MA/AMA 514

Today: TU matrices from graphs (§8.4)

A matrix $A \in \mathbb{R}^{m \times n}$ is <u>totally unimodular</u> (TU) if every square submatrix has determinant $0, \pm 1$.

Ex: $A = \begin{pmatrix} 1 & \\ & 1 \end{pmatrix}$ $A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}$ Non-ex: $A = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}$ $\det(A) = -2$?

A TU, $b \in \mathbb{Z}^m \Rightarrow P = \{x \in \mathbb{R}^n : Ax \le b\}$ is integral

$\Rightarrow$ the LP $\max\{c^T x : x \in P\}$ solves the IP $\max\{c^T x : x \in P \cap \mathbb{Z}^n\}$

<u>Cor</u>: If A is TU, $b \in \mathbb{Z}^m$ and $\ell, u \in \mathbb{Z}^n$ then each of the following polyhedra are integer:

1) $\{x \in \mathbb{R}^n : Ax \le b, x \ge 0\}$
2) $\{x \in \mathbb{R}^n : Ax \le b, \ell \le x \le u\}$
3) $\{x \in \mathbb{R}^n : Ax = b, x \ge 0\}$
4) $\{x \in \mathbb{R}^n : Ax = b, \ell \le x \le u\}$

Why?

1) $\underline{\begin{pmatrix} A \\ -I \end{pmatrix}} x \le \begin{pmatrix} b \\ 0 \end{pmatrix}$ 2) $\underline{\begin{pmatrix} A \\ -I \\ I \end{pmatrix}} x \le \begin{pmatrix} b \\ -\ell \\ u \end{pmatrix}$

3) $\begin{pmatrix} A \\ -A \\ -I \end{pmatrix} x \le \begin{pmatrix} b \\ -b \\ 0 \end{pmatrix}$ 4) $\begin{pmatrix} A \\ -A \\ -I \\ I \end{pmatrix} x \le \begin{pmatrix} b \\ -b \\ -\ell \\ u \end{pmatrix}$

If A is TU so are (1)-(4).

TU matrices from directed graphs

$G=(V,E)$ directed graph

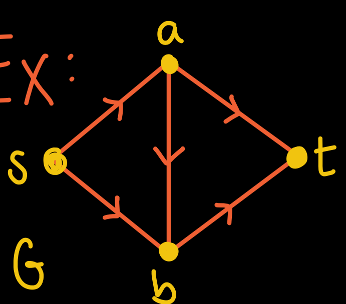
Encode into node edge incidence matrix $A \in \{0, \pm 1\}^{V \times E}$

$$A_{ve} = \begin{cases} 1 & \text{if } e \in \delta^{out}(v) \\ -1 & \text{if } e \in \delta^{in}(v) \\ 0 & \text{o.w.} \end{cases}$$

$e=(v,w)$ for some w $\quad v \xrightarrow{e} w$

$e=(w,v)$ for some w $\quad w \xrightarrow{e} v$

For $x \in \mathbb{R}^E$, $Ax \in \mathbb{R}^V$ with $(Ax)_v = \sum_{e \in \delta^{out}(v)} x_e - \sum_{e \in \delta^{in}(v)} x_e$ netflow out of v

Ex: 

$$A = \begin{array}{c} \\ s \\ a \\ b \\ t \end{array} \begin{array}{c} \begin{array}{ccccc} sa & sb & ab & at & bt \end{array} \\ \begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ -1 & 0 & 1 & 1 & 0 \\ 0 & -1 & -1 & 0 & 1 \\ 0 & 0 & 0 & -1 & -1 \end{pmatrix} \end{array} \qquad Ax = \begin{pmatrix} x_{sa} + x_{sb} \\ -x_{sa} + x_{ab} + x_{at} \\ -x_{sb} - x_{ab} + x_{bt} \\ -x_{at} - x_{bt} \end{pmatrix}$$

has one $+1$, one -1 in each column

Prop: The node-edge incidence matrix of any directed graph is totally unimodular.

Actually show something slightly stronger:

Prop: Let $A \in \{0, \pm 1\}^{n \times m}$ be a matrix with at most one "1" and at most one "-1" in each column. Then A is TU.

