MA/AMA 514

Today: TU matrices from graphs (§8.3, 8.4)

Recall from last time:

A matrix $A \in \mathbb{R}^{m \times n}$ is <u>totally unimodular</u> (TU) if every square submatrix has determinant $0, \pm 1$.

Ex: $A = \begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix}$ $A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}$ Non-ex: $A = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}$ $\det(A) = -2!$

A TU, $b \in \mathbb{Z}^m \Rightarrow P = \{x \in \mathbb{R}^n : Ax \le b\}$ is integral

$\Rightarrow$ the LP $\max\{c^T x : x \in P\}$ solves the IP $\max\{c^T x : x \in P \cap \mathbb{Z}^n\}$

<u>Cor</u>: If A is TU, $b \in \mathbb{Z}^m$ and $\ell, u \in \mathbb{Z}^n$ then each of the following polyhedra are integer:

1) $\{x \in \mathbb{R}^n : Ax \le b,\ x \ge 0\}$
2) $\{x \in \mathbb{R}^n : Ax \le b,\ \ell \le x \le u\}$
3) $\{x \in \mathbb{R}^n : Ax = b,\ x \ge 0\}$
4) $\{x \in \mathbb{R}^n : Ax = b,\ \ell \le x \le u\}$

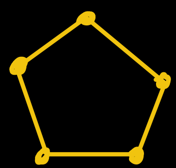
<u>Prop</u>: Let $A \in \{0, \pm 1\}^{n \times m}$ be a matrix with at most one "1" and at most one "-1" in each column. Then A is TU.

§8.3 Connections with bipartite graphs

The incidence matrix A_G of a graph $G=(V,E)$ is a $|V|\times|E|$ matrix with $(A_G)_{ve} = \begin{cases} 1 & \text{if } v\in e \\ 0 & \text{if } v\notin e \end{cases}$

<u>Thm 8.3</u> G is bipartite $\Leftrightarrow$ A_G is totally unimodular

(Proof) ($\Leftarrow$) G not bipartite $\Rightarrow$ G has an odd circuit.

 $\longleftrightarrow$ $\begin{pmatrix} 1&1&0&0&0 \\ 0&1&1&0&0 \\ 0&0&1&1&0 \\ 0&0&0&1&1 \\ 1&0&0&0&1 \end{pmatrix}$

subgraph $\leftrightarrow$ submatrix

Corresponding submatrix has determinant ± 2.

$\Rightarrow$ A_G not TU

($\Rightarrow$) Suppose G is bipartite with $V=U\uplus W$.

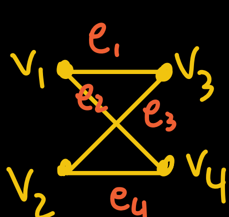
Every column of A has one "1" in rows indexed by U and one "1" in rows indexed by W.

$\begin{array}{c} \\ U \\ W \end{array}\begin{pmatrix} e \\ 1 \\ \hline 1 \end{pmatrix}$

Let A' denote the matrix obtained by scaling the rows of A indexed by W by -1. Then A' has exactly one "1" and one "-1" in each column.

$\Rightarrow$ A' is TU $\Rightarrow$ A_G is TU.

Ex:



$$\begin{array}{c|cccc|c} & e_1 & e_2 & e_3 & e_4 & \\ v_1 & 1 & 1 & 0 & 0 & \multirow{2}{*}{$]\,U$} \\ v_2 & 0 & 0 & 1 & 1 & \\ \hline v_3 & -1 & 0 & -1 & 0 & \multirow{2}{*}{$]\,W$} \\ v_4 & 0 & -1 & 0 & -1 & \end{array}$$

Cor: For any bipartite graph G, the polyhedra are integer:

$$\{x \in \mathbb{R}^E_{\geq 0} : Ax \leq \mathbb{1}_V\} \qquad \{y \in \mathbb{R}^V_{\geq 0} : y^T A \geq \mathbb{1}_E^T\}$$
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Cor: For bipartite graphs, $\{x \in \mathbb{R}^E_{\geq 0} : Ax \leq \mathbb{1}_V\} = \mathrm{Conv}\{\mathbb{1}_M : \underset{\text{matching}}{M \subseteq E}\}$

($\supseteq$) Clear.

($\subseteq$) Let z be a vertex of $\{x \in \mathbb{R}^E_{\geq 0} : Ax \leq \mathbb{1}_V\}$. Then $z \in \mathbb{Z}^E$ and $z_e \geq 0$, $\sum_{e \ni v} z_e \leq 1$ for all $v \in V$, $e \in E$.

In particular, $z \in \{0,1\}^E$, so $z = \mathbb{1}_M$ for some $M \subseteq E$.

The inequalities $\sum_{e \ni v} z_e \leq 1$ imply that M is a matching.

