Math 591 – Real Algebraic Geometry and Convex Optimization
Lecture 14: Matrix Completion and Graph Realizability
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For linearly independent matrices $A_1, \ldots, A_d \in \mathbb{R}^{n\times n}_{\text{sym}}$ and $b \in \mathbb{R}^d$, consider the linear space

$$\mathcal{L} = \left\{X \in \mathbb{R}^{n\times n}_{\text{sym}} : \langle A_i, X\rangle = b_i \text{ for } i = 1, \ldots, d\right\}.$$

Last time we saw that

Theorem. *If $d < \binom{r+2}{2}$ and $\mathcal{L} \cap \mathrm{PSD}_n$ is nonempty, then $\mathcal{L} \cap \mathrm{PSD}_n$ contains a matrix of rank $\leq r$.*

Today we improve on this slightly:

Theorem (Barvinok's improvement)**.** *If $d = \binom{r+2}{2}$, $n \geq r+2 \geq 3$, and $\mathcal{L}\cap\mathrm{PSD}_n$ is nonempty and bounded, then $\mathcal{L} \cap \mathrm{PSD}_n$ contains a matrix of rank $\leq r$.*

Sketch of proof (when $n = r - 2$). If $n = r - 2$, then $d = \binom{n}{2}$ and $\mathcal{L}$ is an affine linearspace of dimension n. Suppose that $\mathcal{L}$ contains a positive definite matrix, so that the intersection $\mathcal{L}\cap\mathrm{PSD}_n$ is full dimensional. Let $S = \mathcal{L}\cap\mathrm{PSD}_n$. Then S is a n-dimensional compact convex set. It is homeomorphic to a n-dimensional ball and its boundary ∂S is homeomorphic to the $(n-1)$-dimensional sphere, $\mathbb{S}^{n-1}$. Suppose that $\mathrm{rank}(X) \geq n - 1$ for all $X \in S$. Then $\mathrm{rank}(X) = n-1$ for all X on the boundary ∂S. If X is an $n\times n$ matrix with rank $n-1$, the kernel of X is a line in $\mathbb{R}^n$, which, by definition, is a point in the real projective space $\mathbb{RP}^{n-1}$. Then $X \mapsto \ker(X)$ gives a continuous, injective map from $\partial S \cong \mathbb{S}^{n-1}$ to $\mathbb{RP}^{n-1}$. However it is topologically impossible to embed $\mathbb{S}^{n-1}$ in to $\mathbb{RP}^{n-1}$ for $n \geq 3$! $\square$

Let's examine the smallest possible case of this:

Example. ($n = 3$, $r = 1$, $d = \binom{r+2}{2} = 3$) Consider a linear space

$$\mathcal{L} = \left\{X \in \mathbb{R}^{3\times 3}_{\text{sym}} : \langle A_1, X\rangle = b_1, \langle A_2, X\rangle = b_2, \langle A_3, X\rangle = b_3\right\}$$

that contains some positive definite matrix and whose intersection with PSD_3 is bounded. For example, if we take $A_i = E_{ii}$ and $b_i = 1$, we get the 3-dimensional affine space

$$\mathcal{L} = \left\{\begin{pmatrix} 1 & x & y \\ x & 1 & z \\ y & z & 1\end{pmatrix} : (x, y, z) \in \mathbb{R}^3\right\}.$$

Then $S = \mathcal{L} \cap \mathrm{PSD}_3$ is homeomorphic to a ball and its boundary ∂S is homemorphic to a sphere $\mathbb{S}^2$. For a matrix $X \in \mathbb{R}^{3\times 3}_{\text{sym}}$ with rank 2, its kernel $\ker(X)$ is a vector of length 3, up to scaling, which is a point in the real projective plane $\mathbb{RP}^2$. Since there is no embedding of $\mathbb{S}^2$ into $\mathbb{RP}^2$, there must be some point on the boundary of S at which the map $X \mapsto \ker(X)$ is not defined. This will be a matrix of rank 1. Indeed, for the specific example above, we find (several) rank-one matrices. One such is given by $(x, y, z) = (1, 1, 1)$.

Application to graph realizability.

Definition. A **weighted graph** G on n vertices is a set of edges $E \subseteq \binom{[n]}{2}$ and a weight function $\rho : E \to \mathbb{R}_+$. We say that G is r-realizable if there exist points $v_1, \ldots, v_n \in \mathbb{R}^r$ whose pairwise distances are the prescribed weights, i.e.

$$||v_i - v_j||_2 \;=\; \rho(ij) \quad \text{for all } ij \in E.$$

We say that G is realizable if it is r-realizable for some $r \in \mathbb{Z}_+$.

