# MA 721 – Final Exam

Spring 2020

# Solutions

| Question | Points | Score |
|---|---|---|
| 1 | 10 | |
| 2 | 5 | |
| 3 | 9 | |
| 4 | 10 | |
| 5 | 10 | |
| 6 | 15 | |
| 7 | 10 | |
| 8 | 20 | |
| 9 | 21 | |
| Total: | 110 | |

1. (10 points) Consider the ring $R = M_2(\mathbb{Z})$ of $2 \times 2$ matrices over $\mathbb{Z}$. Let $M = \mathbb{Z}^2$.

   (a) Show that $M$ is an $R$-module under the action of left multiplication (i.e. identifying $M$ with the space of $2 \times 1$ matrices over $\mathbb{Z}$ and letting $A \cdot v = Av$ for $A \in R$, $v \in M$).

**Solution:** Note that $\mathbb{Z}^2$ is an abelian group under addition. Let $A, B \in M_2(\mathbb{Z})$ and $v, w \in M$. Then

- $(A + B)v = Av + Bv$
- $(AB)v = A(Bv)$
- $A(v + w) = Av + Aw$
- $I_2 v = v$, where $I_2$ is the $2 \times 2$ identity matrix.

   (b) Find a nontrivial submodule $0 \subsetneq N \subsetneq M$.

**Solution:** Consider $N = 2M = \{(2x, 2y) : x, y \in \mathbb{Z}\}$. Note that $0 \subsetneq N \subsetneq M$, since $(2, 2) \in N \backslash \{0\}$ and $(1, 1) \in M \backslash N$.

To see that $N$ is a submodule, let $v, w \in M$ with $2v, 2w \in N$ and $A \in R$. Then $2v + 2w = 2(v + w) \in N$ and $A(2v) = 2(Av) \in N$, showing that $N$ is a submodule.

2. (5 points) Find the Jordan canonical form of the linear transformation given by multiplication by $1 + x$ on the $\mathbb{Q}$-vectorspace $\mathbb{Q}[x]/\langle x^2(x-1)^2 \rangle$. Do not justify your answers.

**Solution:** First we write $\mathbb{Q}[x]/\langle x^2(x-1)^2 \rangle \cong \mathbb{Q}[x]/\langle x^2 \rangle \oplus \mathbb{Q}[x]/\langle (x-1)^2 \rangle$. The Jordan canonical form of multiplication by $x$ is

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Since multiplication by 1 acts as the identity on $\mathbb{Q}[x]/\langle x^2 \rangle \oplus \mathbb{Q}[x]/\langle (x-1)^2 \rangle$, The Jordan canonical form of multiplication by $1 + x$ is

$$\begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 2 \end{pmatrix}.$$

3. (9 points) Fill in the blank with another commonly used term from algebra (not involving the term "module"). Do not justify your answers.

   (a) A module over a field $F$ is the same as a(n) ____________.

   (b) A module over $\mathbb{Z}$ is the same as a(n) ____________.

   (c) An $R$-submodule of a ring $R$ is the same as a(n) ____________.

**Solution:**

(a) $F$-vector space

(b) abelian group

(c) (left) ideal of $R$

4. (10 points) Let $R = M_{n_1}(\mathbb{C}) \times \ldots \times M_{n_r}(\mathbb{C})$, where $n_1, \ldots, n_r \in \mathbb{Z}_{>0}$.
   In terms of $r$, $n_1, \ldots, n_r$, describe each of the following. Do not justify your answers.

   (a) The dimension of $R$ as a $\mathbb{C}$-vectorspace.

   (b) The dimension of the center of $R$, as a $\mathbb{C}$-vectorspace.

   (c) The number of primitive central idempotents in $R$.

   (d) The number of distinct (i.e. non-isomorphic) irreducible $R$-modules.

   (e) The dimensions of the distinct irreducible $R$-modules.
   (This should be a list of numbers with length equal to your answer from (d).)

**Solution:**

(a) $\sum_{i=1}^{r} n_i^2$

(b) $r$

(c) $r$

(d) $r$

(e) $n_1, n_2, \ldots, n_r$.

5. (10 points) Let $G$ be a finite group and $\varphi : G \to \mathrm{GL}_n(\mathbb{C})$ a representation of $G$ with character $\chi$. Suppose that $g \in G$ has order 2. Show that $\chi(g)$ is an integer.

