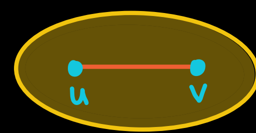
Tropical Geometry: Background

Part 1: Convexity and Polyhedra

Part 2: Polynomial Ideals and Varieties

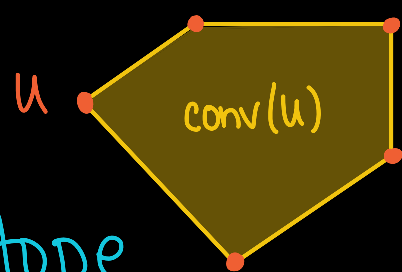
Part 1: Convexity and Polyhedra

A set $X \subseteq \mathbb{R}^n$ is convex if for all $u, v \in X$, $\lambda \in [0,1]$, $\lambda u + (1-\lambda) v \in X$



For $U \subseteq \mathbb{R}^n$, the convex hull of U is the smallest convex set containing U:

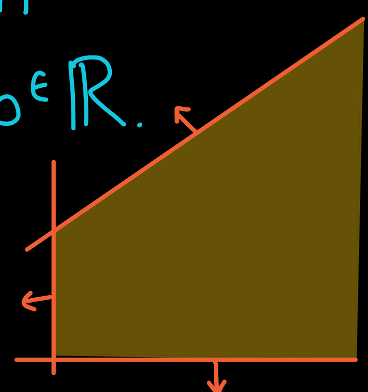
$$\mathrm{conv}(U) = \left\{ \sum_{i=1}^{k} \lambda_i u_i : k \in \mathbb{N},\ u_i \in U,\ \lambda_i \geq 0,\ \sum_{i=1}^{k} \lambda_i = 1 \right\}$$
$$= \bigcap_{C \supseteq U,\ C \text{ convex}} C$$



For $U \subseteq \mathbb{R}^n$ finite, $\mathrm{conv}(U)$ is a polytope.

A (closed) halfspace has the form $\{x \in \mathbb{R}^n : a^T x \leq b\}$ for some $a \in \mathbb{R}^n$, $b \in \mathbb{R}$.

The intersection of finitely-many half-spaces is a polyhedron.

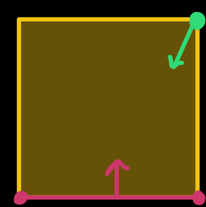


Thm: $\{\text{polytopes}\} = \{\text{bounded polyhedra}\}$

A linear function $x \mapsto w^T x$ defines a <u>face</u> of a polyhedron P, consisting of points minimizing $w^T x$:

$$\text{face}_w(P) = \{ x \in \mathbb{R}^n : w^T x \le w^T y \quad \forall y \in P \}$$

Ex: $P = \{ (x_1, x_2) \in \mathbb{R}^2 : 0 \le x_1 \le 1, 0 \le x_2 \le 1 \}$



$$\text{face}_{(-1,-2)}(P) = (1,1)$$

$$\text{face}_{(0,1)}(P) = \{ (x_1, 0) : 0 \le x_1 \le 1 \}$$

$$\text{face}_{(0,0)}(P) = P$$

The dimension of a face is the dimension of its affine span. $\dim(P) = \dim(\text{aff span}(P))$

A face of dim 0 is called a <u>vertex</u>.

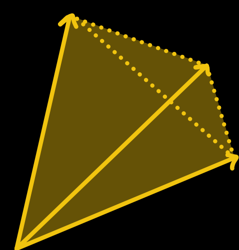
" " 1 " " <u>edge</u>

" " $\dim(P) - 1$ " " <u>facet</u>

A <u>polyhedral cone</u> is a polyhedron that is closed under nonnegative scaling ($x \in P, \lambda \ge 0 \Rightarrow \lambda x \in P$).

Any polyhedron has the form

$$P = \{ x \in \mathbb{R}^n : a_1^T x \ge 0, \ldots, a_m^T x \ge 0 \}$$

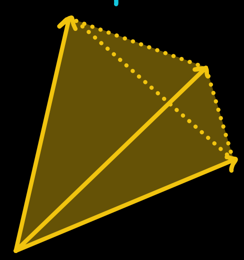


for some $a_1, \ldots, a_m \in \mathbb{R}^n$.

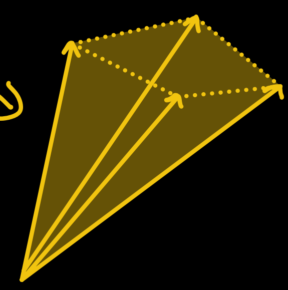
Also as $P = \text{pos}\{v_1, \dots, v_r\} = \{\sum_{i=1}^{r} \lambda_i v_i : \lambda_1, \dots, \lambda_r \geq 0\}$ for some $v_1, \dots, v_r$.

If $v_1, \dots, v_r$ can be chosen to be linearly indep. then P is <u>simplicial</u>.

Simplicial ⇝



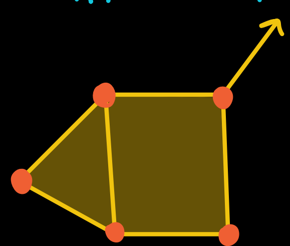
not simplicial ⇝



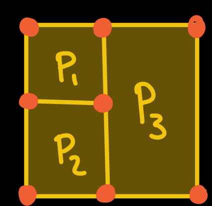
A <u>polyhedral complex</u> is a collection Σ of polyhedra satisfying

1) Every face of an elt. of Σ belongs to Σ.
2) The intersection of any two polyhedra in Σ is empty or a face of both.

Ex:



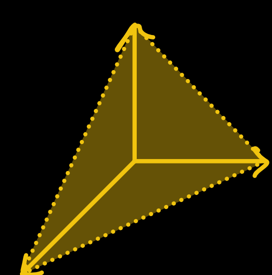
Non-ex:



$P_1 \cap P_3$ not a face of P_3

The support $|\Sigma|$ of Σ is the union of its elements: $|\Sigma| = \bigcup_{P \in \Sigma} P$.

If every elt. of Σ is a polyhedral <u>cone</u> we call Σ a polyhedral fan.



Part 2: Polynomial Ideals and Varieties

Fix a field K. Let $f_1, \dots, f_r \in R = K[x_1, \dots, x_n]$.

The ideal <u>generated</u> by $f_1, \dots, f_r$ is

$$\langle f_1, \dots, f_r \rangle = \left\{ \sum_{i=1}^{r} g_i f_i \; : \; g_i \in R \right\}.$$

<u>Thm</u>: Every ideal of $K[x_1, \dots, x_n]$ is finitely generated.

Given $S \subseteq K^n$, let $\mathcal{I}(S) = \{ f \in R : f(a) = 0 \;\; \forall a \in S \}$ denote the <u>ideal of polynomials vanishing on S</u>.

The (affine) variety of a set $\mathcal{F} \subseteq R$ of polynomials is

$$V(\mathcal{F}) = \{ a \in K^n : f(a) = 0 \text{ for all } f \in \mathcal{F} \}$$

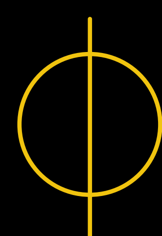
Note: $V(\{f_1, \dots, f_r\}) = V(\langle f_1, \dots, f_r \rangle)$

<u>Hilbert's Nullstellensatz</u>: If K is algebraically closed and $I \subseteq K[x_1, \dots, x_n]$ is an ideal, then

$$\mathcal{I}(V(I)) = \sqrt{I} := \{ f \in K[x_1, \dots, x_n] : f^m \in I \text{ for some } m \in \mathbb{Z} \}$$

Ex: $K = \mathbb{C}$ $\quad I = \langle (x_1)^2 (1 - x_1^2 - x_2^2)^3 \rangle$

$$\mathcal{I}(V(I)) = \sqrt{I} = \langle x_1 (1 - x_1^2 - x_2^2) \rangle$$



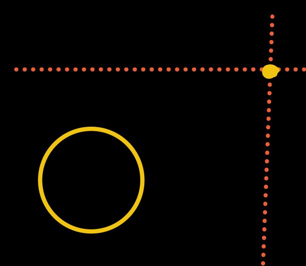
$V(I)$ (in $\mathbb{R}^2$)

A variety $X = V(I)$ is <u>irreducible</u> if it cannot be written as a union of two proper subvarieties $X_1 \cup X_2$ with $X_i \neq X$.

Every variety is the union of finitey many irreducible varieties!

Ex: $I = \langle (x_1-2)(1-x_1^2-x_2^2), (x_2-3)(1-x_1^2-x_2^2) \rangle$

$V(I) = V(\langle x_1-2, x_2-3 \rangle) \cup V(1-x_1^2-x_2^2)$



Such a decomposition $X = X_1 \cup \ldots \cup X_s$ corresponds to a <u>primary decomposition</u> of I:

$I = J_1 \cap \ldots \cap J_s$ where $J_1, \ldots, J_s$ are primary ideals.

This associates a multiplicity to each component $X_1, \ldots, X_s$ of X.

