MA/AMA 514

Today: Matroid Intersection (§10.4, §10.5)

Recall: Independent set matroid axioms $M=(X,\mathcal{I})$

(1) $\mathcal{I}$ nonempty collection of subsets of X

(2) closed under containment ($T\in\mathcal{I}$, $S\subseteq T \Rightarrow S\in\mathcal{I}$)

(3) $S,T\in\mathcal{I}$, $|T|>|S| \Rightarrow \exists t\in T\setminus S$ with $S\cup\{t\}\in\mathcal{I}$

Ex (graphic) $G=(V,E)$ $\mathcal{I}=\{F\subseteq E : (V,F) \text{ acyclic}\}$

Ex (linear) $v_1,\ldots,v_n\in\mathbb{R}^d$ $\mathcal{I}=\{S\subseteq[n] : \{v_i : i\in S\} \text{ lin. indep}\}$

Matroid intersection

Given two matroids on the same groundset, find the largest common indep. set.

Input: $M_1=(X,\mathcal{I}_1)$ and $M_2=(X,\mathcal{I}_2)$

Goal: $\max\{|I| : I\in\mathcal{I}_1\cap\mathcal{I}_2\}$

This can encode the problems of bipartite matching, rainbow forests and edge-disjoint spanning trees.

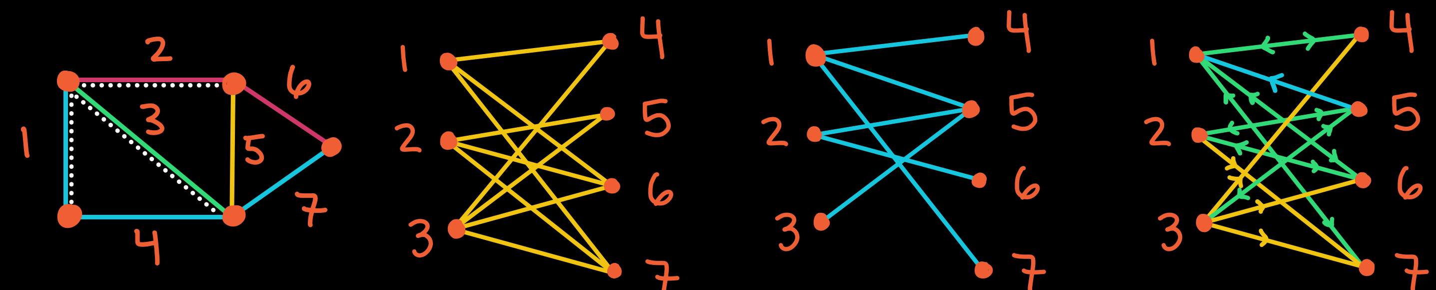
Edmond's Alg for matroid intersection

Given $M_1=(X,\mathcal{I}_1)$, $M_2=(X,\mathcal{I}_2)$ and $Y\in\mathcal{I}_1\cap\mathcal{I}_2$ define directed bipartite graph $H_{M_1,M_2}(Y)$ with vertices $X=Y\uplus X\setminus Y$, arcs

(y,x) if $(Y \setminus \{y\}) \cup \{x\} \in \mathcal{I}_1$ $\left(\Leftrightarrow \{y,x\} \in H_{M_1}(Y)\right)$

(x,y) if $(Y \setminus \{y\}) \cup \{x\} \in \mathcal{I}_2$ $\left(\Leftrightarrow \{y,x\} \in H_{M_2}(Y)\right)$

Ex: M_1 = graphic matroid M_2 = partition matroid of colors



$Y = 123$ $H_{M_1}(Y)$ $H_{M_2}(Y)$ $H_{M_1,M_2}(Y)$

Define $X_1 = \{x \in X \setminus Y \text{ s.t. } Y \cup \{x\} \in \mathcal{I}_1\}$ $= \{6,7\}$ in ex

$X_2 = \{x \in X \setminus Y \text{ s.t. } Y \cup \{x\} \in \mathcal{I}_2\}$ $= \{5\}$ in ex

Algorithm:

Input: $M_1 = (X, \mathcal{I}_1)$, $M_2 = (X, \mathcal{I}_2)$, $Y \in \mathcal{I}_1 \cap \mathcal{I}_2$

Output: $Y' \in \mathcal{I}_1 \cap \mathcal{I}_2$ with $|Y'| > |Y|$, if it exists

Find the shortest directed path P from X_1 to X_2, if one exists.

