MA/AMA 514

Today: Matroid Intersection (§10.4, §10.5)

Recall: Independent set matroid axioms $M=(X,\mathcal{I})$

(1) $\mathcal{I}$ nonempty collection of subsets of X

(2) closed under containment ($T\in\mathcal{I},\ S\subseteq T \Rightarrow S\in\mathcal{I}$)

(3) $S,T\in\mathcal{I},\ |T|>|S| \Rightarrow \exists t\in T\setminus S$ with $S\cup\{t\}\in\mathcal{I}$

Ex (graphic) $G=(V,E)$ $\mathcal{I}=\{F\subseteq E : (V,F) \text{ acyclic}\}$

Ex (linear) $v_1,\dots,v_n\in\mathbb{R}^d$ $\mathcal{I}=\{S\subseteq[n] : \{v_i : i\in S\} \text{ lin. indep}\}$

<u>Partition Matroids</u> ground set X, partition $X=A_1\uplus\dots\uplus A_m$

$M=(X,\mathcal{I})$ $\mathcal{I}=\{S\subseteq X \text{ s.t. } |S\cap A_i|\leq 1 \text{ for all } i\}$

$=$ transversal matroid with sets $A_1,\dots,A_m$

$\Leftrightarrow \exists$ injection $f: S\to[m]$ with $s\in A_{f(s)}$ $\forall s\in S$

<u>Matroid intersection</u>

Given two matroids on the same groundset, find the largest common indep. set.

<u>Input</u>: $M_1=(X,\mathcal{I}_1)$ and $M_2=(X,\mathcal{I}_2)$

<u>Goal</u>: $\max\{|I| : I\in\mathcal{I}_1\cap\mathcal{I}_2\}$

Ex (bipartite matching) $G=(U\uplus W,E)$ bipartite graph

ground set: $X=E$

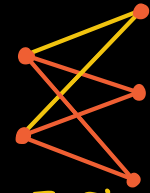
$M_1=$ transversal matroid with sets $A_u=\delta(u)$ for $u\in U$

$M_2=$ " " $A_w=\delta(w)$ for $w\in W$

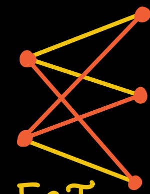
$F \in \mathcal{I}_1 \iff |F \cap \delta(u)| \leq 1$ for all $u \in U$

$F \in \mathcal{I}_2 \iff |F \cap \delta(w)| \leq 1$ for all $w \in W$

$F \in \mathcal{I}_1 \cap \mathcal{I}_2 \iff F$ matching



$F \in \mathcal{I}_1$



$F \in \mathcal{I}_2$

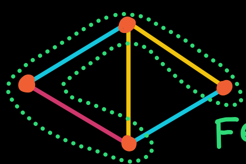
Ex (rainbow forests) $G=(V,E)$ connected graph

Edges E colored with $|V|-1$ colors

$M_1=(E,\mathcal{I}_1)$ graphic matroid of G $(\mathcal{I}_1=\{\text{forests}\})$

$M_2=(E,\mathcal{I}_2)$ trans. matroid with $A_i=\{\text{edges with color } i\}$

$F \in \mathcal{I}_2 \iff F$ has at most one edge of each color



$F \in \mathcal{I}_1 \cap \mathcal{I}_2$

$F \in \mathcal{I}_1 \cap \mathcal{I}_2 \iff F$ is a "rainbow" forest

Ex (disjoint spanning trees) $G=(V,E)$ connected graph

$M_1=(E,\mathcal{I}_1)$ graphic matroid of G $(\mathcal{I}_1=\{\text{forests}\})$

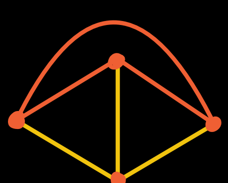
$M_2=(E,\mathcal{I}_2)$ cographic matroid of G $(\mathcal{I}_2=\{S \subseteq E : S \subseteq E \setminus T \text{ for some span. tree } T\})$

$=\{S \subseteq E : (V, E \setminus S) \text{ connected}\}$

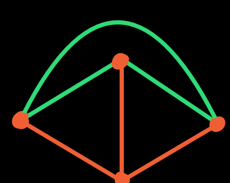
G has two edge-disjoint spanning trees

$\iff \max\{|I| : I \in \mathcal{I}_1 \cap \mathcal{I}_2\} = |V|-1$

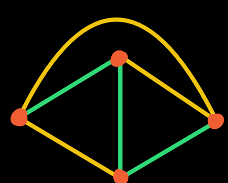
Ex.



