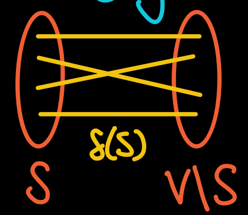
MA/AMA 514

Today: Max Cut and SDPs

Maximum cut of a graph

Let $G=(V,E)$ be a graph with edge weights $\omega: E \to \mathbb{R}_{\geq 0}$



MAX-CUT PROBLEM:

$$\max\left\{ \sum_{e \in \delta(S)} \omega(e) \; : \; S \subseteq V \right\}$$

$$\mathrm{maxcut}(G) = \max\left\{ \sum_{ij \in E} \omega(ij)\left(\frac{1 - x_i x_j}{2}\right) : x \in \{\pm 1\}^n \right\}$$

Idea: replace $x_1, \dots, x_n \in \{\pm 1\}^n$ with unit vectors $v_1, \dots, v_n \in \mathbb{R}^n$

$$GW(G) = \max\left\{ \sum_{ij \in E} \omega(ij)\left(\frac{1 - \langle v_i, v_j \rangle}{2}\right) : v_1, \dots, v_n \in \mathbb{R}^n, \; \|v_i\| = 1 \; \forall i \right\}$$

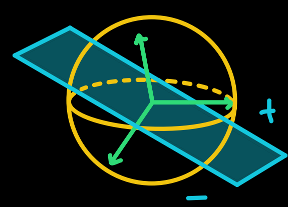
(can be solved with SDP - details next time)

Claim 1: $\mathrm{maxcut}(G) \leq GW(G)$

Idea: Probabilistic Rounding

Suppose $v_1, \dots, v_n \in \mathbb{R}^n$ acheive max in $GW(G)$.

Let $H = \{x : \langle a, x \rangle = 0\}$ be a random hyperplane (through the origin) in $\mathbb{R}^n$.

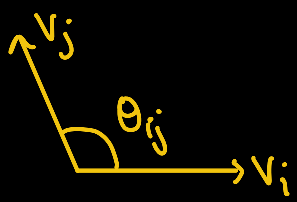


(Equivalent to choosing random $a \in \mathbb{S}^{n-1}$)

Take $S = \{ i \in [n] : \langle v_i, a \rangle > 0 \}$. (vectors on one side of H)

Claim 2: [Expected value of $w(\delta(S))$] $\geq .878\,\text{maxcut}(G)$

(Proof) For $e \in E$, what is the probability that $e \in \delta(S)$?



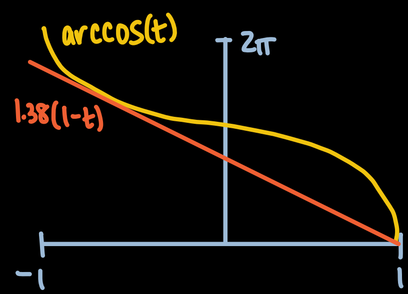
$$\text{Prob}(H \text{ separates } v_i, v_j) = \frac{\theta_{ij}}{\pi} = \frac{\arccos(\langle v_i, v_j \rangle)}{\pi}$$

Expected value of cut:

$$\sum_{ij \in E} w(ij)\frac{\theta_{ij}}{\pi} = \sum_{ij \in E} w(ij)\frac{\arccos(\langle v_i, v_j\rangle)}{\pi} \underset{(*)}{\geq} \sum_{ij \in E} w(ij)\frac{1.38005}{\pi}(1-\langle v_i, v_j\rangle)$$

$$= \underbrace{2\left(\frac{1.38005}{\pi}\right)}_{\approx .878} \underbrace{\sum_{ij \in E} w(ij)\left(\frac{1-\langle v_i, v_j\rangle}{2}\right)}_{= GW(G)}$$

(*) uses $\arccos(t) \geq 1.38005(1-t)$ for all $t \in [-1,1]$



$$\text{Value } .878 \approx \min_{0 \leq \theta \leq \pi} \frac{\theta}{2\pi(1-\cos(\theta))}$$

Construction derandomized by Alon-Spencer (2000)

It is an open question whether one can do better than a .878-approximation in polynomial time!