   (Hint: what are the possibilities for the minimal polynomial and eigenvalues of $\varphi(g)$?)

**Solution:** Since $\varphi$ is a group homomorphism, $\varphi(\mathrm{id}) = I_n$ and

$$I_n = \varphi(\mathrm{id}) = \varphi(g^2) = (\varphi(g))^2.$$

Since the matrix $\varphi(g)$ satisfies the polynomial $x^2 - 1 = 0$, the minimal polynomial of $\varphi(g)$ divides $x^2 - 1 = (x+1)(x-1)$. Since this is squarefree, $\varphi(g)$ is diagonalizable. There is some $U \in \mathrm{GL}_n(\mathbb{C})$, s.t.

$$\varphi(g) = U \begin{pmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{pmatrix} U^{-1}$$

Moreover, the eigenvalues $\lambda_1, \ldots, \lambda_n$ are roots of the minimal polynomial, which can only be $\pm 1$ (since it divides $x^2 - 1$). Since $\lambda_1, \ldots, \lambda_n \in \{\pm 1\} \subset \mathbb{Z}$,

$$\chi(g) = \mathrm{trace}(\varphi(g)) = \lambda_1 + \lambda_2 + \ldots + \lambda_n \in \mathbb{Z}.$$

6. (15 points) Here is the character table of a mystery finite group $G$, with conjugacy classes $\{\mathrm{id}\}, K_2, K_3, K_4$:

| | $\{\mathrm{id}\}$ | $K_2$ | $K_3$ | $K_4$ |
|---|---|---|---|---|
| $\chi_1$ | 1 | 1 | 1 | 1 |
| $\chi_2$ | 1 | 1 | 1 | $-1$ |
| $\chi_3$ | 2 | $(-1+\sqrt{5})/2$ | $(-1-\sqrt{5})/2$ | 0 |
| $\chi_4$ | 2 | $(-1-\sqrt{5})/2$ | $(-1+\sqrt{5})/2$ | 0 |

Make sure to justify your answers to each of the following:

(a) What is the size of $G$?

**Solution:** The size of $G$ is $\sum_{i=1}^{4} (\chi_i(\mathrm{id}))^2 = 1^2 + 1^2 + 2^2 + 2^2 = 1 + 1 + 4 + 4 = 10$.

(b) Is $G$ abelian?

**Solution:** No. In abelian group, every conjugacy class has size one. However this is a group of size 10 with only 4 conjugacy classes.

Alternatively, By Wedderburn's Theorem, the group algebra $\mathbb{C}G$ is isomorphic to $M_1(\mathbb{C}) \times M_1(\mathbb{C}) \times M_2(\mathbb{C}) \times M_2(\mathbb{C})$. Since $M_2(\mathbb{C})$ is not commutative, we see that $\mathbb{C}G$ is not commutative and $G$ is not abelian.

(c) What are the sizes of the conjugacy classes $K_2, K_3, K_4$?

**Solution:** By the second orthogonality theorem

$$\begin{aligned}|G|/|K_2| = \sum_{i=1}^{4} \chi_i(K_2)\overline{\chi_i(K_2)} &= 1^2 + 1^2 + \left(\frac{-1+\sqrt{5}}{2}\right)^2 + \left(\frac{-1-\sqrt{5}}{2}\right)^2 \\ &= 1 + 1 + \frac{1 - 2\sqrt{2} + 5}{4} + \frac{1 + 2\sqrt{2} + 5}{4} \\ &= 1 + 1 + 2 \cdot \frac{6}{4} \\ &= 5\end{aligned}$$

$\Rightarrow |K_2| = |G|/5 = 2.$

Similarly,

$$|G|/|K_3| = \sum_{i=1}^{4} \chi_i(K_3)\overline{\chi_i(K_3)} = 1^2 + 1^2 + \left(\frac{-1-\sqrt{5}}{2}\right)^2 + \left(\frac{-1+\sqrt{5}}{2}\right)^2 = 5$$

$\Rightarrow |K_3| = |G|/5 = 2$

and

$$|G|/|K_4| = \sum_{i=1}^{4} \chi_i(K_4)\overline{\chi_i(K_4)} = 1^2 + (-1)^2 + 0^2 + 0^2 = 2$$

