# Tropical Geometry: Intro & Basics

**What?** The study of varieties (sol'n to poly. equations) via degeneration to a polyhedral complex.

**Why?** Bridge Alg. Geom. $\longleftrightarrow$ Combinatorics

- gives combinatorial tools for algebraic problems (e.g. predicting # solutions, constructing intricate examples with desirable properties)
- gives algebraic tools for combinatorial problems (e.g. ReLU neural networks = trop. rational functions
  valuated matroids = trop. linear spaces)

Etymology: named after work of Brazilian mathematician Imre Simon (1980's)

**Tropical semiring**: $(\mathbb{R}\cup\{\infty\}, \oplus, \odot)$

For $a,b\in\mathbb{R}$, $a\oplus b=\min\{a,b\}$, $a\odot b=a+b$.

For $a\in\mathbb{R}\cup\{\infty\}$, $a\oplus\infty=\infty\oplus a=a$, $a\odot\infty=\infty\odot a=\infty$.

Ex:
$3\oplus 17=3$
$3\odot 17=20$
$(-1)\oplus 2=-1$
$(-1)\odot 2=1$

(Some authors use $a\oplus b=\max\{a,b\}$ convention with semiring $(\mathbb{R}\cup\{-\infty\},\oplus,\odot)$. Isomorphic under $a\longleftrightarrow -a$.)

additive id: $\infty$ multiplicative id: $0$

"semiring"

$(\mathbb{R}\cup\{\infty\},\oplus,\odot)$ has mult. inverses ($a\odot(-a)=0$ for $a\neq\infty$) but not additives inverses ($a\neq\infty \Rightarrow a\oplus b\neq\infty$ for any $b$)

Check: $(\mathbb{R}\cup\{\infty\}, \oplus, \odot)$ is

- associative: $(a\oplus b)\oplus c = a\oplus(b\oplus c) = \min\{a,b,c\}$

  $(a\odot b)\odot c = a\odot(b\odot c) = a+b+c$
- commutative: $a\oplus b = b\oplus a = \min\{a,b\}$, $a\odot b = b\odot a = a+b$
- distributative: $a\odot(b\oplus c) = (a\odot b)\oplus(a\odot c)$

  $$\underset{a+\min\{b,c\}}{a\odot(b\oplus c)} = \underset{\min\{a+b,\, a+c\}}{(a\odot b)\oplus(a\odot c)}$$

<u>Motivation</u>: algebra of "exponents"

<u>Interpretation 1 (log limits)</u> For $\varepsilon>0$ very small

$\varepsilon^a + \varepsilon^b \approx \varepsilon^{\min\{a,b\}}$ and $\varepsilon^a \cdot \varepsilon^b = \varepsilon^{a+b}$

ex: $\varepsilon = 1/10$, $a=-1$, $b=3$ $\quad \varepsilon^a + \varepsilon^b = \left(\frac{1}{10}\right)^{-1} + \left(\frac{1}{10}\right)^3 = 10 + 10^{-3} \approx 10$

More formally: $\lim\limits_{\varepsilon\to 0} \log_\varepsilon(\varepsilon^a + \varepsilon^b) = \min\{a,b\}$

$$\lim_{\varepsilon\to 0} \log_\varepsilon(\varepsilon^a \cdot \varepsilon^b) = a+b$$

<u>Interpretation 2 (valuations)</u>

A <u>valuation</u> on a field $K$ is a function $\mathrm{val}: K \to \mathbb{R}\cup\{\infty\}$ satisfying, for all $a,b\in K$,

1) $\mathrm{val}(a) = \infty$ iff $a=0$
2) $\mathrm{val}(ab) = \mathrm{val}(a) + \mathrm{val}(b)$, and
3) $\mathrm{val}(a+b) \geq \min\{\mathrm{val}(a), \mathrm{val}(b)\}$

