MA/AMA 514

Today: Intro to SDP's

Rest of class: Matroids (§10)

The PSD cone in $\mathbb{R}^{n\times n}_{sym}$.

The set of real symmetric matrices is a real vectorspace of dimension $\binom{n+1}{2}$.

Inner product: $\langle A,B\rangle = \text{trace}(AB)$

Ex: $A=\begin{pmatrix} a_{11} & a_{12} \\ a_{12} & a_{22}\end{pmatrix}$ $B=\begin{pmatrix} b_{11} & b_{12} \\ b_{12} & b_{22}\end{pmatrix}$

$$\langle A,B\rangle = \text{trace}(AB) = \text{trace}\begin{pmatrix} a_{11}b_{11}+a_{12}b_{12} & a_{12}b_{11}+a_{22}b_{12} \\ a_{12}b_{11}+a_{22}b_{12} & a_{12}b_{12}+a_{22}b_{22}\end{pmatrix}$$

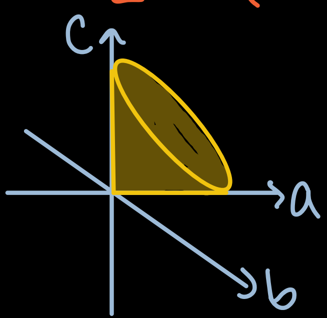
$$= a_{11}b_{11} + 2a_{12}b_{12} + a_{22}b_{22}$$

Linear Alg. Fact: all eigenvalues of $A\in\mathbb{R}^{n\times n}_{sym}$ are real

Call $A\in\mathbb{R}^{n\times n}_{sym}$ positive semidefinite if all of its eigenvalues are nonnegative (denoted "$A\succeq 0$")

Let $PSD_n = \{A\in\mathbb{R}^{n\times n}_{sym} : A\succeq 0\}$.

Ex: $(n=2)$ $\mathbb{R}^{2\times 2}_{sym} = \left\{\begin{pmatrix} a & b \\ b & c\end{pmatrix} : a,b,c\in\mathbb{R}\right\}\cong\mathbb{R}^3$



eigval of $\begin{pmatrix} a & b \\ b & c\end{pmatrix}$ = roots of $(t-a)(t-c)-b^2$

$$= \frac{a+c\pm\sqrt{(a-c)^2+4b^2}}{2}$$

$$\geq 0 \iff a\geq 0,\ c\geq 0,\ ac\geq b^2$$

Characterizations of PSD matrices

For $A \in \mathbb{R}^{n\times n}_{sym}$, TFAE:

1) all eig. val. of $A \geq 0$

2) all principal minors $\det(A_{S,S}) \geq 0$

3) $v^T A v \geq 0$ for all $v \in \mathbb{R}^n$ ← implies that PSD_n is a convex cone?

4) $A = BB^T$ for some $B \in \mathbb{R}^{n\times k}$, $k = \text{rank}(A)$

$$= \sum_{i=1}^{r} c_i c_i^T = (\langle r_i, r_j \rangle)_{1\leq i,j\leq n}$$

where $c_1, \dots, c_k$ = col of B
$r_1, \dots, r_n$ = rows of B

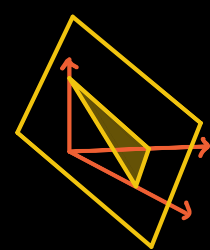
Ex: $A = \begin{pmatrix} 1 & 2 \\ 2 & 5 \end{pmatrix} \succeq 0$ $A = BB^T$ with $B = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$

$$= \begin{pmatrix} 1 \\ 2 \end{pmatrix}\begin{pmatrix} 1 & 2 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \end{pmatrix} = \begin{pmatrix} (1,0)^T(1,0) & (2,1)^T(1,0) \\ (2,1)^T(1,0) & (2,1)^T(2,1) \end{pmatrix}$$

Cor: PSD_n is a convex cone in $\mathbb{R}^{n\times n}_{sym}$ and is self-dual under $\langle A, B \rangle = \text{trace}(AB)$.

$A \in PSD_n \iff \langle A, B \rangle \geq 0$ for all $B \in PSD_n$.

