MA|AMA 514

Today: TU matrices from graphs (§8.3, §8.4)

From last time:

A matrix $A \in \mathbb{R}^{m\times n}$ is <u>totally unimodular</u> (TU) if every square submatrix has determinant $0, \pm 1$.

Ex: $A = \begin{pmatrix} 1 & \\ & 1 \end{pmatrix}$ $A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}$ Non-ex: $A = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}$ Det(A) $= -2$?

<u>Prop</u>: Let $A \in \{0, \pm 1\}^{n\times m}$ be a matrix with at most one "1" and at most one "-1" in each column. Then A is TU.

<u>Cor</u>: The node-edge incidence matrix of any directed graph is totally unimodular.

incidence matrix $A \in \{0,\pm1\}^{V\times E}$ of a directed graph:

$$A_{ve} = \begin{cases} 1 & \text{if } e \in \delta^{out}(v) \\ -1 & \text{if } e \in \delta^{in}(v) \\ 0 & \text{o.w.} \end{cases}$$

$e = (v, w)$ for some w 

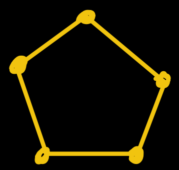
$e = (w, v)$ for some w 

<u>§8.3 Connections with bipartite graphs</u>

The <u>incidence matrix</u> A of a graph $G = (V, E)$ is a $|V| \times |E|$ matrix with $(A)_{ve} = \begin{cases} 1 & \text{if } v \in e \\ 0 & \text{if } v \notin e \end{cases}$

Thm 8.3 G is bipartite $\Leftrightarrow$ A is totally unimodular

(Proof) ($\Leftarrow$) G not bipartite $\Rightarrow$ G has an odd circuit.

 $\longleftrightarrow$ $\begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 \end{pmatrix}$

subgraph $\longleftrightarrow$ submatrix

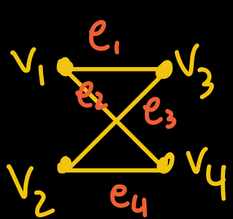
Corresponding submatrix has determinant ± 2.

$\Rightarrow$ A not TU

($\Rightarrow$) Suppose G is bipartite with $V = U \uplus W$.
Consider directed graph D obtained by orienting all edges from $U \to W$. The incidence matrix of D is obtained from that of G by multiplying rows indexed by $w \in W$ by -1.

A_D is TU $\Rightarrow$ $A_G = \begin{pmatrix} I_U & 0 \\ 0 & -I_W \end{pmatrix} A_D$ is TU.

Ex:



$$\begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \end{array} \begin{array}{c} \begin{array}{cccc} e_1 & e_2 & e_3 & e_4 \end{array} \\ \left(\begin{array}{cccc} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ \hline -1 & 0 & -1 & 0 \\ 0 & -1 & 0 & -1 \end{array}\right) \end{array} \begin{array}{l} \\ \left.\begin{array}{c} \\ \\ \end{array}\right] U \\ \left.\begin{array}{c} \\ \\ \end{array}\right] W \end{array}$$

Cor: For any bipartite graph G, the polyhedra are integer:

$\{x \in \mathbb{R}^E_{\geq 0} : Ax \leq \mathbb{1}_V\}$ $\quad$ $\{y \in \mathbb{R}^V_{\geq 0} : y^T A \geq \mathbb{1}_E^T\}$

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Cor: For bipartite graphs, $\{x \in \mathbb{R}^E_{\geq 0} : Ax \leq \mathbb{1}_V\} = \mathrm{Conv}\{\mathbb{1}_M : M \subseteq E \text{ matching}\}$

($\supseteq$) Clear.

($\subseteq$) All vertices integer $\Rightarrow$ equal to $\mathbb{1}_M$ for some matching M

Cor: For bipartite graphs, $\alpha(G) = \max\{\sum_{e \in E} x_e : x \in \mathbb{R}^E_{\geq 0}, Ax \leq \mathbb{1}_V\}$

Similar results for $\nu(G)$, $\tau(G)$, $\rho(G)$.

Cor: For any bipartite graph G, $\nu(G) = \tau(G)$ and $\alpha(G) = \rho(G)$

(By linear programming duality!)

Applications to interval scheduling

Call $A \in \{0,1\}^{m \times n}$ an <u>interval matrix</u> if the 1's in each row are consecutive (transposed version also used: consecutive 1's in each column).

<u>**Prop**</u>: If $A \in \{0,1\}^{m \times n}$ is an interval matrix then A is totally unimodular.

(Proof) Let M be a $k \times k$ submatrix of A^T.

$\Rightarrow M$ has consec. 1's.

$$\begin{pmatrix} 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{pmatrix}$$

Take $U \in \mathbb{R}^{k \times k}$ with $U_{ij} = \begin{cases} 1 & \text{if } j = i \\ -1 & \text{if } j = i-1 \\ 0 & \text{o.w} \end{cases}$ $\qquad U = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix}$

$\Rightarrow \det(U) = 1$ and UM has i^{th} row $m_i - m_{i-1}$ where $m_i = i^{th}$ row of M

Since M had consecutive ones, UM has at most one 1 and at most one -1 in each column $\Rightarrow UM$ totally unimod

$\Rightarrow \det(M) = \det(UM) \in \{0, \pm 1\}$

Ex:
$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & -1 & 1 \end{pmatrix}$$

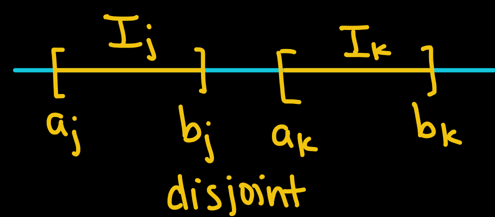
Cor: All vertices of $\{x \in \mathbb{R}^n : Ax \leq 1,\ 0 \leq x \leq 1\}$ are integer

Interval Scheduling

Input: intervals $I_1, \ldots, I_n$ with values $c_1, \ldots, c_n \in \mathbb{R}$

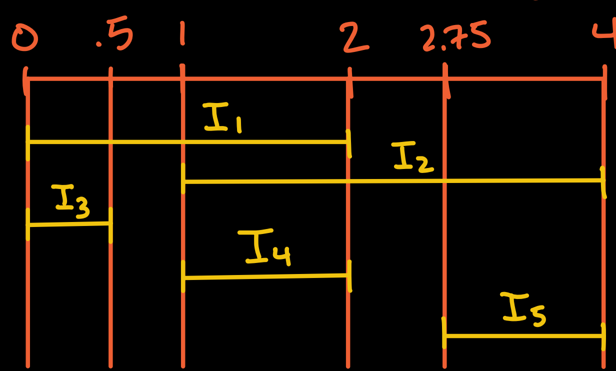
Goal: Select a disjoint subset of intervals $I_{i_1}, \ldots, I_{i_k}$ to maximize value $c_{i_1} + \ldots + c_{i_k}$

$$I_j = [a_j, b_j] \subseteq \mathbb{R}$$



Ex: $I_1 = [0,2]$, $I_2 = [1,4]$, $I_3 = [0,.5]$, $I_4 = [1,2]$, $I_5 = [2.75, 4]$

$c_1 = 4 \quad c_2 = 5 \quad c_3 = 1 \quad c_4 = 2 \quad c_5 = 3$



Some feas. sol. and values

$\{I_1, I_5\} \longrightarrow c_1 + c_5 = 7$

$\{I_2, I_3\} \longrightarrow c_2 + c_3 = 6$

$\{I_3, I_4, I_5\} \longrightarrow c_3 + c_4 + c_5 = 6$

Interval scheduling IP:

variables $x_1, \ldots, x_n \in \{0,1\}$ $\quad x_j = \begin{cases} 1 \leftrightarrow \text{pick } I_j \\ 0 \leftrightarrow \text{don't pick } I_j \end{cases}$

How to encode disjointness of chosen intervals?

