Math 591 – Real Algebraic Geometry and Convex Optimization
Lecture 4: The PSD cone (continued) and nonnegative polynomials
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In order to understand the faces of the PSD cone, we will use the following useful face about exposed faces of general convex cones.

Proposition. *The exposed faces of a convex cone $K \subseteq V$ have the form*

$$F = \{v \in K : \ell(v) = 0\}$$

where $\ell \in K^*$.

Proof. By definition an exposed faces has the form $F = \{v \in K : \ell(v) \geq \ell(w) \text{ for all } w \in K\}$ for some $\ell \in V^*$. Note though that if $\ell(v) > 0$ for some $v \in K$, then for any $\lambda > 1$, $\lambda v \in K$ and $\ell(\lambda v) = \lambda\ell(v) > \ell(v)$. Taking $\lambda \to \infty$ shows that ℓ in unbounded on K and cannot expose a face. Otherwise $\ell(v) \leq 0$ for all $v \in K$. Since $\ell(0) = 0$, the maximum of ℓ over K is zero and $-\ell$ belongs to K^*. Then the face exposed by ℓ is

$$F \;=\; \{v \in K : \ell(v) = 0\} \;=\; \{v \in K : -\ell(v) = 0\}.$$

□

Applying this to $K = \text{PSD}_n$, we find that

Corollary. *The exposed faces of PSD_n are parametrized by linear subspaces $L \subseteq \mathbb{R}^n$. Namely they have the form*

$$\mathcal{F}_L \;=\; \{A \in \text{PSD}_n : L \subseteq \ker(A)\},$$

where $L \subseteq \mathbb{R}^n$ is a subspace and for every $L \subseteq \mathbb{R}^n$, $\mathcal{F}_L$ is a face of PSD_n.

Proof. Recall that every element of $(\text{PSD}_n)^*$ has the form $A \mapsto \langle A, B\rangle$, where $B \in \text{PSD}_n$. Then by the proposition above, every exposed face has the form

$$\{A \in \text{PSD}_n : \langle A, B\rangle = 0\} \;=\; \{A \in \text{PSD}_n : \text{rowspan}(B) \subseteq \ker(A)\},$$

where $B \in \text{PSD}_n$. Furthermore for any subspace $L = \text{span}_{\mathbb{R}}\{v_1, \ldots, v_k\} \subseteq \mathbb{R}^n$, the matrix $B = \sum_{i=1}^k v_i v_i^T$ is positive semidefinite and has rowspan equal to L. □

Note that the PSD cone is invariant under a large group action. Namely, for any $n \times n$ invertible matrix U, the PSD cone is invariant under the map $A \mapsto U^T A U$. To check we see that the image is symmetric and PSD:

- (symmetric) $(U^T A U)^T = U^T A^T (U^T)^T = U^T A U$
- (PSD) $v^T (U^T A U) v = (Uv)^T A (Uv) \geq 0$ for all $v \in \mathbb{R}^n$

If U is an orthogonal matrix (i.e. $U^T U = U U^T = I_n$), then this action even preserves the inner product. That is,

$$\begin{aligned}\langle U^T A U, U^T B U\rangle &= \text{trace}(U^T A U U^T B U) = \text{trace}(U^T A B U) \\ &= \text{trace}(A B U U^T) = \text{trace}(AB) = \langle A, B\rangle.\end{aligned}$$

We can use this acton to give a very explicit understanding of the faces $\mathcal{F}_L$ of the PSD cone. Suppose that U is a matrix with columns $u_1, \ldots, u_n \in \mathbb{R}^n$. Then

$$u_i \in \ker(A) \;\Leftrightarrow\; e_i \in \ker(U^T A U),$$

where e_i is the ith unit coordinate vector. Now suppose $L = \text{span}_{\mathbb{R}}\{u_1, \ldots, u_k\}$. Then

$$\begin{aligned}
\mathcal{F}_L &= \{A \in \text{PSD}_n : L \subseteq \ker(A)\} \\
&= \{U^{-T} B U^{-1} : B \in \text{PSD}_n \text{ and } \text{span}_{\mathbb{R}}\{e_1, \ldots, e_k\} \subseteq \ker(B)\} \\
&= \left\{ U^{-T} \begin{pmatrix} 0 & 0 \\ 0 & \tilde{B} \end{pmatrix} U^{-1} : \tilde{B} \in \text{PSD}_{n-k} \right\} \\
&\cong \text{PSD}_{n-k}.
\end{aligned}$$

This gives a linear isomorphism between $\mathcal{F}_L$ and $\text{PSD}_{n-\dim(L)}$.