(Proof) Let M be a $k \times k$ submatrix of A. We induct on k.

($k=1$) M = an entry of $A \Rightarrow \det(M) = M = 0, \pm 1$ ✓

($k>1$) Case 1: M has a column of 0's $\Rightarrow \det(M) = 0$

Case 2: M has a column w/ exactly one nonzero entry

$M = \begin{pmatrix} \pm 1 & v^T \\ 0 & \\ \vdots & \tilde{M} \\ 0 & \end{pmatrix} \Rightarrow \det(M) = \pm \det(\tilde{M})$ where $\tilde{M}$ is a sq. submatrix of size $k-1 \Rightarrow \det(M) = 0, \pm 1$

Case 3: Every column of M has one 1 and one -1

$\Rightarrow$ the vector $(1, .., 1)$ belongs to the left kernel of M

$\Rightarrow \det(M) = 0$

Applications to Network Flow

$G = (V, E)$ directed graph, $s, t \in V$, $c: E \to \mathbb{Z}_{\geq 0}$

Equations defining s-t flows $f: E \to \mathbb{R}_{\geq 0}$, $f(e) = x_e$

$$\sum_{e \in \delta^{out}(v)} x_e = \sum_{e \in \delta^{in}(v)} x_e \quad \forall v \in V \setminus \{s, t\}$$

Cor: For integer edge capacities $c: E \to \mathbb{Z}_{\geq 0}$, the set of s-t flows equals the polytope

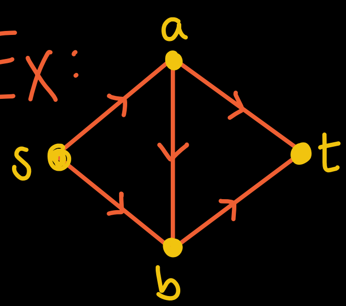
$$P_{flow} = \{ x \in \mathbb{R}^E : \tilde{A}x = 0,\ 0 \leq x \leq c \}$$

where $\tilde{A}$ is the $V \setminus \{s,t\} \times E$ submatrix of A.

Moreover, all vertices of P_{flow} are integer.

(Proof) A TU $\Rightarrow$ $\tilde{A}$ TU $\Rightarrow$ integer vertices

by Cor about $\{x \in \mathbb{R}^n : Ax = b,\ \ell \leq x \leq u\}$

Ex: 

$$A = \begin{array}{c} \\ s \\ a \\ b \\ t \end{array} \begin{array}{c} \begin{array}{ccccc} sa & sb & ab & at & bt \end{array} \\ \begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ -1 & 0 & 1 & 1 & 0 \\ 0 & -1 & -1 & 0 & 1 \\ 0 & 0 & 0 & -1 & -1 \end{pmatrix} \end{array} \qquad Ax = \begin{pmatrix} x_{sa} + x_{sb} \\ -x_{sa} + x_{ab} + x_{at} \\ -x_{sb} - x_{ab} + x_{bt} \\ -x_{at} - x_{bt} \end{pmatrix}$$

$\tilde{A}$ $\qquad\qquad$ $\tilde{A}x$

Cor: For $c: E \to \mathbb{R}_{\geq 0}$, the max value of an s-t flow is acheived by an integer flow $f: E \to \mathbb{Z}_{\geq 0}$

(objective function $= \sum_{e \in \delta^{out}(s)} x_e - \sum_{e \in \delta^{in}(s)} x_e$, linear in x_e)

Cor: Let $D = (V, E)$ be a directed graph and $c: E \to \mathbb{Z}$, $d: E \to \mathbb{Z}$. If there exists a circulation $f: E \to \mathbb{R}$ with $d \leq f \leq c$, then there exists a circulation $f': E \to \mathbb{Z}$ with $d \leq f' \leq c$.

(Proof) The set of circulations on D is

$$P = \{x \in \mathbb{R}^E : Ax = 0,\ d \le x \le c\}$$

where A is the incidence matrix of D.

Since A is TU and each coordinate is bounded, P is an integral polytope. Since $f \in P$, it is nonempty and so has a vertex, f', which is integer.