Check: if $\mathrm{val}(a) \neq \mathrm{val}(b)$, then $\mathrm{val}(a+b) = \min\{\mathrm{val}(a), \mathrm{val}(b)\}$

Ex: $K=\mathbb{C}(t)$ with $\mathrm{val}\left(\frac{p(t)}{q(t)}\right) = (\text{lowest deg of } t \text{ in } p) - (\text{lowest deg of } t \text{ in } q)$

$\mathrm{val}(t+t^2)=1$ $\quad \mathrm{val}\left(\boxed{\frac{t+t^2}{t^{17}+t^{500}}}\right)=-16$ behaves like $t^{-16}$ near $t=0$

"val" measures order of vanishing/growth near $t=0$

Ex: $K=\mathbb{Q}$ with $p$-adic valuation for any prime $p$

$q\in\mathbb{Q}^* \Rightarrow q=p^k\cdot\frac{a}{b}$ where $k\in\mathbb{Z}$, $a,b\in\mathbb{Z} \not\equiv 0 \mod p$

$\Rightarrow \mathrm{val}(q)=k$.

$(p=3)$ $\mathrm{val}\left(\frac{9}{5}\right)=2$, $\mathrm{val}\left(\frac{17}{27}\right)=-3$, $\mathrm{val}\left(\frac{9}{5}+\frac{17}{27}\right)=\mathrm{val}\left(\frac{1}{27}\left(\frac{3^5+5\cdot 17}{5}\right)\right)=-3$

## Tropical polynomials

A tropical polynomial in variables $\underline{x}=(x_1,\dots,x_n)$ has the form

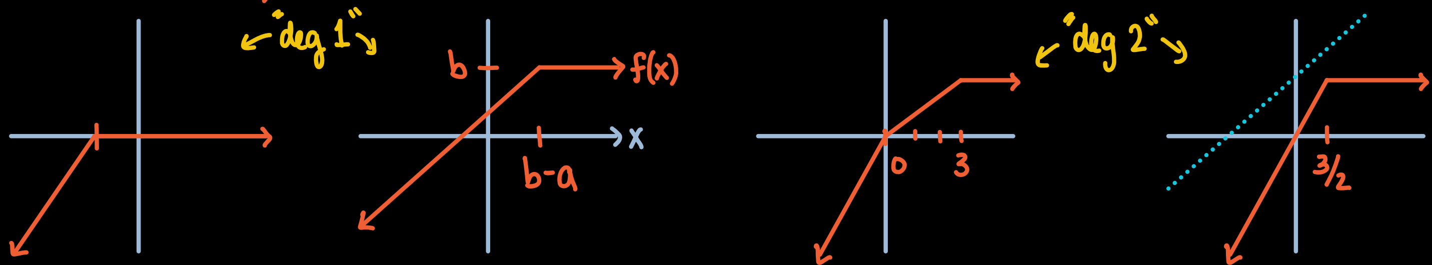
$$F(\underline{x}) = \bigoplus_{\alpha\in\mathcal{A}} \left(c_\alpha \odot x_1^{\odot\alpha_1}\odot \dots \odot x_n^{\odot\alpha_n}\right) = \bigoplus_{\alpha\in\mathcal{A}} \left(c_\alpha\odot \underline{x}^{\odot\alpha}\right)$$

$$= \min_{\alpha\in\mathcal{A}} \{ c_\alpha + \alpha_1 x_1 + \dots + \alpha_n x_n \}$$

where $\mathcal{A}\subseteq\mathbb{Z}^n_{\geq 0}$ is finite and $c_\alpha\in\mathbb{R}$ $\forall\alpha\in\mathcal{A}$.

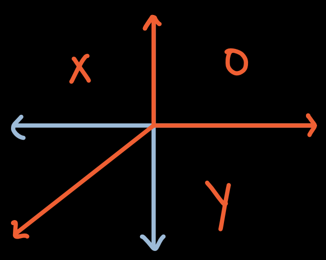
This defines a concave, piecewise-linear function $F:\mathbb{R}^n\to\mathbb{R}$.

