Math 721 – Homework 6

Due Friday, February 28 at 5pm

Good practice problems (do not turn in solutions):
DF 12.3 Exercises 1,2, 17, 18, 22, 23, 24, 32, 33, 37

Problem 1 (DF 12.3 Exercise 19)**.** Prove that all $n \times n$ matrices over F with characteristic polynomial $f(x)$ are similar if and only if $f(x)$ has no repeated factors in its unique factorization in $F[x]$.

Problem 2 (DF 12.3 Exercises 29, 30)**.** Let V be a vectorspace over a field F and $T : V \to V$ a linear transformation whose eigenvalues all lie in F. For any eigenvalue λ of T, the *generalized eigenspace* of T corresponding to λ is the p-primary component of V as a $F[x]$-module corresponding to the prime $p = x - \lambda$. Equivalently, it is the subspace of vectors annihilated by some power of the linear operator $T - \lambda \cdot \mathrm{id}_V$.

Let λ be an eigenvalue T and let W denote the corresponding generalized eigenspace. Suppose that V is finite dimensional.

(a) Show that for any $k \geq 0$ the dimension of the kernel of $T - \lambda \cdot \mathrm{id}$ on the vectorspace $(T - \lambda \cdot \mathrm{id})^k W$ equals the dimension of the kernel of $T - \lambda \cdot \mathrm{id}$ on the vectorspace $(T - \lambda \cdot \mathrm{id})^k V$, and that this equals the number of Jordan blocks of T having eigenvalue λ and size $> k$.

(b) Let $r_k = \dim_F (T - \lambda \cdot \mathrm{id})^k V$. Show that for any $k \geq 1$, the number of Jordan blocks of size k with eigenvalue λ equals $r_{k-1} - 2r_k + r_{k+1}$. (You may use DF 12.1 Exercise 12 without proof.)