Math 721 – Homework 9 – Solutions

Problem 1 (DF 17.1.21)**.** Let $R = k[x, y]$ where k is a field and let I denote the maximal ideal $\langle x, y\rangle$ in R.

(a) Show that the following is a free resolution of k as an R-module:

$$0 \to R \xrightarrow{\alpha} R^2 \xrightarrow{\beta} R \xrightarrow{\pi} k \to 0,$$

where

$$\alpha(f) = (yf, -xf), \quad \beta(f, g) = xf + yg, \quad \text{and} \quad \pi(f) = f + I \in R/I \cong k.$$

(b) Use the resolution in (a) to show that $\operatorname{Tor}_2^R(k, k) \cong k$.

(c) Prove that $\operatorname{Tor}_1^R(k, I) \cong k$. (Hint: Use the long exact sequence corresponding to the short exact sequence $0 \to I \to R \to k \to 0$ and (b).)

(d) Conclude that the torsion-free R module I is not flat.

Proof of (a). Since neither x nor y is a zero-divisor in $k[x, y]$, we see that $\ker(\alpha) = \{0\}$, meaning that α is injective.

Note that $x(yf)+y(-xf) = 0$, showing that $\operatorname{image}(\alpha) \subseteq \ker(\beta)$. For the reverse inclusion, suppose that $(f, g) \in \ker(\beta)$. Then $xf = -yg$. Since x, y are relatively prime, we see that x must divide g and y must divide f. Moreover, $\frac{f}{y} = \frac{-g}{x}$. Call this polynomial $h = \frac{f}{y}$. Then $\alpha(h) = (y \cdot h, -x \cdot h) = (y \cdot \frac{f}{y}, -x \cdot \frac{-g}{x}) = (f, g) \in \operatorname{image}(\alpha)$. Therefore $\operatorname{image}(\alpha) = \ker(\beta)$.

The image of β is exactly $I = \langle x, y\rangle = \{xf + yg : f, g \in R\}$, which is the kernel of the map $\pi : R \to R/I$.

Finally, the projection $\pi : R \to R/I$ is surjective. An element $r + I \in R/I$ is the image of $r \in R$. □

Proof of (b). The exact sequence in (a) is a free-resolution of $k \cong R/I$ as an R-module with $P_2 = R$, $P_1 = R^2$, $P_0 = R$, and $\alpha = d_2$, $\beta = d_1$. To calculate, $\operatorname{Tor}_2^R(k, k)$, we tensor this sequence with k, giving

$$0 \to k \otimes_R R \xrightarrow{1\otimes\alpha} k \otimes_R R^2 \xrightarrow{1\otimes\beta} k \otimes_R R \xrightarrow{1\otimes\pi} k \otimes_R k \to 0,$$

By definition, $\operatorname{Tor}_2^R(k, k) = \ker(1 \otimes \alpha)/\operatorname{image}(1 \otimes 0) = \ker(1 \otimes \alpha)$.

Recall that $k \otimes_R R \cong k$ and that every element can be written as $a \otimes 1$ for some $a \in k$.

$$(1 \otimes \alpha)(a \otimes 1) = a \otimes \alpha(1) = a \otimes (y, -x) = ay \otimes (1, 0) - ax \otimes (0, 1) = (0, 0)$$

The last equation follows from the fact that multiplication of any element $a \in k \cong R/I$ by x or y gives zero. Since $M \otimes_R R \cong M$ for any R-module M, we find that

$$\operatorname{Tor}_2^R(k, k) = \ker(1 \otimes \alpha) = k \otimes_R R \cong k.$$

□

Proof of (c). Tensoring the short exact exact sequence $0 \to I \to R \to k \to 0$ with $k \cong R/I$ gives a long exact sequence

$$\cdots \to \operatorname{Tor}_2^R(k, R) \to \operatorname{Tor}_2^R(k, k) \to \operatorname{Tor}_1^R(k, I) \to \operatorname{Tor}_1^R(k, R) \to \cdots.$$

Since R is a free R-module, it is flat, implying that $\operatorname{Tor}_1^R(k, R) = \operatorname{Tor}_2^R(k, R) = 0$ (see DF Prop. 17.1.16). Therefore the sequence $0 \to \operatorname{Tor}_2^R(k, k) \to \operatorname{Tor}_1^R(k, I) \to 0$ is exact, giving

$$\operatorname{Tor}_1^R(k, I) \cong \operatorname{Tor}_2^R(k, k) \cong k.$$

□

Proof of (d). By part (c), $\operatorname{Tor}_1^R(k, I) \cong k \neq 0$. By DF Prop. 17.1.16, it follows that I is not a flat R-module. □

Problem 2 (DF 17.2.9 +)**.** Let G be an infinite cyclic group with generator σ.