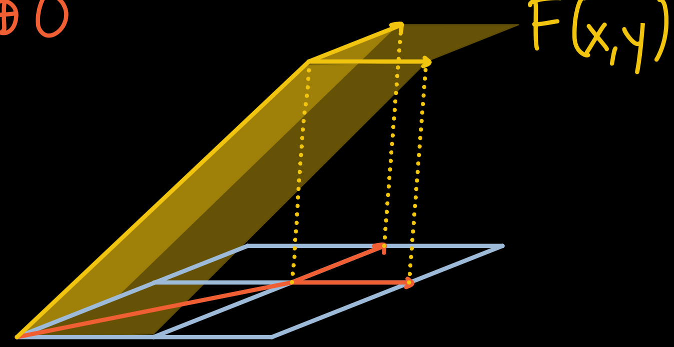
Ex's $(n=1)$




$(1\odot x)\oplus 0$ $\quad (a\odot x)\oplus b$ $\quad x^{\odot 2}\oplus x\oplus 3$ $\quad x^{\odot 2}\oplus(3\odot x)\oplus 3$

Ex $(n=2)$ $F(x,y) = x \oplus y \oplus 0$




## Solutions to polynomial equations

### (n=1) Roots of univariate polynomials

Ex: $F(x) = a \odot x \oplus b \rightsquigarrow \varepsilon^a x + \varepsilon^b = \varepsilon^a (x + \varepsilon^{b-a})$

root: $x = -\varepsilon^{b-a}$ $\quad \log_\varepsilon(|-\varepsilon^{b-a}|) = b-a$

Ex: $F(x) = x^{\odot 2} \oplus x \oplus 3 \rightsquigarrow x^2 + x + \varepsilon^3$

$= (x \oplus \underline{\underline{0}}) \odot (x \oplus \underline{\underline{3}})$

root 1: $\dfrac{-1-\sqrt{1-4\varepsilon^3}}{2} = -\varepsilon^0 + \varepsilon^3 + \varepsilon^6 + 2\varepsilon^9 + \text{h.o.t.}$

$\log_\varepsilon|\text{root } 1| \to 0$ as $\varepsilon \to 0$

root 2: $\dfrac{-1+\sqrt{1-4\varepsilon^3}}{2} = -\varepsilon^3 - \varepsilon^6 - 2\varepsilon^9 + \text{h.o.t.}$

$\log_\varepsilon|\text{root } 2| \to 3$ as $\varepsilon \to 0$

## Tropical Fundamental Theorem of Algebra

Every tropical univariate poly. $F(x) = \bigoplus_{k=0}^{d} (a_k \odot x^{\odot k})$ with $a_d \neq \infty$ coincides (as a function) with a unique product of polynomials of deg 1: $a_d \odot \bigodot_{k=1}^{d} (x \oplus r_k)$ for some $r_1 \leq \ldots \leq r_d \in \mathbb{R}$.

interpret $r_1, \ldots, r_d$ as roots

(You'll prove in hwk!)

WARNING: Different poly expressions can represent the same function!

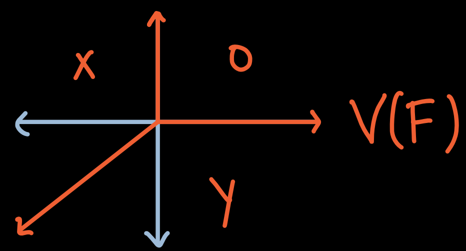
Ex: $x^{\odot 2} \oplus (3 \odot x) \oplus 3 = \min\{2x, 3\}$ as a function of x

$(x \oplus \frac{3}{2}) \odot (x \oplus \frac{3}{2}) = 2\min\{x, \frac{3}{2}\}$ "