$\Rightarrow |K_4| = |G|/2 = 5.$

Alternatively, let $k_i = |K_i| \in \mathbb{Z}_{>0}$. Then $1 + k_2 + k_3 + k_4 = |G| = 10$. Moreover,

$$0 = \langle \chi_1, \chi_2 \rangle = 1 \cdot 1^2 + k_2 \cdot 1^2 + k_3 \cdot 1^2 + k_4 \cdot (1)(-1) = 1 + k_2 + k_3 - k_4$$

It follows that $k_2 + k_3 = 4$ and $k_4 = 5$. Moreoever,

$$\begin{aligned}
0 = \langle \chi_1, \chi_3 \rangle &= 1 \cdot 1 \cdot 2 + k_2 \cdot 1 \cdot \left( \frac{-1+\sqrt{5}}{2} \right) + k_3 \cdot 1 \cdot \left( \frac{-1-\sqrt{5}}{2} \right) + k_4 \cdot (1)(0) \\
&= 2 + (k_2 + k_3) \left( \frac{-1}{2} \right) + (k_2 - k_3) \left( \frac{\sqrt{5}}{2} \right) \\
&= (k_2 - k_3) \left( \frac{\sqrt{5}}{2} \right).
\end{aligned}$$

So $k_2 = k_3 = 2$.

(d) Let $\varphi$ be a representation of $G$ with character $\chi$ where $\chi(g) = \chi_3(g)^2 \chi_4(g)^2$ for all $g \in G$, i.e.

| | $\{\text{id}\}$ | $K_2$ | $K_3$ | $K_4$ |
|---|---|---|---|---|
| $\chi$ | 16 | 1 | 1 | 0 |

How does $\varphi$ decompose into irreducible representations of $G$?

**Solution:** We calculate the inner products

$$\begin{aligned}
\langle \chi, \chi_i \rangle &= \left(\tfrac{1}{10}\right) \left( 1 \cdot \chi(\text{id}) \cdot \chi_i(\text{id}) + \sum_{j=2}^{4} |K_j| \cdot \chi(K_j) \cdot \chi_i(K_j) \right) \\
&= \left(\tfrac{1}{10}\right) \left(1 \cdot 16 \cdot \chi_i(\text{id}) + 2 \cdot 1 \cdot \chi_i(K_2) + 2 \cdot 1 \cdot \chi_i(K_3) + 5 \cdot 0 \cdot \chi_i(K_4)\right)
\end{aligned}$$

$$\begin{aligned}
\langle \chi, \chi_1 \rangle &= \left(\tfrac{1}{10}\right)\big(1 \cdot 16 \cdot 1 + 2 \cdot 1 \cdot 1 + 2 \cdot 1 \cdot 1\big) = \left(\tfrac{1}{10}\right)(16 + 2 + 2) = 2 \\
\langle \chi, \chi_2 \rangle &= \left(\tfrac{1}{10}\right)\big(1 \cdot 16 \cdot 1 + 2 \cdot 1 \cdot 1 + 2 \cdot 1 \cdot 1\big) = \left(\tfrac{1}{10}\right)(16 + 2 + 2) = 2 \\
\langle \chi, \chi_3 \rangle &= \left(\tfrac{1}{10}\right)\big(1 \cdot 16 \cdot 2 + 2 \cdot 1 \cdot (\tfrac{-1+\sqrt{5}}{2}) + 2 \cdot 1 \cdot (\tfrac{-1-\sqrt{5}}{2})\big) = \left(\tfrac{1}{10}\right)(32 - 1 - 1) = 3 \\
\langle \chi, \chi_4 \rangle &= \left(\tfrac{1}{10}\right)\big(1 \cdot 16 \cdot 2 + 2 \cdot 1 \cdot (\tfrac{-1-\sqrt{5}}{2}) + 2 \cdot 1 \cdot (\tfrac{-1+\sqrt{5}}{2})\big) = \left(\tfrac{1}{10}\right)(32 - 1 - 1) = 3
\end{aligned}$$

Therefore $\chi = 2\chi_1 + 2\chi_2 + 3\chi_3 + 3\chi_4$ and $\varphi$ is a direct sum of two copies of $\varphi_1$ and $\varphi_2$ and three copies of each of $\varphi_3$ and $\varphi_4$, where $\varphi_i$ is the irreducible representation corresponding to $\chi_i$.

