R(M_a)$ we must find $\mathcal{P} \in \operatorname{supp}_{\mathcal{T}}(a)$ such that $\rho(\mathcal{P}) = \mathfrak{p}$. By corollary 5.16 it suffices to show that this holds for $\mathcal{T}_{\mathfrak{p}}$, so we replace $\mathcal{T} = \mathcal{T}_{\mathfrak{p}}$ and $R$ with the graded local ring $R_{\mathfrak{p}}$ with maximal ideal $\mathfrak{p}$. We will use lemma 5.25 as inspiration. Set

$$\mathcal{S} \coloneqq \{a^{\otimes m} /\!\!/ \langle f_1, ..., f_n \rangle \mid f_i \in \mathfrak{p} \text{ homogeneous}, \ m, n \in \mathbb{N}\} \cup \{\mathbb{1}\}$$

This is a $\otimes$-multiplicative subset of $\mathcal{T}$, so if we can show that 0 is not contained in $\mathcal{S}$ it will follow from proposition 2.9 that there exists $\mathcal{P} \in \operatorname{Spc}(\mathcal{T})$ such that $\mathcal{S} \cap \mathcal{P} = \emptyset$. If this is the case, we would then have $a \notin \mathcal{P}$ and therefore $\mathbb{1} /\!\!/ f \notin \mathcal{P}$ for any homogeneous $f \in \mathfrak{p}$; hence, $\mathcal{P} \in \operatorname{Supp}_{\mathcal{T}}(a)$ and $\mathfrak{p} \subseteq \rho(\mathcal{P})$, but then $\mathfrak{p} = \rho(\mathcal{P})$ by maximality of $\mathfrak{p}$. ☐

**Corollary 5.47.** *Equality holds for all objects $a \in \mathcal{T}$ in lemma 5.46 if $(\operatorname{Spec}^{\text{h}}(R), \mathcal{V})$ is a support datum or if $\mathcal{T}$ is rigid and end-finite.*

*Proof.* Let $\mathfrak{p} \in \mathcal{V}(a)$. As in the proof of lemma 5.46 we reduce to the case that $\mathcal{T} = \mathcal{T}_{\mathfrak{p}}$. Let $\mathcal{S}$ also be as in the proof of lemma 5.46. The claim follows from lemma 5.46 if we can show that $0 \notin \mathcal{S}$.

Suppose that $(\operatorname{Spec}^{\text{h}}(R), \mathcal{V})$ is a support datum. Then (SD5) holds for $\mathcal{V}$ and therefore

$$\mathcal{V}(a /\!\!/ \langle f_1, ..., f_n \rangle) = \mathcal{V}(a) \cap V(f_1) \cap ... \cap V(f_n)$$

for any homogeneous $f_1, ..., f_n \in \mathfrak{p}$. In particular, $\mathfrak{p} \in \mathcal{V}(s)$ for any $s \in \mathcal{S}$. Since $\mathcal{V}(0) = \emptyset$ and $\mathcal{V}(s)$ is non-empty for all $s \in \mathcal{S}$, it follows that $0 \notin \mathcal{S}$.

Suppose that $M_a$ is Noetherian for all objects $a \in \mathcal{T}$ and $\mathcal{T}$ is rigid. In particular, this means that $a^{\otimes n} = 0$ for positive $n$ if and only if $a = 0$ to begin with. Since $\mathfrak{p} \in \mathcal{V}(a)$, we know that $\mathcal{V}(a) \neq 0$. Therefore, it suffices to show that $a /\!\!/ f \neq 0$ for any homogeneous $f \in \mathfrak{p}$, but this follows from proposition 5.27 by setting $x = a$. ☐

**Corollary 5.48.** *Suppose that $(\operatorname{Spec}^{\text{h}}(R), \mathcal{V})$ is a support datum, or that $\mathcal{T}$ is rigid and end-finite. Then $\rho$ is bijective if and only if $\rho$ is a homeomorphism*

*Proof.* Recall that $\{\operatorname{supp}_{\mathcal{T}}(a)\}_{a \in \operatorname{Obj}(\mathcal{T})}$ is a closed basis for $\operatorname{Spc}(\mathcal{T})$. In either of the assumptions of the corollary statement, $\rho$ is a closed map. The claim follows. ☐

**Proposition 5.49.** *If $\mathcal{T}$ is rigid and $\rho$ is a homeomorphism, then $\rho(\operatorname{supp}(a)) = \mathcal{V}(a)$ for $a \in \operatorname{Obj}(\mathcal{T})$.*

*Proof.* Let $\mathcal{P} \in \operatorname{Spc}(\mathcal{T})$ and $\mathfrak{p} = \rho(\mathcal{P})$. Consider the natural functor $F : \mathcal{T}_{\mathfrak{p}} \to \mathcal{T}/\mathcal{P}$, which we know exists from the proof of lemma 5.46. We want to show that $F$ is an equivalence. From theorem 5.12, there is the cartesian square below,

$$\begin{array}{ccc} \operatorname{Spc}(\mathcal{T}_{\mathfrak{p}}) & \xhookrightarrow{\operatorname{Spc}(q)} & \operatorname{Spc}\mathcal{T} \\ \downarrow{\scriptstyle \rho_{\mathcal{T}_{\mathfrak{p}}}} & & \downarrow{\scriptstyle \rho_{\mathcal{T}}} \\ \operatorname{Spec}^{\mathrm{h}}(R_{\mathfrak{p}}) & \hookrightarrow & \operatorname{Spec}^{\mathrm{h}}(R) \end{array}$$

where $q : \mathcal{T} \to \mathcal{T}_{\mathfrak{p}}$ is the canonical localization. Since $\rho_{\mathcal{T}}$ is a homeomorphism by assumption, it must be that $\rho_{\mathcal{T}_{\mathfrak{p}}}$ is a homeomorphism as well. We know that $\operatorname{Spec}^{\mathrm{h}}(R_{\mathfrak{p}})$ has a unique closed point as $R_{\mathfrak{p}}$ is a local ring, and we know that this point is mapped to $\mathfrak{p} \in \operatorname{Spec}^{\mathrm{h}}(R)$. Therefore, $\operatorname{Spc}(\mathcal{T}_{\mathfrak{p}})$ has a unique closed point $\mathcal{Q}$ which is sent to $\mathcal{P}$ in $\operatorname{Spc}(\mathcal{T})$. As $\mathcal{T}_{\mathfrak{p}}$ is rigid, $\mathcal{Q}$ must be the zero ideal of $\mathcal{T}_{\mathfrak{p}}$ by proposition 5.2. Hence, $\mathcal{P} = q^{-1}(0)$, so $q(\mathcal{P}) = 0$. Therefore, $q$ must induce a functor $\mathcal{T}/\mathcal{P} \to \mathcal{T}_{\mathfrak{p}}$ which is an inverse of $F$. The claim then follows.

□

# 6 Big Categories

So far almost everything we have covered has been under the stipulation that our tt-category $\mathcal{T}$ be essentially small. There are a number of reasons for doing this, but main one is that in order to define the Balmer spectrum we need our collection of prime $\otimes$-ideals to actually be a set. The requirement that $\mathcal{T}$ be essentially small is somewhat restrictive as it leaves out some categories that we would very much like to understand, such as the unbounded derived category $\mathrm{D}(R)$, the big stable module category $\mathsf{StMod}(A)$, and the stable homotopy category $\mathsf{SHC}$. In this section, we develop some machinery for extending the support theory of the Balmer spectrum to larger categories. Much of the content of this section comes from [HPS97] and [Del11], but the author also consulted the following sources: [Kra04], [Kra21], [Nee01], and [NP23]. The author would also like to thank John Palmieri for his helpful comments and discussions.

**Definition 6.1.** Let $\mathcal{C}$ be a locally small category. An object $c \in \mathcal{T}$ is called *compact* if there is an isomorphism

$$\mathrm{Hom}_{\mathcal{T}}(c, \bigoplus_{s\in\mathcal{S}} s) \cong \bigoplus_{s\in\mathcal{S}} \mathrm{Hom}_{\mathcal{T}}(c, s)$$

for every set of objects $\mathcal{S} \subseteq \mathrm{Obj}(\mathcal{T})$ such that the coproduct $\bigoplus_{s\in\mathcal{S}} s$ exists.

**Definition 6.2.** A triangulated category $\mathcal{T}$ is *compactly generated* if

1. $\mathcal{T}$ is closed under small coproducts
2. There exists a small full subcategory $\mathcal{C} \subseteq \mathcal{T}$ such that all $c \in \mathcal{C}$ are compact and for any $b \in \mathcal{T}$, if $\mathrm{Hom}_{\mathcal{T}}(c, b) = 0$ for all $c \in \mathcal{C}$, then $b = 0$.

We denote by $\mathcal{T}^c$ the full subcategory of compact objects in $\mathcal{T}$ and note that it is a thick subcategory.

**Example 6.3.** The category $\mathsf{Vec_k}$ for a field $k$ is compactly generated with finite dimensional spaces as compact objects. If $A$ is a Hopf algebra over $k$ then $\mathsf{Rep_A}$ has finite dimensional $A$-modules as compact objects and is also compactly generated. This comes from the fact that any representation $M$ of $A$ is a directed union of finitely generated submodules.

The following theorem is a consequence of *Brown Representability*; see theorem 1.17 of [Nee01].

**Theorem 6.4** (Neeman)**.** *Let $\mathcal{T}$ be a compactly generated category triangulated category and $\mathcal{T}'$ any triangulated category. Let $F$ be a covariant exact functor that commutes with set indexed coproducts. Then $F$ has a right adjoint $G$.*

*Proof.* See theorem 1.17 and proposition 1.21 of [Nee01]. □

**Definition 6.5.** A *compactly generated tensor triangulated category* is a triangulated $\mathcal{T}$ which is compactly generated equipped with a symmetric monoidal structure $(\otimes, \mathbb{1})$ where $\otimes$ is a coproduct preserving exact functor in both variables and the compact objects $\mathcal{T}^c$ form a $\otimes$-subcategory. Note that this means that $\mathbb{1} \in \mathcal{T}^c$.