(a) Show that the map $\operatorname{aug} : \mathbb{Z}G \to \mathbb{Z}$ defined by

$$\operatorname{aug}\left(\sum_{i\in\mathbb{Z}} a_i\sigma^i\right) = \sum_{i\in\mathbb{Z}} a_i$$

is a $(\mathbb{Z}G)$-module homomorphism, taking the trivial action of G on $\mathbb{Z}$.

(b) Prove that multiplication by $\sigma - 1$ in $\mathbb{Z}G$ gives the following free resolution of $\mathbb{Z}$ as a $(\mathbb{Z}G)$-module:

$$0 \to \mathbb{Z}G \xrightarrow{\sigma-1} \mathbb{Z}G \xrightarrow{\operatorname{aug}} \mathbb{Z} \to 0.$$

(c) Let A be a G-module. Show that $H^0(G, A) \cong A^G$, $H^1(G, A) \cong A/(\sigma - 1)A$, and $H^n(G, A) = 0$ for $n \geq 2$.

(d) Show that $H^1(G, \mathbb{Z}G) \cong \mathbb{Z}$.
(This shows that free modules can have nontrivial cohomology groups.)

Proof of (a). To show that aug is a $\mathbb{Z}G$-module homomorphism, it suffices to show that aug is $\mathbb{Z}$ linear and $g \cdot \operatorname{aug}(x) = \operatorname{aug}(g \cdot x)$ for all $x \in \mathbb{Z}G$.

Let $\sum_{i\in\mathbb{Z}} a_i\sigma^i, \sum_{j\in\mathbb{Z}} b_j\sigma^j \in \mathbb{Z}G$. (By definition, only finitely many a_i and b_j are non-zero.) Let $\alpha, \beta \in \mathbb{Z}$. Then

$$\begin{aligned}
\operatorname{aug}\left(\alpha \cdot \sum_{i\in\mathbb{Z}} a_i\sigma^i + \beta \cdot \sum_{j\in\mathbb{Z}} b_j\sigma^j\right) &= \operatorname{aug}\left(\sum_{i\in\mathbb{Z}} (\alpha \cdot a_i + \beta \cdot b_i)\sigma^i\right) \\
&= \sum_{i\in\mathbb{Z}} (\alpha \cdot a_i + \beta \cdot b_i) \\
&= \alpha \cdot \operatorname{aug}\left(\sum_{i\in\mathbb{Z}} a_i\sigma^i\right) + \beta \cdot \operatorname{aug}\left(\sum_{j\in\mathbb{Z}} b_j\sigma^j\right).
\end{aligned}$$

Similarly, for any $\sigma^k \in G$,

$$\operatorname{aug}\left(\sigma^k \cdot \sum_{i\in\mathbb{Z}} a_i\sigma^i\right) = \operatorname{aug}\left(\sum_{i\in\mathbb{Z}} a_i\sigma^{i+k}\right) = \sum_{i\in\mathbb{Z}} a_i = \sigma^k \cdot \left(\sum_{i\in\mathbb{Z}} a_i\right) = \sigma^k \cdot \operatorname{aug}\left(\sum_{i\in\mathbb{Z}} a_i\sigma^i\right).$$

□

Proof of (b). Let $x = \sum_{i\in\mathbb{Z}} a_i\sigma^i \in \mathbb{Z}G$. Suppose that $x \in \ker(\sigma - 1)$. Then

$$0 = (\sigma-1)\cdot\sum_{i\in\mathbb{Z}} a_i\sigma^i = \sum_{i\in\mathbb{Z}} a_i(\sigma^{i+1}-\sigma^i) = \sum_{i\in\mathbb{Z}}(a_{i-1}-a_i)\sigma^i.$$

Therefore $a_{i-1} = a_i$ for all i. However a_i is non-zero for only finitely-many i. Since all the a_i's must be equal, this implies that $a_i = 0$ for all i and $x = 0$. Therefore multiplication by $\sigma - 1$ defines an injective map on $\mathbb{Z}G$.