7. (10 points) Let $F$ be the field $\mathbb{Z}/2\mathbb{Z}$. Given a set $A$, let $FA$ denote the free $F$-module on $A$ (i.e. the $F$-vectorspace with basis $A$). Consider the maps

$$0 \to F\{f_{123}, f_{124}\} \xrightarrow{d_1} F\{e_{12}, e_{13}, e_{14}, e_{23}, e_{24}, e_{34}\} \xrightarrow{d_2} F\{v_1, v_2, v_3, v_4\} \xrightarrow{d_3} 0 \to \dots$$

given by

$$d_1(f_{ijk}) = e_{ij} + e_{ik} + e_{jk} \quad \text{and} \quad d_2(e_{ij}) = v_i + v_j.$$

(a) Show that this is a cochain complex $\mathcal{C}$.

**Solution:** Note that $d_2 \circ d_1 = 0$ to see this, note that

$$\begin{aligned} d_2(d_1(f_{ijk})) = d_2(e_{ij} + e_{ik} + e_{jk}) &= d_2(e_{ij}) + d_2(e_{ik} + d_2(e_{jk}) \\ &= v_i + v_j + v_i + v_k + v_j + v_k = 2(v_i + v_j + v_k) = 0. \end{aligned}$$

Since $d_3$ is the zero map, $d_3 \circ d_2 = 0$ as well.

(b) Find the dimension, as an $F$-vectorspace, of $H^0(\mathcal{C})$, $H^1(\mathcal{C})$, and $H^2(\mathcal{C})$.

(Hint: first calculate the dimensions of the kernels and images of each $d_i$, and remember the rank-nullity theorem from linear algebra. You can use without proof that the image of $d_2$ is the (3-dimensional) hyperplane $\{a_1v_1 + a_2v_2 + a_3v_3 + a_4v_4 : a_1 + a_2 + a_3 + a_4 = 0\}$.)

**Solution:** Since $d_1(f_{123}))$ and $d_1(f_{124}))$ are not scalar multiples of each other, we see that $d_1$ is injective, meaning that $\ker(d_1) = \{0\}$ and $\dim_F(\text{image}(d_1)) = 2$. By assumption, the image of $d_2$ has dimension 3, and so its kernel has dimension $6 - 3 = 3$. Finally, since $d_3$ is the zero map, its kernel has full dimension 4.

$$\begin{aligned} \dim_F\left(H^0(\mathcal{C})\right) &= \dim_F(\ker(d_1)) = 0 \\ \dim_F\left(H^1(\mathcal{C})\right) &= \dim_F(\ker(d_2)/\text{image}(d_1)) = \dim_F(\ker(d_2)) - \dim_F(\text{image}(d_1)) = 3 - 2 = 1 \\ \dim_F\left(H^2(\mathcal{C})\right) &= \dim_F(\ker(d_3)/\text{image}(d_2)) = \dim_F(\ker(d_3)) - \dim_F(\text{image}(d_2)) = 4 - 3 = 1 \end{aligned}$$

8. (20 points) Consider the $\mathbb{Z}$-module $M = \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z}$ and the sequence of maps

$$0 \xrightarrow{\phi_2} \mathbb{Z}^2 \xrightarrow{\phi_1} \mathbb{Z}^3 \xrightarrow{\phi_0} M \to 0$$

where $\phi_2(0) = (0,0)$, $\phi_1(a,b) = (0, 2a, 3b)$ and $\phi_0(x,y,z) = (x, y + 2\mathbb{Z}, z + 3\mathbb{Z})$.

(a) Show that this is a projective resolution of $M$ as a $\mathbb{Z}$-module.

**Solution:** Note that the modules $0, \mathbb{Z}^2, \mathbb{Z}^3$ are free and therefore projective $\mathbb{Z}$-modules. So it suffices to show that the sequence above is exact.

First note that the map $\phi_1$ is injective, since $\phi_1(a,b) = (0,2a,3b) = (0,0,0)$ implies that $(a,b) = (0,0)$, meaning that the above sequence is exact at $\mathbb{Z}^2$.

The kernel of $\phi_2$ is $\{(x,y,z) : x = 0, y \in 2\mathbb{Z}, z \in 3\mathbb{Z}\}$, which is exactly the image of $\phi_1$, showing exactness at $\mathbb{Z}^3$.

Finally, the map $\phi_0$ is surjective, since $(x, y + 2\mathbb{Z}, z + 3\mathbb{Z}) \in M$ is the image of $(x,y,z) \in \mathbb{Z}^3$, giving exactness at $M$.