**Corollary 6.6.** *If $\mathcal{T}$ be a compactly generated tt-category where $\otimes$ commutes with coproducts, then $\mathcal{T}$ is monoidally closed and has an internal hom structure. In particular, a compactly generated tt-category is closed as a monoidal category.*

*Proof.* Under these hypotheses the functor $- \otimes x : \mathcal{T} \to \mathcal{T}$ satisfies the conditions of theorem 6.4, so $0 \otimes x$ has a right adjoint. By definition 1.21, $\mathcal{T}$ is closed. □

**Definition 6.7.** A tt-category $\mathcal{T}$ is *rigidly compactly generated* if

1. $\mathcal{T}$ is a compactly generated tt-category, and,
2. An object of $\mathcal{T}$ is compact if and only if it is rigid.

**Example 6.8.** Let $\mathcal{T} = \mathrm{D}(\mathsf{Vec}_k)$. Then $\mathcal{T}^c = \mathrm{D}^b(k)$, which is the category of complexes quasi-isomorphic to bounded complexes of finite dimensional vector spaces.

**Example 6.9.**

- The unbounded derived category $\mathrm{D}(R)$ is rigidly compactly generated by $\mathrm{D}^{\mathrm{pf}}(R)$.
- If $G$ is a finite group and $k$ is a field, then $\mathsf{StMod}(kG)$ is rigidly compactly generated with $\mathsf{stmod}(kG) \subset \mathsf{StMod}(kG)$ embedding as the full subcategory of compact rigid objects.
- The stable homotopy category $\mathsf{SHC}$ is rigidly compactly generated by $\mathbb{S}^0$, the sphere spectrum.

## 6.1 Localizing Subcategories

For this section, assume that $\mathcal{T}$ is a rigidly compactly generated tt-category with $\mathcal{T}^c$ as the tt-subcategory of compact objects.

**Definition 6.10.** Let $\mathcal{T}$ be a (tensor)-triangulated category admitting set indexed coproducts. A subcategory $\mathcal{C} \subseteq \mathcal{T}$ is

1. *localizing* if it is a full triangulated subcategory which is closed under set indexed coproducts.
2. a *localizing ideal* if it is localizing, such that for each $a \in \mathcal{C}$, $b \in \mathcal{T}$ we have $a \otimes b \in \mathcal{C}$.

If $\mathcal{S}$ is some collection of objects in $\mathcal{T}$ then we write $\mathrm{loc}(\mathcal{S})$ to denote the smallest localizing subcategory containing the full subcategory of $\mathcal{S}$, and we write $\langle \mathcal{S} \rangle_{loc}$ to denote the smallest localizing ideal containing $\mathcal{S}$.

**Proposition 6.11.** *Let $\mathcal{I}$ be a $\otimes$-ideal in $\mathcal{T}^c$. Then $\langle \mathcal{I} \rangle_{\mathrm{loc}} \cap \mathcal{T}^c = \mathcal{I}$.*

*Proof.* See appendix D of [Nee01]. □

**Proposition 6.12** (Neeman)**.** *Let $\mathcal{T}$ be a triangulated category admitting set indexed coproducts. A localizing subcategory is thick.*

*Proof.* Let $a \otimes b \in \mathcal{C}$, a localizing subcategory of $\mathcal{T}$. We want to show that $a \in \mathcal{C}$. Define

$$\begin{aligned} x &:= (a \oplus b) \oplus (a \oplus b) \oplus \ldots \\ &= (b \oplus a) \oplus (b \oplus a) \oplus \ldots \end{aligned}$$

Note that $x \cong a \oplus x$. Since $\mathcal{C}$ localizes, $x \in \mathcal{C}$. Then we have an exact triangle

$$x \to x \to a \to \Sigma x$$

Since $\mathcal{C}$ is a triangulated subcategory, $a \in \mathcal{C}$. □

**Definition 6.13.** A Bousfield *localization functor* is a pair $(L, \eta)$ where $L : \mathcal{T} \to \mathcal{T}$ is a functor and $\eta : \mathrm{id}_{\mathcal{T}} \to L$ is a natural transformation satisfying:

(a) $L$ is exact.

(b) $L\eta : L \xrightarrow{\sim} L^2$ is an isomorphism and $L\eta = \eta L$, that is to say, $\eta_{La} = L\eta_a$ for all objects $a \in \mathcal{T}$.

(c) $L$ commutes with all coproducts.

We define

$$\ker(L) = \{a \in \mathcal{T} \mid La \cong 0\} \quad \text{and} \quad \operatorname{im}(L) = \{a \in \mathcal{T} \mid \eta_a : La \to a \;\text{ an isomorphism}\}$$

We say that $a$ is *$L$-acyclic* if $a \in \ker L$ and that $a$ is *$L$-local* if $a \in \operatorname{im} L$.

*Remark* 6.14. In subsection 1.2 we spoke of *Verdier localization*, which is different but very related to the Bousfield localization functors defined above. This will be expounded upon in the next section, but for the remainder of this section we will refer to the functors of definition 6.13 simply as localization functors in order to follow [HPS97] more closely.

**Definition 6.15.** For a triangulated subcategory $\mathcal{S} \subseteq \mathcal{T}$ we define two full subcategories:

$$\mathcal{S}^{\perp} \coloneqq \{a \in \mathcal{T} \mid \operatorname{Hom}_{\mathcal{T}}(s, a) = 0 \; \forall s \in \mathcal{S}\}$$

$$^{\perp}\mathcal{S} \coloneqq \{a \in \mathcal{T} \mid \operatorname{Hom}_{\mathcal{T}}(a, s) = 0 \; \forall s \in \mathcal{S}\}$$

These are called the *orthogonal* subcategories with respect to $\mathcal{S}$. We write $\mathcal{I} \perp \mathcal{J}$ if $\mathcal{I}$ and $\mathcal{J}$ are subcategories where $\operatorname{Hom}_{\mathcal{T}}(a, b) = 0$ for all $a \in \mathcal{I}$ and all $b \in \mathcal{J}$, and in this case we say that $\mathcal{I}$ and $\mathcal{J}$ are *orthogonal*.

*Remark* 6.16 (Warning). The notation of definition 6.15 varies in meaning across the literature. Beware that the notation of [Nee01] is the opposite of our notation. In this text, we follow [Kra21]. This is because it is easier for me to remember that $\mathcal{S}$ being on the left of the notation $\mathcal{S}^{\perp}$ means that $s \in \mathcal{S}$ appears on the left side of the hom functor in the definition.

**Lemma 6.17.** *Let $L$ be a localization functor of $\mathcal{T}$. Then $(\ker L)^{\perp} = \operatorname{im} L$ and $^{\perp}(\operatorname{im} L) = \ker L$. The latter of these is a localizing ideal, and the former is what is called a colocalizing ideal, that is, $(\operatorname{im} L)^{op}$ is a localizing ideal of $\mathcal{T}^{op}$.*

*Proof.* We will just prove the left-hand claim as the right-hand claim follows from more or less the same tricks. The fact that these are localizing ideals follows from the fact that $L$ commutes with coproducts and that $L$ is exact.

First suppose that $a$ is $L$-local, so $a \in \operatorname{im} L$. Let $f : x \to a$ be a morphism where $s \in \ker L$. Then we apply $\eta$ to get a commutative diagram:

$$\begin{array}{ccc} s & \xrightarrow{f} & a \\ \downarrow{\scriptstyle \eta_s} & & \downarrow{\scriptstyle \eta_a} \\ 0 \cong Ls & \xrightarrow{Lf} & La \end{array}$$

The right hand vertical map is an isomorphism, so it follows that $f = 0$, i.e. $\operatorname{Hom}_{\mathcal{T}}(s, a) = 0$, so $a \in (\ker L)^{\perp}$.

Now let $a \in (\ker(L))^{\perp}$, so $\operatorname{Hom}_{\mathcal{T}}(s, a) = 0$ for all $s \in \ker(L)$. We can fit $\eta_a : a \to La$ into an exact triangle below:

$$x \xrightarrow{f} a \xrightarrow{\eta_a} La \to \Sigma x$$

We then apply $\eta$ to this triangle to get the diagram below. Note that the bottom row is the image of the triangle above under $L$ and since $L$ is exact the bottom row is exact. Hence, the diagram below is a morphism of triangles.

$$\begin{array}{ccccccc}
x & \xrightarrow{f} & a & \xrightarrow{\eta_a} & La & \longrightarrow & \Sigma x \\
\downarrow{\scriptstyle \eta_x} & & \downarrow{\scriptstyle \eta_a} & & \downarrow{\scriptstyle \eta_{La}=L\eta_a} & & \downarrow{\scriptstyle \eta_{\Sigma x}} \\
Lx & \xrightarrow{Lf} & La & \xrightarrow{L\eta_a} & L^2a & \longrightarrow & \Sigma La
\end{array}$$

Since $L$ is a localizing functor, it follows that $L\eta_a$ is an isomorphism, and therefore $Lx \cong 0$. But then $x \in \ker(L)$ and therefore $f = 0$ since $a \in (\ker(L))^{\perp}$. It follows that $\eta_a$ is an isomorphism, so $a \in \operatorname{im}(L)$. □

**Proposition 6.18.** *Let $L$ be a localization functor for a tt-category $\mathcal{T}$. There is a functor $\Gamma : \mathcal{T} \to \mathcal{T}$ and a natural transformation $\gamma : \Gamma \to \mathrm{id}_{\mathcal{T}}$ such that all $a \in \mathcal{T}$ fit into an exact triangle*

$$\Gamma(a) \xrightarrow{\gamma_a} a \xrightarrow{\eta_a} L(a) \to \Sigma\Gamma(a)$$

*Furthermore, the pair $(\Gamma, \gamma)$ is a colocalizing functor, which is to say that it satisfies the same axioms as $L$ but with the arrows reversed.*

*Sketch.* Let $(L, \eta)$ be a localization functor. Then we can find an exact triangle

$$\Sigma^{-1}La \xrightarrow{d} \Gamma a \xrightarrow{\gamma_a} a \xrightarrow{\eta_a} La$$

We saw in the last proof that application of $L$ to an exact triangle of the form above results in $L\Gamma a \cong 0$, so $\Gamma a \in \ker(L)$. Then let $x \in \ker(L)$. From lemma 6.17 we have that $\operatorname{Hom}(x, La) = 0$ and so by applying the cohomological functor $\operatorname{Hom}_{\mathcal{T}}(x, -)$ to the exact triangle above we see that $\operatorname{Hom}_{\mathcal{T}}(x, \Gamma a) \cong \operatorname{Hom}_{\mathcal{T}}(x, a)$ and therefore $\Gamma a$ is terminal among $L$-acyclic objects over $a$. Now consider a morphism $f : a \to b$ in $\mathcal{T}$. We fit it into a morphism of exact triangles below:

$$\begin{array}{ccccccc}
\Sigma^{-1}La & \xrightarrow{d} & \Gamma a & \xrightarrow{\gamma} & a & \xrightarrow{\eta} & La \\
\downarrow{\scriptstyle \Sigma Lf} & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle Lf} \\
\Sigma^{-1}Lb & \xrightarrow{d} & \Gamma b & \xrightarrow{\gamma} & b & \xrightarrow{\eta} & Lb
\end{array}$$

The map $g$ comes from (TR3), but it is not a priori unique among maps making this diagram commute. However, due to the universal property of $\gamma_b$, $g$ must be unique, and therefore we can say that $\Gamma f := g$. This makes $\Gamma$ into a functor and $\gamma : \Gamma \to \mathrm{id}_{\mathcal{T}}$ into a natural transformation.

