Math 591 – Real Algebraic Geometry and Convex Optimization
Lecture 16: Algebraic boundaries and optimization
Cynthia Vinzant, Spring 2019

For this lecture, we will take K to be a full-dimensional, pointed (i.e. containing no lines), semialgebraic, convex cone inside of a finite dimensional real vector space (like $\mathbb{R}^n$) so that the dual cone K^* is also a full-dimensional, pointed, semialgebraic, convex cone

Recall that the **algebraic boundary** of K, denoted $\partial_{\text{alg}}K$ is the Zariski-closure of the Euclidean boundary of K, ∂K. Since K is a full-dimensional cone and not the whole space, the boundary ∂K is a semi-algebraic set of codimension one. Then the algebraic boundary $\partial_{\text{alg}}K$ also has codimension one and is defined by a single polynomial equation $f(x_1, \dots, x_n) = 0$.

Example. Consider the convex cone

$$K = \{(x, y, z) \in \mathbb{R}^3 : f \geq 0,\ z \geq x,\ z \geq 0\}$$

where $f = (x + z)(x - z)^2 - y^2 z$. Then the algebraic boundary of K is the surface $V_{\mathbb{R}}(f)$.

The condition that the plane $\{(x, y, z) : ax + by + cz\}$ is tangent to ∂K at some point imposes conditions on the coefficients (a, b, c). Namely (a, b, c) must belong to the boundary of the dual cone and thus the *algebraic boundary* of the dual cone,

$$\partial_{\text{alg}}(K^*) \ = \ V_{\mathbb{R}}(4a^4 + 13a^2b^2 + 32b^4 - 4a^3c + 18ab^2c - 27b^2c^2) \cup V_{\mathbb{R}}(a + c).$$

This is the union of the dual varieties $V(f)^*$ and $V(x - z, y)^*$.

Application to optimization. One can think of the algebraic boundary of the dual cone K^* as simultaneously solving the Lagrange multiplier equations for a family of optimization problems over K.

Take $c, a \in (\mathbb{R}^n)^* \cong \mathbb{R}^n$ such that the intersection $K \cap \{x : \langle a, x\rangle = 1\}$ is compact, and consider the optimization problem

$$p^* \ = \ \min\langle c, x\rangle \ \text{ such that } \ \langle a, x\rangle = 1,\ x \in K.$$

Here p^* is the optimal value.

Proposition. *$c - p^*a$ belongs to the boundary of the dual cone K^*.*

Proof. We will show this by showing that $c - \lambda a$ belongs to K^* if and only if $\lambda \leq p^*$.

Let $x \in K\backslash\{0\}$. Since $K \cap \{y : \langle a, y\rangle = 1\}$ in compact, $\langle a, x\rangle$ is strictly positive, and we can consider $\tilde{x} = (\frac{1}{\langle a,x\rangle}) \cdot x$ which belongs to K and satisfies $\langle a, \tilde{x}\rangle = 1$. Then

$$p^* \leq \langle c, \tilde{x}\rangle = \frac{1}{\langle a, x\rangle} \cdot \langle c, x\rangle.$$

If $\lambda \leq p^*$, then multiplying this inequality by $\langle a, x\rangle$ shows that $\lambda\langle a, x\rangle \leq \langle c, x\rangle$. Therefore $\langle c - \lambda a, x\rangle \geq 0$ for all $x \in K$, meaning that $c - \lambda a \in K^*$.

Similarly, if $p^* < \lambda$, then consider the point $x \in K$ achieving the minimum p^*. That is, for which $\langle a, x\rangle = 1$ and $\langle c, x\rangle = p^* < \lambda$. Then $\langle c - \lambda a, x\rangle = \langle c, x\rangle - \lambda\langle a, x\rangle = p^* - \lambda < 0$. This shows that $c - \lambda a$ does not belong to K^*.

