MA/AMA 514

Today: Matroid Intersection (§10.4, §10.5)

Recall: Independent set matroid axioms $M=(X,\mathcal{I})$

(1) $\mathcal{I}$ nonempty collection of subsets of X

(2) closed under containment ($T\in\mathcal{I}$, $S\subseteq T \Rightarrow S\in\mathcal{I}$)

(3) $S,T\in\mathcal{I}$, $|T|>|S| \Rightarrow \exists t\in T\setminus S$ with $S\cup\{t\}\in\mathcal{I}$

Ex (graphic) $G=(V,E)$ $\mathcal{I}=\{F\subseteq E : (V,F) \text{ acyclic}\}$

Ex (linear) $v_1,\dots,v_n\in\mathbb{R}^d$ $\mathcal{I}=\{S\subseteq[n] : \{v_i : i\in S\} \text{ lin. indep}\}$

Matroid intersection

Given two matroids on the same groundset, find the largest common indep. set.

Input: $M_1=(X,\mathcal{I}_1)$ and $M_2=(X,\mathcal{I}_2)$

Goal: $\max\{|I| : I\in\mathcal{I}_1\cap\mathcal{I}_2\}$

This can encode the problems of bipartite matching, rainbow forests and edge-disjoint spanning trees.

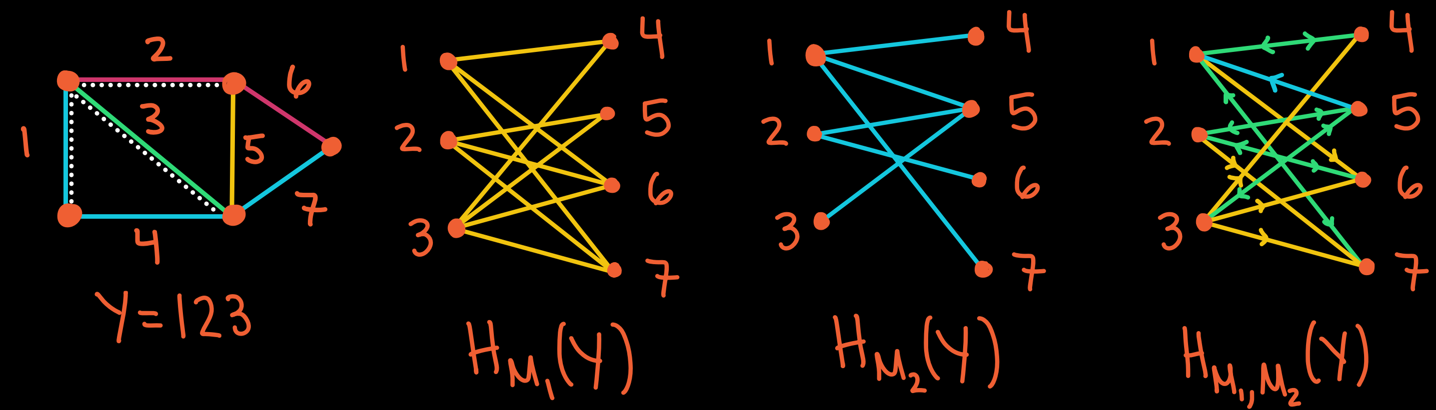
Edmond's Alg for matroid intersection

Given $M_1=(X,\mathcal{I}_1)$, $M_2=(X,\mathcal{I}_2)$ and $Y\in\mathcal{I}_1\cap\mathcal{I}_2$ define directed bipartite graph $H_{M_1,M_2}(Y)$ with vertices $X=Y\uplus X\setminus Y$, arcs

(y,x) if $(Y\setminus\{y\})\cup\{x\}\in\mathcal{I}_1$ $(\Leftrightarrow \{y,x\}\in H_{M_1}(Y))$

(x,y) if $(Y\setminus\{y\})\cup\{x\}\in\mathcal{I}_2$ $(\Leftrightarrow \{y,x\}\in H_{M_2}(Y))$