Since $L$ is an exact functor, we have $\Sigma L \cong L\Sigma$, and since $\eta$ is a morphism of exact functors, it follows that $\eta\Sigma = \Sigma\eta$. We can then construct a morphism of exact triangles below. Basically the same argument to the one used in the previous paragraph shows that $\Gamma\Sigma \cong \Sigma\Gamma$.

$$\begin{array}{ccccccc}
\Sigma^{-1}L\Sigma a & \xrightarrow{\Sigma^{-1}d\Sigma} & \Gamma\Sigma a & \xrightarrow{\gamma\Sigma} & \Sigma a & \xrightarrow{\Sigma\eta} & L\Sigma a \\
\downarrow{\scriptstyle \sim} & & \downarrow & & \| & & \downarrow{\scriptstyle \sim} \\
La & \xrightarrow{-d} & \Sigma\Gamma a & \xrightarrow{\Sigma\gamma} & \Sigma a & \xrightarrow{\Sigma\eta} & \Sigma La
\end{array}$$

To see that $\Gamma$ is exact, let $a \xrightarrow{f} b \xrightarrow{g} c \xrightarrow{h} \Sigma a$ be exact. Let $x$ be the cone of $\Gamma f$. We can then construct a morphism of exact triangles

$$\begin{array}{ccccccc} \Gamma a & \xrightarrow{\Gamma f} & \Gamma b & \xrightarrow{\alpha} & x & \xrightarrow{\beta} & \Sigma\Gamma a \\ \downarrow{\scriptstyle\eta} & & \downarrow{\scriptstyle\eta} & & \downarrow{\scriptstyle k} & & \downarrow{\scriptstyle\Sigma\gamma} \\ a & \xrightarrow{f} & b & \xrightarrow{g} & c & \xrightarrow{h} & \Sigma a \end{array}$$

Note that $x$ is $L$-acyclic since $L$ is exact and $L\Gamma = 0$. If we apply the homological functor $\mathrm{Hom}_{\mathcal{T}}(z, -)$to the morphism of triangles above we see that $k_* : \mathrm{Hom}_{\mathcal{T}}(z, x) \to \mathrm{Hom}_{\mathcal{T}}(z, c)$ is an isomorphism for all $L$-acyclic $z$. Therefore, $x$ is terminal among $L$-acyclic objects over $c$, so by our definition of $\Gamma$ we have $x \cong \Gamma c$ and can identify $k$ with $\gamma$. Similarly, $\alpha$ and $\beta$ are $\Gamma g$ and $\Gamma h$ respectively. It remains to show that $\gamma : \Gamma \to \mathrm{id}_{\mathcal{T}}$ satisfies $\Gamma\gamma : \Gamma^2 \xrightarrow{\sim} \Gamma$ and $\Gamma\gamma = \gamma\Gamma$, and that $\Gamma$ commutes with all products. To see this first claim, we consider the exact triangle below:

$$\Gamma\Gamma \xrightarrow{\gamma\Gamma} \Gamma a \xrightarrow{\eta} L\Gamma \longrightarrow \Sigma\Gamma\Gamma a$$

Note that $L\Gamma a = 0$, so $\Gamma\Gamma a \xrightarrow{\gamma\Gamma} \Gamma a$ is an isomorphism. For the last claim, note that the image of $\Gamma$ is closed under products, so the last claim follows from the universal property of $\gamma$. ☐

**Corollary 6.19.** *With $L$ and $\Gamma$ as above,* $\mathrm{im}\,\Gamma = \ker L$ *and* $\ker\Gamma = \mathrm{im}\,L$*. Additionally,* $\ker L$ *is a localizing ideal and* $\ker\Gamma$ *is a colocalizing ideal.*

*Remark* 6.20. We have now seen that given a localization functor $L : \mathcal{T} \to \mathcal{T}$, there is a functorial triangle $\Gamma a \xrightarrow{\gamma} a \xrightarrow{\eta} La \to \Sigma\Gamma a$. In the literature, this is often called the *gluing triangle*. Triangles of this form are somewhat special, in that the formation of a cone from a morphism is not usually functorial. By this we mean that axiomatically, given a morphism $f : a \to b$ there is always a some exact triangle $a \xrightarrow{f} b \to c \to \Sigma a$ into which $f$ fits, but there are potentially many choices of $c$, all of which are isomorphic but not functorially so.

**Proposition 6.21.** *Let $(\Gamma, L)$ and $(\Gamma', L')$ be pairs of corresponding colocalization/localization functors such that $\ker(L) \subset \ker(L')$ (equivalently, $\ker(\Gamma) \supset \ker(\Gamma')$). Then $L' \cong L'L$ and $\Gamma\Gamma' \cong \Gamma$*

*Proof.* We will only prove $L' \cong L'L$ since $\Gamma\Gamma' \cong \Gamma$ follows from duality. Let $a$ be an object. Then we have the gluing triangle below

$$\Gamma a \xrightarrow{\gamma} a \to La \xrightarrow{\eta} \Sigma\Gamma a$$

Apply $L'$ to get the exact triangle below:

$$L'\Gamma a \xrightarrow{L'\gamma} L'a \xrightarrow{L'\eta} L'La \to L'\Sigma\Gamma a$$

Since $\Gamma a \in \ker(L) \subset \ker(L')$ it follows that $L'a \xrightarrow{L'\eta} L'La$ is an isomorphism, and it is straightforward to check that it is a natural isomorphism. ☐

*Proof.* Follows from lemma 6.17 and proposition 6.18. ☐

**Theorem 6.22.** *Let $\mathcal{L}$ and $\mathcal{C}$ be localizing and colocalizing subcategories, respectively, of $\mathcal{T}$. Suppose that they satisfy the following condition:*

- $\mathcal{L} \perp \mathcal{C}$ *and for every* $a \in \mathrm{Obj}(\mathcal{T})$ *there is an exact triangle* $a' \to a \to a'' \to \Sigma a$ *with* $a' \in \mathcal{C}$ *and* $a'' \in \mathcal{L}$.

*Then there exists a localization functor* $L : \mathcal{T} \to \mathcal{T}$ *and associated colocalization functor* $\Gamma : \mathcal{T} \to \mathcal{T}$ *such that* $\ker L = \operatorname{im} \Gamma = \mathcal{L}$ *and* $\ker \Gamma = \operatorname{im} L = \mathcal{C}$.

*Proof.* Since localizing and colocalizing subcategories are thick, the theorem follows from 9.1.13 of [Nee01]. □

### 6.2 Six functor formalism

In the last section we explored localizing subcategories and localization functors, and it was remarked that these are related to the Verdier localization functors from earlier. We will now explore this relationship with an example taken from [Kra04].

Let $A$ be a finite dimensional Hopf algebra over a field $k$, e.g. $A = kG$ for a finite group $G$. Let $\mathcal{T} = \mathrm{K}(\text{Inj-}A)$ be the homotopy category of complexes of injective $A$-modules and $\mathrm{K}_{ac}(\text{Inj-}A)$ the subcategory of acyclic complexes. We saw in subsection 4.5 that there is an equivalence of tt-categories $\mathsf{StMod}(A) \xrightarrow{\sim} \mathrm{K}_{ac}(\text{Inj-}A)$ by taking the Tate resolution $M \to tM$. There is a Verdier quotient sequence,

$$\mathrm{D}(A) \xleftarrow{Q} \mathrm{K}(\text{Inj-}A) \xleftarrow{I} \mathrm{K}_{ac}(\text{Inj-}A)$$

It can be shown that $Q$ and $I$ possess both left and right adjoints. This is an example of *recollement*, or the *six functor formalism*, and it is traditionally rendered in a diagram of the form below:

$$\mathrm{D}(A) \underset{\xrightarrow{Q_\lambda}}{\overset{\xrightarrow{Q_\rho}}{\xleftarrow{Q}}} \mathrm{K}(\text{Inj-}A) \underset{\xrightarrow{I_\lambda}}{\overset{\xrightarrow{I_\rho}}{\xleftarrow{I}}} \mathrm{K}_{ac}(\text{Inj-}A)$$

We can then take compact objects to get a Verdier quotient in the other direction. This is *Rickard's localization sequence*.

$$\mathrm{D}^{\mathrm{pf}}(A) \xrightarrow{Q_\lambda} \mathrm{D}^{\mathrm{b}}(A) \xrightarrow{I_\lambda} \mathsf{stmod}(A)$$

The same procedure can be done to sheaves on a Noetherian scheme $X$:

$$\mathrm{D}(X) \underset{\longrightarrow}{\overset{\longrightarrow}{\longleftarrow}} \mathrm{K}(\text{Inj-}X) \underset{\longrightarrow}{\overset{\longrightarrow}{\longleftarrow}} \mathrm{K}_{ac}(\text{Inj-}X)$$

Here $\mathrm{K}_{ac}(X)$ is taken as definition for 'big stable category' of quasi-coherent sheaves $S(\mathrm{Qcoh}(X))$, and if we take compact objects we get another Verdier quotient.

$$\mathrm{D}^{\mathrm{pf}}(X) \xrightarrow{Q_\lambda} \mathrm{D}^{\mathrm{b}}(X) \xrightarrow{I_\lambda} \mathrm{Sing}(X)$$

where $\mathrm{Sing}(X)$ is the *singularity category* of our scheme $X$.

