Math 526 – Homework 2

Due 8:30am on Thursday, September 10, 2015

Please indicate any sources you used for a given problem on the solution to that problem. For example, if you worked with another student to get the solution to a problem, please indicate who. You are welcome to work together in small groups (2-4 people), but please try the problems on your own first.

Problem 1. A *binomial* is a polynomial $ax^\alpha + bx^\beta$ with at most two term terms. A *binomial ideal* is an ideal $\langle a_j x^{\alpha(j)} + b_j x^{\beta(j)} \rangle_{j \in J}$ generated by binomials.

(a) Prove that every binomial ideal has a Gröbner basis consisting of binomials.

(b) A trinomial is a polynomial $ax^\alpha + bx^\beta + cx^\gamma$ with at most three terms. Does every ideal generated by trinomials have a Gröbner basis of trinomials?

Problem 2. Hassett 3.3 and 3.8

Problem 3. Hassett 3.4 and 3.5

Problem 4. Consider the subset $V \subset \mathbb{R}^3$ and ideal $I \subset \mathbb{R}[x, y, z]$ defined as

$$V = \{(t, t^2, t^3) : t \in \mathbb{R}\} \quad \text{and} \quad I = \langle y - x^2, z - x^3 \rangle.$$

(a) Let $>_1$ be the graded reverse lexicographic order on $\mathbb{R}[x, y, z]$ with $x >_1 y >_1 z$ and let $>_2$ be lexicographic order with $z >_2 y >_2 x$. For $j = 1, 2$, compute a Gröbner basis and the standard monomials M_j of I with respect to $>_j$.

(b) Show that $I = \mathcal{I}(V)$.

(c) Define a linear isomorphism $\phi : \mathbb{R}\{M_1\} \to \mathbb{R}\{M_2\}$, where $\mathbb{R}\{M_j\}$ is the linear span of the monomials M_j, such that for every $f \in \mathbb{R}\{M_1\}$, $f \equiv \phi(f)$ modulo the ideal I.

Problem 5.

(a) Show that every Zariski closed subset of $\mathbb{R}^n$ is closed in the Euclidean topology and that every Zariski open subset of $\mathbb{R}^n$ is open in the Euclidean topology on $\mathbb{R}^n$.

(b) Give an example of subset of $\mathbb{R}^n$ that is closed in the Euclidean topology but not closed in the Zariski topology.

(c) Is the projection $\pi : \mathbb{R}^2 \to \mathbb{R}$ with $\pi(a_1, a_2) = a_1$ a closed map in the Zariski topology? That is, is the projection $\pi(S)$ of every Zariski closed set $S \subset \mathbb{R}^2$ Zariski closed?