How do we compute $GW(G)$? Semidefinite Programs!

The PSD Cone in $\mathbb{R}^{n\times n}_{sym}$.

The set of real symmetric matrices is a real vectorspace of dimension $\binom{n+1}{2}$.

Inner product: $\langle A,B\rangle = \text{trace}(AB)$

Ex: $A=\begin{pmatrix} a_{11} & a_{12} \\ a_{12} & a_{22}\end{pmatrix}$ $B=\begin{pmatrix} b_{11} & b_{12} \\ b_{12} & b_{22}\end{pmatrix}$

$$\langle A,B\rangle = \text{trace}(AB) = \text{trace}\begin{pmatrix} a_{11}b_{11}+a_{12}b_{12} & a_{12}b_{11}+a_{22}b_{12} \\ a_{12}b_{11}+a_{22}b_{12} & a_{12}b_{12}+a_{22}b_{22}\end{pmatrix}$$

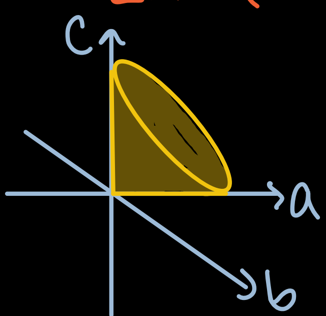
$$= a_{11}b_{11} + 2a_{12}b_{12} + a_{22}b_{22}$$

<u>Linear Alg. Fact</u>: all eigenvalues of $A\in\mathbb{R}^{n\times n}_{sym}$ are real

Call $A\in\mathbb{R}^{n\times n}_{sym}$ <u>positive semidefinite</u> if all of its eigenvalues are nonnegative (denoted "$A\succeq 0$")

Let $PSD_n = \{A\in\mathbb{R}^{n\times n}_{sym} : A\succeq 0\}$.

Ex: $(n=2)$ $\mathbb{R}^{2\times 2}_{sym} = \left\{\begin{pmatrix} a & b \\ b & c\end{pmatrix} : a,b,c\in\mathbb{R}\right\} \cong \mathbb{R}^3$



eigval of $\begin{pmatrix} a & b \\ b & c\end{pmatrix}$ = roots of $(t-a)(t-c)-b^2$

$$= \frac{a+c\pm\sqrt{(a-c)^2+4b^2}}{2}$$

$$\geq 0 \iff a\geq 0,\ c\geq 0,\ ac\geq b^2$$

Characterizations of PSD matrices

For $A \in \mathbb{R}^{n\times n}_{sym}$, TFAE:

1) all eig. val. of $A \geq 0$

2) all principal minors $\det(A_{I,I}) \geq 0$

3) $v^T A v \geq 0$ for all $v \in \mathbb{R}^n$ $\Leftarrow$ implies that PSD_n is a convex cone?

4) $A = BB^T$ for some $B \in \mathbb{R}^{n\times k}$, $k = \text{rank}(A)$

$= \sum_{i=1}^{r} c_i c_i^T = (\langle r_i, r_j \rangle)_{1\leq i,j\leq n}$ where $c_1, \dots, c_k$ = col of B

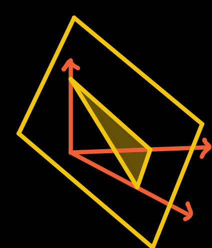
$r_1, \dots, r_n$ = rows of B

Ex: $A = \begin{pmatrix} 1 & 2 \\ 2 & 5 \end{pmatrix} \succeq 0$ $\quad A = BB^T$ with $B = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$

$= \begin{pmatrix} 1 \\ 2 \end{pmatrix}\begin{pmatrix} 1 & 2 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \end{pmatrix} = \begin{pmatrix} (1,0)^T(1,0) & (2,1)^T(1,0) \\ (2,1)^T(1,0) & (2,1)^T(2,1) \end{pmatrix}$

Cor: PSD_n is a convex cone in $\mathbb{R}^{n\times n}_{sym}$ and is self-dual under $\langle A, B \rangle = \text{trace}(AB)$.

