

MA/AMA 514 :

Today: Matroids: Examples, Operations, & Polytopes
(§10.2, 10.3, 10.7)

From last time:

A matroid $M = (X, \mathcal{B})$ consists of a finite set X and collection $\mathcal{B}$ of subsets of X s.t.

(1) $\mathcal{B}$ is nonempty

(2) $A, B \in \mathcal{B}, a \in A \setminus B \Rightarrow \exists b \in B \setminus A$ s.t.

$$B \cup \{a\} \setminus \{b\} \in \mathcal{B}$$

Ex 1 (Graphic) $G = (V, E)$ connected graph

Take $X = E, \mathcal{B} = \{\text{spanning trees of } G\}$

Ex 2 (Linear) $v_1, \dots, v_n$ spanning $\mathbb{R}^d$ (or any
vec. space)

$X = \{1, \dots, n\}, \mathcal{B} = \{S \subseteq X : \{v_i : i \in S\} \text{ is a basis for } \mathbb{R}^d\}$

Terminology

An element $S \in \mathcal{B}$ is called a basis of M .

A subset S of X is independent

if $S \subseteq B$ for some $B \in \mathcal{B}$, dependent otherwise

The rank of a subset $S \subseteq X$ is

$$r_M(S) = \max\{|I| : I \subseteq S, I \text{ independent}\}$$

Then $r_M(\emptyset) = 0$, $r_M(S) \leq r_M(T)$ when $S \subseteq T$ and

$$r_M(S) + r_M(T) \geq r_M(S \cap T) + r_M(S \cup T) \quad \forall S, T \subseteq X$$

r_M is "submodular"

Matroid operations

Let $M = (X, \mathcal{B})$ be a matroid and $S \subseteq X$.

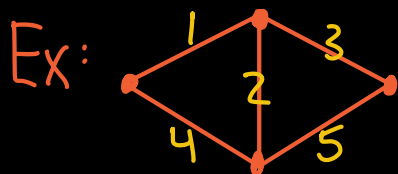
We can get new matroids via the following operations:

(Deletion) $M \setminus S = (X \setminus S, \mathcal{B}')$ where $\mathcal{B}' = \{B \in \mathcal{B} : B \cap S = \emptyset\}$

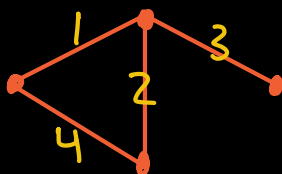
(Contraction) For S independent in M ,

$$M/S = (X \setminus S, \mathcal{B}')$$

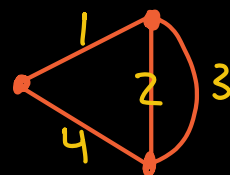
where $\mathcal{B}' = \{B \setminus S : B \in \mathcal{B}, S \subseteq B\}$



$M = \text{graphic matroid}$



deletion: $M \setminus 5$



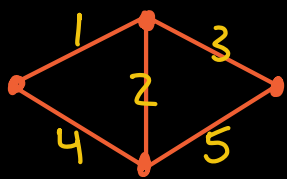
contraction: $M/5$

(Duality) $M^* = (X, \mathcal{B}')$ where $\mathcal{B}' = \{X \setminus B : B \in \mathcal{B}\}$

All of these are matroids! (You check)

Ex 1' (Cographic) $G = (V, E)$ connected graph

Take $X = E$, $\mathcal{B} = \{E \setminus T : T \text{ sp. tree of } G\}$



$$\begin{aligned} \mathcal{B} &= \{12, 13, \dots, 45\} \\ &= \binom{[5]}{2} \setminus \{14, 35\} \end{aligned}$$

$$S \text{ indep} \iff E \setminus S \text{ acyclic}$$

$$\text{rank} = |E| - |V| + 1$$

- Remarks:
- $\text{rank}(M \setminus S) = \text{rank}(M)$
 - $\text{rank}(M/S) = \text{rank}(M) - |S|$ (for S indep)
 - $\text{rank}(M^*) = |X| - \text{rank}(M)$

(Contraction) For arbitrary $S \subseteq X$, define $M/S = (M^* \setminus S)^*$

(Truncation) For $k < \text{rank}(M)$, the k -truncation of M is $M = (X, \mathcal{B}')$ where $\mathcal{B}' = \{S \subseteq X : |S| = k, S \text{ independent in } M\}$

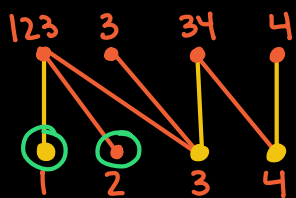
Ex: For graphic M , $\{\text{bases of } k\text{-trunc.}\} = \{k\text{-forests}\}$

Ex 3 (Transversal) Fix subsets $A_1, \dots, A_m \subseteq X$.

A subset $S = \{s_1, \dots, s_n\} \subseteq X$ is a partial transversal of $A_1, \dots, A_m$ if $\exists$ distinct indices $i_1, \dots, i_n$ s.t. $s_j \in A_{i_j} \quad \forall j=1, \dots, n$.

= set of $S \subseteq X$ covered by some matching in bipartite graph with edges (x, A_i) when $x \in A_i$

e.g. $X = \{1, 2, 3, 4\}$ $A_1 = \{1, 2, 3\}$ $A_2 = \{3\}$ $A_3 = \{3, 4\}$ $A_4 = \{4\}$



$\{1, 3, 4\}$ is a partial transversal

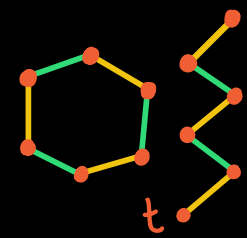
$\{1, 2\}$ is NOT

Called a "partition matroid" when $A_1, \dots, A_m$ form a partition of X

Claim: $\{\text{Partial transversals}\}$ form the independent sets of a matroid on X . ($\{\text{Maximal partial trans.}\} = \mathcal{B}$)

(Proof) Easy to check $\mathcal{I}$ nonempty and closed under containment. Suppose S, T are partial transversals

with $|S| < |T|$. Let M, M' be the matchings of the bipartite graph on $X \cup \{A_1, \dots, A_m\}$ covering S, T resp. Every connected comp. of $M \Delta M'$ is a cycle or path



with alternating edges. Since $|M'| = |T| > |S| = |M|$ there must be a path with more edges from M' . Since it has odd length, one endpoint t belongs to X . Moreover t is covered by M' and not M , so $t \in T \setminus S$. Let $M'' = M' \Delta P$.

Then M'' is a matching covering $S \cup \{t\}$. □

Matroid polytope (§10.7)

Two polytopes assoc. to $M = (X, B)$:

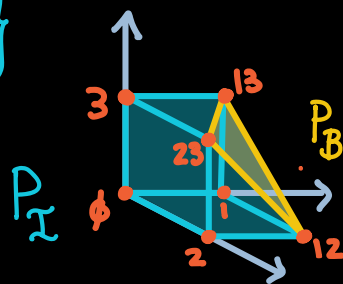
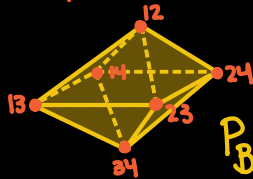
independence polytope: $P_I(M) = \text{conv}\{\mathbb{1}_S : S \subseteq X \text{ indep}\}$

basis polytope: $P_B(M) = \text{conv}\{\mathbb{1}_S : S \in B\}$

Note: $P_B(M) = P_I(M) \cap \{\sum_i z_i = \text{rank}(M)\}$

Ex: $X = \{1, 2, 3\}$ $B = \{\{1, 2\}, \{1, 3\}, \{2, 3\}\}$

Ex: $X = \{1, 2, 3, 4\}$ $B = \left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} \right\}$



$$P_B \subseteq \{z \in \mathbb{R}^4 : z_1 + \dots + z_4 = 2\}$$

"Obvious" inequalities that hold on $P_I(M)$

(1) $z_i \geq 0$ for all $i \in X$

(2) $\sum_{i \in S} z_i \leq r_M(S)$ for all $S \subseteq X$

Linear programs: Given $w: X \rightarrow \mathbb{R}$

$$\text{(Primal)} \quad \max w^T z \quad \text{s.t.} \quad z \geq 0, \quad \sum_{i \in S} z_i \leq r_M(S) \quad \forall S \subseteq X$$

$$\text{(Dual)} \quad \min \sum_{S \subseteq X} r_M(S) y_S \quad \text{s.t.} \quad y \geq 0, \quad \sum_{S \ni i} y_S \geq w_i \quad \forall i \in X$$

Thm 10.14 : For any $w: X \rightarrow \mathbb{Z}$, opt sol. are integer.

$$\text{Cor 10.14a: } P_I(M) = \{z \in \mathbb{R}^X \text{ satisfying (1) \& (2)}\}$$