MA/AMA 514

Today: Stable sets & colorings

Stable sets and colorings

For a graph $G=(V,E)$, recall

$$\alpha(G) = \max\{|S| : S \subseteq V \text{ stable}\}$$

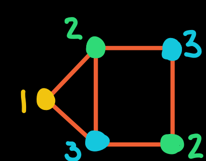
$$\omega(G) = \max\{|S| : S \subseteq V \text{ clique}\}$$

$$\chi(G) = \min\{k : G \text{ has a } k\text{-coloring}\}$$

S stable $\iff$ $\forall u,v \in S$, $\{u,v\} \notin S$

S clique $\iff$ $\forall u,v \in S$, $\{u,v\} \in S$

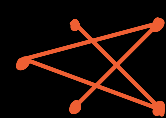
A <u>k-coloring of G</u> is a function $c: V \to \{1,..,k\}$ s.t. $c(u) \neq c(v)$ for all edges $\{u,v\} \in E$.



3-coloring

<u>Remarks</u>

- $\omega(G) = \alpha(\bar{G})$ where $e \in E(\bar{G}) \Leftrightarrow e \notin E(G)$
- $\omega(G) \leq \chi(G)$
- NP-hard to compute any of these!

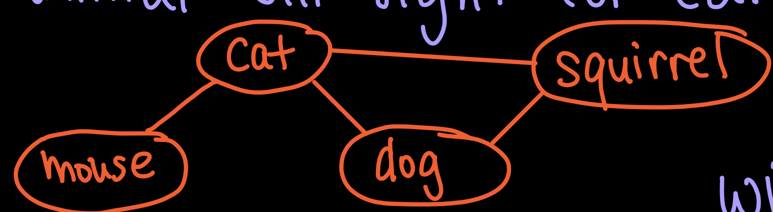


$\bar{G}$

Applications of graph coloring

<u>Appl. 7.2</u>: (Storage of goods) Given n animals needing transport, assign to fewest # carriages so that no animal will fight (or eat each other!)

e.g.



Take $G=(V,E)$ with $V=\{\text{animals}\}$

and E = pairs of animals with a conflict.

$\Rightarrow \chi(G)$ = smallest # of carriages needed

Appl. 7.4 (Scheduling classes)

n different classes (1 hr each), m teachers

Every teacher can teach some subset of classes.

Goal: What in min # class times needed to schedule all class/teacher pairs?

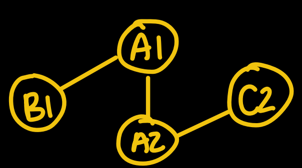
Take $G=(V,E)$ with V = compatible teacher/class pairs

E = pairs with common teacher or class

e.g.

Teacher \ Class	A	B	C
1	✓	✓	✗
2	✓	✗	✓

k-coloring of G = feasible schedule with k class times



$\leftarrow$ 2 colorable : 1^{st} slot 1 teaches A, 2 teaches C

2^{nd} slot 1 teaches B, 2 teaches A

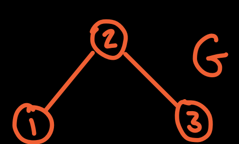
Study $\chi(G)$ through $\omega(G) = \alpha(\bar{G})$

Stable-set polytope

For $|V|=n$, identify subsets $S \subseteq V$ with $\mathbb{1}_S \subseteq \{0,1\}^n$ $(\mathbb{1}_S)_i = \begin{cases} 1 & \text{if } i \in S \\ 0 & \text{if } i \notin S \end{cases}$

The stable set polytope of a graph G is

$$\mathrm{STAB}(G) = \mathrm{conv}\{\mathbb{1}_S : S \subseteq V \text{ stable set of } G\}$$

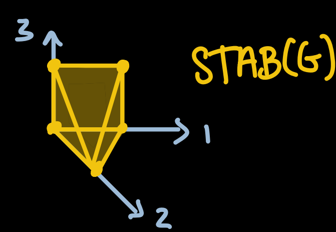
Ex: 

Stable sets :

$\emptyset \to (0,0,0)$ $\{3\} \to (0,0,1)$

$\{1\} \to (1,0,0)$ $\{1,3\} \to (1,0,1)$

$\{2\} \to (0,1,0)$



Some valid inequalities for STAB(G)

$$\left.\begin{array}{l}\left.\begin{array}{ll} x_i \geq 0 & \forall\, i \in [n] \\ x_i + x_j \leq 1 & \forall\, ij \in E \end{array}\right\} \text{define FRAC}(G) \\ \displaystyle\sum_{i \in K} x_i \leq 1 \quad \forall \text{ cliques } K \end{array}\right\} \text{define QSTAB}(G)$$

Then $\text{STAB}(G) \subseteq \text{QSTAB}(G) \subseteq \underbrace{\text{FRAC}(G)}_{\text{describable in polynomial time}}$.

All these containments can be strict!

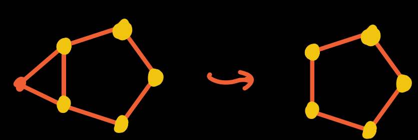
No complete inequality description known for general graphs!

