Math 514 - Homework 3

Due on Thursday, October 17

You are welcome to talk with other students in the class about problems but should write up solutions on your own. Solutions can be handwritten or typed but need to be legible and submitted via Gradescope by the end of the day on Thursday. You should justify all your answers in order to receive full credit.

Problem 1. Exercise 2.23 from Chapter 2 of Schrijver's notes.

Problem 2. Exercise 2.26 from Chapter 2 of Schrijver's notes.

Problem 3. Exercise 2.27 from Chapter 2 of Schrijver's notes.

Problem 4. Let $G = (V, E)$ be a graph with no isolated vertices and let A denote the incidence matrix of G. This is a $|V| \times |E|$ matrix with entries

$$A_{v,e} = \begin{cases} 1 & \text{if } v \in e \\ 0 & \text{if } v \notin e. \end{cases}$$

Consider the linear programs

$$\begin{aligned} \alpha_{\text{LP}} &= \max\{\mathbf{1}_V^T \mathbf{x} \ : \ \mathbf{x} \in \mathbb{R}^V_{\geq 0},\ A^T \mathbf{x} \leq \mathbf{1}_E\} \\ \tau_{\text{LP}} &= \min\{\mathbf{1}_V^T \mathbf{y} \ : \ \mathbf{y} \in \mathbb{R}^V_{\geq 0},\ \mathbf{y}^T A \geq \mathbf{1}_E^T\} \\ \nu_{\text{LP}} &= \max\{\mathbf{1}_E^T \mathbf{x} \ : \ \mathbf{x} \in \mathbb{R}^E_{\geq 0},\ A\mathbf{x} \leq \mathbf{1}_V\} \\ \rho_{\text{LP}} &= \min\{\mathbf{1}_E^T \mathbf{y} \ : \ \mathbf{y} \in \mathbb{R}^E_{\geq 0},\ \mathbf{y}^T A^T \geq \mathbf{1}_V^T\} \end{aligned}$$

where $\mathbf{1}_V$ and $\mathbf{1}_E$ denote the all-ones vectors in $\mathbb{R}^V$ and $\mathbb{R}^E$, respectively.

(i) Show that $\alpha(G) \leq \alpha_{\text{LP}} = \rho_{\text{LP}} \leq \rho(G)$.

(ii) Show that $\nu(G) \leq \nu_{\text{LP}} = \tau_{\text{LP}} \leq \tau(G)$.

(iii) Give an example of a graph G for which all of the inequalities in (i) and (ii) are strict.