Math 591 – Real Algebraic Geometry and Convex Optimization

Lecture 2: Cones and Duality

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From last time, we take V to be a vector space over $\mathbb{R}$, V^* to be the dual vector space of linear functionals $\ell : V \to \mathbb{R}$. For any convex set $C \subseteq V$, we say that $\ell \in V^*$ exposes the face $F \subseteq C$ where

$$F = \{v \in C : \ell(v) \geq \ell(w) \text{ for all } w \in C\}.$$

Many linear functions give the same face. For example, multiplying ℓ by a positive scalar results in the same face. The question we start with today is: what is this collection of linear functions?

Definition. Given a convex set $C \subset V$ and point $v \in C$, the (outer) **normal cone** of C at v is

$$N_C(v) \;=\; \{\ell \in V^* : \ell(v) \geq \ell(w) \text{ for all } w \in C\},$$

and the (outer) **normal cone** of C at a face F is

$$N_C(F) \;=\; \bigcap_{v \in F} N_C(v) \;=\; \{\ell \in V^* : \ell(v) \geq \ell(w) \text{ for all } v \in F, w \in C\}.$$

Example. Consider the square $C = [-1, 1]^2 \subset \mathbb{R}^2$ and the points $u = (1, 1)$, $v = (0, 1)$, and $w = (0, 0)$. Note that any linear function $\ell : \mathbb{R}^2 \to \mathbb{R}$ has the form $\ell(x, y) = ax + by$ for some $a, b \in \mathbb{R}$. This explicitly identifies $(\mathbb{R}^2)^*$ with $\mathbb{R}^2$ via $\ell \leftrightarrow (a, b)$. Then computing the normal cones at these points we find:

$$N_C(u) = \{\ell(x, y) = ax + by : a \geq 0, b \geq 0\} \cong \mathbb{R}^2_{\geq 0}$$
$$N_C(v) = \{\ell(x, y) = ax + by : a = 0, b \geq 0\} \cong \{0\} \times \mathbb{R}_{\geq 0}$$
$$N_C(w) = \{\ell(x, y) = ax + by : a = 0, b = 0\} \cong \{(0, 0)\}$$

Proposition. *The normal cone $N_C(v) \subset V^*$ is convex.*

Proof. Let $\ell_1, \ell_2 \in N_C(v)$ and $\lambda \in [0, 1]$. To show $\lambda\ell_1 + (1 - \lambda)\ell_2 \in N_C(v)$, we take any $w \in C$. By definition $\ell_i(v) \geq \ell_i(w)$. Scaling and adding these inequalities appropriately we find that

$$\lambda\ell_(v) + (1 - \lambda)\ell_2(v) \geq \lambda\ell_1(w) + (1 - \lambda)\ell_2(w).$$

Since this holds for all w, $\lambda\ell_1 + (1 - \lambda)\ell_2 \in N_C(v)$. □

Note that the normal cone $N_C(v)$ is also invariant under nonnegative scaling, i.e. for any $\ell \in \mathcal{V}^*$, we have $\ell \in N_C(v) \Leftrightarrow \lambda\ell \in N_C(v)$ for $\lambda \in \mathbb{R}_{\geq 0}$. This together with convexity shows that $N_C(v)$ is a *convex cone* (and justifies its name).

(Aside on convex cones)

Definition. A subset $K \subseteq V$ is a **convex cone** if K is convex and invariant under nonnegative scaling (i.e. $\mathbb{R}_{\geq 0}K \subseteq K$). Equivalently if K if for all $u, v \in K$ and $\lambda, \mu \in \mathbb{R}_{\geq 0}$, the point $\lambda u + \mu v$ belongs to K.

For much of the terminology of convex sets has a conical version.

Definition. A conic combination of points $v_1, \ldots, v_k \in V$ is a point of the form $\sum_{i=1}^{k} \lambda_i v_i$ where $\lambda_i \geq 0$ and the **conical hull** of a set $S \subseteq V$ is the set of all conic combinations of finitely-many points from S:

$$\left\{ \sum_{i=1}^{k} \lambda_i v_i : k \in \mathbb{N}, v_1, \ldots, v_k \in S, \lambda_1, \ldots, \lambda_k \in \mathbb{R}_{\geq 0} \right\}.$$

Another natural cone associated to a convex set at a point is the *tangent cone*.

Definition. The **tangent cone** of C at v is

$$T_C(v) \;=\; \{w \in V : v + \epsilon w \in C \text{ for sufficiently small } \epsilon > 0\}.$$

Note that $T_C(v)$ is also a convex cone. If $w_1, w_2 \in T_C(v)$, then $v + \epsilon_1 w_1$ and $v + \epsilon_2 w_2$ belong to C for some $\epsilon_1, \epsilon_2 > 0$. Then for any $\lambda, \mu \in \mathbb{R}_{\geq 0}$ we need to show $\lambda w_1 + \mu w_2 \in T_C(v)$. This can be done for finding ϵ so that the point $v + \epsilon(\lambda w_1 + \mu w_2)$ belongs to the convex hull of the three points $v, v + \epsilon_1 w_1, v + \epsilon_2 w_2$.