To relate this back to Bousfield localizations, if we have a Verdier quotient $\mathcal{T}/\mathcal{S} \xleftarrow{Q} \mathcal{T} \xleftarrow{I} \mathcal{S}$ such that $Q$ and $I$ both have right adjoints $Q_\rho, I_\rho$ and left adjoints $Q_\lambda, I_\lambda$ then we have that $QI = 0$ and therefore

$$I_\lambda Q_\lambda = 0 = I_\rho Q_\rho$$

There are also functorial distinguished triangles within $\mathcal{T}$:

$$II_\rho a \longrightarrow a \longrightarrow Q_\rho Q a \longrightarrow \Sigma II_\rho a$$

$$Q_\lambda Q a \longrightarrow a \longrightarrow II_\lambda a \longrightarrow \Sigma Q_\lambda Q a$$

These structures arise naturally in a number of contexts, as seen in the examples above. The Bousfield localizations of the last section are an axiomatization of this scenario, and was first developed in the form that we presented in [HPS97]. For more information on the six functor formalism and localizations see chapter 9 of [Nee01], chapter 4 of [Kra21].

### 6.3 Lavish Support

The Balmer spectrum $\mathrm{Spec}(\mathcal{T})$ was defined in the context that $\mathcal{T}$ was an essentially small category, that is, when $\mathcal{T}$ is equivalent to a category whose objects and morphisms actually form sets as opposed to proper classes. This situation is somewhat limiting, as it does not include many categories that we care about, such as the unbounded derived category of a scheme or ring, the big stable module category, or the stable homotopy category. We would like to be able to say something about these big categories using the language tensor triangulated geometry. The idea is to look at the compact objects $\mathcal{T}^c$ within the larger tt-category $\mathcal{T}$ and create an extension of the support data $(\mathrm{Spc}(\mathcal{T}^c), \mathrm{supp})$ to all of $\mathcal{T}$.

**Theorem 6.23.** *Let $\mathcal{T}$ be rigidly compactly generated tt-category. Let $X$ be a spectral topological space and let $\sigma : \mathrm{Obj}(\mathcal{T}) \to \mathscr{P}(X)$ be a function assigning every object of $\mathcal{T}$ a subset of $X$. Assume that the pair $(X, \sigma)$ satisfies the following axioms:*

*(LS0)* $\sigma(a) = \emptyset$ *if and only if* $a \cong 0$.

*(LS1)* $\sigma(\mathbb{1}) = X$

*(LS2)* $\sigma(\bigoplus_{i\in I} a_i) = \bigcup_{i\in I} \sigma(a_i)$ *for any small family of objects* $\{a_i\}_{i\in I} \subseteq \mathcal{T}$.

*(LS3)* $\sigma(\Sigma a) = \sigma(a)$

*(LS4)* $\sigma(b) \subset \sigma(a) \cup \sigma(c)$ *for every exact triangle* $a \to b \to c \to \Sigma a$

*(LS5)* $\sigma(a \otimes b) = \sigma(a) \cap \sigma(b)$ *for all* $b \in \mathcal{T}$ *and arbitrary* $a \in \mathcal{T}^c$.

*(LS6)* $\sigma(a)$ *is closed in* $X$ *with quasi-compact complement* $X \setminus \sigma(a)$ *for all* $a \in \mathcal{T}^c$

*(LS7)* *For every closed subset* $Z \subset X$ *with quasi-compact open complement, there exists a compact object* $a \in \mathcal{T}^c$ *with* $\sigma(a) = Z$.

*Then $(X, \sigma|_{\mathcal{T}^c})$ is a classifying support datum for $\mathcal{T}^c$, and so the induced map $X \to \mathrm{Spc}(\mathcal{T}^c)$ is a homeomorphism.*

**Definition 6.24.** If $(X, \sigma)$ is a pair satisfying the properties of theorem 6.23 then we call $(X, \sigma)$ a *lavish support datum* of $\mathcal{T}$.

*Remark* 6.25. Theorem 6.23 was published in [Del11]. The same result was independently announced by Julia Pevtsova and Paul Smith. The term *lavish support* comes from [NP23].

**Definition 6.26.** Let $(X, \sigma)$ be a lavish support datum and let $Y \subset X$. Analogous to definition 3.28 we introduce the following notation:

$$\mathcal{C}_Y \coloneqq \{a \in \mathcal{T}^c \mid \sigma(a) \subset Y\} \subset \mathcal{T}^c$$
$$\mathcal{T}_Y \coloneqq \langle \mathcal{C}_Y \rangle_{\mathrm{loc}} \subset \mathcal{T}$$

**Lemma 6.27.**

(a) *The subcategory $\mathcal{C}_Y \subset \mathcal{T}^c$ is a radical thick $\otimes$-ideal and $\mathcal{C}_Y = (\mathcal{T}_Y)^c$*

(b) *If $a \in \mathcal{T}_Y$, then $\sigma(a) \subset Y$.*

*Proof.* The claim that $\mathcal{C}_Y$ is a thick $\otimes$-ideal is just lemma 3.14 since we are staying within $\mathcal{T}^c$. To see that it is radical, let $a \in \mathcal{T}^c$ with $a^{\otimes n} \in \mathcal{C}_Y$ for some $n$. Then $\Sigma(a^{\otimes n}) \subset Y$, but then $\sigma(a) \subset Y$ by (LS5), so $a \in \mathcal{C}_Y$. It remains to show that $\mathcal{C}_Y = (\mathcal{T})^c$, but this follows from proposition 6.11.

For (b), the axioms (LS0) through (LS6) tell us that $\{a \in \mathcal{T} \mid \sigma(a) \subset \mathcal{T}\}$ is a localizing subcategory of $\mathcal{T}$. This subcategory obviously contains $\mathcal{Y}$, so it must contain $\mathcal{T}_Y = \langle \mathcal{C}_Y \rangle_{\text{loc}}$. ☐

**Lemma 6.28.** *Let $\mathcal{S} \subset \mathcal{T}^c$ be a self-dual collection compact objects, meaning that*

$$\mathcal{S} = \mathcal{S}^\vee := \{a^\vee \mid a \in \mathcal{S}\}$$

*Let $\sigma(\mathcal{S}) := \bigcup_{a\in\mathcal{S}} \sigma(a)$. Then the thick $\otimes$-ideal in $\mathcal{T}^c$ generated by $\mathcal{S}$ consists of the compact objects supported on $\sigma(S)$, i.e.*

$$\langle \mathcal{S} \rangle = \mathcal{C}_{\sigma(\mathcal{S})}$$

*Proof.* For brevity we denote $Y := \sigma(\mathcal{S})$. By theorem 6.22, there exist (co-)localization functors $(\Gamma_{\langle\mathcal{S}\rangle}, L_{\langle\mathcal{S}\rangle})$ and $(\Gamma_{\mathcal{C}_Y}, L_{\mathcal{C}_Y})$ such that $\operatorname{im}\Gamma_{\langle\mathcal{S}\rangle} = \operatorname{im}\Gamma_{\mathcal{C}_Y} = \ker L_{\langle\mathcal{S}\rangle} = \langle\mathcal{S}\rangle_{\text{loc}}$ and $\ker L_{\mathcal{C}_Y} = \mathcal{T}_Y$. From proposition 6.11, we can recover our original thick $\otimes$-ideals:

$$\langle\mathcal{S}\rangle = (\operatorname{im}\Gamma_{\langle\mathcal{S}\rangle})^c \quad \text{and} \quad \mathcal{C}_Y = (\operatorname{im}\Gamma_{C_Y})^c$$

A natural isomorphism between $\Gamma_{\langle\mathcal{S}\rangle}$ and $\Gamma_{\mathcal{C}_Y}$ will therefore imply $\langle\mathcal{S}\rangle = \mathcal{C}_Y$.

Note that $\langle\mathcal{S}\rangle \subset \mathcal{C}_Y$ by construction, so $\langle\mathcal{S}\rangle_{\text{loc}} \subset \mathcal{T}_Y$. By proposition 6.21, it follows that $\Gamma_{\langle\mathcal{S}\rangle}\Gamma_{\mathcal{C}_Y} \cong \Gamma_{\langle\mathcal{S}\rangle}$. Now let $a$ be an object of $\mathcal{T}$ and apply $\Gamma_{\langle\mathcal{S}\rangle}$ to the gluing triangle coming from $\Gamma_{\mathcal{C}_Y}$. We then have an exact triangle,

$$\Gamma_{\langle\mathcal{S}\rangle}\Gamma_{\mathcal{C}_Y} a \cong \Gamma_{\langle\mathcal{S}\rangle} a \to \Gamma_{\mathcal{C}_Y} a \to L_{\langle\mathcal{S}\rangle}\Gamma_{\mathcal{C}_Y} a \to \Sigma\Gamma_{\langle\mathcal{S}\rangle} a$$

Therefore we have the desired isomorphism if we can show that $L_{\langle\mathcal{S}\rangle}\Gamma_{\mathcal{C}_Y} a \cong 0$. For brevity, we write $b := L_{\langle\mathcal{S}\rangle}\Gamma_{\mathcal{C}_Y} a$. By (LS0) it suffices to show that $\sigma(B) = \emptyset$.

