Math 591 – Real Algebraic Geometry and Convex Optimization

Lecture 10: An application to stable sets
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Take graph $G = ([n], E)$ on vertices $[n] = \{1, \ldots, n\}$ and edges $E \subseteq \{\{i,j\} : i \neq j \in [n]\}$. We say that $S \subseteq E$ is a **stable set** (or **independent set**) of G if $\{i, j\} \notin E$ for every pair $i, j \in S$. We can identify each stable set S with its *characteristic vector* $\mathbb{1}_S = \sum_{i=1}^n e_i$ in $\{0,1\}^n$ with ith coordinate 1 if $i \in S$ and 0 otherwise. Let STAB_G denote the set of indicator vectors of stable sets of G:

$$\text{STAB}_G \;=\; \{\mathbb{1}_S \;:\; S \text{ is a stable set of } G\}.$$

The (aptly named) *maximum stable set problem* is to find the maximum size stable set:

$$\begin{aligned}\alpha(G) \;&=\; \max\{|S| \;:\; S \text{ is a stable set of } G\} \\ &=\; \max \sum_{i=1}^n x_i \;\text{ such that } x \in \text{STAB}_G.\end{aligned}$$

Example. For $G = ([4], \{12, 13, 23, 34\})$, the stable sets are

$$\emptyset, \{1\}, \{2\}, \{3\}, \{4\}, \{1,4\}, \{2,4\}$$

and we see $\alpha(G) = 2$.

Example. For the five-cycle, $C_5 = ([5], \{12, 23, 34, 45, 15\})$, the stable sets are $\emptyset$, singletons $\{i\}$, and $\{i, j\}$ where $i - j \not\equiv \pm 1 \bmod 5$ and we have $\alpha(C_5) = 2$.

Note that the maximum stable set of the complement of G, $G^c = ([n], \binom{[n]}{2} \backslash E)$ corresponds to the maximum size of a *clique* in G, i.e. the maximum size of a subset $S \subset E$ for which every pair is connected.

In general, computing $\alpha(G)$ is NP-Hard and so becomes quickly intractable as n grows. Finding a stable set of G produces a lower bound for $\alpha(G)$. It would be nice to produce an upper bound for $\alpha(G)$.

Translation to polynomial optimization. We can write STAB_G as the variety of some collection of polynomials, namely

$$\{x_i^2 - x_i : i \in [n]\} \;\cup\; \{x_i x_j : \{i,j\} \in E\}.$$

To check: a point $p \in \mathbb{R}^n$ satisfies $p_i^2 - p_i = 0$ for all i if and only if $p \in \{0,1\}^n$, meaning that $p = \mathbb{1}_S$ for some $S \subseteq [n]$. Also, $p_i p_j = 0$ for all $\{i, j\} \subset E$ if and only if S does not contain any i, j with $\{i, j\} \in E$.

Let I_G denote the ideal *generated by these polynomials*, i.e.

$$I_G \;=\; \left\{ \sum_{i \in [n]} g_i (x_i^2 - x_i) + \sum_{ij \in E} h_{ij} x_i x_j \;:\; g_i, h_{ij} \in \mathbb{R}[x_1, \ldots, x_n] \right\}.$$

Clearly, every polynomial in I_G vanishes on STAB_G. In fact, one can check that I_G contains *all* polynomials that vanish on STAB_G.

Example. For $G = ([4], \{12, 13, 23, 34\})$, the ideal I_G is generated by the polynomials $x_i^2 - x_i$ for each $i = 1, \ldots, 4$ and x_1x_2, x_1x_3, x_2x_3, x_3x_4. If variety consists of the seven points

$$\mathbb{1}_\emptyset = (0,0,0,0),\ \mathbb{1}_{\{1\}} = (1,0,0,0),\ \mathbb{1}_{\{2\}} = (0,1,0,0),\ \mathbb{1}_{\{3\}} = (0,0,1,0),\ \mathbb{1}_{\{4\}} = (0,0,0,1),$$
$$\mathbb{1}_{\{1,4\}} = (1,0,0,1),\ \text{and}\ \mathbb{1}_{\{2,4\}} = (0,1,0,1).$$

Then

$$\begin{aligned}
\alpha(G) &= \max \sum_{i=1}^n x_i \text{ such that } x \in \mathrm{STAB}_G \\
&= \min_{c \in \mathbb{R}} c \ \text{ such that } c - \sum_{i=1}^n x_i \geq 0 \text{ on } \mathrm{STAB}_G \\
&= \min_{c \in \mathbb{R}} c \ \text{ such that } c - \sum_{i=1}^n x_i \in \mathrm{SOS}_n + I_G
\end{aligned}$$

The last equality comes from the fact that the variety of I_G is finite, so *every* polynomial that is nonnegative on $V(I_G) = \mathrm{STAB}_G$ can be written as a sum of squares plus some element of the ideal I_G. (Recall that we can just take sums of squares of indicator polynomials).

