MA/AMA 514

Today: TU matrices from graphs (§8.4)

A matrix $A \in \mathbb{R}^{m \times n}$ is <u>totally unimodular</u> (TU) if every square submatrix has determinant $0, \pm 1$.

Ex: $A = \begin{pmatrix} 1 & 1 \end{pmatrix}$ $A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}$ Non-ex: $A = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}$ Det(A) = -2?

A TU, $b \in \mathbb{Z}^m \Rightarrow P = \{x \in \mathbb{R}^n : Ax \leq b\}$ is integral

$\Rightarrow$ the LP $\max\{c^T x : x \in P\}$ solves the IP $\max\{c^T x : x \in P \cap \mathbb{Z}^n\}$

<u>Cor</u>: If A is TU, $b \in \mathbb{Z}^m$ and $\ell, u \in \mathbb{Z}^n$ then each of the following polyhedra are integer:

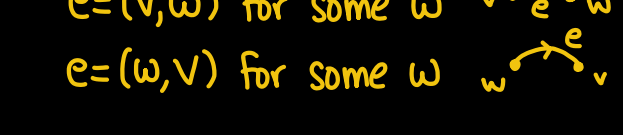
1) $\{x \in \mathbb{R}^n : Ax \leq b, x \geq 0\}$
2) $\{x \in \mathbb{R}^n : Ax \leq b, \ell \leq x \leq u\}$
3) $\{x \in \mathbb{R}^n : Ax = b, x \geq 0\}$
4) $\{x \in \mathbb{R}^n : Ax = b, \ell \leq x \leq u\}$

<u>TU matrices from directed graphs</u>

$G = (V, E)$ directed graph

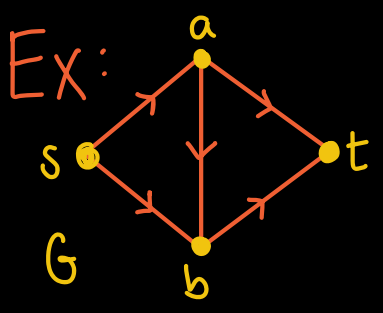
Encode into node edge incidence matrix $A \in \{0, \pm 1\}^{V \times E}$

$$A_{ve} = \begin{cases} 1 & \text{if } e \in \delta^{out}(v) \quad e = (v, w) \text{ for some } w \\ -1 & \text{if } e \in \delta^{in}(v) \quad e = (w, v) \text{ for some } w \\ 0 & \text{o.w.} \end{cases}$$



For $x \in \mathbb{R}^E$, $Ax \in \mathbb{R}^V$ with $(Ax)_v = \sum_{e \in \delta^{out}(v)} x_e - \sum_{e \in \delta^{in}(v)} x_e$ ← netflow out of v

Ex:



$$A = \begin{array}{c} \\ s \\ a \\ b \\ t \end{array} \begin{array}{c} \begin{array}{ccccc} sa & sb & ab & at & bt \end{array} \\ \begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ -1 & 0 & 1 & 1 & 0 \\ 0 & -1 & -1 & 0 & 1 \\ 0 & 0 & 0 & -1 & -1 \end{pmatrix} \end{array} \qquad Ax = \begin{pmatrix} x_{sa} + x_{sb} \\ -x_{sa} + x_{ab} + x_{at} \\ -x_{sb} - x_{ab} + x_{bt} \\ -x_{at} - x_{bt} \end{pmatrix}$$

↑ has one +1, one -1 in each column

Prop: The node-edge incidence matrix of any directed graph is totally unimodular.

Actually show something slightly stronger:

Prop: Let $A \in \{0, \pm 1\}^{n \times m}$ be a matrix with at most one "1" and at most one "-1" in each column. Then A is TU.

(Proof) Let M be a $k \times k$ submatrix of A. We induct on k.

($k=1$) M = an entry of $A \Rightarrow \det(M) = M = 0, \pm 1$ ✓

($k>1$) Case 1: M has a column of 0's $\Rightarrow \det(M) = 0$

Case 2: M has a column w/ exactly one nonzero entry

$M = \begin{pmatrix} \pm 1 & v^T \\ 0 & \\ \vdots & \tilde{M} \\ 0 & \end{pmatrix} \Rightarrow \det(M) = \pm \det(\tilde{M})$ where $\tilde{M}$ is a sq. submatrix of size $k-1 \Rightarrow \det(M) = 0, \pm 1$

Case 3: Every column of M has one 1 and one -1

$\Rightarrow$ the vector $(1,..,1)$ belongs to the left kernel of M

$\Rightarrow \det(M) = 0$

Applications to Network Flow

$G=(V,E)$ directed graph, $s,t \in V$, $c: E \to \mathbb{Z}_{\geq 0}$

Equations defining s-t flows $f: E \to \mathbb{R}_{\geq 0}$, $f(e) = x_e$

$$\sum_{e \in \delta^{out}(v)} x_e = \sum_{e \in \delta^{in}(v)} x_e \quad \forall v \in V \setminus \{s,t\}$$

