Math 591 – Real Algebraic Geometry and Convex Optimization

Lecture 17: Spectrahedra and hyperbolic polynomials
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Recall that a **spectrahedron** is a set $C \subseteq \mathbb{R}^n$ of the form

$$C = \{x \in \mathbb{R}^n : A_0 + \sum_{i=1}^{n} x_i A_i \succeq 0\}$$

where $A_0, A_1, \ldots, A_n \in \mathbb{R}^{d\times d}_{\text{sym}}$. Since the cone of PSD matrices is convex and semialgebraic, C is also. One of the motivating questions for today is:

Question. What convex semialgebraic sets are spectrahedra?

Let's consider some examples:

Example. Consider the disk $\{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}$. This is convex and semialgebraic. In this case, we see that this is a spectrahedron by writing down a description

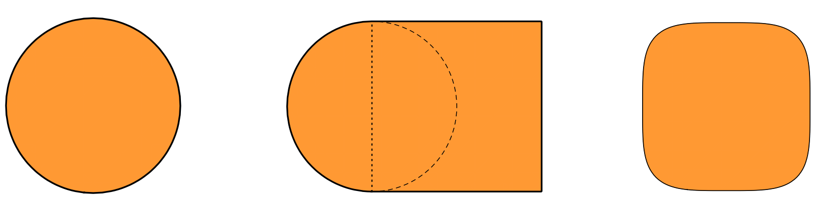
$$\left\{(x, y) \in \mathbb{R}^2 : \begin{pmatrix} 1-x & y \\ y & 1+x \end{pmatrix}\right\}.$$

Example. One of our examples from the beginning of class was the union of the disk and the square $\{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\} \cup \{(x, y) \in \mathbb{R}^2 : 0 \leq x \leq 2, -1 \leq y \leq 1\}$. This is *not* a spectrahedron because it is not a *basic closed* semialgebraic set.

Indeed, the matrix $A_0 + \sum_{i=1}^{n} x_i A_i$ is positive semidefinite if and only if all of its principal minors are nonnegative. This gives a description of C as a the set of points x satisfying finitely-many polynomial inequalities (i.e. as a basic closed semialgebraic set).

Example. Consider the fourth norm ball $\{(x, y) \in \mathbb{R}^2 : x^4 + y^4 \leq 1\}$, which is a convex, basic closed semialgebraic set. It's not clear whether or not this has a description as spectrahedron.

To summarize, are the following set spectrahedra?



YES NO ??

To address the last example, we will consider a special property of the algebraic boundary of spectrahedra. Note that the polynomial $f = \det(A_0 + \sum_{i=1}^{n} x_i A_i)$ necessarily vanishes on the boundary of the set C.

For simplicity, let us suppose that the origin belongs to the interior of C and the matrix A_0 is positive definite. Then we can write $A_0 = UU^T$ for some full rank matrix $U \in \mathbb{R}^{d\times d}$. Then for any vector $v \in \mathbb{R}^n$, consider the univariate polynomial $f(tv) = f(tv_1, \ldots, tv_n)$ obtained by restricting f to the line through the origin in direction v. Then

$$\begin{aligned} f(tv) &= \det\left(A_0 + \sum_{i=1}^{n} tv_iA_i\right) \\ &= \det\left(UU^{-1}(A_0 + \sum_{i=1}^{n} tv_iA_i)U^{-T}U^T\right) \\ &= \det(U)\det\left(U^{-1}A_0U^{-T} + t\cdot U^{-1}\sum_{i=1}^{n} v_iA_iU^{-T}\right)\det(U^T) \\ &= \det(U)^2\det\left(I + tB\right) \end{aligned}$$

where $B = U^{-1}(\sum_{i=1}^n v_iA_i)U^{-T}$. Since B is a real symmetric matrix, its eigenvalues are real. In particular, the roots of $f(tv)$ are $-1/\lambda_1, \ldots, -1/\lambda_d$ where $\lambda_1, \ldots, \lambda_d$ are the eigenvalues of B. In particular these are all real.