Restricting the *degree* of the sums of squares involved will result in a semidefinite program and provide an upper bound for $\alpha(G)$. For any $k \in \mathbb{N}$, we can define

$$\Theta_{2k}(G) = \min_{c \in \mathbb{R}} c \ \text{ such that } \ c - \sum_{i=1}^n x_i \in \mathrm{SOS}_{n, \leq 2k} + I_G.$$

Then $\alpha(G) \leq \Theta_{2k}(G) \leq \ldots \leq \Theta_4(G) \leq \Theta_2(G)$.

It is convenient to work in the quotient ring $\mathbb{R}[x_1, \ldots, x_n]/I_G$, with elements $f + I_G$ where $f \in \mathbb{R}[x_1, \ldots, x_n]$. Then $f + I_G = g + I_G$ if and only if $f - g \in I_G$. Note that the quotient ring $\mathbb{R}[x_1, \ldots, x_n]/I_G$ is a finite-dimensional vector space over $\mathbb{R}$. For example, it is spanned by the elements $\{\prod_{i \in S} x_i + I_G : S \in \mathrm{STAB}(G)\}$. For $k \in \mathbb{N}$, let m_k be the vector with entries $\prod_{i \in S} x_i + I_G \in \mathbb{R}[x_1, \ldots, x_n]/I_G$, where $S \in \mathrm{STAB}(G)$ and $|S| \leq k$. This lets us write

$$\Theta_{2k}(G) = \min_{c \in \mathbb{R}} c \ \text{ such that } \ c - \sum_{i=1}^n x_i \equiv m_k^T A m_k \bmod I_G, \quad \text{where } A \succeq 0.$$

Note that the equation $c - \sum_{i=1}^n x_i \equiv m_k^T A m_k \bmod I_G$ imposes linear conditions on the matrix A, making this a semidefinite program.

Example. For $G = ([4], \{12, 13, 23, 34\})$, I_G is generated by the polynomials $x_i^2 - x_i$ for each $i = 1, \ldots, 4$ and x_1x_2, x_1x_3, x_2x_3, x_3x_4. Indeed, in this case, we see that $\Theta_2(G) = \alpha(G) = 2$ because we can write

$$\begin{aligned}
2 - x_1 - x_2 - x_3 - x_4 &= (1 - x_1 - x_2)^2 + (1 - x_3 - x_4)^2 - \sum_{i=1}^4 (x_i^2 - x_i) - 2x_1x_2 - 2x_3x_4 \\
&\equiv (1 - x_1 - x_2)^2 + (1 - x_3 - x_4)^2 \mod I_G.
\end{aligned}$$

In the 1979, Lovász introduced this relaxation and showed that it worked very well for a class of graphs called *perfect graphs*. This was the first major use of semidefinite programming to provide bounds for a problem in combinatorial optimization!

A graph $G = ([n], E)$ is **perfect** if G has no induced odd cycles of length at least five or their complements. (Alternatively, for G and all of its induced subgraphs have the property that the coloring number equals the size of the largest clique.)

Theorem (Lovász)**.** *If G is perfect, then $\Theta_2(G) = \alpha(G)$.*

We will talk about the proof next time.

Example. Note that $G = ([4], \{12, 13, 23, 34\})$ is a perfect graph, and indeed $\Theta_2(G) = \alpha(G)$. But the five cycle $C_5 = ([5], \{12, 23, 34, 45, 15\})$ is not! We might wonder what $\Theta_2(C_5)$ is.

In fact, Lovász proved a stronger statement, namely that if G is perfect, then for *every* linear form $\ell = \sum_i a_i x_i$,

$$\max_{x \in \mathrm{STAB(G)}} \ell(x) \;=\; \min_{c \in \mathbb{R}} c \;\text{ such that }\; c - \ell(x) \in \mathrm{SOS}_{n,\leq 2} + I_G.$$

But in Lovász did not actually prove this as stated. He proved the dual version.

