MA 721 – Midterm II

Spring 2020

Time: 90 mins.

1. Write out solutions to all questions on your own paper and send me an electronic copy by email or on Moodle by midnight, March 24, 2020.
2. You do not need to rewrite the problem statements, but make sure that all problems are clearly labeled and in order.
3. Unless stated otherwise, justify your answers to receive full credit. Your answers do not have to be in complete sentences, but they do need to be understandable.
4. No outside assistance (from notes, books, internet, people, etc.) is allowed.
5. You will have 90 minutes to write up solutions, starting when you look at any page after this one. (This does not include scanning/sending time.)
6. Please write out the following on the top of your exam solutions and sign your name after it:

   I affirm that I have not used any outside resources while taking this exam and that I have completed the exam in a continuous 90-minute time period.

Question	Points	Score
1	10	
2	10	
3	15	
4	10	
5	10	
6	15	
Total:	70	

1. (10 points) Here is a matrix in rational canonical form:

$$A = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 2 \end{pmatrix}.$$

   (a) What are its invariant factors?
   (b) What is its Jordan canonical form?
   (c) What is its minimal polynomial?
   (d) What is its characteristic polynomial?

2. (10 points) Consider the $\mathbb{Q}$-vectorspace

$$V = \mathbb{Q}[x]/\langle (x-\lambda)^{d_1} \rangle \oplus \mathbb{Q}[x]/\langle (x-\lambda)^{d_2} \rangle \oplus \ldots \oplus \mathbb{Q}[x]/\langle (x-\lambda)^{d_k} \rangle$$

   where $0 < d_1 \leq d_2 \leq \ldots \leq d_k$ and $\lambda \in \mathbb{Q}$. Let $T : V \to V$ be the linear transformation given by multiplication by x. For each of the following, do not justify your answers:

   (a) What is the minimal polynomial of T?

   (b) What is the characteristic polynomial of T?

   (c) What is the dimension of the kernel of $T - \lambda \cdot \mathrm{id}$?
   (Here id denotes the identity map $V \to V$.)

   (d) Suppose $p(x) \in \mathbb{Q}[x]$ and $p(\lambda) \neq 0$. Let $L : V \to V$ be the linear map given by multiplication by $p(x)$. What is the dimension of the kernel of L?

3. (15 points) For each pair of principal ideal domain R and R-module M, give the **rank**, **invariant factors**, and **elementary divisors** of M. Do not justify your answers.

   (a) $R = \mathbb{Z}$, $M = \mathbb{Z}/10\mathbb{Z} \oplus \mathbb{Z}/15\mathbb{Z} \oplus \mathbb{Z}/35\mathbb{Z}$

   (b) $R = \mathbb{Z}$, $M = \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z} \oplus \mathbb{Z}, \mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z})$

   (c) $R = \mathbb{Z}$, $M = \mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/2\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/4\mathbb{Z}$

   (d) $R = \mathbb{Z}/5\mathbb{Z}$, $M = \mathbb{Z}/5\mathbb{Z} \oplus \mathbb{Z}/5\mathbb{Z}$

   (e) $R = \mathbb{Z}$, $M = \mathbb{Z}^2/N$ where $N \subseteq \mathbb{Z}^2$ is the submodule generated by $\{(2,1),(1,-1)\}$

4. (10 points) Here is a primary decomposition of an ideal in $\mathbb{Q}[x, y, z]$:

$$I = \langle x^2z, xy^2, xyz, xz^2\rangle = \langle x\rangle \cap \langle y^2, z\rangle \cap \langle x, y, z\rangle^3.$$

   (a) List the associated primes of I and label each as isolated or embedded.

   (b) For each associated prime P, give an element $f \in R$ for which $P = \text{rad}(I : \langle f\rangle)$.

5. (10 points) Let R be a Noetherian ring and $Q \subseteq R$ an ideal.

   (a) Using the definition, show that Q is primary if and only if every zero-divisor of R/Q is nilpotent. (Recall that an element r is nilpotent if $r^k = 0$ for some k.)

   (b) For $R = \mathbb{Q}[x, y]$ and $Q = \langle x^2, xy\rangle$ give an example of a zero-divisor in R/Q that is *not* nilpotent.

6. (15 points) For each, answer **True** or **False**. Do not justify your answer.

   (a) A linear transformation $T : V \to V$ on a finite dimensional vectorspace V over an algebraically closed field is a direct sum of its generalized eigenspaces.

   (b) If two matrices have the same minimal and characteristic polynomials, then they are similar.

   (c) If R is a Noetherian ring and $I \subseteq R$ is an ideal, then R/I is also Noetherian.

   (d) If I is an ideal of a principal ideal domain R, then I has no embedded primes.

   (e) The intersection of prime ideals is prime.