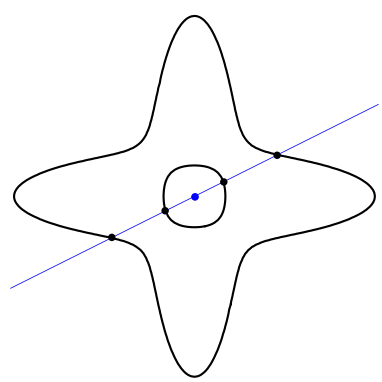
Corollary. *If $C = \{x \in \mathbb{R}^n : A_0 + \sum_{i=1}^n x_iA_i\}$ where $A_i \in \mathbb{R}^{d\times d}_{\text{sym}}$ and A_0 is positive definite and g is the minimal polynomial vanishing on the boundary of C, then for every $v \in \mathbb{R}^n$, the polynomial $g(tv) \in \mathbb{R}[t]$ is real rooted.*

Proof. The polynomial g must be a factor of $f = \det(A_0 + \sum_{i=1}^n x_iA_i)$. Therefore for every $v \in \mathbb{R}^n$, $g(tv)$ is a factor of $f(tv)$ and must also be real rooted. □

Example. For the example $f = 1 - x^2 - y^2$ and $v = (v_1, v_2) \in \mathbb{R}^2$, we see that $f(tv) = f(tv_1, tv_2) = 1 - t^2(v_1^2 + v_2^2)$. When $v \neq (0,0)$, this has roots $t = \pm 1/\sqrt{v_1^2 + v_2^2} \in \mathbb{R}$.

Non-example. On the other hand, consider the polynomial $f = 1 - x^4 - y^4$, whose variety bounds the fourth-norm ball consider above. For $v = (v_1, v_2) \in \mathbb{R}^2$, we see that $f(tv) = f(tv_1, tv_2) = 1 - t^4(v_1^4 + v_2^4)$. When $v \neq (0,0)$, this has roots $t = \pm\omega/\sqrt[4]{v_1^4 + v_2^4} \in \mathbb{R}$ where $\omega \in \{\pm 1, \pm i\}$. This polynomial is not real rooted! Therefore the fourth-norm ball is *not* a spectrahedron.

Example. The following is a picture of the variety $V_{\mathbb{R}}(f)$ of a polynomial $f \in \mathbb{R}[x, y]$ of degree 4 with the the property that $f(0,0) \neq 0$ and $f(tv)$ is real-rooted for all $v \in \mathbb{R}^2$.



Any line through the origin intersects $V_{\mathbb{R}}(f)$ in four real points.

Polynomials with this "real-rootedness" property are very special. To discuss them further we move the setting of homogeneous polynomials. Recall that a polynomial $f = \sum_\alpha c_\alpha x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ is homogeneous of degree d if $\sum_{i=1}^n \alpha_i = d$ for all α with $c_\alpha \neq 0$. In particular, f then has the property that $f(\lambda x_1, \ldots, \lambda x_n) = \lambda^d f(x_1, \ldots, x_n)$. We will use $\mathbb{R}[x_1, \ldots, x_n]_d$ to denote the space of homogeneous polynomials of degree d.

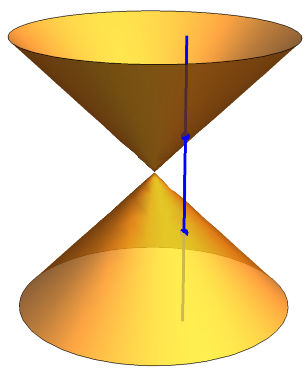
Definition. A polynomial $f \in \mathbb{R}[x_1, \ldots, x_n]_d$ is **hyperbolic with respect to** $e \in \mathbb{R}^n$ if $f(e) \neq 0$ and for every $v \in \mathbb{R}^n$, the polynomial $f(te - v) \in \mathbb{R}[t]$ is real rooted.

Since f is homogeneous, an equivalent condition is that $f(e - sv) \in \mathbb{R}[s]$ is real rooted for all $v \in \mathbb{R}^n$. To see this note that

$$f(e - sv) = f\left(s \cdot \left(\frac{1}{s} \cdot e - v\right)\right) = s^d f\left(\frac{1}{s} \cdot e - v\right)$$

So the roots of $f(e - sv)$ are the reciprocals of the roots of $f(te - v)$. In particular, one is real rooted if and only if the other is.

Example. The polynomial $x^2 + y^2 - z^2$ is hyperbolic with respect to the vector $(0, 0, 1)$.



We see that any vertical line intersects $V_{\mathbb{R}}(x^2 + y^2 - z^2)$ in two real points (which is a single double point when the line passes through the origin).

Example. The polynomial $f = \det(A(x))$ if hyperbolic with respect to $e \in \mathbb{R}^n$ if $A(x) = \sum_{i=1}^n x_i A_i$ where $A_i \in \mathbb{R}^{d \times d}_{\text{sym}}$ and $A(e)$ is positive definite. Just as before, if $A(e) = UU^T$, then

$$f(te - v) = \det(tA(e) - A(v)) = \det(U)^2 \cdot \det(tI - U^{-1}A(v)U^{-T}),$$

whose roots are the eigenvalues of the real symmetric matrix $U^{-1}A(v)U^{-T}$.