Let $t_1 < t_2 < \dots < t_m$ be the set of all endpoints $\bigcup_{j=1}^{n} \{a_j, b_j\}$.

<u>Claim</u>: $\{I_{i_1}, \dots, I_{i_k}\}$ are disjoint

$\iff$ for all $j = 1, \dots, m$, at most one I_{i_ℓ} contains t_j

(Proof)($\Rightarrow$) $t_j \in I_{i_1} \cap I_{i_2} \Rightarrow I_{i_1}, I_{i_2}$ not disjoint!

($\Leftarrow$) If $I_1 \cap I_2$ not disjoint, $\max(I_{i_1} \cap I_{i_2}) = b_{i_1}$ or b_{i_2}
$= t_j$ for some j

Interval scheduling solved by

(IP) $\max\ c^T x$ s.t. $0 \le x \le 1,\ x \in \mathbb{Z}^n$

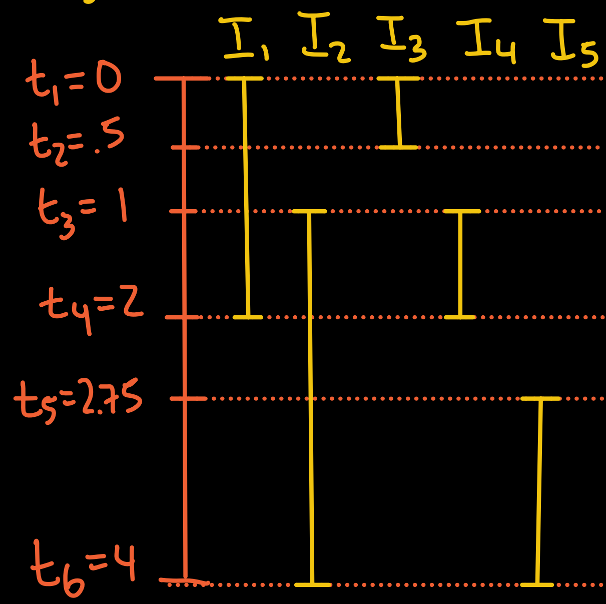
$$\sum_{i \text{ s.t. } t_j \in I_i} x_i \le 1 \quad \text{for all } j = 1, \dots, m$$

Encode as $Ax \le 1$

with $A \in \{0,1\}^{m \times n}$, $A_{ji} = \begin{cases} 1 & \text{if } t_j \in I_i \\ 0 & \text{if } t_j \notin I_i \end{cases}$

Ex (above)

$$A = \begin{array}{c} \\ t_1 \\ t_2 \\ t_3 \\ t_4 \\ t_5 \\ t_6 \end{array} \begin{array}{c} \begin{array}{ccccc} I_1 & I_2 & I_3 & I_4 & I_5 \end{array} \\ \begin{pmatrix} 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 \end{pmatrix} \end{array}$$



Note: since $t_1 < t_2 < \dots < t_m$, $t_j \in I_i, t_k \in I_i \Rightarrow t_\ell \in I_i\ \forall\ j \le \ell \le k$

$\Longrightarrow$ the ones in each column of A are consecutive

<u>Cor</u>: The LP relaxation

(LP) $\max \; c^T x$ s.t. $0 \le x \le 1$, $Ax \le 1$

solves interval scheduling (in polynomial time).

(Proof) Opt. val. attained at a vertex x^*

$\Rightarrow x^*$ integer $\Rightarrow x^* \in \{0,1\}^n$

$\Rightarrow$ take $I_{i_1}, \dots, I_{i_k}$ where $\{i_1, \dots, i_k\} = \{i : x_i^* = 1\}$