Math 591 – Homework 2

Due 3pm on Thursday, February 7, 2019

Please indicate any sources you used for a given problem on the solution to that problem. For example, if you worked with another student to get the solution to a problem, please indicate who. You are welcome to work together in small groups (2-4 people), but you should try the problems on your own first.

Problem 1. Let K denote the convex cone of quadratic polynomials in $\mathbb{R}[x]$ that are nonnegative on $[-1, 1]$, i.e.

$$K \;=\; \mathcal{P}([-1,1]) \;=\; \{(a,b,c) \in \mathbb{R}^3 : ax^2 + bx + c \geq 0 \text{ for all } x \in [-1,1]\}.$$

(a) Describe the dual cone K^*.

(b) Draw the both K and K^* in the planes "last coordinate"$= 1$.

(c) Give a semialgebraic description of K.

Problem 2. For $A \in \mathbb{R}^{n\times n}_{\text{sym}}$, consider the maximization problem

$$\max_{y\in\mathbb{R}} \; y \quad \text{s.t.} \quad A - yI \in \text{PSD}_n,$$

where I is the $n \times n$ identity matrix.

(a) Write a minimization problem of which this is the dual.

(b) Show that the primal and dual problems attain the same value. What is this optimal value in terms of the matrix A?

Problem 3.

(a) Show that if any polynomial $f \in \mathbb{R}[x, y]$ vanishes on the half circle $V_{\mathbb{R}}(x^2 + y^2 - 1) \cap \{(x, y) : x \leq 0\}$, then f vanishes on all of $V_{\mathbb{R}}(x^2 + y^2 - 1)$. *Hint: you can parametrize the circle by* $\left\{\left(\frac{t^2-1}{t^2+1}, \frac{2t}{t^2+1}\right) : t \in \mathbb{R}\right\}$.

(b) Show that the set

$$\{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\} \cup \{(x, y) \in \mathbb{R}^2 : 0 \leq x \leq 2, y^2 \leq 1\}$$

is not a basic closed semialgebraic set.

Problem 4. The *Newton polytope* of a polynomial $f = \sum_\alpha c_\alpha x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ is the convex hull of the exponent vectors of f:

$$\text{Newt}(f) \;=\; \text{conv}(\{\alpha : c_\alpha \neq 0\} \;\subset \mathbb{R}^n.$$

For $w \in \mathbb{R}^n$, it can also be useful to consider the w-initial form of f, $\text{in}_w(f) = \sum_{\alpha\in\mathcal{A}} c_\alpha \underline{x}^\alpha$, where $\mathcal{A} = \{\alpha \in \text{Newt}(f) : w^T\alpha \geq w^T\beta \text{ for all } \beta \in \text{Newt}(f)\}$. Show the following:

(a) If $f = \sum_{i=1}^k h_i^2$, then $\text{Newt}(h_i) \subseteq \frac{1}{2}\text{Newt}(f)$.

(b) The Motzkin polynomial $M(x, y) \;=\; 1 - 3x^2y^2 + x^4y^2 + x^2y^4$ is not a sum of squares. *Hint: look at the coefficient of x^2y^2 in any sum-of-squares representation.*