Since $\mathcal{T}_Y$ is a triangulated category and two of the objects in the triangle above are in $\mathcal{T}_Y$, it must be that $b \in \mathcal{T}_Y$, so $\sigma(b) \subset Y$ by lemma 6.27. Let $s \in \mathcal{S}$ and let $c$ be a compact object. Since $\mathcal{T}^c$ is rigid and $s \in \mathcal{T}^c$, so by remark 1.25 the functor $s^\vee \otimes -$ is both a left and right adjoint to $s \otimes -$. Hence,

$$\operatorname{Hom}_{\mathcal{T}}(c, s^\vee \otimes b) \cong \operatorname{Hom}_{\mathcal{T}}(c \otimes s, b) \cong 0$$

The right isomorphism comes from the fact that $c \otimes s \in \langle\mathcal{S}\rangle_{\text{loc}}$ and $b \in \operatorname{im} L_{\langle\mathcal{S}\rangle} = (\ker L_{\langle\mathcal{S}\rangle})^\perp = \langle\mathcal{S}\rangle_{\text{loc}}^\perp$. But then since $c$ is an arbitrary compact object, it follows that $s_\vee \otimes b \cong 0$, so $\sigma(s^\vee \otimes b) = \emptyset$. Finally, we employ the fact that $\mathcal{S}$ is self-dual to get

$$\sigma(b) \subset Y = \sigma(\mathcal{S}) = \sigma(\mathcal{S}^\vee)$$

which allows

$$\sigma(b) = \sigma(\mathcal{S}^\vee) \cap \sigma(b) = \bigcup_{s\in\mathcal{S}} \sigma(s^\vee) \cap \sigma(b) = \bigcup_{s\in\mathcal{S}} \sigma(s^\vee \otimes b) = \emptyset$$

where the third equality above comes from (LS5). ☐

*Proof of theorem 6.23.* It follows immediately from the axioms of a lavish support datum that the pair $(X, \sigma|_{\mathcal{T}^c})$ is a support datum for $\mathcal{T}^c$, so we only need to show that $(X, \sigma|_{\mathcal{T}^c})$ is classifying. Since $X$ is spectral, we only need to show that there is a mutually inverse bijection between Thomason subsets of $X$ and thick $\otimes$-ideals of $\mathcal{T}^c$:

$$\begin{array}{ccc} Y & \longmapsto & C_Y \\ \sigma(\mathcal{I}) & \longleftarrow & \mathcal{I} \end{array}$$

Recall that $\mathcal{C}_Y = \{a \in \mathcal{T}^c \mid \sigma(a) \subset Y\}$ and $\sigma(\mathcal{I}) = \bigcup_{a \in \mathcal{I}} \sigma(a)$. These maps are well defined as $\mathcal{C}_Y$ is a thick $\otimes$-ideal by lemma 6.27, and $\sigma(\mathcal{I})$ is Thomason due to axiom (LS6) for any subcategory $\mathcal{I} \subset \mathcal{T}^c$.

Let $\mathcal{I}$ be a thick $\otimes$-ideal in $\mathcal{T}^c$. By lemma 6.28 and proposition 3.35, $\mathcal{I} = \mathcal{C}_{\sigma(\mathcal{I})}$ where we take $\mathcal{S}$ to be $\mathcal{I}$.

On the other hand, let $Y$ be a Thomason subset, so $Y = \bigcup_i Y_i$ where each $Y_i$ is as closed subset with quasi-compact complement $X \setminus Y_i$ (equivalently, $Y$ is open in the Hochster dual of $X$). Then

$$\sigma(\mathcal{C}_Y) = \bigcup_{a \in \mathcal{C}_Y} \sigma(a) = \bigcup_{\sigma(a) \subset Y} \sigma(a) \subseteq Y$$

For the other containment, axiom (LS7) yields an object $x_i \in \mathcal{T}^c$ such that $\sigma(x_i) = Y_i$ for all $i$. This means $x_i \in \mathcal{C}_{Y_i} \subset \mathcal{C}_Y$, and therefore $Y = \bigcup_i \sigma(x_i) \subset \sigma(\mathcal{C}_Y)$, so $\sigma(\mathcal{C}_Y) = Y$. □

**Example 6.29.** Let $G$ be a finite group and $k$ a field. Recall that the stable module category $\mathsf{stmod}(kG)$ forms the full tt-subcategory of rigid compact objects within $\mathsf{StMod}(kG)$. It turns out that the support varieties of section 4.2 form a lavish support datum. See [BCR95] and [BCR96]. All the axioms except (LS5) and the backwards direction of (LS9), are straightforward to prove. The former requires theorem 10.8 of [BCR96] and the latter by Chouinard's theorem.

# A Triangulated Categories

## A.1 The Octohedral Axiom

Here is an undiagrammatic presentation of TR4. Given a composition $X \xrightarrow{f} Y \xrightarrow{g} Z$ and exact triangles $(X, Y, Z'; f_1, p_1, d_1)$, $(X, Z, Y'; g \circ f, p_2, d_2)$, and $(Y, Z, X'; g, p_3, d_3)$, there exist morphisms $Z' \xrightarrow{a} Y'$ and $Y' \xrightarrow{b} X'$ such that

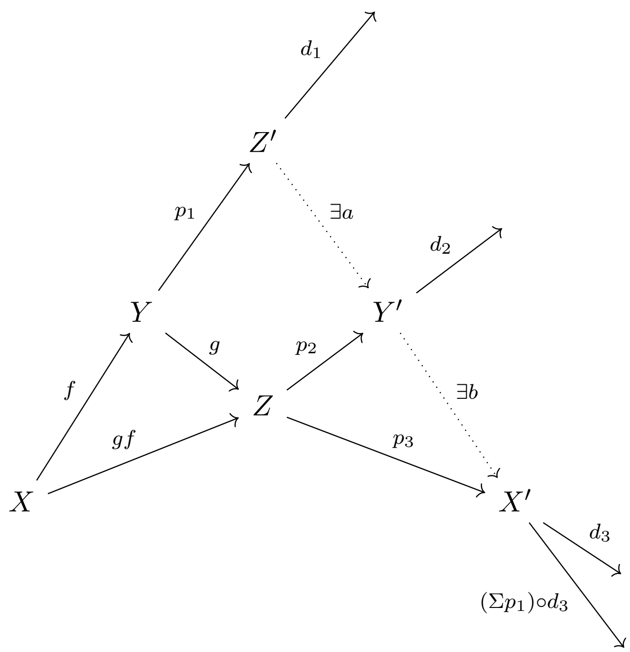
(a) $(Z', Y', X'; a, b, \Sigma p_1 \circ d_3)$ is exact,

(b) the triple $(\mathrm{id}_X, g, a)$ is a morphism of triangles $(X, Y, Z'; f, p_1, d_1) \to (X, Z, Y'; g \circ f, p_2, d_2)$, and

(c) the triple $(f, \mathrm{id}_Z, b)$ is a morphism of triangles $(X, Z, Y'; g \circ f, p_2, d_2) \to (Y, Z, X'; g, p_3, d_3)$.

Here one should think of $Z' = \mathrm{cone}(f)$, $Y' = \mathrm{cone}(gf)$ and $X' = \mathrm{cone}(g)$.

Here is yet another presentation, utilizing a different diagram.




This view emphasizes some of the intuition of the octohedral axiom a little more clearly. Since exact triangles are supposed to be thought of as playing the role of exact sequences, we might want to think of our cones as quotients, i.e. $Z' = Y/X$, $Y' = Z/X$. For $X'$ we have $X' = Z/Y$ from the triangle $Y \to Z \to X' \to ...$ and $X' = Y'/Z'$ from the triangle $Z' \to Y' \to X' \to ....$ Then,

$$(Z/X)/(Y/X) \cong (Y')/(Z') \cong X' \cong Z/Y$$

This looks a lot like the third isomorphism theorem, and so one way to view the octohedral axiom is as a coherence condition enforcing the kind of quotient isomorphisms that we expect to see in algebraic settings.

## A.2 Definitions and Elementary Results

Much of the exposition here is adapted from [Stacks, Section 05QN].

**Definition A.1.** Let $\mathcal{T}$ be a triangulated category and $\mathcal{A}$ an abelian category. An additive functor $H : \mathcal{T} \to \mathcal{A}$ is called *homological* if for every exact triangle $(X, Y, Z; f, g, h)$ the sequence given by the image

$$H(X) \to H(Y) \to H(Z)$$

is exact in $\mathcal{A}$. An additive functor $H : \mathcal{T}^{\text{op}} \to \mathcal{A}$ is called *cohomological* if the corresponding opposite functor $\mathcal{T} \to \mathcal{A}^{\text{op}}$ is homological.

If $H : \mathcal{T} \to \mathcal{A}$ is homological then we'll write $H_n(X) \coloneqq H(\Sigma^n X)$ and $H_0(X) \coloneqq H(X)$. Then, this means that for every exact triangle $(X, Y, Z; f, g, h)$ we get a long exact sequence

$$\ldots \to H_{-1}(Z) \to H_0(X) \to H_0(Y) \to H_0(Z) \to H_0(\Sigma X) = H_1(X) \to \ldots$$

The long exact sequence associated to $(X, Y, Z; f, g, h)$ by $H$ is called the *long exact sequence* associated to the triangle by $H$.

**Definition A.2.** Let $\mathcal{T}$ be a triangulated category and $\mathcal{A}$ an abelian category. A $\delta$*-functor* between $\mathcal{A}$ and $\mathcal{T}$ is functor $G : \mathcal{A} \to \mathcal{T}$ and functorial assignment from short exact sequences $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ in $\mathcal{A}$ to exact triangles in $\mathcal{T}$. Explicitly, for any $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ there is a morphism $\delta_{f,g} : G(C) \to \Sigma G(A)$ such that

i. $(G(A), G(B), G(C); G(f), G(g), \delta_{f,g})$ is an exact triangle, and

ii. For any morphism of short exact sequences $\phi : (A \xrightarrow{f} B \xrightarrow{g} C) \to (A' \xrightarrow{f'} B' \xrightarrow{g'} C')$ the diagram

$$\begin{array}{ccc} G(C) & \xrightarrow{\delta_{fg}} & \Sigma G(A) \\ {\scriptstyle G(\phi_C)}\downarrow & & \downarrow{\scriptstyle \Sigma G(\phi_A)} \\ G(C') & \xrightarrow[\delta_{f'g'}]{} & \Sigma G(A') \end{array}$$

The two definitions above are of critical importance as they axiomatize the relationship between short exact sequences and their derived long exact counterparts. As one should expect, for any $A \in \mathcal{T}$ the functor $\text{Hom}_{\mathcal{T}}(A, -)$ is homological and $\text{Hom}_{\mathcal{T}}(-, A)$ is cohomological.

