Math 591 – Homework 4

Due 3pm on Thursday, March 7, 2019

Please indicate any sources you used for a given problem on the solution to that problem. For example, if you worked with another student to get the solution to a problem, please indicate who. You are welcome to work together in small groups (2-4 people), but you should try the problems on your own first.

Problem 1. Given a nonnegative univariate polynomial $f \in \mathbb{R}[x]_{\leq 8}$, the set of Gram matrices of f is

$$\mathrm{Gram}(f) = \left\{X \in \mathrm{PSD}_5 : m_4^T X m_4 = f\right\}$$

where $m_4 = (1, x, x^2, x^3, x^4)^T$. Note that $m_4^T X m_4 = f$ imposes $9 = \dim(\mathbb{R}[x]_{\leq 8})$ affine linear constraints on X.

(a) What is the smallest r for which $\mathrm{Gram}(f)$ is guaranteed to have a matrix of rank $\leq r$ by the PSD matrix-completion theorems from class? That is, what is the smallest r for which the intersection of a codimension-9 affine linear space and PSD_5 is guaranteed to have a matrix of rank $\leq r$ (assuming it is non-empty)?

(b) Show that if f is nonnegative, then f is a sum of *two* squares $f = h_1^2 + h_2^2$ and $\mathrm{Gram}(f)$ has a matrix of rank ≤ 2.
Hint: factor f as $(h_1 + ih_2) \cdot (h_1 - ih_2)$ where $h_1, h_2 \in \mathbb{R}[x]_{\leq 4}$.

Problem 2. Let $\mathbb{S}^{n-1}$ denote the sphere $\{x \in \mathbb{R}^n : x_1^2 + \ldots + x_n^2 = 1\}$.

(a) Show that $\{X \in \mathrm{PSD}_n : \mathrm{rank}(X) = 1,\ \langle I, X\rangle = 1\} = \{xx^T : x \in \mathbb{S}^{n-1}\}$.

(b) Show that for any matrices $A, B \in \mathbb{R}^{n\times n}_{\mathrm{sym}}$ with $n \geq 3$, the image of the map $\mathbb{S}^{n-1} \to \mathbb{R}^2$ given by $x \mapsto (x^T Ax, x^T Bx)$ is convex.

Problem 3. Let $\ell_{12}, \ell_{13}, \ell_{23} \in \mathbb{R}_+$ and for each i, j, let $d_{ij} = \ell_{ij}^2$.

(a) Show that there is a triangle with side lengths $\ell_{12}, \ell_{13}, \ell_{23}$ if and only if the matrix

$$\begin{pmatrix} 2d_{13} & d_{13} + d_{23} - d_{12} \\ d_{13} + d_{23} - d_{12} & 2d_{23} \end{pmatrix}$$

is positive semidefinite.

(b) Use (a) to show the triangle inequality. That is, show that there is triangle with side lengths $\ell_{12}, \ell_{13}, \ell_{23}$ if and only if $\ell_{jk} \leq \ell_{ij} + \ell_{ik}$ for all distinct choices of i, j, k.
Hint: Plug in $d_{ij} = \ell_{ij}^2$ and factor the 2×2 determinant.

Optional Bonus Problem. Find a sum-of-squares certificate that $V_{\mathbb{R}}(x^4 + y^4 - x^2)$ is contained in the ball $\{(x, y) : x^2 + y^2 \leq 5/4\}$.