Math 591 – Homework 1

Due 3pm on Thursday, January 24, 2019

Please indicate any sources you used for a given problem on the solution to that problem. For example, if you worked with another student to get the solution to a problem, please indicate who. You are welcome to work together in small groups (2-4 people), but you should try the problems on your own first.

Recall that $F \subseteq C$ is a **face** of a convex set C if it is convex and for all $u, v \in C$, $\lambda \in (0,1)$,

$$\lambda u + (1-\lambda)v \in F \;\Rightarrow\; u, v \in F.$$

An **extreme point** of C is a point $p \in C$ for which $\{p\}$ is a face of C.

Problem 1. For a convex set $C \subseteq V$, show that for any $\ell \in V^*$, the set $F = \{v \in C : \ell(v) \geq \ell(w) \text{ for all } w \in C\}$ is a face of C.

(*Bonus Problem*): Find (sufficient) conditions on C and/or V so that the following is true: If the maximum of a linear functional $\ell : V \to \mathbb{R}$ is attained on C, then it is attained by an extreme point of C.

Problem 2. For convex sets $A, B \subset V$, show that the following are also convex:

(a) $T(A)$ where W is an $\mathbb{R}$-vector space and $T : V \to W$ is a linear map

(b) $A \times B = \{(a, b) : a \in A, b \in B\} \subset V \times V$

(c) $A + B = \{a + b : a \in A, b \in B\}$

Problem 3. Suppose that $C, K \subset V$ are convex cones. Show the following:

(a) If $C \subseteq K$ then $C^* \supseteq K^*$.

(b) $(C + K)^* = C^* \cap K^*$

(c) If $L \subset V$ is a linear subspace, then the convex cone dual to L is

$$L^\perp = \{\ell \in V^* : \ell(v) = 0 \text{ for all } v \in L\}.$$

(d) If K is contained in a linear subspace L, then $L^\perp$ belongs to the *lineality space* of K^* , that is, $K^* + L^\perp = K^*$.

(e) If L belongs to the lineality space of K, i.e. $K + L = K$, then $K^* \subseteq L^\perp$.

Recall that for a subspace $L \subseteq \mathbb{R}^n$, $\mathcal{F}_L = \{A \in \mathrm{PSD_n} : L \subseteq \ker(A)\}$ is a face of PSD_n.

Problem 4. (Barvinok II.12.4.1) Let L_1, L_2 be subspaces of $\mathbb{R}^n$. Prove that $\mathcal{F}_{L_1}$ is an exposed face of $\mathcal{F}_{L_2}$ if and only if $L_2 \subset L_1$.