MA/AMA 514

Today: Approximation Algorithms

Non-bipartite graphs

$\max\{|M| : M \subseteq E \text{ matching}\} \le \min\{|U| : U \subseteq V \text{ vertex cover}\}$

Why? Every $e \in M$ covered by some $u \in U$.
Since M is a matching, map $e \to u$ is injective.

This "$\le$" can be strict. E.g. $G = K_3$

Blossom algorithm computes max size matching in poly. time $O(|V|^5)$. ($\to O(|V|^3)$ with careful implementation)

Computing min vertex cover is NP-hard!

In fact, NP-hard to compute within a factor of 1.4!

Today: Computing within a factor of 2 in poly. time. That is, β s.t. $\beta \le |U^*| \le 2\beta$

Approximation Algorithm 1: Fractional Rounding

Input: $G = (V, E)$ undirected (not necessarily bipartite)

Goal: min vertex cover $S \subseteq V$ (with $|e \cap S| \ge 1 \ \forall e \in E$)

Formulation as an IP:

$$\min \sum_{v \in V} y_v \quad \text{s.t. } 0 \le y \le 1,\ y_u + y_v \ge 1\ \ \forall \{u,v\} \in E,\ y \in \mathbb{Z}^V$$

Let y^*_{IP} = opt. sol. for (IP) $= \mathbb{1}_S$ for min vertex cover $S \subseteq V$

y^*_{LP} = " " LP relaxation

(Rounding) Define $\tilde{y} \in \{0,1\}^V$ by $\tilde{y}_v = \begin{cases} 1 & \text{if } (y^*_{LP})_v \ge 1/2 \\ 0 & \text{if } (y^*_{LP})_v < 1/2 \end{cases}$

Claim 1: $\tilde{y}$ is feasible for (IP) (i.e. $\tilde{y} = \mathbb{1}_S$, $S \subseteq V$ a vertex cover)

(Proof) For $\{u,v\} \in E$, $(y^*_{LP})_u + (y^*_{LP})_v \ge 1$ $\Rightarrow$ at least one $\ge 1/2$

$\Rightarrow \tilde{y}_u + \tilde{y}_v \ge 1$

Claim 2: $\mathbb{1}^T y^*_{LP} \underset{(1)}{\le} \mathbb{1}^T y^*_{IP} \underset{(2)}{\le} \mathbb{1}^T \tilde{y} \underset{(3)}{\le} 2 y^*_{LP}$

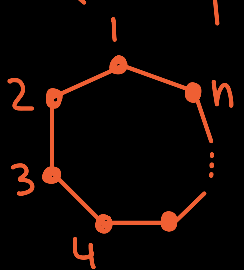
(Proof) (1) minimizing over larger set

(2) $\tilde{y}$ feasible for (IP), $m^*_{IP} = \min \mathbb{1}^T y$ over all feas. y

(3) $\sum_{v \in V} \tilde{y}_v \le \sum_{v \in V} 2 (y^*_{LP})_v = 2 \mathbb{1}^T y^*_{LP}$

Ex (n-cycle) $\min \sum_{i=1}^{n} y_i$ s.t. $0 \le y \le 1$, $y_i + y_{i+1} \ge 1\ \forall i = 1, \dots, n-1$

$y_1 + y_n \ge 1$



$y^*_{LP} = (1/2, \dots, 1/2)$ $\mathbb{1}^T y^*_{LP} = n/2$

round

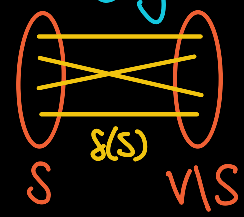
$\tilde{y} = (1, \dots, 1)$ $\mathbb{1}^T \tilde{y} = n$