<u>Thm</u>: $\text{STAB}(G) = \text{FRAC}(G) \iff G$ bipartite

($\Rightarrow$ for bipartite G, can find $\alpha(G)$ in polynomial time)

<u>Thm</u> (Chvátal-Fulkerson) $\text{STAB}(G) = \text{QSTAB}(G) \iff G$ <u>perfect</u>

A graph G is perfect $\iff$ $\omega(H) = \chi(H)$ for all induced subgraphs H of G

Ex:  Non-ex:  $\omega(H) = 2 < \chi(H) = 3$

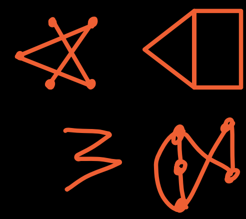
<u>Thm</u> (Lovasz 1972) G perfect $\iff \overline{G}$ perfect

<u>Thm</u> (Chudnovsky, Robertson, Seymour, Thomas 2002)

A graph G is <u>perfect</u> if neither G nor $\overline{G}$ has an odd circuit of length ≥ 5 as an induced subgraph.

Examples of Perfect Graphs:

- bipartite graphs and their complements

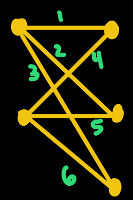


- line graphs of bipartite graphs

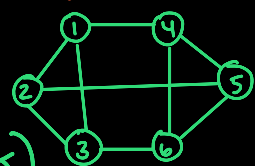
$G=(V,E) \to$ line graph of G has vertices $= E$

edges $= \{e_1, e_2\}$ with $e_1 \cap e_2 \neq \emptyset$

Ex: G



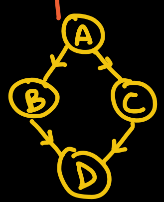
$L(G)$



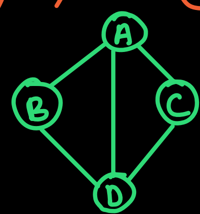
- comparability graphs

partially ordered set $(S,<) \to G=(V,E)$ $\quad V = S$, $E = \left\{ \{s,t\} : \begin{matrix} s<t \text{ or} \\ t<s \end{matrix} \right\}$

Ex:



Partial order



comp. graph

Max stable sets of perfect graphs via SDP

Idea: $G=([n],E)$, $S \subseteq [n]$ stable set, $v = \mathbb{1}_S \in \mathbb{R}^n$

$\leadsto$ Consider $\begin{pmatrix} 1 \\ v \end{pmatrix} \begin{pmatrix} 1 & v^T \end{pmatrix} = \begin{pmatrix} 1 & v^T \\ v & X \end{pmatrix} \in \mathbb{R}^{(n+1)\times(n+1)}_{sym}$

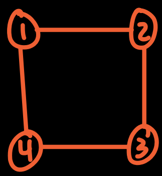
Then $X_{ii} = v_i^2 = v_i$ $(v_i \in \{0,1\})$ and $X_{ij} = 0$ for $ij \in E$

The Lovász theta body of G is

$$TH(G) = \left\{ x \in \mathbb{R}^n : \exists Y \in \mathbb{R}^{n\times n} \text{ s.t. } \begin{pmatrix} 1 & x^T \\ x & Y \end{pmatrix} \in PSD_{n+1} \right. \\ \left. Y_{ii} = x_i \ \forall i \in [n] \text{ and } Y_{ij} = 0 \ \forall ij \in E \right\}$$

Remarks:

- $TH(G)$ is convex and contains $STAB(G)$
- Can solve $\max\{ \sum_{i=1}^n x_i : x \in TH(G)\}$ in polynomial time (SDP!)

Ex:  $TH(G) = \left\{ x \in \mathbb{R}^4 : \exists\, y_{13}, y_{24} \in \mathbb{R} \text{ s.t. } \begin{pmatrix} 1 & x_1 & x_2 & x_3 & x_4 \\ x_1 & x_1 & 0 & y_{13} & 0 \\ x_2 & 0 & x_2 & 0 & y_{24} \\ x_3 & y_{13} & 0 & x_3 & 0 \\ x_4 & 0 & y_{24} & 0 & x_4 \end{pmatrix} \in PSD_5 \right\}$

Thm (Lovász) $STAB(G) = TH(G) \Leftrightarrow G$ is perfect

Idea of proof: $TH(G) \subseteq QSTAB(G)$

If vertices $\{1, \dots, r\}$ form a clique in G and $x \in TH(G)$

$\Rightarrow \exists\, Y$ s.t.

$$\left(\begin{array}{cccc|ccc} 1 & x_1 & \cdots & x_r & x_{r+1} & \cdots & x_n \\ x_1 & x_1 & & 0 & & & \\ \vdots & & \ddots & & & * & \\ x_r & 0 & & x_r & & & \\ \hline x_{r+1} & & & & & & \\ \vdots & & * & & & * & \\ x_n & & & & & & \end{array}\right) \in PSD_{n+1}$$

$$\Rightarrow \begin{pmatrix} 1 & x_1 & \cdots & x_r \\ x_1 & x_1 & & 0 \\ \vdots & & \ddots & \\ x_r & 0 & & x_r \end{pmatrix} \in PSD_{r+1} \overset{(*)}{\Longleftrightarrow} \underbrace{1 - \begin{pmatrix} x_1 \\ \vdots \\ x_r \end{pmatrix}^T \begin{bmatrix} 1/x_1 & & 0 \\ & \ddots & \\ 0 & & 1/x_r \end{bmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_r \end{pmatrix}}_{1 - \sum_{i \in K_r} x_i \geq 0} \geq 0$$

$\Rightarrow x \in QSTAB(G)$.

$(*)$ Schur complements

If C positive definite, then $\begin{pmatrix} A & B^T \\ B & C \end{pmatrix}$ PSD $\Leftrightarrow A - B^T C^{-1} B$ PSD

Cor: For perfect graphs G, there is a polynomial time algorithm to compute $\alpha(G)$ and $\omega(G) = \chi(G)$.

Open Question: Is there a combinatorial algorithm to find $\alpha(G)$ for perfect graphs G?

(Known in some special cases!)