Example. Let G be the weighted graph with $n = 4$, $E = \{12, 23, 34, 14\}$, and $\rho(i(i+1)) = 2$ for $i = 1, 2, 3$ and $\rho(14) = 3$. Then G is 2-realizable but not 1-realizable.

The weighted graph G' with the same vertices and edges, but weights $\rho'(ij) = 2$ for all $ij \in E$ is 1-realizable, by points $v_1 = v_3 = 0$ and $v_2 = v_4 = 2$ on the real line.

Some weighted graphs are not realizable in any dimension. For example the graph on $n = 3$ vertices with all edges $E = \{12, 13, 23\}$ with weights $\rho(12) = \rho(13) = 1$ and $\rho(23) = 3$ does not satisfy the triangle inequality, and so cannot come from distances between points in $\mathbb{R}^r$.

Observation. If G is realizable, then it is $(n-1)$-realizable. To see this, note that if G is realized by $v_1, \ldots, v_n \in \mathbb{R}^r$ for some $r \geq n$, then the affine span of $v_1, \ldots, v_n$ is (at most) $(n-1)$-dimensional and there is an isometry between this affine span and $\mathbb{R}^{n-1}$.

If $v_1, \ldots, v_n \in \mathbb{R}^r$, then the $n \times n$ matrix

$$X = \begin{pmatrix} -v_1^T- \\ -v_2^T- \\ \vdots \\ -v_n^T- \end{pmatrix} \begin{pmatrix} | & | & & | \\ v_1 & v_2 & \ldots & v_n \\ | & | & & | \end{pmatrix} = \begin{pmatrix} \langle v_1, v_1\rangle & \langle v_1, v_2\rangle & \ldots & \langle v_1, v_n\rangle \\ \langle v_1, v_2\rangle & \langle v_2, v_2\rangle & \ldots & \langle v_2, v_n\rangle \\ \vdots & & \ddots & \vdots \\ \langle v_1, v_n\rangle & \langle v_2, v_n\rangle & \ldots & \langle v_n, v_n\rangle \end{pmatrix}$$

is positive semidefinite with rank $\leq r$. Moreover, for any $ij \in E$,

$$\begin{aligned} \rho(ij)^2 = ||v_i - v_j||_2^2 &= \langle v_i - v_j, v_i - v_j\rangle \\ &= \langle v_i, v_i\rangle - 2\langle v_i, v_j\rangle + \langle v_j, v_j\rangle \\ &= X_{ii} - 2X_{ij} + X_{jj}. \end{aligned}$$

Proposition. *The weighted graph G is r-realizable if and only if there is a matrix $X \in \mathrm{PSD}_n$ of rank $\leq r$ satisfying the affine linear equations*

$$\rho(ij)^2 = X_{ii} - 2X_{ij} + X_{jj} \quad \textit{for all } ij \in E.$$

Proof. ($\Rightarrow$) By the argument above, given a realization $v_1, \ldots, v_n$, the matrix $X = V^T V$ where V is the $r \times n$ matrix with columns $v_1, \ldots, v_n$ satisfies the desired conditions.

($\Leftarrow$) If X is positive semidefinite of rank $\leq r$, then it can be factored as $X = V^T V$ where $V \in \mathbb{R}^{r \times n}$. Let $v_1, \ldots, v_n \in \mathbb{R}^r$ be the columns of V. Then $||v_i - v_j||^2 = X_{ii} - 2X_{ij} + X_{jj} = \rho(ij)^2$, meaning that $v_1, \ldots, v_n$ are a realization of G. □

Corollary. *If G has less than $\binom{r+2}{2}$ edges, then G is realizable if and only if it is r-realizable.*

Proof. The linear space

$$\mathcal{L} = \{X \in \mathbb{R}^{n \times n}_{\text{sym}} : X_{ii} - 2X_{ij} + X_{jj} = \rho(ij)^2 \text{ for all } ij \in E\}$$

has codimension $|E|$, since every edge imposes a linearly independent constraint. The graph G is realizable if and only if $L \cap \mathrm{PSD}_n$ is non-empty and G is r-realizable if and only if $L \cap \mathrm{PSD}_n$ contains a matrix of rank $\leq r$. So this follows from our first theorem on PSD matrix completion. □

For example, any realizable graph with at most 5 edges it 2-realizable and any realizable graph with at most 9 edges is 3-realizable.

0.1. **A smaller representation.** Since distances are preserved under translation, we can assume, without loss of generality, that the last vector v_n is the origin. (Explicitly, we can replace a representation $v_1, \ldots, v_n$ with $v_1 - v_n, \ldots, v_{n-1} - v_n, v_n - v_n$.)

Then, if $v_1, \ldots, v_{n-1} \in \mathbb{R}^r$, the $(n-1) \times (n-1)$ matrix

$$X = \begin{pmatrix} -v_1^T- \\ -v_2^T- \\ \vdots \\ -v_{n-1}^T- \end{pmatrix} \begin{pmatrix} | & | & & | \\ v_1 & v_2 & \ldots & v_{n-1} \\ | & | & & | \end{pmatrix} = \begin{pmatrix} \langle v_1, v_1 \rangle & \langle v_1, v_2 \rangle & \ldots & \langle v_1, v_{n-1} \rangle \\ \langle v_1, v_2 \rangle & \langle v_2, v_2 \rangle & \ldots & \langle v_2, v_{n-1} \rangle \\ \vdots & & \ddots & \vdots \\ \langle v_1, v_{n-1} \rangle & \langle v_2, v_{n-1} \rangle & \ldots & \langle v_{n-1}, v_{n-1} \rangle \end{pmatrix}$$

is positive semidefinite with rank $\leq r$ that satisfies

$$X_{ii} = \langle v_i, v_i \rangle = \rho(in)^2 \ \text{ for } \ in \in E$$

and

$$X_{ii} - 2X_{ij} + X_{jj} = \rho(ij)^2 \ \text{ for } \ ij \in E, \ i, j \neq n.$$

Example. Consider the weighted graph G on vertices $\{1, 2, 3, 4\}$ with edges $E = \{12, 13, 23, 24, 34\}$ where $\rho(ij) = 1$ for all $ij \in E$. This graph is realizable if and only if there exists a matrix $X \in \mathrm{PSD}_3$ with

$$\begin{aligned} X_{22} = 1, \ X_{33} = 1 \\ X_{11} - 2X_{12} + X_{22} = 1 \\ X_{11} - 2X_{13} + X_{33} = 1 \\ X_{22} - 2X_{23} + X_{33} = 1 \end{aligned}$$

Solving this equations shows that

$$X = \begin{pmatrix} X_{11} & \frac{1}{2}X_{11} & \frac{1}{2}X_{11} \\ \frac{1}{2}X_{11} & 1 & \frac{1}{2} \\ \frac{1}{2}X_{11} & \frac{1}{2} & 1 \end{pmatrix},$$

which is positive semidefinite if and only if $X_{11} \in [0, 3]$. For $X_{11} \in \{0, 3\}$ is has rank two, and there is a realization of the graph in $\mathbb{R}^2$ with the distance between v_1 and v_4 equal to $\sqrt{X_{11}}$.

For $X_{11} \in (0, 3)$, this matrix has rank three and there is a realization of the graph in $\mathbb{R}^3$ with the distance between v_1 and v_4 equal to $\sqrt{X_{11}}$.

This allows us to translate between graph realization problems and matrix completion problems. Barvinok's improvement for matrix completion gives the following:

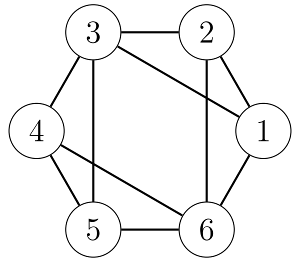
Proposition. *If G has $\binom{r+2}{2}$ edges and G is not the union of the complete graph K_{r+2} with some isolated vertices, then G is realizable if and only if it is r-realizable.*

Proof. Suppose that G is connected. Since G is not a complete graph, the number of edges implies that there are at least $r + 3$ vertices, implying that $n - 1 \geq r + 2$. The linear space

$$\mathcal{L} = \{X \in \mathbb{R}^{(n-1)\times(n-1)}_{\text{sym}} : X_{ii} = \rho(in)^2 \text{ for } in \in E, X_{ii} - 2X_{ij} + X_{jj} = \rho(ij)^2 \text{ for } ij \in E\}$$

has codimension $|E| = \binom{r+2}{2}$. Then G is realizable if and only if $\mathcal{L} \cap \mathrm{PSD}_{n-1}$ is non-empty. If G is connected, then $\mathcal{L} \cap \mathrm{PSD}_{n-1}$ is bounded. Therefore by Barvinok's improvement, $\mathcal{L} \cap \mathrm{PSD}_{n-1}$ contains a matrix of rank $\leq r$, which gives a realization of G in $\mathbb{R}^r$. □

Example. For example, consider the following graph on six vertices with $10 = \binom{3+2}{2}$ edges and all weights $\rho(ij) = 1$ for $ij \in E$.



This realized by vectors in $\mathbb{R}^6$ by taking $v_i = \frac{1}{\sqrt{2}} e_i$ for each $i = 1, \ldots, 6$. Then by the proposition above, it has some realization in $\mathbb{R}^3$.