Math 721 – Homework 8 Solutions

Problem 1 (DF 17.1 Exercise 1)**.** Give the details of the proof of the following proposition:

Proposition 17.1.1. A homomorphism $\alpha : \mathcal{A} \to \mathcal{B}$ of cochain complexes induces group homomorphisms from $H^n(\mathcal{A}) \to H^n(\mathcal{B})$ for $n \geq 0$ on their respective cohomology groups.

Proof. Consider the cochain complex $\mathcal{A}$ given by abelian groups $\{A_n\}$ and maps $\phi_n : A_{n-1} \to A_n$ and the cochain complex $\mathcal{B}$ given by groups $\{B_n\}$ and maps $\psi_n : B_{n-1} \to B_n$.

Let $\alpha : \mathcal{A} \to \mathcal{B}$ be a homomorphism of cochain complexes. By definition, this means a collection of group homomorphisms $\alpha_n : A_n \to B_n$ for every n, the following diagram commutes:

$$\begin{array}{ccccc} A_{n-1} & \xrightarrow{\phi_n} & A_n & \xrightarrow{\phi_{n+1}} & A_{n+1} \\ \downarrow{\scriptstyle \alpha_{n-1}} & & \downarrow{\scriptstyle \alpha_n} & & \downarrow{\scriptstyle \alpha_{n+1}} \\ B_{n-1} & \xrightarrow{\psi_n} & B_n & \xrightarrow{\psi_{n+1}} & B_{n+1} \end{array}$$

By definition the nth cohomology group of $\mathcal{A}$ is $H^n(\mathcal{A}) = \ker(\phi_{n+1})/\operatorname{image}(\phi_n)$, and similarly $H^n(\mathcal{B}) = \ker(\psi_{n+1})/\operatorname{image}(\psi_n)$.

Suppose $a \in \ker(\phi_{n+1})$. Then $\phi_{n+1}(a) = 0$, and so commutativity of the right square above gives

$$0 = \alpha_{n+1}(\phi_{n+1}(a)) = \psi_{n+1}(\alpha_n(a)).$$

So $\alpha_n(a) \in \ker(\psi_{n+1})$. This shows that α_n restricts to a map $\ker(\phi_{n+1}) \to \ker(\psi_{n+1})$. This gives a natural homomorphism $\alpha_n' : \ker(\phi_{n+1}) \to H^n(\mathcal{B}) = \ker(\psi_{n+1})/\operatorname{image}(\psi_n)$ given by $a \mapsto \alpha_n(a) + \operatorname{image}(\psi_n)$.

To see that this extends to a well-defined map from $H^n(\mathcal{A}) = \ker(\phi_{n+1})/\operatorname{image}(\phi_n)$ to $H^n(\mathcal{B})$, it suffices to check that the image of $\operatorname{image}(\phi_n)$ under α_n is contained in $\operatorname{image}(\psi_n)$. To see this, suppose $a \in \operatorname{image}(\phi_n)$. Then there is some $a' \in A_{n-1}$ for which $a = \phi_n(a')$. Then by commutativity of the left square above,

$$\alpha_n(a) = \alpha_n(\phi_n(a')) = \psi_n(\alpha_{n-1}(a')) \in \operatorname{image}(\psi_n).$$

This shows that the kernel of the homomorphism α_n' contains $\operatorname{image}(\phi_n)$, and so it induces a well-defined homomorphism from $H^n(\mathcal{A}) = \ker(\phi_{n+1})/\operatorname{image}(\phi_n)$ to $H^n(\mathcal{B})$ given by

$$a + \operatorname{image}(\phi_n) \mapsto \alpha_n(a) + \operatorname{image}(\psi_n).$$

□

Problem 2 (DF 17.1 Exercise 3)**.** Suppose

$$\begin{array}{ccccccccc} & & A & \xrightarrow{\alpha} & B & \xrightarrow{\beta} & C & \longrightarrow & 0 \\ & & \big\downarrow{\scriptstyle f} & & \big\downarrow{\scriptstyle g} & & \big\downarrow{\scriptstyle h} & & \\ 0 & \longrightarrow & A' & \xrightarrow{\alpha'} & B' & \xrightarrow{\beta'} & C' & & \end{array}$$

is a commutative diagram of R-modules with exact rows. Show the following:

(a) If $c \in \ker(h)$ and $\beta(b) = c$, then $g(b) \in \ker(\beta')$ and $g(b) = \alpha'(a')$ for some $a' \in A'$.

(b) The map $\delta : \ker(h) \to A'/\operatorname{image}(f)$ given by $\delta(c) = a' \mod \operatorname{image}(f)$ is a well-defined R-module homomorphism.