Ex: $I = \langle (x_1+1)^2 (x_1+2)(x_1+3)^4 \rangle$

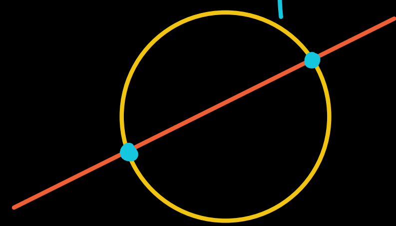
$= \langle (x_1+1)^2 \rangle \cap \langle x_1+2 \rangle \cap \langle (x_1+3)^4 \rangle$

$V(I) = V(x_1+1) \cup V(x_1+2) \cup V(x_1+3)$

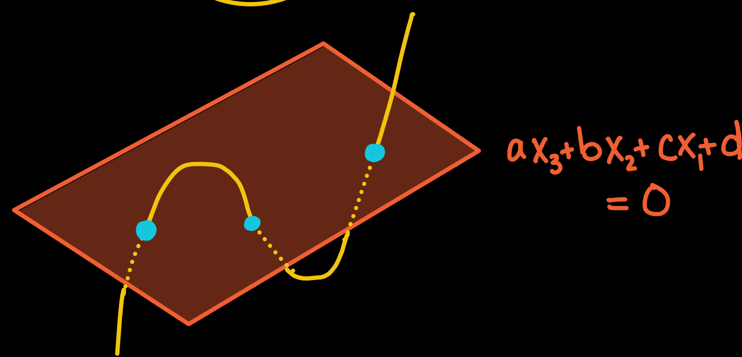
mult: 2 1 4

Every irreducible variety has a <u>dimension</u> and a <u>degree</u> (with many equiv. definitions).
If K is algebraically closed then an alg. var. $X \subseteq K^n$ has dim m and deg d if a generic $(n-m)$-dim'l affine-linear space $L \subseteq K^n$ intersects X in exactly d pts.

Ex: $V(1-x_1^2-x_2^2) \subseteq \mathbb{C}^2$
has dim 1, deg 2



Ex: $V(x_2-x_1^2,\ x_3-x_1^3) \subseteq \mathbb{C}^3$
has dim 1, deg 3



Varieties in other spaces:

<u>affine space</u> $\mathbb{A}_K^n = \{(a_1,\dots,a_n) : a_1,\dots,a_n \in K\}$

<u>projective space</u> $\mathbb{P}_K^n = \{[a_0:\dots:a_n] : [a] = a/\sim,\ a \in K^{n+1}$
$a \sim b$ if $a = \lambda b$ for some $\lambda \in K^*\}$

<u>algebraic torus</u> $\mathbb{T}_K^n = \{(a_1,\dots,a_n) \in \mathbb{A}_K^n : a_i \neq 0\ \forall i\}$

"torus" because for $K=\mathbb{C}$, $\mathbb{C}^*$ is topologically a circle S^1
$\Rightarrow \mathbb{T}_K^n = (\mathbb{C}^*)^n$ " " $(S^1)^n$ = a torus

There are natural inclusions $\mathbb{T}_K^n \hookrightarrow \mathbb{A}_K^n \hookrightarrow \mathbb{P}_K^n$
$a \mapsto a \mapsto [1:a]$

Varieties in $\mathbb{A}^n_K$ defined by $F \subseteq K[x_1,\dots,x_n]$, as above.

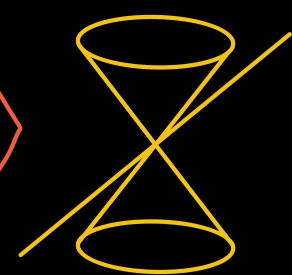
Varieties in $\mathbb{P}^n_K$ defined by homogenous polynomials.

$f = \sum_\alpha c_\alpha x^\alpha \in K[x_0,\dots,x_n]$ is **homogeneous** of degree d if $\alpha_0 + \dots + \alpha_n = d$ for all α with $c_\alpha \neq 0$.

Then $f(\lambda a) = \lambda^d f(a) \Rightarrow$ condition $f(x)=0$ well-defined on $\mathbb{P}^n_K$.

An ideal I is **homogeneous** if it is generated by homogeneous polynomials. These define **projective varieties** $V(I) \subseteq \mathbb{P}^n_K$

Ex: $I = \langle (x_1 - 2x_0)(x_0^2 - x_1^2 - x_2^2),\ (x_2 - 3x_0)(x_0^2 - x_1^2 - x_2^2) \rangle$



Varieties in $\mathbb{T}^n_K$ defined by Laurent polynomials/ideals in $K[x_1^\pm,\dots,x_n^\pm]$

$$= K[x_1, y_1, \dots, x_n, y_n] / \langle x_1 y_1 - 1, \dots, x_n y_n - 1 \rangle$$

These are called very affine varieties:

$$V(I) = \{ a \in \mathbb{T}^n_K : f(a) = 0 \text{ for } f \in I \}.$$

Ex: $\langle 1 - x_1^2 - x_2^2 \rangle = \langle x_1^{-2} x_2^{-2} - x_2^{-2} - x_1^{-2} \rangle$ in $\mathbb{C}[x_1^\pm, x_2^\pm]$