Math 582D – Homework 1

Due Sunday, January 19, 2025

You are welcome to talk with other students in the class about problems but should write up solutions on your own. Please indicate any sources you used to find the solution to a given problem. Solutions can be handwritten or typed but need to be legible and submitted via Gradescope by the end of the day on Sunday. You should justify all your answers in order to receive full credit.

Problem 1 (Univariate tropical polynomials)**.**

(a) (Fundamental theorem of algebra). Prove that every tropical polynomial,

$$F(x) = (c_d \odot x^{\odot d}) \oplus (c_{d-1} \odot x^{\odot d-1}) \oplus \ldots \oplus (c_1 \odot x) \oplus c_0 \quad \text{with} \quad c_d \neq \infty,$$

agrees as a function with a unique tropical product $c_d \odot \left(\bigodot_{k=1}^{d} (x \oplus r_k) \right)$ where $r_1 \leq \ldots \leq r_d$.

(b) (Quadratic formula). Find an explicit formula for this factorization when

$$F(x) = (a \odot x^{\odot 2}) \oplus (b \odot x) \oplus c.$$

Problem 2. Let $P \subset \mathbb{R}^n$ be a polytope. The *(inner) normal fan* $\mathcal{N}_P$ of P is the polyhedral fan consisting of cones

$$\mathcal{N}_P(F) = \{w \in \mathbb{R}^n : F \subseteq \text{face}_w(P)\}$$

as F runs over all faces of P.

(a) Show that $\mathcal{N}_P$ is indeed a polyhedral fan.

(b) For each of the following polytopes $P \subset \mathbb{R}^n$, compute the polyhedral *complex* whose cells are $\{w \in \mathbb{R}^{n-1} : (w, 1) \in \mathcal{N}_P(F)\}$ as F runs over the faces of P.

(i) $P = \text{conv}\{(0,1), (1,0), (2,2), (3,2)\} \subset \mathbb{R}^2$

(ii) $P = \text{conv}\{(0,0,1), (1,0,0), (2,0,1), (0,1,0), (1,1,0), (0,2,1)\} \subset \mathbb{R}^3$

Problem 3. Let K be an algebraically closed field and $I \subseteq K[x_1, \ldots, x_n]$ be an ideal. Show that the variety of I in $\mathbb{T}_K^n = (K^*)^n$ is empty if and only if I contains a monomial $\mathbf{x}^\alpha$ for some $\alpha \in \mathbb{Z}_{\geq 0}^n$.