The dual version. Let us first go through the dual problem of computing Θ_2.

The dual cone of $\mathrm{SOS}_{n,\leq 2} + I_G$ is the set of linear functions $L : \mathbb{R}[x_1, \ldots, x_n]_{\leq 2}/I_G \to \mathbb{R}$ that are nonnegative on squares. Note that

$$\{1 + I_G\} \cup \{x_i + I_G : i \in [n]\} \cup \{x_i x_j + I_G : ij \notin E\}$$

is an $\mathbb{R}$-basis for $\mathbb{R}[x_1, \ldots, x_n]_{\leq 2}/I_G$. Therefore a linear function $L : \mathbb{R}[x_1, \ldots, x_n]_{\leq 2}/I_G \to \mathbb{R}$ is determined by

$$L(1 + I_G) = y_\emptyset,\; L(x_i + I_G) = y_i \;\text{ for }\; i \in [n], \;\text{ and }\; L(x_i x_j + I_G) = y_{ij} \text{ for } ij \notin E$$

Then L belongs to the dual cone of $\mathrm{SOS}_{n,\leq 2} + I_G$ if and only if L is nonnegative on squares $h^2 + I_G$ where $h + I_G \in \mathbb{R}[x_1, \ldots, x_n]_{\leq 1}/I_G$, which happens if and only if the $(n+1) \times (n+1)$ matrix $L(m_1 m_1^T)$ is positive semidefinite, where $m_1 = (1, x_1 + I_G, \ldots, x_n + I_G)$. Denote this matrix by

$$\mathcal{M}_G(y) \;=\; L(m_1 m_1^T).$$

Since $L(x_i^2 + I_G) = L(x_i + I_G) = y_i$ and $L(x_i x_j + I_G) = L(0 + I_G) = 0$ for $ij \in E$, this is given by

$$\begin{aligned}
&\mathcal{M}_G(y)_{00} = L(1 + I_G) = y_\emptyset \\
&\mathcal{M}_G(y)_{0i} = L(x_i + I_G) = y_i \\
&\mathcal{M}_G(y)_{ii} = L(x_i^2 + I_G) = L(x_i + I_G) = y_i \\
&\mathcal{M}_G(y)_{ij} = L(x_i x_j + I_G) = L(0 + I_G) = 0 \text{ for } ij \in E, \text{ and} \\
&\mathcal{M}_G(y)_{ij} = L(x_i x_j + I_G) = y_{ij} \text{ for } ij \notin E.
\end{aligned}$$

Example. For example, for $G = ([4], \{12, 13, 23, 34\})$, we have

$$L(m_1 m_1^T) \;=\; \mathcal{M}_G(y) \;=\; \begin{pmatrix} y_\emptyset & y_1 & y_2 & y_3 & y_4 \\ y_1 & y_1 & 0 & 0 & y_{14} \\ y_2 & 0 & y_2 & 0 & y_{24} \\ y_3 & 0 & 0 & y_3 & 0 \\ y_4 & y_{14} & y_{24} & 0 & y_4 \end{pmatrix}$$

We can compute Θ_2 by computing the *dual* of the sums of squares problem above. This translates to:

$$\Theta_2(G) \;=\; \max \sum_{i=1}^{n} y_i \quad \text{such that} \quad y_\emptyset = 1 \text{ and } \mathcal{M}_G(y) \in \mathrm{PSD}_{n+1}$$

Note that for any stable set S, $\mathbb{1}_S \mathbb{1}_S^T$ gives a feasible matrix $\mathcal{M}_G(y)$ with objective value $|S|$, so we immediately get $\alpha(G) \leq \Theta_2(G)$, just as before.

In Lovász's original proof, he defines the *theta body* of a graph to be

$$\mathrm{TH}(G) \;= \{(y_1, \ldots, y_n) \in \mathbb{R}^n : \mathcal{M}_G(y) \succeq 0 \text{ with } y_\emptyset = 1\}$$

This is the projection of the feasible set above onto $(y_1, \ldots, y_n)$. It is convex and contains $\mathrm{STAB}(G)$.

Theorem (Lovász)**.** *The graph G is perfect if and only if* $\mathrm{TH(G)} = \mathrm{conv}(\mathrm{STAB}(G))$.