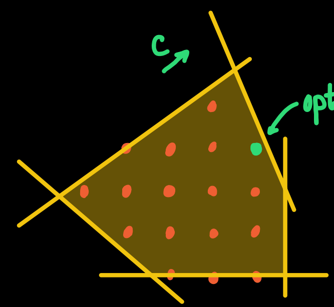
MA/AMA 514

Today: Integer programming (§8.1) & total unimodularity (8.2)

§8.1 Integer linear programming



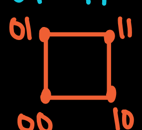
An integer program is a problem of the form
$$\max\{c^T x : Ax \leq b,\ x \in \mathbb{Z}^n\}$$
(maximize a linear function over the integer points $P \cap \mathbb{Z}^n$ of a polyhedron $P = \{x \in \mathbb{R}^n : Ax \leq b\}$)

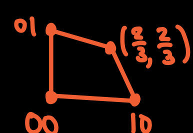
In general, these are NP-Hard to solve.

The associated LP gives an upper bound
$$\max\{c^T x : Ax \leq b,\ x \in \mathbb{Z}^n\} \leq \max\{c^T x : Ax \leq b\}$$

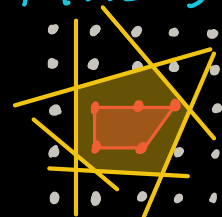
Usually this bound is strict!

<u>Def</u> A polytope is integral (or "an integer polytope") if all of its vertices have integer coordinates.

Ex's: 

$P_{matching}(G)$, $P_{perfect\,matching}(G)$ Non-ex: 

For integral polytopes P,
$$\max\{c^T x : x \in P\} = \max\{c^T x : x \in P \cap \mathbb{Z}^n\}$$
(Maximum is attained by some vertex of $P \Rightarrow$ in $P \cap \mathbb{Z}^n$)



Note: For any polytope P, $P \cap \mathbb{Z}^n$ is finite
$\Rightarrow P^I = \operatorname{conv}(P \cap \mathbb{Z}^n)$ also a polytope!

A polytope P is integral $\iff P^I = P$

Thm 3.7 For any bipartite graph G,

$\{x \in \mathbb{R}^E : x_e \geq 0 \ \forall e \in E, \ \sum_{e \ni v} x_e = 1 \ \forall v \in V\}$

is integral (and so equals $P_{\text{perfect matching}}(G)$).

Proved by you in HW4!

Totally Unimodular Matrices

A matrix $A \in \mathbb{R}^{m \times n}$ is totally unimodular (TU) if every square submatrix has determinant $0, \pm 1$.

Ex: $A = \begin{pmatrix} 1 & \\ & 1 \end{pmatrix}$ $A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}$ Non-ex: $A = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}$ $\det(A) = -2$!

Thm 8.1 Let $A \in \mathbb{R}^{m \times n}$ be totally unimodular and $b \in \mathbb{Z}^m$. Every vertex of the polyhedron $P = \{x \in \mathbb{R}^n : Ax \leq b\}$ is an integer vector.

(Proof) Let z be a vertex of P. Recall that A_z is the submatrix of A given by the subset of rows a_i for which $a_i^T z = b_i$ and that $\text{rank}(A_z) = n$.

$\Rightarrow A_z$ has a nonsingular $n \times n$ submatrix A'.

(That is, there are n linearly indep. rows $a_{i_1}, \ldots, a_{i_n}$ of A s.t. $a_{i_k}^T z = b_{i_k}$ for all $k = 1, \ldots, n$. Take $A' = \begin{pmatrix} - a_{i_1} - \\ \vdots \\ - a_{i_n} - \end{pmatrix}$, $b' = \begin{pmatrix} b_{i_1} \\ \vdots \\ b_{i_n} \end{pmatrix}$)

Then $A'z = b' \in \mathbb{Z}^n$ and $z = (A')^{-1} b'$.

A totally unimodular $\Rightarrow \det(A') = \pm 1 \Rightarrow (A')^{-1} \in \mathbb{Z}^{n \times n}$

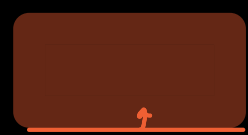
$\Rightarrow z = (A')^{-1} b' \in \mathbb{Z}^n$

<u>Lemma</u>: For $A \in \{0, \pm 1\}^{m \times n}$ the following operations do not change whether or not A is TU:

1) Multiplying a row/col by -1
2) Permuting rows/col
3) Adding a row or col with at most one ± 1 entry (and all other entries $= 0$)
4) Duplicating rows/cols
5) Transposing

You check!

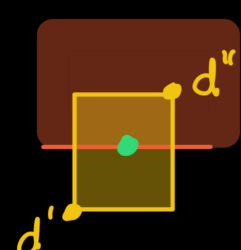
Note: Some polyhedra don't have vertices



Call a polyhedron integer if $\max\{c^T x : x \in P\}$ is attained by an integer vector whenever it is finite

<u>Cor 8.1a</u> For totally unimodular $A \in \mathbb{Z}^{m \times n}$ and $b \in \mathbb{Z}^m$ the polyhedron $P = \{x \in \mathbb{R}^n : Ax \leq b\}$ is integer.

(Proof) Suppose $x^* \in P$ attains $\max\{c^T x : x \in P\}$.



