Math 721 – Homework 1 Solutions

Problem 1 (Kernels and images)**.** Let R be a ring with $1 \neq 0$.

(a) If $\varphi : M \to N$ is a homomorphism of R-modules, show that the kernel and image of φ are submodules of M and N respectively.

(b) For each of the following R-modules homomorphisms, describe the kernel and image as simply as possible. (You do not need to check that they are homomorphisms.)

(i) $R = \mathbb{Q}[x, y]$, $M = R^2$, $N = R$, $\varphi(f, g) = xf + yg$

(ii) $R = \mathbb{Z}$, $M = R^2$, $N = R$, $\varphi(a, b) = 4a + 6b$

(ii) $R = \mathrm{Mat}_{2\times2}(\mathbb{Q})$, $M = \mathrm{Mat}_{2\times3}(\mathbb{Q})$, $N = \mathrm{Mat}_{2\times4}(\mathbb{Q})$, $\varphi(A) = AU$ where

$$U = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

Proof. (a) Suppose $\varphi : M \to N$ is a homomorphism of R-modules. Note that $\ker(\varphi)$ is nonempty, since it contains 0. Moreover, for any $x, y \in \ker(\varphi) \subseteq M$ and $r \in R$,

$$\varphi(x + ry) = \varphi(x) + r\varphi(y) = 0 + r0 = 0.$$

Therefore $x + ry \in \ker(\varphi)$, showing that $\ker(\varphi)$ is a submodule of M.

Similarly, $\varphi(M)$ is nonempty, since it contains $\varphi(0) = 0$. For any $x, y \in \varphi(M)$, there exist $a, b \in M$ with $\varphi(a) = x$ and $\varphi(b) = y$. Then for $r \in R$,

$$\varphi(a + rb) = \varphi(a) + r\varphi(b) = x + ry,$$

showing that $x + ry \in \varphi(M)$ and that $\varphi(M)$ is a submodule of N. □

Proof. (b) (i) For $R = \mathbb{Q}[x, y]$, $M = R^2$, $N = R$, $\varphi(f, g) = xf + yg$, an element $(f, g) \in R^2$ belongs to the kernel of φ exactly when $xf + yg = 0$. This gives $xf = -yg$, meaning that x divides yg and y divides xf. Since neither x nor y divide each other (and R is a unique factorization domain) there must be some $h \in R$ with $f = -yh$ and $g = xh$. Therefore $(f, g) = (-yh, hx) = h(-y, x)$. This shows that

$$\ker(\varphi) = R(-y, x).$$

The image of φ is the ideal of R generated by x and y.

(ii) For $R = \mathbb{Z}$, $M = R^2$, $N = R$, $\varphi(a, b) = 4a + 6b$, the kernel of φ is $\{(a, b) : 4a + 6b = 0\}$. If $4a + 6b = 0$, then $2a = -3b$. Since 2 and 3 are relatively prime, there must exist $c \in \mathbb{Z}$ with $a = -3c$ and $b = 2c$. Then $(a, b) = c(-3, 2)$, showing that

$$\ker(\varphi) = \mathbb{Z}(-3, 2).$$

Then image of φ is the ideal of $\mathbb{Z}$ generated by 4 and 6. Since $\mathbb{Z}$ is a principal ideal domain, we see that this is generated by $\gcd(4, 6) = 2$, showings that $\varphi(M) = 2\mathbb{Z}$.

(iii) Let $R = \mathrm{Mat}_{2\times2}(\mathbb{Q})$, $M = \mathrm{Mat}_{2\times3}(\mathbb{Q})$, $N = \mathrm{Mat}_{2\times4}(\mathbb{Q})$, $\varphi(A) = AU$ where

$$U = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

Note that

$$\varphi\left(\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix}\right) = \begin{pmatrix} a_{11} & a_{12} & a_{13} & 0 \\ a_{21} & a_{22} & a_{23} & 0 \end{pmatrix}$$

The only way that $\varphi(A) = 0_N$ is that $A = 0_M$, so $\ker(\varphi) = \{0_M\}$. The image of φ is the set of 2×4 matrices with last column equal to zero. □

Problem 2 (Hom_R)**.** Let R be a commutative ring with $1 \neq 0$ and let A, B and M be R-modules. Show the following isomorphisms of R-modules:

(a) $\mathrm{Hom}_R(R, M) \cong M$,

(b) $\mathrm{Hom}_R(A \times B, M) \cong \mathrm{Hom}_R(A, M) \times \mathrm{Hom}_R(B, M)$, and

(c) $\mathrm{Hom}_R(R^n, M) \cong \underbrace{M \times \cdots \times M}_{n}$.

