Math 591 – Real Algebraic Geometry and Convex Optimization

Lecture 15: Algebraic boundaries and duality

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For this lecture, we will take K to be a full dimensional, pointed (i.e. containing no lines), semialgebraic, convex cone inside of a finite dimensional real vector space (like $\mathbb{R}^n$).

Definition. The **algebraic boundary** of K, denoted $\partial_{\text{alg}}K$ is the Zariski-closure of the Euclidean boundary of K, ∂K.

Since K is a full-dimensional cone and not the whole space, the boundary ∂K is a semi-algebraic set of codimension one. Then the algebraic boundary $\partial_{\text{alg}}K$ also has codimension one and is defined by a single polynomial equation $f(x_1, \ldots, x_n) = 0$.

We say that a polynomial $f = \sum_\alpha c_\alpha x^\alpha \in \mathbb{R}[x_1, \ldots, x_n]$ is **homogeneous** of degree d if $\sum_{i=1}^n \alpha_i$ for all α with $c_\alpha \neq 0$. Note that this implies that $f(\lambda x_1, \ldots, \lambda x_n) = \lambda^d f(x)$. Moreover, taking the partial derivatives with respect to λ and then restricting to $\lambda = 1$ gives Euler's identity:

$$\sum_{i=1}^n x_i \frac{\partial f}{\partial x_i} \quad = \quad d \cdot f.$$

Since ∂K is invariant under positive scaling ($x \mapsto \lambda x$ for $\lambda \in \mathbb{R}_+$), the algebraic boundary $\partial_{\text{alg}}K$ will be invariant under real scaling ($x \mapsto \lambda x$ for $\lambda \in \mathbb{R}$), implying that the defining polynomial f will be homogeneous of some degree d.

Example. For $K = \{(x, y, z) \in \mathbb{R}^3 : z^2 \geq x^2 + y^2, z \geq 0\}$, $\partial_{\text{alg}}(K)$ equals $V(x^2 + y^2 - z^2)$.

Example. For $K = (\mathbb{R}_{\geq 0})^n$, the boundary of K is $\{x \in (\mathbb{R}_{\geq 0})^n : x_i = 0 \text{ for some } i\}$. Then $\partial_{\text{alg}}(K)$ equals $V(\prod_{i=1}^n x_i) = \{x \in \mathbb{R}^n : x_i = 0 \text{ for some } i\}$.

Example. For $K = \text{PSD}_n$, the boundary of K is $\{X \in \text{PSD}_n : \text{rank}(X) \leq n - 1\}$. Then $\partial_{\text{alg}}(K)$ equals $V(\det(X)) = \{X \in \mathbb{R}^{n\times n}_{\text{sym}} : \text{rank}(X) \leq n - 1\}$.

A motivating question for the day is

Question. How are the algebraic boundaries of K and K^* related?

Note that by definition

$$K^* = \{a \in \mathbb{R}^n : \langle a, x\rangle \geq 0 \text{ for all } x \in K\}$$

Since K is pointed, the intersection of all these hyperplanes $\{x : \langle a, x\rangle\}$ where $a \in K^*$ is the origin $\mathbf{0} = (0, \ldots, 0)$. Therefore the boundary of K^* is

$$\partial K^* = \left\{a \in K^* : \langle a, x_0\rangle = 0 \text{ for some } x_0 \in \partial K\backslash\{\mathbf{0}\}\right\}.$$

Since $\langle a, x\rangle \geq 0$ for all $x \in K$, the plane $\{x \in \mathbb{R}^n : \langle a, x\rangle = 0\}$ must be *tangent* to the cone K at the point $x = x_0$. In particular, if f vanishes on ∂K, then $\nabla f(x_0)$ must be a scalar multiple of a. (Note if $\nabla f(x_0) = \mathbf{0}$, this scalar might be zero.)

Proposition. *If $\partial_{\text{alg}}K = V(f)$ and $\nabla f(p) \neq 0$ for all $p \in \partial K\backslash\{\mathbf{0}\}$, then*

$$\partial K^* = \left\{\nabla f(p) \;:\; p \in \partial K\right\}.$$

Proof. Note that $\mathbf{0} \in K^*$. Since f is homogeneous of degree ≥ 2, $\mathbf{0} = \nabla f(\mathbf{0})$. (Note that if f has degree one, then K would not be pointed.)

If $a \in \partial K^* \backslash \{\mathbf{0}\}$, the argument above shows that $a = \lambda \cdot \nabla f(p)$ for some $\lambda \in \mathbb{R}$ and $p \in \partial K$. Since $a \neq \mathbf{0}$, λ and $\nabla f(p)$ are also non-zero. Since f is homogeneous of some degree $d \geq 2$, the entries of ∇f are homogeneous of degree $d-1 \geq 1$. Then $a = \lambda \cdot \nabla f(p) = \nabla f(q)$, where $q = \lambda^{1/(d-1)} p \in \partial K$.