Choose $d', d'' \in \mathbb{Z}^n$ s.t. $d' \leq x^* \leq d''$ (coord-wise)

Consider $Q = \{x \in \mathbb{R}^n : Ax \leq b,\ d' \leq x \leq d''\}$

$$= \left\{ x \in \mathbb{R}^n : \begin{bmatrix} A \\ -I \\ I \end{bmatrix} x \leq \begin{bmatrix} b \\ -d' \\ d'' \end{bmatrix} \right\}$$

The matrix $\begin{bmatrix} A \\ -I \\ I \end{bmatrix}$ must also be totally unimodular and $\begin{bmatrix} b \\ d' \\ d'' \end{bmatrix} \in \mathbb{Z}^{m+2n} \Rightarrow Q$ is an integer polytope

$\Rightarrow \max\{c^T x : x \in Q\}$ acheived by some integer vertex $\tilde{x}$.

Since $Q \subseteq P$, $c^T\tilde{x} = \max\{c^T x : x \in Q\} \leq \max\{c^T x : x \in P\} = c^T x^*$

But $x^* \in Q$, so $c^T \tilde{x} \geq c^T x^*$. Together this gives $c^T\tilde{x} = c^T x^*$.

That is, $\tilde{x} \in P \cap \mathbb{Z}^n$ acheives $\max\{c^T x : x \in P\}$.

A matrix $A \in \mathbb{R}^{m \times n}$ is <u>unimodular</u> if $\operatorname{rank}(A) = m$ and every $m \times m$ submatrix has $\det = 0, \pm 1$

e.g. $A = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 1 & 2 \end{pmatrix}$ unimodular, not TU

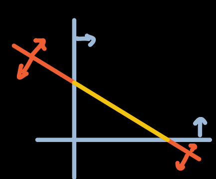
Note: $A \in \mathbb{R}^{m \times n}$ is TU $\Leftrightarrow$ $[I_m \; A]$ is unimodular

<u>Thm 8.2</u> Let $A \in \mathbb{Z}^{m \times n}$ with $\operatorname{rank}(A) = m$.

A unimodular $\Leftrightarrow$ $\forall b \in \mathbb{Z}^m$, $P_b = \{x \in \mathbb{R}^n_{\geq 0} : Ax = b\}$ is an integer polyhedron.

(Proof) ($\Rightarrow$) For A unimodular, $b \in \mathbb{Z}^m$,

$P_b = \{x \in \mathbb{R}^n_{\geq 0} : Ax = b\} = \left\{x \in \mathbb{R}^n : \begin{pmatrix} A \\ -A \\ -I \end{pmatrix} x \leq \begin{pmatrix} b \\ -b \\ 0 \end{pmatrix}\right\}$



Let $D = \begin{pmatrix} A \\ -A \\ -I \end{pmatrix}$, $f = \begin{pmatrix} b \\ -b \\ 0 \end{pmatrix}$ and let z be a vertex of P_b.

All the inequalities $a_i^T z \leq b_i$ and $-a_i^T z \leq -b_i$

hold with equality, so the matrix D_z contains all rows of A and $-A$. Since $\operatorname{rank}(A)=m$ the matrix D_z has an $n \times n$ submatrix B from m rows of A and $n-m$ rows of $-I$.

$\Rightarrow \det(B) =$ an $m \times m$ minor of $A \Rightarrow \det(B) = \pm 1$

If f' is the corresponding subvector of f, then $Bz = f'$ and $z = B^{-1}f \in \mathbb{Z}^n$

($\Leftarrow$) Suppose $\forall b \in \mathbb{Z}^m$, $P_b = \{x \in \mathbb{R}^n_{\geq 0} : Ax = b\}$ is integer.

Let B be an $m \times m$ submatrix of A with $\det(B) \neq 0$ w.l.o.g. take $B = (A_1, \dots, A_m)$ and let $v \in \mathbb{Z}^m$.

Claim: $B^{-1}v \in \mathbb{Z}^m$.

Take $u \in \mathbb{Z}^m$ s.t. $u + B^{-1}v > 0$ and take $z = u + B^{-1}v$ and $b = Bz = Bu + v \in \mathbb{Z}^m$.

Then $z' = \binom{z}{0} \in \mathbb{R}^n$ is a vertex of $P_b = \{x \in \mathbb{R}^n_{\geq 0} : Ax = b\}$.

First, note that $A = (B\ B')$ and $Az' = A\binom{z}{0} = Bz = b$.

There are n linearly indep. rows in $D = \begin{pmatrix} A \\ -A \\ -I \end{pmatrix}$ whose inequalities are tight: $\begin{pmatrix} B & B' \\ 0 & I_{n-m} \end{pmatrix}\binom{z}{0} = \binom{b}{0}$.

$\Rightarrow z \in \mathbb{Z}^m \Rightarrow B^{-1}v = z - u \in \mathbb{Z}^m$. $\square$ claim

Taking $v = e_1, \dots, e_n$ shows $B^{-1} \in \mathbb{Z}^{m \times m} \Rightarrow \det(B^{-1}) = \frac{1}{\det(B)} \in \mathbb{Z}$

Also $B \in \mathbb{Z}^{m \times m} \Rightarrow \det(B) \in \mathbb{Z} \Rightarrow \det(B) = \pm 1$. $\square$

Cor (Hoffman-Kruskal Thm) Let $A \in \mathbb{Z}^{m \times n}$.

A is totally unimodular

$\iff \forall b \in \mathbb{Z}^m,\ P = \{x \in \mathbb{R}^n_{\geq 0} : Ax \leq b\}$ is integer.

(See notes for proof)

Summary: For $A \in \mathbb{Z}^{m \times n}$

A unimodular $\iff \forall b \in \mathbb{Z}^m,\ \{x \in \mathbb{R}^n_{\geq 0} : Ax = b\}$ is integer

A is TU $\iff \forall b \in \mathbb{Z}^m,\ \{x \in \mathbb{R}^n_{\geq 0} : Ax \leq b\}$ is integer