Example. Let's again consider the square $C = [-1, 1]^2 \subset \mathbb{R}^2$ and the points $u = (1, 1)$, $v = (0, 1)$, and $w = (0, 0)$. Then we find that

$$T_C(u) = (\mathbb{R}_{\leq 0})^2$$
$$T_C(v) = \mathbb{R} \times \mathbb{R}_{\leq 0}$$
$$T_C(w) = \mathbb{R}^2.$$

It gets slightly more interesting if instead a point on the boundary of the disk

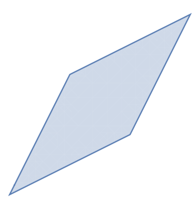
$$\tilde{C} = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}.$$

For $v = (0, 1)$, we find that the tangent cone is not closed. Specifically,

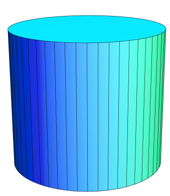
$$T_{\tilde{C}}(v) \;=\; \{0, 0\} \cup (\mathbb{R} \times \mathbb{R}_{<0}).$$

Question. What are the normal and tangent cones of the following convex sets?:

- a slanty quadrilateral where v is the top right vertex



- a cylinder $C = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 \leq 1, -1 \leq z \leq 1\}$ at each of the three points $(1, 0, 1)$, $(1, 0, 0)$ and $(0, 0, 1)$



For every point $w \in T_C(v)$ and every $\ell \in N_C(v)$, we see that $\ell(w) \leq 0$. Indeed if $v + \epsilon w \in C$ and $\ell(w) > 0$, then

$$\ell(v + \epsilon w) = \ell(v) + \epsilon \ell(w) > \ell(v)$$

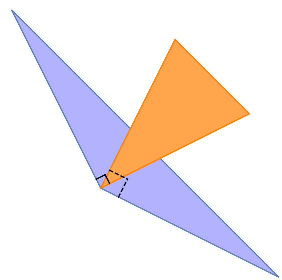
which implies that $\ell \notin N_C(v)$.

When $\dim(V) < \infty$, this becomes if and only if (up to closure). Then $-\overline{T_C(v)}$ and $N_C(v)$ are *dual cones*.

Definition. The **dual cone** of a convex cone $K \subseteq V$ is

$$K^* \ = \ \{\ell \in V^* : \ell(v) \geq 0 \text{ for all } v \in K\}.$$

One can check that this is a convex cone in V^*.



Example. Consider the nonnegative orthant $K = (\mathbb{R}_{\geq 0})^n$ We can write linear functions on $\mathbb{R}^n$ as $\ell(x) = \sum_{i=1}^n a_i x_i$. Then

$$\ell(x) = \sum_{i=1}^{n} a_i x_i \geq 0 \ \text{ on } \ K \quad \Leftrightarrow \quad a_1 \geq 0, \ldots, a_n \geq 0.$$

Certainly if each a_i is nonnegative, then for any $p \in (\mathbb{R}_{\geq 0})^n$, $\ell(p) = \sum_{i=1}^n a_i p_i$ which is nonnegative, since $a_i p_i \geq 0$. Conversely, if $a_i < 0$ for some a_i, then the evaluation at the ith coordinate vector e_i is negative, $\ell(e_i) = a_i < 0$. Therefore

$$K^* \ = \ \{\ell(x) = \sum_{i=1}^{n} a_i x_i : a_1 \geq 0, \ldots, a_n \geq 0\} \ \cong \ (\mathbb{R}_{\geq 0})^n.$$

So the dual cone of the nonnegative orthant is again the nonnegative orthant. When K and K^* are the same (or more precisely, linearly isomorphic) we say that K is “self-dual”.

What about the dual of the dual, $(K^*)^*$?

First, let's remark that for vector spaces, there is a natural linear isomorphism between V and $((V^*)^*)$, namely

$$v \in V \quad \leftrightarrow \quad \ell \mapsto \ell(v) \quad \text{(a linear functional on } V^*\text{)}.$$

From now on we will identify V and $(V^*)^*$ in this way.

With this out of the way, $(K^*)^*$ is a convex cone in V and we see immediately that it contains K:

$$K \subseteq (K^*)^* = \{v \in V : \ell(v) \geq 0 \text{ for all } \ell \in K^*\}.$$

When $\dim(V) = n < \infty$, then $V = \mathbb{R}^n$ comes equipped with the Euclidean topology. Then $((K^*)^*$ is closed and we see that up to closure, all the nonnegative linear functions on K^* come from points evaluations at points in K.

Proposition. *If* $\dim(V) < \infty$ *and* K *is closed, then* $(K^*)^* = K$.

Question. Consider the real vector space $V = \mathbb{R}[x]_{\leq 2}$ of univariate polynomials of degree ≤ 2 and the convex cone of nonnegative polynomails

$$\mathrm{Pos} = \{f = ax^2 + bx + c : f(p) \geq 0 \text{ for all } p \in \mathbb{R}\}.$$

What is $(\mathrm{Pos})^*$?