Then $P: x_0 \to y_1 \to x_1 \to \dots \to y_m \to x_m$ where $y_j \in Y$, $x_j \in X$, $x_0 \in X_1$, $x_m \in X_2$.

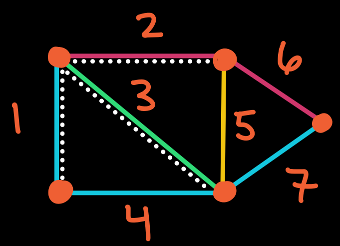
Output $Y' = (Y \setminus \{y_1, \dots, y_m\}) \cup \{x_0, \dots, x_m\}$

(If $X_1 \cap X_2$ is nonempty, say $x \in X_1 \cap X_2$, the we take P to be the path of length 0 and $Y' = Y \cup \{x\}$.)

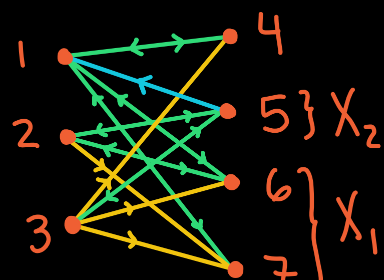
Claim 1: If P exists, $Y' \in \mathcal{I}_1 \cap \mathcal{I}_2$

Claim 2: If no such P exists, $|Y| = \max\{|I| : I \in \mathcal{I}_1 \cap \mathcal{I}_2\}$

Ex: M_1 = graphic matroid $\quad M_2$ = partition matroid of colors



$Y = 123$

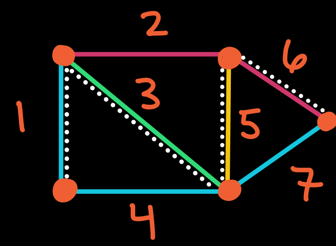


$H_{M_1,M_2}(Y)$

$P: 6 \to 2 \to 5$ shortest X_1–X_2 path

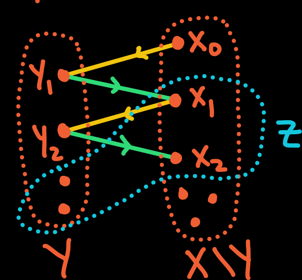
$x_0 \to y_1 \to x_1$

$Y' = (Y \setminus 2) \cup 56$
$= 1356$



(Proof of Claim 1) We will show $Y' \in \mathcal{I}_1$. ($Y' \in \mathcal{I}_2$ symmetrically)

Take $Z = (Y \setminus \{y_1, \dots, y_m\}) \cup \{x_1, \dots, x_m\} = Y' \setminus \{x_0\}$.



The edges $\{y_i, x_i\}$ for $i = 1, \dots, m$ are a perfect matching on $Y \Delta Z$. The graph $H_{M_1}(Y)$ on $Y \Delta Z$ has no edges $\{y_i, x_j\}$ with $j > i$, as this would give a shorter path from X_1 to X_2. It follows that there is a unique matching on $Y \Delta Z$. By Lemma 10.2, $Z = Y' \setminus \{x_0\}$ is independent.

Since $x_0 \in X_1$, $Y \cup \{x_0\} \in \mathcal{I}_1 \Rightarrow \text{rank}_{M_1}(Y \cup Z \cup \{x_0\}) \geq |Y| + 1$.

However $\text{rank}_{M_1}(Y \cup Z) = |Y| = |Z|$ since $Y \cup \{x_i\} \notin \mathcal{I}_1$ for $i = 1, \dots, m$.
(This would give a shorter X_1-X_2 path!)

