MA/AMA 514

Today: Max Cut

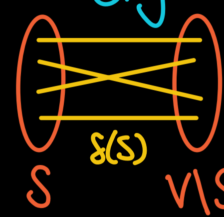
Wed: Semidefinite Programming

Maximum cut of a graph

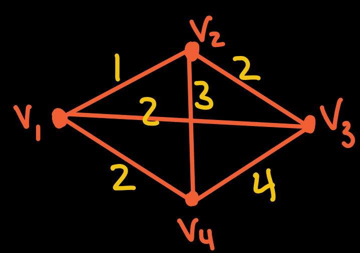
Let $G=(V,E)$ be a graph with edge weights $w: E \to \mathbb{R}_{\geq 0}$

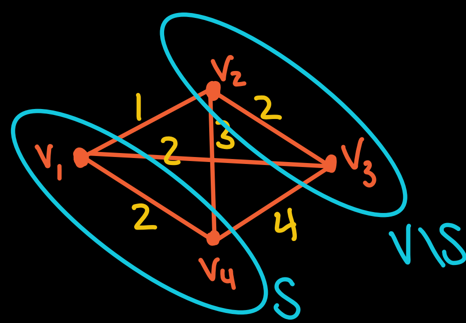
MAX-CUT PROBLEM:

$$\max \left\{ \sum_{e \in \delta(S)} w(e) \; : \; S \subseteq V \right\}$$



Ex:





$w(\delta(S)) = 1+2+3+4$
$= 10$

- Applications in clustering and image segmentation (want elements in S to be "close" to each other)

- Can be solved in polynomial time for special classes of graphs: planar (Hadlock 1975)
  no K_5 minor (Barahona 1983)

- NP hard to solve for general graphs (unlike min-cut!)
  $\Longrightarrow$ try to find polynomial-time algorithms that give approximations of optimal solution

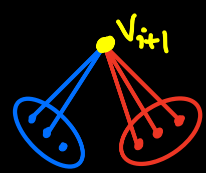
Erdös (1967) ½-approximation algorithm

Order vertices $V = \{v_1, \dots, v_n\}$ and color red/blue:

For $i = 1, \dots, n$, let $S_i = \{v_j : j \le i,\ v_j \text{ blue}\}$

$$T_i = \{v_1, \dots, v_i\} \setminus S_i = \{v_j : j \le i,\ v_j \text{ red}\}$$

Color v_{i+1} blue if $w(E(v_{i+1}, S_i)) < w(E(v_{i+1}, T_i))$ and red otherwise. Take $S = S_n$.



Claim: $w(\delta(S)) \ge \frac{1}{2} \operatorname{maxcut}(G)$ ← Ex

(Proof) In fact $w(\delta(S)) \ge \frac{1}{2} w(E) \ge \frac{1}{2} \operatorname{maxcut}(G)$.

Every edge $e = \{i, j\}$ considered exactly once (at step $\max\{i, j\}$). At each step i, $\ge \frac{1}{2}$ edge weight is cut. Formally:

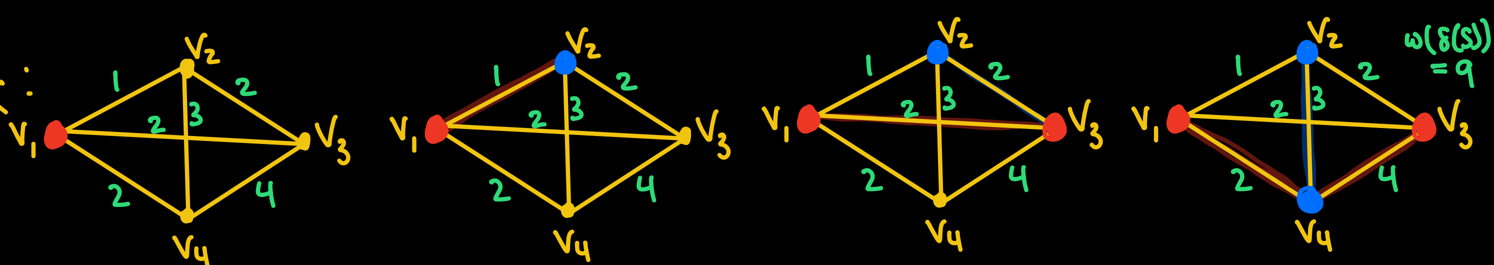
$$w(\{ij \in E : i > j\}) = w(E(i, S_{i-1})) + w(E(i, T_{i-1}))$$

color v_i s.t. larger set is cut!

$$\le 2\, w(\{ij \in \delta(S) : i > j\})$$

Summing over all $i = 1, \dots, n$, gives $w(E) \le 2\, w(\delta(S))$. □

Ex:



Håstad (1997) It is NP-Hard to find a cut $S \subseteq V$ with $w(\delta(S)) \ge \frac{16}{17} \operatorname{maxcut}(G)$.

Goemans-Williamson (1994)

There is a polynomial time alg. to find a cut with weight $\geq .878\,\mathrm{MAXCUT}(G)$

uses non-linear convex optimization called semidefinite programming

Goemans-Williamson Max-Cut Formulation

Given a graph $G=(V,E)$ with n vertices $V=[n]$, associate $S\subseteq[n]$ with $\chi_S = \begin{cases} 1 & \text{if } i\in S \\ -1 & \text{if } i\notin S. \end{cases}$

Claim: $\mathrm{maxcut}(G) = \max\left\{ \sum_{ij\in E} w(ij)\left(\frac{1-x_ix_j}{2}\right) : x\in\{\pm1\}^n \right\}$

(Proof) For $x\in\{\pm1\}^n$, $x=\chi_S$ for some $S\subseteq[n]$.