**Lemma A.3.** *If $\mathcal{T}$ is a triangulated category and $(X, Y, Z; f, g, h)$ is an exact triangle then $g \circ f, h \circ g, \Sigma f \circ h$ are all the zero map.*

*Proof.* From TR1 we know that $(X, X, 0; \text{id}_X, 0, 0)$ is exact. Then by TR3 we know that the dashed map below exists making the diagram commute

$$\begin{array}{ccccccc} X & \xrightarrow{\text{id}_X} & X & \longrightarrow & 0 & \longrightarrow & \Sigma X \\ \| & & \downarrow{\scriptstyle f} & & \downarrow & & \| \\ X & \xrightarrow{f} & Y & \xrightarrow{g} & Z & \xrightarrow{h} & \Sigma X \end{array}$$

but the dashed map must be the zero map, and so by commutativity $g \circ f = 0$. The other compositions follow from rotation of the triangle. $\square$

**Proposition A.4.** *For any object $A \in \mathcal{T}$ the functor $\mathrm{Hom}_{\mathcal{T}}(A, -)$ is homological and $\mathrm{Hom}_{\mathcal{T}}(-, A)$ is cohomological.*

*Proof.* Since $\mathcal{T}$ is an additive category $\mathrm{Hom}_{\mathcal{T}}(-, A)$ is an additive functor. By the lemma previous $\mathrm{Hom}_{\mathcal{T}}(-, A)$ takes exact triangles to chain complexes of abelian groups, so it remains to show exactness. Let $(X, Y, Z; f, g, h)$ be an exact triangle. Then using T3 and rotation, given $\phi \in \mathrm{Hom}_{\mathcal{T}}(A, Z)$ we can find $\psi$ such that the diagram below commutes:

$$\begin{array}{ccccccc} 0 & \longrightarrow & A & \longrightarrow & A & \longrightarrow & 0 \\ \downarrow & & \downarrow{\scriptstyle \exists\psi} & & \downarrow{\scriptstyle \phi} & & \downarrow \\ X & \xrightarrow{f} & Y & \xrightarrow{g} & Z & \xrightarrow{h} & \Sigma X \end{array}$$

Therefore, $g \circ \psi = \phi$. We can do the same for each position in the triangle, and therefore we have exactness of the long exact sequence induced by $\mathrm{Hom}_{\mathcal{T}}(A, -)$. The proof for $\mathrm{Hom}_{\mathcal{T}}(-, A)$ is analogous. $\square$

**Proposition A.5.** *If $(\alpha, \beta, \gamma) : (X, Y, Z; f, g, h) \to (X', Y', Z'; f', g', h')$ is a morphisms of exact triangles such that any two of $\alpha, \beta, \gamma$ are isomorphisms, then the third is also an isomorphism.*

*Proof.* Without loss of generality assume that $\alpha$ and $\gamma$ are isomorphisms. Then let $A \in \mathrm{Obj}(\mathcal{T})$. Abbreviate $\mathrm{Hom}_{\mathcal{T}}(A, -)$ as $H_A$. Then all the maps in the diagram below are isomorphisms, save for the middle one:

$$\begin{array}{ccccccccc} H_A(\Sigma Z) & \longrightarrow & H_A(X) & \longrightarrow & H_A(Y) & \longrightarrow & H_A(Z) & \longrightarrow & H_A(\Sigma X) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ H_A(Z') & \longrightarrow & H_A(X') & \longrightarrow & H_A(Y') & \longrightarrow & H_A(Z') & \longrightarrow & H_A(\Sigma X') \end{array}$$

Then, by the 5-lemma, the middle map is an isomorphism, so $\mathrm{Hom}_{\mathcal{T}}(A, Y) \cong \mathrm{Hom}_{\mathcal{T}}(A, Y')$ via $\mathrm{Hom}_{\mathcal{T}}(\beta, A)$ for any arbitrary $A$. By Yoneda's lemma it follows that $Y \xrightarrow{\beta} Y'$ is an isomorphism. $\square$

*Remark* A.6. This proof actually give us a little more than advertised. It says that if we have a morphism of (not necessarily exact) triangles $(\alpha, \beta, \gamma) : (X, Y, Z; f, g, h) \to (X', Y', Z'; f', g', h')$ such that the long exact sequences coming from $\mathrm{Hom}_{\mathcal{T}}(W, -)$ on the triangles are exact for all $W \in \mathrm{Obj}(\mathcal{T})$, then any two of $\alpha, \beta, \gamma$ being isomorphisms implies that the third is an isomorphism. We will call such triangles *special triangles* It's worth pointing this out since this conclusion is slightly stronger as there are triangles for which this condition holds that are not exact.

**Corollary A.7.** *Given triangles $(X, Y, Z; f, g, h)$ and $(X', Y', Z'; f', g', h')$, the triangle*

$$(X \oplus X', Y \oplus Y', Z \oplus Z'; f + f', g + g', h + h')$$

*is exact if and only if both $(X, Y, Z; f, g, h)$ and $(X', Y', Z'; f', g', h')$ are exact.*

*Proof.* Assume that the two individual triangles are exact. By T2 there exists $Q$ such that $(X \oplus X', Y \oplus Y', Q; f + f', g'', h'')$ is exact. By TR3 there are morphisms from $(X, Y, Z; f, g, h)$ and $(X', Y', Z'; f', g', h')$ into $(X \oplus X', Y \oplus Y', Q; f + f', g'', h'')$ induced by the inclusion maps from the components of the triangles into their direct sums. This in turn induces a map

$$(X \oplus X', Y \oplus Y', Z \oplus Z'; f + f', g + g', h + h') \xrightarrow{(\mathrm{id},\mathrm{id},\alpha)} (X \oplus X', Y \oplus Y', Q; f + f', g'', h'')$$

By proposition A.5, $\alpha$ is an isomorphism and therefore the direct sum of triangles is exact.

Now suppose that the direct sum is exact. We will show that $(X, Y, Z; f, g, h)$ is exact, and by symmetry the result will follow. Using TR2 and TR3 there is an exact triangle $(X, Y, Q; f, g'', h'')$ and a morphism of exact triangles $(\pi_X, \Pi_Y, p) : (X \oplus X', Y \oplus Y', Z \oplus Z'; f + f', g + g', h + h') \to (X, Y, Q; f, g'', h'')$. We can then get a morphism of triangles $(X, Y, Z; f, g, h) \to (X, Y, Q; f, g'', h'')$ by precomposing with the inclusion maps coming from $X, Y, Z$. The long exact sequence on $(X \oplus X', Y \oplus Y', Z \oplus Z'; f + f', g + g', h + h')$ coming from $\mathrm{Hom}_{\mathcal{T}}(W, -)$ will split into the direct sum of exact complexes coming from $(X, Y, Z; f, g, h)$ and $(X', Y', Z'; f', g', h')$ and therefore $(X, Y, Z; f, g, h)$ satisfies the condition in *remark A.6*. Therefore, the morphism $(X, Y, Z; f, g, h) \to (X, Y, Q; f, g'', h'')$ of triangles is a morphism on the third component since the other two components are the identity map. Hence, $(X, Y, Z; f, g, h)$ is isomorphism to an exact triangle and is therefore itself exact. ☐

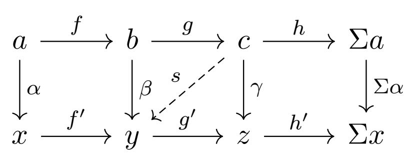
**Corollary A.8.** *For any two objects $A$ and $B$ of $\mathcal{T}$, the triangle $(A, A \oplus B, B; (\mathrm{id}_A, 0), (0, \mathrm{id}_B), 0)$ is exact.*

*Proof.* Apply corollary A.7 to the exact triangles $(A, A, 0; \mathrm{id}_A, 0, 0)$ and $(0, B, B; 0, \mathrm{id}_B, 0)$. ☐

**Corollary A.9.** *If $\mathcal{T}'$ is a triangulated subcategory of $\mathcal{T}$ and $a, b$ are objects in $\mathcal{T}'$, then $a \oplus b \in \mathcal{T}'$.*

*Proof.* As $a, b$ in $\mathcal{T}'$ it follows from corollary A.8 and remark 1.7 that $a \oplus b \in \mathcal{T}'$. ☐

**Lemma A.10.** *Let the diagram below is a morphism of exact triangles.*




*Suppose that $\beta \circ f = 0$. Then $\exists s : c \to y$ such that $s \circ g = \beta$.*

*Proof.* Abbreviate the functor $\mathrm{Hom}(-, y)$ as $\mathrm{H}_y(-)$. Note that $\beta^*(\mathrm{id}_y) = \beta$ and that $f^* \circ \beta^* = 0$ as $\beta \circ f = 0$, i.e. $\beta \in \ker(f^*)$. Since the top row of the diagram is exact, it follows that $\ker(f^*) = \mathrm{im}(g^*)$ so $\exists s \in \mathrm{H}_y(c)$ such that $g^*(s) = \beta$, so $s \circ g = \beta$ are we are done. ☐

# B Spectral Spaces

Here we will develop a little more machinery to do with spectral spaces. Specifically, we want just enough to prove the theorem below:

**Theorem B.1.** *If $f : X \to Y$ is a spectral map of spectral spaces and an isomorphism on the specialization orders of $X$ and $Y$, then $f$ is a homeomorphism.*

The content presented here follows [DST19], which is the most comprehensive reference on spectral spaces that the author knows of.

## B.1 The Constructible Topology

The specialization order carries a lot of data about a spectral space, but not all of it, meaning that the partial order coming from a spectral space is not enough to reconstruct the space. One needs slightly more information, and luckily there is another natural topology that may be associated to a spectral space $X$.

**Definition B.2.** Given a topological space $X$ we define $\overline{\mathcal{K}}(X) \coloneqq \{Y \subset X \mid X \setminus Y \in \mathcal{K}^\circ(X)\}$, i.e. the collection of complements of quasi-compact opens.

*Remark* B.3. With this notation, $\mathcal{K}^\circ(X) = \overline{\mathcal{K}}(X_{\text{inv}})$ and $\overline{\mathcal{K}}(X) = \mathcal{K}^\circ(X_{\text{inv}})$.

**Definition B.4.** Let $X$ be a spectral space. We define the *constructible topology* $X_{\text{con}}$ to be the topology obtained by taking $\mathcal{K}^\circ(X) \cup \overline{\mathcal{K}}(X)$ as an open subbase. Equivalently, $X_{\text{con}}$ is the smallest topology refining both $X$ and $X_{\text{inv}}$.

An element of $\mathcal{K}^\circ(X_{\text{con}})$ is called *constructible*, and we will denote $\mathcal{K}(X) \coloneqq \mathcal{K}^\circ(X_{\text{con}})$.

**Fact B.5.** Sets of the form $V \cap U$ where $V \in \overline{\mathcal{K}}(X)$ and $U \in \mathcal{K}^\circ(X)$ are closed and open in $X_{\text{con}}$ and form a basis of open subsets.