Remark. The example above shows that if K is a spectrahedral cone, then the minimal polynomial vanishing on the boundary of K is hyperbolic with respect to any point in the interior of K. This shows that we have some flexibility in the choice of point e.

Before making this precise, we need a little more terminology.

Definition. If $f \in \mathbb{R}[x_1, \ldots, x_n]_d$ is hyperbolic with respect to $e \in \mathbb{R}^n$, then for any $x \in \mathbb{R}^n$, we call the roots

$$\lambda_1(x) \geq \ldots \geq \lambda_d(x)$$

of $f(te - x)$ the **eigenvalues** of x (with respect to f and e).

Definition. The **hyperbolicity cone** of f with respect to e is

$$C_e(f) \;=\; \{x \in \mathbb{R}^n : \text{ all roots of } f(te - x) \text{ are nonnegative}\} \;=\; \{x \in \mathbb{R}^n : \lambda_d(x) \geq 0\}.$$

We will see that this is actually a convex cone (justify its name).

Example. Consider $f = \prod_{i=1}^n x_i$ and $e = (1, \ldots, 1)$. Then f is hyperbolic with respect to e and for any $x \in \mathbb{R}^n$,

$$\begin{aligned} \text{eigenvalues of } x &= \text{roots of } f(te - x) = \prod_{i=1}^n (t - x_i) \\ &= \{x_1, \ldots, x_n\} \end{aligned}$$

The hyperbolicity cone is the set of $x \in \mathbb{R}^n$ for which all of the eigenvalues are nonnegative which is $(\mathbb{R}_{\geq 0})^n$.

Example. Consider the polynomial

$$f = \det \begin{pmatrix} x_{11} & x_{12} & \ldots & x_{1d} \\ x_{12} & x_{22} & \ldots & x_{2d} \\ \vdots & & \ddots & \\ x_{1d} & \ldots & & x_{dd} \end{pmatrix}$$

in $\mathbb{R}[x_{ij} : 1 \leq i \leq j \leq d]_d$ and the point e corresponding to the identity matrix (with $x_{ii} = 1$ and $x_{ij} = 0$ for $i \neq j$). Then f is hyperbolic with respect to e and for any point $X \in \mathbb{R}^{d \times d}_{\text{sym}}$,

$$\begin{aligned} \text{eigenvalues of } X \text{ (w.r.t } f) &= \text{roots of } f(te - x) = \det(tI - X) \\ &= \text{eigenvalues of } X \text{ (as a matrix)} \end{aligned}$$

The hyperbolicity cone is the set of $X \in \mathbb{R}^{d \times d}_{\text{sym}}$ for which all of the eigenvalues are nonnegative which is the convex cone of positive semidefinite matrix PSD_d.

Both of these examples suggest the following theorem, which was proved by Gårding in 1959. Gårding started studying hyperbolic polynomials in the context of partial differential equations, which inspired their name.

Theorem. *If f is hyperbolic with respect to e, then $C_e(f)$ is a convex cone and f is hyperbolic with respect to any point in its interior.*

We discuss the proof of this next class. An important lemma will be the following characterization of the hyperbolicity cone.

Lemma. *The interior of $C_e(f)$ is the connected component of $\mathbb{R}^n \backslash V_{\mathbb{R}}(f)$ containing e.*

For example, if $f = x^2 + y^2 - z^2$ and $e = (0, 0, 1)$, then removing $V_{\mathbb{R}}(f)$ from $\mathbb{R}^3$, leaves three connected components (two open, convex cones and the rest). The component containing e is the open convex cone $\{(x, y, z) \in \mathbb{R}^3 : z > \sqrt{x^2 + y^2}\}$.

One of the main motivations for studying hyperbolic polynomial and their hyperbolicity cones is that they provide a natural generalization for linear and semidefinite programming.

Definition. A **hyperbolic program** is a convex optimization problem of the form

$$\min \langle c, x \rangle \quad \text{such that } x \in C_e(f) \text{ and } \langle a_i, x \rangle = b_i \text{ for } i = 1, \ldots, m.$$

When $f = \prod_{i=1}^n x_i$ and $C_e(f) = (\mathbb{R}_{\geq 0})^n$, then this is a linear program. When $f = \det(X)$ and $C_e(f) = \text{PSD}_d$, then this is a semidefinite program. We will see that some of the techniques for solving linear and semidefinite programs extend to this more general context and can be understood by understanding the roots of univariate polynomials.