For $ij\in E$,

$$1-x_ix_j = \begin{cases} 0 & \text{if } i\in S,\ j\in S & \leftarrow 1-(1)(1)=0 \\ 0 & \text{if } i\notin S,\ j\notin S & \leftarrow 1-(-1)(-1)=0 \\ 2 & \text{if } ij\in\delta(S) & \leftarrow 1-(1)(-1)=1+1=2 \end{cases}$$

$$\Rightarrow \sum_{ij\in\delta(S)} w(ij) = \sum_{ij\in E} w(ij)\left(\frac{1-x_ix_j}{2}\right)$$ $\square$ Claim

<u>Idea</u>: replace $x_1,\dots,x_n\in\{\pm1\}^n$ with unit vectors $v_1,\dots,v_n\in\mathbb{R}^n$

$$GW(G) = \max\left\{ \sum_{ij\in E} w(ij)\left(\frac{1-\langle v_i,v_j\rangle}{2}\right) : v_1,\dots,v_n\in\mathbb{R}^n,\ \|v_i\|=1\ \forall i \right\}$$

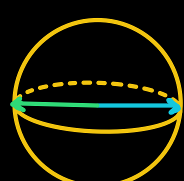
(can be solved with <u>SDP</u> - details next time)

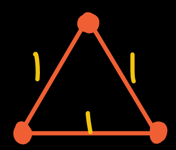
<u>Claim 1</u>: $\mathrm{maxcut}(G)\leq GW(G)$

For $x=(x_1,\dots,x_n)\in\{\pm1\}^n$, take $v_i=(x_i,0,\dots,0)\in\mathbb{R}^n$ for $i=1,\dots,n$.

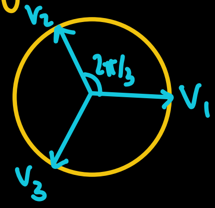
Then $\|v_i\|=1$ and $\langle v_i,v_j\rangle = x_ix_j$.

Note: Inequality can be strict!



Ex: 

$G = K_3$



$\langle v_i, v_j \rangle = \|v_i\| \cdot \|v_j\| \cos\left(\frac{2\pi}{3}\right) = -1/2$

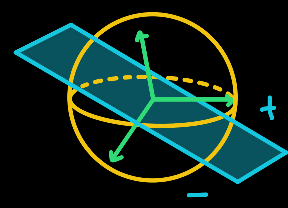
$$\sum_{ij \in E} w(ij)\left(\frac{1 - \langle v_i, v_j \rangle}{2}\right) = \sum_{ij \in E} 1\left(\frac{3}{4}\right) = \frac{9}{4} > \overset{\text{maxcut}}{2}$$

But not $\underset{\sim}{\text{too}}$ strict:

Idea: Probabilistic Rounding

Suppose $v_1, \ldots, v_n \in \mathbb{R}^n$ acheive max in $GW(G)$.

Let $H = \{x : \langle a, x \rangle = 0\}$ be a random hyperplane (through the origin) in $\mathbb{R}^n$.

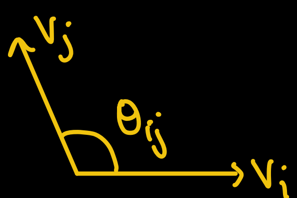


(Equivalent to choosing random $a \in \mathbb{S}^{n-1}$)

Take $S = \{ i \in [n] : \langle v_i, a \rangle > 0 \}$. (vectors on one side of H)

<u>Claim 2</u>: [Expected value of $\omega(\delta(S))$] $\geq .878\ \text{maxcut}(G)$

(Proof) For $e \in E$, what is the probability that $e \in \delta(S)$?



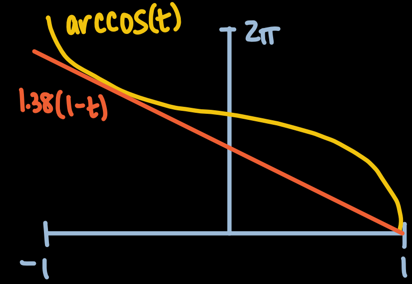
$$\text{Prob}(H \text{ separates } v_i, v_j) = \frac{\theta_{ij}}{\pi} = \frac{\arccos(\langle v_i, v_j \rangle)}{\pi}$$

Expected value of cut:

$$\sum_{ij \in E} w(ij) \frac{\theta_{ij}}{\pi} = \sum_{ij \in E} w(ij) \frac{\arccos(\langle v_i, v_j \rangle)}{\pi} \underset{(*)}{\geq} \sum_{ij \in E} w(ij) \frac{1.38005}{\pi}(1 - \langle v_i, v_j \rangle)$$

$$= \underbrace{2\left(\frac{1.38005}{\pi}\right)}_{\approx .878} \underbrace{\sum_{ij \in E} w(ij)\left(\frac{1 - \langle v_i, v_j \rangle}{2}\right)}_{= GW(G)}$$

(*) uses $\arccos(t) \geq 1.38005(1-t)$ for all $t \in [-1,1]$



Value $.878 \approx \min_{0 \leq \theta \leq \pi} \dfrac{\theta}{2\pi(1-\cos(\theta))}$

Construction derandomized by Alon-Spencer (2000)

It is an open question whether one can do better than a .878-approximation in polynomial time!