Math 721 – Homework 2 Solutions

Problem 1 (DF Exercise 16 +)**.** Suppose that R is a commutative ring with $1 \neq 0$ and let I and J be ideals of R, so R/I and R/J are naturally R-modules.

(a) Prove that every element of $R/I \otimes_R R/J$ can be written as a simple tensor $(1 \mod I) \otimes (r \mod J)$.

(b) Prove that there is an R-module isomorphism $R/I \otimes_R R/J \to R/(I+J)$ mapping $(r \mod I) \otimes (s \mod J)$ to $(rs \mod I+J)$.

(c) Give an example of R, I, J and an element in $(R/I)^2 \otimes_R (R/J)^2$ that cannot be written as a simple tensor. Make sure to justify your answer.

Proof of (a). Consider an element $\sum_{k=1}^n (a_k + I) \otimes (b_k + J)$ where $a_k, b_k \in R$. Using the relations on tensors, we find that

$$\begin{aligned}
\sum_{k=1}^n (a_k + I) \otimes (b_k + J) &= \sum_{k=1}^n a_k(1 + I) \otimes (b_k + J) \\
&= \sum_{k=1}^n (1 + I) \otimes a_k(b_k + J) \\
&= \sum_{k=1}^n (1 + I) \otimes (a_k b_k + J) \\
&= (1 + I) \otimes \left(\sum_{k=1}^n a_k b_k + J \right) \\
&= (1 + I) \otimes (r + J)
\end{aligned}$$

where $r = \sum_{k=1}^n a_k b_k \in R$. □

Proof of (b). Consider the map $\varphi : (R/I) \times (R/J) \to R/(I+J)$ given by

$$\varphi(a + I, b + J) = ab + I + J.$$

First let us check that φ is well-defined. If $a + I = c + I$ and $b + J = d + J$, then $a - c \in I$ and $b - d \in J$. Then

$$ab - cd = (a - c)b + c(b - d) \in I + J,$$

showing that

$$\varphi(a + I, b + J) = ab + I + J = cd + I + J = \varphi(c + I, d + J).$$

The map φ is bilinear. To see this note, let $a, b, c, d, r_1, r_2 \in R$. We can check that

$$\begin{aligned}
\varphi(r_1(a + I) + r_2(c + I), b + J) &= \varphi(r_1 a + r_2 c + I, b + J) \\
&= (r_1 a + r_2 c)b \\
&= r_1 ab + r_2 cb \\
&= r_1 \varphi(a + I, b + J) + r_2 \varphi(c + I, b + J)
\end{aligned}$$

Therefore there is an R-module homomorphism $\Phi : R/I \otimes_R R/J \to R/(I+J)$ with $\Phi(a+I, b+J) = ab + I + J$.

Note that Φ is surjective because for any $r \in R$, $\Phi((1+I)\otimes(r+J)) = r+I+J \in R/(I+J)$.

To see that Φ is injective, suppose that some element $X \in R/I \otimes_R R/J$ belongs to the kernel of Φ. By part (a), we can write $X = (1+I) \otimes (r+J)$ for some $r \in R$. Then $0 = \varphi((1+I) \otimes (r+J)) = r + I + J$. Therefore $r \in I + J$. In particular, there exist $a \in I, b \in J$ with $r = a + b$. Then

$$\begin{aligned} X = (1+I) \otimes (r+J) &= (1+I) \otimes (a+b+J) \\ &= (1+I) \otimes (a+J) \\ &= (1+I) \otimes a(1+J) \\ &= a(1+I) \otimes (1+J) \\ &= (a+I) \otimes (1+J) \\ &= (0+I) \otimes (1+J) \\ &= 0 \cdot (0+I) \otimes (1+J) \\ &= 0_{R/I\otimes_R R/J} \end{aligned}$$

□

Proof of (c). Consider $R = \mathbb{Q}$ and $I = J = \{0\}$. Then $R/I = R/J = \mathbb{Q}$ and $\mathbb{Q}^2 \otimes \mathbb{Q}^2$ is the four dimensional $\mathbb{Q}$-vectorspace

$$\mathbb{Q}^2 \otimes \mathbb{Q}^2 = \{ae_1 \otimes e_1 + be_1 \otimes e_2 + ce_2 \otimes e_1 + de_2 \otimes e_2 : a, b, c, d \in \mathbb{Q}\}$$

Simple tensors have the form

$$(v_1e_1 + v_2e_2) \otimes (w_1e_1 + w_2e_2) = v_1w_1e_1 \otimes e_1 + v_1w_2e_1 \otimes e_2 + v_2w_1e_2 \otimes e_1 + v_2w_2e_2 \otimes e_2.$$

Organizing the coefficients into a 2×2 matrix, we find that

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} v_1w_1 & v_1w_2 \\ v_2w_1 & v_2w_2 \end{pmatrix} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \begin{pmatrix} w_1 & w_2 \end{pmatrix}.$$