$A \in PSD_n \iff \langle A, B \rangle \geq 0$ for all $B \in PSD_n$.



Recall: One standard form for LP:

Given $a_1, \dots, a_m \in \mathbb{R}^n$, $b_1, \dots, b_m \in \mathbb{R}$, $c \in \mathbb{R}^n$

$\max\{c^T x : x \geq 0, \; a_i^T x = b_i \text{ for } i = 1, \dots m\}$

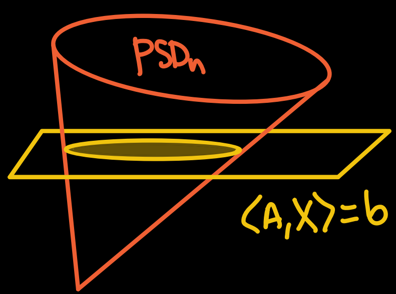
Transform inequality "$a_i^T x \leq b_i$" to equality "$a_i^T x + s_i = b_i$" by adding additional variable s_i and requiring $s_i \geq 0$

Semidefinite Programming

A semidefinite program (SDP) has the form

$$(\text{SDP}) \quad \max\{\langle C, X\rangle : X \in \text{PSD}_n,\ \langle A_i, X\rangle = b_i,\ i=1,\dots,m\}$$

where $A_1,\dots,A_m \in \mathbb{R}^{n\times n}_{sym}$, $b_1,\dots,b_m \in \mathbb{R}$, $C \in \mathbb{R}^{n\times n}_{sym}$



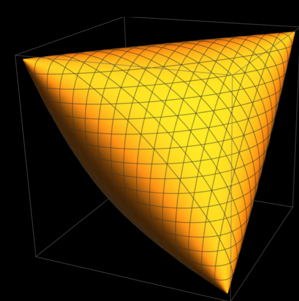
Can be solved* in polynomial time using interior point methods

Ex: $(n=3, m=3)$ $A_1 = \begin{pmatrix} 1&0&0\\0&0&0\\0&0&0\end{pmatrix}$ $A_2 = \begin{pmatrix} 0&0&0\\0&1&0\\0&0&0\end{pmatrix}$ $A_3 = \begin{pmatrix} 0&0&0\\0&0&0\\0&0&1\end{pmatrix}$

$C = \begin{pmatrix} 0&-1&-1\\-1&0&-1\\-1&-1&0\end{pmatrix}$ $\quad b_1 = 1 \quad b_2 = 1 \quad b_3 = 1$

$$\max\{\langle C, X\rangle : X \succeq 0,\ \langle A_i, X\rangle = b_i,\ i=1,2,3\}$$

$$= \max\left\{-2x_{12} - 2x_{13} - 2x_{23} : \begin{pmatrix} 1 & x_{12} & x_{13}\\ x_{12} & 1 & x_{23}\\ x_{13} & x_{23} & 1\end{pmatrix} \succeq 0\right\}$$



acheived by $(x_{ij}^*) = (\frac{-1}{2}, \frac{-1}{2}, \frac{-1}{2})$

Goemans-Williamson approx for MAXCUT(G)

$$GW(G) = \max\left\{\sum_{ij\in E} \omega(ij)\left(\frac{1-\langle v_i, v_j\rangle}{2}\right) : \begin{array}{l} v_1,\dots,v_n \in \mathbb{R}^n \\ \|v_i\| = 1\ \forall i \end{array}\right\}$$

For $i=1,\dots,n$, take $A_i = e_i e_i^T$ (that is, $(A_i)_{jk} = \begin{cases} 1 & \text{if } j=k=i \\ 0 & \text{o.w} \end{cases}$)

$b_1 = \dots = b_n = 1$, $C \in \mathbb{R}^{n\times n}_{sym}$ with $C_{jk} = \begin{cases} -\omega(jk) & \text{if } jk \in E \\ 0 & \text{o.w} \end{cases}$