Recall: One standard form for LP:



Given $a_1, \dots, a_m \in \mathbb{R}^n$, $b_1, \dots, b_m \in \mathbb{R}$, $c \in \mathbb{R}^n$

$$\max\{c^T x : x \geq 0,\ a_i^T x = b_i \text{ for } i = 1, \dots m\}$$

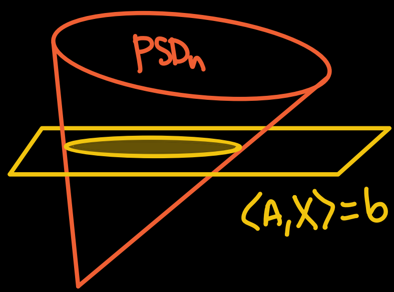
Transform inequality "$a_i^T x \leq b_i$" to equality "$a_i^T x + s_i = b_i$" by adding additional variable s_i and requiring $s_i \geq 0$

Semidefinite Programming

A semidefinite program (SDP) has the form

$$\text{(SDP)} \quad \max\{\langle C, X\rangle : X \in PSD_n,\ \langle A_i, X\rangle = b_i,\ i=1,\dots,m\}$$

where $A_1,\dots,A_m \in \mathbb{R}^{n\times n}_{sym}$, $b_1,\dots,b_m \in \mathbb{R}$, $C \in \mathbb{R}^{n\times n}_{sym}$



- Can be solved* in polynomial time using interior point methods
- Dual SDP bounds value of primal:

$$\text{(Dual SDP)} \quad \min\left\{b^T y : \sum_{i=1}^m y_i A_i - C \succeq 0\right\}$$

Weak duality: $X \succeq 0,\ \langle A_i, X\rangle = b_i,\ \sum_{i=1}^m y_i A_i - C \succeq 0 \Rightarrow \langle C, X\rangle \le b^T y$

Why? $$\langle C, X\rangle = \underbrace{\left\langle C - \sum_{i=1}^m y_i A_i,\ X\right\rangle}_{= -\langle \sum y_i A_i - C,\ X\rangle \le 0} + \left\langle \sum_{i=1}^m y_i A_i,\ X\right\rangle$$

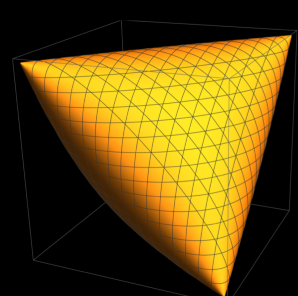
$$\le \left\langle \sum_{i=1}^m y_i A_i,\ X\right\rangle = \sum_{i=1}^m y_i \langle A_i, X\rangle = \sum_{i=1}^m b_i y_i = b^T y$$

Ex: $(n=3, m=3)$ $A_1 = \begin{pmatrix} 1&0&0\\0&0&0\\0&0&0\end{pmatrix}$ $A_2 = \begin{pmatrix} 0&0&0\\0&1&0\\0&0&0\end{pmatrix}$ $A_3 = \begin{pmatrix} 0&0&0\\0&0&0\\0&0&1\end{pmatrix}$

$C = \begin{pmatrix} 0&-1&-1\\-1&0&-1\\-1&-1&0\end{pmatrix}$ $b_1 = 1$ $b_2 = 1$ $b_3 = 1$

$$\max\{\langle C, X\rangle : X \succeq 0,\ \langle A_i, X\rangle = b_i,\ i=1,2,3\}$$

$$= \max\left\{-2x_{12} - 2x_{13} - 2x_{23} : \begin{pmatrix} 1 & x_{12} & x_{13}\\ x_{12} & 1 & x_{23}\\ x_{13} & x_{23} & 1\end{pmatrix} \succeq 0\right\}$$



acheived by $(x_{ij}^*) = \left(\frac{-1}{2}, \frac{-1}{2}, \frac{-1}{2}\right)$

Dual SDP: $\min \left\{ y_1+y_2+y_3 : \begin{pmatrix} y_1 & 1 & 1 \\ 1 & y_2 & 1 \\ 1 & 1 & y_3 \end{pmatrix} \succeq 0 \right\}$

$\min = 3$ acheived by $y^* = (1,1,1)^T$.

Duality: $\langle C, X^* \rangle = \left\langle \begin{pmatrix} 0 & -1 & -1 \\ -1 & 0 & -1 \\ -1 & -1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & -1/2 & -1/2 \\ -1/2 & 1 & -1/2 \\ -1/2 & -1/2 & 1 \end{pmatrix} \right\rangle = 3 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}^T \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = b^T y^*$

$\Longrightarrow$ both X^* and y^* are optimal!