In particular, the set of simple tensors in $\mathbb{Q}^2 \otimes \mathbb{Q}^2$ is given by

$$\left\{ ae_1 \otimes e_1 + be_1 \otimes e_2 + ce_2 \otimes e_1 + de_2 \otimes e_2 : \det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = 0 \right\}.$$

This shows the tensor $e_1 \otimes e_1 + e_2 \otimes e_2$ (corresponding to $(a, b, c, d) = (1, 0, 0, 1)$) cannot be written as a simple tensor. □

Definition. Let R be a commutative ring with identity $1_R \neq 0$. An *R-algebra* is a ring A with identity $1_A \neq 0$ and a ring homomorphism $f : R \to A$ with

(1) $f(1_R) = 1_A$ and
(2) $f(r)a = af(r)$ for all $r \in R$ and $a \in A$.

In particular, if R is a subring of A contained in its center with $1_R = 1_A$, then A is an R-algebra with the map $f : R \to A$ given by $f(r) = r$.

Problem 2. Let R, A, B be rings with R contained in the center of A and the center of B and with coinciding (nonzero) multiplicative identities $1_R = 1_A = 1_B \neq 0$.

(a) Show that the multiplication $(a \otimes b)(a' \otimes b') = aa' \otimes bb'$ makes $A \otimes_R B$ into an R-algebra. (In the proof of Proposition 10.4.21, it is shown that this multiplication is well-defined. This completes the proof of the statement of this proposition.)
(b) Show that $\mathbb{Z}[i] \otimes_{\mathbb{Z}} \mathbb{R} \cong \mathbb{C}$ as rings.

Proof of (a). Note that we already know that $A \otimes_R B$ is group under $+$ and an R-module. We claim that it is a ring with identity $1 \otimes 1$. As noted in the book, the map $A \times B \times A \times B \to A \otimes_R B$ given by $(a, b, a', b') \mapsto (aa' \otimes bb')$ is R-bilinear, and so extends R-linearly to a map $\Phi : A \otimes_R B \times A \otimes_R B \to A \otimes_R B$ with $\Phi((a \otimes b), (a' \otimes b')) = aa' \otimes bb'$,

(Associativity of multiplication) First, let us check this on simple tenors: For $a_1, a_2, a_3 \in A$ and $b_1, b_2, b_3 \in B$,

$$\begin{aligned}\big((a_1 \otimes b_1) \cdot (a_2 \otimes b_2)\big) \cdot (a_3 \otimes b_3) &= (a_1 a_2 \otimes b_1 b_2) \cdot (a_3 \otimes b_3) \\ &= a_1 a_2 a_3 \otimes b_1 b_2 b_3 \\ &= (a_1 \otimes b_1) \cdot (a_2 a_3 \otimes b_2 b_3) \\ &= (a_1 \otimes b_1) \cdot \big((a_2 \otimes b_2) \cdot (a_3 \otimes b_3)\big).\end{aligned}$$

Now suppose $\sum_i r_i, \sum_j s_j, \sum_k t_k$ are tensors in $A \otimes_R B$ where each r_i, s_j, t_k are simple tensors. Associativity of simple tensors shows that $\Phi\big(\Phi(r_i, s_j), t_k\big) = \Phi\big(r_i, \Phi(s_j, t_k)\big)$. Then

$$\begin{aligned}\Phi\big(\Phi((\sum_i r_i), (\sum_j s_j)), \sum_k t_k\big) &= \Phi\big(\sum_{i,j} \Phi(r_i, s_j), \sum_k t_k\big) \\ &= \sum_{i,j,k} \Phi\big(\Phi(r_i, s_j), t_k\big) \\ &= \sum_{i,j,k} \Phi\big(r_i, \Phi(s_j, t_k)\big) \\ &= \Phi\big(\sum_i r_i, \sum_{j,k} \Phi(s_j, t_k)\big) \\ &= \Phi\big(\sum_i r_i, \Phi(\sum_j s_j, \sum_k t_k)\big)\end{aligned}$$

(Distributivity of multiplication over addition) This follows from the bilinearity of the multiplication operator Φ. For any tensors $r, s, t \in A \otimes_R B$,

$$\Phi\,(r + s, t) = \Phi(r, t) + \Phi(s, t) \quad \text{and} \quad \Phi\,(r, s + t) = \Phi(r, s) + \Phi(r, t).$$

(Indentity) We claim that $1 \otimes 1$ is the identity in $A \otimes_R B$, where 1 denotes the common identity of R, A, and B. Note that

$$\begin{aligned}
(1 \otimes 1) \cdot \left(\sum_{i=1}^{n} a_i \otimes b_i \right) &= \sum_{i=1}^{n} (1 \otimes 1) \cdot (a_i \otimes b_i) \\
&= \sum_{i=1}^{n} 1 \cdot a_i \otimes 1 \cdot b_i \\
&= \sum_{i=1}^{n} a_i \otimes b_i \\
&= \sum_{i=1}^{n} a_i \cdot 1 \otimes b_i \cdot 1 \\
&= \sum_{i=1}^{n} (a_i \otimes b_i) \cdot (1 \otimes 1) \\
&= \left(\sum_{i=1}^{n} a_i \otimes b_i \right) \cdot (1 \otimes 1)
\end{aligned}$$