Putting this together we see that $c - p^*a$ belongs to K^* but $c - (p^* + \epsilon)a$ does not belong to K^* for $\epsilon > 0$. Therefore $c - p^*a$ is on the boundary of K^* □

Corollary. *If g is a polynomial that vanishes on ∂K^*, then the optimal value p^* is a root of the univariate polynomial $g(c - ta) \in \mathbb{R}[t]$.*

Example. Consider the cone $K = \{(x,y,z) \in \mathbb{R}^3 : (x+z)(x-z)^2 - y^2 z \geq 0,\ z \geq x,\ z \geq 0\}$ from above and the optimization problem

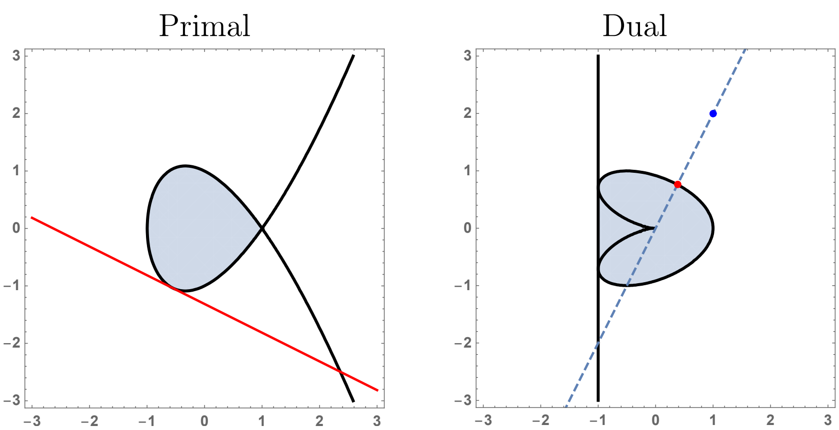
$$p^* \ = \ \min ax + by \ \text{ such that } (x,y,z) \in K,\ z = 1$$

The polynomial $g = (4a^4 + 13a^2b^2 + 32b^4 - 4a^3c + 18ab^2c - 27b^2c^2) \cdot (a+c)$ vanishes on the boundary of the dual cone K^*, meaning that if p^* is the optimal value given by the objective function $ax+by$, then $g(a,b,-p^*) = 0$. This lets us solve for p^* in terms of a and b:

$$p^* \in \left\{ a,\ \frac{-2a^3 + 9ab^2 \pm 2\sqrt{(a^2+6b^2)^3}}{27b^2} \right\}.$$

To visualize this in the $c = 1$ plane, we note that since g is homogeneous, $g(a,b,-p^*) = 0$ if and only if $g(\frac{-a}{p^*}, \frac{-b}{p^*}, 1) = 0$ meaning that $-1/p^*$ is a root of the univariate polynomial $g(ta, tb, 1) \in \mathbb{R}[t]$.

For example, for $(a,b) = (1,2)$, the minimal value of $ax+by$ is $p^* = -71/27$. The line $1x + 2y = -71/27$ is tangent to the boundary of K in the plane $z = 1$. Moreover, for $t = -71/27$, the point $(t, 2t, 1)$ belongs to the algebraic boundary of K^*, shown below in the plane $c = 1$.



$K \cap \{z = 1\}$ and $\{1x + 2y = p^*\}$ $\qquad$ $K^* \cap \{c = 1\}$ and $\{(t, 2t) : t \in \mathbb{R}\}$

By intersecting with an affine plane, we can visualize a three-dimensional cone via a two-dimensional convex set. Similarly, we can visualize a four-dimensional cone via a three-dimensional set.