Cor: For bipartite graphs, $\alpha(G) = \max\{\sum_{e \in E} x_e : x \in \mathbb{R}^E_{\geq 0}, Ax \leq \mathbb{1}_V\}$

Similar results for $\nu(G)$, $\tau(G)$, $\rho(G)$.

Cor: For any bipartite graph G, $\nu(G) = \tau(G)$ and $\alpha(G) = \rho(G)$

(By linear programming duality!)

TU matrices from directed graphs

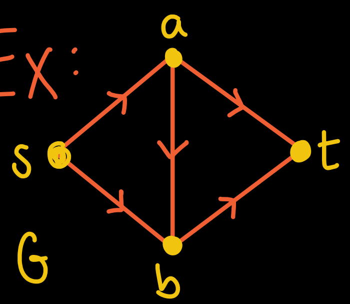
$G = (V, E)$ directed graph

Encode into node edge incidence matrix $A \in \{0, \pm 1\}^{V \times E}$

$$A_{ve} = \begin{cases} 1 & \text{if } e \in \delta^{out}(v) \quad e = (v,w) \text{ for some } w \quad v \xrightarrow{e} w \\ -1 & \text{if } e \in \delta^{in}(v) \quad e = (w,v) \text{ for some } w \quad w \xrightarrow{e} v \\ 0 & \text{o.w.} \end{cases}$$

For $x \in \mathbb{R}^E$, $Ax \in \mathbb{R}^V$ with $(Ax)_v = \sum_{e \in \delta^{out}(v)} x_e - \sum_{e \in \delta^{in}(v)} x_e$ ← netflow out of v

Ex:



$$A = \begin{array}{c} \\ s \\ a \\ b \\ t \end{array} \begin{array}{c} \begin{array}{ccccc} sa & sb & ab & at & bt \end{array} \\ \begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ -1 & 0 & 1 & 1 & 0 \\ 0 & -1 & -1 & 0 & 1 \\ 0 & 0 & 0 & -1 & -1 \end{pmatrix} \end{array} \qquad Ax = \begin{pmatrix} x_{sa} + x_{sb} \\ -x_{sa} + x_{ab} + x_{at} \\ -x_{sb} - x_{ab} + x_{bt} \\ -x_{at} - x_{bt} \end{pmatrix}$$

↳ has one +1, one -1 in each column

Prop: The node-edge incidence matrix of any directed graph is totally unimodular.

Applications to Network Flow

$G = (V, E)$ directed graph, $s, t \in V$, $c : E \to \mathbb{Z}_{\geq 0}$

Equations defining s-t flows $f : E \to \mathbb{R}_{\geq 0}$, $f(e) = x_e$

$$\sum_{e \in \delta^{out}(v)} x_e = \sum_{e \in \delta^{in}(v)} x_e \quad \forall v \in V \setminus \{s, t\}$$

Cor: For integer edge capacities $c : E \to \mathbb{Z}_{\geq 0}$, the set of s-t flows equals the polytope

$$P_{flow} = \{ x \in \mathbb{R}^E : \tilde{A}x = 0,\ 0 \leq x \leq c \}$$

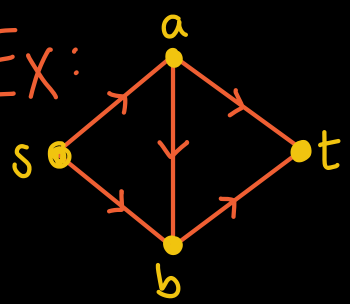
where $\tilde{A}$ is the $V \setminus \{s,t\} \times E$ submatrix of A.

Moreover, all vertices of P_{flow} are integer

Moreover, all vertices of P_{flow} are integer.

(Proof) A TU $\Rightarrow \tilde{A}$ TU $\Rightarrow$ integer vertices

by Cor about $\{x\in\mathbb{R}^n : Ax=b,\ \ell\le x\le u\}$

Ex: 

$$A = \begin{array}{cc} & \begin{array}{ccccc} sa & sb & ab & at & bt \end{array} \\ \begin{array}{c} s \\ a \\ b \\ t \end{array} & \begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ -1 & 0 & 1 & 1 & 0 \\ 0 & -1 & -1 & 0 & 1 \\ 0 & 0 & 0 & -1 & -1 \end{pmatrix} \end{array} \qquad Ax = \begin{pmatrix} x_{sa}+x_{sb} \\ -x_{sa}+x_{ab}+x_{at} \\ -x_{sb}-x_{ab}+x_{bt} \\ -x_{at}-x_{bt} \end{pmatrix}$$

$\tilde{A}$ $\qquad$ $\tilde{A}x$

<u>Cor</u>: For $c:E\to\mathbb{R}_{\ge 0}$, the max value of an s-t flow is acheived by an integer flow $f:E\to\mathbb{Z}_{\ge 0}$

(objective function $= \sum_{e\in\delta^{out}(s)} x_e - \sum_{e\in\delta^{in}(s)} x_e$, linear in x_e)