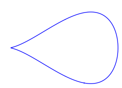
Math 591 – Homework 3

Due 5pm on Friday, February 22, 2019

Please indicate any sources you used for a given problem on the solution to that problem. For example, if you worked with another student to get the solution to a problem, please indicate who. You are welcome to work together in small groups (2-4 people), but you should try the problems on your own first.

Problem 1. Consider $m = \begin{pmatrix} 1 & xy & x^2y & xy^2 \end{pmatrix}$. Find a 4×4 positive semidefinite matrix Y so that $\langle A, Y \rangle < 0$ for every matrix A for which $1 - 3x^2y^2 + x^4y^2 + x^2y^4 = m^T A m$.

Problem 2. Show that x is nonnegative on the variety $V(f)$ where $f = x^4 - x^3 + y^2$ but that x cannot be written as a sum of squares plus a polynomial multiple of f.



Hint: consider the lowest exponent of x.

Problem 3. (Univariate representations)

(a) Show that every polynomial $f \in \mathbb{R}[x]$ that is nonnegative on $[0, \infty)$ can be written as $\sigma_0 + x\sigma_1$ where σ_0, σ_1 are sums of squares with $\deg(\sigma_0), \deg(x\sigma_1) \leq \deg(f)$.

(b) Write down a spectrahedral description of the closure of the convex hull of $\{(t, t^2, t^3, t^4) : t \in [0, \infty)\}$. That is, find a matrix $\mathcal{M}(y)$, whose entries are affine linear functions of $y_1, \ldots, y_4$ so that

$$\overline{\text{conv}(\{(t, t^2, t^3, t^4) : t \in [0, \infty)\})} = \{(y_1, y_2, y_3, y_4) : \mathcal{M}(y) \succeq 0\}.$$

Problem 4. Let $G = ([5], E)$ be the five cycle with edges $E = \{\{i, j\} : i \equiv j \pm 1 \bmod 5\}$.

(a) Write down the problem

$$\min \ c \ \text{ such that } \ c - \sum_{i=1}^{5} x_i \in \text{SOS}_{5,\leq 2} + I_G$$

explicitly as a semidefinite program

$$\min \langle C, X \rangle \text{ such that } \langle A_i, X \rangle = b_i \text{ for } i = 1, \ldots, m$$

for some m and real symmetric matrices $C, A_1, \ldots, A_m \in \mathbb{R}^{6\times 6}_{\text{sym}}$ and $b \in \mathbb{R}^m$. Here I_G denotes the the ideal generated by $x_i^2 - x_i$ for $i = 1, \ldots, 5$ and $x_i x_j$ for $\{i, j\} \in E$. *Hint: write down the condition that $c - \sum_{i=1}^{5} x_i = m_1^T X m_1 \bmod I_G$ as a set of affine constraints on the matrix X where $m_1 = \begin{pmatrix} 1 & x_1 & \ldots & x_5 \end{pmatrix}^T$.*

(b) Write down the dual semidefinite program.

(c) *Optional bonus:* Show that the optimal value is strictly greater than 2.