Similarly, for any point $p \in \partial K$, by convexity $x \mapsto \langle \nabla f(p), x \rangle$ is nonnegative on K, meaning that $\nabla f(p) \in K^*$ and $p \cdot \nabla f(p) = d \cdot f(p) = 0$, implying that $\nabla f(p) \in \partial K^*$. $\square$

Example. For $K = \{(x,y,z) \in \mathbb{R}^3 : z^2 \geq x^2 + y^2, z \geq 0\}$, we had that $\partial_{\text{alg}} K = V(f)$ where $f = x^2 + y^2 - z^2$ and $\nabla f = (2x, 2y, -2z)$. Then

$$\begin{aligned} \partial K^* &= \big\{(a,b,c) \in \mathbb{R}^3 \ : \ ax + by + cz \geq 0 \text{ for all } (x,y,z) \in K \text{ and} \\ &\qquad\qquad ax_0 + by_0 + cz_0 = 0 \text{ for some } (x_0,y_0,z_0) \in \partial K\big\} \\ &= \big\{(2x_0, 2y_0, -2z_0) : (x_0,y_0,z_0) \in \partial K\big\} \end{aligned}$$

More, any point $(x_0,y_0,z_0) \in \partial K$ satisfies $f(x_0,y_0,z_0)$, so $(a,b,c) = \nabla f(x_0,y_0,z_0)$ satisfies

$$a^2 + b^2 - c^2 = (2x_0)^2 + (2y_0)^2 - (2z_0)^2 = 4 \cdot f(x_0,y_0,z_0) = 0.$$

Indeed in this case $\partial_{\text{alg}} K^* = V(a^2 + b^2 - c^2)$. (In fact, K is self-dual, and this is the same as the algebraic boundary of K in different coordinates!)

To understand the boundary more generally, we need another definition.

Definition. Let $W \subset \mathbb{R}^n$ be a real variety that is invariant under scaling ($x \in W \Rightarrow \lambda x \in W$). The **tangent space** of W at a point $p \in W$ is defined as

$$T_p W = \{x \in R^n : \langle x, \nabla f(p) \rangle = 0 \text{ for all } f \in \mathcal{I}(W)\}.$$

We say that p is a **regular** or **nonsingular** point of W if $p \in W$ and the dimension of the tangent space $\dim(T_p W)$ equals the dimension of W.

This is easiest when W is a hypersurface, meaning that the ideal of polynomials vanishing on W is the set of polynomial multiples of some $f \in \mathbb{R}[x_1, \ldots, x_n]$. If $\mathcal{I}(W) = \langle f \rangle = \{h \cdot f : h \in \mathbb{R}[x_1, \ldots, x_n]\}$, then one can check that

$$T_p W = \{x \in \mathbb{R}^n : \langle x, \nabla f(p) \rangle = 0\}.$$

In particular, this is a hyperplane with $\nabla f(p) \neq \mathbf{0}$ and all of $\mathbb{R}^n$ when $\nabla f(p) = \mathbf{0}$.

Since $\dim(W) = n - 1$, $p \in W$ is a regular point of W if $\nabla f(p) \neq \mathbf{0}$ and a singular point if $\nabla f(p) = \mathbf{0}$.

Example. For $W = V(f)$ where $f = x^2 + y^2 - z^2$, we have $\nabla f = (2x, 2y, -2z)$, so $\nabla f(p) = \mathbf{0}$ only for $p = \mathbf{0}$. Therefore the only singular point of W is $\mathbf{0} = (0,0,0)$.

Definition. The **dual variety** of W, denoted W^*, equals the Zariski-closure of the set of hyperplanes containing the tangent space of W at some regular point. That is,

$$W^* \ = \ \overline{\{a \in \mathbb{R}^n : T_p W \subseteq \{x : \langle a, x \rangle\} \text{ for some regular point } p \in W\}}.$$

Just as in convex duality, under some mild assumptions, the dual variety of the dual variety is the original.

Theorem (Biduality)**.** *If W is invariant under scaling and irreducible (i.e. not a union of proper subvarieties), then $(W^*)^* = W$.*

If W is a hypersurface, $\mathcal{I}(W) = \langle f \rangle$, then at every regular point $p \in W$, the tangent space T_pW is a hyperplane defined by $\langle \nabla f(p), x \rangle = 0$. Then T_pW is contained in $\{x : \langle a, x \rangle = 0$ if and only if $a = \lambda \nabla f(p)$ for some $\lambda \in \mathbb{R}$.

Corollary. *If $\mathcal{I}(W) = \langle f \rangle$, then*

$$W^* = \overline{\{\nabla f(p) : p \textit{ is a regular point of } W\}}^{Zar}.$$

Example. Consider

$$W = \{(\lambda, \lambda t, \lambda t^2) : \lambda, t \in \mathbb{R}\} = \{(x, y, z) \in \mathbb{R}^3 : xz - y^2 = 0\}$$

Then $\mathcal{I}(W) = \langle f \rangle$ where $f = xz - y^2$ and $\nabla f = (z, -2y, x)$. At a point $p = (1, t, t^2)$ of W, we find that

$$T_pW = \{(x, y, z) \in \mathbb{R}^3 : t^2 x - 2ty + z = 0\} = \mathrm{span}_{\mathbb{R}}\{(1, t, t^2), (0, 1, 2t)\}.$$

We can also check that the tangent space of W at λp equals the tangent space of W at p. Then

$$\begin{aligned} W^* &= \overline{\{(a, b, c) \in \mathbb{R}^3 : a + bt + ct^2 = 0 \text{ and } b + 2ct = 0 \text{ for some } t \in \mathbb{R}\}}^{Zar} \\ &= \overline{\{(a, b, c) \in \mathbb{R}^3 : a + bt + ct^2 \text{ has a double root}\}}^{Zar} \\ &= V(b^2 - 4ac) \end{aligned}$$

This generalizes to the following:

Example. Consider

$$\begin{aligned} W &= \{(\lambda, \lambda t, \lambda t^2, \ldots, t^d) : \lambda, t \in \mathbb{R}\} \\ &= \{((x_0, \ldots, x_n) \in \mathbb{R}^{n+1} : x_{i+1} x_{i-1}^2 = x_i^2 \text{ for } i = 1, \ldots, n-1\} \end{aligned}$$

One can check that at a point $p = (1, t, t^2, \ldots, t^d)$ of W, we find that

$$T_pW = \mathrm{span}_{\mathbb{R}}\{(1, t, t^2, \ldots, t^d), (0, 1, 2t, \ldots, dt^{d-1})\}.$$

Then just as before,

$$\begin{aligned} W^* &= \overline{\{(a_0, a_1, \ldots, a_d) \in \mathbb{R}^{n+1} : \sum_{i=0}^{n} a_i t^i = 0 \text{ and } \sum_{i=0}^{n} i \cdot a_i t^{i-1} = 0 \text{ for some } t \in \mathbb{R}\}}^{Zar} \\ &= \overline{\{(a_0, a_1, \ldots, a_d) \in \mathbb{R}^{n+1} : \sum_{i=0}^{n} a_i t^i \text{ has a double root}\}}^{Zar}. \end{aligned}$$

Proposition. *If $\partial_{\mathrm{alg}} K$ has no singular points besides $\mathbf{0} = (0, \ldots, 0)$, then the algebraic boundary of the dual cone, $\partial_{\mathrm{alg}} K^*$, equals the dual variety of the algebraic boundary, $(\partial_{\mathrm{alg}} K)^*$.*

Proof. If $\partial_{\mathrm{alg}} K = V(f)$ has no singular points besides $\mathbf{0} = (0, \ldots, 0)$, then

$$\partial K^* = \{\nabla f(p) \ : \ p \in \partial K\}.$$

and

$$\partial_{\mathrm{alg}} K^* = \overline{\{\nabla f(p) \ : \ p \in \partial K\}}^{Zar} = \overline{\{\nabla f(p) \ : \ p \in V(f)\}}^{Zar} = V(f)^* = (\partial_{\mathrm{alg}} K)^*.$$

□

However if $\partial_{\mathrm{alg}} K$ has a nonzero singular point on its boundary, then things can be more difficult.

Example. Let K be the conic hull of $\{(1,t,t^2) : t \in [-1,1]\}$. A semialgebraic description of K is $\{(x,y,z) \in \mathbb{R}^3 : xz \geq y^2, x \geq z\}$. Then

$$\partial_{\mathrm{alg}} K = V((xz-y^2)(x-z)).$$

Note that this has two singular points in $V(xz-y^2) \cap V(x-z)$, namely $(1,1,1)$ and $(1,-1,1)$ on the boundary of K.

