MA/AMA 514

Today: Matroid Intersection (§10.4, §10.5)

Recall: Independent set matroid axioms $M=(X,\mathcal{I})$

(1) $\mathcal{I}$ nonempty collection of subsets of X

(2) closed under containment ($T\in\mathcal{I},\ S\subseteq T \Rightarrow S\in\mathcal{I}$)

(3) $S,T\in\mathcal{I},\ |T|>|S| \Rightarrow \exists t\in T\setminus S$ with $S\cup\{t\}\in\mathcal{I}$

Ex (graphic) $G=(V,E)$ $\mathcal{I}=\{F\subseteq E : (V,F) \text{ acyclic}\}$

Ex (linear) $v_1,\dots,v_n\in\mathbb{R}^d$ $\mathcal{I}=\{S\subseteq[n] : \{v_i : i\in S\} \text{ lin. indep}\}$

Partition Matroids ground set X, partition $X=A_1\uplus\dots\uplus A_m$

$M=(X,\mathcal{I})$ $\mathcal{I}=\{S\subseteq X \text{ s.t. } |S\cap A_i|\le 1 \text{ for all } i\}$ $\Leftrightarrow$ $\exists$ bijection $f: S\to[m]$ with $s\in A_{f(s)}$ $\forall s\in S$

= transversal matroid with sets $A_1,\dots,A_m$

Matroid intersection

Given two matroids on the same groundset, find the largest common indep. set.

Input: $M_1=(X,\mathcal{I}_1)$ and $M_2=(X,\mathcal{I}_2)$

Goal: $\max\{|I| : I\in\mathcal{I}_1\cap\mathcal{I}_2\}$

Ex (bipartite matching) $G=(U\uplus W,E)$ bipartite graph

ground set: $X=E$

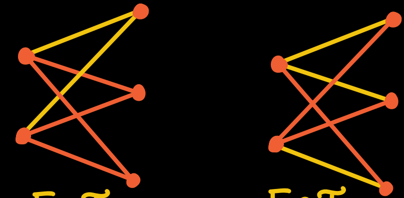
M_1 = transversal matroid with sets $A_u=\delta(u)$ for $u\in U$

M_2 = " " $A_w=\delta(w)$ for $w\in W$

$F \in \mathcal{I}_1 \Leftrightarrow |F \cap \delta(u)| \le 1$ for all $u \in U$

$F \in \mathcal{I}_2 \Leftrightarrow |F \cap \delta(w)| \le 1$ for all $w \in W$

$F \in \mathcal{I}_1 \cap \mathcal{I}_2 \Leftrightarrow F$ matching



$F \in \mathcal{I}_1$ $\quad$ $F \in \mathcal{I}_2$

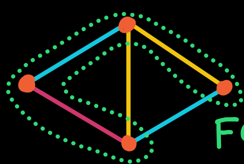
Ex (rainbow forests) $G=(V,E)$ connected graph

Edges E colored with $|V|-1$ colors

$M_1 = (E, \mathcal{I}_1)$ graphic matroid of G $(\mathcal{I}_1 = \{\text{forests}\})$

$M_2 = (E, \mathcal{I}_2)$ trans. matroid with $A_i = \{\text{edges with color } i\}$

$F \in \mathcal{I}_2 \Leftrightarrow F$ has at most one edge of each color



$F \in \mathcal{I}_1 \cap \mathcal{I}_2$

$F \in \mathcal{I}_1 \cap \mathcal{I}_2 \Leftrightarrow F$ is a "rainbow" forest

Ex (disjoint spanning trees) $G=(V,E)$ connected graph

$M_1 = (E, \mathcal{I}_1)$ graphic matroid of G $(\mathcal{I}_1 = \{\text{forests}\})$

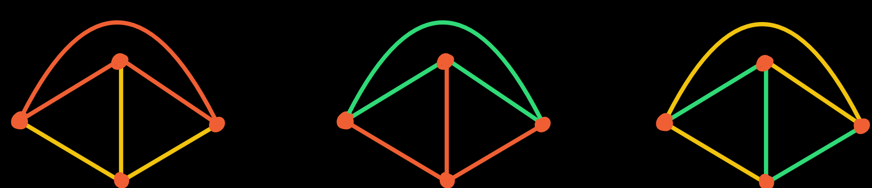
$M_2 = (E, \mathcal{I}_2)$ cographic matroid of G $(\mathcal{I}_2 = \{S \subseteq E : S \subseteq E \setminus T \text{ for some span. tree } T\})$

$= \{S \subseteq E : (V, E \setminus S) \text{ connected}\}$

G has two edge-disjoint spanning trees

$\Leftrightarrow \max\{|I| : I \in \mathcal{I}_1 \cap \mathcal{I}_2\} = |V|-1$

Ex.



$F \in \mathcal{I}_1 \setminus \mathcal{I}_2$ $\quad$ $F \in \mathcal{I}_2 \setminus \mathcal{I}_1$ $\quad$ $F \in \mathcal{I}_1 \cap \mathcal{I}_2$

We will give a polynomial time algorithm for matroid intersection. First we need two technical lemmas:

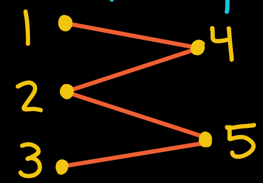
Two exchange lemmas (§10.4)

Let $M=(X,\mathcal{I})$ be a matroid and $Y\in\mathcal{I}$.