Ex: M_1 = graphic matroid M_2 = partition matroid of colors



Define $X_1=\{x\in X\setminus Y \text{ s.t. } Y\cup\{x\}\in\mathcal{I}_1\}$ $=\{6,7\}$ in ex

$X_2=\{x\in X\setminus Y \text{ s.t. } Y\cup\{x\}\in\mathcal{I}_2\}$ $=\{5\}$ in ex

<u>Algorithm</u>:

Input: $M_1=(X,\mathcal{I}_1)$, $M_2=(X,\mathcal{I}_2)$, $Y\in\mathcal{I}_1\cap\mathcal{I}_2$

Output: $Y'\in\mathcal{I}_1\cap\mathcal{I}_2$ with $|Y'|>|Y|$, if it exists

Find the shortest directed path P from X_1 to X_2, if one exists.

Then $P: x_0\to y_1\to x_1\to\dots\to y_m\to x_m$ where $y_j\in Y$, $x_j\in X$, $x_0\in X_1$, $x_m\in X_2$.

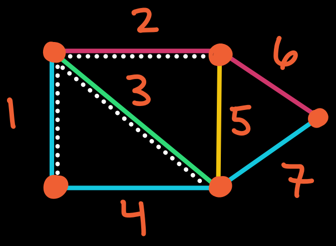
Output $Y'=(Y\setminus\{y_1,\dots,y_m\})\cup\{x_0,\dots,x_m\}$

(If $X_1\cap X_2$ is nonempty, say $x\in X_1\cap X_2$, the we take P to be the path of length 0 and $Y'=Y\cup\{x\}$.)

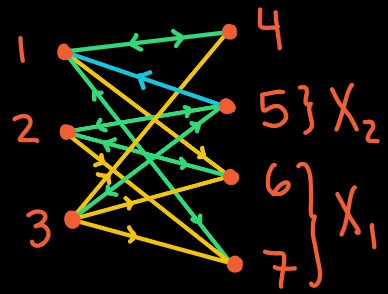
<u>Claim 1</u>: If P exists, $Y'\in\mathcal{I}_1\cap\mathcal{I}_2$

<u>Claim 2</u>: If no such P exists, $|Y|=\max\{|I|:I\in\mathcal{I}_1\cap\mathcal{I}_2\}$

Ex: M_1 = graphic matroid $\quad M_2$ = partition matroid of colors



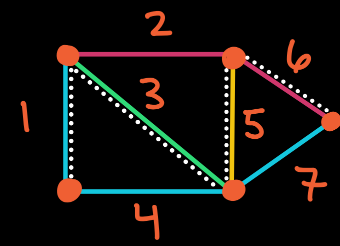
$Y = 123$



$H_{M_1,M_2}(Y)$

P: $6 \to 2 \to 5$ shortest X_1–X_2 path
$x_0 \to y_1 \to x_1$

$Y' = (Y \setminus 2) \cup 56$
$= 1356$



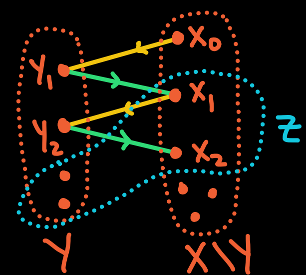
Need two technical lemmas about $H_M(Y)$:
Let $M = (X, \mathcal{I})$ be a matroid and $Y \in \mathcal{I}$.

<u>Lemma 10.1</u> If $Y, Z \in \mathcal{I}$ with $|Y| = |Z|$, then $H_M(Y)$ has a perfect matching on $Y \triangle Z$.

<u>Lemma 10.2</u> Let $Y \in \mathcal{I}$ and $Z \subseteq X$ with $|Y| = |Z|$. If $H_M(Y)$ contains a unique perfect matching on $Y \triangle Z$ then $Z \in \mathcal{I}$.