Maximum cut of a graph

Let $G=(V,E)$ be a graph with edge weights $w: E \to \mathbb{R}_{\geq 0}$

MAX-CUT PROBLEM:

$$\max \left\{ \sum_{e \in \delta(S)} w(e) \; : \; S \subseteq V \right\}$$



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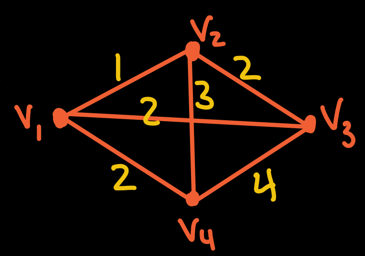
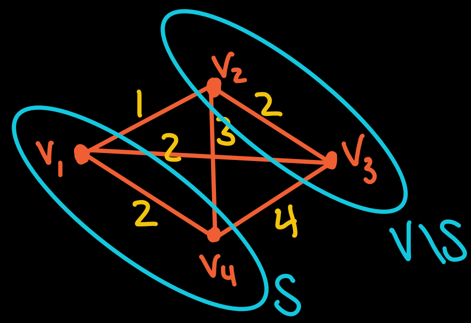




$w(\delta(S)) = 1+2+3+4$
$= 10$

- Applications in clustering and image segmentation (want elements in S to be "close" to each other)

- Can be solved in polynomial time for special classes of graphs: planar (Hadlock 1975)
  no K_5 minor (Barahona 1983)

- NP hard to solve for general graphs (unlike min-cut!)
  $\Rightarrow$ try to find polynomial-time algorithms that give approximations of optimal solution

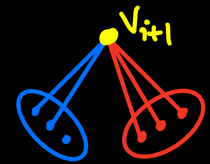
Erdös (1967) $\frac{1}{2}$-approximation algorithm

Order vertices $V = \{v_1, \dots, v_n\}$ and color red/blue:

For $i = 1, \dots, n$, let $S_i = \{v_j : j \le i,\ v_j \text{ blue}\}$

$$T_i = \{v_1, \dots, v_i\} \setminus S_i = \{v_j : j \le i,\ v_j \text{ red}\}$$

Color v_{i+1} blue if $\omega(E(v_{i+1}, S_i)) < \omega(E(v_{i+1}, T_i))$ and red otherwise. Take $S = S_n$.



Claim: $\omega(\delta(S)) \ge \frac{1}{2} \operatorname{maxcut}(G)$

(Proof) In fact $\omega(\delta(S)) \ge \frac{1}{2} \omega(E) \ge \frac{1}{2} \operatorname{maxcut}(G)$.

Every edge $e = \{i, j\}$ considered exactly once (at step $\max\{i, j\}$). At each step i, $\ge \frac{1}{2}$ edge weight is cut. Formally:

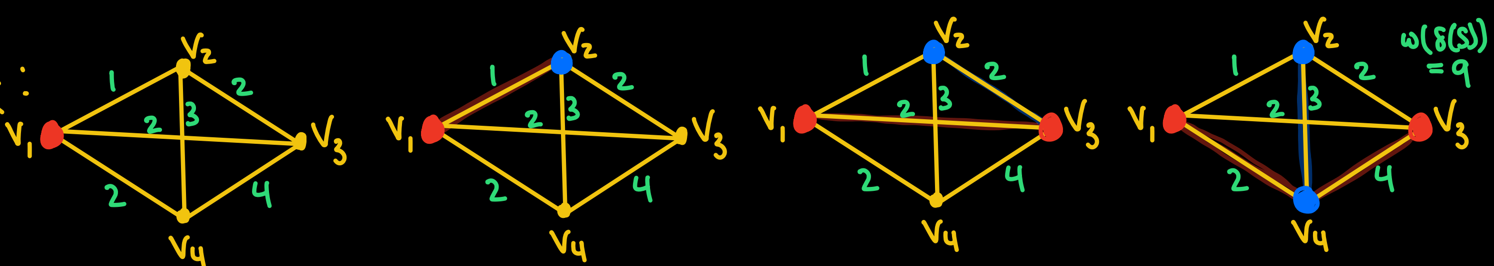
$$\omega(\{ij \in E : i > j\}) = \omega(E(i, S_{i-1})) + \omega(E(i, T_{i-1}))$$

color v_i s.t. larger set is cut!

