Math 721 – Worksheet 02/20/2020

Problem 1. Consider the matrix

$$A = \begin{pmatrix} 0 & 0 & 0 & -4 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

(a) Find a polynomial $a(x) \in \mathbb{Q}[x]$ for which A is the companion matrix.
(b) Factor $a(x)$ over $\overline{\mathbb{Q}}[x]$
(c) Find the Jordan canonical form J of A.
(d) Find an explicit matrix P for which $J = P^{-1}AP$.

Problem 2. Consider the matrix

$$J = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

(a) Find the rational canonical form A of J.
(b) Find an explicit matrix P for which $A = P^{-1}JP$.
(c) What are the minimal and characteristic polynomials of J?

Problem 3. Suppose that A is a 6×6 matrix over $\mathbb{Q}$ fo which $A^3 = 0$ but $A^2 \neq 0$.
(a) What are the possible Jordan canonical forms of A (over $\overline{\mathbb{Q}}$)?
(b) What are the minimal and characteristic polynomials these matrices?

Problem 4. Suppose that M an $n \times n$ is a block diagonal matrix

$$M = \begin{pmatrix} A_1 & 0 & 0 & \dots & 0 \\ 0 & A_2 & 0 & \dots & 0 \\ 0 & 0 & A_3 & & 0 \\ \vdots & \vdots & & \ddots & \vdots \\ 0 & 0 & 0 & \dots & A_k \end{pmatrix}$$

where for each i, A_i is a $d_i \times d_i$ matrix where $n = \sum_{i=1}^k d_i$.
(a) What are the minimal and characteristic polynomials of M in terms of the minimal and characteristic polynomials of $A_1, \dots, A_k$?
(b) How does the Jordan canonical form of M relate to the Jordan canonical forms of $A_1, \dots, A_k$?

Answers (without explanation)

*Note, in the rational canonical form, the order of the basis of $\overline{\mathbb{Q}}[x]/\langle (x-\lambda)^k \rangle$ is taken to be $\{(x-\lambda)^{k-1}, \dots, x-\lambda, 1\}$ (not the reverse, which I may have incorrectly used in class).

Problem 1 Solutions

(a) $a(x) = x^4 - 4x^2 + 4$ for which A is the companion matrix.

(b) $a(x) = (x^2-2)^2 = (x-\sqrt{2})^2 \cdot (x+\sqrt{2})^2$.

(c) $$\begin{pmatrix} \sqrt{2} & 1 & 0 & 0 \\ 0 & \sqrt{2} & 0 & 0 \\ 0 & 0 & -\sqrt{2} & 1 \\ 0 & 0 & 0 & -\sqrt{2} \end{pmatrix}$$

(d) $P = Q^{-1}$ where $Q = \begin{pmatrix} 0 & 1 & 2\sqrt{2} & 6 \\ 1 & \sqrt{2} & 2 & 2\sqrt{2} \\ 0 & 1 & -2\sqrt{2} & 6 \\ 1 & -\sqrt{2} & 2 & -2\sqrt{2} \end{pmatrix}$

This is the change of basis matrix from $\overline{\mathbb{Q}}$-basis $\{1, x, x^2, x^3\}$ of $\overline{\mathbb{Q}}[x]/\langle a(x) \rangle$ to the $\overline{\mathbb{Q}}$-basis $\{(x-\sqrt{2}, 0), (1, 0), (0, x+\sqrt{2}), (0, 1)\}$ of $\overline{\mathbb{Q}}[x]/(x-\sqrt{2})^2 \oplus \overline{\mathbb{Q}}[x]/(x+\sqrt{2})^2$.

Problem 2 Solutions

(a) $A = \begin{pmatrix} 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & -3 \\ 0 & 0 & 0 & 0 & 0 & 1 & 3 \end{pmatrix}$

(b) $P = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 3 & 6 \\ 0 & 0 & 0 & 1 & 2 & 3 & 4 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}$

(c) minimal polynomial $= x^2(x-1)^3$
characteristic polynomial $= x^2(x-1)^5$

Problem 3 Solutions

(a) $$\begin{pmatrix} 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&1&0\\ 0&0&0&0&0&1\\ 0&0&0&0&0&0 \end{pmatrix}, \begin{pmatrix} 0&0&0&0&0&0\\ 0&0&1&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&1&0\\ 0&0&0&0&0&1\\ 0&0&0&0&0&0 \end{pmatrix}, \text{ and } \begin{pmatrix} 0&1&0&0&0&0\\ 0&0&1&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&1&0\\ 0&0&0&0&0&1\\ 0&0&0&0&0&0 \end{pmatrix}$$

(b) For each, the minimal polynomial is x^3 and the characteristic polynomial is x^6.

Problem 4 Solutions

(a) The minimal polynomial of M is the least common multiple of the minimal polynomials of $A_1, \ldots, A_k$.

The characteristic polynomial of M is the product of the characteristic polynomials of $A_1, \ldots, A_k$.

(b) The Jordan canonical form of M is a block diagonal matrix consisting of the Jordan canonical forms of $A_1, \ldots, A_k$.