Goemans-Williamson approx for MAXCUT(G)

From last time: Given a graph $G = ([n], E)$ with weights $\omega : E \to \mathbb{R}_{\geq 0}$,

$$GW(G) = \max \left\{ \sum_{ij \in E} \omega(ij) \left(\frac{1 - \langle v_i, v_j \rangle}{2} \right) : \begin{array}{l} v_1, \dots, v_n \in \mathbb{R}^n \\ \|v_i\| = 1 \ \forall i \end{array} \right\}$$

Use vectors $v_1, \dots, v_n$ acheiving max and probalistic rounding to find cut $S \subseteq V$ with $\omega(\delta(S)) \geq .878 \,\mathrm{maxcut}(G)$ (in expectation)

For $i = 1, \dots, n$, take $A_i = e_i e_i^T$ (that is, $(A_i)_{jk} = \begin{cases} 1 & \text{if } j=k=i \\ 0 & \text{o.w} \end{cases}$)

$b_1 = \dots = b_n = 1$, $C \in \mathbb{R}^{n \times n}_{sym}$ with $C_{jk} = \begin{cases} -\omega(jk) & \text{if } jk \in E \\ 0 & \text{o.w} \end{cases}$

Claim: $GW(G) = \frac{1}{2}\omega(E) + \frac{1}{4}\alpha_{SDP}$ where

$$\alpha_{SDP} = \max \{ \langle C, X \rangle : X \succeq 0, \underbrace{\langle A_i, X \rangle = b_i}_{\Leftrightarrow X_{ii} = 1}, i = 1, \dots, n \}.$$

(Proof) A matrix $X \in \mathbb{R}^{n \times n}_{sym}$ is PSD
$\iff \exists\, v_1, \dots, v_n \in \mathbb{R}^n$ s.t. $X_{ij} = \langle v_i, v_j \rangle \ \forall i,j$

$$\left(\begin{array}{l} X \succeq 0 \\ \iff X = BB^T \\ \iff X_{ij} = \langle v_i, v_j \rangle \ \forall i,j \\ \quad v_i = i^{th} \text{ row of } B \end{array} \right)$$

Note: $X_{ii} = 1 \iff \langle v_i, v_i \rangle = \|v_i\|^2 = 1$.

$$\langle C, X \rangle = 2 \sum_{ij \in E} (-w(ij)) X_{ij} = -2 \sum_{ij \in E} w(ij) \langle v_i, v_j \rangle$$

$$GW(G) = \max \left\{ \sum_{ij \in E} w(ij) \left(\frac{1 - \langle v_i, v_j \rangle}{2} \right) : v_1, \dots, v_n \in \mathbb{R}^n, \ \|v_i\| = 1 \ \forall i \right\}$$

$$= \frac{1}{2} w(E) + \frac{1}{2} \max \left\{ \sum_{ij \in E} \underbrace{(-w(ij)) \langle v_i, v_j \rangle}_{= C_{ij} X_{ij}} : \begin{array}{l} v_1, \dots, v_n \in \mathbb{R}^n \\ \|v_i\| = 1 \ \forall i \end{array} \right\}$$

$$= \frac{1}{2} w(E) + \frac{1}{2} \max \left\{ \frac{1}{2} \langle C, X \rangle : X \in PSD_n, \ X_{ii} = 1 \ \forall i \right\}$$

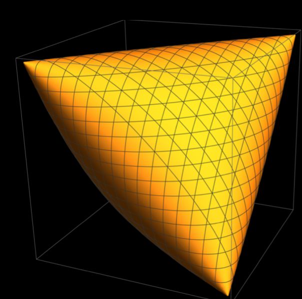
Ex: $(n=3, m=3)$ $\quad G = K_3 \rightsquigarrow C = \begin{pmatrix} 0 & -1 & -1 \\ -1 & 0 & -1 \\ -1 & -1 & 0 \end{pmatrix}$

$$A_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad A_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad A_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

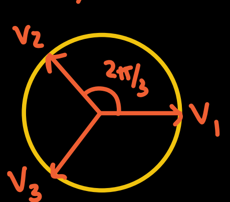
$$b_1 = 1 \qquad b_2 = 1 \qquad b_3 = 1$$

$$\alpha_{SDP} = \max \{ \langle C, X \rangle : X \succeq 0, \langle A_i, X \rangle = b_i, \ i = 1,2,3 \}$$

$$= \max \left\{ -2x_{12} - 2x_{13} - 2x_{23} : \begin{pmatrix} 1 & x_{12} & x_{13} \\ x_{12} & 1 & x_{23} \\ x_{13} & x_{23} & 1 \end{pmatrix} \succeq 0 \right\} = 3$$



$$X^* = \begin{pmatrix} \langle v_1, v_1 \rangle & \langle v_1, v_2 \rangle & \langle v_1, v_3 \rangle \\ \langle v_2, v_1 \rangle & \langle v_2, v_2 \rangle & \langle v_2, v_3 \rangle \\ \langle v_3, v_1 \rangle & \langle v_3, v_2 \rangle & \langle v_3, v_3 \rangle \end{pmatrix} = \begin{pmatrix} 1 & -1/2 & -1/2 \\ -1/2 & 1 & -1/2 \\ -1/2 & -1/2 & 1 \end{pmatrix}$$

for 

$$GW(K_3) = \frac{3}{2} + \frac{3}{4} = \frac{9}{4}$$