(R-algebra) Consider the function $f : R \to A \otimes_R B$ given by $f(r) = r \otimes 1$. Note that $f(1) = 1 \otimes 1 = 1_{A \otimes_R B}$. Furthermore, we can check that f is a ring homomorphism. For any $r, s \in R$,

$$f(r+s) = (r \otimes 1) + (s \otimes 1) = (r+s) \otimes 1 = f(r) + f(s)$$

and

$$f(r \cdot s) = (r \otimes 1) \cdot (s \otimes 1) = (r \cdot s) \otimes 1 = f(r) \cdot f(s).$$

Finally we can check that the image of f belongs to the center of $A \otimes_R B$.

$$\begin{aligned}
f(r) \cdot \sum_{k=1}^{n} a_k \otimes b_k = (r \otimes 1) \cdot \left(\sum_{k=1}^{n} a_k \otimes b_k \right) &= \sum_{k=1}^{n} (r \otimes 1)(a_k \otimes b_k) \\
&= \sum_{k=1}^{n} r a_k \otimes b_k \\
&= \sum_{k=1}^{n} a_k r \otimes b_k \\
&= \sum_{k=1}^{n} (a_k \otimes b_k)(r \otimes 1) \\
&= \sum_{k=1}^{n} (a_k \otimes b_k)(r \otimes 1) \\
&= \left(\sum_{k=1}^{n} a_k \otimes b_k \right) \cdot (r \otimes 1)
\end{aligned}$$

This shows that $A \otimes_R B$ is as R-algebra. $\square$

Proof of (b). Consider the map $\varphi : \mathbb{Z}[i] \times \mathbb{R} \to \mathbb{C}$ defined by $\varphi(z, r) = rz$. We claim that φ is $\mathbb{Z}$-bilinear. To see this, suppose $z, w \in \mathbb{Z}[i]$ and $r, s \in \mathbb{R}$. Then

$$\varphi(z + w, r) = r(z + w) = rz + rw = \varphi(z, r) + \varphi(w, r)$$

and

$$\varphi(z, r + s) = (r + s)z = rz + sz = \varphi(z, r) + \varphi(z, s).$$

This extends to a unique $\mathbb{Z}$-module homomorphism $\Phi : \mathbb{Z}[i] \otimes_{\mathbb{Z}} \mathbb{R} \to \mathbb{C}$ with $\Phi(z \otimes r) = rz$.

Now consider the map $\Psi : \mathbb{C} \to \mathbb{Z}[i] \otimes_{\mathbb{Z}} \mathbb{R}$ given by $\Psi(a+ib) = 1 \otimes a + i \otimes b$. We claim that this is also a $\mathbb{Z}$-module homomorphism (i.e. group homomorphism). To check, note that for $a, b, c, d \in \mathbb{R}$,

$$\Psi(a+ib+c+id) = 1 \otimes (a+c) + i \otimes (b+d) = 1 \otimes a + 1 \otimes c + i \otimes b + i \otimes d = \Psi(a+ib) + \Psi(c+id).$$

Note that $\Phi \circ \Psi = \mathrm{id}_{\mathbb{C}}$:

$$\Phi(\Psi(a + ib)) = \Phi(1 \otimes a + i \otimes b) = \Phi(1 \otimes a) + \Phi(i \otimes b) = a + ib.$$

Also $\Psi \circ \Phi = \mathrm{id}_{\mathbb{Z}[i] \otimes_{\mathbb{Z}} \mathbb{R}}$. To see this, note that for $a_j, b_j \in \mathbb{Z}$, $r_j \in \mathbb{R}$,

$$\begin{aligned}
\Psi(\Phi(\sum_j (a_j + ib_j) \otimes r_j)) &= \Psi(\sum_j \Phi((a_j + ib_j) \otimes r_j)) \\
&= \Psi(\sum_j r_j(a_j + ib_j)) \\
&= \Psi((\sum_j r_j a_j) + i(\sum_j r_j b_j)) \\
&= 1 \otimes (\sum_j r_j a_j) + i \otimes (\sum_j r_j b_j) \\
&= \sum_j (a_j \otimes r_j) + \sum_j (ib_j \otimes r_j) \\
&= \sum_j (a_j + ib_j) \otimes r_j.
\end{aligned}$$

Therefore Φ gives a $\mathbb{Z}$-module isomorphism between $\mathbb{Z}[i] \otimes_{\mathbb{Z}} \mathbb{R}$ and $\mathbb{C}$ whose inverse is Ψ. $\square$