There must be some $z \in (Y \cup \{x_0\}) \setminus Z$ for which $Z \cup \{z\} \in \mathcal{I}_1$.
Since $\text{rank}_{M_1}(Y \cup Z) = |Z|$, the only possibility in $z = x_0$. $\square$

<u>Claim 2</u>: If no such P exists, $|Y| = \max\{|I| : I \in \mathcal{I}_1 \cap \mathcal{I}_2\}$

Lemma: For any $Y \in \mathcal{I}_1 \cap \mathcal{I}_2$ and $U \subseteq X$,

$$|Y| \leq \text{rank}_{M_1}(U) + \text{rank}_{M_2}(X \setminus U).$$

"=" implies $|Y| = \max\{|I| : I \in \mathcal{I}_1 \cap \mathcal{I}_2\}$

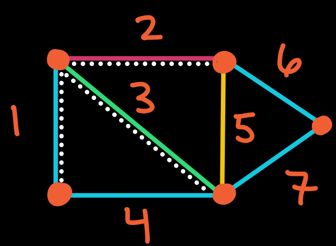
(Proof) $|Y| = |\underbrace{Y \cap U}_{\in \mathcal{I}_1}| + |\underbrace{Y \cap (X \setminus U)}_{\in \mathcal{I}_2}| \leq \text{rank}_{M_1}(U) + \text{rank}_{M_2}(X \setminus U)$

$\square$ Lemma

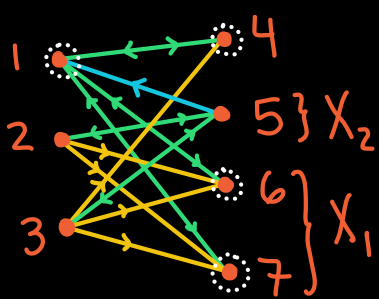
(Idea of proof of Claim 2) If there is no X_1-X_2 path take $U = \{z \in X : z \text{ not reachable from } X_1\}$.

One can show $\text{rank}_{M_1}(U) = |Y \cap U|$ and $\text{rank}_{M_2}(X \setminus U) = |Y \cap (X \setminus U)|$. See notes for details. $\square$

Ex: M_1 = graphic matroid M_2 = partition matroid of colors



$Y = 123$



$H_{M_1,M_2}(Y)$

no X_1-X_2 path!

reachable from X_1: 1 4 6 7

$\rightarrow U = 235 \quad X \setminus U = 1467$

$Y \cap U = 23 \quad Y \cap (X \setminus U) = 1$

$\text{rank}_{M_1}(235) = |\{2,3\}| \quad \text{rank}_{M_2}(1467) = |\{1\}|$

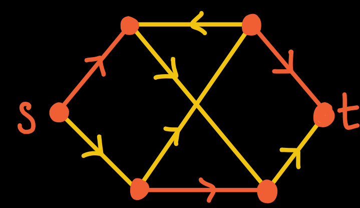
Then $|Y| = \text{rank}_{M_1}(U) + \text{rank}_{M_2}(X \setminus U) \Rightarrow Y$ optimal!

Intersection of 3 matroids

Thm: Given matroids $M_1 = (X, \mathcal{I}_1)$, $M_2 = (X, \mathcal{I}_2)$, $M_3 = (X, \mathcal{I}_3)$ it is NP-Hard to find $\max\{|I| : I \in \mathcal{I}_1 \cap \mathcal{I}_2 \cap \mathcal{I}_3\}$

(Proof via reduction to Hamiltonia path)

Let $D=(V,A)$ and $s,t\in V$. It is NP-hard to decide if there exists an s-t path in D that visits every vertex exactly once.



Ham. s-t path

Define matroids:

$M_1=(A,\mathcal{I}_1)$ = graphic matroid of (undir) graph

$M_2=(A,\mathcal{I}_2)$ = partition matroid with

$\mathcal{I}_2=\{A'\subseteq A : |\delta^{out}(v)\cap A'|\le 1\ \forall v\neq t, |\delta^{out}(t)\cap A'|=0\}$

$M_3=(A,\mathcal{I}_3)$ = partition matroid with

$\mathcal{I}_3=\{A'\subseteq A : |\delta^{in}(v)\cap A'|\le 1\ \forall v\neq s, |\delta^{in}(s)\cap A'|=0\}$

D has a Ham. s-t path $\Leftrightarrow \max\{|A'| : A'\in\mathcal{I}_1\cap\mathcal{I}_2\cap\mathcal{I}_3\}=|V|-1$.