(b) Describe the cochain complex and maps obtained from applying $\mathrm{Hom}_{\mathbb{Z}}(-, \mathbb{Z}/2\mathbb{Z})$ to this resolution and use it to calculate $\mathrm{Ext}^n_R(M, \mathbb{Z}/2\mathbb{Z})$ for all $n \geq 0$.

**Solution:** Applying $\mathrm{Hom}_{\mathbb{Z}}(-, \mathbb{Z}/2\mathbb{Z})$ we get a cochain complex

$$0 \longrightarrow \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Z}/2\mathbb{Z}) \xrightarrow{d_0} \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}^3, \mathbb{Z}/2\mathbb{Z}) \xrightarrow{d_1} \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}^2, \mathbb{Z}/2\mathbb{Z}) \xrightarrow{d_2} 0$$

where $d_i(f) = f \circ \phi_i$.

For $n = 0$,

$$\begin{aligned}\mathrm{Ext}^0_{\mathbb{Z}}(M, \mathbb{Z}/2\mathbb{Z}) &\cong \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Z}/2\mathbb{Z}) \\ &\cong \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}, \mathbb{Z}/2\mathbb{Z}) \oplus \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}, \mathbb{Z}/2\mathbb{Z}) \oplus \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/3\mathbb{Z}, \mathbb{Z}/2\mathbb{Z}) \\ &\cong (\mathbb{Z}/2\mathbb{Z})^2\end{aligned}$$

Note that for $n \geq 2$, $d_n$ is the zero map, so $\ker(d_{n+1}) = \mathrm{image}(d_n) = \{0\}$, showing that

$$\mathrm{Ext}^n_{\mathbb{Z}}(M, \mathbb{Z}/2\mathbb{Z}) = \ker(d_{n+1})/\mathrm{image}(d_n) = 0.$$

It only remains to find $\mathrm{Ext}^1_{\mathbb{Z}}(M, \mathbb{Z}/2\mathbb{Z}) = \ker(d_2)/\mathrm{image}(d_1)$. Note that $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}^k, \mathbb{Z}/2\mathbb{Z})$ is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^k$. Specifically, let $f_1, f_2, f_3 \in \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}^3, \mathbb{Z}/2\mathbb{Z})$ with $f_1(x, y, z) = x + 2\mathbb{Z}$, $f_2(x, y, z) = y + 2\mathbb{Z}$, and $f_3(x, y, z) = z + 2\mathbb{Z}$. Then

$$\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}^3, \mathbb{Z}/2\mathbb{Z}) = \{u_1 f_1 + u_2 f_2 + u_3 f_3 : u_1, u_2, u_3 \in \mathbb{Z}\} \cong (\mathbb{Z}/2\mathbb{Z})^3$$

To find their image under $d_1$ note that

$$\begin{aligned}f_1(\phi_1(a, b)) &= f_1(0, 2a, 3b) = 0 \\ f_2(\phi_1(a, b)) &= f_2(0, 2a, 3b) = 2a = 0 \\ f_3(\phi_1(a, b)) &= f_3(0, 2a, 3b) = 3b = b\end{aligned}$$

So the image of $d_1$ is $\mathbb{Z}/2\mathbb{Z} \cdot g_2$ where $g_2(a, b) = b$.

Since $d_2$ is the zero-map, the kernel of $d_2$ is all of

$$\ker(d_2) = \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}^2, \mathbb{Z}/2\mathbb{Z}) = \{u_1 g_1 + u_2 g_2 : u_1, u_2 \in \mathbb{Z}\} \cong (\mathbb{Z}/2\mathbb{Z})^2$$

where $g_1(a, b) = a + 2\mathbb{Z}$. Then

$$\mathrm{Ext}^1_{\mathbb{Z}}(M, \mathbb{Z}/2\mathbb{Z}) = \ker(d_2)/\mathrm{image}(d_1) \cong \mathbb{Z} g_1 \cong \mathbb{Z}/2\mathbb{Z}.$$

(c) Describe the chain complex and maps obtained from applying $\mathbb{Z}/2\mathbb{Z} \otimes_{\mathbb{Z}} -$ to this resolution and use them to calculate $\mathrm{Tor}_n^R(\mathbb{Z}/2\mathbb{Z}, M)$ for all $n \geq 0$.

(Your final answers to (b) and (c) should be expressions as finite abelian groups, not involving quotients, "ker", or "image")

**Solution:** Applying $\mathbb{Z}/2\mathbb{Z} \otimes_{\mathbb{Z}} -$ we get a chain complex

$$0 \xrightarrow{d_2} \mathbb{Z}/2\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}^2 \xrightarrow{d_1} \mathbb{Z}/2\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}^3 \xrightarrow{d_0} \mathbb{Z}/2\mathbb{Z} \otimes_{\mathbb{Z}} M \to 0.$$

where $d_n = 1 \otimes \phi_n$. For $n = 0$,

$$\begin{aligned}
\mathrm{Tor}_0^{\mathbb{Z}}(\mathbb{Z}/2\mathbb{Z}, M) &\cong \mathbb{Z}/2\mathbb{Z} \otimes_{\mathbb{Z}} M \\
&\cong (\mathbb{Z}/2\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}) \oplus (\mathbb{Z}/2\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/2\mathbb{Z}) \oplus (\mathbb{Z}/2\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/3\mathbb{Z}) \\
&\cong (\mathbb{Z}/2\mathbb{Z}) \oplus (\mathbb{Z}/2\mathbb{Z}) \oplus 0 \\
&\cong (\mathbb{Z}/2\mathbb{Z})^2.
\end{aligned}$$

Note that for $n \geq 2$, $d_n$ is the zero map, so $\ker(d_{n+1}) = \mathrm{image}(d_n) = \{0\}$, showing that

$$\mathrm{Tor}_n^{\mathbb{Z}}(M, \mathbb{Z}/2\mathbb{Z}) = \ker(1 \otimes d_n)/\mathrm{image}(1 \otimes d_{n+1}) = 0.$$

It remains to find

$$\mathrm{Tor}_1^{\mathbb{Z}}(M, \mathbb{Z}/2\mathbb{Z}) = \ker(1 \otimes d_1)/\mathrm{image}(1 \otimes d_2) \cong \ker(1 \otimes d_1)$$

Note that every element of $\mathbb{Z}/2\mathbb{Z} \otimes \mathbb{Z}^k \cong (\mathbb{Z}/2\mathbb{Z})^k$ can be written as $\sum_{i=1}^k a_i \otimes e_i$ where $a_i \in \mathbb{Z}/2\mathbb{Z}$ and $e_i$ is the $i$ coordinate vector. Then

$$\begin{aligned}
1 \otimes d_1(1 \times (a, b)) &= 1 \otimes (0, 2a, 3b) = 1 \otimes (0, 2a, 0) + 1 \otimes (0, 0, 3b) \\
&= 2a \cdot 1 \otimes (0, 1, 0) + 3b \cdot 1 \otimes (0, 0, 1) \\
&= 3b \cdot 1 \otimes (0, 0, 1)
\end{aligned}$$

This equals zero if and only if $b \in 2\mathbb{Z}$. Then

$$\mathrm{Tor}_1^{\mathbb{Z}}(M, \mathbb{Z}/2\mathbb{Z}) \cong \{1 \otimes (a, 0) = a \cdot 1 \otimes (1, 0) : a \in \mathbb{Z}\} \cong \mathbb{Z}/2\mathbb{Z}.$$

9. (21 points) For each, answer **True** or **False**. Do not justify your answer.

   (a) For any submodule $N$ of a module $M$, there exists another submodule $N'$ of $M$ so that $M = N \oplus N'$.

   (b) The tensor algebra, $\mathcal{T}(M)$, of a module $M$ over a commutative ring $R$ is commutative.

   (c) Every matrix $A \in M_n(\mathbb{C})$ is similar to one of the form $N + D$ where $N$ is nilpotent and $D$ is diagonal.

   (d) Every finitely-generated torsion-free module over a PID is free.

   (e) The module $\mathbb{Q}[x]/\langle x \rangle \oplus \mathbb{Q}[x]/\langle x - 1 \rangle$ can be generated (as a $\mathbb{Q}[x]$-module) by a single element.

   (f) Every radical ideal in a Noetherian ring is the intersection of finitely many prime ideals.

   (g) Let $G$ be a finite group. Every function $G \to \mathbb{C}$ that is constant on conjugacy classes is the character of some representation of $G$.

**Solution:**

(a) False.

(b) False.

(c) True.

(d) True.

(e) True.

(f) True.

(g) False.