*Remark* B.6. The constructible topology alone does not carry enough information to determine the two spectral spaces from which it arose, but it has the advantage of being easier to work with. In particular, $X_{\text{con}}$ is Hausdorff since open sets in $X_{\text{con}}$ separate points essentially by definition. Additionally, we will use the *finite intersection property* to determine that $X_{\text{con}}$ is compact.

**Definition B.7.** Let $X$ be a set. A collection of sets $\mathcal{U} \subset \mathcal{P}(X)$ is said to have the finite intersection property if the intersection of any finite subcollection of sets in $\mathcal{U}$ is non-empty.

**Proposition B.8.** *A topological space $X$ is quasi-compact if and only if every collection of closed subsets satisfying the finite intersection property has non-empty total intersection.*

*Proof.* Let $X$ be a topological space. Suppose that $X$ is quasi-compact. Suppose for the sake of contradiction that we may find a collection of closed sets $\mathcal{C} = \{C_i\}_{i\in I}$ satisfying the finite intersection property such that $\cap_{i\in I} C_i = \emptyset$. Then, let $U_i \coloneqq X \setminus C_i$, so we have

$$\begin{aligned} X &= X \setminus \bigcap_{i\in I} C_i \\ &= \bigcup_{i\in I} U_i \end{aligned}$$

so the $U_i$ form an open cover of $X$. By quasi-compactness, we may find a finite subcover $U_1, ..., U_n$ of $X$. But then by Demorgans law,

$$\bigcap_{i=1}^{n} C_i = X \setminus \bigcup_{i=1}^{n} U_i = X \setminus X = \emptyset$$

which contradicts the finite intersection property.

The reverse direction more or less follows the above argument in reverse. □

**Theorem B.9.** *Let $X$ be a spectral space. The constructible topology $X_{\text{con}}$ is Hausdorff, totally disconnected, and compact.*

*Proof.* Let $x$ and $y$ be distinct points in $X$. Since $X$ is sober, it follows that there exists an open set $U$ containing $x$ but not $y$. We may replace $U$ with a quasi-compact open set since these kinds of sets form a basis for $X$. Then by definition of $X_{\text{con}}$ we have that $U$ and $U^c := X \setminus U$ are both closed and open. Then $y \in U^c$ is an open set, so $X$ is Hausdorff but since $U^c$ is also open and closed we have that $x$ and $y$ are in different connected components, so $X$ is totally disconnected.

From fact B.5 we know that the collection below forms a closed basis for $X_{\text{con}}$.

$$\mathcal{B} = \{U \cup V \mid U \in \mathcal{K}^{\circ}(X) \text{ and } V \in \overline{\mathcal{K}}(X)\}$$

Therefore, to prove that $X_{\text{con}}$ is compact it suffices to show that every subset $\mathcal{U} \subseteq \mathcal{B}$ that satisfies the finite intersection property has nonempty intersection. Here we really do mean compact as opposed to quasi-compact, as $X_{\text{con}}$ is actually Hausdorff.

Using Zorn's lemma we may assume that $\mathcal{U}$ is maximal among subsets of $\mathcal{B}$ which have the finite intersection property. Define $P$ to be the intersection of all elements of $\mathcal{U}$ having quasi-compact complement, i.e.

$$I := \bigcap_{V \in \mathcal{U} \cap \overline{\mathcal{K}}(X)} V$$

<u>Claim:</u>

(a) If $A, B \in \mathcal{B}$ with $A \cup B \in \mathcal{U}$ then $A \in \mathcal{U}$ or $B \in \mathcal{U}$.

(b) If $U \in \mathcal{K}^{\circ}(X)$ then $U \in \mathcal{U}$ if and only if $I \cap U \neq \emptyset$.

*Proof of claim.* To show (a), assume that $A, B \in \mathcal{U}$.

By maximality of $\mathcal{U}$ we may find $A_1, ..., A_n, B_1, ..., B_k \in \mathcal{U}$ such that

$$A \cap A_1 \cap ... \cap A_n = B \cap B_1 \cap ... \cap B_k = \emptyset$$

But then this would imply

$$(A \cup B) \cap A_1 \cap ... \cap A_n \cap B_1 \cap ... \cap B_k = \emptyset$$

This forces $A \cup B \notin \mathcal{U}$.

Now for (b). Let $U \in \mathcal{K}^{\circ}(X)$. If $U \in \mathcal{U}$ then the set $\{U\} \cup \{\mathcal{U} \cap \overline{\mathcal{K}}(X)\}$ is contained in $\mathcal{U}$, and therefore has the finite intersection property as well by maximality of $\mathcal{U}$. Since $U$ was assumed to be quasi-compact we have that any collection of closed subsets in $U$ satisfying the finite intersection property has non-empty total intersection, so therefore $U \cap I \neq \emptyset$. On the other hand, if $U \notin \mathcal{U}$ then $X \setminus U \in \mathcal{U}$ by (a) since $X = U \cup (X \setminus U) \in \mathcal{U}$. Therefore, $X \setminus U \in \mathcal{U} \cap \overline{\mathcal{K}}(X)$. Hence $I \subseteq X \setminus U$. □

Now wwe want to show that $I$ is an irreducible subset of $X$. Let $U_1, U_2 \in \mathcal{K}^\circ(X)$ where $I \cap U_1 \neq \emptyset \neq I \cap_2$. By (b) above we have that $U_1$ and $U_2$ are in $\mathcal{U}$. Spectral spaces by definition have the property that the intersection of two quasi-compact sets is again quasi-compact, so therefore $U_1 \cap U_2 \in \mathcal{K}^\circ(X) \subseteq \mathcal{B}$. By maximality of $\mathcal{U}$, we have that $U_1 \cap U_2 \in \mathcal{U}$, so by using (b) again we see that $I \cap U_1 \cap U_2 \neq \emptyset$, so $I$ is irreducible.

Now we know that $I$ is closed and irreducible. If we set $U = X$ in (b) we see that $I \neq \emptyset$. Since $X$ is sober, there is a unique point $x \in X$ such that $\overline{\{x\}} = I$. We want to show that $x \in U \cup V \in \mathcal{U}$ for all choices of $U \in \mathcal{K}^\circ(X)$ and $V \in \overline{\mathcal{K}}(X)$. By (a) above we know that $U \in \mathcal{U}$ or $V \in \mathcal{U}$. In the second case, $x \in I \subseteq V$ since $P$ is closed irreducible and $V$ is closed. In the first case (b) yields $I \cap U \neq \emptyset$, and since all non-empty open subsets of an irreducible space are dense, it follows that $x \in U$. Therefore, $x \in \bigcap \mathcal{U}$, and the claim is proved. □

**Proposition B.10.** *A set map between spectral spaces $f : X \to Y$ is a spectral map if and only if it is continuous in both the spectral topology and the constructible topology.*

*Proof.* First, we need two preliminary facts:

Let $U \subset X$. Note that $U \in \mathcal{K}(X)$, i.e. is compact and open in the constructible topology, if and only if $U$ is closed and open in $X_{\text{con}}$. This is immediate from the fact that $X_{\text{con}}$ is compact and Hausdorff.

Now we want to show that $U \subset X$ is quasi-compact and open if and only if $U$ is open and constructible. Let $U$ be quasi-compact and open. Then $U$ is open in $X_{\text{con}}$ since $X$ is coarser than $X_{\text{con}}$, but $U$ must also be closed in the constructible topology since $X \setminus U$ is open in $X_{\text{inv}}$ and $X_{\text{con}}$ refines $X_{\text{inv}}$. By the paragraph above, $U$ is constructible. For the converse it suffices to show that constructible subsets of $X$ are quasi-compact in $X$ but this is immediate since constructible subsets of $X$ are compact in $X_{\text{con}}$ which is finer than $X$.

Now we are ready for the actual proof. Recall that a continuous map is called spectral if the pre-image of quasi-compact sets are quasi-compact. Suppose that $f$ is continuous for both topologies. Then for every $V \in \mathcal{K}^\circ(Y)$, the set $f^{-1}(V) \subseteq X$ is open by continuity for the spectral topologies and constructible by continuity of the constructible topologies, and is therefore open (by the paragraph above). Conversely, assume that $f$ is spectral. Since spectral spaces have their quasi-compact opens as a basis, it follows that $f$ is already continuous in the spectral topologies, so it remains to show that $f$ is continuous in the constructible topology. Since $\mathcal{K}^\circ(Y) \cup \overline{\mathcal{K}}(Y)$ is a subbasis for the constructible topology on $Y$, it suffices to show that $f^{-1}(U)$ and $f^{-1}(V)$ are constructibly open if $U \in \mathcal{K}^\circ(Y)$ and $V \in \overline{\mathcal{K}}(V)$. This is immediate from the fact that $f^{-1}(U) \subset \mathcal{K}^\circ(X) \subseteq \mathcal{K}^\circ(X_{\text{con}})$ and $f^{-1}(V) \subset \overline{\mathcal{K}}(X_{\text{con}})$. □

*Remark* B.11. Since $X_{\text{con}} = (X_{\text{inv}})_{\text{con}}$, proposition B.10 says that a map between spectral spaces is spectral if and only if it is continuous in the original topology, the constructible topology, and the inverse topology. In particular, $f : X \to Y$ it is spectral if and only $f_{\text{inv}} : X_{\text{inv}} \to Y_{\text{inv}}$ is spectral.

**Proposition B.12.** *If $X$ is a spectral space, then a subset $E \subseteq X$ is closed in $X$ if and only if $E$ is constructible and stable under specialization.*

*Proof.* The forwards direction is easy as closed sets are stable under specialization and $X_{\text{con}}$ is finer than $X$.

Let $x \in \overline{E}$. Then let $\mathcal{U} \coloneqq \{U_i\}$ be the collection of quasi-compact neighborhoods of $x$. It is an elementary fact that $\bigcup_i U_i = \{y \in X \mid x \in \overline{\{y\}}\}$, so if we can find some point $y$ in $\bigcap_i(U_i \cap E)$ then $x \in \overline{\{y\}}$. Since $x \in \overline{E}$, we have that $U_i \cap E \neq \emptyset$ for all $i$. Note that by the axioms of a spectral space $U_i \cap U_j \in \mathcal{U}$ for all $i, j$ and each such intersection is closed in the constructible topology. Therefore, $\{E\} \cup \mathcal{U}$ has the finite intersection property, so by quasi-compactness of $X_{\text{con}}$ we have that $\bigcap(U_i \cap E) \neq \emptyset$. It follows that there exists $y \in E$ such that $x \in \overline{\{y\}}$. If $E$ is stable under specialization, then $x \in E$ making $E$ closed. □

Now we are ready for the main result of this section.