Example. Consider the cone

$$\partial_{\text{alg}} K = \left\{ (t,x,y,z) \in \mathbb{R}^4 : \begin{pmatrix} t & x & y \\ x & t & z \\ y & z & t \end{pmatrix} \succeq 0 \right\}.$$

The algebraic boundary of K is given by the determinant of the defining 3×3 matrix

$$\partial_{\text{alg}} K \ = \ V(\det \begin{pmatrix} t & x & y \\ x & t & z \\ y & z & t \end{pmatrix}) \ = \ V(t^3 - t(x^2+y^2+z^2) + 2xyz).$$

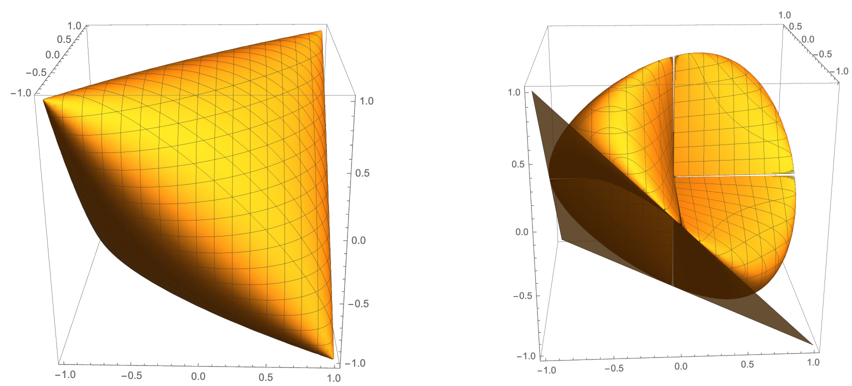
This hypersurface is singular alone the lines spanned by each of the four points $(x,y,z,t) = (1,1,1,1)$, $(1,-1,-1,1)$, $(-1,1,-1,1)$, and $(-1,-1,1,1)$. The algebraic boundary of the dual cone is the dual hypersurface of $\partial_{\text{alg}}K$ along with the four hyperplanes dual to these lines:

$$\partial_{\text{alg}}(K^*) \;=\; (\partial_{\text{alg}}K)^* \cup \text{ four hyperplanes } \;=\; V(g)$$

where

$$g = (a^2b^2 + a^2c^2 + b^2c^2 - 2abcd)(a+b+c+d)(a-b-c+d)(-a+b-c+d)(-a-b+c+d).$$

The intersection of K with the plane $t = 1$ along with the variety of the quartic and one linear factor of g in the plane $d = 1$ is shown here:



$K \cap \{t = 1\}$ $\qquad$ $K^* \cap \{d = 1\}$

Consider the minimization problem

$$p^* \;=\; \min ax + by + cz \text{ such that } (x, y, z, 1) \in K.$$

Despite the quartic factor of g, the roots of the polynomial $g(a, b, c, -t)$ are *rational* in a, b, c. Namely we find that

$$p^* \in \left\{ \frac{-a^2b^2 - a^2c^2 - b^2c^2}{2abc},\; a+b+c,\; a-b-c,\; -a+b-c,\; -a-b+c \right\}.$$

In general, one cannot hope for the optimal value to be a rational function of the input data.

An important case of this theory is a spectrahedral cone

$$K \;=\left\{ y \in \mathbb{R}^d \;: \sum i = 1^d y_i A_i \in \text{PSD}_n \right\}$$

defined by $A_1, \ldots, A_d \in \mathbb{R}^{n\times n}_{\text{sym}}$. Its dual cone is

$$K^* \;=\; \{(\langle A_i, X\rangle)_{i=1,\ldots,d} : X \in \text{PSD}_n\}\,.$$

In the example above with $d = 4$,$n = 3$, we had that

$$K^* = (2X_{12}, 2X_{13}, 2X_{23}, X_{11} + X_{22} + X_{33}) : X \in \text{PSD}_3\}.$$

In this case, if K has extreme rays of rank r, then we expect that the dual variety of the rank-r locus of K

$$\left(\overline{(K \cap \{\text{rank} \le r\})}^{Zar}\right)^*$$

will appear as a component of the algebraic boundary of K^*. For generic choices of $A_i \in \mathbb{R}^{d\times d}_{\text{sym}}$, the degrees of the hypersurfaces (which depend on d, n, and r are given by the *algebraic degree of semidefinite programming*.