One can check that the dual variety of $V((xz-y^2)(x-z))$ is a union

$$(\partial_{\mathrm{alg}} K)^* = V(b^2-4ac) \cup V(b, a+c).$$

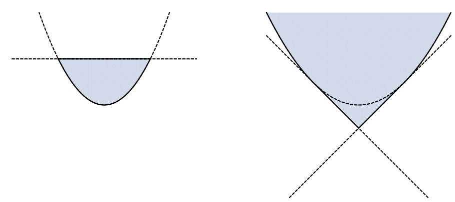
But the dual cone

$$K^* = \{(a,b,c) \in \mathbb{R}^3 : a + bt + ct^2 \geq 0 \text{ for } t \in [-1,1]\}.$$

Its algebraic boundary has three components

$$\partial_{\mathrm{alg}}(K^*) = V((b^2-4ac)(a+b+c)(a-b+c)) = V(b^2-4ac) \cup V(a+b+c) \cup V(a-b+c).$$

Here are the cones K and K^* in the planes $x=1$ and $a=1$, respectively:



Theorem (Sinn, 2014)**.** *Let* $\mathrm{Exr}(K)$ *denote the extreme rays of* K*, then the dual variety of the Zariski-closure of the set of extreme rays belongs to the algebraic boundary of the dual cone* K^**:*

$$\left(\overline{\mathrm{Exr}(K)}^{Zar}\right)^* \subset \partial_{\mathrm{alg}}(K^*).$$

In fact, $\left(\overline{\mathrm{Exr}(K)}^{Zar}\right)^*$ *is an irreducible component of* $\partial_{\mathrm{alg}}(K^*)$*. Moreover*

$$(\partial_{\mathrm{alg}} K)^* = \overline{\mathrm{Exr}(K^*)}^{\mathrm{Zar}}.$$

Example. In the example above, where K be the conic hull of $\{(1,t,t^2) : t \in [-1,1]\}$

$$\begin{aligned} \mathrm{Exr}(K) &= \{(\lambda, \lambda t, \lambda t^2) : t \in [-1,1], \lambda \in \mathbb{R}_{\geq 0}\}, \\ \overline{\mathrm{Exr}(K)}^{Zar} &= V(xz-y^2), \\ \left(\overline{\mathrm{Exr}(K)}^{Zar}\right)^* &= V(b^2-4ac), \end{aligned}$$

which is a component of $\partial_{\mathrm{alg}}(K^*)$.

For the second equation, one can check that the dual variety of $\partial_{\mathrm{alg}} K$ is

$$(\partial_{\mathrm{alg}} K)^* = V(b^2-4ac) \cup V(b, a+c).$$

The extreme rays of K^* are

$$\mathrm{Exr}(K^*) = \{(\lambda, 0, -\lambda) : \lambda \in \mathbb{R}_{\geq 0}\} \cup \{(a,b,c) \in \mathbb{R}^3 : b^2 = 4ac, c \geq 0\}.$$

Example. Consider the polynomial $\det(X)$ for a square non-symmetric matrix of variables.

$$\frac{\partial \det(X)}{\partial X_{ij}} \;=\; (-1)^{i+j} \cdot \det(X_{[n]\backslash i,[n]\backslash j}).$$

We can differentiate X with respect to any variable, and we can organize the partial derivatives into a matrix. Then

$$\nabla \det(X) \;=\; X^{\mathrm{adj}}$$

Then M is a regular point of $\det(X)$ if and only if $\nabla \det(M)$ is non-zero if and only if $\mathrm{rank}(M) = n-1$. Moreover

$$\nabla \det(M) = vw^T$$

where $v, w \in \mathbb{R}^n$ are the right and left kernel of M, respectively. Indeed, if $\det(M) = 0$, then

$$M \cdot M^{\mathrm{adj}} = M^{\mathrm{adj}} \cdot M = \det(M) \cdot I_n = 0 \cdot I_n.$$

In particular, the columnspan of M^{adj} belongs to the right kernel of M and the rowspan of M^{adj} belongs to the left kernel of M.

The dual variety of $V(\det(X))$ is therefore

$$V(\det(X))^* \;=\; \{Y \in \mathbb{R}^{n\times n} : \mathrm{rank}(Y) \leq 1\}.$$

Example. For $K = \mathrm{PSD}_n$, the algebraic boundary

$$\partial_{\mathrm{alg}} K = V(\det(X))$$
$$(\partial_{\mathrm{alg}} K)^* = \{Y \in \mathbb{R}^{n\times n}_{\mathrm{sym}} : \mathrm{rank}(Y) \leq 1\}.$$

The dual cone is again the PSD cone, $K^* = \mathrm{PSD}_n$, whose extreme rays are the rank-one positive semidefinite matrices. Again we see that the Zariski-closure agrees with $(\partial_{\mathrm{alg}} K)^*$:

$$\mathrm{Exr}(K^*) = \{Y \in \mathrm{PSD}_n : \mathrm{rank}(Y) \leq 1\}$$
$$\overline{\mathrm{Exr}(K^*)}^{Zar} = \{Y \in \mathbb{R}^{n\times n}_{\mathrm{sym}} : \mathrm{rank}(Y) \leq 1\}.$$