$F \in \mathcal{I}_1 \setminus \mathcal{I}_2$



$F \in \mathcal{I}_2 \setminus \mathcal{I}_1$



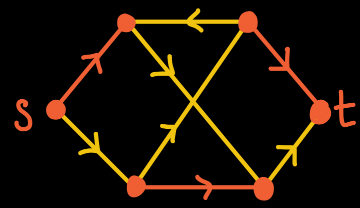
$F \in \mathcal{I}_1 \cap \mathcal{I}_2$

Intersection of 3 matroids

Thm: Given matroids $M_1=(X,\mathcal{I}_1)$, $M_2=(X,\mathcal{I}_2)$, $M_3=(X,\mathcal{I}_3)$ it is NP-Hard to find $\max\{|I| : I \in \mathcal{I}_1 \cap \mathcal{I}_2 \cap \mathcal{I}_3\}$

(Proof via reduction to Hamiltonian path)

Let $D=(V,A)$ and $s,t \in V$. It is NP-hard to decide if there exists an s-t path in D that visits every vertex exactly once.



Ham. s-t path

Define matroids:

$M_1=(A,\mathcal{I}_1)$ = graphic matroid of (undir) graph

$M_2=(A,\mathcal{I}_2)$ = partition matroid with

$\mathcal{I}_2=\{A'\subseteq A : |\delta^{out}(v)\cap A'|\leq 1 \ \forall v\neq t, |\delta^{out}(t)\cap A'|=0\}$

$M_3=(A,\mathcal{I}_3)$ = partition matroid with

$\mathcal{I}_3=\{A'\subseteq A : |\delta^{in}(v)\cap A'|\leq 1 \ \forall v\neq s, |\delta^{in}(s)\cap A'|=0\}$

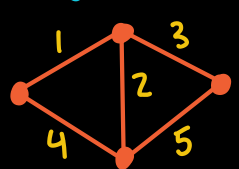
D has a Ham. s-t path $\Leftrightarrow \max\{|A'| : A'\in \mathcal{I}_1\cap\mathcal{I}_2\cap\mathcal{I}_3\}=|V|-1$.

We will give a polynomial time algorithm for matroid intersection generalizing augmenting path algorithms for max matchings in bipartite graphs

Let $M=(X,\mathcal{I})$ be a matroid and $Y\in\mathcal{I}$.

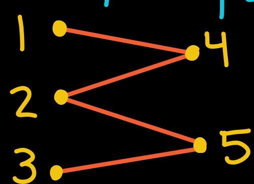
Define $H_M(Y)$ to be the bipartite graph with vertices $Y \uplus (X\setminus Y)$ and edges $\{y,x\}$ for $y\in Y$, $x\in X\setminus Y$ with $Y\setminus\{y\}\cup\{x\}\in\mathcal{I}$

Ex:



M = graphic matroid

$Y=\{1,2,3\}$



$Y\setminus\{1\}\cup\{5\}$
$Y\setminus\{3\}\cup\{4\}$
not indep.

Edmond's Alg for matroid intersection

Given $M_1=(X,\mathcal{I}_1)$, $M_2=(X,\mathcal{I}_2)$ and $Y\in\mathcal{I}_1\cap\mathcal{I}_2$ define directed bipartite graph $H_{M_1,M_2}(Y)$ with vertices $X=Y\uplus X\backslash Y$, arcs

(y,x) if $(Y\backslash\{y\})\cup\{x\}\in\mathcal{I}_1$ $(\Leftrightarrow \{y,x\}\in H_{M_1}(Y))$

(x,y) if $(Y\backslash\{y\})\cup\{x\}\in\mathcal{I}_2$ $(\Leftrightarrow \{y,x\}\in H_{M_2}(Y))$

Idea: look for directed path in $H_{M_1,M_2}(Y)$, use it to "augment" Y to a larger set $Y'\in\mathcal{I}_1\cap\mathcal{I}_2$.