MA/AMA 514:

Today: Matroids: Examples, Operations, & Polytopes

(§10.2, 10.3, 10.7)

From last time:

A <u>matroid</u> $M = (X, \mathcal{B})$ consists of a finite set X and collection $\mathcal{B}$ of subsets of X s.t.

(1) $\mathcal{B}$ is nonempty

(2) $A, B \in \mathcal{B}$, $a \in A \setminus B \Rightarrow \exists\, b \in B \setminus A$ s.t.

$$B \cup \{a\} \setminus \{b\} \in \mathcal{B}$$

<u>Ex 1</u> (Graphic) $G = (V, E)$ connected graph

Take $X = E$, $\mathcal{B} = \{\text{spanning trees of } G\}$

<u>Ex 2</u> (Linear) $v_1, \dots, v_n$ spanning $\mathbb{R}^d$ (or any vec. space)

$X = \{1, \dots, n\}$, $\mathcal{B} = \{S \subseteq X : \{v_i : i \in S\} \text{ is a basis for } \mathbb{R}^d\}$

<u>Terminology</u>

An element $S \in \mathcal{B}$ is called a <u>basis</u> of M.

A subset S of X is <u>independent</u> if $S \subseteq B$ for some $B \in \mathcal{B}$, <u>dependent</u> otherwise

The rank of a subset $S \subseteq X$ is

$$r_M(S) = \max\{|I| : I \subseteq S,\ I \text{ independent}\}$$

Then $r_M(\emptyset)=0$, $r_M(S) \leq r_M(T)$ when $S \subseteq T$ and

$$r_M(S) + r_M(T) \geq r_M(S \cap T) + r_M(S \cap T) \quad \forall S,T \subseteq X$$

r_M is "submodular"

Matroid operations

Let $M=(X,\mathcal{B})$ be a matroid and $S \subseteq X$.

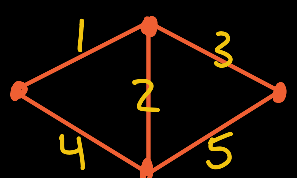
We can get new matroids via the following operations:

(Deletion) $M \backslash S = (X \backslash S, \mathcal{B}')$ where $\mathcal{B}' = \{B \in \mathcal{B} : B \cap S = \emptyset\}$

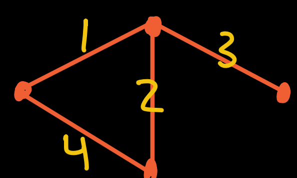
(Contraction) For S independent in M,

$M/S = (X \backslash S, \mathcal{B}')$ where $\mathcal{B}' = \{B \backslash S : B \in \mathcal{B}, S \subseteq B\}$

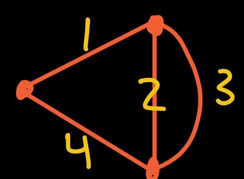
Ex:



M = graphic matroid



deletion: $M \backslash 5$



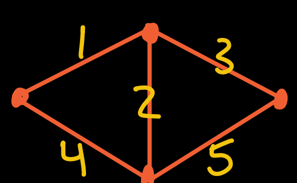
contraction: $M/5$

(Duality) $M^* = (X, \mathcal{B}')$ where $\mathcal{B}' = \{X \backslash B : B \in \mathcal{B}\}$

All of these are matroids! (You check)

Ex 1' (Cographic) $G=(V,E)$ connected graph

Take $X=E$, $\mathcal{B} = \{E \backslash T : T \text{ sp. tree of } G\}$



$\mathcal{B} = \{12, 13, \ldots, 45\}$
$= \binom{[5]}{2} \backslash \{14, 35\}$

S indep $\Leftrightarrow$ $E \backslash S$ acyclic

rank $= |E| - |V| + 1$

Remarks: • $\mathrm{rank}(M\setminus S) = \mathrm{rank}(M)$

• $\mathrm{rank}(M/S) = \mathrm{rank}(M) - |S|$ (for S indep)

• $\mathrm{rank}(M^*) = |X| - \mathrm{rank}(M)$

(Contraction) For arbitrary $S \subseteq X$, define $M/S = (M^* \setminus S)^*$

(Truncation) For $k < \mathrm{rank}(M)$, the k-truncation of M is $M = (X, \mathcal{B}')$ where $\mathcal{B}' = \{S \subseteq X : |S| = k,\ S \text{ independent in } M\}$

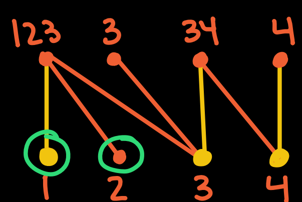
Ex: For graphic M, $\{\text{bases of } k\text{-trunc.}\} = \{k\text{-forests}\}$

Ex 3 (Transversal) Fix subsets $A_1, \ldots, A_m \subseteq X$.