**Theorem B.13.** *Let $f : X \to Y$ be a spectral map of between spectral spaces. Then $f$ isomorphism on the specialization orders of $X$ and $Y$ if and only if $f$ is a homeomorphism.*

*Proof.* The spectral map $f$ being an order isomorphism means that it must be both continuous and bijective. By proposition B.10 $f$ is also continuous in the spectral topology, and since $X_{\text{con}}$ and $Y_{\text{con}}$ are compact Hausdorff spaces, it follows that $f$ is a homeomorphism on the constructible topologies. Since continuous maps preserve specialization and $f$ is a closed map in the constructible topology, it follows from proposition B.12 that $f$ is a closed map. Thus, $f$ is a bijective closed continuous map, and is therefore a homeomorphism. The converse is obvious. □

## References


[Hoc69] M. Hochster. “Prime Ideal Structure in Commutative Rings”. In: *Trans. Amer. Math. Soc.* 142 (1969), pp. 43–60. [28, 29]

[Fox79] Hans-Bjørn Foxby. “Bounded complexes of flat modules”. In: *Journal of Pure and Applied Algebra* 15.2 (1979), pp. 149–172. ISSN: 0022-4049. DOI: https://doi.org/10.1016/0022-4049(79)90030-6. URL: https://www.sciencedirect.com/science/article/pii/0022404979900306. [33]

[Joh82] P.T. Johnstone. *Stone Spaces*. Cambridge Studies in Advanced Mathematics. Cambridge University Press, 1982. ISBN: 9780521337793. [27]

[Hop87] Michael J. Hopkins. *Global methods in homotopy theory*. Ed. by E. Rees and J. D. S.Editors Jones. London Mathematical Society Lecture Note Series. Cambridge University Press, 1987, pp. 73–96. [3, 43]

[DHS88] Ethan S. Devinatz, Michael J. Hopkins, and Jeffrey H. Smith. “Nilpotence and Stable Homotopy Theory I”. In: *Annals of Mathematics* 128.2 (1988), pp. 207–241. ISSN: 0003486X, 19398980. URL: http://www.jstor.org/stable/1971440 (visited on 08/31/2025). [3, 43]

[Hap88] Dieter Happel. *Triangulated Categories in the Representation of Finite Dimensional Algebras*. London Mathematical Society Lecture Note Series. Cambridge University Press, 1988. [15]

[Ric89] Jeremy Rickard. “Derived categories and stable equivalence”. In: *Journal of Pure and Applied Algebra* 61.3 (1989), pp. 303–317. ISSN: 0022-4049. DOI: https://doi.org/10.1016/0022-4049(89)90081-9. [49]

[Nee92] Amnon Neeman. “The chromatic tower for $D(R)$”. In: *Topology* 31.3 (1992), pp. 519–532. ISSN: 0040-9383. [3, 43]

[Wei94] C. A. Weibel. *Introduction to homological algebra*. Vol. 38. Cambridge Studies in Advanced Mathematics. Cambridge University Press, 1994. [5, 12]

[BCR95] David J. Benson, Jon F. Carlson, and J. Rickard. "Complexity and varieties for infinitely generated modules". In: *Mathematical Proceedings of the Cambridge Philosophical Society* 118.2 (1995), pp. 223–243. DOI: 10.1017/S0305004100073618. [84]

[Eis95] David Eisenbud. *Commutative Algebra with a View Towards Algebraic Geometry*. Cambridge Studies in Advanced Mathematics. Springer-Verlag, New York Inc, 1995. [39]

[BCR96] David J. Benson, Jon F. Carlson, and J. Rickard. "Complexity and varieties for infinitely generated modules, II". In: *Mathematical Proceedings of the Cambridge Philosophical Society* 120.4 (1996), pp. 597–615. DOI: 10.1017/S0305004100001584. [84]

[BCR97] David J. Benson, Jon F. Carlson, and Jeremy Rickard. "Thick subcategories of the stable module category". eng. In: *Fundamenta mathematicae* 153.1 (1997), pp. 59–80. ISSN: 0016-2736. [43, 71]

[FS97] Eric M. Friedlander and Andrei Suslin. "Cohomology of finite group schemes over a field". In: *Inventiones Mathematicae* 127.2 (1997), pp. 209–270. [71]

[HPS97] M. Hovey, J. H. Palmieri, and N. P. Strickland. "Axiomatic stable homotopy theory". In: *Mem. Amer. Math. Soc.* 128(610) (1997). [3, 76, 78, 82]

[Tho97] R. W. Thomason. "The classification of triangulated subcategories". In: *Compositio Mathematica* 105.1 (1997), pp. 1–27. DOI: 10.1023/A:1017932514274. [3, 25, 27, 34, 43]

[Ben98] David J. Benson. *Representations and Cohomology, Vol. II: Cohomology of Groups and Modules*. Cambridge Studies in Advanced Mathematics. Cambridge University Press, 1998. [48]

[BS01] Paul Balmer and Marco Schlichting. "Idempotent Completion of Triangulated Categories". In: *Journal of Algebra* 236.2 (2001), pp. 819–834. ISSN: 0021-8693. [44, 45]

[Nee01] Amnon Neeman. *Triangulated Categories*. Annals of Mathematics Studies. Princeton University Press, 2001. [5, 11, 76–78, 81, 82]

[Kra04] Henning Krause. *The stable derived category of a noetherian scheme*. 2004. arXiv: math/0403526 [math.AG]. URL: https://arxiv.org/abs/math/0403526. [76, 81]

[Bal05] Paul Balmer. "The spectrum of prime ideals in tensor triangulated categories". In: *J. Reine Angew. Math.* 588 (2005). [2, 3, 5, 20, 23, 31, 39, 42, 43, 45]

[Huy06] D. Huybrechts. *Fourier-Mukai Transforms in Algebraic Geometry*. Oxford University Press, Apr. 2006. DOI: 10.1093/acprof:oso/9780199296866.001.0001. URL: https://doi.org/10.1093/acprof:oso/9780199296866.001.0001. [13]

[BKS07] Aslak Bakke Buan, Henning Krause, and Øyvind Solberg. "Support varieties: an ideal approach". In: *Homology, Homotopy and Applications* 9 (2007), pp. 45–74. [20, 30, 42, 43]

[FP07] Eric M. Friedlander and Julia Pevtsova. "$\pi$-supports for modules for finite group schemes". eng. In: *Duke mathematical journal* 139.2 (2007), pp. 317–368. ISSN: 0012-7094. [43, 71]

[Lur09] Jacob Lurie. *Higher Topos Theory*. 2009. [44]

[Bal10] Paul Balmer. "Spectra, spectra, spectra–Tensor triangular spectra versus Zariski spectra of endomorphism rings". In: *Algebraic and Geometric Topology - ALGEBR GEOM TOPOL* 10 (2010). [2, 44, 48–50, 57, 61, 71]

[BIK11] David J. Benson, Srikanth Iyengar, and Henning Krause. *Representations of Finite Groups: Local Cohomology and Support*. Oberwolfach Seminars. Birkhäuser Basel, 2011. DOI: https://doi.org/10.1007/978-3-0348-0260-4. [15, 18, 48]

[Del11] Ivo Dell'Ambrogio. *Tensor triangular geometry and KK-theory*. 2011. arXiv: 1001.2637 [math.KT]. URL: https://arxiv.org/abs/1001.2637. [76, 82]

[Mal14] Cary Malkiewich. *The Stable Homotopy Category*. 2014. [15]

[KP15] Joachim Kock and Wolfgang Pitsch. *Hochster duality in derived categories and point-free reconstruction of schemes*. 2015. arXiv: 1305.1503 [math.AG]. URL: https://arxiv.org/abs/1305.1503. [20, 27, 30]

[Stacks] The Stacks Project Authors. *Stacks Project*. 2018. [86]

[BKS19] Paul Balmer, Henning Krause, and Greg Stevenson. "Tensor-triangular fields: ruminations". In: *Selecta Mathematica* 25.1 (Feb. 2019). ISSN: 1420-9020. DOI: 10.1007/s00029-019-0454-2. URL: http://dx.doi.org/10.1007/s00029-019-0454-2. [50]

[DST19] Max Dickmann, Niels Schwartz, and Marcus Tressl. *Spectral Spaces*. New Mathematical Monographs. Cambridge University Press, 2019. [89]

[Kra21] Henning Krause. *Homological Theory of Representations*. Cambridge Studies in Advanced Mathematics. Cambridge University Press, 2021. [11, 15, 49, 58, 60, 76, 78, 82]

[NVY21] Daniel K. Nakano, Kent B. Vashaw, and Milen T. Yakimov. *Noncommutative tensor triangular geometry*. 2021. arXiv: 1909.04304 [math.CT]. URL: https://arxiv.org/abs/1909.04304. [10]

[Lau22] Eike Lau. *The Balmer spectrum of certain Deligne-Mumford stacks*. 2022. arXiv: 2101.01446 [math.AG]. URL: https://arxiv.org/abs/2101.01446. [72, 73]

[BHS23] Tobias Barthel, Drew Heard, and Beren Sanders. *Stratification in tensor triangular geometry with applications to spectral Mackey functors*. 2023. arXiv: 2106.15540 [math.AT]. URL: https://arxiv.org/abs/2106.15540. [25]

[NP23] Cris Negron and Julia Pevtsova. "Hypersurface support and prime ideal spectra for stable categories". In: *Annals of K-Theory* 8.1 (May 2023), pp. 25–79. ISSN: 2379-1683. DOI: 10.2140/akt.2023.8.25. URL: http://dx.doi.org/10.2140/akt.2023.8.25. [76, 82]

[CFH24] Lars Winther Christensen, Hans-Bjørn Foxby, and Henrik Holm. *Derived Category Methods in Commutative Algebra*. Springer Cham, 2024. DOI: 10.1093/acprof:oso/9780199296866.001.0001. URL: https://doi.org/10.1093/acprof:oso/9780199296866.001.0001. [13, 33]

[nLa25] nLab authors. *homotopy category of chain complexes*. Revision 6. May 2025. [13]