MA/AMA 514:

Today: Matroids & the greedy algorithm

Problem: Given a finite set X, collection $\mathcal{B}$ of subsets of X, and weights $w: X \to \mathbb{R}$, find the minimum weight element of $\mathcal{B}$, where

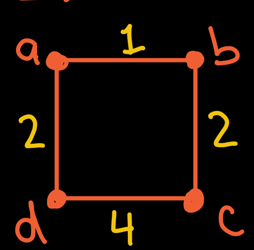
$$w(S) = \sum_{x \in S} w(x).$$

Greedy Algorithm Start: $S = \emptyset$

While $S \notin \mathcal{B}$, choose $x \in X \setminus S$ of minimum weight so that $S \cup \{x\} \subseteq B$ for some $B \in \mathcal{B}$ and update $S \to S \cup \{x\}$.

If there is no such x, stop and output S.

Ex (failure!) $X = \{ab, bc, cd, ad\}$ $\mathcal{B} = \{\{ab, cd\}, \{ad, bc\}\}$ perfect matchings



Greedy alg: Chooses ab in step 1, forced to take cd in step 2

Output: $S = \{ab, cd\}$, $w(S) = 5$

True opt: $T = \{ad, bc\}$, $w(T) = 4$

Structures for which the greedy alg. always finds the optimal solution are called matroids.

Matroids

A matroid $M=(X,\mathcal{B})$ consists of a finite set X and collection $\mathcal{B}$ of subsets of X s.t.

(1) $\mathcal{B}$ is nonempty

(2) $A,B\in\mathcal{B}$, $a\in A\setminus B \Rightarrow \exists\, b\in B\setminus A$ s.t.

$$B\cup\{a\}\setminus\{b\}\in\mathcal{B}$$

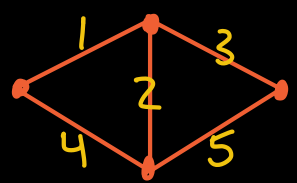
Terminology:

An element $S\in\mathcal{B}$ is called a basis of M.
By (2), all bases of M have the same size, which is called the rank of M.

A subset S of X is independent if $S\subseteq B$ for some $B\in\mathcal{B}$, dependent otherwise

Ex 1 (Graphic) $G=(V,E)$ connected graph
Take $X=E$, $\mathcal{B}=\{$spanning trees of $G\}$

rank $= \#V - 1$ "Independent" = acyclic

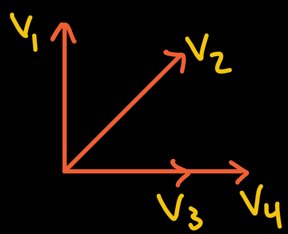


$\mathcal{B} = \{123, 125, \ldots, 345\}$

Ex 2 (Linear) $v_1, \ldots, v_n$ spanning $\mathbb{R}^d$ (or any vec. space)

$X = \{1, \ldots, n\}$, $\mathcal{B} = \{S \subseteq X : \{v_i : i \in S\}$ is a basis for $\mathbb{R}^d\}$

rank $= d$ "Independent" = linearly indep.



$\rightarrow \mathcal{B} = \{12, 13, 14, 23, 24\}$

Other axiomatizations of matroids

($\mathcal{I}$) Independent sets: $(X, \mathcal{I})$

(1) $\mathcal{I}$ nonempty collection of subsets of X

(2) closed under containment ($T \in \mathcal{I}$, $S \subseteq T \Rightarrow S \in \mathcal{I}$)

(3) $S \in \mathcal{I}$, $|T| > |S| \Rightarrow \exists t \in T \setminus S$ with $S \cup \{t\} \in \mathcal{I}$

Ex (linear): $v_1, \ldots, v_n \in \mathbb{R}^d$, $\mathcal{I} = \{S : \{v_i : i \in S\}$ linearly indep.$\}$

$\mathcal{B} \to \mathcal{I}$: $\mathcal{I} = \{S \subseteq X : S \subseteq B$ for some $B \in \mathcal{B}\}$

$\mathcal{I} \to \mathcal{B}$: $\mathcal{B} = \{S \in \mathcal{I} : S \not\subseteq T$ for any $T \in \mathcal{I} \setminus \{S\}\}$

Bases = inclusion maximal indep. subsets

$(\mathcal{C})$ Circuits: $(X, \mathcal{C})$ $\quad \mathcal{C}$ collection of subsets of X

(1) $\emptyset \notin \mathcal{C}$

(2) $S, T \in \mathcal{C}$, $S \subseteq T \Rightarrow S = T$

(3) $S, T \in \mathcal{C}$, $S \neq T$, $a \in S \cap T \Rightarrow (S \cup T) \setminus \{a\}$ contains a set in $\mathcal{C}$

$\mathcal{I} \to \mathcal{C}$: $\quad \mathcal{C} = \{S \subseteq X : S \notin \mathcal{I},\ T \in \mathcal{I} \text{ for any } T \subsetneq S\}$

$\mathcal{C} \to \mathcal{I}$: $\quad \mathcal{I} = \{S \subseteq X : T \nsubseteq S \text{ for all } T \in \mathcal{C}\}$

Circuits = inclusion minimal dependent subsets

See Thm 10.2 for more

Connections to the greedy algorithm

Thm 10.1 Suppose $|A| = |B|$ for all $A, B \in \mathcal{B}$ and $\mathcal{B} \neq \emptyset$.

$(X, \mathcal{B})$ is a matroid

$\iff$ for all $w : X \to \mathbb{R}$, the greedy algorithm finds a set $S \in \mathcal{B}$ of minimum weight

(Proof) $(\Leftarrow)$ Let $A, B \in \mathcal{B}$ and $a \in A \setminus B$.

Define $w : X \to \mathbb{R}$ by

$$w(x) = \begin{cases} -1 & \text{if } x = a \text{ or } x \in A \cap B \\ 0 & \text{if } x \in B \setminus A \\ 2 & \text{o.w.} \end{cases}$$

Since $A' = \{a\} \cup (A \cap B) \subseteq A$, A' is independent so the greedy alg. picks A' in the first $|A'|$ steps.
$\Rightarrow$ output has the form $A' \cup S \in \mathcal{B}$.

If $S \not\subseteq B \setminus A$. Then
$w(A' \cup S) = -|A'| + 2|S \setminus B| \geq -|A'| + 2 = -|A \cap B| + 1$.
But $w(B) = -|A \cap B| < w(A' \cup S)$, contradicting the assumption that output $A' \cup S$ has min weight.
Therefore $S \subseteq B \setminus A$. By $|A' \cup S| = |B|$, $S = (B \setminus A) \setminus \{b\}$ for some $b \in B \setminus A$. Then $B \cup \{a\} \setminus \{b\} = A' \cup S \in \mathcal{B}$.

($\Longrightarrow$) Call $S \subseteq X$ "greedy" if S is contained in a minimum weight basis.

<u>Claim</u>: If $S \subseteq X$ is greedy and $a \in X \setminus S$ has min. weight s.t. $S \cup \{a\}$ independent, then $S \cup \{a\}$ is greedy. (Analogous to Thm 1.11)

(Proof) Let B = min weight basis containing S.
If $a \in B \rightarrow$ done
If $a \notin B$, let A = basis containing $S \cup \{a\}$.

Know $\exists\, b \in B$ s.t. $B' = B \cup \{a\} \setminus \{b\} \in \mathcal{B}$

Since $S \cup \{b\} \subseteq B \in \mathcal{B}$, $w(b) \geq w(a)$.

Then

$$w(B') = w(B) + w(a) - w(b) \leq w(B).$$

$\Rightarrow B'$ also a basis of minimum weight!

$\square_{\text{Claim}}$

By Claim, at every step in greedy alg.

$S \subseteq X$ is greedy $\Rightarrow S \subseteq B$ for some min weight $B \in \mathcal{B}$

For output, $\nexists\, x$ with $S \cup \{x\} \in \mathcal{B} \Rightarrow S = B$