MA/AMA 514

Today: Approximation Algorithms

Non-bipartite graphs

$\max\{|M| : M \subseteq E \text{ matching}\} \leq \min\{|U| : U \subseteq V \text{ vertex cover}\}$

Why? Every $e \in M$ covered by some $u \in U$.

Since M is a matching, map $e \to u$ is injective.

This "$\leq$" can be strict. E.g. $G = K_3$

Blossom algorithm computes max size matching in poly. time $O(|V|^5)$. ($\to O(|V|^3)$ with careful implementation)

Computing min vertex cover is NP-hard!

In fact, NP-hard to compute within a factor of 1.4?

Today: Computing within a factor of 2 in poly. time. That is, β s.t. $\beta \leq |U^*| \leq 2\beta$

Approximation Algorithm 1: Fractional Rounding

Input: $G = (V, E)$ undirected (not necessarily bipartite)

Goal: min vertex cover $S \subseteq V$ (with $|e \cap S| \geq 1 \ \forall e \in E$)

Formulation as an IP:

$$\min \sum_{v \in V} y_v \quad \text{s.t. } 0 \le y \le 1,\ y_u + y_v \ge 1 \ \forall \{u,v\} \in E,\ y \in \mathbb{Z}^V$$

Let y^*_{IP} = opt. sol. for (IP) $= 1_S$ for min vertex cover $S \subseteq V$

y^*_{LP} = " " LP relaxation

(Rounding) Define $\tilde{y} \in \{0,1\}^V$ by $\tilde{y}_v = \begin{cases} 1 & \text{if } (y^*_{LP})_v \ge 1/2 \\ 0 & \text{if } (y^*_{LP})_v < 1/2 \end{cases}$

Claim 1: $\tilde{y}$ is feasible for (IP) (i.e. $\tilde{y} = 1_S$, $S \subseteq V$ a vertex cover)

(Proof) For $\{u,v\} \in E$, $(y^*_{LP})_u + (y^*_{LP})_v \ge 1$ $\Rightarrow$ at least one $\ge 1/2$

$\Rightarrow \tilde{y}_u + \tilde{y}_v \ge 1$

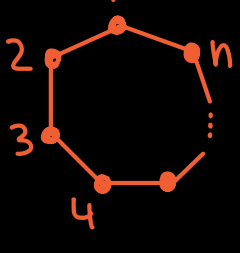
Claim 2: $\mathbb{1}^T y^*_{LP} \underset{(1)}{\le} \mathbb{1}^T y^*_{IP} \underset{(2)}{\le} \mathbb{1}^T \tilde{y} \underset{(3)}{\le} 2 y^*_{LP}$

(Proof) (1) minimizing over larger set

(2) $\tilde{y}$ feasible for (IP), $m^*_{IP} = \min \mathbb{1}^T y$ over all feas. y

(3) $\sum_{v \in V} \tilde{y}_v \le \sum_{v \in V} 2 (y^*_{LP})_v = 2 \mathbb{1}^T y^*_{LP}$

Ex (n-cycle) $\min \sum_{i=1}^{n} y_i$ s.t. $0 \le y \le 1$, $y_i + y_{i+1} \ge 1$ $\forall i = 1, \dots, n-1$, $y_1 + y_n \ge 1$



round ↳ $y^*_{LP} = (1/2, \dots, 1/2)$ $\mathbb{1}^T y^*_{LP} = n/2$

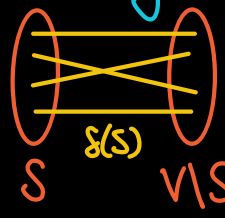
$\tilde{y} = (1, \dots, 1)$ $\mathbb{1}^T \tilde{y} = n$

Maximum cut of a graph

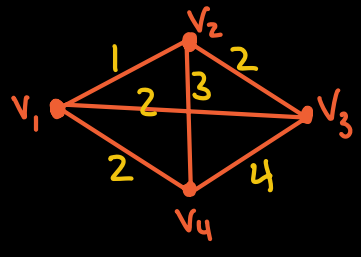
Let $G=(V,E)$ be a graph with edge weights $w: E \to \mathbb{R}_{\geq 0}$

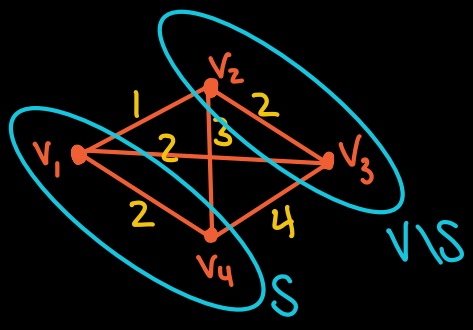
MAX-CUT PROBLEM:

$$\max \left\{ \sum_{e \in \delta(S)} w(e) : S \subseteq V \right\}$$



Ex:





$w(\delta(S)) = 1+2+3+4 = 10$

- Applications in clustering and image segmentation (want elements in S to be "close" to each other)
- Can be solved in polynomial time for special classes of graphs: planar (Hadlock 1975)
  no K_5 minor (Barahona 1983)
- NP hard to solve for general graphs (unlike min-cut!)
  $\Rightarrow$ try to find polynomial-time algorithms that give approximations of optimal solution

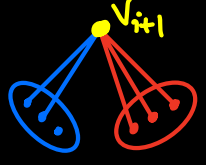
Erdös (1967) ½-approximation algorithm

Order vertices $V=\{v_1,\dots,v_n\}$ and color red/blue:

For $i=1,\dots,n$, let $S_i=\{v_j : j\le i,\ v_j \text{ blue}\}$

$T_i=\{v_1,\dots,v_i\}\setminus S_i=\{v_j : j\le i,\ v_j \text{ red}\}$

Color v_{i+1} blue if $w(E(v_{i+1},S_i)) < w(E(v_{i+1},T_i))$ and red otherwise. Take $S=S_n$.



Claim: $w(\delta(S)) \ge \frac{1}{2}\,\text{maxcut}(G)$

(Proof) In fact $w(\delta(S)) \ge \frac{1}{2}w(E) \ge \frac{1}{2}\,\text{maxcut}(G)$.

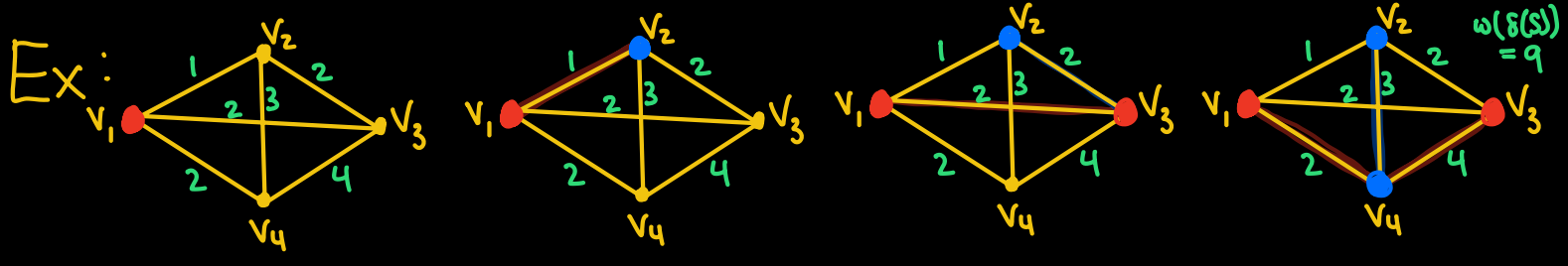
Every edge $e=\{i,j\}$ considered exactly once (at step $\max\{i,j\}$). At each step i, $\ge\frac{1}{2}$ edge weight is cut. Formally:

$$w(\{ij\in E : i>j\}) = w(E(i,S_{i-1})) + w(E(i,T_{i-1})) \le 2\,w(\{ij\in\delta(S) : i>j\})$$

(color v_i s.t. larger set is cut!)

Summing over all $i=1,\dots,n$, gives $w(E)\le 2\,w(\delta(S))$. □

Ex:



Håstad (1997) It is NP-Hard to find a cut $S\subseteq V$ with $w(\delta(S)) \ge \frac{16}{17}\,\text{maxcut}(G)$.

Goemans-Williamson (1994) There is a polynomial time alg. to find a cut with weight $\geq .878\,\text{MAXCUT}(G)$

uses non-linear convex optimization called semidefinite programming

Goemans-Williamson Max-Cut Formulation

Given a graph $G=(V,E)$ with n vertices $V=[n]$, associate $S\subseteq[n]$ with $\chi_S = \begin{cases} 1 & \text{if } i\in S \\ -1 & \text{if } i\notin S. \end{cases}$

$$\text{Claim: } \text{maxcut}(G) = \max\left\{ \sum_{ij\in E} w(ij)\left(\frac{1-x_ix_j}{2}\right) : x\in\{\pm1\}^n \right\}$$

(Proof) For $x\in\{\pm1\}^n$, $x=\chi_S$ for some $S\subseteq[n]$.

For $ij\in E$, $1-x_ix_j = \begin{cases} 0 & \text{if } i\in S, j\in S \quad \leftarrow 1-(1)(1)=0 \\ 0 & \text{if } i\notin S, j\notin S \quad \leftarrow 1-(-1)(-1)=0 \\ 2 & \text{if } ij\in \delta(S) \quad \leftarrow 1-(1)(-1)=1+1=2 \end{cases}$

$$\Rightarrow \sum_{ij\in\delta(S)} w(ij) = \sum_{ij\in E} w(ij)\left(\frac{1-x_ix_j}{2}\right) \qquad \square_{\text{Claim}}$$

Idea: replace $x_1,\dots,x_n\in\{\pm1\}^n$ with unit vectors $v_1,\dots,v_n\in\mathbb{R}^n$

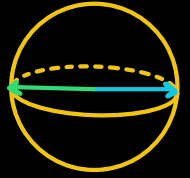
$$GW(G) = \max\left\{ \sum_{ij\in E} w(ij)\left(\frac{1-\langle v_i,v_j\rangle}{2}\right) : v_1,\dots,v_n\in\mathbb{R}^n,\ \|v_i\|=1\ \forall i \right\}$$

(can be solved with SDP - details next time)

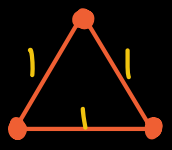
Claim 1: $\text{maxcut}(G) \leq GW(G)$

For $x=(x_1,\dots,x_n)\in\{\pm1\}^n$, take $v_i=(x_i,0,\dots,0)\in\mathbb{R}^n$ for $i=1,\dots,n$.

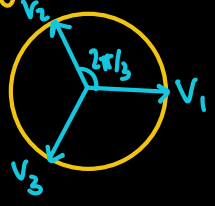
Then $\|v_i\|=1$ and $\langle v_i,v_j\rangle = x_ix_j$.



Note: Inequality can be strict!

Ex: 

$G = K_3$



$\langle v_i, v_j \rangle = \|v_i\| \cdot \|v_j\| \cos\left(\frac{2\pi}{3}\right) = -1/2$

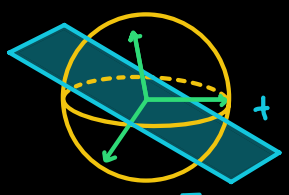
$\sum_{ij \in E} w(ij)\left(\frac{1-\langle v_i, v_j\rangle}{2}\right) = \sum_{ij \in E} 1\left(\frac{3}{4}\right) = \frac{9}{4} > 2$ (max cut)

But not too strict:

Idea: Probabilistic Rounding

Suppose $v_1, \dots, v_n \in \mathbb{R}^n$ acheive max in $GW(G)$.

Let $H = \{x : \langle a, x\rangle = 0\}$ be a random hyperplane (through the origin) in $\mathbb{R}^n$.

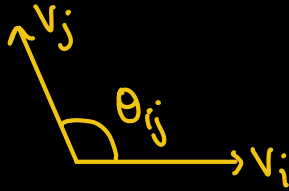


(Equivalent to choosing random $a \in S^{n-1}$)

Take $S = \{i \in [n] : \langle v_i, a\rangle > 0\}$. (vectors on one side of H)

Claim 2: $[\text{Expected value of } w(\delta(S))] \geq .878 \cdot \text{maxcut}(G)$

(Proof) For $e \in E$, what is the probability that $e \in \delta(S)$?



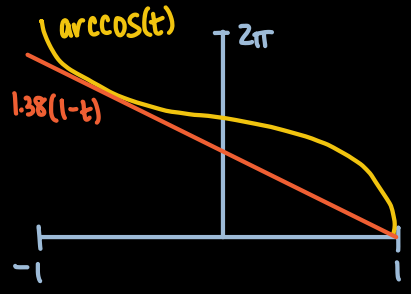
$\text{Prob}(H \text{ separates } v_i, v_j) = \frac{\theta_{ij}}{\pi} = \frac{\arccos(\langle v_i, v_j\rangle)}{\pi}$

Expected value of cut:

$$\sum_{ij \in E} w(ij)\frac{\theta_{ij}}{\pi} = \sum_{ij \in E} w(ij)\frac{\arccos(\langle v_i, v_j\rangle)}{\pi} \underset{(*)}{\geq} \sum_{ij \in E} w(ij)\frac{1.38005}{\pi}(1-\langle v_i, v_j\rangle)$$

$$= \underbrace{2\left(\frac{1.38005}{\pi}\right)}_{\approx .878} \underbrace{\sum_{ij \in E} w(ij)\left(\frac{1-\langle v_i, v_j\rangle}{2}\right)}_{= GW(G)}$$

(*) uses $\arccos(t) \geq 1.38005(1-t)$ for all $t \in [-1,1]$



$$\text{Value } .878 \approx \min_{0 \leq \theta \leq \pi} \frac{\theta}{2\pi(1-\cos(\theta))}$$

Construction derandomized by Alon-Spencer (2000)

It is an open question whether one can do better than a .878-approximation in polynomial time!