(d) Give an example to show that it is not always that case that $\mathrm{Hom}_R(M, R) \cong M$.

Proof. (a) Consider the map $\Psi : \mathrm{Hom}_R(R, M) \to M$ given by $\Psi(\varphi) = \varphi(1)$ where $\varphi \in \mathrm{Hom}_R(R, M)$ and 1 is the unit in R. Note that $\varphi(1) \in M$. We claim that Ψ is an isomorphism.

(Homomorphism) Let $\varphi, \psi \in \mathrm{Hom}_R(R, M)$. Then

$$\Psi(\varphi + \psi) = (\varphi + \psi)(1) = \varphi(1) + \psi(1) = \Psi(\varphi) + \Psi(\psi).$$

and for any $r \in R$,

$$\Psi(r\varphi) = (r\varphi)(1) = r\varphi(1) = r\Psi(\varphi).$$

(Bijection) First, note that Ψ is injective. If $\varphi(1) = \psi(1)$, then since φ and ψ are homomorphisms, for every $r \in R$,

$$\varphi(r) = r\varphi(1) = r\psi(1) = \psi(r),$$

meaning that φ and ψ are the same function on R. To see that it is surjective, for $m \in M$, define the map $\varphi_m : R \to M$ by $\varphi_m(r) = rm$. We claim that φ_m is indeed an R-module homomorphism. To check, we see that it is a function from R to M and satisfies

$$\varphi_m(rx + y) = (rx + y)m = rxm + ym = r\varphi_m(x) + \varphi_m(y)$$

for any $x, y, r \in R$. Therefore $\varphi_m \in \mathrm{Hom}_R(R, M)$. Moreover $\Psi(\varphi_m) = \varphi_m(1) = m$. □

Proof. (b) Consider the map $\Psi : \mathrm{Hom}_R(A, M) \times \mathrm{Hom}_R(B, M) \to \mathrm{Hom}_R(A \times B, M)$ given by

$$\Psi(\alpha, \beta)(a, b) = \alpha(a) + \beta(b) \in M$$

where $\alpha \in \mathrm{Hom}_R(A, M)$, $\beta \in \mathrm{Hom}_R(B, M)$, and $(a, b) \in A \times B$. First, note that $\varphi = \Psi(\alpha, \beta)$ is an element of $\mathrm{Hom}_R(A \times B, M)$. Given $(a, b), (c, d) \in A \times B$ and $r \in R$,

$$\begin{aligned} \varphi((a, b) + r(c, d)) = \varphi(a + rc, b + rd) &= \alpha(a + rc) + \beta(b + rd) \\ &= \alpha(a) + r\alpha(c) + \beta(b) + r\beta(d) \\ &= (\alpha(a) + \beta(b)) + r(\alpha(c) + \beta(d)) \\ &= \varphi(a, b) + r\varphi(c, d). \end{aligned}$$

Therefore $\Psi(\alpha, \beta) = \varphi \in \mathrm{Hom}_R(A \times B, M)$. We claim that Ψ is an isomorphism.

(Homomorphism) Let $(\alpha,\beta),(\gamma,\delta)\in \operatorname{Hom}_R(A,M)\times \operatorname{Hom}_R(B,M)$ and $r\in R$. For any $a\in A$ and $b\in B$,

$$\begin{aligned}\Psi((\alpha,\beta)+r(\gamma,\delta))(a,b)=\Psi(\alpha+r\gamma,\beta+r\delta)(a,b)&=(\alpha+r\gamma)(a)+(\beta+r\delta)(b)\\&=\alpha(a)+r\gamma(a)+\beta(b)+r\delta(b)\\&=(\alpha(a)+\beta(b))+r(\gamma(a)+\delta(b))\\&=\Psi(\alpha,\beta)(a,b)+r\Psi(\gamma,\delta)(a,b).\end{aligned}$$

So as elements of $\operatorname{Hom}_R(A\times B,M)$, $\Psi((\alpha,\beta)+r(\gamma,\delta))=\Psi(\alpha,\beta)+r\Psi(\gamma,\delta)$, giving that Ψ is a homomorphism.