A subset $S = \{s_1, \ldots, s_n\} \subseteq X$ is a partial transversal of $A_1, \ldots, A_m$ if $\exists$ distinct indices $i_1, \ldots, i_n$ s.t. $s_j \in X_{i_j}$ $\forall j = 1, \ldots, n$.

= set of $S \subseteq X$ covered by some matching in bipartite graph with edges $(x, \text{"}A_i\text{"})$ when $x \in A_i$

e.g. $X = \{1,2,3,4\}$ $A_1 = \{1,2,3\}$ $A_2 = \{3\}$ $A_3 = \{3,4\}$ $A_4 = \{4\}$



$\{1,3,4\}$ is a partial transversal

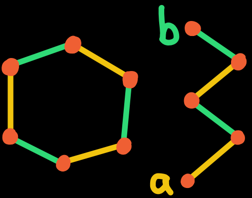
$\{1,2\}$ is NOT

Called a "partition matroid" when $A_1, \ldots, A_m$ form a partition of X

Claim: {Partial transversals} form the independent sets of a matroid on X. ({Maximal partial trans.} $= \mathcal{B}$)

(Proof) Suppose S, T are partial transversals of max size and $a \in S \setminus T$. Let M, M' be the matchings of the

the bipartite graph on $X \cup \{\text{“}A_1\text{”}, \dots, \text{“}A_m\text{”}\}$ covering S, T resp.
Every connected comp. of $M \Delta M'$ is a cycle or path

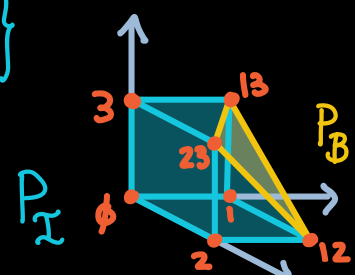


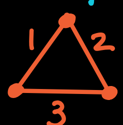
with an even # of edges. (No augmenting paths!)
Since a is uncovered by M', it is one end point of an even path P. Let b be the other end pt.
Then $b \in T \setminus S$ (covered by M', not M). Let $M'' = M' \Delta P$.
Then M'' is a matching of max size, covering $T \cup \{a\} \setminus \{b\}$. $\square$

Matroid polytope (§10.7)

Two polytopes assoc. to $M = (X, \mathcal{B})$:
independence polytope: $P_{\mathcal{I}}(M) = \operatorname{conv}\{\mathbb{1}_S : S \subseteq X \text{ indep}\}$
basis polytope: $P_{\mathcal{B}}(M) = \operatorname{conv}\{\mathbb{1}_S : S \in \mathcal{B}\}$

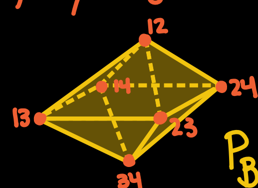
Note: $P_{\mathcal{B}}(M) = P_{\mathcal{I}}(M) \cap \{\sum_i z_i = \operatorname{rank}(M)\}$



Ex: 

$X = \{1, 2, 3\}$ $\mathcal{B} = \{12, 13, 23\}$

Ex: $X = \{1, 2, 3, 4\}$ $\mathcal{B} = \binom{[4]}{2}$



$P_{\mathcal{B}} \subseteq \{z \in \mathbb{R}^4 : z_1 + \dots + z_4 = 2\}$

"Obvious" inequalities that hold on $P_{\mathcal{I}}(M)$
(1) $z_i \geq 0$ for all $i \in X$
(2) $\sum_{i \in S} z_i \leq r_M(S)$ for all $S \subseteq X$

Linear programs: Given $w: X \to \mathbb{R}$

(Primal) $\max w^T z$ s.t. $z \geq 0$, $\sum_{i \in S} z_i \leq r_M(S)$ $\forall S \subseteq X$

(Dual) $\min \sum_{S \subseteq X} r_M(S) y_S$ s.t. $y \geq 0$, $\sum_{S \ni i} y_S \geq w_i$ $\forall i \in X$

<u>Thm 10.14</u>: For any $w: X \to \mathbb{Z}$, opt sol. are integer.

<u>Cor 10.14a</u>: $P_I(M) = \{ z \in \mathbb{R}^X$ satisfying (1) & (2)$\}$