(c) (*The Snake Lemma*) There is an exact sequence

$$\ker(f) \to \ker(g) \to \ker(h) \xrightarrow{\delta} \operatorname{coker}(f) \to \operatorname{coker}(g) \to \operatorname{coker}(h),$$

where $\operatorname{coker}(f) = A'/\operatorname{image}(f)$ denotes the *cokernel* of f and similarly for g and h.

(d) If α is injective and β' is surjective (i.e. the two rows in the above commutative diagram can be extended to short exact sequences) then the sequence in (c) can be extended to an exact sequence

$$0 \to \ker(f) \to \ker(g) \to \ker(h) \xrightarrow{\delta} \operatorname{coker}(f) \to \operatorname{coker}(g) \to \operatorname{coker}(h) \to 0.$$

Proof of (a). Suppose that $c \in \ker(h)$ and $\beta(b) = c$. By commutativity of the diagram above,

$$\beta'(g(b)) = h(\beta(b)) = h(c) = 0,$$

showing that $g(b)$ belongs to the kernel of β'. Moreover, since the bottom row is exact, $\ker(\beta') = \operatorname{image}(\alpha')$, so there exists some $a' \in A'$ for which $g(b) = \alpha'(a')$. □

Proof of (b). Consider the map $\delta : \ker(h) \to A'/\operatorname{image}(f)$ given by $\delta(c) = a' \mod \operatorname{image}(f)$. First, we show that this is well-defined. Let $c \in \ker(h)$ with $\beta(b) = \beta(\widetilde{b}) = c$ where $g(b) = \alpha'(a')$ and $g(\widetilde{b}) = \alpha'(\widetilde{a'})$. We need to show that $a' - \widetilde{a'} \in \operatorname{image}(f)$.

Note that

$$\beta(b - \widetilde{b}) = \beta(b) - \beta(\widetilde{b}) = c - c = 0,$$

$b - \widetilde{b} \in \ker(\beta) = \operatorname{image}(\alpha)$. Therefore there exists some $a \in A$ for which $\alpha(a) = b - \widetilde{b}$.

Furthermore, by the commutativity of the diagram above, $g(\alpha(a)) = \alpha'(f(a))$. Using this, we find that

$$\alpha'(a' - \widetilde{a'}) = \alpha'(a') - \alpha'(\widetilde{a'}) = g(b) - g(\widetilde{b}) = g(b - \widetilde{b}) = g(\alpha(a)) = \alpha'(f(a)).$$

Since α' is injective, this implies that $a' - \widetilde{a'} = f(a) \in \operatorname{image}(f)$, and thus the map δ is well-defined.

To see that δ is an R-module homomorphism, suppose that $x, y \in \ker(h)$ with $\beta(b_1) = x$, $\beta(b_2) = y$ and $g(b_i) = \alpha'(a_i')$ for each i. Let $r \in R$.

Because β is an R-module homomorphism,

$$\beta(b_1 + rb_2) = \beta(b_1) + r\beta(b_2) = x + ry.$$

Moreover, since both g and α' are R-module homomorphisms, we find that

$$g(b_1 + rb_2) = g(b_1) + rg(b_2) = \alpha'(a_1') + r\alpha'(a_2') = \alpha'(a_1' + ra_2').$$

So $\delta(x + ry) = (a_1' + ra_2') + \text{image}(f) = \delta(x) + r\delta(y)$, as desired. □

Proof of (c). First, note that if $a \in \ker(f)$, then $f(a) = 0$ and so

$$0 = \alpha'(0) = \alpha'(f(a)) = g(\alpha(a))$$

giving that $\alpha(a) \in \ker(g)$, so α restricting to a homomorphism $\alpha_r : \ker(f) \to \ker(g)$. By an analogous argument, β restricts to a homomorphism $\beta_r : \ker(g) \to \ker(h)$.

(Exactness at $\ker(g)$) Note that by definition and exactness of rows,

$$\ker(\beta_r) = \ker(\beta) \cap \ker(g) = \text{image}(\alpha) \cap \ker(g).$$

Furthermore, for an element $a \in A$,

$$\begin{aligned} \alpha(a) \in \ker(g) &\Leftrightarrow 0 = g(\alpha(a)) = \alpha'(f(a)) \\ \text{(by injectivity of } \alpha'\text{)} \quad &\Leftrightarrow f(a) = 0 \\ &\Leftrightarrow a \in \ker(f). \end{aligned}$$

Therefore

$$\ker(\beta_r) = \text{image}(\alpha) \cap \ker(g) = \alpha(\ker(f)) = \text{image}(\alpha_r).$$

(Exactness at $\ker(h)$) Let $c \in \ker(h)$.