(Bijection) Suppose that $\Psi(\alpha,\beta)(a,b)=\Psi(\gamma,\delta)(a,b)$ for all $(a,b)\in A\times B$. Taking points $(a,0_B)$ with $a\in A$ shows that

$$\alpha(a)=\Psi(\alpha,\beta)(a,0_B)=\Psi(\gamma,\delta)(a,0_B)=\gamma(a)$$

for all $a\in A$, giving that $\alpha=\gamma$. Similarly evaluating at points $(0_A,b)$ shows that $\beta=\delta$. Therefore Ψ is injective.

For surjectivity, suppose that $\varphi\in \operatorname{Hom}_R(A\times B,M)$. Let $\alpha:A\to M$ and $\beta:B\to M$ be defined by $\alpha(a)=\varphi(a,0_B)$ and $\beta(b)=\varphi(0_A,b)$, respectively, where 0_A and 0_B are the additive identities of A and B. We claim that $\alpha\in \operatorname{Hom}_R(A,M)$, $\beta\in \operatorname{Hom}_R(B,M)$ and $\Psi(\alpha,\beta)=\varphi$.

First note that for $a,c\in A$ and $r\in R$,

$$\alpha(a+rc)=\varphi(a+rc,0_B)=\varphi((a,0_B)+r(c,0_B))=\varphi(a,0_B)+r\varphi(c,0_B)=\alpha(a)+r\alpha(c).$$

Similarly, for $b,d\in B$ and $r\in R$,

$$\beta(b+rd)=\varphi(0_A,b+rd)=\varphi((0_A,b)+r(0_A,d))=\varphi(0_A,b)+r\varphi(0_A,d)=\beta(b)+r\beta(d).$$

Finally, note that for any $(a,b)\in A\times B$, $(a,b)=(a,0_B)+(0_A,b)$. Then

$$\Psi(\alpha,\beta)(a,b)=\alpha(a)+\beta(b)=\varphi(a,0_B)+\varphi(0_A,b)=\varphi(a,b).$$

□

Proof. (c) We proceed by induction on n. By part (a), this holds with $n=1$. Suppose that it holds for some $n\in\mathbb{Z}_+$. Note that $R^{n+1}\cong R\times R^n$. Then by part (b),

$$\operatorname{Hom}_R(R^{n+1},M)\cong \operatorname{Hom}_R(R,M)\times \operatorname{Hom}_R(R^n,M)\cong M\times(\underbrace{M\times\cdots\times M}_{n})\cong \underbrace{M\times\cdots\times M}_{n+1}.$$

This shows the claim for all $n\in\mathbb{Z}_+$. □

Proof. (d) Let $R=\mathbb{Z}$ and $M=\mathbb{Z}/2\mathbb{Z}$ and consider $\varphi\in \operatorname{Hom}_R(M,R)$. Note that $\varphi(0)=0\varphi(1)=0$. If $\varphi(1)=n\in\mathbb{Z}$, then

$$0=\varphi(0)=\varphi(1+1)=\varphi(1)+\varphi(1)=2\varphi(1).$$

Since $\mathbb{Z}$ has no zero-divisors, $2\varphi(1)=0$ implies $\varphi(1)=0$. Therefore the only $\mathbb{Z}$-module homomorphism from $\mathbb{Z}/2\mathbb{Z}$ to $\mathbb{Z}$ is the map φ given by $\varphi(a)=0$ for all $a\in\mathbb{Z}/2\mathbb{Z}$. Then there is only one element in $\operatorname{Hom}_{\mathbb{Z}}(\mathbb{Z}/2\mathbb{Z},\mathbb{Z})$, whereas $M=\mathbb{Z}/2\mathbb{Z}$ has two. □