$$\le 2\,\omega(\{ij \in \delta(S) : i > j\})$$

Summing over all $i = 1, \dots, n$, gives $\omega(E) \le 2\,\omega(\delta(S))$. $\square$

Ex:



Håstad (1997)

It is NP-Hard to find a cut $S \subseteq V$ with $\omega(\delta(S)) \ge \frac{16}{17} \operatorname{maxcut}(G)$.

Goemans-Williamson (1994) There is a polynomial time alg. to find a cut with weight $\geq .878\,\mathrm{MAXCUT}(G)$

uses non-linear convex optimization called semidefinite programming

Goemans-Williamson Max-Cut Formulation

Given a graph $G=(V,E)$ with n vertices $V=[n]$, associate $S\subseteq[n]$ with $\chi_S = \begin{cases} 1 & \text{if } i\in S \\ -1 & \text{if } i\notin S. \end{cases}$

Claim: $\mathrm{maxcut}(G) = \max\left\{ \sum_{ij\in E} w(ij)\left(\frac{1-x_ix_j}{2}\right) : x\in\{\pm1\}^n \right\}$

(Proof) For $x\in\{\pm1\}^n$, $x=\chi_S$ for some $S\subseteq[n]$.

For $ij\in E$,
$$1-x_ix_j = \begin{cases} 0 & \text{if } i\in S,\ j\in S \quad \leftarrow 1-(1)(1)=0 \\ 0 & \text{if } i\notin S,\ j\notin S \quad \leftarrow 1-(-1)(-1)=0 \\ 2 & \text{if } ij\in\delta(S) \quad \leftarrow 1-(1)(-1)=1+1=2 \end{cases}$$

$$\Rightarrow \sum_{ij\in\delta(S)} w(ij) = \sum_{ij\in E} w(ij)\left(\frac{1-x_ix_j}{2}\right)$$ $\square$ Claim

Idea: replace $x_1,\dots,x_n\in\{\pm1\}^n$ with unit vectors $v_1,\dots,v_n\in\mathbb{R}^n$

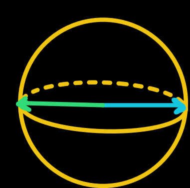
$$GW(G) = \max\left\{ \sum_{ij\in E} w(ij)\left(\frac{1-\langle v_i,v_j\rangle}{2}\right) : v_1,\dots,v_n\in\mathbb{R}^n,\ \|v_i\|=1\ \forall i \right\}$$

(can be solved with SDP - details next time)

Claim 1: $\mathrm{maxcut}(G)\leq GW(G)$

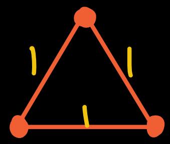
For $x=(x_1,\dots,x_n)\in\{\pm1\}^n$, take $v_i=(x_i,0,\dots,0)\in\mathbb{R}^n$ for $i=1,\dots,n$.

Then $\|v_i\|=1$ and $\langle v_i,v_j\rangle = x_ix_j$.

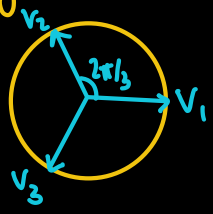


Note: Inequality can be strict!

Ex:



$G = K_3$



$$\langle v_i, v_j \rangle = \|v_i\| \cdot \|v_j\| \cos\left(\frac{2\pi}{3}\right) = -1/2$$

$$\sum_{ij \in E} w(ij)\left(\frac{1-\langle v_i, v_j\rangle}{2}\right) = \sum_{ij \in E} 1\left(\frac{3}{4}\right) = \frac{9}{4} > \overset{\text{maxcut}}{2}$$

But not <u>too</u> strict: