Math 591 – Real Algebraic Geometry and Convex Optimization
Lecture 13: Matrix Completion and Spectrahedra
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Let $A_1, \ldots, A_d \in \mathbb{R}^{n\times n}_{\text{sym}}$ be linear independent over $\mathbb{R}$. This defines a linear map

$$T : \mathbb{R}^{n\times n}_{\text{sym}} \to \mathbb{R}^d \quad \text{by} \quad T(X) = (\langle X, A_i\rangle)_{i=1,\ldots,d}.$$

If we want to reconstruct X from $T(X)$ and we don't have any addition information, then we need $d = \binom{n+1}{2} = \dim_{\mathbb{R}}(\mathbb{R}^{n\times n}_{\text{sym}})$. But if, in addition, we know that X was low rank, then we may be able to take d smaller.

Question. What is the lowest rank of a matrix in $T^{-1}(b)$ for $b \in \mathbb{R}^d$? For *generic* $b \in \mathbb{R}^d$? What about the lowest PSD matrix?

Example. ($n = d = 2$) Consider the map $T : \mathbb{R}^{2\times 2}_{\text{sym}} \to \mathbb{R}^2$ given by $T\begin{pmatrix} a & b \\ b & c\end{pmatrix} = (a, b)$. What is the lowest rank of a matrix in $T^{-1}(a, b)$? Then we can check that

$$\text{lowest rank in } T^{-1}(a,b) = \begin{cases} 0 & \text{if } a = b = 0 \\ 1 & \text{if } a \neq 0 \\ 2 & \text{if } a = 0, b \neq 0. \end{cases}$$

If $a > 0$, then we can even find a positive semidefinite matrix of rank one in $T^{-1}(a, b)$.

Example. (Connection with sums of squares) Consider the linear map

$$T : \mathbb{R}^{N\times N}_{\text{sym}} \to \mathbb{R}[x_1, \ldots, x_n]_{\leq 2d} \quad \text{given by} \quad T(X) = m_d^T X m_d$$

where m_d is the vector of all monomials of degree $\leq d$ in $\mathbb{R}[x_1, \ldots, x_n]$. The finding the smallest rank of a positive semidefinite matrix in $T^{-1}(f)$ is equivalent to finding the smallest number of squares in a sum-of-squares representation of $f \in \mathbb{R}[x_1, \ldots, x_n]_{\leq 2d}$.

To get an idea of what we might expect, let us first study this problem over $\mathbb{C}$. To do this, we need some information about the dimension of an algebraic variety.

The dimension of algebraic varieties and matrix completion over $\mathbb{C}$.

Definition. The **dimension** of an algebraic variety $V \subset \mathbb{C}^N$, denoted $\dim(V)$, equals

- the largest k such that a generic affine linear space $L \subset \mathbb{C}^N$ of codimension k (dimension $N - k$) intersects V, i.e. $L \cap V \neq \emptyset$, and
- the largest k such that for some $i_1 < \ldots < i_k \in \{1, \ldots, N\}$ the image of V under the projection $\pi(x_1, \ldots, x_N) = (x_{i_1}, \ldots, x_{i_k})$ is Zariski-dense in $\mathbb{C}^k$, i.e. $\overline{\pi(V)}^{Zar} = \mathbb{C}^k$.

It turns out, but is non-trivial to prove, that these two definitions are equivalent.

Example. Consider the variety $V = V(x^2 - y, x^3 - z, y^3 - z^2) = \{(t, t^2, t^3) : t \in \mathbb{C}\}$ in $\mathbb{C}^3$. A generic affine linear space of codimension one in $\mathbb{C}^3$ is an affine linear space, which has the form

$$L \;=\; \{(x, y, z) \in \mathbb{C}^3 : a + bx + cy + dz = 0\}$$

for $a, b, c, d \in \mathbb{C}$, then the intersection

$$L \cap V = \{(t, t^2, t^3) \in \mathbb{C}^3 : a + bt + ct^2 + dt^3 = 0\}$$

is non-empty, since the polynomial $a + bt + ct^2 + dt^3$ has some root. However a affine linear space of codimension two is just a line. We can check that a generic affine line in $\mathbb{C}^3$ does not intersect V. (One way to see this is that two randomly chosen cubics in $\mathbb{R}[t]_{\leq 3}$ will not have a common root!) Therefore $\dim(V) = 1$.

To see this, we can also consider the second definition of dimension. Since V is a variety, $\overline{V}^{Zar} = V$, which is not all of $\mathbb{C}^3$, so $\dim(V) < 3$. Similarly, there is a polynomial that vanishes on the projection of V onto any *pair* of coordinates (namely one of the three polynomials listed in its description). Therefore $\dim(V) < 2$. However, taking $\pi(x, y, z) = x$, we see that $\pi(V) = \mathbb{C}$. So, again, $\dim(V) = 1$.

For us, the variety of interest will be

$$\mathcal{M}_r \;=\; \{X \in \mathbb{C}^{n\times n}_{\text{sym}} : \text{rank}(X) \leq r\}.$$

Recall that this is the variety defined by the vanishing of the $(r+1) \times (r+1)$ minors of X.

Warm-up Question. What is the dimension of $\mathcal{M}_1$ in $\mathbb{C}^{n\times n}_{\text{sym}}$?

For $n = 2$, we have $\mathcal{M}_1 = \left\{\begin{pmatrix} a & b \\ b & c \end{pmatrix} : ac = b^2\right\}$. Since not every matrix has rank ≤ 1, $\mathcal{M}_1$ is not the whole space, and so has dimension < 3. We saw that for the projection $\pi\begin{pmatrix} a & b \\ b & c \end{pmatrix} = (a, b)$, we have

$$\pi(\mathcal{M}_1) = \{(a, b) : a \neq 0\} \;\text{ and }\; \overline{\pi(\mathcal{M}_1)}^{Zar} = \mathbb{C}^2.$$

So $\dim(\mathcal{M}_1) = 2$.

For $n = 3$, we have

$$\mathcal{M}_1 = \left\{\begin{pmatrix} a & b & c \\ b & d & e \\ c & e & f \end{pmatrix} : \text{ all } 2 \times 2 \text{ minors } = 0\right\}.$$

Note that for any choice of $(a, b, c) \in \mathbb{C}^3$ with $a \neq 0$, there is a unique way to extend to a rank-one matrix. (Namely, taking $d = b^2/a$, $e = bc/a$, $f = c^2/a$). This shows that $\dim(\mathcal{M}_1) = 3$.

More generally we have the following:

Proposition. *The set of $n \times n$ symmetric matrices of rank $\leq r$ has dimension*

$$\dim(\mathcal{M}_r) = \binom{n+1}{2} - \binom{n-r+1}{2} = \binom{r+1}{2} + r(n-r).$$

Sketch of proof. For a generic choice of X_{ij} with $1 \leq i \leq r$ and $i \leq j \leq n$, there is a unique way to extend to a rank r matrix X. (Specifically, we need that the $r \times r$ matrix $(X_{ij})_{1\leq i,j\leq r}$ has full rank r.) The number of these entries is the value claimed. One can see this as either the total number of entries minus the number in the bottom right $(n-r) \times (n-r)$ corner or as the number in the top left $r \times r$ block plus the number in the first r rows and last $(n-r)$ columns. □

As an immediate corollary of our first definition of dimension we find the following:

Corollary. *For generic $A_1, \ldots, A_d \in \mathbb{C}^{n\times n}_{\text{sym}}$ and $b \in \mathbb{C}^d$, there exists a matrix of rank $\leq r$ in the affine linear space $\{X \in \mathbb{C}^{n\times n}_{\text{sym}} : \langle A_i, X\rangle = b_i, i = 1, \ldots, d\}$ if and only if $d \leq \dim(\mathcal{M}_r)$.*

Example. $(n = 4, r = 2)$. We find that variety of 4×4 symmetric matrices of rank ≤ 2 has dimension 7 in $\mathbb{C}^{4\times 4}_{\text{sym}} \cong \mathbb{C}^{10}$. Therefore for $d \leq 7$ the affine linear space defined by $\langle A_i, X\rangle = b_i$ for generic A_i and b contains a matrix of rank ≤ 2.

Back to $\mathbb{R}$ and PSD_n.

Fix $A_1, \ldots, A_d \in \mathbb{R}^{n\times n}_{\text{sym}}$ to be linearly independent over $\mathbb{R}$ and $b \in \mathbb{R}^d$. Then affine linear space

$$\mathcal{L} = \{X \in \mathbb{R}^{n\times n}_{\text{sym}} : \langle X, A_i\rangle = b_i,\ i = 1, \ldots d\}$$

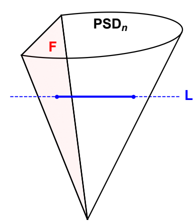
has codimension d in $\mathbb{R}^{n\times n}_{\text{sym}}$ (and so dimension $\binom{n+1}{2} - d$).

Proposition. *If $d < \binom{r+2}{2}$ and $\mathcal{L} \cap \text{PSD}_n$ is non-empty, then $\mathcal{L} \cap \text{PSD}_n$ contains a matrix of rank $\leq r$.*

Note that in terms of the map $T(X) = (\langle A_i, X\rangle)_{i=1,\ldots,d}$ this says that for any b in $T(\text{PSD}_n)$, $T^{-1}(b)$ contains a positive semidefinite matrix of rank $\leq r$.

Proof. Let S denote the intersection $\mathcal{L} \cap \text{PSD}_n$. Then S is convex, non-empty, and contains no lines. It follows that S has some extreme point $\tilde{X}$. Let $m = \text{rank}(\tilde{X})$.

By properties of the facial structure of PSD_n, $\tilde{X}$ lies in the relative interior of a face $\mathcal{F}$ that is linearly isomorphic to PSD_m. Furthermore, since $\tilde{X}$ is an extreme point in S, it must be the only point in the intersection of $\mathcal{F}$ with $\mathcal{L}$.



Since the intersection of $\mathcal{F}$ with $\mathcal{L}$ is a single point, their dimensions cannot add to more than the dimension of the whole space, $\mathbb{R}^{n\times n}_{\text{sym}}$. That is,

$$\dim(\mathcal{F}) + \dim(\mathcal{L}) \leq \dim(\mathbb{R}^{n\times n}_{\text{sym}}).$$

Since $\dim(\mathcal{F}) = \dim(\mathbb{R}^{m\times m}_{\text{sym}}) = \binom{m+1}{2}$, and the codimension of $\mathcal{L}$ is $d = \dim(\mathbb{R}^{n\times n}_{\text{sym}}) - \dim(\mathcal{L})$, this gives that $\binom{m+1}{2} \leq d < \binom{r+2}{2}$. Therefore $m < r + 1$ and $m \leq r$. □

In the proof of this, we actually showed the following:

Corollary. *Any extreme point of $\mathcal{L} \cap \text{PSD}_n$ has rank $\leq r$ where $\binom{r+1}{2} \leq d$.*

Example. $(n = 4, r = 2)$. If $\mathcal{L}$ is an affine space of codimension d in $\mathbb{R}^{4\times 4}_{\text{sym}}$ so that $\mathcal{L} \cap \text{PSD}_4$ is non-empty, then $\mathcal{L}$ is guaranteed to have a positive semidefinite matrix of rank ≤ 2 for $d < \binom{2+2}{2} = 6$.

Over $\mathbb{C}$, we could impose 7 (generic) affine linear constraints and still find a matrix of rank ≤ 2, where as this says to guarantee a real and PSD matrix of rank ≤ 2, we can only impose 5 affine linear constraints.

Next time, we'll use stronger techniques, to show that you can actually impose one more constraint and still find a PSD matrix of rank $\leq r$: