MA/AMA 514

Today: TU matrices from graphs (§8.3, §8.4)

From last time:

A matrix $A \in \mathbb{R}^{m \times n}$ is <u>totally unimodular</u> (TU) if every square submatrix has determinant $0, \pm 1$.

Ex: $A = \begin{pmatrix} 1 & \\ & 1 \end{pmatrix}$ $\quad A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}$ $\quad$ Non-ex: $A = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}$ $\quad$ Det(A) $= -2$?

<u>Prop</u>: Let $A \in \{0, \pm 1\}^{n \times m}$ be a matrix with at most one "1" and at most one "-1" in each column. Then A is TU.

<u>Cor</u>: The node-edge incidence matrix of any directed graph is totally unimodular.

incidence matrix $A \in \{0, \pm 1\}^{V \times E}$ of a directed graph:

$$A_{ve} = \begin{cases} 1 & \text{if } e \in \delta^{out}(v) \quad e = (v, w) \text{ for some } w \\ -1 & \text{if } e \in \delta^{in}(v) \quad e = (w, v) \text{ for some } w \\ 0 & \text{o.w.} \end{cases}$$

<u>Cor</u>: For $c: E \to \mathbb{Z}_{\geq 0}$, the max value of an s-t flow is acheived by an integer flow $f: E \to \mathbb{Z}_{\geq 0}$

(objective function $= \sum_{e \in \delta^{out}(s)} x_e - \sum_{e \in \delta^{in}(s)} x_e$, linear in x_e)

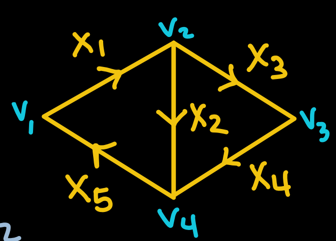
Cor: Let $D=(V,E)$ be a directed graph and $c:E\to\mathbb{Z}$, $d:E\to\mathbb{Z}$. If there exists a circulation $f:E\to\mathbb{R}$ with $d\le f\le c$, then there exists a circulation $f':E\to\mathbb{Z}$ with $d\le f'\le c$.

(Proof) The set of circulations on D is

$$P=\{x\in\mathbb{R}^E : Ax=0,\ d\le x\le c\}$$

where A is the incidence matrix of D.
Since A is TU and each coordinate is bounded, P is an integral polytope. Since $f\in P$, it is nonempty and so has a vertex, f', which is integer.

Ex:

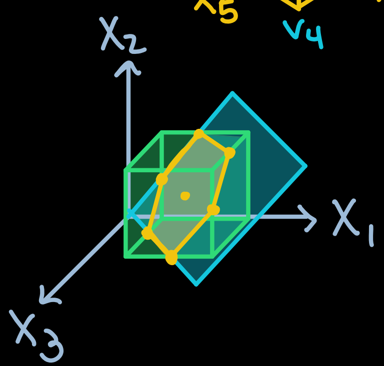


$$A_D=\begin{array}{c c} & \begin{array}{ccccc} a_1 & a_2 & a_3 & a_4 & a_5 \end{array} \\ \begin{array}{c} v_1 \\ v_2 \\ v_3 \\ v_4 \end{array} & \begin{pmatrix} 1 & 0 & 0 & 0 & -1 \\ -1 & 1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 \\ 0 & -1 & 0 & -1 & 1 \end{pmatrix} \end{array}$$

Circulation $\Longleftrightarrow$

$x_1=x_5 \quad x_1=x_2+x_3$

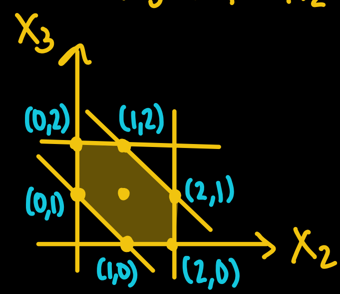
$x_3=x_4 \quad x_2+x_4=x_5$



Bounds: $1\le x_1\le 3$

$0\le x_2\le 2$

$0\le x_3\le 2$



Applications to interval scheduling

Call $A \in \{0,1\}^{m \times n}$ an interval matrix if the 1's in each row are consecutive (transposed version also used: consecutive 1's in each column).

Prop: If $A \in \{0,1\}^{m\times n}$ is an interval matrix then A is totally unimodular.

(Proof) Let M be a $k \times k$ submatrix of A^T.

$\Rightarrow M$ has consec. 1's.

$$\begin{pmatrix} 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{pmatrix}$$

Take $U \in \mathbb{R}^{k\times k}$ with $U_{ij} = \begin{cases} 1 & \text{if } j=i \\ -1 & \text{if } j=i-1 \\ 0 & \text{o.w} \end{cases}$ $\qquad U = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix}$

$\Rightarrow \det(U) = 1$ and UM has i^{th} row $m_i - m_{i-1}$ where $m_i = i^{th}$ row of M

Since M had consecutive ones, UM has at most one 1 and at most one -1 in each column $\Rightarrow UM$ totally unimod

$\Rightarrow \det(M) = \det(UM) \in \{0, \pm 1\}$

Ex:
$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & -1 & 1 \end{pmatrix}$$

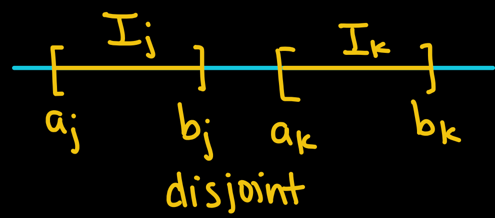
Cor: All vertices of $\{x \in \mathbb{R}^n : Ax \le 1,\ 0 \le x \le 1\}$ are integer.

Interval Scheduling

Input: intervals $I_1, \dots, I_n$ with values $c_1, \dots, c_n \in \mathbb{R}$

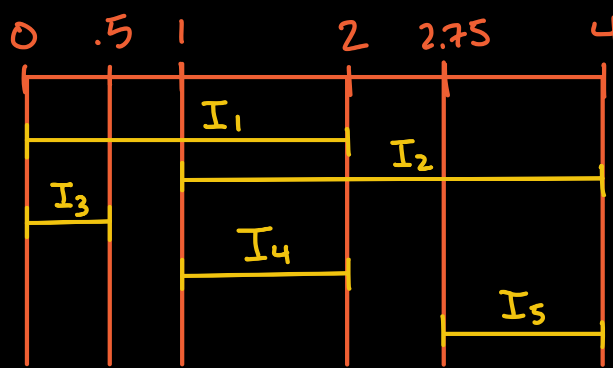
Goal: Select a disjoint subset of intervals $I_{i_1}, \dots, I_{i_k}$ to maximize value $c_{i_1} + \dots + c_{i_k}$

$$I_j = [a_j, b_j] \subseteq \mathbb{R}$$



Ex: $I_1 = [0,2]$, $I_2 = [1,4]$, $I_3 = [0,.5]$, $I_4 = [1,2]$, $I_5 = [2.75, 4]$

$c_1 = 4$ $\quad c_2 = 5$ $\quad c_3 = 1$ $\quad c_4 = 2$ $\quad c_5 = 3$



Some feas. sol. and values

$\{I_1, I_5\} \longrightarrow c_1 + c_5 = 7$

$\{I_2, I_3\} \longrightarrow c_2 + c_3 = 6$

$\{I_3, I_4, I_5\} \longrightarrow c_3 + c_4 + c_5 = 6$

Interval scheduling IP:

variables $x_1, \dots, x_n \in \{0,1\}$ $\quad x_j = \begin{cases} 1 \iff \text{pick } I_j \\ 0 \iff \text{don't pick } I_j \end{cases}$

How to encode disjointness of chosen intervals?

Let $t_1 < t_2 < \dots < t_m$ be the set of all endpoints $\bigcup_{j=1}^{n} \{a_j, b_j\}$.

Claim: $\{I_{i_1}, \dots, I_{i_k}\}$ are disjoint

$\iff$ for all $j = 1, \dots, m$, at most one I_{i_ℓ} contains t_j

(Proof) ($\Rightarrow$) $t_j \in I_{i_1} \cap I_{i_2} \Rightarrow I_{i_1}, I_{i_2}$ not disjoint!

($\Leftarrow$) If $I_1 \cap I_2$ not disjoint, $\max(I_{i_1} \cap I_{i_2}) = b_{i_1}$ or b_{i_2}
$= t_j$ for some j

Interval scheduling solved by

(IP) $\max\ c^T x$ s.t. $0 \le x \le 1,\ x \in \mathbb{Z}^n$

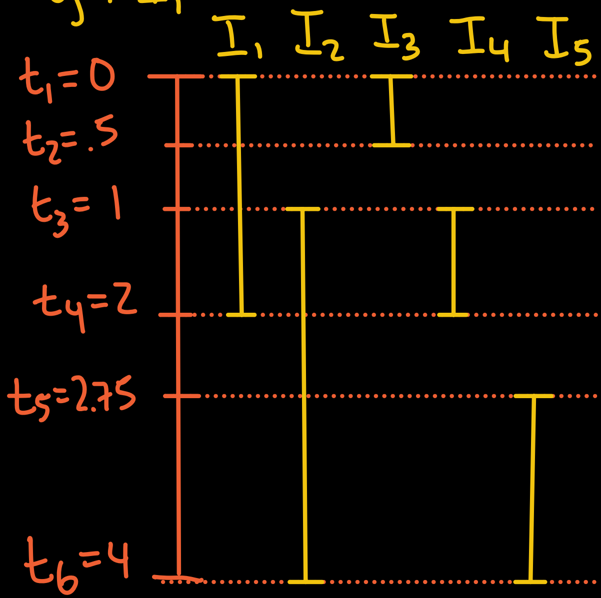
$$\sum_{i \text{ s.t. } t_j \in I_i} x_i \le 1 \quad \text{for all } j = 1, \dots, m$$

Encode as $Ax \le 1$

with $A \in \{0,1\}^{m \times n}$, $A_{ji} = \begin{cases} 1 & \text{if } t_j \in I_i \\ 0 & \text{if } t_j \notin I_i \end{cases}$

Ex (above)

$$A = \begin{array}{c} \\ t_1 \\ t_2 \\ t_3 \\ t_4 \\ t_5 \\ t_6 \end{array} \begin{array}{c} \begin{array}{ccccc} I_1 & I_2 & I_3 & I_4 & I_5 \end{array} \\ \begin{pmatrix} 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 \end{pmatrix} \end{array}$$



Note: since $t_1 < t_2 < \dots < t_m$, $t_j \in I_i, t_k \in I_i \Rightarrow t_\ell \in I_i\ \forall j \le \ell \le k$

$\Longrightarrow$ the ones in each column of A are consecutive

Cor: The LP relaxation

(LP) $\max\ c^T x$ s.t. $0 \le x \le 1,\ Ax \le 1$

solves interval scheduling (in polynomial time).

(Proof) Opt. val. attained at a vertex x^*

$\Rightarrow x^*$ integer $\Rightarrow x^* \in \{0,1\}^n$

$\Rightarrow$ take $I_{i_1}, \dots, I_{i_k}$ where $\{i_1, \dots, i_k\} = \{i : x_i^* = 1\}$

Remark: As noted at the end of lecture, the proof shows that it is enough to test the right endpts, i.e. to take $t_1 < \ldots < t_m$ to be the elements of $\{b_1, \ldots, b_n\}$.