Cor: For integer edge capacities $c: E \to \mathbb{Z}_{\geq 0}$, the set of s-t flows equals the polytope

$$P_{flow} = \{ x \in \mathbb{R}^E : \tilde{A}x = 0,\ 0 \leq x \leq c \}$$

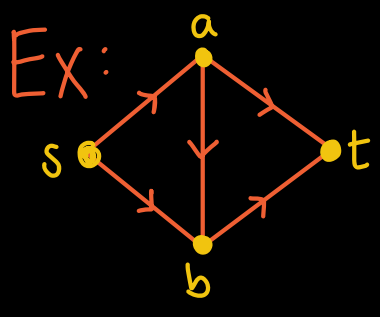
where $\tilde{A}$ is the $V \setminus \{s,t\} \times E$ submatrix of A.

Moreover, all vertices of P_{flow} are integer.

(Proof) A TU $\Rightarrow$ $\tilde{A}$ TU $\Rightarrow$ integer vertices

↑ by Cor about $\{x \in \mathbb{R}^n : Ax = b,\ l \leq x \leq u\}$

Ex:



$$A = \begin{array}{c} \\ s \\ a \\ b \\ t \end{array} \begin{array}{c} \begin{array}{ccccc} sa & sb & ab & at & bt \end{array} \\ \left(\begin{array}{ccccc} 1 & 1 & 0 & 0 & 0 \\ -1 & 0 & 1 & 1 & 0 \\ 0 & -1 & -1 & 0 & 1 \\ 0 & 0 & 0 & -1 & -1 \end{array} \right) \end{array}$$

$\tilde{A}$

$$Ax = \begin{pmatrix} x_{sa} + x_{sb} \\ -x_{sa} + x_{ab} + x_{at} \\ -x_{sb} - x_{ab} + x_{bt} \\ -x_{at} - x_{bt} \end{pmatrix}$$

$\tilde{A}x$

Cor: For $c: E \to \mathbb{Z}_{\geq 0}$, the max value of an s-t flow is acheived by an integer flow $f: E \to \mathbb{Z}_{\geq 0}$

(objective function $= \sum_{e \in \delta^{out}(s)} x_e - \sum_{e \in \delta^{in}(s)} x_e$, linear in x_e)

Cor: Let $D=(V,E)$ be a directed graph and $c: E \to \mathbb{Z}$, $d: E \to \mathbb{Z}$. If there exists a circulation $f: E \to \mathbb{R}$ with $d \leq f \leq c$, then there exists a circulation $f': E \to \mathbb{Z}$ with $d \leq f' \leq c$.

(Proof) The set of circulations on D is

$$P = \{x \in \mathbb{R}^E : Ax = 0, \quad d \leq x \leq c\}$$

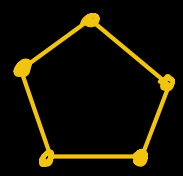
where A is the incidence matrix of D.
Since A is TU and each coordinate is bounded, P is an integral polytope. Since $f \in P$, it is nonempty and so has a vertex, f', which is integer.

§8.3 Connections with bipartite graphs

The incidence matrix A_G of a graph $G=(V,E)$ is a $|V| \times |E|$ matrix with $(A_G)_{ve} = \begin{cases} 1 & \text{if } v \in e \\ 0 & \text{if } v \notin e \end{cases}$

Thm 8.3 G is bipartite $\Leftrightarrow$ A_G is totally unimodular

(Proof) ($\Leftarrow$) G not bipartite $\Rightarrow$ G has an odd circuit.



$$\longleftrightarrow \begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 \end{pmatrix}$$

subgraph $\longleftrightarrow$ submatrix

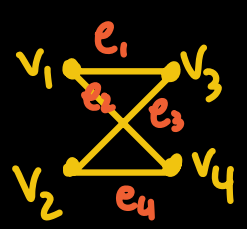
Corresponding submatrix has determinant ± 2.

$\Rightarrow A_G$ not TU

($\Rightarrow$) Suppose G is bipartite with $V = U \uplus W$.
Consider directed graph D obtained by orienting all edges from $U \to W$. The incidence matrix A_D is obtained from A_G by multiplying rows indexed by $w \in W$ by -1.

A_D is TU $\Rightarrow A_G = \begin{pmatrix} I_U & 0 \\ 0 & -I_W \end{pmatrix} A_D$ is TU.

Ex:



$$\begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \end{array} \begin{array}{c} \begin{array}{cccc} e_1 & e_2 & e_3 & e_4 \end{array} \\ \left(\begin{array}{cccc} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ \hline -1 & 0 & -1 & 0 \\ 0 & -1 & 0 & -1 \end{array}\right) \end{array} \begin{array}{l} \\ \left.\begin{array}{c} \\ \\ \end{array}\right] U \\ \left.\begin{array}{c} \\ \\ \end{array}\right] W \end{array}$$