Example. Consider $L = \text{span}_{\mathbb{R}}\{(1,1,1)\} \subset \mathbb{R}^3$. Then

$$\mathcal{F}_L = \{A \in \text{PSD}_3 : (1,1,1)A = 0\} = \left\{ A = \begin{pmatrix} a & b & -a-b \\ b & c & -b-c \\ -a-b & -b-c & a+2b+c \end{pmatrix} \in \text{PSD}_3 \right\}.$$

For a matrix A of this form, we find that for

$$U = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \quad \text{we have} \quad U^T A U = \begin{pmatrix} 0 & 0 & 0 \\ 0 & a & b \\ 0 & b & c \end{pmatrix}.$$

Therefore $\mathcal{F}_L$ is isomorphic to $\mathcal{F}_{\text{span}\{e_1\}}$, which is isomorphic to PSD_2.

To complete our understanding of the facial structure of PSD_n, we need the following:

Theorem (Ramana, Goldman 1995)**.** *All faces of* PSD_n *are exposed.*

Sketch of proof. For a face F of PSD_n, consider the linear space $L \subseteq \mathbb{R}^n$ obtained by intersecting $\ker(A)$ over all $A \in F$. Then certainly $F \subseteq \mathcal{F}_L$. Moreover, for A in the *relative interior* of F, one can show that $\ker(A) = L$ and thus that A is in the relative interior of $\mathcal{F}_L$. □

Here we used the **relative interior** of a face F, which is the interior of F relative to its affine span.

Corollary. *The faces of* PSD_n *are linearly isomorphic to* PSD_r*, with dimension* $\binom{r+1}{2}$*, for* $r = 0, \ldots, n$*.*

For example, the six-dimensional cone PSD_3 has faces of dimensions $0, 1, 3$ and 6.
Note the for the PSD cone, we saw that

$$\text{PSD}_n = \{A : \langle A, vv^T \rangle \geq 0 \text{ for all } v\} \quad \text{and} \quad (\text{PSD}_n)^* = \text{conv}\{vv^T : v \in \mathbb{R}^n\} = \text{PSD}_n.$$

Using similar ideas we can understand other nonnegative polynomials.

Nonnegative polynomials. For any $d \in \mathbb{N}$, consider the vector space $V = \mathbb{R}[x]_{\leq 2d} \cong \mathbb{R}^{2d+1}$ of univariate polynomials of degree $\leq 2d$ and within V define

$$P_{1,\leq 2d} = \{f \in \mathbb{R}[x]_{\leq 2d} \ : \ f(p) \geq 0 \text{ for all } p \in \mathbb{R}\}$$

to be the convex cone of nonnegative polynomials. What is the dual cone $(P_{1,\leq 2d})^*$?

For any point $p \in \mathbb{R}$, define the linear function $\mathrm{ev}_p : \mathbb{R}[x]_{\leq 2d} \to \mathbb{R}$ by $\mathrm{ev}_p(f) = f(p)$. Then

$$\begin{aligned} f \in P_{1,\leq 2d} \;\Leftrightarrow\; & \mathrm{ev}_p(f) \geq 0 \;\text{ for all }\; p \in \mathbb{R} \\ \Leftrightarrow\; & \sum_{i=1}^{k} \lambda_i \mathrm{ev}_p(f) \geq 0 \;\text{ for all }\; k \in \mathbb{N}, p_1, \ldots, p_k \in \mathbb{R}, \lambda_1, \ldots, \lambda_k \in \mathbb{R}_{\geq 0}. \end{aligned}$$

This shows that $P_{1,\leq 2d}$ is already defined as the dual cone of the conical hull of point evaluations, namely

$$P_{1,\leq 2d} \;=\; (\text{conical hull}\{\mathrm{ev}_p : p \in \mathbb{R}\})^*,$$

and so dualizing gives that

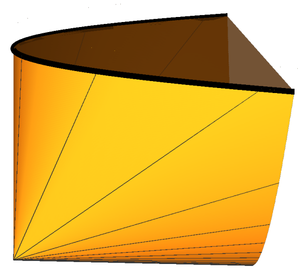
$$(P_{1,\leq 2d})^* \;=\; \overline{\text{conical hull}\{\mathrm{ev}_p : p \in \mathbb{R}\}}.$$

To make this concrete, we can write $f \in \mathbb{R}[x]_{\leq 2d}$ as $f = a_0 + a_1 x + \ldots + a_{2d}x^{2d}$, then

$$\mathrm{ev}_p(f) \;=\; \langle (a_0, a_1, a_2, \ldots, a_{2d}), (1, p, p^2, \ldots, p^{2d}) \rangle.$$

This identifies $(P_{1,\leq 2d})^*$ with (the closure of) the conical hull of $\{(1, p, p^2, \ldots, p^{2d}) : p \in \mathbb{R}\}$ in $\mathbb{R}^{2d+1}$.

For example, for $d = 1$, this is the cone over a parabola at height 1:



It is interesting to note that in the closure we pick up the ray spanned $(0, 0, 1)$, which corresponds to the linear function $a_2 x^2 + a_1 x + a_0 \mapsto a_2$.

These cones having interesting connections with the PSD cone and we will return to them later in the class.