Claim: $GW(G) = \frac{1}{2}w(E) + \frac{1}{4}\alpha_{SDP}$ where

$$\alpha_{SDP} = \max\left\{\langle C, X\rangle : X \succeq 0,\ \underbrace{\langle A_i, X\rangle = b_i,\ i=1,\dots,n}_{\Leftrightarrow\ X_{ii}=1}\right\}.$$

(Proof) A matrix $X \in \mathbb{R}^{n\times n}_{sym}$ is PSD $\Leftrightarrow \exists\, v_1,\dots,v_n \in \mathbb{R}^n$ s.t. $X_{ij} = \langle v_i, v_j\rangle\ \forall i,j$

$$\left(\begin{array}{l} X \succeq 0 \\ \Leftrightarrow X = BB^T \\ \Leftrightarrow X_{ij} = \langle v_i, v_j\rangle\ \forall i,j \\ \quad v_i = i^{th} \text{ row of } B \end{array}\right)$$

Note: $X_{ii} = 1 \Leftrightarrow \langle v_i, v_i\rangle = \|v_i\|^2 = 1$.

$$\langle C, X\rangle = 2\sum_{ij\in E}(-w(ij))X_{ij} = -2\sum_{ij\in E} w(ij)\langle v_i, v_j\rangle$$

$$GW(G) = \frac{1}{2}w(E) + \frac{1}{2}\max\left\{\sum_{ij\in E}\underbrace{(-w(ij))\langle v_i, v_j\rangle}_{=C_{ij}X_{ij}} : \begin{array}{c} v_1,\dots,v_n \in \mathbb{R}^n \\ \|v_i\|=1\ \forall i \end{array}\right\}$$

$$= \frac{1}{2}w(E) + \frac{1}{2}\max\left\{\frac{1}{2}\langle C, X\rangle : X \in PSD_n,\ X_{ii} = 1\ \forall i\right\}$$

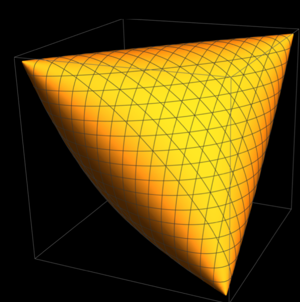
Ex: $(n=3, m=3)$ $G = K_3 \rightsquigarrow C = \begin{pmatrix} 0 & -1 & -1 \\ -1 & 0 & -1 \\ -1 & -1 & 0 \end{pmatrix}$

$$A_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad A_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad A_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$b_1 = 1 \qquad b_2 = 1 \qquad b_3 = 1$$

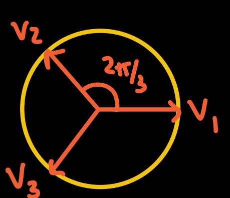
$$\alpha_{SDP} = \max\{\langle C, X\rangle : X \succeq 0,\ \langle A_i, X\rangle = b_i,\ i=1,2,3\}$$

$$= \max\left\{-2x_{12} - 2x_{13} - 2x_{23} : \begin{pmatrix} 1 & x_{12} & x_{13} \\ x_{12} & 1 & x_{23} \\ x_{13} & x_{23} & 1 \end{pmatrix} \succeq 0\right\} = 3$$



$$X^* = \begin{pmatrix} \langle v_1, v_1\rangle & \langle v_1, v_2\rangle & \langle v_1, v_3\rangle \\ \langle v_2, v_1\rangle & \langle v_2, v_2\rangle & \langle v_2, v_3\rangle \\ \langle v_3, v_1\rangle & \langle v_3, v_2\rangle & \langle v_3, v_3\rangle \end{pmatrix} = \begin{pmatrix} 1 & -1/2 & -1/2 \\ -1/2 & 1 & -1/2 \\ -1/2 & -1/2 & 1 \end{pmatrix}$$

for



$$GW(K_3) = \frac{3}{2} + \frac{3}{4} = \frac{9}{4}$$