Note that the image of $\sigma - 1$ belongs to the kernel of aug. To see this, note that

$$\operatorname{aug}\left((\sigma-1)\cdot\sum_{i\in\mathbb{Z}} a_i\sigma^i\right) = \operatorname{aug}\left(\sum_{i\in\mathbb{Z}}(a_{i-1}-a_i)\sigma^i\right) = \sum_{i\in\mathbb{Z}}(a_{i-1}-a_i) = 0.$$

For the reverse inclusion, suppose that $y = \sum_{j\in\mathbb{Z}} b_j\sigma^j \in \mathbb{Z}G$ belongs to the kernel of aug, i.e. $\sum_{j\in\mathbb{Z}} b_j$. For $y \neq 0$, let $M = \max\{j : b_j \neq 0\}$ and $m = \min\{j : b_j \neq 0\}$. We will show that $y \in \operatorname{image}(\sigma-1)$ by induction on $M - m$.

If $M - m = 0$, then $y = b_m\sigma^m$ with $b_m \neq 0$ and $y \notin \ker(\operatorname{aug})$. If $M - m = 1$, then $y = b_m\sigma^m + b_{m+1}\sigma^{m+1}$ and $b_m + b_{m+1} = 0$. Then $y = (\sigma-1)b_{m+1}\sigma^m \in \operatorname{image}(\sigma-1)$.

Now suppose $M - m \geq 2$ and consider the element $z = y - b_M\sigma^{M-1}(\sigma-1)$. Then $\operatorname{aug}(z) = \operatorname{aug}(y) + b_M\cdot\operatorname{aug}(\sigma-1) = 0 + 0 = 0$, so $z \in \ker(\operatorname{aug})$. Moreover, $z = \sum_{j\in\mathbb{Z}} c_j\sigma_j$ where $c_j = b_j$ for all $j \neq M-1, M$ and $c_M = 0$. Therefore

$$\max\{j : c_j \neq 0\} - \min\{j : c_j \neq 0\} = \max\{j : c_j \neq 0\} - m < M - m.$$

By induction $z \in \operatorname{image}(\sigma-1)$. Then $y = z + b_M\sigma^{M-1}(\sigma-1) \in \operatorname{image}(\sigma-1)$.

Finally, for all $a \in \mathbb{Z}$, $a\sigma \in \mathbb{Z}G$ and $\operatorname{aug}(a\sigma) = a$, so the map aug is surjective. □

Proof of (c). Let A be a G-module. Recall that one definition of the group $H^n(G, A)$ is $\operatorname{Ext}^n_{\mathbb{Z}G}(\mathbb{Z}, A)$. To find this, we use the free resolution of $\mathbb{Z}$ as a $\mathbb{Z}G$-module given in (b), take $\operatorname{Hom}_{\mathbb{Z}G}(-, A)$ (and drop the "$\mathbb{Z}$" term) to get the cochain complex:

$$0 \to \operatorname{Hom}_{\mathbb{Z}G}(\mathbb{Z}G, A) \xrightarrow{\sigma-1} \operatorname{Hom}_{\mathbb{Z}G}(\mathbb{Z}G, A) \xrightarrow{d_2} 0.$$

Noting that $\operatorname{Hom}_{\mathbb{Z}G}(\mathbb{Z}G, A) \cong A$ gives

$$0 \to A \xrightarrow{\sigma-1} A \xrightarrow{d_2} 0.$$

Then $H^0(G, A) = \ker(\sigma-1) \cong \{a \in A : (\sigma-1)a = 0\} = \{a \in A : \sigma a = a\} = A^G$. We see that $H^1(G, A) = \ker(d_2)/\operatorname{image}(\sigma-1) = A/(\sigma-1)A$. Finally for $n \geq 2$, $H^n(G, A) = \ker(d_{n+1})/\operatorname{image}(d_n) = 0$. □

Proof of (d). By part (c), $H^1(G, \mathbb{Z}G) \cong \mathbb{Z}G/(\sigma-)\mathbb{Z}G$. By part (b), $(\sigma-1)\mathbb{Z}G$ equals the kernel of the $\mathbb{Z}G$-module homomorphism aug. Together with the first isomorphism theorem, this gives

$$H^1(G, \mathbb{Z}G) \;\cong\; \mathbb{Z}G/(\sigma-1)\mathbb{Z}G \;=\; \mathbb{Z}G/\ker(\operatorname{aug}) \;\cong\; \operatorname{image}(\operatorname{aug}) = \mathbb{Z}.$$

□