Math 721 – Worksheet 01/14/20

(10:15am) Form into 4 groups of size 3-4.
(10:17am) As a group, read through the definition of tensor product and try Problem 0.
(10:30am) Discuss as a class.
(10:35am) Students in Group X work on Problem X.
If it's too hard, try the warm-up. If you finish, try another problem.
(11:05am) Groups present their findings. (~5 minutes per group)

Let R be a commutative ring with $1 \neq 0$ and suppose M and N are both (left) R-modules.

Definition. The *tensor product* of M and N over R, $M \otimes_R N$, is the set of formal sums

$$M \otimes_R N = \left\{ \sum_{i=1}^{k} m_i \otimes n_i \ : \ k \in \mathbb{Z}_{\geq 0},\ m_i \in M,\ n_i \in N \right\}$$

subject to the relations

$$\begin{aligned} (m_1 + m_2) \otimes n &= (m_1 \otimes n) + (m_2 \otimes n), \\ m \otimes (n_1 + n_2) &= (m \otimes n_1) + (m \otimes n_2), \text{ and} \\ rm \otimes n &= m \otimes rn \end{aligned}$$

for all $m_1, m_2, m \in M$, $n_1, n_2, n \in N$ and $r \in R$. This is an R-module under the action

$$r \left(\sum_{i=1}^{k} m_i \otimes n_i \right) = \sum_{i=1}^{k} rm_i \otimes n_i = \sum_{i=1}^{k} m_i \otimes rn_i.$$

Example. Consider $R = \mathbb{R}$ and $M = N = \mathbb{R}^2 = \{ae_1 + be_2 : a, b \in \mathbb{R}\}$. The tensor product $\mathbb{R}^2 \otimes_{\mathbb{R}} \mathbb{R}^2$ is a 4-dimensional vectorspace with basis

$$\{e_1 \otimes e_1,\ e_1 \otimes e_2,\ e_2 \otimes e_1,\ e_2 \otimes e_2\}.$$

Problem 0. Write the element $(e_1 + e_2) \otimes (e_1 + 2e_2) + (e_1 - e_2) \otimes (3e_1)$ of $\mathbb{R}^2 \otimes_{\mathbb{R}} \mathbb{R}^2$ as an $\mathbb{R}$-linear combination of the basis above.

Show each of the following R-module isomorphisms:

	Warm-up
Problem 1. $M \otimes_R R \cong M$	$R \otimes_R R \cong R$
Problem 2. $R^m \otimes_R R^n \cong R^{mn}$	$m = 2$ and $n = 2$
Problem 3. $\mathbb{Q}^2 \otimes_{\mathbb{Q}} \mathbb{R} \cong \mathbb{R}^2$	$\mathbb{Q} \otimes_{\mathbb{Q}} \mathbb{R} \cong \mathbb{R}$
Problem 4. $\mathbb{Z}/m\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/n\mathbb{Z} \cong \mathbb{Z}/d\mathbb{Z}$ where $d = \gcd(m, n)$	$m = 2$ and $n = 2, 3$