Suppose that $\delta(c) = 0$, meaning that $a' \in \text{image}(f)$. Then

$$g(b) \in \alpha'(\text{image}(f)) = g(\text{image}(\alpha)) = g(\ker(\beta))$$

In particular, $g(b) = g(\widetilde{b})$ for some $\widetilde{b} \in \ker(\beta)$. Then $b - \widetilde{b} \in \ker(g)$ and

$$\beta(b - \widetilde{b}) = \beta(b) - \beta(\widetilde{b}) = \beta(b) = c.$$

Therefore $c \in \beta(\ker(g))$, showing $\ker(\delta) \subseteq \text{image}(\beta_r)$.

Similarly, if $c \in \beta(\ker(g))$, then $c = \beta(b)$ for some $b \in \ker(g)$. Since $c \in \ker(h)$, by part (a), there exists $a' \in A'$ with $0 = g(b) = \alpha'(a')$. Since α' is injective, this implies that $0 = a' = \delta(c)$. So $\text{image}(\beta_r) \subseteq \ker(\delta)$.

(Exactness at coker f) The map $\text{coker}(f) \to \text{coker}(g)$ is the homomorphism $a' + \text{image}(f) \mapsto \alpha'(a') + \text{image}(g)$ induced by α'. To see that this is well-defined, it suffices to show that $\alpha'(\text{image}(f)) \subseteq \text{image}(g)$, which follows from $\alpha'(\text{image}(f)) = \alpha'(f(A)) = g(\alpha(A))$.

The kernel of this map is

$$\begin{aligned} \{a' + \text{image}(f) : &\alpha'(a') \in \text{image}(g)\} \\ &= \{a' + \text{image}(f) : \alpha'(a') = g(b) \text{ for some } b \in B\} \\ &= \{a' + \text{image}(f) : \alpha'(a') = g(b) \text{ for some } b \text{ with } \beta(b) \in \ker(h)\}. \\ &= \text{image}(\delta). \end{aligned}$$

To see the second-to-last equality, note that if $g(b) \in \text{image}(\alpha') = \ker(\beta')$, then $h(\beta(b)) = \beta'(g(b)) = 0$, so $\beta(b) \in \ker(h)$.

(Exactness at coker g) The map $\text{coker}(g) \to \text{coker}(h)$ is the homomorphism $b' + \text{image}(g) \mapsto \beta'(b') + \text{image}(h)$ induced by α'. To see that this is well-defined, it suffices to show that $\beta'(\text{image}(g)) \subseteq \text{image}(h)$, which follows from $\beta'(\text{image}(g)) = \beta'(g(B)) = h(\beta(B))$.

The kernel of this map is

$$\begin{aligned}
&\{b' + \text{image}(g) : \beta'(b') \in \text{image}(h)\} \\
&\quad = \{b' + \text{image}(g) : \beta'(b') = h(c) \text{ for some } c \in C\} \\
\text{(by surjectivity of } \beta) &\quad = \{b' + \text{image}(g) : \beta'(b') = h(\beta(b)) \text{ for some } b \in B\} \\
&\quad = \{b' + \text{image}(g) : \beta'(b') = \beta'(g(b)) \text{ for some } b \in B\} \\
&\quad = \{b' + \text{image}(g) : b' - g(b) \in \ker(\beta') \text{ for some } b \in B\} \\
\text{(taking } \widetilde{b'} = b' - g(b)) &\quad = \{\widetilde{b'} + \text{image}(g) : b'' \in \ker(\beta')\} \\
\text{(by exactness of bottom row)} &\quad = \{\widetilde{b'} + \text{image}(g) : b'' \in \text{image}(\alpha')\} \\
&\quad = \{\alpha'(a') + \text{image}(g) : a' \in A'\}.
\end{aligned}$$

This is exactly the image of the map coker $f \to$ coker g. □

Proof of (d). (Injectivity of $\ker(f) \to \ker(g)$) Suppose that the map α is injective. Then the restriction of α to $\ker(g)$ must also be injective.

(Surjectivity of $\text{coker}(g) \to \text{coker}(h)$) Suppose that the map β' is sujective. For any $c' + \text{image}(h) \in \text{coker}(h)$, there exists $b' \in B'$ with $\beta'(b') = c'$. Therefore $c' + \text{image}(h)$ is the image of $b' + \text{image}(g)$ under the map $\text{coker}(g) \to \text{coker}(h)$ induced by β'.

With the Snake Lemma from part(c), this gives that the follow sequence is exact:

$$0 \to \ker(f) \to \ker(g) \to \ker(h) \xrightarrow{\delta} \text{coker}(f) \to \text{coker}(g) \to \text{coker}(h) \to 0.$$

□