(n arbitrary) The <u>tropical hypersurface</u> $V(F)$ of a tropical polynomial $F(\underline{x}) = \bigoplus_{\alpha\in A} c_\alpha \odot \underline{x}^{\odot\alpha}$ is the complement domains of linearity of $F$. That is

$$V(F) = \{ x \in \mathbb{R}^n : \text{the minimum } \bigoplus_{\alpha \in A} (c_\alpha \odot x^{\odot\alpha}) \text{ is is attained at least twice} \}$$

Ex $(n=2)$ $F(x,y) = x \oplus y \oplus 0$




## Fundamental Thm (for hypersurfaces)

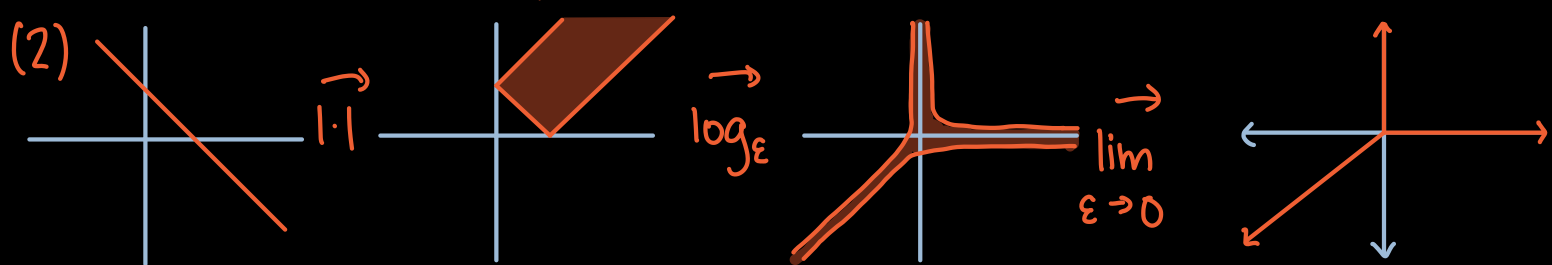
For $f_t = \sum_{\alpha\in A} c_\alpha \underline{x}^\alpha \in \mathbb{C}(t)[x_1,\dots,x_n]$, define $F(w) = \mathrm{trop}(f) = \bigoplus_{\alpha\in A} (\mathrm{val}(c_\alpha) \odot \underline{w}^{\odot\alpha})$.

Then the following sets coincide:

1) tropical hypersurface $V(F) = \{ w \in \mathbb{R}^n : \text{min attained} \geq \text{twice} \}$
2) logarithmic limit set of $V(f)$:
$$\lim_{\varepsilon \to 0} \{ (\log_\varepsilon |x_1|, \dots, \log_\varepsilon |x_n|) : \underline{x} \in (\mathbb{C}^*)^n, f_\varepsilon(\underline{x}) = 0 \}$$
3) image of $V(f)$ under val (up to closure)
$$\overline{\{ (\mathrm{val}(x_1), \dots, \mathrm{val}(x_n)) : \underline{x} \in \overline{\mathbb{C}(t)}^{alg}, f_t(\underline{x}) = 0 \}}$$
← Euclidean closure in $\mathbb{R}^n$

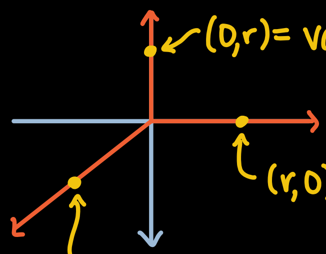
Ex $(n=2)$ $f(x,y) = x+y-1 \rightsquigarrow F(w,z) = w \oplus z \oplus 0$

(2)




$\{(x,y) : x+y-1=0\}$ $\{(x,y) : x+y-1=0\}$

(3)



$(0,r) = \mathrm{val}(1-t^r, t^r)$ for $r \geq 0$

$(r,0) = \mathrm{val}(t^r, 1-t^r)$ for $r \geq 0$

$(r,r) = \mathrm{val}(t^r, -t^r+1)$ for $r \leq 0$