(See §10.4 for proofs)

(Proof of Claim 1) We will show $Y' \in \mathcal{I}_1$. ($Y' \in \mathcal{I}_2$ symmetrically)
Take $Z = (Y \setminus \{y_1, \ldots, y_m\}) \cup \{x_1, \ldots, x_m\} = Y' \setminus \{x_0\}$.
The edges $\{y_i, x_i\}$ for $i = 1, \ldots, m$ are a perfect matching on $Y \triangle Z$. The graph $H_{M_1}(Y)$ on $Y \triangle Z$ has no edges $\{y_i, x_j\}$ with $j > i$, as this would give a shorter path from X_1 to X_2. It follows that there is a unique matching on $Y \triangle Z$. By Lemma 10.2, $Z = Y' \setminus \{x_0\}$ is independent.



Since $x_0 \in X_1$, $Y \cup \{x_0\} \in \mathcal{I}_1 \Rightarrow \text{rank}_{M_1}(Y \cup Z \cup \{x_0\}) \geq |Y|+1$.
However $\text{rank}_{M_1}(Y \cup Z) = |Y| = |Z|$ since $Y \cup \{x_i\} \notin \mathcal{I}_1$ for $i=1,\dots,m$.
(This would give a shorter X_1-X_2 path!)
There must be some $z \in (Y \cup \{x_0\}) \setminus Z$ for which $Z \cup \{z\} \in \mathcal{I}_1$.
Since $\text{rank}_{M_1}(Y \cup Z) = |Z|$, the only possibility in $z = x_0$. □

<u>Claim 2</u>: If no such P exists, $|Y| = \max\{|I| : I \in \mathcal{I}_1 \cap \mathcal{I}_2\}$

Lemma: For any $Y \in \mathcal{I}_1 \cap \mathcal{I}_2$ and $U \subseteq X$,

$$|Y| \leq \text{rank}_{M_1}(U) + \text{rank}_{M_2}(X \setminus U).$$

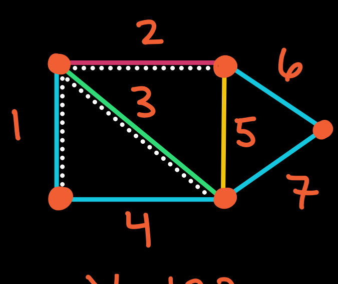
↳ "=" implies $|Y| = \max\{|I| : I \in \mathcal{I}_1 \cap \mathcal{I}_2\}$

(Proof) $|Y| = \underbrace{|Y \cap U|}_{\in \mathcal{I}_1} + \underbrace{|Y \cap (X \setminus U)|}_{\in \mathcal{I}_2} \leq \text{rank}_{M_1}(U) + \text{rank}_{M_2}(X \setminus U)$

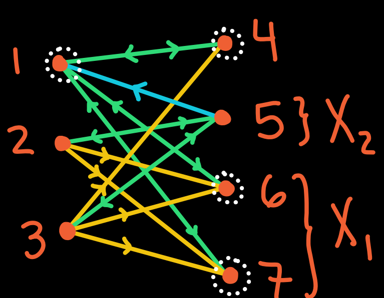
□Lemma

(Idea of proof of Claim 2) If there is no X_1-X_2 path take $U = \{z \in X : z \text{ not reachable from } X_1\}$.
One can show $\text{rank}_{M_1}(U) = |Y \cap U|$ and $\text{rank}_{M_2}(X \setminus U) = |Y \cap (X \setminus U)|$. See notes for details. □

Ex: M_1 = graphic matroid M_2 = partition matroid of colors



$Y = 123$



$H_{M_1,M_2}(Y)$

no X_1-X_2 path!
reachable from X_1: 1467
$\rightarrow U = 235$ $X \setminus U = 1467$
$Y \cap U = 23$ $Y \cap (X \setminus U) = 1$

$\mathrm{rank}_{M_1}(235) = |\{2,3\}|$ $\mathrm{rank}_{M_2}(1467) = |\{1\}|$

$\Rightarrow$ Any $Y' \in \mathcal{I}_1 \cap \mathcal{I}_2$ has ≤ 2 elt from $\{2,3,5\}$ and ≤ 1 from $\{1,4,6,7\}$

Then $|Y| = \mathrm{rank}_{M_1}(U) + \mathrm{rank}_{M_2}(X \setminus U)$ $\Rightarrow Y$ optimal!