Define $H_M(Y)$ to be the bipartite graph with vertices $Y \uplus (X\setminus Y)$ and edges $\{y,x\}$ for $y\in Y$, $x\in X\setminus Y$ with $Y\setminus\{y\}\cup\{x\}\in\mathcal{I}$

Ex: $M=$ graphic matroid

$Y=\{1,2,3\}$

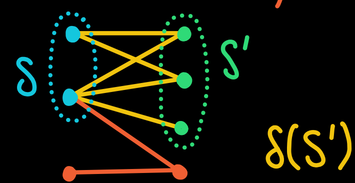


$Y\setminus\{1\}\cup\{5\}$
$Y\setminus\{3\}\cup\{4\}$
not indep.

Lemma 10.1 If $Y,Z\in\mathcal{I}$ with $|Y|=|Z|$, then $H_M(Y)$ has a perfect matching on $Y\triangle Z$.

(Proof) Suppose not. By Hall's marriage theorem, $\exists\, S\subseteq Y\setminus Z$ and $S'\subseteq Z\setminus Y$ with $|S|<|S'|$ and $\delta(S')\subseteq S$. Since $(Y\cap Z)\cup S\subseteq Y$ and $(Y\cap Z)\cup S'\subseteq Z$, both are in $\mathcal{I}$ and $\exists\, z\in S'$ s.t.

$$T:=(Y\cap Z)\cup S\cup\{z\}\in\mathcal{I}.$$



Similarly, we can add elt's of Y to T until we get $\exists\, U\in\mathcal{I}$ with $T\subseteq U\subseteq T\cup Y$ and $|U|=|Y|$.

$\Rightarrow$ for some $y\in Y\setminus T$, $U=(Y\setminus\{y\})\cup\{z\}\Rightarrow\{y,z\}$ an edge. $\divideontimes$

$(y\notin S,\ z\in S')$

Lemma 10.2 Let $Y\in\mathcal{I}$ and $Z\subseteq X$ with $|Y|=|Z|$. If $H_M(Y)$ contains a unique perfect matching on $Y\triangle Z$ then $Z\in\mathcal{I}$.

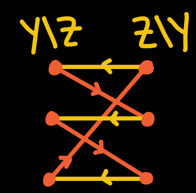
(Proof) By induction on $k=|Z\setminus Y|$. $(k=0)$ $Z=Y\in\mathcal{I}$.

$(k\geq 1)$ Let N be the unique perfect matching on $Y\Delta Z$.

Claim: the restriction of $H_M(Y)$ to $Y\Delta Z$ has a leaf $y\in Y\setminus Z$.

That is $\exists y\in Y\setminus Z$ s.t. $|\{z\in Z\setminus Y : Y\setminus\{y\}\cup\{z\}\in\mathcal{I}\}|=1$.

(Proof) Define walk on restriction. Start at any $w\in Y\Delta Z$.

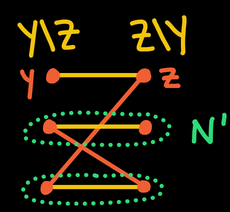


At $z\in Z\setminus Y$, walk along edge $\{z,y\}\in N$.

At $y\in Y\setminus Z$, walk along edge $\{y,z\}\notin N$, if it exists.

Either walk terminates at a leaf $y\in Y\setminus Z$ or results in an even cycle $C \Rightarrow N\Delta C$ is another perfect matching on $Y\Delta Z$. □ Claim

Let $y\in Y\setminus Z$ be a leaf and $z\in Z\setminus Y$ with $\{y,z\}\in N$.



Take $Z'=(Z\setminus\{z\})\cup\{y\}$ and $N'=N\setminus\{\{y,z\}\}$.

Then $|Y\Delta Z'|<|Y\Delta Z|$ and N' is unique matching on $Y\Delta Z'$ $\Rightarrow$ $Z'\in\mathcal{I}$ (by induction).

Know: $(Y\setminus\{y\})\cup\{z\}\in\mathcal{I}$ since $\{y,z\}\in H_M(Y)$.

We can add an elt. $w\in Y\setminus\{y\}\cup\{z\}$ to $Z'\setminus\{y\}\in\mathcal{I}$ s.t. $S=(Z'\setminus\{y\})\cup\{w\}\in\mathcal{I}$. If $w=z$, $Z=S\in\mathcal{I}$. ✓

Otherwise $w\in Y\setminus\{y\}$. Since $S\in\mathcal{I}$ with $|S|>|Y\setminus\{y\}|$, $\exists z'\in S\setminus Y$ s.t. $(Y\setminus\{y\})\cup\{z'\}\in\mathcal{I}$, contradicting y a leaf!

Edmond's Alg for matroid intersection

Given $M_1 = (X, \mathcal{I}_1)$, $M_2 = (X, \mathcal{I}_2)$ and $Y \in \mathcal{I}_1 \cap \mathcal{I}_2$ define directed bipartite graph $H_{M_1, M_2}(Y)$ with vertices $X = Y \uplus X \setminus Y$, arcs

(y, x) if $(Y \setminus \{y\}) \cup \{x\} \in \mathcal{I}_1$ $\left(\Leftrightarrow \{y, x\} \in H_{M_1}(Y)\right)$

(x, y) if $(Y \setminus \{y\}) \cup \{x\} \in \mathcal{I}_2$ $\left(\Leftrightarrow \{y, x\} \in H_{M_2}(Y)\right)$

Idea: look for directed path in $H_{M_1, M_2}(Y)$, use it to "augment" Y to a larger set $Y' \in \mathcal{I}_1 \cap \mathcal{I}_2$.